Theory of Elementary Particles homework IV

- submission via the UTOL (online learning management system). Multiple files can be uploaded multiple times until the deadline (in early August).
- We request that the file name includes the problem number, II-1***.pdf or ****-IV-2-IX-1.jpeg, so the file names can be used to double check who did what at the end of grading; thanks for coorporation! The UTOL shows who had submitted the file (student ID and name), so the file name will not have to contain your name or ID number.
- Reports do not have to be neatly written or type-set just for the reason that the reports have to be readable for me.
- Pick up any problems that are suitable for your study. **You are not expected to work on all of them!**
- A sample solution has been prepared and is made available in the form of a PDF file on the problems with "$\star$" (e.g., III-1, III-5). The PDF is posted to you through the UTOL in return for an early submission of a report on that problem during the semester.

1. **A Consequence of QED Ward Identity** [B] $\star$

   Wavefunction renormalization constant Z_2 of a Dirac fermion with a pole mass $p^2 = m^2$ in QED is given by

$$1 + \delta_{Z2} =: Z_2 = \left. \frac{(1-A)}{(1-A)^2 + 2(A-1)p^2 \frac{\partial A}{\partial p^2} - 2(M+B)\frac{\partial B}{\partial p^2}} \right|_{p^2=m^2}, \tag{1}$$

   where $A(p^2, M^2)$ and $B(p^2, M^2)$ characterize the fermion self-energy

$$-i\Sigma(p, M) := -i\left[A(p^2, M^2)\not{p} + B(p^2, M^2)\right]. \tag{2}$$

   At 1-loop ($\mathcal{O}(e^2)$) level, the fermion self-energy (Figure 1 (a)) is given by

$$\begin{aligned} -i\Sigma^{(1)}(p, M) &= \frac{-i(eQ)^2}{16\pi^2}\int_0^1 dx\left[-2(1-x)\not{p} + 4M\right]\ln\left(\frac{(1-x)\Lambda^2 + xM^2 - x(1-x)p^2}{xM^2 - x(1-x)p^2}\right), \\ &=: -i\left[A^{(1)}(p^2, M^2)\,\not{p} + B^{(1)}(p^2, M^2)\right] \end{aligned} \tag{3}$$

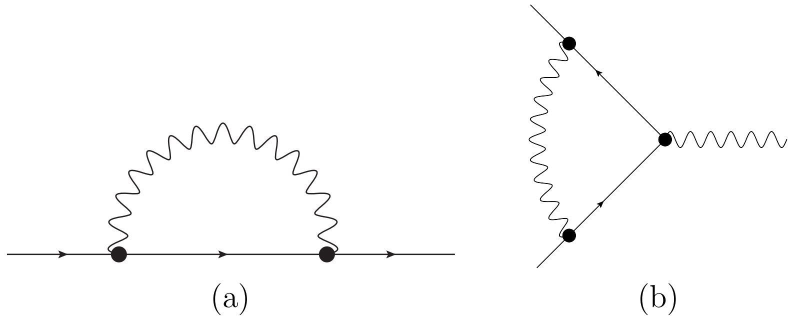

Figure 1: Fermion self-energy and fermion-photon vertex corrections at 1-loop.

in the higher covariant derivative regularization for the photon propagator in the unrenormalized perturbative calculation, and the wavefunction renormalization constant becomes

$$\delta^{(1)}_{Z2} = \left[A^{(1)} + 2M^2 \frac{\partial A^{(1)}}{\partial p^2} + 2M \frac{\partial B^{(1)}}{\partial p^2} \right] \Bigg|_{p^2=M^2} \tag{4}$$

at this $\mathcal{O}(e^2)$ level.

On the other hand, fermion–fermion–photon vetex $-ieQ\Gamma^\mu$—including quantum corrections—is known to be cast into the form

$$-ieQ\Gamma^\mu = -ieQ\left[V_1 \gamma^\mu - \frac{V_2}{4m} \left[\gamma^\mu, \gamma^\nu\right] q_\nu \right] + (***) \times (\not{p} - m) + \left(\not{p}' - m\right) \times (***)\,; \tag{5}$$

here, we assume that the momentum of the fermion coming from below in Figure 1 (b) is p, that of the fermion going out to the above p', and the photon comes from the right with momentum $q = p' - p$. As a result of tough calculation (see Peskin–Schroeder, and also the week-8 9 lecture note of the QFT II course), one will find, in higher covariant derivative regularization, that

$$\begin{aligned} V_1^{(1)} &= \frac{(eQ)^2}{8\pi^2} \int dx dy \Bigg\{ \quad \ln\left(\frac{(1-x-y)\Lambda^2 + (x+y)^2 M^2 - xyq^2}{(x+y)^2M^2 - xyq^2} \right) \\ &\qquad + \left[\left\{ 1 - 4(1-x-y) + (1-x-y)^2 \right\} M^2 + (1-x)(1-y)q^2 \right] \times \\ &\qquad \left[\frac{1}{(x+y)^2M^2 - xyq^2} - \frac{1}{(1-x-y)\Lambda^2 + (x+y)^2M^2 - xyq^2} \right] \Bigg\}, \end{aligned} \tag{6}$$

$$\begin{aligned} V_2^{(1)} &= \frac{(eQ)^2}{16\pi^2} \int dx dy \; (1-x-y)(x+y)4M^2 \times \\ &\qquad \left[\frac{1}{(x+y)^2M^2 - xyq^2} - \frac{1}{(1-x-y)\Lambda^2 + (x+y)^2M^2 - xyq^2} \right] \end{aligned} \tag{7}$$

Here, $dxdy$ integral should be carried out in a trianglular region determined by $0 \leq x, y \leq 1$, $x + y \leq 1$.

Just like the wavefunction renormalization constant Z_2 characterizes partial information of self-energy diagrams, a parameter $Z_1 := 1/V_1(q^2 = 0)$ is used to capture partial information of vertex corrections $ie\Gamma^\mu$. At 1-loop,

$$\delta_{Z1}^{(1)} = (Z_1 - 1)^{\text{1-loop}} = \left[\frac{1}{1 + V_1^{(1)}(q^2 = 0)} - 1 \right]^{\text{1-loop}} = -V_1^{(1)}(q^2 = 0). \tag{8}$$

Problem: It is known from Ward identity in QED that $Z_1 = Z_2$ at all order in perturbation theory. Verify this relation at 1-loop level. [that is, show that $\delta_{Z2}^{(1)} = \delta_{Z1}^{(1)}$.]

See [Peskin–Schröder] section 7.1, if necessary. It is also good to know that Mathematica is sometimes quite useful.