Symmetries of QFTs in the Language of Path Integral

★ Traditional story (for a continuous symmetry)

A QFT has an infinitesimal sym. transf. $\phi(x) \to \phi(x) + \varepsilon(x)(\Delta\phi)(x)$

if (& def) its action & the path-integral measure are invariant under the infinitesimal transf.

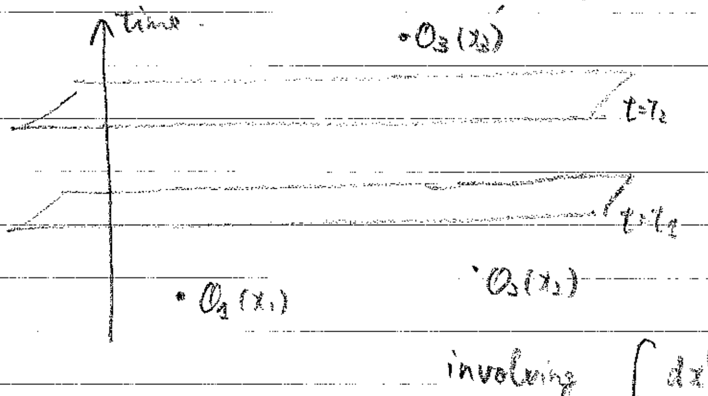
$\Rightarrow$ $\delta_\varepsilon S = -\int d^D x\, \varepsilon(x)(\partial_\mu J^\mu)(x)$ $\left(\exists J^\mu\right)$ when $\varepsilon(x)$ replaced by $\varepsilon(x)$

$\Rightarrow$ (Schwinger-Dyson eq. just like in hw I-2?)

$$
\begin{aligned}
&-i\langle\Omega|T\{(\partial_\mu J^\mu)(y)\, \mathcal{O}_1(x_1)\cdots \mathcal{O}_n(x_n)\}|\Omega\rangle \\
&+ \delta^D(y-x_1)\langle\Omega|T\{(\Delta\mathcal{O}_1)(x_1)\,\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\}|\Omega\rangle \\
&+\cdots+\delta^D(y-x_n)\langle\Omega|T\{\mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\cdots(\Delta\mathcal{O}_n)(x_n)\}|\Omega\rangle = 0
\end{aligned}
$$

$(\partial_\mu J^\mu)(y)$ is regarded zero so long as it stays away from other local operators.

$\Rightarrow$ $\left(\int d^d\vec{x}\, J^0(\vec{x})\right)_{\text{at } t=t_1}$ and $\left(\int d^d\vec{x}\, J^0(\vec{x})\right)_{\text{at } t=t_2}$ have the same effect



★ A quick lesson

Correlation functions $\langle\Omega|T\{\cdots\}|\Omega\rangle$ involving $\int_{\Sigma_{d=D-1}} dx^{\mu_1}\wedge dx^{\mu_2}\cdots\wedge dx^{\mu_d}\, J^{\mu_0}\,\epsilon_{\mu_0\mu_1\cdots\mu_d} =: Q(\Sigma_d)$

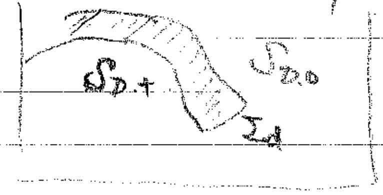
do not depend on continuous deformations of Σ_d.

(Σ_d: codim-1 submfd in the spacetime)

mfd: manifold

☆ A short-cut derivation (both continuous / discrete sym.)

Σ_d : codim-1 submfd in a $D=d+1$-dim spacetime.



sym transf. $\phi \to \phi'$

$$\begin{cases} S[\phi'] = S[\phi] \text{ if done everywhere} \\ \mathcal{D}\phi = \mathcal{D}\phi' \quad \text{———} (***) \end{cases}$$

Do the field redef. only in $\mathcal{S}_{D,+}$:

$$\int \mathcal{D}\phi \; e^{iS[\phi]} \, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) = \langle \Omega | T\{ \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \} | \Omega \rangle$$

$$\|$$

$$\int \underset{y \in \mathcal{S}_{D,0}}{\mathcal{D}\phi(y)} \; \underset{x_1 \in \mathcal{S}_{D,0}}{\mathcal{O}_1(x_1)} \cdots \underset{y \in \mathcal{S}_{D,+}}{\mathcal{D}\phi'(y)} \cdots \underset{x_n \in \mathcal{S}_{D,+}}{\mathcal{O}'_n(x_n)} \; e^{iS[\phi,\phi']}$$

just rename $\phi(z)$ with $z \in \mathcal{S}_{D,+}$ as $\phi'(z)$

$$\|$$

field redef. (change integration coordinates) in $\mathcal{S}_{D,+}$

$$\int \underset{y, x_1 \in \mathcal{S}_{D,0}}{\mathcal{D}\phi(y) \, \mathcal{O}_1(x_1)} \cdots \underset{\substack{y \in \mathcal{S}_{D,+} \\ (\text{use } (***))}}{\mathcal{D}\phi(y)} \cdots \mathcal{O}'_n(x_n) \; e^{iS[\phi]} \, e^{-i\alpha Q(\Sigma_d)}$$

$$S[\phi \text{ in } \mathcal{S}_{D,0}, \phi' \text{ in } \mathcal{S}_{D,+}] =: S[\phi] - \underset{\text{def.}}{\alpha Q(\Sigma_d)}$$

$$\|$$

$$\boxed{\langle \Omega | T\{ \underset{x\text{'s in } \mathcal{S}_{D,0}}{\mathcal{O}_1(x_1) \cdots} \; \underset{x\text{'s in } \mathcal{S}_{D,+}}{\cdots \mathcal{O}'_n(x_n)} \; e^{-i\alpha Q(\Sigma_d)} \} | \Omega \rangle}$$

The correlation function remains the same
when the sym transf is applied to the local operators in $\mathcal{S}_{D,+}$
and the opposite sym transf. operator $e^{-i\alpha Q(\Sigma_d)}$ is also inserted.

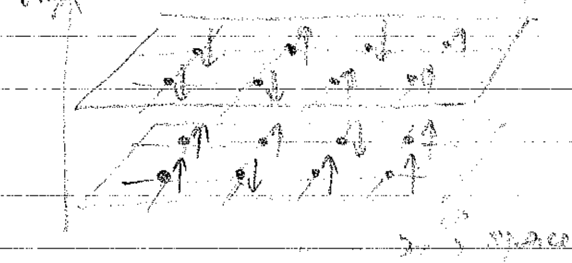
Example $\quad \mathcal{L} = \bar{\psi} i \gamma^\mu (\partial_\mu \psi) \qquad \psi' = e^{i\alpha}\psi \quad \bar{\psi}' = e^{-i\alpha}\bar{\psi}$

$$S[\psi \text{ in } \mathcal{S}_{D,0} ; \psi' \text{ in } \mathcal{S}_{D,+}] = S[\psi] - \alpha \int_{\Sigma_d} d^d x \, (\bar{\psi}\gamma^n\psi)(x)$$

(derivative on the step fcn is the δ-fcn supported on Σ_d)

This idea works also for a set-up

with a discrete sym. & latticized space-time.



☆ An insertion of sym. transf. operator on $\Sigma_{d=D-1}$

⟺ A gauge-field background that is flat everywhere except Σ_d.

(e.g. $A_0(x) = \alpha \cdot \delta(t-t_0)$) flat $\overset{\text{def}}{\Longleftrightarrow}$ field strength is zero → trivial holonomy

better characterization for a discrete sym.

☆ Full gauging goes beyond allowing (taking superposition of) all kinds of sym. transf. operator insertion.

[• non-flat gauge-field config. is also integrated over
• large gauge coupling allows highly non-flat config.]

☆ Higher-dim analogue (p-form symmetry $p > 0$)

Ex. { the Maxwell theory (photon without $e^{\pm}$)
pure Yang-Mills theory (gluon without quarks)

$\partial_\mu F^{\mu\nu} = 0$ (by eq. of motion) ⇒ topological charge operator $\int_{\Sigma_2} dx^\mu dx^\nu F^{\kappa\lambda} \varepsilon_{\mu\nu\kappa\lambda} = Q(\Sigma_2)$

more generally

$$Q(\Sigma_{d-1}) = \int_{\Sigma_{d-1}} (*F)$$

F: 2-form

$(*F)$: $(d+1)-2 = (d-1)$-form.

In the Maxwell theory, the objects charged under $Q(\Sigma_{d-1}) = \int (*F)$ are Wilson loops / lines.

$$W_C^{(q)} := e^{iq\int_C A(s)\,ds} = e^{i\int (q\delta^{(d)}(C))\wedge A}$$

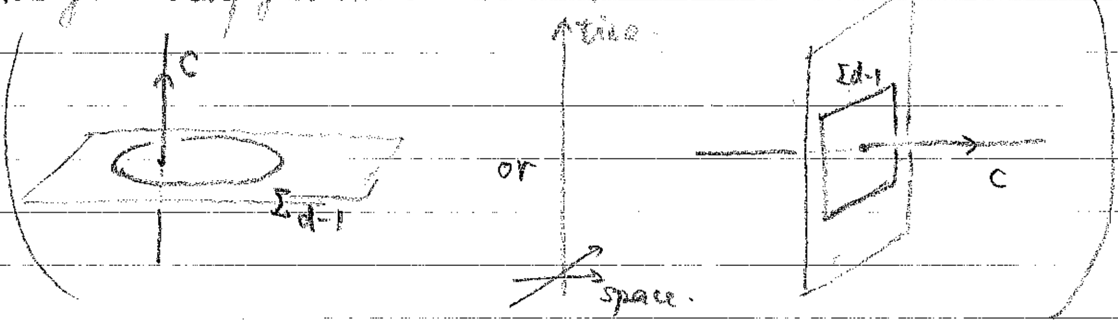
$\delta^{(d)}(C)$: d-form whose support is C (δ-fcn like)

(gauge-inv. if C is a loop, or C is a line that comes in from and goes out to infinity (where we do not do gauge transf.))

$$\langle \cdots W_C^{(q)} \cdots \rangle = \int \mathcal{D}\phi\,\mathcal{D}\psi\,\mathcal{D}A \left(e^{iS[\phi,\psi,A]} \cdots e^{i\int (q\delta^{(d)}(C))\wedge A} \cdots \right) \Big/ Z.$$

so insertion of $W_C^{(q)}$ is the same as implementing a source of the gauge field.

imagine configuration like this



- The correlation fcns involving $Q(\Sigma_{d-1})$ & local op's & Wilson (loops / lines) are topological with respect to Σ_{d-1}. (use $\partial_\mu F^{\mu\nu} = 0 \Leftrightarrow d(*F) = 0$)
- $(*F)$ ~ electric field in $Q(\Sigma_{d-1})$ measures the electric charge of the source $q\delta^{(d)}(C)$ in $W_C^{(q)}$

$W_C^{(R)}$

In a pure Yang-Mills theory (gauge group $SU(N)$)

Wilson loops/lines are $\mathrm{Tr}_R\left[P\left(\exp\left(i\int_C A\right)\right)\right]$. R: a repr. of $SU(N)$ (fund. repr. (dim $=N$), rk=2 anti-sym repr (dim $=\frac{N(N-1)}{2}$), rk-3 anti-sym (dim $=\binom{N}{3}$), ⋮)

$P\left(\exp\left(i\int_C A\right)\right)$:= path-ordered exponential.

parametrize the curve C by $s\in\mathbb{R}$.

$(x^\mu(s)\in C)$

Instead of the naive/simple exponential

$$\exp\left(i\int_C A\right) = 1 + i\int_C ds\,\frac{dx^\mu}{ds}A_\mu(x(s)) + \frac{i^2}{2!}\left(\int_C ds\,\frac{dx^\mu}{ds}A_\mu(x(s))\right)^2 + \cdots$$

it is defined by

$$P\exp\left(i\int_C A\right) := 1 + i\int_C ds\,\frac{dx^\mu}{ds}A_\mu(x(s)) + i^2\int ds_1\int^{s_1\geq s_2} ds_2\,\frac{dx^{\mu_1}}{ds_1}\frac{dx^{\mu_2}}{ds_2}A_{\mu_1}(x(s_1))A_{\mu_2}(x(s_2))$$

path-ordered ⟶ do not commute

⇒ Under the gauge transformation by $g(x)\in SU(N)$,

$$P\left(\exp\left(i\int_C A\right)\right) \to g(x(s_{max}))\,P\left(\exp\left(i\int_C A\right)\right)\,g(x(s_{min}))^{-1}$$

Wilson loops/lines are gauge inv. for $^\forall g(x(s_{max})) = g(x(s_{min}))$ as we take the trace.

The charge operators: $\mathbb{Z}_N$-valued ($\mathbb{Z}_N$: center of the group $SU(N)$)

$Q(\Sigma_{d-1})$

⟺ gauge transformation that is flat in $(\mathbb{R}^{d,1}\setminus\Sigma_{d-1})$

that has the twist by a center element $\left(g(x(s_{max}))\in g(x(s_{min}))\times\mathbb{Z}_N\right)$

around the codim-2 defect Σ_{d-1}.

$\exp\left(\alpha\int_{\Sigma_{d-1}}(*F)\right)$ is too naive for $Q(\Sigma_{d-1})$ ($\partial_\mu F^{\mu\nu}=0$ but $\partial_\mu F^{\mu\nu}\neq 0$)

The Wilson loops are changed under the change of $Q^n(\Sigma_{d-1})$ $(n \in \mathbb{Z}_N)$

because $W_C^{(R)} = \mathrm{Tr}_R\left[P\left(\exp\left(i\int_C A\right)\right)\right] \Rightarrow \mathrm{Tr}_R\left[g(x(s_{max}))\, P\left(\exp\left(i\int_C A\right)\right) g(x(s_{min}))^{-1}\right]$

$$= \mathrm{tr}_R\left[\left(g(s_{min})^{-1} g(s_{max})\right) P\left(\exp\left(i\int_C A\right)\right)\right] = R(n) \times W_C^{(R)}$$

($g(s_{min})^{-1} g(s_{max})$ $\in$ center $\mathbb{Z}_N \subset SU(N)$)

$$\mathbb{Z}_N \ni n \leftrightarrow e^{2\pi i n/N} \mathbb{1}_{N\times N} \in SU(N) \longmapsto e^{2\pi i (nm/N)}$$

(rk-m anti-sym repr)

$$\boxed{Q^n(\Sigma_{d-1})\, W_C^{(R_m)} = e^{2\pi i\, nm/N}\, W_C^{(R_m)}}$$

Wilson loops of $SU(N)$ pure Yang-Mills theory

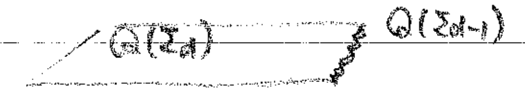
used as an order parameter of the confining/deconfining phase transition.

In QED or QCD (instead of the Maxwell theory or the pure YM theory)

$Q(\Sigma_{d-1})$ is not a topological operator.

QED: $\partial_\mu F^{\mu\nu} \neq 0$ QCD gauge transf twisted by a center : non-trivial also on matter fields.

Instead the combination



is a topological operator of QED