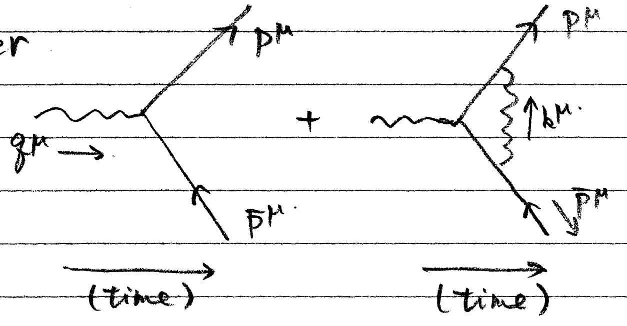
§5 Soft and Collinear Divergence

§5.1 Divergence in Virtual Corrections

Consider



$\gamma^* \to f + \bar{f}$ in QED

(or $\gamma^* \to q + \bar{q}$ in QCD)

[higher cov. deriv. regularizat'n + on-shell renormalization]

$$(-ieQ_f\Gamma^\mu) = -ie_0 Q_f\gamma^\mu - ie_0 Q_f\frac{(Q_f e_0)^2}{(4\pi)^2}\left\{2\gamma^\mu F_1^{(1)}(q^2) + [\gamma^\mu,\gamma^\nu]q_\nu(////)(q^2)\right\} + O(e_0^5)$$

$$F_1^{(1)}(q^2) = 2\int_{\triangle} dx\,dy\left(\ln\left(\frac{(x+y)^2 m_f^2}{(x+y)^2m_f^2 - xyq^2}\right) + \frac{m_f^2(1-4z+z^2)+q^2(1-x)(1-y)}{m_f^2(x+y)^2-q^2xy+m_g^2 z} - \frac{m_f^2(1-4z+z^2)}{m_f^2(x+y)^2+m_g^2 z}\right)$$

Peskin-Schroeder eq.(6.56)

e_0: QED gauge coupling on-shell scheme $\qquad z := 1-x-y$

Q_f: electric change of the fermion f

m_f: mass of f. $\qquad m_g$: photon/gluon propagator $\frac{-i\eta_{\mu\nu}}{(k^2-m_g^2+i\epsilon)}$ mass (regulator)

[dimensional regularization + $\overline{MS}$ scheme]

$$(-ieQ_f\Gamma^\mu) = -ie_\mu Q_f\gamma^\mu - ie_\mu Q_f\frac{(Q_f e_\mu)^2}{(4\pi)^2}\left\{2\gamma^\mu F_1^{(1)}(q^2) + \cdots\right\}$$

$$F_1^{(1)}(q^2) = 2\int_{\triangle} dx\,dy\left(\ln\left(\frac{\mu^2}{(x+y)^2m_f^2 - xyq^2}\right) + \frac{m_f^2(1-4z+z^2)+q^2(1-x)(1-y)}{m_f^2(x+y)^2-q^2xy+m_g^2 z}\right)$$

Either way, we still need to integrate over $dx\,dy$.

QCD (1-loop by gluon instead of a γ) $\qquad (Q_f e)^2$ is replaced by $(C_2(\rho)g^2)$

$\sum_a \rho(t^a)\rho(t^a) =: C_2(\rho)\cdot\mathbb{1}$. $\qquad \boxed{C_2 = \frac{4}{3} \text{ of } SU(3) \text{ triplet}}$

★ integral over small x region. (fixed y)

$$F_1^{(1)}(q^2) \supset dy \int_0^{x_*} dx \, \frac{m_f^2\{1-4(1-y)+(1-y)^2\} + q^2(1-y)}{m_f^2(y^2+2xy) - q^2 xy + m_g^2(1-y-x) \quad + O(x^2)}$$

$$= dy \, \frac{m_f^2\{1-4(1-y)+(1-y)^2\} + q^2(1-y)}{(2y m_f^2 - q^2 y - m_g^2)} \ln\left(\frac{m_f^2(y^2+2x_*y) - q^2 x_* y + m_g^2(1-y-x_*)}{m_f^2 y^2 + m_g^2(1-y)}\right)$$

If $q^2 \gg m_f^2$ $\quad (e^+ + e^- \to \gamma^* \to f + \bar{f} \quad q^2 = s.)$

$$F_1^{(1)}(q^2) \gtrsim - dy \frac{(1-y)}{y} \ln\left(\frac{-q^2 y}{m_f^2 y^2 + m_g^2(1-y)}\right)$$

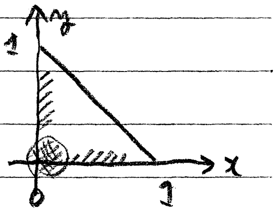
lange log. if $m_f \neq 0$

log divergence if $m_f = 0$

[$m_g \neq 0$ regularization]

★ integral over small y region (fixed x)

the same.



★ integral over small-x & small-y region.

$$F^{(1)}(q^2) \supset \int dx\, dy \frac{q^2}{m_f^2 (x+y)^2 - q^2 xy + m_g^2} \sim \int_0 d\lambda \, \lambda \frac{1}{\lambda^2}$$

diverges even if $m_f \neq 0$.

($m_g \neq 0$ regularization)

Even after removing the UV divergence by renormalization,

yet another divergence remains in the amplitudes.

Origin of those divergence

$$\frac{d^4k}{(2\pi)^4}\int dx\,dy\,dz\,\frac{2\,\delta(x+y+z-1)\quad \left(p^\nu,\ \bar{p}^\nu,\ k^\nu,\ m_f\ \text{etc. no } x, y, z\right)}{\left\{x\left[(p-k)^2-m_f^2\right]+y\left[(\bar{p}+k)^2-m_f^2\right]+z\left[k^2-\ {}^2\right]\right\}^3}$$

$$\{\cdots\} =: D(k)$$

The divergence is from an integral over a finite range in $\left\{\begin{matrix} x, y, z \\ k^\nu \end{matrix}\right\}$.

$\Rightarrow \exists$ poles in the integrand. (not from UV divergence)

| that cannot be avoided by deforming the integration contour.

$\exists$ simultaneous solution (x_*, y_*, z_*, k_*^μ) to all of this:

- $D(x_*, y_*, z_*, k_*^\mu) = 0$ ⇐ pole
- $\left(\frac{\partial}{\partial k^\mu} D\right)(x_*, y_*, z_*, k_*) = 0$] Two roots of $D(k)$ forming a double root. → no contour deformat'n.
- $x_* = 0$ or $x_* = 1$ or $(p-k_*)^2 - m_f^2 = 0$
- $y_* = 0$ or $y_* = 1$ or $(\bar{p}+k_*)^2 - m_f^2 = 0$
- $z_* = 0$ or $z_* = 1$ or $k_*^2 = 0$

] so that contour deformation cannot avoid the pole.

$D(x, y, z, k)$ linear ↗ ↖ quadratic

(1st & 3rd & 4th conditions ⇒ 5th condition is automatically satisfied.)

simultaneous solution to the five conditions.

(x_*, y_*, z_*, k_*) : called <u>pinch (hyper) surface</u>.

☆ $\boxed{x_* = y_* = 0 \ \& \ k_*^2 = 0}$ $\quad x_*(k_*-p)^\mu + y_*(k_*+\bar{p})^\mu + z_* k_*^\mu = 0 \Longrightarrow \boxed{k_*^\mu = 0}$

Introduce a scaling parameter $\lambda \ll 1$ in the (x, y, k^ν) space around $(x_*, y_*, k_*^\nu) = (0, 0, 0^\nu)$

$$(x, y, k^\nu) \sim (\lambda x_0, \lambda y_0, \lambda k_0^\nu)$$

- $dx\,dy\,d^4k \sim d^5\Omega\, d\lambda\, \lambda^5$
- $$[D(x,y,k)]^3 \cong \Big(\lambda x_0 \cdot [\underset{\underset{(\underset{\underset{0}{\|}}{p^2-m_f^2}) - 2\underset{\underset{\lambda}{\wr}}{p\cdot k} + \underset{\underset{\lambda^2}{\wr}}{k^2}}{\wr}}{(p-k)^2-m_f^2}] + \lambda y_0 \cdot [\underset{\underset{(\underset{\underset{0}{\|}}{\bar{p}^2-m_f^2}) + 2\underset{\underset{\lambda}{\wr}}{\bar{p}\cdot k} + \underset{\underset{\lambda^2}{\wr}}{k^2}}{\wr}}{(\bar{p}+k)^2-m_f^2}] + 1 \cdot \underset{\underset{\lambda^2 k_0^2}{\wr}}{k^2}\Big)^3 \sim \lambda^6$$

so $\displaystyle\int d^5\Omega \left(\int_0 \frac{d\lambda\, \lambda^5}{\lambda^6} \sim \text{log divergence} \right)$

associated with $k^\nu \sim (k_*^\nu = 0^\nu)$ $\underline{\underline{\text{soft}}} \left\{ \begin{matrix} \text{photon} \\ \text{gluon} \end{matrix} \right\}$.

☆ $\boxed{x_* = 0 \ \& \ (\bar{p}+k_*)^2 - m_f^2 = 0 \ \& \ k_*^2 = 0}$ $\quad \overset{\nearrow 0}{x_*}(k_*-p)^\mu + y_*(k_*+\bar{p})^\mu + z_* k_*^\mu = 0$

$$\Longrightarrow \frac{(-k_*^\mu)}{y_*} = \frac{\bar{p}^\mu - (-k_*)^\mu}{(1-y_*)}$$

$k_*^\mu \ /\!/ \ [\bar{p}-(-k_*)]^\mu$ and both on-shell
$\rightarrow$ possible only if $m_f = 0$.

(photon/gluon collinear to a massless $\bar{f}$.)

$$\left\{ \begin{matrix} (-k_*^\mu) = y_*(\bar{p}^\mu) \\ [\bar{p}-(-k_*)]^\mu = (1-y_*)(\bar{p}^\mu) \end{matrix} \right\}$$

light cone components of a four vector

$$(l^0+l^3, \ l^0-l^3, \ \vec{l}_T) =: (l^+, l^-, \vec{l}_T)$$

Introduce a scaling parameter $\lambda \ll 1$.

$$\bar{p}^\mu = (E, -E, \vec{0}) \qquad (-k^\mu) = \left(y_0 E - \frac{k_0^- + \lambda^2 k_0^+}{2}, -y_0 E - \frac{-k_0^- + \lambda^2 k_0^+}{2}, -\lambda \vec{k}_{T,0} \right) \quad x = \lambda^2 x_0$$

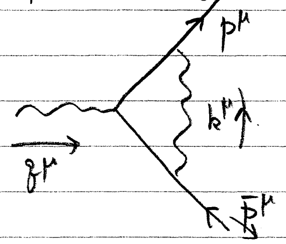
$$\Longrightarrow (\bar{p}+k)^2 - m_f^2 \simeq O(\lambda^2) \qquad k^2 \simeq O(\lambda^2) \qquad \text{so } \underline{D(x,y,k) \sim O(\lambda^2)}$$

- $dx\,dy\,d^4k \sim [dk_0^-\, dy_0]\left[dk^+\, d^2\vec{k}_T\, dx \sim d^3\Omega\, d\lambda\, \lambda^5\right] \frac{1}{2}$

so $\displaystyle dy_0\, dk_0^- \int d^3\Omega \left(\int \frac{d\lambda\, \lambda^5}{\lambda^6} \sim \text{log divergence} \right)$

$\boxed{\text{Recap}}$

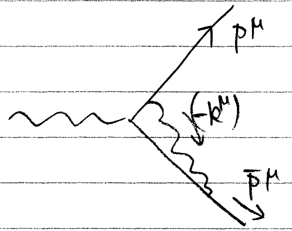
soft divergence



$k^\mu \approx 0$ (soft)

$$\left.\begin{array}{l}\dfrac{(\not{p}-\not{k})+m}{(p-k)^2-m_f^2}\\[2ex] \dfrac{-(\not{\bar{p}}+\not{k})+m}{(\bar{p}+k)^2-m_f^2}\end{array}\right\} \Rightarrow \boxed{\begin{array}{c}\text{nearly on-shell}\\ \text{(small virtuality)}\end{array}}$$

collinear divergence (for massless fermion)



$$\left\{\begin{array}{l}(-k^\mu)\\ {[\bar{p}-(-k^\mu)]^\mu}\end{array}\right. \begin{array}{l}\text{almost parallel}\\ \quad\text{almost on-shell}\end{array} \Leftarrow \boxed{\begin{array}{c}\text{nearly on-shell}\\ \text{intermediate}\\ \text{states}\end{array}}$$

(collinear)

Identify the pinch hypersurface for a given Feynman graph,

and then compare (by the power counting)

the enhancement around the poles against

the smallness of the measure.

UV divergence : $\dfrac{1}{(\text{large virtuality})} \times (\#\ \text{UV DOF} \to \infty)$

vs IR divergence : $\dfrac{1}{(\underline{\text{small}}\ \text{virtuality})} \times (\#\ \text{IR DOF}\ \underline{\text{small}})$

available if $\exists$ enough states w/. very small excitation energy.

(massless states)

higher space-time dim. $\Rightarrow$ severer UV divergence

lower space-time dim. $\Rightarrow$ severer IR divergence.

(In a theory wher interactions always involve derivatives (momentum). (e.g. Nambu–Goldstone boson), we also need to pay attention to the numerator when doing the power counting.)

§5.2 Divergence in Real Emissions

$e^+(\vec{p}_2) + e^-(\vec{p}_1) \to \gamma^* \to f(\vec{p}_3) + \bar{f}(\vec{p}_4) + \gamma(\vec{k})$ or $q + \bar{q} + g$ in QED/QCD

kinematics in the center of mass frame. $s = (2E)^2$. $(m_f, m_e \ll \sqrt{s})$

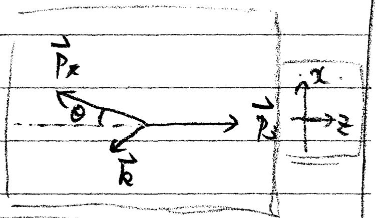
There is a choice of 3 spatial coordinates (z, x, y axes) so

$$p_3^\mu \cong (Ex_3,\ Ex_3,\ 0,\ 0) \quad \leftarrow m_f \simeq 0$$

$$p_4^\mu \cong (Ex_4,\ -Ex_4\cos\theta,\ Ex_4\sin\theta,\ 0)$$

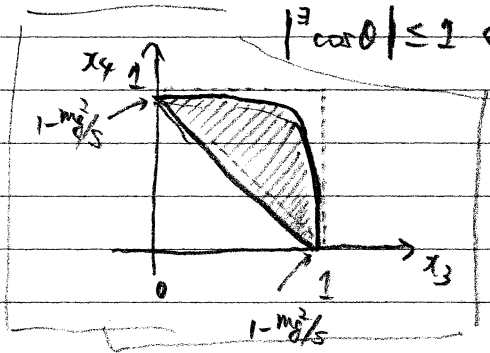
$$k^\mu \cong (\underset{\substack{\Uparrow \\ \text{energy}}}{Ex_5},\ \underbrace{-Ex_3+Ex_4\cos\theta,\ -Ex_4\sin\theta,\ 0}_{\text{spatial momenta}})$$

(z) (x) (y)



- energy conservation : $x_3 + x_4 + x_5 = 2$. $(x_3, x_4, x_5 \geq 0)$
- $(k^\mu)^2 = m_g^2 \iff 2E^2 x_3 x_4 \cos\theta = E^2\{x_3^2 + x_4^2 - (2 - x_3 - x_4)^2\} + m_g^2$

$$|{}^\exists\cos\theta| \leq 1 \iff \left[(x_3+x_4-1) + m_g^2/s\right]\left[(1-x_3)(1-x_4) - m_g^2/s\right] \geq 0$$



The 3-particle final state phase space

$\Rightarrow$ 5-dim. $\left(d^3\vec{p}_3\, d^3\vec{p}_4\, d^3\vec{k}\ \delta^4(p_3+p_4+k-p_1-p_2)\right)$

$\Uparrow$

dx_3, dx_4 and (3 more angles) (ψ, χ, φ)

(z, x, y above) vs Lab (z'', x'', y'')

translation

$$\frac{d^3\vec{p}_3}{(2\pi)^3}\frac{1}{(2Ex_3)}\frac{d^3\vec{p}_4}{(2\pi)^3}\frac{1}{(2Ex_4)}\frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2E_k} \overset{dPS_{\delta^4}}{=}$$

$$(2\pi)^4\delta^4(p_3+p_4+k-p_1-p_2)$$

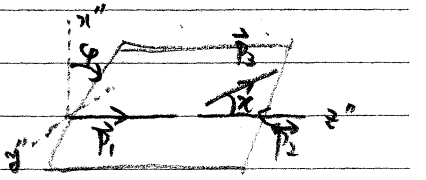
in terms of

$dx_3\, dx_4\, d\varphi\, d\psi\, d\chi$

angle ψ
rotatn. in x-y plane

$\vec{p}_3, \vec{p}_4, \vec{p}_5$ plane (z x) plane $\Rightarrow$ $\vec{p}_3, \vec{p}_1, \vec{p}_3$ plane

+



(φ, χ) to describe $\hat{p}_3$ relative to the Lab frame

$\Rightarrow$ doable. hw IX-1 or QFT II hw VII-3

No.

Date . .

$\overline{\Sigma|\mathcal{M}|^2}$ spin-average over the initial states e^-, e^+

spin-sum over the final states $f, \bar{f}, \gamma/g$

sum over the color of $g, f, \bar{f}$

$\Rightarrow$ doable. QFT II hw VII-3 (in the case $m_g = 0$)

$$d\sigma(e^-e^+\to\gamma^*\to q\bar{q}g) \cong \frac{1}{2s}\, dPS_{3g}\, \overline{(\Sigma|\mathcal{M}|^2)} \qquad \sigma_0 = \frac{4\pi\alpha_e^2}{3s} \qquad R_0 = (N_c\times)\,Q_f^2$$

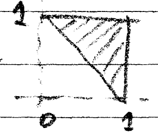
$$= \sigma_0 R_0 \frac{\alpha_s}{2\pi} C_2 \frac{dx_3\, dx_4}{(1-x_3)(1-x_4)}\left[(x_3^2+x_4^2) + \beta_g\left\{2(x_3+x_4) - \frac{(1-x_3)^2+(1-x_4)^2}{(1-x_3)(1-x_4)}\right\} + 2\beta_g^2\right]. \quad \text{——}(*)$$

$$\beta_g := m_g^2/s.$$

- The 5-dim differential cross section: too complicated to write down here. simpler expression (as above) after integrating over (ψ, χ, φ)

- G. Fox and S. Wolfram Nucl. Phys. B149 (1979) 423 for the formula (*) (better to use Mathematica to derive (*) by oneself)

Set $m_g = 0$ ($\beta_g = 0$) and integrate over $dx_3\, dx_4$



$$\propto \frac{dx_3\, dx_4}{(1-x_3)(1-x_4)} \Rightarrow \text{diverges.} \quad (x_3 \simeq 1 \text{ or/and } x_4 \simeq 1)$$

$x_3 \simeq 1 \Leftrightarrow \vec{p}_4 \simeq -x_4\vec{p}_3$. $\vec{k} \simeq -(1-x_4)\vec{p}_3$ g is collinear to $\bar{f}$

$x_4 \simeq 1 \Leftrightarrow \vec{p}_3 \simeq -x_3\vec{p}_4$ $\vec{k} \simeq -(1-x_3)\vec{p}_4$ g is collinear to f.

($x_3 \simeq 1$ and $x_4 \simeq 1$) $\Leftrightarrow |\vec{k}| \ll E$. g is soft.

With $m_g \neq 0$.

$$\int_{\substack{x_3+x_4 \geq 1-m_g^2/s \\ (1-x_3)(1-x_4) \geq m_g^2/s}} d\sigma = \sigma_0 R_0 \frac{\alpha_s}{2\pi} C_2\left[\ln^2\beta_g + 3\ln\beta_g - \frac{\pi^2}{3} + 5\right]$$

We will use this "regularized" result later.

calculation details found in a book R. Field "Applications of Perturbative QCD" §2

PLUS

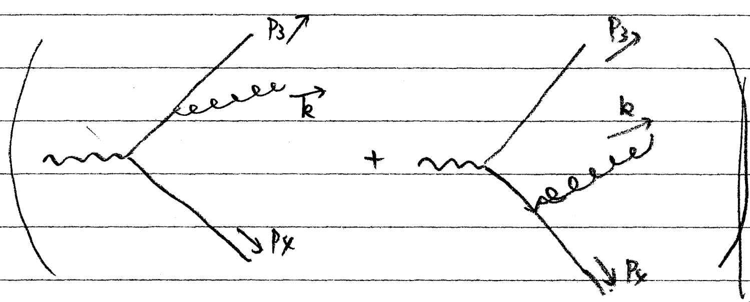
The divergence $\left(\frac{1}{(1-x_3)(1-x_4)} \right)$ is from $\overline{\Sigma|\mathcal{M}|^2}$ (not from dPS_{34}).

$$2p_3 \cdot k = 2E^2 x_3 x_5 + 2|\vec{p}_3|^2 - 2|\vec{p}_3||\vec{p}_4| \cos\theta = \cdots = (2E)^2(1-x_4) - m_g^2$$

$$2p_4 \cdot k = 2E^2 x_4 x_5 + 2|\vec{p}_4|^2 - 2|\vec{p}_3||\vec{p}_4| \cos\theta = \cdots = (2E)^2(1-x_3) - m_g^2$$

$$\Rightarrow (p_3+k)^2 - m_f^2 = s(1-x_4) \qquad (p_4+k)^2 - m_f^2 = s(1-x_3)$$

$\left(\text{even when } \begin{matrix} m_f \neq 0 \\ m_g \neq 0 \end{matrix} \right)$



$$\boxed{\begin{aligned} &\bar{u}(\vec{p}_3)(-ig t^a \gamma^\kappa) \frac{i[(\not{p}_3+\not{k})+m_f]}{(p_3+k)^2-m_f^2}(-ieQ_f\gamma^\mu)v(\vec{p}_4) \\ &+ \bar{u}(\vec{p}_3)(-ieQ_f\gamma^\mu)\frac{i[(\not{p}_4-\not{k})+m_f]}{(p_4+k)^2-m_f^2}(-ig t^a\gamma^\kappa)v(\vec{p}_4) \end{aligned}}$$

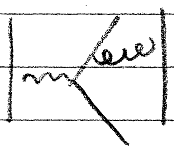
$(p_3+k)^\mu$ is close to on-shell if k^μ is soft. or collinear to light-like p_3^μ.
$(p_4+k)^\mu$ is 〃 〃 〃 if k^μ is soft or collinear to 〃 〃 p_4^μ

an intermediate state w/ small virtuality ⇒ enhanced rate. ($|\mathcal{M}|^2$)

memo: when $m_f = 0$ & $m_g = 0$ $\overline{\Sigma|\mathcal{M}|^2} \propto \frac{1}{(1-x_3)(1-x_4)}$ not $\frac{1}{(1-x_3)^2}$ or $\frac{1}{(1-x_4)^2}$

$\frac{1}{(1-x_4)^2}$ is absent because...

- $\bar{u}(\vec{p}_3)\gamma^\kappa(\not{p}_3+\not{k}) = -\underset{\approx 0 \; \left(\begin{matrix} \bar{u}(\vec{p}_3)\not{p}_3 = 0 & k \parallel p_3 \\ \text{Dirac eq.} & @ x_4 \sim 1 \end{matrix} \right)}{\bar{u}(\vec{p}_3)(\not{p}_3+\not{k})} + \bar{u}(\vec{p}_3)\{\gamma^\kappa, (\not{p}_3+\not{k})\} = \bar{u}(\vec{p}_3)\, 2(p_3+k)^\kappa$.

- $\left|\right.$  $\left.\right|^2 \propto \frac{2(p_3+k)^\kappa}{(p_3+k)^2} \underset{\substack{\uparrow \\ g \text{ spin sum}}}{(-\eta_{\kappa\lambda})} \frac{2(p_3+k)^\lambda}{(p_3+k)^2} = \frac{4(p_3+k)^2}{((p_3+k)^2)^2}$

§6 Cancellation of IR Divergence

§6.1 Observable Observables

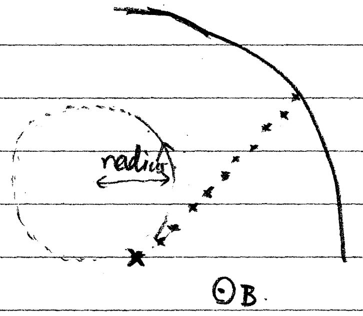
soft γ emission : given a detector arrangement (calorimeters, scintillators, photomultiplier tubes)

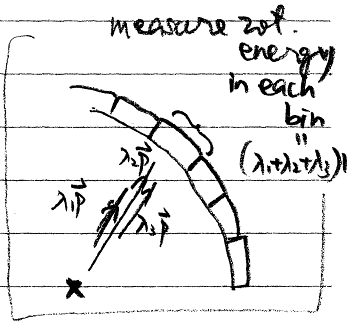
any photon below some energy threshold cannot ionize materials in the detector $\Rightarrow$ no cross section data available.

[collinear relativistic bunch of particles] : calorimeter : limited angular resolution

some charged some neutral

tracker: too large curvature radius of a charged particle Track cannot be measured.





(also limited angular resolution) $\frac{|p_T|}{eQ\,|B|}$ = (curvature radius) > lower bound.

so the tracks w/ $\lambda_1\vec p_1$, $\lambda_2\vec p_2$ .. may not be resolved, $|\lambda_i\vec p|$ may not be measured by the tracker.

$\Rightarrow$ bunch of relativistic particles as a whole can be measured (the sum of their momenta)

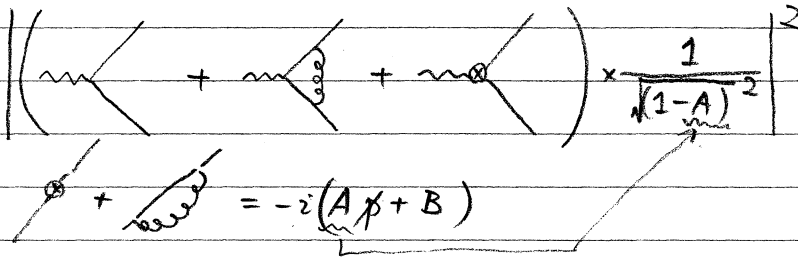
$\Rightarrow$ $\sigma(e^-e^+\to f\bar f) + \sigma(e^-e^+\to f\bar f + \gamma_{\text{soft/coll to }f\text{/coll to }\bar f})$ is measurable.

but $\sigma(e^-e^+\to f\bar f)$ is not.

$\sigma(e^-e^+\to f\bar f + \gamma_{\text{non soft non-coll }f\text{ non coll }\bar f})$ is measurable.

but $\sigma(e^-e^+\to f\bar f\gamma)$ is not.

$$\sigma(e^-e^+\to q\bar q)_{m_g\neq 0} = \sigma_0 R_0\left(1+\frac{\alpha_s}{2\pi}C_2\left(-\ln^2(\beta_g)-3\ln(\beta_g)-\frac{7}{2}+\frac{\pi^2}{3}\right)+\mathcal{O}(\alpha_s^2)\right)$$



$$\left|\left(\;\cdots + \cdots + \cdots\;\right)\times\frac{1}{\sqrt{1-A}^{\,2}}\right|^2$$ is used.

$$\cdots + \cdots = -i(A\not p + B)$$

(see. R.Field's Textbook §2 for details of the calculation.)

[diverges @ $\beta_g\to 0$. $\beta_g := m_g^2/s$]

(§5.2. of this lecture)

$$+)\;\sigma(e^-e^+\to q\bar q g)_{m_g\neq 0} = \sigma_0 R_0 \times\frac{\alpha_s}{2\pi}C_2\left(+\ln^2\beta_g+3\ln(\beta_g)+5-\frac{\pi^2}{3}\right)+\mathcal{O}(\alpha_s^2)$$

$$\sigma(e^-e^+\to q\bar q+\text{any}) = \sigma_0 R_0\left(1+\frac{\alpha_s}{2\pi}C_2\times\frac{3}{2}+\mathcal{O}(\alpha_s)\right)$$

$C_2=\frac{4}{3}\Leftarrow\frac{(N_c^2-1)}{2N_c}$ @ $N_c=3$

finite when $m_g\to 0$. double log $(\ln(\beta_g))^2$, single log $\ln(\beta_g)$ divergence cancel in the observable observable

☆ $$\sigma(e^-e^+\to q\bar q g)\Big|_{\substack{(1-x_3)\geq\delta\\(1-x_4)\geq\delta}} \cong \sigma_0 R_0\frac{\alpha_s}{2\pi}C_2\iint_{\substack{1\leq x_3+x_4\\(1-x_3)\geq\delta\\(1-x_4)\geq\delta}} dx_3dx_4\frac{x_3^2+x_4^2}{(1-x_3)(1-x_4)}$$

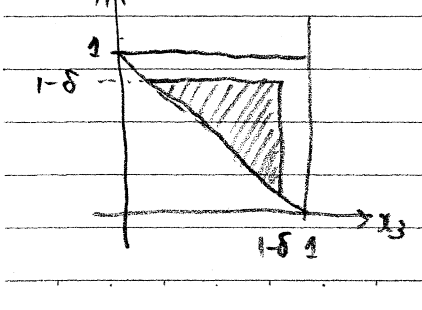
is also finite if $\delta>0$.

$\Rightarrow 2(\ln(\delta))^2+3\ln(\delta)+\mathcal{O}(1)$

☆ $$\sigma\left(e^-e^+\to q\bar q \text{ or } q\bar q g \text{ w/ } \begin{matrix}(1-x_3)\leq\delta\\ \text{or}\\ (1-x_4)\leq\delta\end{matrix}\right) = \sigma_0R_0\left[1+\frac{\alpha_s}{2\pi}C_2\left(\frac{3}{2}-\text{integral above}\right)+\mathcal{O}(\alpha_s^2)\right]$$

$\hookrightarrow -2(\ln(\delta))^2-3\ln(\delta)+\frac{3}{2}-\mathcal{O}(1)$

both observable observables have sensible theoretical calculated (finite) results.



memo: The cross section of $e^-e^+\to q+\bar q$ without a hard gluon is not as large as σ_0R_0 (the $\mathcal{O}(\alpha_s)$ correction term has $-$ sign. $\frac{\alpha_s}{2\pi}C_2\left(-2\ln^2\delta\right)$).

memo: With $0<\delta\ll 1$, we still have observable observables but their theoretical calculation needs higher order in $(\alpha_s\ln^2\delta)$ for a better precision.

§ 6.2 Unitarity / Cutkosky Rule

☆ <u>unitarity of the S-matrix</u> $\boxed{\sum_\beta (S_{\beta.\gamma})^{cc} S_{\beta\alpha} = \delta_{\gamma.\alpha}}$ ($\alpha.\beta.\gamma$: Fock state labels.)

$$S_{\beta\alpha} = \mathbb{1}_{\beta.\alpha} + (2\pi)^4 \delta^4(p^\beta - p^\alpha)\, i\mathcal{M}_{\beta.\alpha} \quad \Updownarrow$$

$$i\mathcal{M}_{\gamma.\alpha} - i(\mathcal{M}_{\alpha.\gamma})^{c.c.} + \Sigma_\beta (2\pi)^4\delta^4(p^\beta - p^\alpha)(\mathcal{M}_{\beta.\gamma})^{cc}(\mathcal{M}_{\beta.\alpha}) = 0$$

$$\Longleftrightarrow \boxed{(i)^{-1}[\mathcal{M}_{\gamma.\alpha} - (\mathcal{M}_{\alpha\gamma})^{c.c}] = \Sigma_\beta (2\pi)^4\delta^4(p^\beta - p^\alpha)\,(\mathcal{M}_{\beta.\gamma})^{cc}(\mathcal{M}_{\beta.\alpha})} \text{ — } (*)$$

for $\gamma = \alpha$ $\quad 2\,\mathrm{Im}(\mathcal{M}_{\alpha.\alpha}) = \Sigma_\beta (2\pi)^4\delta^4(p^\beta - p^\alpha)\,|\mathcal{M}_{\beta.\alpha}|^2$

optical theorem [photon amplitude $e^{-i\omega t + in(k)kx}$]

$\Downarrow$

(absorption rate) $= 2\mathrm{Im}(n(k))\,\omega = \mathrm{Im}$ (forward amplitude)

The unitarity relation $(*)$: for the <u>full</u> scattering amplitude $\mathcal{M}_{\beta.\alpha}$

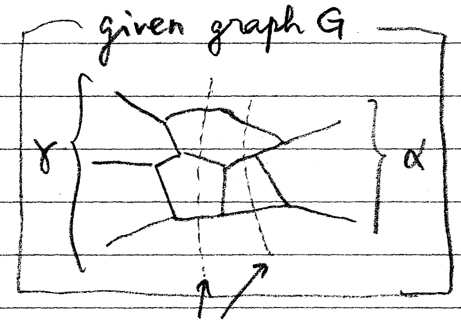
(all the perturbative contributions and non-perturbative effects <u>summed</u> up.)

☆ The amplitude $\mathcal{M}^G_{\beta.\alpha}$ for a Feynman graph G :

<u>For each</u> G of the process $\alpha \to \beta$, $^\exists$ a relation similar to $(*)$.

$$\boxed{(i)^{-1}\left[\mathcal{M}^G_{\gamma.\alpha} - (\mathcal{M}^{G^\dagger}_{\alpha.\gamma})^{cc}\right] = \sum_{cut}^{\text{of } G} \int dPS_{\beta(cut)} \left(\mathcal{M}^{\binom{\text{post cut}}{G}}_{\gamma.\beta}\ \left(\begin{smallmatrix}+i\varepsilon \\ \to -i\varepsilon\end{smallmatrix}\right)\right) \mathcal{M}^{\binom{\text{pre-cut}}{G}}_{\beta.\alpha}}$$

called the Cutkosky rule. (derivation: suppl. note "unitarity")



[examples of such curves]

- list up all the topologically different ways to draw a curve over the graph G between the final state (γ)-side and the initial state (α)-side

$\Rightarrow$ [The graph G is cut into two graphs post-cut (γ-side) and pre-cut (α-side)

The collection of propagators cut by the curve

$\Rightarrow$ a family of Fock states

- sum the expression on the RHS over the different curves.

memo: when $\mathcal{L} \supset c_j \mathcal{O}_j$ is used in a process $\alpha \to \gamma$.

use $\mathcal{L} \supset c_j^{cc} \mathcal{O}_j^{\dagger}$ to generate the process $\gamma \to \alpha$.

("graph $G^{\dagger}$" for $\gamma \to \alpha$ cf. graph G for $\alpha \to \gamma$)

$$\boxed{\left(\mathcal{M}^{G^{\dagger}}_{\alpha.\gamma}\right)^{cc} = \mathcal{M}_{\gamma.\alpha}\left(\overset{+i\varepsilon}{\to} -i\varepsilon\right)}$$

(at least $(c_j^{cc})^{cc}$ on the LHS vs c_j on the RHS)

(more systematic argument for this relation. ⇒ see "suppl. notes unitarity")

☞ another version of the Cutkosky rule.

$$\boxed{(i)^{-1}\left[\mathcal{M}^{G}_{\gamma.\alpha}(+i\varepsilon) - \mathcal{M}^{G}_{\gamma.\alpha}(-i\varepsilon)\right] = \sum_{cut}^{\text{of } G} \int dPS'_{\beta(cut)}\, \overset{\binom{post}{cut}}{\mathcal{M}_{\gamma.\beta}(-i\varepsilon)} \cdot \overset{\binom{pre}{cut}}{\mathcal{M}_{\beta.\alpha}(+i\varepsilon)}}$$

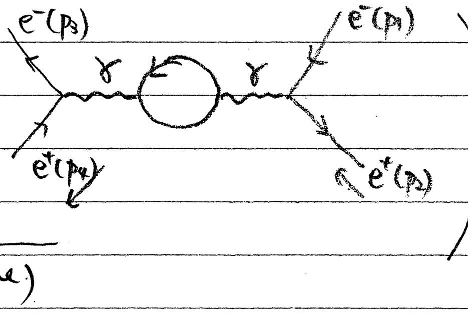
$\Uparrow$

discontinuity of $\mathcal{M}^{G}_{\gamma.\alpha}$ with respect to $(p_\alpha)^2 \pm i\varepsilon$.

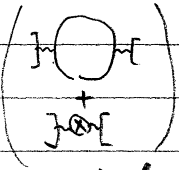
(non-analyticity)

$$\boxed{(i)^{-1}\left[\mathcal{M}^{G}_{\gamma.\alpha} - \left(\mathcal{M}^{G^{\dagger}}_{\alpha.\gamma}\right)^{cc}\right] = \sum_{cut}^{\text{of } G} \int dPS'_{\beta(cut)} \left(\mathcal{M}_{\beta.\gamma}^{\binom{post}{cut\, G}^{\dagger}}\right)^{cc} \left(\mathcal{M}_{\beta.\alpha}^{\binom{pre}{cut\, G}}\right)}$$

An example (see also hw Ⅲ-2, Ⅸ-5, Ⅸ-6)

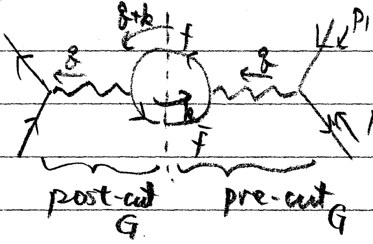
$G =$ 

$$\boxed{i\mathcal{M}^G = \begin{array}{l} ie(\bar{u}(\vec{p}_3)\gamma^\mu v(\vec{p}_4)) \\ ie(\bar{v}(\vec{p}_2)\gamma^\nu u(\vec{p}_1)) \\ \left(\frac{-i}{s+i\varepsilon}\right)^2 i s \eta_{\mu\nu} \Pi^{(1)}_{ren}(s+i\varepsilon) \end{array}}$$



$1PI =: i(q^2\eta_{\mu\nu} - q_\mu q_\nu)\Pi^{(1)}_{ren}$

- single photon in the s-channel cannot be on-shell
- The only possible cut (curve) $\Rightarrow$



The intermediate state

$$\left\{\begin{array}{l} f: (q^0+k^0, \vec{k}) = (q+k)^\mu \\ \bar{f}: (-k^0, -\vec{k}) = -k^\mu \end{array}\right\}$$

both are on-shell $\Leftrightarrow$ $-k^0 = \sqrt{|\vec{k}|^2+m_f^2}$

$(q^0+k^0) = \sqrt{|\vec{k}|^2+m_f^2}$

$$\Rightarrow (\text{RHS}) = \int \frac{d^3\vec{k}_f}{(2\pi)^3}\frac{1}{2E_{\vec{k}}}\frac{d^3\vec{k}_{\bar{f}}}{(2\pi)^3}\frac{1}{2E_{\vec{k}_f}}(2\pi)^4\delta^3(\vec{k}_f+\vec{k}_{\bar{f}})\delta(E_{\vec{k}_f}+E_{\vec{k}_{\bar{f}}}-q^0)$$

$$\underbrace{[ie\bar{u}(\vec{p}_3)\gamma^\mu v(\vec{p}_4)]\frac{-i}{s+i\varepsilon}(-ieQ_f)}_{\substack{\times(i)^{-1} \\ \text{post cut}}}\ \overset{\text{fermion loop}}{(-)}\text{Tr}[\gamma_\mu((\not{q}+\not{k})+m_f)\gamma_\nu(\not{k}+m_f)]\ \underbrace{(-ieQ_f)\frac{-i}{s+i\varepsilon}\times [ie\bar{v}(\vec{p}_2)\gamma^\nu u(\vec{p}_1)]\times i^{-1}}_{\text{pre-cut.}}$$

$$= \int\frac{d^3\vec{k}_f}{(2\pi)^3}\frac{1}{(q^2)}(2\pi)\frac{E_{k_f}}{2|k_f|}\delta\left(|k_f| - \sqrt{\frac{s}{4}-m_f^2}\right)\times$$

$$(-)\frac{e^2[\bar{u}(\vec{p}_3)\gamma^\mu v(\vec{p}_4)][\bar{v}(\vec{p}_2)\gamma^\nu u(\vec{p}_1)]}{s^2}\ (+4)e^2Q_f^2\left[(q+k)_\mu k_\nu + k_\mu(q+k)_\nu - \eta_{\mu\nu}(k\cdot(q+k)-m_f^2)\right]$$

$[\quad]_{\mu\nu} =$ ▨ $(q^2\eta_{\mu\nu} - q_\mu q_\nu) \Rightarrow$ ▨ $= [\ ]_{\mu\nu}\eta^{\mu\nu}\frac{1}{3q^2}$

$$= \frac{\overset{\swarrow vol(S^2)}{4\pi}}{(2\pi)^2}\frac{1}{8}\sqrt{1-\frac{4m_f^2}{s}}\times(-)\frac{e^2[\bar{u}(\vec{p}_3)\gamma^\mu v(\vec{p}_4)][\bar{v}(\vec{p}_2)\gamma^\nu u(\vec{p}_1)]}{s^2}\times$$

$$4(q^2\eta_{\mu\nu} - q_\mu q_\nu)\frac{e^2Q_f^2}{3q^2}\left(-2k\cdot(q+k)+4m_f^2 \Rightarrow s+2m_f^2\right)$$

~~independently~~

$$(\mathrm{LHS}) = \frac{1}{i}\left(\mathcal{M}^{q}(q^2+i\varepsilon) - \mathcal{M}^{q}(q^2-i\varepsilon)\right)$$

$$= (-)\,\frac{e^2[\bar{u}(\vec{p}_3)\gamma^\mu v(\vec{p}_4)][\bar{v}(\vec{p}_2)\gamma^\nu u(\vec{p}_1)]}{s^2}\;\bar{i}\,(q^2\eta_{\mu\nu} - q_\mu q_\nu)\left(\Pi^{(1)}_{\mathrm{ren}}(q^2+i\varepsilon) - \Pi^{(1)}_{\mathrm{ren}}(q^2-i\varepsilon)\right)$$

where $\Pi^{(1)}_{\mathrm{ren}}(q^2+i\varepsilon) = \dfrac{e^2Q_f^2}{2\pi^2}\displaystyle\int_0^1 dx\; x(1-x)\ln\left(\frac{m_f^2 - x(1-x)(q^2+i\varepsilon)}{m_f^2 + x(1-x)\mu^2}\right)$

↰ computed this before

(details "$m_f^2 + x(1-x)\mu^2$" depend on renormalization conditions.)

- $x(1-x)$ takes values in $[0, 1/4]$ over $x \in [0,1]$
- small $q^2 = s$ $(\sqrt{s} < 2m_f) \Rightarrow m_x^2 - x(1-x)s > 0$

above the threshold of $f+\bar{f}$ creation $(\sqrt{s} > 2m_f)$

$$\begin{cases} m_f^2 - x(1-x)(s-i\varepsilon) & : \text{(negative)} + i\,\text{(positive)} \\ m_f^2 - x(1-x)(s+i\varepsilon) & : \text{(negative)} + i\,\text{(negative)} \end{cases}$$

$$\left(\text{for} \quad \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{m_f^2}{s}} \le x \le \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{m_f^2}{s}}\right)$$

$$\Rightarrow i\left(\Pi^{(1)}_{\mathrm{ren}}(s+i\varepsilon) - \Pi^{(1)}_{\mathrm{ren}}(s-i\varepsilon)\right) = \frac{e^2Q_f^2}{2\pi^2}\int_{\frac{1}{2}-\sqrt{} =: x_-}^{\frac{1}{2}+\sqrt{} =: x_+} dx\; x(1-x)\; i \times \{(-\pi i) - (+\pi i)\}$$

$$= \frac{e^2Q_f^2}{2\pi^2} \times 2\pi \times \left\{\frac{x_+^2 - x_-^2}{2} - \frac{x_+^3 - x_-^3}{3}\right\} = \frac{e^2Q_f^2}{\pi}\,\frac{s+2m_f^2}{6s}\sqrt{1-\frac{4m_f^2}{s}}$$

$\boxed{\text{Compare}}$ common $(-)\dfrac{e^2[\bar{u}\gamma^\mu v][\bar{v}\gamma^\nu u]}{s^2} \times (q^2\eta_{\mu\nu} - q_\mu q_\nu) \times$

$\boxed{\text{LHS}}$ $\boxed{\text{RHS}}$

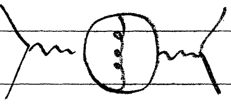
$$\frac{e^2Q_f^2}{\pi}\,\frac{s+2m_f^2}{6s}\sqrt{1-\frac{4m_f^2}{s}} \quad \text{vs} \quad \frac{e^2Q_f^2}{2\pi}\sqrt{1-\frac{4m_f^2}{s}}\,\frac{1}{3s}(s+2m_f^2)$$

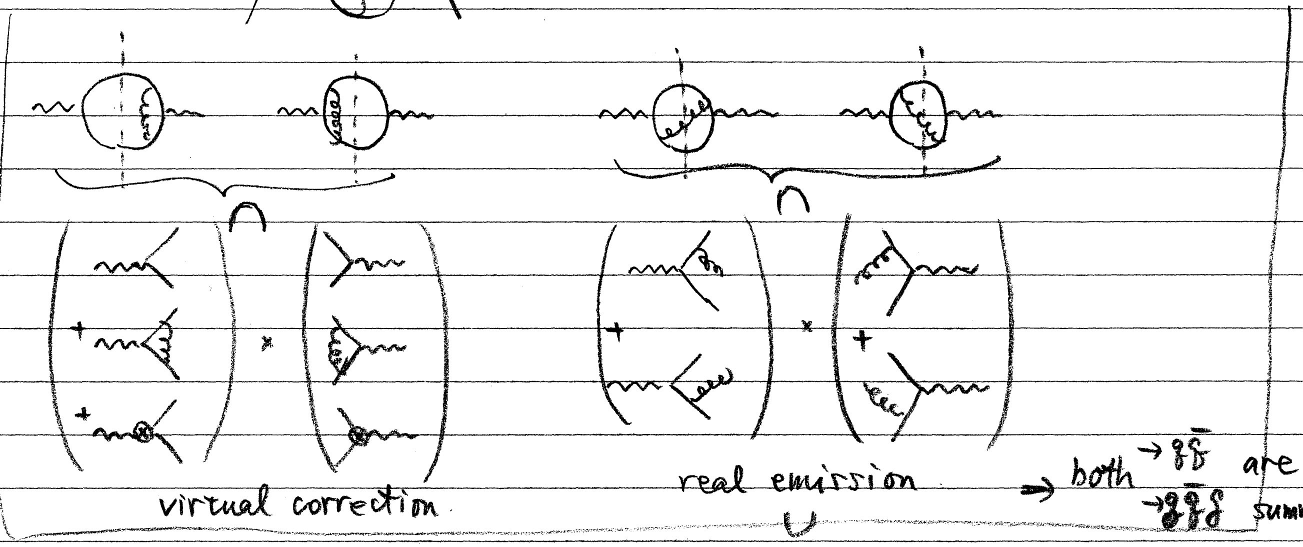
<u>agree.</u>

Apply the optical thm (unitarity relation) to
$\sigma_{tot}(e^- e^+ \to$ any hadronic).

$$\boxed{\sigma_{tot} = \frac{1}{2s}\int dPS\; \mathcal{M}^{cc}\mathcal{M} = \frac{1}{2s}\frac{1}{i}\left(\mathcal{M}_{\alpha\alpha} - \mathcal{M}_{\alpha\alpha}^{cc}\right) = \frac{1}{2s}\frac{1}{i}\left(\mathcal{M}_{\alpha\alpha}(s+i\epsilon) - \mathcal{M}_{\alpha\alpha}(s-i\epsilon)\right)}$$

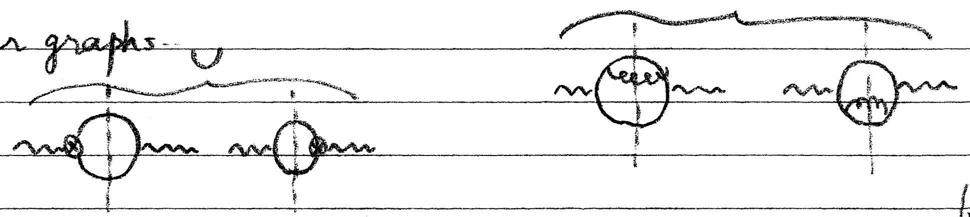
We will see that the forward amplitude $\mathcal{M}_{\alpha\alpha}(s+i\epsilon)$ is finite — (**)
$\Rightarrow$ the finiteness of σ_{tot} follows.

The graph  for $\mathcal{M}_{e\bar e, e\bar e}$ contributes to σ_{tot} through

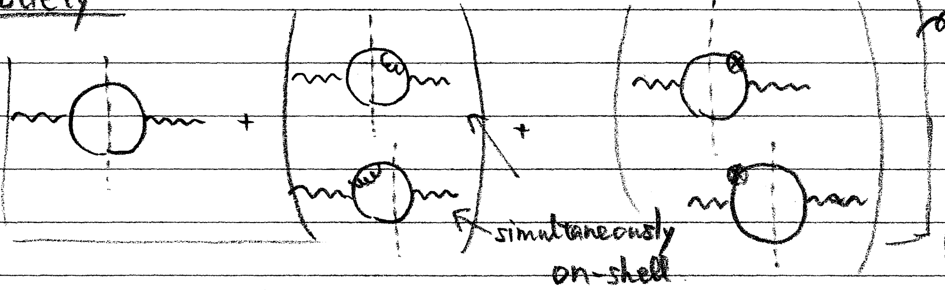


$\Rightarrow$ both $\to q\bar q$ / $\to q\bar q g$ are summed

Other graphs ∪



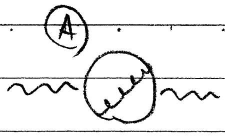
subtlety

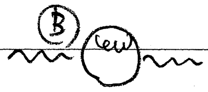


1PI —(1PI)— $=: -i(A\not p + B)$

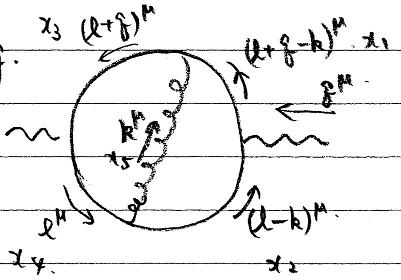
resum first
$$\frac{i}{(1-A)\not p - (m+B) + i\epsilon}$$
$\Downarrow$
$$\frac{u(\vec p)\bar u(\vec p)}{(1-A)}(2\pi)\delta^+\!\left(p^2 - \frac{(m+B)^2}{(1-A)^2}\right)$$
apply the Cutkosky rule later.

$(\ast\ast)$

Ⓐ Is  (with UV regulator) IR-finite when $m_f = m_g = 0$?

Ⓑ Is  (〃 〃) IR-finite 〃 〃 ?

e.g.



Ⓐ

$\boxed{\text{analysis step 1}}$ list up pinch hypersurface

- for each i=1,2,…,5 either $x_i=0$ or $l_i^2=0$.
- for each independent loop momenta l_a^μ (k^μ and l^μ in the graph Ⓐ)

$$\frac{\partial}{\partial l_a^\mu}\left(\Sigma_i x_i l_i^2\right)=0 \quad \text{——} (\ast\ast\ast)$$

$\boxed{\text{Step 2}}$: power counting to compare (small measure) vs (singular integrand)

$$D = x_1(l+q-k)^2 + x_2(l-k)^2 + x_3(l+q)^2 + x_4 l^2 + x_5 k^2$$

e.g. $\boxed{x_1\sim0,\ x_2\sim0,\ x_3\sim0,\ x_4\sim0,\ (x_5\sim1)\ \&\ k^2\sim0}$

$\left((\ast\ast\ast) \text{ for } l_a^\mu=l^\mu : \text{trivial} \qquad (\ast\ast\ast) \text{ for } l_a^\mu=k^\mu \Rightarrow \boxed{k^\mu=0} \right) \Rightarrow \left(\begin{array}{l} k_*^\mu=0 \\ l_*^\mu = \text{free} \end{array} \right)$ $\boxed{\text{soft gluon}}$

power counting $k^\mu\sim O(\lambda)$ $x_1\sim x_4\sim O(\lambda^2)$ so $D(x,l,k)\sim\lambda^2$.

(measure) $d^4k\, d^4l\, d^4x \Rightarrow \lambda^4\times\lambda^0\times\lambda^{4\times2}=\lambda^{12}$.

(denominator) $D^5 \Rightarrow \lambda^{5\cdot2}=\lambda^{10}$

$\int_0^{\text{cut}}\frac{d\lambda^{12}}{\lambda^{10}}$ is finite.

e.g. $\boxed{x_1\sim0,\ x_2\sim0,\ (l+q)^2\sim0,\ l^2\sim0,\ k^2\sim0}$

$(\ast\ast\ast)$ for $k^\mu \Rightarrow \boxed{k_*^\mu=0}$ $\qquad (\ast\ast\ast)$ for $l^\mu \Rightarrow x_3(l+q)^\mu + x_4 l^\mu = 0$.

$\Rightarrow (l+q)^\mu$, & l^μ both light-like and opposite direction.

$\Rightarrow$ impossible because $(l+q)^\mu - l^\mu = q^\mu$ is time-like.

e.g. $\boxed{x_1\sim0,\ x_3\sim0,\ (l-k)^2\sim0,\ l^2\sim0,\ k^2\sim0}$

$(\ast\ast\ast)$ for $l^\mu \Rightarrow x_2(l-k)^\mu + x_4 l^\mu = 0$

$(\ast\ast\ast)$ for $k^\mu \Rightarrow x_2(k-l)^\mu + x_5 k^\mu = 0$

$\Rightarrow$ one parameter ($x_2+x_4+x_5=1$)

$\boxed{x_2^2+x_2x_4+x_4^2=x_2+x_4}$

l^μ, k^μ parallel, light like

$\boxed{(x_2+x_4)l_*^\mu = x_2 k_*^\mu \ \&\ x_2 l_*^\mu = (1-x_4)k_*^\mu}$

$\boxed{\text{collinear } q \ \& \ g}$

$\left(\begin{array}{l} l^-, k^- \sim O(\lambda^2),\ \vec{l}_T, \vec{k}_T \sim O(\lambda) \\ x_1, x_3 \sim O(\lambda^2) \end{array} \right)$

$\Rightarrow$ (denominator) $D^5 \Rightarrow \lambda^{5\cdot2}=\lambda^{10}$

(numerator) $(d^4k\Rightarrow\lambda^4)\,(d^4l\Rightarrow\lambda^4)\,(d^4x\Rightarrow\lambda^4)$ tot λ^{12}

$\int_0^{\text{cut}}\frac{d\lambda^{12}}{\lambda^{10}}$ is finite.

§ ~~10~~ 6.3 OPE and Non-perturbative corrections

$e^- + e^+ \to q\bar{q}g$ etc. : hadron production in fact.

In which sense is perturbative QCD approach justified for $s \gg \Lambda_{QCD}^2$?

The optical thm : insertion of a complete system of the Hilbert space either in q, g or in hadron.

does not rely on perturbation.

But $\sigma_{tot}(s) \propto e^2 \left[i\Pi(s+i\varepsilon) - i\Pi(s-i\varepsilon) \right]$ and we need to evaluate

$$(2\pi)^4 \delta^4(q-q')\ i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\ \Pi(q^2+i\varepsilon)$$

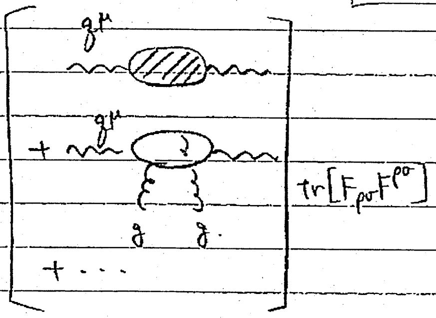
$$= (-ie)^2 \iint \langle\Omega| T\{ J^\mu(x) J^\nu(y) \} |\Omega\rangle\ e^{iq'\cdot x}\ e^{-iq\cdot y}\ d^4x\, d^4y$$

Remember that the operator product expansion (OPE) expands

$$(-ie)^2 \int d^4x\ T\{ J^\mu(x) J^\nu(0) \}\ e^{iq\cdot x} = i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\ \Pi(q^2+i\varepsilon)\ \mathbb{1}$$

$$+\ i(\qquad \vdots \qquad) \times \left[\mathcal{O}\left(\frac{1}{q^{2p}}\right) \right] \mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma})(0)$$

$$+\ \cdots\cdots$$



☆ Is OPE defined at non-perturbative level?

maybe if local operators are defined at non-perturbative level.

☆ The perturbative computation $\Pi_{pert}(q^2)$ for $\Pi(q^2)$ may have zero convergence radius in α_s

still $\left\{ (\text{Borel resummation of } \Pi_{pert}) + \text{minimum non-P contributions to add} \right\} =: [\Pi_{pert}]_B$ may become well-defined.

☆ also quite likely that $\langle\Omega| \mathrm{tr}(F_{\rho\sigma}F^{\rho\sigma}) |\Omega\rangle \sim (\Lambda_{QCD})^4 \neq 0$.

This will give rise to (yet another) contribution supp. relatively by $\left(\frac{\Lambda_{QCD}^4}{s^2}\right)$.

from Peskin–Schroeder $\boxed{\dfrac{\sigma(e^-e^+\to hadron)}{\left[\sigma(e^-e^+\to\mu^-\mu^+)=\frac{4\pi\alpha_e^2}{3s}\right]} =: R(s)}$ data vs theory

$R=R_0\cdot K$. $\boxed{R_0=\sum_f N_c Q_f^2}$ $\boxed{K \simeq 1+\frac{\alpha_s(\mu)}{\pi}+\cdots}$

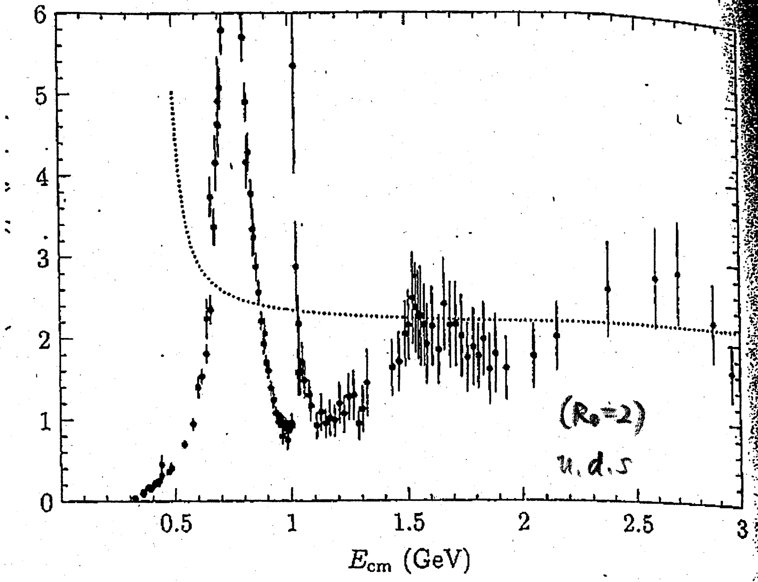
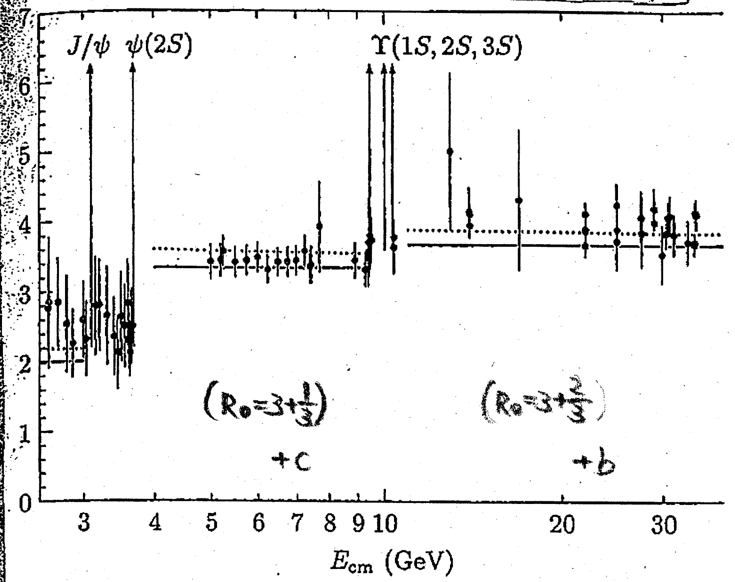


Figure 18.8. Experimental measurements of the total cross section for the reaction $e^+e^- \to$ hadrons at energies below 3 GeV, compared to the prediction of perturbative QCD for 3 quark flavors. The data are taken from the compilation of M. Swartz, *Phys. Rev.* D53, 5268 (1996). Complete references to the various results are given there.



igure 5.3. Experimental measurements of the total cross section for action $e^+e^- \to$ hadrons, from the data compilation of M. Swartz, F ev. D53, 5268 (1996). Complete references to the various experiments ven there. The measurements are compared to theoretical predictions I uantum Chromodynamics, as explained in the text. The solid line is nple prediction (5.16).

:tions, which reflect the form of the proton wavefunction and are de ed by soft QCD dynamics. However, we saw in Section 17.5 that effects D perturbation theory cause the parton distributions to change their fo . function of the m

$e^+e^- \to \mu^+\mu^-$: Helicity Structure

r a reaction is generally easy to c

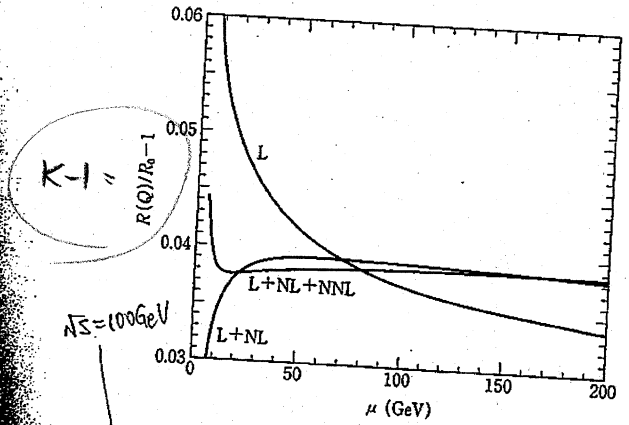


図 5.6 $R=\dfrac{\sigma(e^+e^-\to ハドロン)}{\sigma(e^+e^-\to\mu\bar{\mu}:\mathrm{QED})}$ の QCD 効果のスケール依存性[5]

図には $\dfrac{R(Q=100\mathrm{GeV})}{R_0}-1$, $R_0=\dfrac{11}{3}$ を書いてある.

L=対数第 1 近似 (leading log approximation), NL=第 2 近似 (next-to-LLA), NNL=第 3 近似 (next next-to-LLA).

from Nagashima p.139.

By using

$\alpha_s(\mu)$ with $\mu \sim (\sqrt{s}=Q)$

$\Leftarrow$ K^L can be a good approximation.

$\overset{\shortparallel}{1+\frac{\alpha_s(\mu)}{\pi}}$ to $K(Q)=K(Q)^{L+NL+N^2L}+\cdots$

(cf. lecture note §3.3)