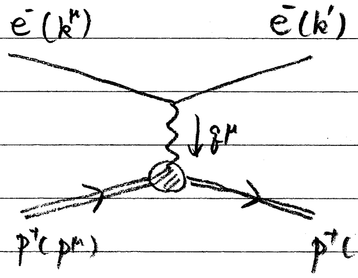
§7 Parton Distribution and Collinear Factorization

§7.1 Deep Inelastic Scattering

☆ experimental foundation for QCD (around 1970)

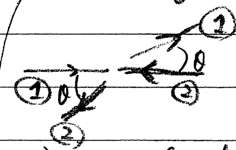
☆ another example of IR safe observable in QCD.

- $e^- + p^+ \longrightarrow e^- + p^+$ elastic scattering $\Rightarrow$ the same set of particles in the in-state and the out-state.



In 2 particle $\to$ 2 particle scatterings

(elastic processes in the center of mass frame.) $\Longleftrightarrow$



just angle changes. no change in $|\vec{p}|$, E

$$(p+q)^2 = (p')^2 = p^2$$

$$\Rightarrow 2(p\cdot q) + q^2 = 0$$

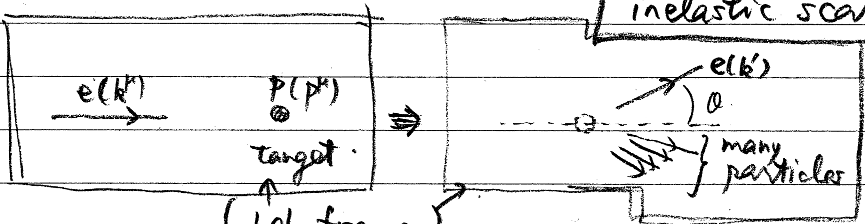
$$\longrightarrow \left(\frac{-q^2}{2p\cdot q} = 1\right)$$

$$\frac{d\sigma}{d\Omega} =: \left(\frac{d\sigma}{d\Omega}\right)_{\substack{\text{point}\\ \text{particle}\\ \text{elastic scatt.}}} \times F(-q^2)$$

$F(-q^2) \longrightarrow$ form factor. describing how much $\left(\frac{d\sigma}{d\Omega}\right)$ differs from $\left(\frac{d\sigma}{d\Omega}\right)_{\text{pt. particle}}$.

- $e(k^\mu) + p(p^\mu) \longrightarrow e(k'^\mu) + (\text{anything})$ — inelastic scattering

(free $\vec{p}, \vec{k}, \vec{k}' \Rightarrow$ 9 DOF
− Lorentz sym $\Rightarrow$ 6 DOF)
$\Rightarrow$ 3 DOF

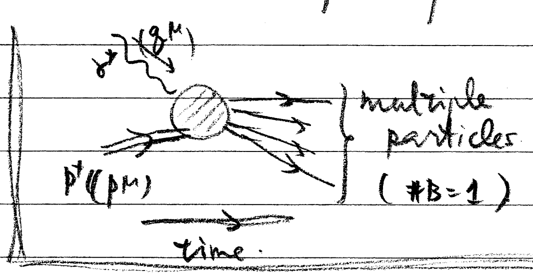


(Lab frame)

Three kinematical variables

$\boxed{\begin{array}{l} \cdot\ s = (k+p)^2 \simeq 2k\cdot p + m_p^2 \simeq 2k\cdot p \\ \cdot\ (q+p)^2 = q^2 + 2q\cdot p + m_p^2 =: W^2 \\ \cdot\ q^2 \end{array}}$

The non-trivial part of this scattering



✓ $x := \frac{-q^2}{2p\cdot q}$ is not necessarily $= 1$ in inelastic scatterings.

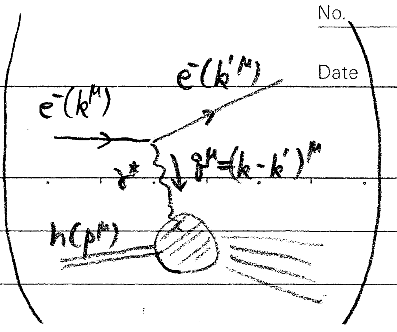
- for a fixed s, data fitting by

$$\frac{d\sigma}{dx\,dQ^2} = \frac{2\pi\alpha_e^2}{xQ^4}\left[2(1-y)\,\sigma^{(L)} + \{1+(1-y)^2\}\,\overline{\sigma}^{(T)}\right]$$

$$\left(Q^2 := -q^2,\quad x := \frac{-q^2}{2p\cdot q},\quad y := \frac{p\cdot q}{p\cdot k}\right)$$

$\Rightarrow \boxed{\sigma^{(L)} \ll \overline{\sigma}^{(T)}}$ and $\boxed{\overline{\sigma}^{(T)} \approx \text{fn of } x,\ \text{not much on } Q^2}$ Bjorken scaling

A brief note on the kinematics in DIS



☆ $\boxed{q^2 < 0}$ $\because$ $[k^2 = (k')^2 = m_e^2]$ & $[k' = k - q]$

$\Rightarrow -2k\cdot q + q^2 = 0.$ In the e^--rest frame $\begin{cases} k^\mu = (m_e, \vec{0}) \\ k'^\mu = (E_e, \vec{k}') \\ q^\mu = (m_e - E_e, -\vec{k}') \end{cases}$

$\boxed{q^2 < 0} \Leftarrow \boxed{k\cdot q = m_e(m_e - E_e) < 0}$

☆ $\boxed{x \leq 1}$ $\because$ no $[\gamma^* + h(p) \longrightarrow (\text{at least one baryon}) + \text{any}]$ scattering.

baryon number conservation

if (the center-of-mass energy)$^2 = (p+q)^2 < m_p^2$.

$\Rightarrow \cancel{m_p^2} + 2p\cdot q + q^2 \geq \cancel{m_p^2} \Leftrightarrow \left[0 \leq -1 + \frac{2p\cdot q}{Q^2} = -1 + \frac{1}{x}\right]$

$\underbrace{(\gamma^* + h \to \text{one proton})}_{\text{elastic scattering}} \Leftarrow (q+p)^2 = m_p^2 \Leftrightarrow x = 1$

☆ at high energy (m_e, m_p negligible)

$$s\, y\, x \simeq (2k\cdot p)\frac{(q\cdot p)}{(k\cdot p)}\frac{Q^2}{2p\cdot q} = Q^2.$$

☆ To read the data of the original SLAC-MIT DIS experiments.

use the proton (target) rest frame.

$(m_e \simeq 0)$
$$\begin{cases} k^\mu \simeq (E, E, 0, 0) \\ k'^\mu \simeq (E', E'\cos\theta, E'\sin\theta, 0) \\ q^\mu = (E-E', E-E'\cos\theta, -E'\sin\theta, 0) \end{cases}$$
$$\{ p^\mu = (m_N, 0, 0, 0)$$

At a fixed e^- beam energy E,

$$dQ^2 dy = d\cos\theta\, dE' \times (2E')$$

$$\frac{d^2\sigma}{dQ^2 dy} = \frac{d^2\sigma}{d\Omega dE'}\frac{2\pi}{2E'}(E', \theta; E)$$

dictionary

$E\, m_N = (k\cdot p)$

$E'/E = 1 - y$

$(E-E')m_N = (q\cdot p)$

$4EE'\sin^2\theta/2$

$2EE'(1-\cos\theta) = Q^2$

$m_N^2 + 2Em_N - 2m_N E' - 4EE'\sin^2\theta/2 = (q+p)^2 =: W^2$

(A) Phys. Rev. Lett. 23 ('69) 930

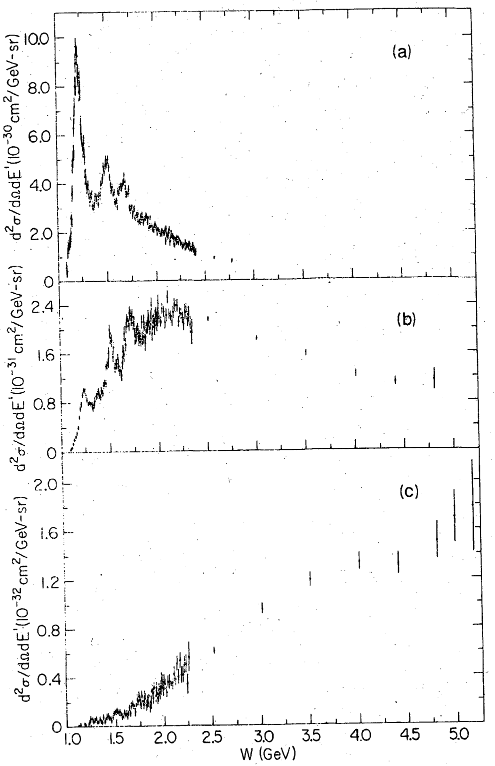


FIG. 2. Three representative radiatively corrected spectra at (a) $\theta = 6°$, $E = 7$ GeV; (b) $\theta = 6°$, $E = 16$ GeV, and (c) $\theta = 10°$, $E = 17.7$ GeV. The ranges of q^2 covere are (a) $0.2 \leq q^2 \leq 0.5$ (GeV/c)^2; (b) $0.7 \leq q^2 \leq 2.6$ (GeV/c)^2; and (c) $1.6 \leq q^2 \leq 7.3$ (GeV/c)^2. The elastic peaks are not shown.

Phys. Rev. Lett. 23 ('69) 935

(B) $\theta = 6°$ or $10°$

log.

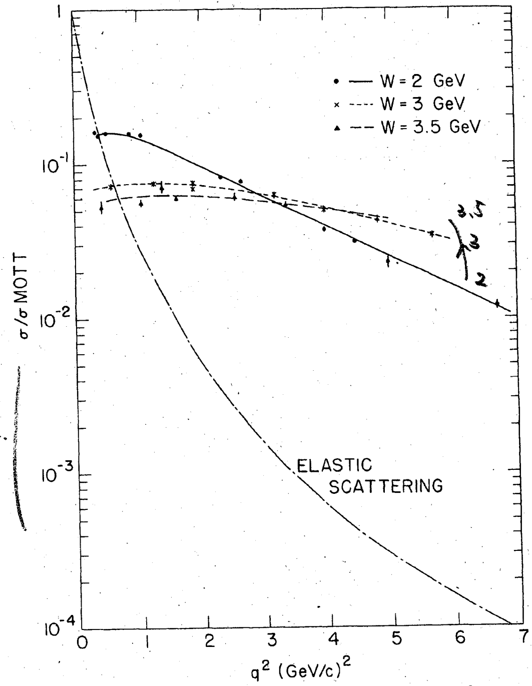


FIG. 1. $(d^2\sigma/d\Omega dE')/\sigma_{\rm Mott}$, in GeV^{-1}, vs q^2 for W = 2, 3, and 3.5 GeV. The lines drawn through the data are meant to guide the eye. Also shown is the cross section for elastic e-p scattering divided by $\sigma_{\rm Mott}$, $(d\sigma/d\Omega)/\sigma_{\rm Mott}$, calculated for $\theta = 10°$, using the dipole form factor. The relatively slow variation with q^2 of the inelastic cross section compared with the elastic cross section is clearly shown.

Phys. Rev. D5 ('72) 528

(C)

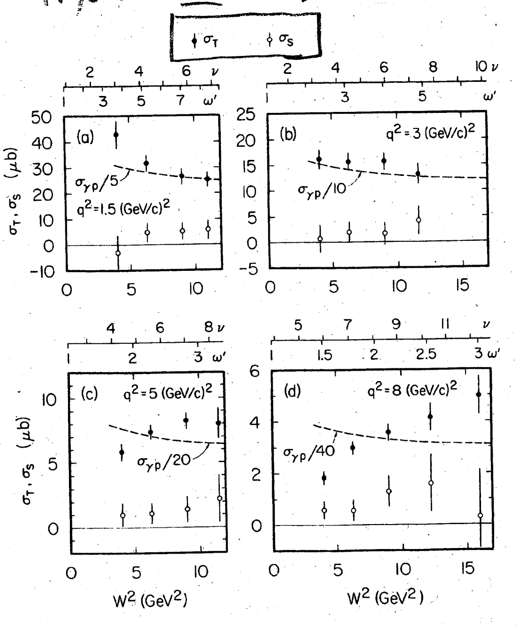


FIG. 8. The values of σ_T and σ_S given in Table III are shown at constant q^2 as a function of W^2 (or ν) for q^2 = 1.5, 3, 5, and 8 (GeV/c)^2. Also shown is the ν dependence of the total photoabsorption cross section.

Phys. Rev. Lett. 23 ('69) 935.

(D)

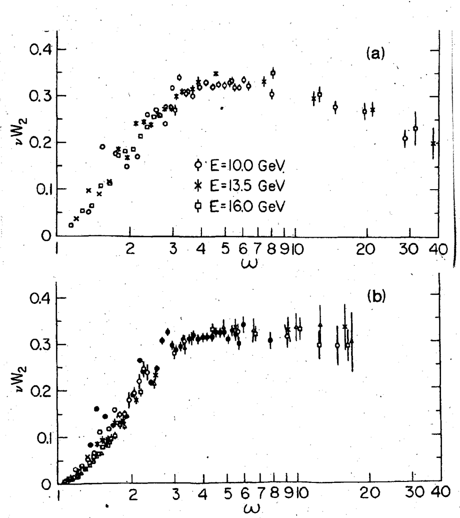


FIG. 2. νW_2 vs $\omega = 2M\nu/q^2$ is shown for various assumptions about $R = \sigma_S/\sigma_T$. (a) 6° data except for 7-GeV spectrum for $R = 0$. (b) 10° data for $R = 0$. (c) 6° data except for 7-GeV spectrum for $R = \infty$. (d) 10° data for $R = \infty$. (e) 6°, 7-GeV spectrum for $R = 0$ and $R = \infty$.

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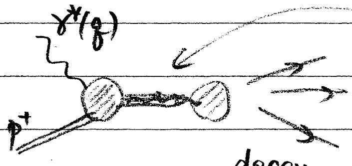
How to read the data?

(A) resonances observed @ $W^2 = (q+p)^2$ a little above m_N^2



resonance (excitation of p^+) states of mass W.

Those resonant peaks (at given masses W) dwindle as

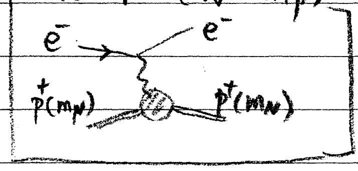
(a) $\Rightarrow$ (b) @ given W, (fixed $\theta = 6^\circ$) $E \nearrow$ then $\left(E' \nearrow \text{ but } (E-E') \nearrow\right) \rightsquigarrow 2q\cdot p \nearrow \; Q^2 \nearrow \rightsquigarrow x \nearrow 1$

(b) $\Rightarrow$ (c) @ given W, $\left(E \simeq \begin{matrix} 16\,\text{GeV} \\ \sim 17.7\,\text{GeV} \end{matrix}\right)$ $\theta \nearrow$ then $(Q^2 \nearrow, \; 2q\cdot p \nearrow) \rightsquigarrow (x \nearrow 1)$

No resonance when a proton is probed by γ^* with $Q^2 \gg 1\,\text{GeV}^2$

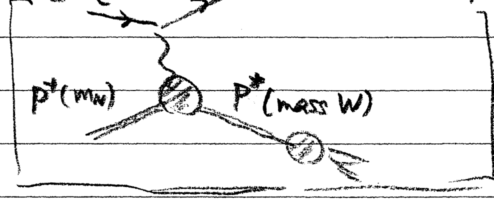
(B) $\left[(A) \simeq \frac{d^2\sigma}{dQ^2 dy} \; y\text{-dependence} \simeq E'\text{-dependence}; \; (B): Q^2\text{-dependence} \sim \theta\text{-dependence} \right]$

$x = 1 \quad (W = m_p)$



but as $W \nearrow$ 2GeV, 3GeV, 3.5GeV

?? (in the resonance picture)



elastic scattering.

$\left[\sigma \propto |F(Q^2)|^2 \right.$ form factor $\approx$ Fourier Transf. of change distribution $\left.\right]$

$$F(Q^2) \sim \frac{1}{(1 + Q^2 \cdot (\text{size})^2)^p}$$

$$\left[d\sigma \propto |F(Q^2)|^2 \, ? \quad F(Q^2) \sim \int \langle p^* | J^\mu(x) e^{-iq\cdot x} | p^+ \rangle \, d^4x \right]$$

The (resonance) $\times$ (form factor) picture breaks down as $W \nearrow$.

no sign of a spatially spread object.

PLUS

(C) Fig. 8 of Phys. Rev. D5 ('72) 528.

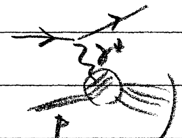
$$\frac{d^2\sigma}{dQ^2dy}(W^2,Q^2,y) =: \frac{\alpha_e}{2\pi}\frac{Q^2}{Q^4}\frac{(W^2-m_p^2)}{(2k\cdot p)}\left(\frac{\varepsilon}{1-\varepsilon}\,\sigma^S_{\gamma^*p}(W^2,Q^2)+\frac{1}{1-\varepsilon}\,\overline{\sigma^T_{\gamma^*p}}(W^2,Q^2)\right)$$

$$\left\{\frac{\varepsilon}{1-\varepsilon} := \frac{2(p\cdot k)(p\cdot k')-m_N^2(k\cdot k')}{\left(m_N^2-\frac{(p\cdot q)^2}{q^2}\right)2(k\cdot k')} \xrightarrow{\text{drop } m_N^2} 2\frac{(p\cdot k)((p\cdot k)-(p\cdot q))}{(p\cdot q)^2}=\frac{2(1-y)}{y^2}\right.$$

$$\left.\underset{\text{Lab frame}}{\Rightarrow}\ \frac{4EE'-Q^2}{2\left(1+\frac{(E-E')^2}{Q^2}\right)Q^2}=\frac{1}{2\left(1+\frac{(E-E')^2}{Q^2}\right)\tan^2(\theta/2)}\right\}$$

[For each (W,Q), the data on the LHS (as a function of y) is fit by the two numbers σ^S and $\overline{\sigma^T}$ (ε is a function of y)]

The result of the fit $\Rightarrow$ $\boxed{\sigma^S \ll \overline{\sigma^T}}$ (Callan–Gross relation.)

memo: a thought behind this fit $\Rightarrow$ 

as γ^*+p^+ scattering

$\Rightarrow$ factor out $\frac{e^2}{Q^4}$, adjust $\left\{\frac{1}{2k\cdot p}\int dPS_{e,\mathrm{any}} \Rightarrow \frac{1}{(W^2-m_p^2)}\int dPS_{\mathrm{any}}\right\}$

γ^* flux via

$$[k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(k\cdot k')] = \left[\left(p_\mu - \frac{(p\cdot q)q_\mu}{q^2}\right)\left(p_\nu-\frac{(p\cdot q)q_\nu}{q^2}\right)\frac{1}{\left(m_N^2-\frac{(p\cdot q)^2}{q^2}\right)}\right]\frac{\varepsilon}{1-\varepsilon}Q^2$$

$$+\frac{1}{2}\left[\left(p_\mu-\frac{(p\cdot q)q_\mu}{q^2}\right)\left(p_\nu-\frac{(p\cdot q)q_\nu}{q^2}\right)\frac{1}{\left(m_N^2-\frac{(p\cdot q)^2}{q^2}\right)}+\left(-\eta_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\right]\frac{1}{1-\varepsilon}Q^2.$$

$$=: [d^{(S)}_{\mu\nu}]\frac{\varepsilon}{1-\varepsilon}Q^2 + \frac{1}{2}[2d^{(T)}_{\mu\nu}]\frac{1}{1-\varepsilon}Q^2.$$

↳ transversely polarized γ^*

(D) Fig. 2 of Phys. Rev. Lett. 23 ('69) 935.

$$e^2(m_N W_1) := (W^2-m_p^2)\frac{\overline{\sigma^T}}{2\pi} \qquad e^2\left(\frac{W_2}{m_N}\right) = (W^2-m_p^2)\frac{Q^2}{m_p^2Q^2+(q\cdot p)^2}\frac{\overline{\sigma^T}+\sigma^S}{2\pi}$$

$(E-E')W_2 = (q\cdot p)\frac{W_2}{m_p}$ is a function of $(W^2,Q^2) = \left(Q^2\left(\frac{1}{x}-1\right),Q^2\right)$

↳ plotted data

for a given $\omega := 1/x$,

$\left[\begin{array}{l}(a):\ \theta=6^\circ\\ (b):\ \theta=10^\circ\end{array}\right.$ fixed$\left.\right]$ $\Rightarrow$ (varying E ⇝ varying Q^2)

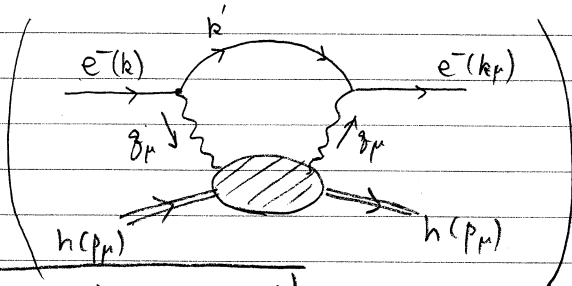
but $(q\cdot p)W_2$ depends on (W^2,Q^2) only through x.

(Bjorken scaling)

§ 7.2 DIS Structure Functions

(Ignore m_e, m_p)

$$\sigma(e^- + p^+ \to e^- + \text{anything}) \simeq \frac{1}{2S}\, 2\,\mathrm{Im}\left(\mathcal{M}(e^-p^+ \to e^-p^+)\right) \qquad \text{optical thm}$$



The unknown part

$J_f^\mu = (\bar{\Psi}_f \gamma^\mu \Psi_f)$ $(f = u, d, s, \cdots)$

$$\int d^4x \int d^4y\, (-ieQ_f)^2 \langle h(p)|\, T\{J_f^\mu(x)\, J_f^\nu(y)\}\, |h(p)\rangle\, e^{iq'\cdot x}\, e^{-iq\cdot y}$$

$\parallel$ translational invariance of the ME.

$$\int d^4x \int d^4y\, (-ieQ_f)^2 \langle h(p)|\, T\{J_f^\mu(0)\, J_f^\nu(y-x)\}\, |h(p)\rangle\, e^{iq'\cdot x}\, e^{-iq\cdot y}$$

$$= \underbrace{\int d^4x\, e^{i(q'-q)\cdot x}}_{(2\pi)^4\delta^4(q'-q)} \underbrace{\int d^4(y-x)\, (-ieQ_f)^2 \langle h(p)|\, T\{J_f^\mu(0)\, J_f^\nu(y-x)\}\, |h(p)\rangle\, e^{-iq\cdot(y-x)}}_{(i e^2 T^{\mu\nu}(p,q))}$$

Compton tensor ($\gamma^* + h$ scattering amplitude.)

If the e^- beam is not polarized, and the spin of $e^-(k')$ is not measured,

$$\sigma(e^-p^+ \to e^- + X) \simeq (\text{homework IX-5}) \simeq$$

$$\boxed{\begin{aligned} &\frac{1}{4p\cdot k}\int dQ^2 \int dy\, \frac{\alpha_e^2}{Q^4}\, 2\left[k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(k\cdot k')\right] 2\,\mathrm{Im}(T^{\mu\nu}) \\ &= \int dQ^2 \int dx\, \frac{y^2\alpha_e^2}{Q^6}\left[k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(k\cdot k')\right] 2\,\mathrm{Im}(T^{\mu\nu}) \end{aligned}}$$

From the gauge invariance of QED,

$$q_\mu T^{\mu\nu} = 0 \qquad T^{\mu\nu} q_\nu = 0$$

$$\Rightarrow \boxed{T^{\mu\nu} = (4\pi)\left\{\left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1 + \frac{1}{(p\cdot q)}\left[p^\mu - \frac{(p\cdot q)}{q^2} q^\mu\right]\left[p^\nu - \frac{(p\cdot q)}{q^2} q^\nu\right] T_2\right\}}$$

parametrized by 2 functions.

$$F_1 := 2\,\mathrm{Im}(T_1) \qquad F_2 := 2\,\mathrm{Im}(T_2)$$

Then

$$\boxed{\frac{d\sigma_{DIS}}{dQ^2 dx}} = \frac{y^2 \alpha_e^2}{Q^6}(4\pi)\left[k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(k\cdot k')\right]\left\{\left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)F_1 + \frac{1}{p\cdot q}[\quad]^\mu[\quad]^\nu F_2\right\}$$

$$= \frac{y^2\alpha_e^2}{Q^6}(4\pi)\left(\left[2(k\cdot k') + \underbrace{\left\{\frac{2(q\cdot k)(q\cdot k')}{q^2} - (k\cdot k')\right\}}_{\to 0}\right]F_1 + \frac{F_2}{(p\cdot q)}\left[\begin{array}{l} 2(p\cdot k)(p\cdot k') \\ -2\frac{p\cdot q}{q^2}\underbrace{\{(p\cdot k)(q\cdot k') + (p\cdot k')(q\cdot k) - (p\cdot q)(k\cdot k')\}}_{\to 0} \\ +\frac{(p\cdot q)^2}{(q^2)^2}\underbrace{\{2(q\cdot k)(q\cdot k') - q^2(k\cdot k')\}}_{\to 0} \end{array}\right]\right)$$

use $0 \simeq (k')^2 = (k-q)^2 \simeq -2k\cdot q + q^2 \Rightarrow k\cdot q \simeq q^2/2$ (Ward-Takahashi id.)

$q^2 = (k-k')^2 \simeq -2k\cdot k' \Rightarrow k\cdot k' \simeq -q^2/2$

$$= \frac{4\pi\alpha_e^2}{Q^4}\left(y^2 F_1 + \frac{(1-y)}{x}F_2\right)$$

$\left(F_1 = m_p W_1, \quad F_2 = \frac{W_2}{m_p}(q\cdot p) \text{ in fact}\right)$

) reorganized

$$= \boxed{\frac{4\pi\alpha_e^2}{Q^4}\left[\frac{(1-y)}{x}(F_2 - 2xF_1) + \{1+(1-y)^2\}F_1\right]}$$

$F_1(x, Q^2)$, $F_2(x, Q^2)$: structure functions.

(experimental data) $\Rightarrow$
- Bjorken scaling: F_1, F_2 depend primarily on x not on Q^2
  $\leftrightarrow$ point-like constituent.
- Callan-Gross relation: $(F_2 - 2xF_1) \ll F_2$
  $\leftrightarrow$ spin-$\frac{1}{2}$ constituent (quark)

$$\overline{|\mathcal{M}|^2} \propto \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2} \propto \frac{(k\cdot p)^2 + (k'\cdot p)^2}{Q^4} = \frac{(k\cdot p)^2}{Q^4}\{1+(1-y)^2\}$$

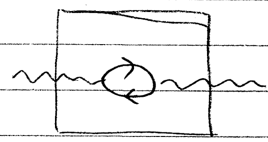
$\uparrow$ ($ef_{1/2} \to ef_{1/2}$) $\qquad \leftarrow$ (homework IX-3)

§ 7.3 Evaluation of $T^{\mu\nu}$ by OPE

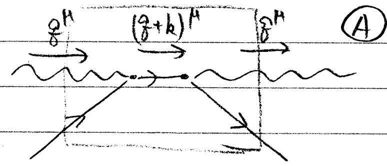
$$T^{\mu\nu} := i Q_f^2 \int d^4y \, \langle h(\vec p)| T\{J_f^\mu(0)\, J_f^\nu(y)\} | h(\vec p)\rangle \, e^{-iq\cdot y} \quad \text{w/ space-like } q^2<0$$

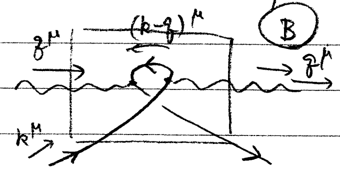
(instead of $\Pi^{\mu\nu} := i e^2 Q_f^2 \int d^4y \, \langle\Omega|T\{J_f^\mu(0)\, J_f^\nu(y)\}|\Omega\rangle \, e^{-iq\cdot y}$ [w/ $q^2>0$ (for $e^+e^-\to$ hadrons)]
vacuum polarization)

☆ $i\int d^4y \, e^{-iq\cdot y} \, T\{J^\mu(0)\, J^\nu(y)\}$ OPE

• begins with $(q^2\eta^{\mu\nu} - q^\mu q^\nu)\Pi_{ren}(q^2)\, \mathbb{1}_{operator}$ 

But $\mathrm{Im}(\Pi_{ren}(q^2)) = 0$ if $q^2<0$.

• 

+ 

A: $i\int d^4y \, e^{-iq\cdot y}\int \frac{d^4k}{(2\pi)^4}\, \bar\psi(-\frac y2)\gamma^\mu \frac{i(\cancel{q}+\cancel{k})\, e^{i(k+q)\cdot y}}{(q+k)^2+i\epsilon}\gamma^\nu\psi(+\frac y2)$

$= (-)\int d^4y \int\frac{d^4k}{(2\pi)^4}\, e^{ik\cdot y}\, e^{\frac y2\cdot(\overleftarrow\partial - \overrightarrow\partial)} \frac{\bar\psi(0)\gamma^\mu(\cancel q+\cancel k)\gamma^\nu\psi(0)}{(q+k)^2+i\epsilon}$

$(2\pi)^4\delta^4(k - \frac{i}{2}\overbrace{(\overleftarrow\partial-\overrightarrow\partial)}^{\overleftrightarrow{\partial}})$

$= (-)\,\bar\psi(0)\frac{\gamma^\mu\, i(\cancel q + \frac i2 \overleftrightarrow{\cancel\partial})\gamma^\nu}{(q+\frac i2\overleftrightarrow\partial)^2+i\epsilon}\psi(0)$

expand $[q^2 + i q\cdot\overleftrightarrow\partial - \frac14(\overleftrightarrow\partial)^2]$
with respect to $(q\cdot\overleftrightarrow\partial / q^2)$

(equation of motion
$\cancel D\psi=0 \Rightarrow \cancel D\cancel D\psi = (D^2 + \frac{ig}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu})\psi = 0$

B: $i\int d^4y \, e^{-iq\cdot y}\int\frac{d^4k}{(2\pi)^4}\, \bar\psi(+\frac y2)\gamma^\nu\frac{i(\cancel k - \cancel q)\, e^{-i(k-q)\cdot y}}{(k-q)^2+i\epsilon}\gamma^\mu\psi(-\frac y2)$

$= \cdots = (+)\,\bar\psi(0)\frac{\gamma^\nu(\cancel q - \frac i2\overleftrightarrow{\cancel\partial})\gamma^\mu}{(q-\frac i2\overleftrightarrow\partial)^2+i\epsilon}\psi(0)$

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OPE $\quad i\int d^4y\ e^{-iq\cdot y}\, T\{J_f^\mu(-y/2)\, J_f^\nu(+y/2)\}$

$$= (q^2\eta^{\mu\nu} - q^\mu q^\nu)\,\Pi_{ren.}(q^2)\,\mathbb{1}$$

local operator

$$+ \sum_{j=2}^{\infty} C^{\mu\nu}_{\lambda_1\cdots\lambda_j}(q)\ \left[\bar\psi_f \gamma^{\lambda_1}\left(\tfrac{i}{2}\overleftrightarrow{\partial}\right)^{\lambda_2}\cdots\left(\tfrac{i}{2}\overleftrightarrow{\partial}\right)^{\lambda_j}\psi_f\right](0) \quad \rightarrow \boxed{\text{twist} = (2+j)-j = 2}$$

$$+ (\mu \leftrightarrow \nu \text{ anti symmetric part})$$

$$+ (\text{coeff.}) \times \text{operator such as } (\bar\psi_f \gamma^\kappa F^{\rho\sigma}\psi_f)(0) \cdots \quad \text{twis} = 5-1 = 4$$

$$(\text{twist}) := (\text{naive operator dim}) - \text{spin}\left(\text{repr. of } SO(3,1)\right)$$

Insert those local operators in $\langle h(\vec p)|\qquad |h(\vec p)\rangle$.

(A_j for $f = u, d, s$ may be different)

$$\langle h(\vec p)|\left[\bar\psi\gamma^{\lambda_1}\left(\tfrac{i}{2}\overleftrightarrow{D}\right)^{\lambda_2}\cdots\left(\tfrac{i}{2}\overleftrightarrow{D}\right)^{\lambda_j}\psi\right]_{\text{sym. traceless}} |h(\vec p)\rangle =: p^{\lambda_1}p^{\lambda_2}\cdots p^{\lambda_j} A_j$$

non-perturbative information

Hermitian op $\Rightarrow A_j \in \mathbb{R}$.

Use the explicit expressions of $C^{\mu\nu}_{\lambda_1\cdots\lambda_j}(q)$

to obtain (homework IX-4)

$$\boxed{\left(\underset{\mu\nu\text{ sym, twist-2}}{T^{\mu\nu}}\right) = \left\{\left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + \frac{1}{(p\cdot q)}\left[p^\mu - q^\mu\frac{p\cdot q}{q^2}\right]\left[p^\nu - q^\nu\frac{(p\cdot q)}{q^2}\right]2x\right\} \times \sum_{j=1}^{\infty}\left[1+(-)^j\right]\left(\frac{1}{x}\right)^j\left(+\frac{A_j}{2}\right)(Q_f^2)}$$

$\Rightarrow$ consistent with the Ward-Takahashi identity

$\Rightarrow$ Callan-Gross relation. $\quad (F_2 = 2xF_1)$

- local operator $\mathcal{O}$ of naive mass-dimension Δ and spin S
  $\langle \text{state}(\vec p)|\mathcal{O}|\text{state}(\vec p)\rangle \sim p^S \Lambda_{QCD}^{(\Delta - S)}$
- OPE $\quad \Sigma_i (\text{energy scale}_Q)^{\text{fixed}-\Delta_i}\, \mathcal{O}_i$

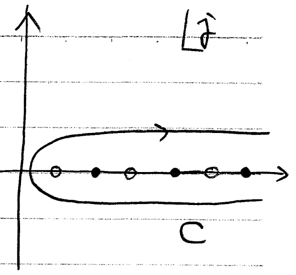
$$\Sigma_i\, Q^{\text{fixed}-\Delta_i} \times p^{S_i}\Lambda_{QCD}^{(\Delta_i - S_i)} = Q^{\text{fixed}}\, \Sigma_i \left(\frac{p}{Q}\right)^{S_i}\left(\frac{\Lambda_{QCD}}{Q}\right)^{\overbrace{\Delta_i - S_i}^{\text{twist}}}$$

- If $\langle \text{vac}|\mathcal{O}|\text{vac}\rangle$ is used instead. $\Rightarrow$ only $(S_i = 0)$ contributes $\Rightarrow$ expansion by Δ_i.

PLUS

$$T_1 = \frac{+1}{8\pi} \sum_{j=1}^{\infty} [1+(-)^j] \frac{1}{x^j} \sum_f A_{j,f} Q_f^2 \quad \Leftarrow \text{ expansion @ } 1 \ll x$$

$$= \frac{1}{8\pi} \int_C \frac{dj}{2i} \left[-\frac{1+e^{-\pi i j}}{\sin(\pi j)} \right] \frac{1}{x^j} A_f^+(j) Q_f^2 \quad \text{continuation to } x<1.$$



- $A^+(j)$: holomorphic fn on j. $\quad A^+(j \in 2\mathbb{N}) = A_{j \in 2\mathbb{N}}$.
- $\dfrac{1+e^{-\pi i j}}{\sin(\pi j)}$: pole @ $j \in 2\mathbb{Z}$. residue $= \dfrac{2}{\pi}$.

(assume that $A^+(j)$ has poles/cuts only @ $j \in \mathbb{R}$, and the residue is real.)

$$\boxed{2\,\mathrm{Im}(T_1) = \frac{1}{4\pi} \int_{-i\infty}^{+i\infty} \frac{dj}{2i} \frac{1}{x^j} A_f^+(j) Q_f^2}$$

$$\boxed{\int_0^{+\infty} dx \, [2\,\mathrm{Im}(T_1)](x) \, x^{j-1} = \frac{1}{4} A_f^+(j) Q_f^2 .}$$

Mellin transformation $\displaystyle\int_0^{+\infty} dx \, f(x) \, x^{j-1} =: \hat{f}(j)$

inverse Mellin transformation $\displaystyle f(x) = \int_{-i\infty}^{+i\infty} \frac{dj}{2\pi i} \left(\frac{1}{x}\right)^j \hat{f}(j)$

(Fourier transformation $\ln(x) \leftrightarrow j/\sqrt{-1}$.)

(Laplace transformation $\ln(x) \leftrightarrow j$)

Structure functions are given by the inverse Mellin transform of the proton matrix elements of twist-2 quark operators.

§ 7.4 Parton Distribution Functions (PDF)

For a hadron h and a quark field Ψ_q $(q = u, d, s, \cdots)$.

$$\boxed{f_{q/h}(\xi) := \frac{1}{4\pi}\int_{-\infty}^{+\infty} dk\, e^{ik\xi} \langle h(\vec{p})| \left[\overline{\Psi}_q(-\tfrac{\bar{n}}{2}k)\, \bar{\cancel{n}}\, \Psi_q(+\tfrac{\bar{n}}{2}k)\right] |h(\vec{p})\rangle}$$

$(\xi > 0)$

- $\bar{n}^\mu$: a light-like future-pointing vector s.t. $\bar{n}_\mu p^\mu = 1$ $\left(\dim_{=-1}\right)$
- The idea: $\displaystyle\int_{\mathbb{R}} dk\, e^{ik\xi} \int \frac{d^3\vec{k}}{(2\pi)^3} \underbrace{a^\dagger_{\vec{k}} a_{\vec{k}}}_{\text{quark}} e^{i\underline{k}\cdot(-\frac{\bar{n}}{2}k)} e^{-i\underline{k}\cdot(+\frac{\bar{n}}{2}k)}$ $\leftarrow$ $\underline{k}^\mu = (E_{\vec{k}}, \vec{k})$

  is supported on $\vec{k}$'s where $\underline{k}^\mu = \xi p^\mu + \left(\text{orthogonal to } \bar{n}_\nu\right)$

  $\left(\displaystyle\int dk\, e^{ik\xi} \int \frac{d^3\vec{k}}{(2\pi)^3} \underbrace{(b_{\vec{k}} b^\dagger_{\vec{k}})}_{\text{antiquark}} e^{-i\underline{k}\cdot(-\frac{\bar{n}}{2}k)} e^{i\underline{k}\cdot(+\frac{\bar{n}}{2}k)} = 0 \text{ when } \xi > 0 \right)$

- (Wilson line, gauge invariance, $\bar{n}^\mu$-dependence $\Rightarrow$ later)

The anti-quark PDF

$$f_{\bar{q}/h}(\xi) := \frac{1}{4\pi}\int_{-\infty}^{+\infty} dk\, e^{ik\xi} \langle h(\vec{p})| \left[\overline{\Psi}^c_{(q)}(-\tfrac{\bar{n}}{2}k)\, \bar{\cancel{n}}\, \Psi^c_{(q)}(+\tfrac{\bar{n}}{2}k)\right] |h(\vec{p})\rangle$$

$(\xi > 0)$

$$= \frac{-1}{4\pi}\int_{-\infty}^{+\infty} dk\, e^{ik\xi} \Big(\langle h(\vec{p})| \left[\overline{\Psi}_q(+\tfrac{\bar{n}}{2}k)\, \bar{\cancel{n}}\, \Psi_q(-\tfrac{\bar{n}}{2}k)\right] |h(\vec{p})\rangle - \lim_{\vec{p}'\to\vec{p}} \langle h(\vec{p}')|h(\vec{p})\rangle \langle \Omega| \overline{\Psi}_q(+\tfrac{\bar{n}}{2}k)\, \bar{\cancel{n}}\, \Psi_q(-\tfrac{\bar{n}}{2}k)|\Omega\rangle \Big)$$

$k' := -k$

$$= \frac{-1}{4\pi}\int_{-\infty}^{+\infty} dk'\, e^{-ik'\xi} \Big(\langle h(\vec{p})| \left[\overline{\Psi}_q(-\tfrac{\bar{n}}{2}k)\, \bar{\cancel{n}}\, \Psi_q(+\tfrac{\bar{n}}{2}k)\right] |h(\vec{p})\rangle - \lim_{\vec{p}'\to\vec{p}} \langle h(\vec{p}')|h(\vec{p})\rangle \langle \Omega| \overline{\Psi}_q(-\tfrac{\bar{n}}{2}k)\, \bar{\cancel{n}}\, \Psi_q(+\tfrac{\bar{n}}{2}k)|\Omega\rangle \Big)$$

- Dirac spinor $\Psi_q = \begin{pmatrix}\psi \\ \bar{\psi}^c\end{pmatrix}$ Weyl spinor decomposit'n $\Rightarrow \Psi^c = \begin{pmatrix}\psi^c \\ \bar{\psi}\end{pmatrix}$

- Extend the def of $f_{q/h}(\xi)$ and $f_{\bar{q}/h}(\xi)$ to $\xi < 0$ by

$$f_{q/h}(\xi) := \frac{1}{4\pi}\int_{-\infty}^{+\infty} dk\, e^{ik\xi} \Big(\langle h(\vec{p})| \left[\overline{\Psi}_q(-\tfrac{\bar{n}}{2}k)\, \bar{\cancel{n}}\, \Psi_q(+\tfrac{\bar{n}}{2}k)\right] |h(\vec{p})\rangle - \lim_{\vec{p}'\to\vec{p}} \langle h(\vec{p}')|h(\vec{p})\rangle \times \langle \Omega| \overline{\Psi}_q(-\tfrac{\bar{n}}{2}k)\, \bar{\cancel{n}}\, \Psi_q(\tfrac{\bar{n}}{2}k)|\Omega\rangle \Big)$$

$f_{\bar{q}/h}(\xi) :=$ do the same with $\Psi^c_{(q)}$ instead of Ψ_q.

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- It is one's freedom to define anything new (such as $f_{g/h}(\xi)$ and $f_{\bar{g}/h}(\xi)$ for $\xi<0$)

- The new def. for $f_{g/h}(\xi)$ with $(\xi>0)$ is the same as before.

$$(\xi>0)\quad \left.\begin{aligned} &\underbrace{\int_{\mathbb{R}} dk\, e^{ik\xi}\int\frac{d^3\underline{k}}{(2\pi)^3}\, e^{i\underline{k}\cdot(-\frac{\bar{n}}{2})k}\, e^{-i\underline{k}\cdot(\frac{\bar{n}}{2}k)}}_{(\neq 0 \text{ if } \underline{k}^\mu=\xi p^\mu+(\bar{n}\text{-orthog}))} \langle\Omega|\, a^\dagger_{\underline{k}}\, \underbrace{a_{\underline{k}}|\Omega\rangle}_{\Rightarrow 0} \\ &\underbrace{\int_{\mathbb{R}} dk\, e^{ik\xi}\int\frac{d^3\vec{k}}{(2\pi)^3}\, e^{-i\underline{k}\cdot(-\frac{\bar{n}}{2}k)}\, e^{i\underline{k}\cdot(\frac{\bar{n}}{2}k)}}_{=0 \text{ when } \xi>0}\ \underbrace{\langle\Omega|\, b_{\underline{k}}\, b^\dagger_{\underline{k}}\,|\Omega\rangle}_{\neq 0} \end{aligned}\right\} \text{both}=0 \text{ if } \xi>0.$$

- benefit : $\exists$ relation $\underline{f_{g/h}(-\xi) = -f_{\bar{g}/h}(\xi) \quad (\forall \xi\in\mathbb{R})}$.

Now, for $\forall j=1,2,\cdots$

$$\int_0^{+\infty} d\xi\ \xi^{j-1}\left(f_{g/h}(\xi)+(-1)^j f_{\bar{g}/h}(\xi)\right)$$

$$=\frac{1}{4\pi}\int_0^{\infty}d\xi\ \xi^{j-1}\int_{-\infty}^{+\infty}dk\left(e^{ik\xi}+(-1)^{j-1}e^{-ik\xi}\right)\left(\begin{aligned}&\langle h(\vec{p}')|\,\bar{\Psi}_g(-\tfrac{\bar{n}}{2}k)\,\bar{\cancel{n}}\,\Psi_g(\tfrac{\bar{n}}{2}k)\,|h(\vec{p})\rangle\\ &-\langle h(\vec{p}')|h(\vec{p})\rangle\langle\Omega|\,\bar{\Psi}_g(-\tfrac{\bar{n}}{2}k)\,\bar{\cancel{n}}\,\Psi_g(+\tfrac{\bar{n}}{2}k)\,|\Omega\rangle\end{aligned}\right)_{(\vec{p}'\to\vec{p}.)}$$

$$=\int_0^{+\infty}d\xi\int_{-\infty}^{+\infty}\frac{dk}{4\pi}\left((-i\partial_k)^{j-1}\left(e^{ik\xi}+e^{-ik\xi}\right)\right)\left(\text{ the same as above}\right)$$

$$=\int_0^{\infty}d\xi\int_{\mathbb{R}}\frac{dk}{4\pi}\left(e^{ik\xi}+e^{-ik\xi}\right)(i\partial_k)^{j-1}\left(\text{the same as above}\right)$$

$$=\int_{\mathbb{R}}d\xi\int_{\mathbb{R}}\frac{dk}{4\pi}\ e^{ik\xi}\,(i\partial_k)^{j-1}\left(\text{the same as above}\right)$$

$$=\frac{1}{2}\,(i\partial_k)^{j-1}\left(\langle h(\vec{p}')|\,\bar{\Psi}_g(-\tfrac{\bar{n}}{2}k)\,\bar{\cancel{n}}\,\Psi_g(+\tfrac{\bar{n}}{2}k)\,|h(\vec{p})\rangle-\langle h(\vec{p}')|h(\vec{p})\rangle\langle\Omega|\,\bar{\Psi}_g(-\tfrac{\bar{n}}{2}k)\,\bar{\cancel{n}}\,\Psi_g(\tfrac{\bar{n}}{2}k)\,|\Omega\rangle\right)_{\substack{\vec{p}'\to\vec{p}\\ k=0}}$$

$$=\frac{1}{2}\left(\langle h(\vec{p}')|\left(\bar{\Psi}_g\,\bar{\cancel{n}}\,\left(\tfrac{i}{2}\bar{n}\cdot\overleftrightarrow{\partial}\right)^{j-1}\Psi_g\right)|h(\vec{p})\rangle-\langle h(\vec{p}')|h(\vec{p})\rangle\langle\Omega|\left(\bar{\Psi}_g\,\bar{\cancel{n}}\,\left(\tfrac{i}{2}\bar{n}\cdot\overleftrightarrow{\partial}\right)^{j-1}\Psi_g\right)|\Omega\rangle\right)_{\vec{p}'\to\vec{p}}$$

$$=\frac{1}{2}\,\langle h(\vec{p})|\left(\bar{\Psi}_g\,\bar{\cancel{n}}\left(\tfrac{i}{2}\bar{n}\cdot\overleftrightarrow{\partial}\right)^{j-1}\Psi_g\right)_{\text{normal ordered}}|h(\vec{p})\rangle \quad \Big\downarrow (\text{use } \bar{n}\cdot p=1)$$

$$=\frac{1}{2}\,A_{j,g}.$$

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(We have seen in §7.3) $\Rightarrow$ $\boxed{F_1(x) \overset{\text{OPE@tree}}{=} \int_{-i\infty}^{+i\infty} \frac{dj}{2\pi i} \frac{1}{x^j} \frac{1}{4}\left(\sum_f^{u,d,\dots} Q_f^2 A_f^+(j)\right)}$

(We have just seen now) $\boxed{\left(f_{f/h}(x) + f_{\bar{f}/h}(x)\right) = \int_{-i\infty}^{+i\infty} \frac{dj}{2\pi i} \frac{1}{x^j} \frac{1}{2} A_f^+(j)}$

(for $x \geq 0$ as we use Mellin transformation)

$$\Rightarrow \boxed{F_1(x) \underset{\text{@tree}}{=} \frac{1}{2} \sum_{f=u,\dots}^{A.S.} Q_f^2 \left\{f_{f/h}(x) + f_{\bar{f}/h}(x)\right\}} \quad x \geq 0.$$

(memo) $\int_0^\infty \left(f_f(x) + f_{\bar{f}}(x)\right) x^{j-1} dx \Big|_{j=1}$ is $A_f^+(j)\Big|_{j=1}$ ← ("the" almost holomorphic function of j that reproduces $A_{j,f}$ @ $j=2,4,6,8,\dots$)

(For h = proton $A_{j,f} = 2$ $\binom{j=1}{f=u}$, $A_{j,f} = 1$ $\binom{j=1}{f=d}$) $\int_0^\infty \left(f_f(x) - f_{\bar{f}}(x)\right) x^{j-1} dx \Big|_{j=1}$ is $A_{j=1,f}$

$\boxed{\text{They are not necessarily the same.}}$

$\boxed{A_f^-(j) : \text{"the" almost holomorphic function of } j \text{ that reproduces } A_{j,f} \text{ @ } j=1,3,5,7,\dots}$

$\boxed{\text{Carlson's theorem}}$ If $f(t)$ is a complex analytic function of t s.t. (i) $\exists k<\pi$, $|f(t)| \leq e^{k|t|}$ @ large $|t|$ region and (ii) $\exists n \in \mathbb{Z}$, $f(t)=0$ @ $t=n, n+1, n+2, \dots$ then $f(t)=0$.

Refs.
(Collins "Introduction to Regge theory and High-energy Physics" §2.5+§2.7)
(Forshaw-Ross "Quantum Chromodynamics and the Pomeron" §1.3)

Apply this theorem for a hypothetical modification to $\left\{\begin{matrix} A_f^+(j) \\ A_f^-(j) \end{matrix}\right.$ with $\left.\begin{matrix} t=\frac{j}{2} \\ t=\frac{j-1}{2} \end{matrix}\right\}$.

The condition (ii) allows $\left\{\begin{matrix} A_f^+(j) + ⓐ \sin\left(\frac{\pi}{2}j\right) =: A_f^+(j)^{\text{new}} \\ A_f^-(j) + ⓐ \cos\left(\frac{\pi}{2}j\right) =: A_f^-(j)^{\text{new}} \end{matrix}\right\}$, but

there can be at most one value of ⓐ where the condition (i) is satisfied.

$\Rightarrow A_f^\pm(j)$ is "the" choice in this sense.

The OPE analysis for $F_1(x)$ and $F_2(x)$ at $\mathcal{O}(\alpha_s)$

the gluon pdf also contributes to F_1 and F_2.

(The Callan-Gross relation no longer holds at this level. $F_2(x) = 2xF_1(x)$)

PLUS

parton model is (an idea / hypothesis; an easy prescription) for ~~high-energy interactions~~ of a hadron, h.

- replace $\langle h(\vec{p})| \cdots |h(\vec{p})\rangle$ by

$$\boxed{\sum_i \int_0^1 \frac{d\xi}{\xi} \langle q_i(\xi\vec{p})| \cdots |q_i(\xi\vec{p})\rangle \cdot f_{q_i/h}(\xi) \quad (q_i = u, d, u^c, d^c, g, \cdots)}$$

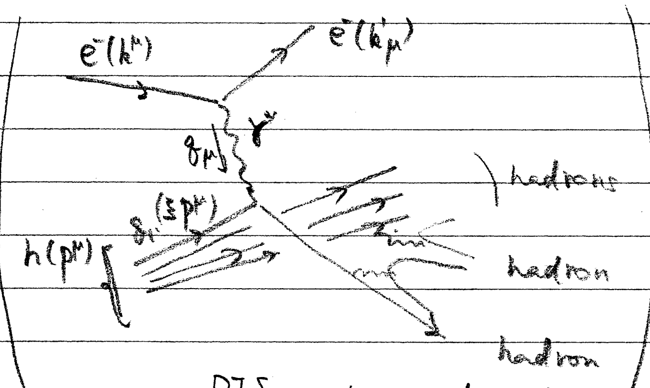
- other constituents of the hadron do ~~not~~ do very much

($d\xi\, f_{q/h}(\xi)$: the number of q in h whose "longitudinal momentum fraction is $\in [\xi, \xi + d\xi]$".)

$\frac{1}{\xi} \Rightarrow$ ($d\sigma \propto \frac{1}{4p'\cdot p_h} \int dPS\, |\mathcal{M}|^2$ $\qquad \frac{1}{4p'\cdot p_h \xi} = \frac{1}{4p'\cdot p_q}$
using $|h(\vec{p})\rangle$

$$\boxed{\text{so} \quad \sigma(h + ⓐ \to X) = \sum_i \int d\xi\, f_{q_i/h}(\xi)\, \sigma(q_i + ⓐ \to X)}$$

[A couple of checks to see that this prescription is (roughly) consistent with what we have discussed so far: hw IX-3, 7.]



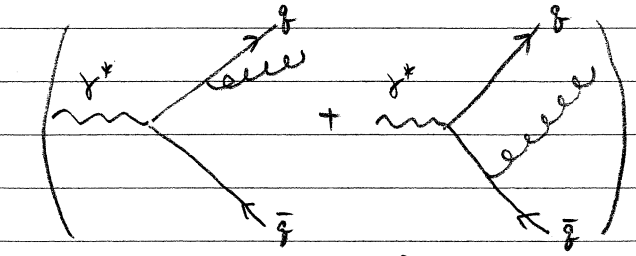
DIS $\boxed{\text{by the quark model}}$

- numerical fit result for individual $f_{q_i/p}(\xi)$, $q_i = u, u^c, d, d^c, \cdots$ [see elsewhere]

- How to combine the e^--p^+ DIS data with other experimental data to extract those $f_{q_i/h}(\xi)$?
$\Rightarrow \boxed{\text{hw E-1}}$

§7.5 Initial State Radiation (ISR) / Final State Radiation (FSR)

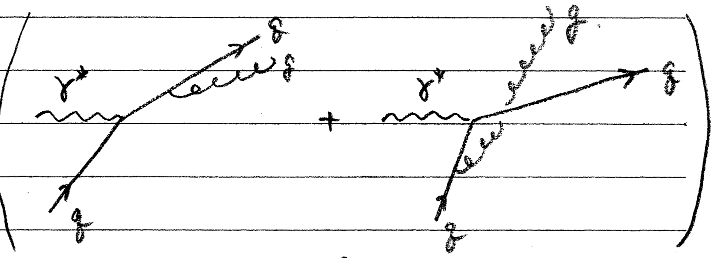
$(\gamma^* \to q+\bar{q})$ $(e^+e^- \to \text{hadron})$



$\Rightarrow$ IR divergent $\int d\sigma(\to q\bar{q}g)$

where $(p_g^\mu \parallel p_q^\mu)$ or $(p_g^\mu \parallel p_{\bar{q}}^\mu)$

$(\gamma^* + q \to q)$ $(e^-p^+ \to e^- + \text{hadron})$



divergent $\int d\sigma(q_{in} \to q g_{out})$

where $(p_g^\mu \parallel p_q^\mu(\text{out}))$ or $(p_g^\mu \parallel p_q^\mu(\text{in}))$

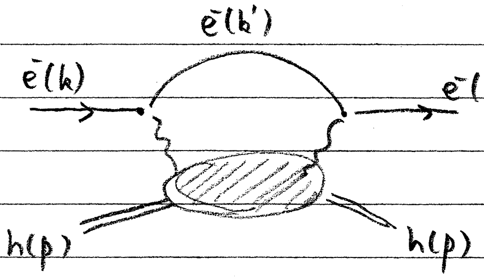
$\Uparrow$ ISR.

☆ $\int d\sigma_{ISR}$ also diverges (the same reason as in FSR)

☆ un-observable (as in FSR) because the radiation goes down the beam pipe.

Does divergence cancel in observables in DIS ?

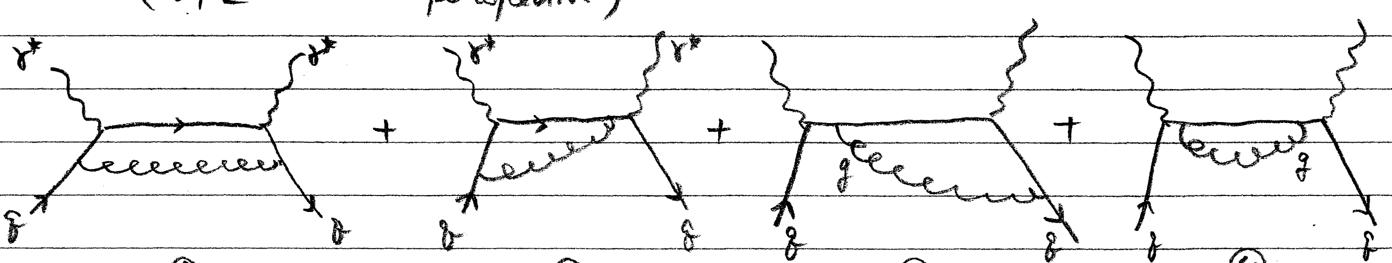
§7.6 DGLAP equation (Dokshitzer - Gribov - Lipatov - Altarelli - Parisi) and Collinear Factorization



$$\sigma_{DIS} = \frac{1}{2S} 2\,\mathrm{Im}\left(\mathcal{M}(e^-h \to e^-h)\right)$$

$$\mathrm{Im}(\mathcal{M}) \Leftarrow \mathrm{Im}\left(T^{\mu\nu}(\gamma^* + h \to \gamma^* + h)\right)$$

In the (parton-model perspective / OPE perspective) $O(\alpha_s)$ correction is from those



① ② ③ ④

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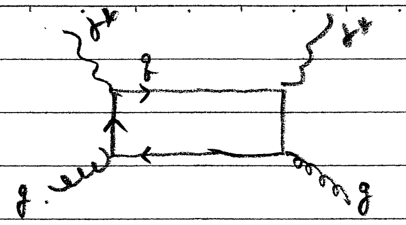
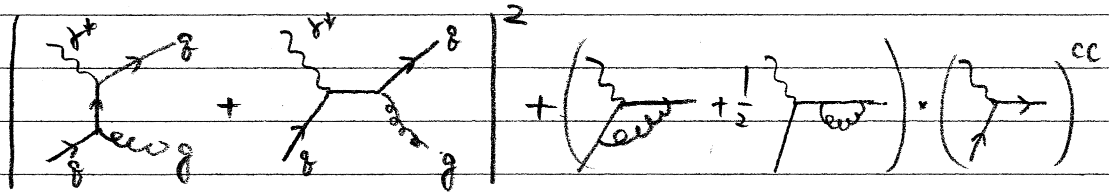
also... 

we will not talk about this during the class though.

Interpretation



$\Rightarrow$ ① + ② + ③ + ④ cut in the middle (② + ④ cut)

$\Uparrow$

$\left(\gamma^* + q \to q + g \right.$ real g emission $\left. \right)$

+ c.c. (③ + ④ cut)

$\Uparrow$

(vertex corrct'n / wave fcn renormalizat'n)

(Q: finite if all the FSR, ISR, vertex correct'n, wave fcn renormalizat'n are summed up?)

For now : use the parton model as an input.

(QPE perspective
PDF = $\langle h(p^\mu) |$ operator $| h(p^\mu) \rangle$
also keep in mind.)

$O(\alpha_s)$

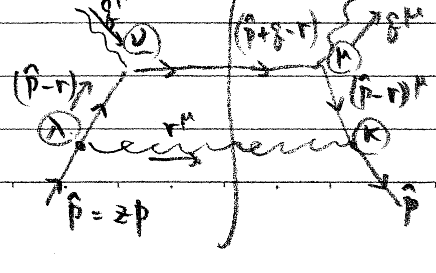
$$\Delta\left(2\,\mathrm{Im}\left(T^{\mu\nu}\right)\right) = \int_0^1 \frac{dz}{z}\, f_q(z) \times (-1) \times i\mathcal{M}(\gamma^* q \to \gamma^* q)\Big|_{\substack{\text{spin}(q)\ \text{average} \\ \text{cut propagator}\ \frac{i}{(p^2 - m^2 + i\varepsilon)} \Rightarrow 2\pi\delta^+(p^2 - m^2)}}$$

(Cutkosky rule)

from graph ①

$$\int_0^1 \frac{dz}{z} f_q(z)\,(-1) \int \frac{d^4 r}{(2\pi)^4}\, \mathrm{Tr}_{\text{spin}}\Big[(-ig t^a \gamma^\kappa) \frac{i}{(\hat{\cancel{p}} - \cancel{r}) + i\varepsilon} (iQ_q \gamma^\mu)(\hat{\cancel{p}} + \cancel{q} - \cancel{r})(iQ_q \gamma^\nu) \frac{i}{(\hat{\cancel{p}} - \cancel{r}) + i\varepsilon} (-ig t^b \gamma^\lambda)$$

$$\times \hat{\cancel{p}}\Big] \frac{1}{2}\left(-\eta_{\kappa\lambda}\delta_{ab}\right)(2\pi)\delta^+(r^2)\,(2\pi)\delta^+\left((\hat{p}+q-r)^2\right)$$



(graph ② ~ ④ : similar)

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need to pay attention to ISR

$\Rightarrow$ contribution from r^μ almost parallel to $(\hat{p}^\mu = z p^\mu)$

~~use the Lorentz frame where $\hat{p}^\mu(g)$ and $q^\mu(\gamma^*)$ collision axis is the "z" (3rd) direct'n.~~

$$\hat{p}^\mu = z(P, P, 0, 0) = z p^\mu$$
$$q^\mu = (Q^0, Q^3, 0, 0)$$

$$r^\mu =: z(1-\chi)p^\mu + (0, 0, \vec{r}_T) + (\Delta r)^\mu$$

parametrize r^μ by $\chi, \vec{r}_T, r^-$ $\quad \left(\Delta r^\mu \parallel (1,-1,\vec{0}) \right.$ $\left. = r^-(1,-1,\vec{0}) \right)$

pay attention to r^μ where $r^\mu \simeq z(1-\chi)p^\mu$

(but $r^\mu \neq z(1-\chi)p^\mu$)

$$\delta^+(r^2) = \delta^+\left(|\vec{r}_T|^2 - 4z(1-\chi)\, r^- P\right) \qquad x := -q^2/2p\cdot q$$

$$\delta^+\left((\hat{p}+q-r)^2\right) \simeq \delta^+\left((z\chi p^\mu + q^\mu)^2\right) = \frac{1}{2p\cdot q}\,\delta(z\chi - x)$$

$\Rightarrow \left(d^4 r = 2P\, z\, d\chi\, dr^-\, d^2\vec{r}_T\right)$ is reduced to $\frac{d(p\cdot r)}{2p\cdot q}\pi$.

graph 1,

$$-\frac{1}{4\pi}\, 2\,\mathrm{Im}(T^{\mu\nu})\,\eta_{\mu\nu} \supset \frac{1}{4\pi}\int_0 \frac{dz}{z} f_g(z) \int \frac{d(p\cdot r)}{2p\cdot q}\frac{\pi}{(2\pi)^2}\frac{Q_q^2 g^2 C_2(R)}{2} \times$$

$$\mathrm{Tr}_{4\times 4}\left[\gamma^\kappa (\hat{\not{p}}-\not{r})\gamma^\mu(\hat{\not{p}}+\not{q}-\not{r})\gamma_\mu(\hat{\not{p}}-\not{r})\gamma_\kappa \hat{\not{p}}\right]\frac{1}{(-2\hat{p}\cdot r)^2}$$

$$\left(\mathrm{Tr}[\,\cdots\,] = 4\,\mathrm{Tr}\left[(\hat{\not{p}}-\not{r})(\hat{\not{p}}+\not{q}-\not{r})(\hat{\not{p}}-\not{r})\hat{\not{p}}\right] \right.$$
$$\left. = 16\,(r\cdot\hat{p})\; 2p\cdot q\,(z-x) \right)$$

use $\hat{p}^2 = 0$, $r^2 = 0$ $(\delta^+(r^2))$

$\delta^+(\hat{p}+q-r) \to (\hat{p}\cdot r + q\cdot r)2 = 2\hat{p}\cdot q + q^2$

$$= \int_0^1 \frac{dz}{z}\left(Q_q^2 f_g(z)\right)\frac{\alpha_s C_2(R)}{8\pi}\int\frac{d(p\cdot r)}{(p\cdot q)}\, 4\,\frac{p\cdot q}{(\hat{p}\cdot r)}(z-x)$$

$$= \int_0^1 \frac{dz}{z}\left(Q_q^2 f_g(z)\right)\frac{\alpha_s C_2(R)}{2\pi}\int\frac{d(p\cdot r)}{(p\cdot r)}\underbrace{\left(1-\frac{x}{z}\right)}_{1}$$

$$\hookrightarrow \underbrace{\left(1-\frac{x}{z}\right)}_{4} + \underbrace{\frac{2x/z}{1-x/z}}_{2\,3} = \underbrace{\frac{1+(x/z)^2}{1-x/z}}_{1\sim 4}$$

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remember that

⊙ $T^{\mu\nu} =: (4\pi)\left\{\left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1 + \frac{1}{(p\cdot q)}\left[p^\mu - \frac{(p\cdot q)}{q^2} q^\mu\right]\left[p^\nu - \frac{(p\cdot q)}{q^2} q^\nu\right] T_2\right\}$

$$F_1 := 2\,\mathrm{Im}(T_1) \qquad F_2 := 2\,\mathrm{Im}(T_2)$$

$$\Rightarrow -\frac{1}{4\pi} 2\,\mathrm{Im}(T^{\mu\nu})\,\eta_{\mu\nu} = 3F_1 - \frac{1}{2x}F_2 \underset{\text{tree level}}{\supset} Q_q^2 f_q(x)$$

⊙ parton model @ Tree level

$$\left.\begin{aligned} F_1(x, Q^2) &\supset \frac{1}{2} Q_q^2 f_q(x) \\ F_2(x, Q^2) &\supset \frac{2x}{2} Q_q^2 f_q(x) \end{aligned}\right\}$$

With quantum effects at α_s^0 (tree), α_s^1 (1-loop) $\cancel{\alpha_s^2}$

$$\boxed{\left(3F_1 - \frac{1}{2x}F_2\right)(x, Q^2) \supset Q_q^2 f_q(x) + Q_q^2 \frac{\alpha_s C_2(R)}{2\pi} \int \frac{d(p\cdot r)}{(p\cdot r)} \int_x^1 \frac{dz}{z} f_q(z) \frac{1+(x/z)^2}{1-(x/z)}}$$

$\boxed{\text{+ contributions from } \bar{q},\ g}$

- dz only $\in [x, 1]$ $\quad \delta^+(r^2)$ for $r^\mu = (z(1-x)P + r^-,\ z(1-x)P - r^-,\ \vec{r}_T^{\,2})$

  $\Rightarrow$ only when $\boxed{1 \geq x\ \&\ r^- \geq 0}$

  $\delta^+((p+q-r)^2) \Rightarrow \delta(zx - x) \rightarrow x/z \leq 1$

- gluons with hierarchically small $p\cdot r = 2pr^-$ contribute equally well

  (i.e. gluons collinear to the initial state hadron)

  [The expression above is reliable only if $(p\cdot r) \ll p\cdot q$ or $(-q^2)$
  for $\delta^+((p+q-r)^2) \Rightarrow \delta(zx - x)$]

- The vertex correction & wave function renormalisation contribute only through $\int dz\, f_q(z)\, \delta(z-x) \times (\text{divergent})$

- How to avoid those two divergent contributions?

  $\sigma_{tot}(e^+e^- \to \gamma^* \to \text{any})$: do not distinguish q from bunch of $\left(\begin{smallmatrix}\text{all collinear} \\ q, g \cdots\end{smallmatrix}\right)$

  DIS : need to rethink what is "$f_q(z)$".

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(idea / analogy) renormalized gauge coupling const.

$$\frac{4\pi}{g^2(E)} = \frac{4\pi}{g_0^2} + \frac{b}{2\pi}\ln\left(\frac{E^2}{\Lambda^2}\right) \Rightarrow \underbrace{\left[\frac{4\pi}{g_0^2} + \frac{b}{2\pi}\ln\left(\frac{\mu_R^2}{\Lambda^2}\right)\right]}_{\overset{!!}{=}\ 4\pi/g^2(\mu_R^2)} + \frac{b}{2\pi}\ln\left(\frac{E^2}{\mu_R^2}\right)$$

$\Uparrow$ physical observable

(renormalized coupling: quantum effects of high energy (above μ_R) DOF have been taken into account)

Now

$$\left(3F_1(x;Q^2) - \frac{1}{2x}F_2(x;Q^2)\right) \supset Q_q^2 f_q(x;\mu_F) + Q_q^2\frac{\alpha_s C_2(R)}{2\pi}\int_{\mu_F^2}^{(p\cdot q)}\frac{d(p\cdot r)}{(p\cdot r)}\int_x^1\frac{dz}{z} f_q(z;\mu_F^2)\frac{1+(x/z)^2}{1-(x/z)}$$

quantum contributions due to a gluon w/ $(p\cdot r) < \mu_F^2$ are grouped into the "tree-level" term

$f_q(z;\mu_F^2)$: the number density of q whose virtuality $(\hat{p}-r)^2 = -2\hat{p}\cdot r$ is below $2z\mu_F^2$

under an understanding that such a "q" no longer emit a gluon only to have such a low virtuality.

"sweeping under the carpet" but.

- take into account (high-energy DOF) into (the coupling constants $(g^2(\mu_R))$ in $\mathcal{L}$)
- take into account ($\text{DOF}_{\text{almost onshell}}$) into (the operator matrix element $f_q(z;\mu_F^2)$)

variation under the same theme

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The DGLAP equation

The DIS structure function should not depend on the choice of μ_F

$$\Rightarrow \boxed{\frac{\partial f_q(x;\mu_F^2)}{\partial \ln(\mu_F^2)} = \frac{\alpha_s C_2(R)}{2\pi}\int_x^1 \frac{dz}{z}\Big(\underbrace{\frac{1+(x/z)^2}{1-(x/z)}}_{P(x/z)\ \text{splitting function}} + \underbrace{\cdots}_{\propto\delta(z-x)}\Big) f_q(z;\mu_F^2)}$$

$\propto \delta(z-x)$ ← from (vertex correction, wave fcn renormalization)

a quark with $\hat{p}^\mu = z p^\mu$ with virtuality $< \mu_F^2$
populates a quark w/ $\hat{p}^\mu = x p^\mu$ with a little higher virtuality
at the rate $\frac{\alpha_s C_2(R)}{2\pi} P(x/z)$

(experimental data ⇒ see next page)

DGLAP eq: multiplicative convolution w.r.p.t. x.

↓ Mellin transform

$$\boxed{\frac{\partial \tilde{f}_q(j;\mu_F^2)}{\partial \ln(\mu_F^2)} = \frac{\alpha_s C_2(R)}{2\pi}\gamma(j)\,\tilde{f}_q(j;\mu_F^2)}$$

$$\boxed{\tilde{f}(j) = \int_0 \frac{dx}{x} x^j f(x)}$$

$$\boxed{\gamma(j) := \int \frac{d\xi}{\xi}\xi^j P(\xi)}$$

memo: $\tilde{f}_q(j) + \tilde{f}_{\bar{q}}(j) = \frac{1}{2}A_j$ $\quad \langle h(p)|(\bar{\psi}\gamma^{\mu_1}(\tfrac{i}{2}\overleftrightarrow{D})^{\mu_2}\cdots(\tfrac{i}{2}\overleftrightarrow{D})^{\mu_j}\psi)|h(p)\rangle = p^{\mu_1}\cdots p^{\mu_j}A_j$

back to OPE

$$\int T\{J^\mu(0)J^\nu(y)\}e^{-iq\cdot y}d^4y = \Sigma_j C_j(q^2;\mu_R^2)\left[\bar{\psi}\gamma^{\mu_1}(\tfrac{i}{2}\overleftrightarrow{D})^{\mu_2}\cdots(\tfrac{i}{2}\overleftrightarrow{D})^{\mu_j}\psi\right]_{\mu_R}$$

composite op. renormalized @ μ_R

$A_j \Rightarrow A_j(\mu^2)$ (reduced) matrix elements of operators renormalized @ μ

$\boxed{\mu_R = \mu_F}$
both UV DOF & IR DOF should be taken into account sooner or later

⇐

quantum effects below μ
⇓
should be captured in the operator matrix elements
$A_j(\mu^2)$ or $\tilde{f}_q(j;\mu^2)$ or $f_q(x;\mu^2)$

quantum effects above μ
⇓
captured in the OPE coeff. $C_j(q^2;\mu^2)$

PLUS

$$\sigma^{+}_{r,NC}(x,Q^2) := \frac{d^2\sigma^{e^+p}_{NC}}{dx\,dQ^2} \times \frac{Q^4 x}{2\pi\alpha_e^2}\,\frac{1}{(1+(1-y)^2)}$$

[shifted (by $\times 2^i$) for different values x_i to avoid overlap]

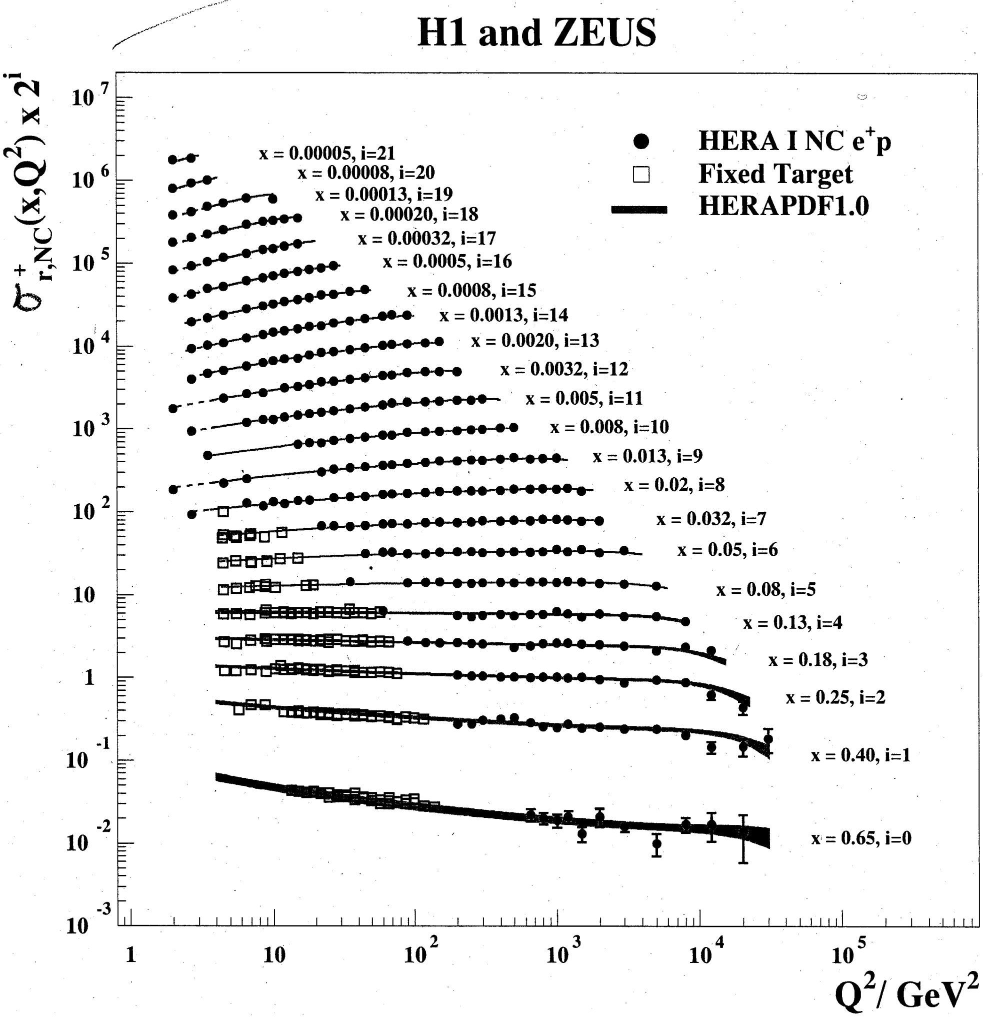

Figure 9. HERA combined NC e^+p reduced cross section and fixed-target data as a function of Q^2. The error bars indicate the total experimental uncertainty. The HERAPDF1.0 fit is superimposed. The bands represent the total uncertainty of the fit. Dashed lines are shown for Q^2 values not included in the QCD analysis.

see above

[At small x : more partons as $Q^2 \nearrow$.
At $x \sim \mathcal{O}(1)$: such partons depleted as $Q^2 \nearrow$]

(hep-ex/0911.0884)

[a quark with larger x splits into a quark with smaller-x and a gluon]