Supplementary notes: cluster decomposition and interference from non-connected graphs

To digest discussions in QFT associated with such keywords as connected Feynman diagrams and the cluster decomposition principle, think of a scattering process whose initial state α and final state β are

$$\alpha = \mu^-(\vec{p}_1) + \mu^+(\vec{p}_2), \tag{1}$$

$$\beta = e^-(\vec{p}_3) + \bar{\nu}_e(\vec{p}_4) + \nu_\mu(\vec{p}_5) + e^+(\vec{p}_6) + \nu_e(\vec{p}_7) + \bar{\nu}_\mu(\vec{p}_8). \tag{2}$$

Suppose that there are three conserved charges: the muon number, electron number and electric charge,

$$\#\mu^- + \#\nu_\mu - \#\mu^+ - \#\bar{\nu}_\mu,$$
$$\#e^- + \#\nu_e - \#e^+ - \#\bar{\nu}_e,$$
$$\#e^- + \#\mu^- - \#e^+ - \#\mu^+.$$

Then the only charge-conserving subprocesses are $\mu^- \to (e^- + \bar{\nu}_e + \nu_\mu) =: \beta^-$ and $\mu^+ \to (e^+ + \nu_e + \bar{\nu}_\mu) =: \beta^+$. Using

$$S_{\beta^-,\mu^-} =: (2\pi)^4 \delta^4(p_{\beta^-} - p_{\mu^-}) i\mathcal{M}^C_{\beta^-,\mu^-}, \tag{3}$$

$$S_{\beta^+,\mu^+} =: (2\pi)^4 \delta^4(p_{\beta^+} - p_{\mu^+}) i\mathcal{M}^C_{\beta^+,\mu^+}, \tag{4}$$

we define $\mathcal{M}^C_{\beta,\alpha}$ as follows:

$$S_{\beta,\alpha} =: (2\pi)^4 \delta^4(p_{\beta^-} - p_{\mu^-})(2\pi)^4 \delta^4(p_{\beta^+} - p_{\mu^+}) i\mathcal{M}^C_{\beta^-,\mu^-} i\mathcal{M}^C_{\beta^+,\mu^+} + (2\pi)^4 \delta^4(p_\beta - p_\alpha) i\mathcal{M}^C_{\beta,\alpha}. \tag{5}$$

S. Weinberg's QFT textbook (vol. I, §4.3) explains the following logical inference. Suppose that $\mathcal{M}^C_{\beta,\alpha}(p_{1\cdots 8})$ as a function of eight momenta $p_{1,\cdots,8}$ does not contain a delta-function singularity; then the contribution from the $\mathcal{M}^C_{\beta,\alpha}$ term is very small when doing a theoretical computation for μ^- decay at Laboratory A and μ^+ decay at Laboratory B that are far from each other; it thus follows that the μ^- decay at Lab A and the μ^+ decay at Lab B take place independently from each other. We do believe, based on experience in experiments, that there is such independence between the experimental results carried out simultaneously but separated from each other in space. So, this empirical fact sets a constraint on theories that $\mathcal{M}^C_{\beta,\alpha}$ defined through the procedure above would not have delta-function singularity of the momenta involved in β and α. This constraint is called the cluster decomposition principle.

When we compute the S-matrix perturbatively (i.e., by using Feynman diagrams), the amplitudes for the graphs with two connected components (one for $\mu^- \to \beta^-$ and the other for $\mu^+ \to \beta^+$) give rise to contributions to $\mathcal{M}^C_{\beta^-,\mu^-}\mathcal{M}^C_{\beta^+,\mu^+}$. So $\mathcal{M}^C_{\beta,\alpha}$ only contains perturbative amplitudes for the graphs that are connected. Because the perturbative contributions to $\mathcal{M}^C_{\beta,\alpha}$ (form the connected graphs) do not contain delta-function singularity of the momenta $p_{1,\cdots,8}$, we can confirm that the cluster decomposition principle is satisfied as long as perturbative contributions are concerned.

Think of the same process $\alpha = \mu^- + \mu^+ \to \beta$, now, at one laboratory. Imagine a $\mu^+ + \mu^-$ collider experiment; when the $\mu^\pm$ beams are boosted enough, their lifetime becomes long enough to do such an experiment. So, we think of boosted beams so that large fraction of the particles $\mu^\pm$ in the beams reach the collision point inside the detector without decaying; the $\mu^\pm$ particles either (ii) collide/scatter against each other, (i) decay independently from each other, accidentally around the collision point, or (iii) pass through the detector without doing anything. Let us try to sort things out to see how $\mathcal{M}^C_{\beta^-,\mu^-}$, $\mathcal{M}^C_{\beta^+,\mu^+}$ and $\mathcal{M}^C_{\beta,\alpha}$ determine theoretically[1] the processes (i) and (ii). We start off by dealing with the processes (i) and (ii) independently without thinking about interference between them.

Suppose that the beam 1 (μ^-) [resp. beam 2 (μ^+)] proceeds in the $(1, 0, -\theta)$ direction [resp. $(-1, 0, -\theta)$ directon] in the $(z, x, y) \in \mathbb{R}^3$ coordinate system; $0 < \theta \ll 1$. The both the beam 1 and beam 2 have width w in the x direction, and the beam 1 [resp. beam 2] has height h_1 [resp. h_2] in the y direction. So, the cross section of the beam 1 [resp. beam 2] has area $A_1 = wh_1$ [resp. $A_2 = wh_2$]. Let its flux be Φ_1 [resp. Φ_2]. We assume that $\mu^\pm$ in the two beams are both relativistic, so we use $v_i = c = 1$. Now, in this set-up, the rate (event counts per unit time) of the $\mu^- + \mu^+$ scattering (with a cross section $d\sigma$) is

$$(\Phi_1 A_1)(\Phi_2 A_2)\frac{1}{w\theta}d\sigma; \tag{6}$$

on the other hand,

$$(\Phi_1 A_1)(\Phi_2 A_2)\frac{1}{2w\theta}\Gamma^2(\Delta t_{\rm rslv})(\Delta \ell_{\rm rslv})^3 \tag{7}$$

[1]An idea of μ^-–μ^+ collider has been considered by experts to probe physics beyond the Standard Model by using $\mathcal{M}_{\beta\alpha}$ with $\alpha = \mu^- + \nu^+$, but with the final state totally different from β here. Discussion here, on the other hand, in not intended for applications to such practical purposes, but to address theoretical questions such as how to deal with momentum delta functions and interference. In practice, it is not easy to detect neutrinos in a collider detectors, and determine the interaction point of a neutrino final state precisely.

is the rate of μ^- decay and μ^+ decay processes taking place at unresolvablly close distance in space and time [if it is hard to derive (6, 7), read the additional notes at the end]. Here, Γ is the decay rate of μ^- and μ^+, and $\Delta t_{\rm rslv}$ [resp. $(\Delta \ell_{\rm rslv})$] is the resolution in the time [resp. the position in the detector] a decay took place. We assume that $\Delta \ell_{\rm rslv} \ll h_i, w$ for simplicity.[2]

We have obtained a factor

$$\frac{\Gamma^2}{2}(\Delta t_{\rm rslv})(\Delta \ell_{\rm rslv})^3 + \int d\sigma \tag{8}$$

multiplied to a common factor. Instead of looking at only the total rate of the process $\alpha \to \beta$, we may also have a look at differential cross section, by resolving the momentum configuration of the final state particles. The momenta $\vec{p}_{3\cdots 8}$ of the two decay processes are subject to four extra momentum conservations relatively to the momenta of a scattering process. So, we may split the region of integration

$$\frac{1}{(2E_1)(2E_2)2}\int d\Pi_{\beta^-} d\Pi_{\beta^+} (2\pi)^4 \delta^4(p_\beta - (p_1 + p_2)), \tag{9}$$

where

$$\int d\Pi_{\beta^\mp} = \int \frac{d^3 p_{3/6}}{(2\pi)^3}\frac{1}{2E_{3/6}}\frac{d^3 p_{4/7}}{(2\pi)^3}\frac{1}{2E_{4/7}}\frac{d^3 p_{5/8}}{(2\pi)^3}\frac{1}{2E_{5/8}}, \tag{10}$$

into the region where the extra momentum conservation $\delta^4(p_{\beta^-} - p_1)$ holds within the accuracy of measurements, and the other region. Only the second term in (8) contributes to the latter region, while both terms in (8) do in the former (on-shell) region. There is no theoretical subtlety in the latter region.

Let us focus on the former region, but now with interference taken into account. Without relying on classical intution, and without ignoring the fact that the intial state particle is not absolutely stable, the formula of the scattering cross section near the $p_{\beta^-} \sim p_1$ region should be

$$\begin{aligned}\sigma_{p_{\beta^-}\sim p_1} &= \frac{1}{(2E_1)(2E_2)2}\int_{p_{\beta^-}\sim p_1} d\Pi_{\beta^-} d\Pi_{\beta^+} (2\pi)^4 \delta^4(p_\beta - (p_1 + p_2)) \\ &\qquad \left| \mathcal{M}^C_{\beta,\alpha} + (2\pi)^4 \delta^4(p_{\beta^-} - p_1) i \mathcal{M}^C_{\beta^-,\mu^-} \mathcal{M}^C_{\beta^+,\mu^+} \right|^2;\end{aligned} \tag{11}$$

The $|\mathcal{M}^C_{\beta\alpha}|^2$ contribution on the right hand side of (11) yields $\int d\sigma$ in (8).

[2]This is absolutely incorrect in the $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ decay process! We still do so, because the discussion here is not intended for practical applications.

The $|\mathcal{M}^C_{\beta^-\mu^-}\mathcal{M}^C_{\beta^+\mu^+}|^2$ contribution on the right hand side of (11) is

$$\frac{1}{2E_1}\int d\Pi_{\beta^-}(2\pi)^4\delta^4(p_{\beta^-}-p_1)|\mathcal{M}^C_{\beta^-\mu^-}|^2 \qquad (12)$$
$$\frac{1}{2E_2}\int d\Pi_{\beta^+}(2\pi)^4\delta^4(p_{\beta^+}-p_2)|\mathcal{M}^C_{\beta^+\mu^+}|^2 \;\times\; \frac{(2\pi)^4\delta^4(p_{\beta^-}-p_1)}{2}.$$

It might look as if we have enocountered a new problem of how to deal with the square of a delta function, but that is not right in fact. Already at the moment we try to derive the formula of the decay rate, $\Gamma = (2E)^{-1}\int d\Pi(2\pi)^4\delta^4(p_{\rm out}-p_{\rm in})|\mathcal{M}|^2$, we had to struggle with the question how to interpret the square of a delta function. See Weinberg's QFT textbook vol. I §3.4. There is an argument there that one set of $(2\pi)^4\delta(p_{\beta^-}-p_1)$ should be used to restrict the final state phase space, and the other set of $(2\pi)^4\delta(p_{\beta^-}-p_1)$ should be interpreted as $(\Delta t)\times$ volume; we obtain the rate Γ by factoring out (Δt). For more explanation (e.g., on volume), see Weinberg's textbook. In the present context, Δt should be used as the time resolution $\Delta t_{\rm rslv}$ between the two decay. If we allow ourselves to interpret volume here as $(\Delta\ell_{\rm rslv})^3$, which does not look terribly bad, the first term of (8) is reproduced.

It is now straightforward to compute the interference term. It is

$$\frac{1}{(2E_1)(2E_2)2}\int d\Pi_{\beta^-}d\Pi_{\beta^+}\;(2\pi)^4\delta^4(p_{\beta^+}-p_2)(2\pi)^4\delta^4(p_{\beta^-}-p_1) \qquad (13)$$
$$\left[\mathcal{M}^{C*}_{\beta,\alpha}\;(i\mathcal{M}^C_{\beta^-,\mu^-}\mathcal{M}^C_{\beta^+,\mu^+})+\text{h.c.}\right].$$

While the two terms $\Gamma^2/2(\Delta t_{\rm rslv})(\Delta\ell_{\rm rslv})^3$ and $\int d\sigma$ are acompanied by the parameters that are determined by experimental details (time and spacial resolution in the point of decay events, fraction of the angles around the collision point covered by detectors and the resolution of measurement of the energy and momenta of outgoing particles), this interference term is determined entirely by the theory parameters.

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Here is a reasoning for (6) and (7). The beam 1 and the beam 2 overlap over the region of width w in the x direction; the overlap region in the (z,y) plane has a shape of a parallelogram; its edges along the beam 1 direction and the beam 2 direction, respectively, are $\ell_1 = h_2/(2\theta)$ and $\ell_2 = h_1/(2\theta)$. The area of the parallelogram in the (z,y) plane is $\ell_1\ell_2(2\theta) = h_1h_2/(2\theta)$, and the volume of the beam overlap region is $wh_1h_2/(2\theta)$ ($\theta^2 \ll 1$ is ignored).

Let us begin with (6). The number of μ^- particles coming into the beam overlap region per unit time is $\Phi_1 A_1$. The probability of each of those μ^- to scatter against the μ^+ particles in the beam 2 before moving out of the overalp region is $d\sigma \times \Phi_2 \ell_1$. So, the number of $\mu^- + \mu^+$ scattering events per unit time is

$$(\Phi_1 A_1) d\sigma \Phi_2 \ell_1 = (\Phi_1 A_1) d\sigma \Phi_2 \frac{h_2}{(2\theta)} = (\Phi_1 A_1) d\sigma (\Phi_2 A_2) \frac{1}{2\theta w}. \tag{14}$$

Let us turn to (7) next. Think of the number of events within a time duration T of the decay of μ^- particles in beam 1 taking place in the beam overlap region. It is $\Phi_1 A_1 \ell_1 \Gamma T$. The probability that one μ^+ in the beam 2 decays within the time difference $(\Delta t_{\rm rslv})$ and space difference $(\Delta \ell_{\rm rslv})^3$ from an event of μ^- decay in the beam overlap region is $\Phi_2 (\Delta \ell_{\rm rslv})^3 \Gamma (\Delta t_{\rm rslv})$. So, within the time duration T, the expected number of events of simultaneous decays of μ^- and μ^+ is given by

$$\begin{aligned} \Phi_1 A_1 \ell_1 \Gamma T \times \Phi_2 (\Delta \ell_{\rm rslv})^3 \Gamma (\Delta t_{\rm rslv}) &= \Phi_1 A_1 \Gamma^2 T \Phi_2 A_2 (\Delta \ell_{\rm rslv})^3 (\Delta t_{\rm rslv}) \frac{\ell_1}{w h_2} \\ &= T \times (\Phi_1 A_1) \Gamma^2 (\Phi_2 A_2)\ (\Delta \ell_{\rm rslv})^3 (\Delta t_{\rm rslv}) \frac{1}{w 2\theta}. \end{aligned} \tag{15}$$