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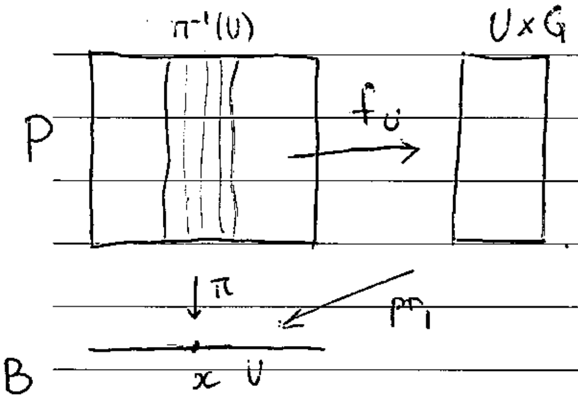
Lecture 1: Bundles and connections

1. Principal bundles.

$$\begin{array}{c} P \\ \downarrow \pi \\ B \end{array}$$ surjective map between manifolds

G : Lie group acting on P from the right

Def 1: (P, B, π, G) is called a principal G-bundle if :



$\forall x \in B$, $\exists$ open $\underset{x}{\underset{\Psi}{U}} \subset B$ and ~~such~~ an isomorphism

$f_U : \pi^{-1}(U) \longrightarrow U \times G$

such that :

(1)
$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "f_U"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "pr_1"] \\ & B & \end{tikzcd}$$
commutes

and (2) $f_U(\bar{x} \cdot g) = f_U(\bar{x}) \cdot g$

G-equivariance

Def 2: Two principal bundles (P, B, π, G) and (P', B, π', G) are equivalent if $\exists$ an isomorphism $h : P \to P'$, s.t.

G-equivariant

$$\begin{tikzcd} P \arrow[rr, "h"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & B & \end{tikzcd}$$

commutes.

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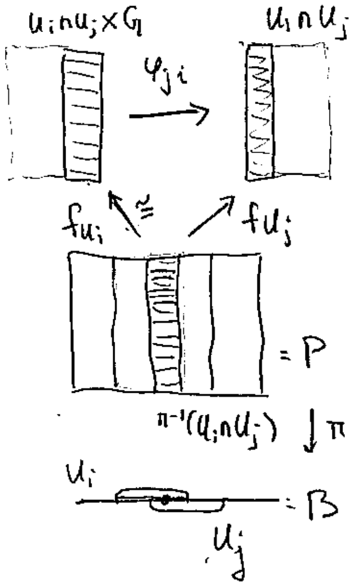
transition functions:



$f_{U_i} : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times G$

$f_{U_j} : \pi^{-1}(U_j) \xrightarrow{\cong} U_j \times G$

trivializations; then

$\varphi_{ji} := f_{U_j} \circ f_{U_i}^{-1} : (U_i \cap U_j) \times G \xrightarrow{\cong} (U_i \cap U_j) \times G$

is well defined $\to$ transition function

Note φ_{ji} is G-equivariant

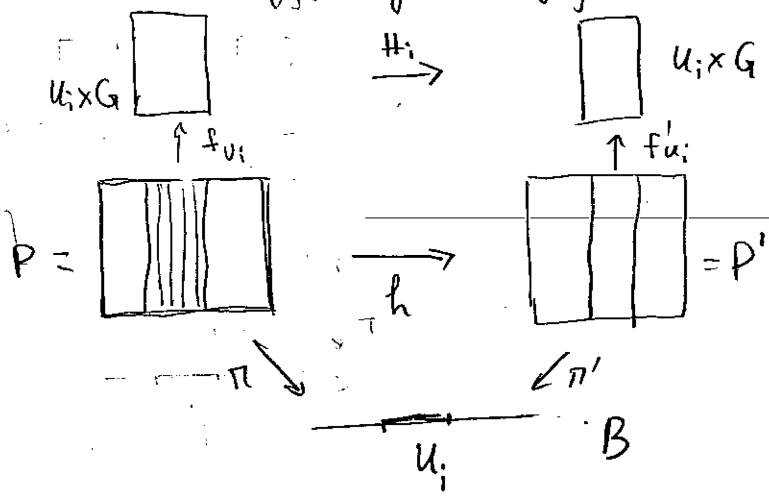
$$\Rightarrow \quad \varphi_{ji}(x, \underset{\substack{\parallel \\ e \cdot g}}{g}) = \varphi_{ji}(x, e) \cdot g = (x, g_{ji}(x)) \cdot g = (x, g_{ji} \cdot g)$$

where $g_{ji} : U_i \cap U_j \longrightarrow G$ is a collection of maps, called gluing cocycle

Note that

(1) $g_{ji}(x) g_{ij}(x) = e$, for $x \in U_i \cap U_j$

(2) $g_{ji}(x) g_{ik}(x) g_{kj}(x) = e$, for $x \in U_i \cap U_j \cap U_k$.



$H_i := f'_{U_i} \circ h \circ f_{U_i} : U_i \times G \xrightarrow{\cong} U_i \times G$

~~$\Rightarrow H_i(x, g) = (x, h_i(x) g)$~~

where $h_i : U_i \to G$ is some map.

Assume $P \cong P'$.

Easy to check that

this defines equivalence between cocycles $\to$ $\boxed{g'_{ji}(x) = h_j(x)\, g_{ji}(x)\, h_i^{-1}(x)}$

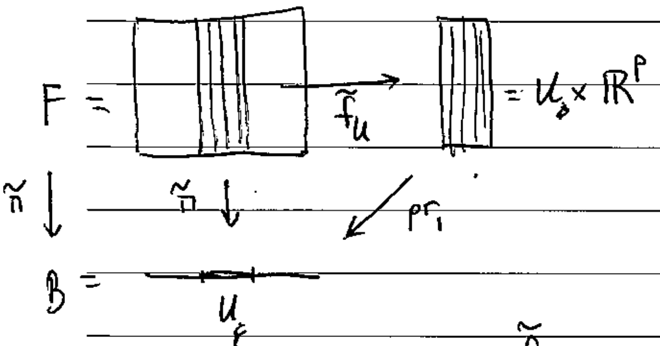
Thm 1. There is a one-to-one corresp. between

$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{principal } G\text{-bundles on } B \end{array} \right\} \xleftrightarrow{\text{and}} \left\{ \begin{array}{c} \text{equivalence classes of} \\ \text{gluing cocycles} \end{array} \right\}$

2. Vector bundles

$$\begin{array}{c} F \\ \tilde{\pi} \downarrow \\ B \end{array}$$ surjective map between manifolds, s.t. $\forall x \in B$, $\tilde{\pi}^{-1}(x)$ is a vector space

Def3: $(F, B, \tilde{\pi})$ is called a vector bundle if



$\forall x \in B$, $\exists$ open $U \subset B$, $x \in U$ and a homeomorphism

$\tilde{f}_U : \tilde{\pi}^{-1}(U) \longrightarrow U \times \mathbb{R}^p$

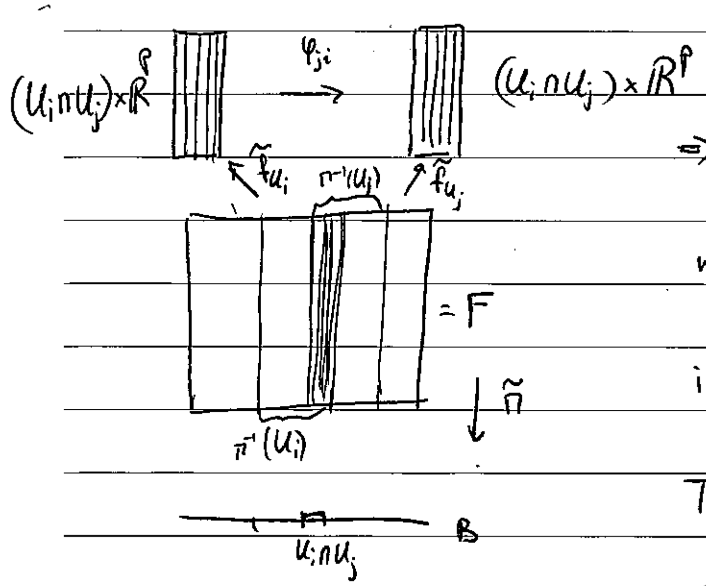
such that

(1)
$$\begin{array}{ccc} \pi^{-1}(\tilde{U}) & \xrightarrow{\tilde{f}_U} & U \times \mathbb{R}^p \\ & {\scriptstyle \tilde{\pi}} \searrow \quad \swarrow {\scriptstyle pr_1} & \\ & \tilde{U} & \end{array}$$
commutes

and (2) The induced map

$\tilde{f}_U^x : \tilde{\pi}^{-1}(x) \longrightarrow \mathbb{R}^p$

is a linear isomorphism.



Similarly $\varphi_{ji} := \tilde{f}_{U_j} \circ \tilde{f}_{U_i}^{-1}$ defines trans. functions

$\Rightarrow \varphi_{ji}(x, v) = (x, g_{ji}(x) \cdot v)$

where $g_{ji} : U_i \cap U_j \longrightarrow GL_p(\mathbb{R})$

is some collection of maps.

They satisfy the cocycle condition for $G := GL_p(\mathbb{R})$

$\Rightarrow$ we can construct a principal G-bundle P on B.

Thm 2. There is a one-to-one correspondence between

$$\left\{\begin{array}{l}\text{equivalence classes}\\ \text{of vector bundles}\\ \text{on } B \text{ (of rank } p)\end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{l}\text{equivalence classes of}\\ \text{principal } GL_p\text{-bundles}\\ \text{on } B\end{array}\right\}.$$

<u>Example</u>. 1) Hopf fibration

$$P = S^3 = \{|z_1|^2 + |z_2^2| = 1\} \subset \mathbb{C}^2$$

$$\downarrow \pi \qquad\qquad \pi(z_1, z_2) = [z_1 : z_2]$$

$$B = \mathbb{P}^1 = (\mathbb{C}^2 \setminus \{0\}) / \mathbb{C}^* \qquad G = S^1 = \{|\lambda| = 1\} \subset \mathbb{C}$$

$$(z_1, z_2)\cdot\lambda = (z_1\lambda, z_2\lambda)$$

$$U_0 = \{z_1 \neq 0\} \cong \mathbb{C}, \quad U_\infty = \{z_2 \neq 0\} \cong \mathbb{C}$$

$$z = \frac{z_2}{z_1} \qquad\qquad t = \frac{z_1}{z_2}$$

$$\pi^{-1}(U_0) \xleftarrow{\cong} U_0 \times S^1 \qquad\qquad \pi^{-1}(U_\infty) \xrightarrow{\cong} U_\infty \times S^1$$

$$(\lambda, z\lambda) \leftarrow (z, \lambda) : f_{U_0}^{-1} \qquad\qquad f_{U_\infty}(z_1, z_2) = \left(\frac{z_1}{z_2}, \frac{z_2}{|z_2|}\right)$$

$$f_{U_0}(z_1, z_2) = \left(\frac{z_2}{z_1}, \frac{z_1}{|z_1|}\right)$$

$$|z_1|\cdot z \quad \frac{1}{|z_2|} = g_{\infty 0}(z)$$

$$f_{U_0}((z_1, z_2)\cdot\lambda) = \left(\frac{z_2}{z_1}, \frac{z_1\lambda}{|z_1|}\right) = f_{U_0}(z_1, z_2)\cdot\lambda \qquad h_\infty(z) = \frac{1}{|z|}, \quad h_0(t) = 1$$

S^1-equivariant

$$g_{0\infty}$$

$$\varphi_{\infty 0}(z, \lambda) = f_{U_\infty} \circ f_{U_0}^{-1}(z, \lambda) = f_{U_\infty}(\lambda, z\lambda) = \qquad 1\cdot t\cdot\frac{1}{|t|}$$

$$= \left(\frac{1}{z}, \frac{z\lambda}{|z|}\right)$$

$$\Rightarrow \boxed{g_{\infty 0}(z) = \frac{z}{|z|}}$$

$$S^3 \times_{S^1} \mathbb{C} \cong \mathcal{O}(-1)$$

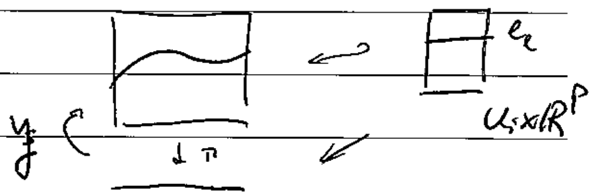
$$h_\infty(t) = |t|, \quad h_0(z) = 1$$

$$h_\infty g_{\infty, 0} h_0^{-1} = |t| \frac{z}{|z|}$$

<u>Remark</u>: If P is a principal G-bundle and $\rho : G \to GL_p(\mathbb{R})$ is a repr. of G in $\mathbb{R}^p$ then $F = P \times \mathbb{R}^p / (\bar{x}\cdot g, v) \sim (\bar{x}, \rho(g)\cdot v)$ is a vector bundle w/ ~~transition functions~~ gluing cocycle: $\rho \circ g_{ji} : U_i \cap U_j \to GL_p(\mathbb{R})$.

3. Connections

$F \xrightarrow{\tilde{\pi}} B$ vector bundle



Def: A connection on F is a linear map

$$\nabla : \Gamma(F) \longrightarrow \Gamma(T_B^* \otimes F)$$

sections of a vector bundle.

satisfying the Leibnitz rule:

$$\nabla(f \cdot y) = df \otimes y + f\, \nabla y.$$

Local expression: $\{U_i\}$ open cover of B, $f_{U_i} : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^p$ trivialization

Note that $f_{U_i} \circ y(z) = (z, y_i(z))$

where $y_i(z) = \begin{bmatrix} y_i^1(z) \\ \vdots \\ y_i^p(z) \end{bmatrix} \in \mathbb{R}^p$, $y_i(z) = \sum_{\ell=1}^{p} y_i^\ell(z)\, e_\ell$

and $y_j(z) = g_{ji}(z) \cdot y_i(z)$, $z \in U_i \cap U_j$

A_i – matrix of 1-forms on U_i s.t.

$$\nabla e_\ell = \sum_{k=1}^{p} A_{i,k\ell} \otimes e_k \qquad \text{connection matrix}$$

then $\nabla y_i = dy_i + A_i \cdot y_i$

$\Rightarrow \{A_i\}_i$ must satisfy $A_i = dg_{ij} \cdot g_{ij}^{-1} + g_{ij} \cdot A_j \cdot g_{ij}^{-1}$.

Ass

Def: y is called a horizontal section if $\nabla y = 0$.

Locally: $dy_i + A_i \cdot y_i = 0$

$z = (z^1, \dots, z^m)$ local coords on U_i

$$A_i = \sum_{a=1}^{m} A_{i,a}(z)\, dz^a$$

$\Updownarrow$

$$\frac{\partial y_i}{\partial z_a} = - A_{i,a}(z) \cdot y_i \quad , a = 1, 2, \dots, m$$

$$\frac{\partial^2 y_i}{\partial z_b \partial z_a} = - \frac{\partial A_a}{\partial z_b} \cdot y_i - A_a (-A_b \cdot y)$$

$$= \left(- \frac{\partial A_a}{\partial z_b} + A_a A_b \right) \cdot y = \left(- \frac{\partial A_b}{\partial z^a} + A_b A_a \right) \cdot y$$

i.e. integrability condition

$$\boxed{\frac{\partial A_a}{\partial z^b} - \frac{\partial A_b}{\partial z^a} = [A_a, A_b] \quad , \text{for all } 1 \le a, b \le m}$$

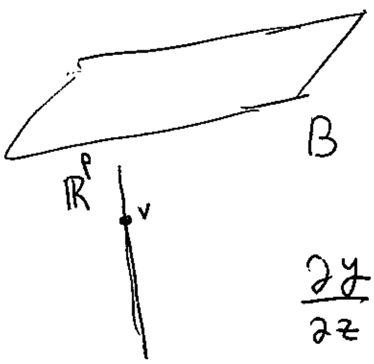
Def: ∇ is called flat if the above equation is satisfied.

Thm. [Frobenius] Assume the system is compatible, i.e., ∇ is flat, then $dy = -A \cdot y$, $y(z_0) = V$ has a unique solution in a neighborhood of z_0

Assume now that $\dim_{\mathbb{C}} B = 1$.

A connection then is locally given by

$$\nabla = d + A(z)\, dz$$



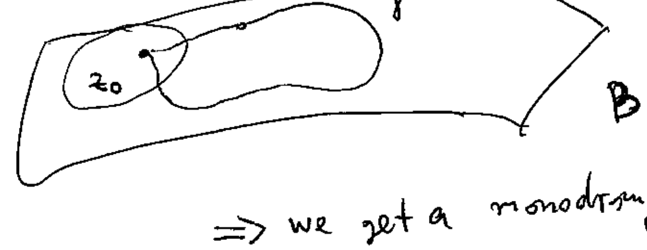
$\Rightarrow \nabla y = 0$ means

$$\frac{\partial y}{\partial z} = -A(z) \cdot y .$$

The Cauchy problem

$$\begin{cases} y'(z) = -A(z) \cdot y \\ y(z_0) = V \end{cases}$$

has a unique solution in a neighborhood of z_0



$\Rightarrow$ we get a monodromy repres. $\chi : \pi_1(B) \longrightarrow GL_p(\mathbb{R})$

Question: Given χ, can we construct a flat connection w/ monodromy repr. χ ?

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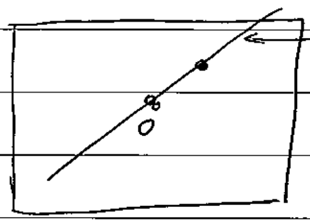
Example 2: $\mathcal{O}(-1)$

$$F \subset \mathbb{P}^1 \times \mathbb{C}^2 \quad , \{([z_1,z_2], x_1, x_2) : z_1 x_2 = z_2 x_1\} \quad (= Bl_0 \mathbb{C}^2$$

$\tilde{\pi} \downarrow \quad \swarrow pr$

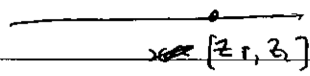
$\mathbb{P}^1$



line = fiber $\pi^{-1}([z_1,z_2])$

$\mathbb{C}^2$

$\mathbb{C} \cong U_0 = \{z_1 \neq 0\} \subset \mathbb{P}^1$



$\mathbb{P}^1$

$z = \frac{z_2}{z_1}$

$f_{U_0} : \pi^{-1}(U_0) \longrightarrow U_0 \times \mathbb{C}$ $\qquad \mathbb{C} \cong U_\infty = \{z_2 \neq 0\} \subset \mathbb{P}^1$

$f_{U_0}([z_1,z_2], x_1, x_2) = \left(\frac{z_2}{z_1}, x_1\right)$ $\qquad t = \frac{z_1}{z_2}$

$f_{U_\infty}([z_1,z_2], x_1, x_2) = ([z_1,z_2], x_2)$

$f_{U_0}^{-1}(z, \lambda) = ([1, z], \lambda, \lambda z)$

$$\varphi_{\infty 0}([z_1,z_2],\lambda) = f_{U_\infty}\left([z_1,z_2], \lambda, \lambda \frac{z_2}{z_1}\right) = \left([z_1,z_2], \frac{z_2}{z_1}\lambda\right)$$

$$\Rightarrow g_{\infty 0}([z_1,z_2]) = \frac{z_2}{z_1} \in \mathbb{C}^*$$

or in terms of $z = \frac{z_2}{z_1}$ local coord. on $U_0 \cap U_\infty$:

$$\boxed{g_{\infty 0}(z) = z}$$

Remark: $g_{\infty 0}(z) := z^{-k}$; then we get a bundle on $\mathbb{P}^1$

$k \in \mathbb{Z}$

called $\mathcal{O}(k)$. $\qquad T_{\mathbb{P}^1} = \mathcal{O}(2)$.

$h_0(z)$, $h_\infty(t)$ s.t. $h_\infty(z) = z^k h_0(z)$ if $k>0$

$z \in \mathbb{C}$ $\quad t \in \mathbb{C}$ $\quad$ holom. at $t=\infty$ $\quad$ defined at $z=\infty$ $\Rightarrow \lim_{z\to\infty} h_0(z) = 0 \Rightarrow h_0$ must be const. 0

$t = \frac{1}{z}$

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~~3. Konnektionen.~~

$$\frac{\partial y}{\partial z_2} = -A_2(z) \cdot y$$

Lecture 2: Merom. connections w/ reg. singular pts.

1. Fuchsian and regular singular points.

F is $\mathbb{C}$-analytic v.b. $/B$ $\quad \dim_{\mathbb{C}} B = 1$

∇ - merom. connection on B, i.e.,

$$\nabla_{\partial/\partial z}\, y = \frac{\partial y}{\partial z} - B(z)\cdot y, \qquad B(z) \text{ is } \text{meromorphic on } B$$

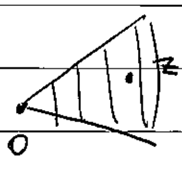
Assume $\mathcal{O} \subset B$ is a neighborhood of $z_0 = 0$. Horizontal sections:

$$\frac{dy}{dz} = B(z)\cdot y \qquad (4.2)$$

<u>Def</u>: $z=0$ is a ~~regular~~ Fuchsian singular point if $B(z)$ has a pole of order ≤ 1. Moreover, if all singular points of $B(z)$ are Fuchsian, then ∇ is called Fuchsian connection.

<u>Def 2</u>: $z=0$ is <u>a regular singular point</u> of ∇ if every horizontal section y satisfies:

$\exists\, C, N$, s.t. $\quad |y(z)| \leq C \cdot |z|^{-N}$



for every $z \in$ some sector w/ vertex $z=0$ and angle $< 2\pi$.

<u>Example</u>: $\quad \dfrac{dy}{dz} = \begin{bmatrix} \frac{1}{z} & 1 \\ 0 & 0 \end{bmatrix} \cdot y, \qquad B = \mathbb{P}^1$

$z=0$ is Fuchsian $\quad\Big|\quad y_1' = \frac{1}{z} y_1 + y_2 \qquad y_2 = c$

$\qquad\qquad\qquad\quad\;\; \Big|\quad y_2' = 0 \qquad\qquad\qquad y_1 = $ [illegible]

$$Y(z) = \begin{bmatrix} z & z\ln z \\ 0 & 1 \end{bmatrix}$$

<u>Ex2.</u> $\frac{dy}{dz} = -\frac{y}{z^2}$ not Fuchsian and not regular

$$y = e^{\frac{1}{z}}$$

<u>Thm 1.</u> A Fuchsian singular point is always regular.

<u>Rem:</u> $\frac{dy}{dz} = \begin{bmatrix} 0 & 1 \\ \frac{1}{z^2} & -\frac{1}{z} \end{bmatrix} \cdot y$, $Y(z) = \begin{bmatrix} z & \frac{1}{z} \\ 1 & -1/z^2 \end{bmatrix}$ $\to$ regular

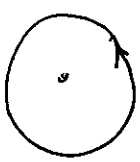
not Fuchsian

2. Monodromy.

$$\mathring{O} = O \setminus \{0\}$$

if γ is a loop in $\mathring{O}$, analytic continuation of $Y(z)$ along γ gives a fundamental matrix $Y'(z) = Y(z) \cdot G_\gamma$ where $G_\gamma \in GL_p(\mathbb{C})$.

$$\Rightarrow \quad \chi_0 : \underbrace{\pi_1(\mathring{O}, z_0)}_{\mathbb{Z}\cdot\gamma} \longrightarrow GL_p(\mathbb{C})$$



γ

$\sigma := G_\gamma$ - monodromy matrix of Y.

<u>Rem:</u> If $\tilde{Y} = Y \cdot S$ is another fundam. matrix then $\tilde{\sigma} = S^{-1} \sigma S$.

If λ is an eigenvalue of a matrix H then we

G - monodromy matrix of $Y(z)$

Put $E = \frac{1}{2\pi i} \ln G$, where the eigen-values of E

$\rho^1, \dots, \rho^p$ are s.t. $0 \leq \mathrm{Re}(\rho^i) < 1$

We define $z^E = \exp(E \ln z)$. Analytic continuation along γ transforms z^E into $z^E \cdot G$

Lemma 1. The fundamental matrix has a decomposition

$$Y(z) = M(z)\, z^E,$$

where $M(z)$ is single valued in $\dot{O}$.

Example: $\frac{dy}{dz} = \begin{bmatrix} \frac{1}{z} & 1 \\ 0 & 0 \end{bmatrix} \cdot y$

$$Y(z) = \begin{bmatrix} z & z \ln z \\ 0 & 1 \end{bmatrix} \xrightarrow{\sigma} \begin{bmatrix} z & z(\ln z + 2\pi i) \\ 0 & 1 \end{bmatrix} = Y(z) \cdot \begin{bmatrix} 1 & (2\pi i) \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow G = \begin{bmatrix} 1 & 2\pi i \\ 0 & 1 \end{bmatrix} \Rightarrow E = \frac{1}{2\pi i} \log G = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow Y(z) = \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \cdot z^{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}.$$

Lemma 2. The elements a_{ij} of the matrix z^E have the form

$$a_{ij} = \sum_{l=1}^{p} z^{\rho^l} P_{ij}^l (\ln z)$$

polynomial of degree $\leq$ size of the largest Jordan block of E.

Pf. Wlog $E = \rho I + N$ is a Jordan block, i.e.,
I - identity
N - uper triangular

$$z^E = z^{\rho} \cdot z^N = z^{\rho} \exp(N \log z) =$$
$$= \sum_{k=1}^{\infty} z^{\rho} \frac{(\ln z)^k}{k!} N^k = \sum_{k=1}^{p'-1} z^{\rho} (\ln z)^k \frac{N^k}{k!}$$

3. Scalar equations.

(3.1) $u^{(p)} + q_1(z) u^{(p-1)} + \dots + q_p(z) u = 0$

locally near $z=0$

Def: $z=0$ is regular singular at $z=0$ if every solution $u(z)$ satisfies: $|u(z)| < C|z|^{-N}$ for some C, N
$z \to 0$ in a sector $< 2\pi$

Def: $z=0$ is Fuchsian if $q_i(z) = \frac{r_i(z)}{z^i}$, where $r_i(z)$ is holomorphic at $z=0$.

Thm. Fuchsian $\Longleftrightarrow$ regular.

pf: $\Rightarrow$) easy: put $y_1 = u$, $y_2 = zu'$, —, $y_p = z^{p-1} u^{(p-1)}$

$$y_1' = u' = \frac{1}{z} y_2$$
$$y_2' = u' + zu'' = \frac{1}{z}(y_2 + y_3)$$
$\vdots$
$$y_p' = \frac{p-1}{z} y_p + \frac{1}{z}(-r_1 \cdot y_1 - r_2 \cdot y_2 - \dots - r_{p-1} y_{p-1})$$

$\Rightarrow$ all poles are at most 1.

$$B(z) = \begin{bmatrix} 0 & z^{-1} & & & \\ & & & & \\ -r_1(z) & -r_2(z) & \dots & -r_{p-1}(z) & \frac{p-1}{z} \end{bmatrix} \frac{1}{z}$$

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$\Leftarrow$) $u_1, \dots, u_p$ a fundamental system of solutions of (3.1)

monodromy $Y(z) = [u_1, \dots, u_p] \longrightarrow [u_1, \dots, u_p] \cdot G$

monodromy matrix

note $Y(z) = M(z) \cdot z^E$

where $M(z) = [m_1(z), \dots, m_p(z)]$, $m_i(z)$ are single valued in $\dot{O}$.

The singularity is regular $\Rightarrow$ $M(z)$ is meromorphic.

May assume that E is uper triangular. Set $\rho = E_{11}$;

then $(z^E)_{11} = z^\rho$ $\Rightarrow$ $u_1(z) = m_1(z) \cdot z^\rho$

$\Rightarrow$ $u_1(z) = v(z) \cdot z^{\rho}$, v - holom. at $z=0$ and $v(0) \neq 0$

Induction on p: $p=1$, $u' + q_1(z) \cdot u = 0$,

$u(z) = z^\rho v(z)$ $\Rightarrow$ $q_1(z) = -\partial_z(\ln u) = -\frac{\rho}{z} - \frac{v'(z)}{v(z)}$ $\Rightarrow$ Fuchsian

Substitute $u(z) = x(z) \cdot u_1(z)$: set all integral coeff. to 1 for simplicity:

$$x^{(p)} + \left(q_1(z) + \frac{u_1'}{u_1}\right) x^{(p-1)} + \dots + \left(q_j(z) + \frac{u_1'}{u_1} + \dots + \frac{u_1^{(j)}}{u_1}\right) x^{(p-j)} + \dots +$$

$$\underbrace{\left(q_p(z) + \frac{u_1'}{u_1} + \dots + \frac{u_1^{(p)}}{u_1}\right)}_{=0} \cdot x = 0$$

Since $x = 1$ is a solution $\Rightarrow$

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In particular,

$q_p(z)$ has a pole of order $\leq p$.

Put $\tilde{u}(z) = \chi'(z)$ $\Rightarrow$ we get a diff. equation for $\tilde{u}$ of order $p-1$ w/ a regular singular point at $z=0$.

In particular, by induction

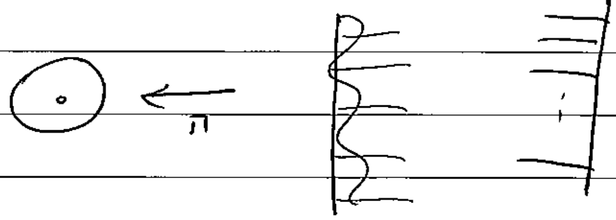
$$\text{order of pole of } \Big(q_j + \underbrace{c_1 \cdot \frac{u'}{u} + c_2 \frac{u''}{u} + \dots + c_j \frac{u^{(j)}}{u}}_{\text{pole at most of order } j} \Big) \leq j$$

Lecture 3: Levelts theory

1. The universal cover of $\mathring{O}$.

$\mathring{O}$ - punctured disk $\{0<|z|<\delta\}$, $\pi_1(\mathring{O}, z_0) = \mathbb{Z}\sigma$

$$\begin{array}{ll} \mathcal{O}^* \leftarrow \text{right half-plane} = \{u \in \mathbb{C} \mid \operatorname{Re} u > \ln \delta\} \\ \pi \downarrow \qquad z = \exp u \\ \mathring{O} \end{array}$$



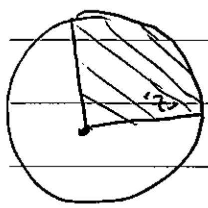
$\bar{z} = \pi^* z$; then $\ln \bar{z}$ is well defined

and $\begin{array}{c} \mathcal{O}^* \\ \downarrow \pi \\ \mathring{O} \end{array}$ is a principal $\mathbb{Z}$-bundle.

Note $\sigma^* u = u + 2\pi i$. For any function

$f(z)$ we define $(\sigma^* f)(z)$

$z=0$ is a regular sing. point $\iff \exists\, Q \in \mathbb{Z}$ s.t. $\forall$ solutions $y(z)$ $\forall$ sector $S \subset \mathring{O}$



$$\frac{y(\bar{z})}{|\bar{z}|^Q} \to 0 \quad \text{as } \underset{z \in S}{z \to 0}$$

Def: We say that $y(z)$ has a polynomial growth.

Def: Evaluation (Levelt's) $\varphi : X \to \mathbb{Z} \cup \{\infty\}$

$$\varphi(y) := \sup \left\{ l \;\middle|\; \lim_{\substack{z \to 0 \\ z \in S}} \frac{y(\bar{z})}{|z|^\lambda} = 0 \text{ for all } \lambda < l \right\}$$

$$\varphi(0) := \infty$$

Given a matrix $M = (f_{ij})_{1 \le i,j \le p}$ we define

$$\varphi(M) = \min_{i,j} \varphi(f_{ij}).$$

Ex. $0 \le \operatorname{Re} \rho^i < 1$ where ρ^i are the eigenvalues of $E = \frac{1}{2\pi i} \ln G$

$\varphi(z^E) = 0.$

~~Exercise:~~

Proposition 1. The evaluation φ has the following properties:

a) $\varphi(y_1 + y_2) \ge \min(\varphi(y_1), \varphi(y_2))$

with equality if $\varphi(y_1) \neq \varphi(y_2)$

b) $\varphi(cy) = \varphi(y)$ for $c \in \mathbb{C} \setminus \{0\}$

c) $\varphi^*(\sigma^* y) = \varphi(y)$ (monodromy invariance).

Pf: c) $\sigma^* z^a = \exp(2\pi\sqrt{-1}\, a)\, \bar{z}^a$

$\sigma^* \ln \bar{z} = \ln \bar{z} + 2\pi i$

On the other hand $y(\bar{z}) = \sum_{j,\ell} f_{j,\ell}(z)\, \bar{z}^{\rho_j} (\ln \bar{z})^{b_\ell}$

$\Rightarrow \varphi(\sigma^* y) \le \varphi(y)$. Similarly $\varphi((\sigma^*)^{-1} y) \le \varphi(y)$. □

From a) and b) we get that $\{\varphi(X)\}$ is a finite

set: $\{\infty > \varphi^1 > \varphi^2 > \dots > \varphi^m\}$.

Def: [Levelt's filtration] $0 \subset X^1 \subset X^2 \subset \dots \subset X^m = X$

where $X^\ell = \{ y \in X \mid \varphi(y) \ge \varphi^\ell \}$.

The filtration is σ^*-invariant

Def: $k_\ell := \dim (X^\ell / X^{\ell-1})$. Note σ^* acts on $X^\ell/X^{\ell-1}$

$$ {}^\ell\sigma^* = \sigma^* \Big|_{X^\ell/X^{\ell-1}} $$

Let $e_1^1 \dots e_{k_1}^1$ base for X^1 s.t. σ^* is upper triangular

Take $\tilde{e}_1^2 \dots \tilde{e}_{k_2}^2$ base for X^2/X^1 s.t. ${}^2\sigma^*$ is upper triangular

and lift (arbitrary) to $\{e_1^2, \dots, e_{k_2}^2\} \in X^2$. Continuing this way we get a fundamental matrix:

$$ Y(z) = [e_1^1 \dots e_{k_1}^1, e_1^2 \dots e_{k_2}^2, \dots, e_1^m \dots e_{k_m}^m] = [e_1, e_2, \dots, e_p] $$

the corresp. monodromy matrix G is upper-triangular.

The following properties hold:

1) φ takes all possible values $\varphi^1, \dots, \varphi^m$ w/ multipl. $k_1, \dots, k_m$

2) $\varphi(e_{\ell+1}) \leq \varphi(e_\ell)$

3) σ^* is upper triangular

Def: Any basis $\{e_1, e_2, \dots, e_p\}$ of X satisfying 1), 2), and 3) then it is called Levelt's basis.

Exercise: If σ^* is a Jordan block, then a Jordan basis is a Levelt's basis. Any other Levelt's basis is obtained by conjugation by an upper triangular matrix.

Pf: $\{e_1, \dots, e_e\}$ Jordan basis

$\sigma^* e. \; --$

If $e = \{e_1, \dots, e_p\}$ is a Levelt's basis then

$$A := \begin{bmatrix} \varphi(e_1) & & 0 \\ & \ddots & \\ 0 & & \varphi(e_p) \end{bmatrix}, \quad G = \sigma^*, \quad E = \frac{1}{2\pi i} \ln G$$

$0 \le \operatorname{Re} \rho^i < 1$

eigen-values of E

Lemma 5.1. Let $\tilde{C} = z^A C z^{-A}$; then $\tilde{G}$ and $\tilde{E}$ are holom. at $z = 0$, and $\varphi(z^A \bar{z}^E z^{-A}) = 0$.

Pf. if $C = (c_{ij})$ if $c_{ij} = 0$ for $i > j$ (upper triangular matrix)

$$\Rightarrow \tilde{c}_{ij} = \begin{cases} z^{\varphi_i - \varphi_j} \cdot c_{ij} & \text{for } i \le j \\ 0 & \text{for } i > j \end{cases}$$

$\Rightarrow \tilde{G}$ and $\tilde{E}$ are holomorphic.

$$z^A \bar{z}^E z^{-A} = \begin{bmatrix} \bar{z}^{\rho_1} & & * \\ & \ddots & \\ 0 & & \bar{z}^{\rho_p} \end{bmatrix}$$

$$E = \begin{bmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_p \end{bmatrix} + \underset{\substack{\uparrow \\ \text{upper triangular}}}{N}$$

$$z^E = z^R \cdot \underset{\substack{\uparrow \\ \text{growth } 0}}{z^N}$$

Thm 1. If (e) is a Levelt's basis; then

$$Y(z) = U(z)\, z^{A} \bar{z}^{E} \qquad \text{w/ } U(z) \text{ is holomorphic at } z=0.$$

Pf. We already saw that $U(z) z^{A}$ is single-valued $\Rightarrow$ $U(z)$ is single-valued.

Put $r = \max_i \rho_i$ and choose $\varepsilon > 0$ s.t. $2\varepsilon + r < 1$.

We want to show that $\lim_{z\to 0} U(z)\, \bar{z}^{r+2\varepsilon} = 0$.

$$U(z)\, \bar{z}^{r+2\varepsilon} = Y_e(z)\, z^{-\cancel{E}}\, \tilde{z}^{-\cancel{A}}\, \bar{z}^{r+2\varepsilon} =$$

$$= \underbrace{\left(Y_e(z)\, z^{-A+\varepsilon}\right)}_{N_1} \cdot \underbrace{\left(z^{A}\, \tilde{z}^{-E}\, z^{-A}\right)}_{N_2} \tilde{z}^{r+\varepsilon}$$

By definition $e_i\, z^{-\varphi(e_i)+\varepsilon} \to 0$ as $z \to 0$ $\Rightarrow$ $\lim N_1(\bar{z}) = 0$ by definition

$\bar{z}^{-E+r}$ $\rightsquigarrow$ has entries $a_{ij} = \sum_{\ell=1}^{p} \bar{z}^{r-\rho_\ell} P^{\ell}_{ij}(\ln z)$

$\Rightarrow \varphi(a_{ij}) \geq 0$

$$\Rightarrow \lim_{z\to 0} \left(z^{A}\, \tilde{z}^{-E+r}\, z^{-A}\right) \cdot z^{\varepsilon} = 0. \quad \square$$

Def: Weak Levelt's basis

- If only one eigenvalue then same as Levelt

$$X = \underset{\lambda_1}{X_1} \oplus \cdots \oplus \underset{\lambda_s}{X_s}$$ eigenspace decomposition w/ respect to σ^*

Put $\sigma_i^{*\prime} = \sigma^*|_{X_i}$.

Construct Levelt's basis for each X_i

$$\text{Weak Levelt } [X] = \bigsqcup_{i=1}^{s} \text{Levelt } [X_i]$$

Exercise 5.5. Show that $WL(X)$ is associated w/ Levelt of X as follows:

$$\varphi\{e_1, \ldots, e_p\} = \psi\ell \quad \text{w/ mult. } k_\ell$$

Thm 1 holds for a weak Levelt basis.

Example: $\dfrac{dy}{dz} = \begin{pmatrix} 0 & 1 \\ z^{-2} & -z^{-1} \end{pmatrix} \cdot y$

$$Y(z) = \begin{vmatrix} z & z^{-1} \\ 1 & -z^{-2} \end{vmatrix}$$

$$\varphi\begin{pmatrix} z \\ 1 \end{pmatrix} = 0, \quad \varphi\begin{pmatrix} z^{-1} \\ -z^{-2} \end{pmatrix} = -2 \quad \Rightarrow \quad \text{Levelt's basis}$$

$$\Rightarrow Y(z) = \begin{bmatrix} z & z \\ 1 & -1 \end{bmatrix} \cdot z^{\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}}.$$

Example: $\dfrac{dy}{dz} = \begin{bmatrix} z^{-1} & 1 \\ 0 & 0 \end{bmatrix} \cdot y$, $\quad Y(z) = \begin{bmatrix} z & z\ln z \\ 0 & 1 \end{bmatrix}$

$$\varphi\begin{pmatrix} z \\ 0 \end{pmatrix} = 1, \quad \varphi\begin{vmatrix} z\ln z \\ 1 \end{vmatrix} = 0$$

$$Y(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot z^{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \cdot z^{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}$$

Thm 2. The matrix $U(0)$ is invertible if and only if the system is Fuchsian at $z=0$.

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Lecture 4: The Levelt theorem (Lecture 6 in Bolibrukh)

(1) $\frac{dy}{dz} = B(z)\cdot y$ near $z=0$, where we have a regular singular point

X - space of solutions and $\sigma^*: X \to X$ is the monodromy

weak Levelt ~~decomposition~~ basis:

$X = \bigoplus_{i=1}^{s} X_i$, X_i - (generalized) eigenspaces of σ^*

Recall $\varphi: X \to \mathbb{Z} \cup \{\infty\}$

$\varphi(y) = \sup\{ l \mid \lim_{z\to 0} \left|\frac{y(z)}{z^\lambda}\right| = 0 \text{ for all } \lambda < l \}$

e.g. $\varphi(z^{1.2}) = 1$, $\varphi(z^{1+i}) = 1$

$\varphi_i^1 > \varphi_i^2 > \dots > \varphi_i^{m_i}$

$0 \subset X_i^1 \subset X_i^2 \subset \dots \subset X_i^{m_i} = X_i$

$X_i^l = \{ y \in X_i \mid \varphi(y) \geq \varphi_i^l \}$

$Y(z) = [\, Y_1(z) \;\; Y_2(z) \dots Y_s(z) \,]$

where $Y_i(z) = [\, Y_{i,1}, \; Y_{i,2}, \dots Y_{i,m_i}]$ columns of $Y_i(z)$ form a basis of X_i

form a basis of X_i^1 s.t. matrix of σ^* is uper triang.,

project to a basis of X_i^2/X_i^1, matrix of σ^* in X_i^2/X_i^1 is upper triangular

- - -

matrix of σ^* $\downarrow$

$$G = \begin{bmatrix} G_1 & & 0 \\ & \ddots & \\ 0 & & G_s \end{bmatrix}$$

matrix of σ_i^* $\downarrow$ $\quad X_i^1 \subset X_i^2 \subset \cdots \subset X_i^{m_i}$:

$$G_i = \begin{bmatrix} G_i^{11} & G_i^{12} & \cdots & G_i^{1,m_i} \\ 0 & G_i^{22} & & \vdots \\ & & \ddots & \\ & & & G_i^{m_i m_i} \end{bmatrix}$$

$G_i^{\ell\ell}$ is the matrix of ${}^{\ell}\sigma_i^*$: $X_i^{\ell}/X_i^{\ell-1} \circlearrowleft$

it is upper triangular w/ diagonal entries λ_i, $1 \le i \le s$

$$\Rightarrow \quad E = \frac{1}{2\pi\sqrt{-1}} \ln G = \begin{bmatrix} E_1 & & 0 \\ & \ddots & \\ 0 & & E_s \end{bmatrix}$$

$$E_i = \begin{bmatrix} E_i^{11} & E_i^{12} & \cdots & E_i^{1m_i} \\ & \ddots & & \vdots \\ & & E_i & \\ O & & & E_i^{m_i m_i} \end{bmatrix}, \quad E_i^{\ell\ell} = \rho_i \cdot I + N_i^{\ell\ell}$$

$\rho_i = \frac{1}{2\pi\sqrt{-1}} \ln \lambda_i$ $\quad$ upper triangular

$0 \le \operatorname{Re} \rho_i < 1$

$$A = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_s \end{bmatrix}, \quad A_i = \begin{bmatrix} \varphi_i^1 I & & 0 \\ & \ddots & \\ 0 & & \varphi_i^{\ell} I \end{bmatrix}$$

$$\varphi_i^{\ell} = \varphi(Y_{i,\ell})$$

G ~~and~~ and $E = \frac{1}{2\pi i} \log G$

$\Rightarrow$ the monodromy matrix of $Y(z)$ ~~is~~ are upper triangular. More precisely,

Put

$$A_i := \begin{bmatrix} \psi_i^1 \cdot I & & 0 \\ & \ddots & \\ 0 & & \psi_i^{m_i} I \end{bmatrix}, \quad A := \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_s \end{bmatrix}$$

$$E = \begin{bmatrix} E_1 & & 0 \\ & \ddots & \\ 0 & & E_s \end{bmatrix}, \quad E_i = \begin{bmatrix} E^i_{11} & \cdots & E^i_{1 m_i} \\ & \ddots & \vdots \\ 0 & & E^i_{m_i m_i} \end{bmatrix}$$

$E^i_{\ell\ell} = \rho^i + N^i_{\ell\ell}$ ← upper triangular

$\rho^i = \frac{1}{2\pi i} \log \lambda^\ell_i, \ 0 \le \rho^i < 1$

then, as we proved last time

$$(2) \qquad Y(z) = U(z)\, z^A z^E,$$

where $U(z)$ is holomorphic at $z = 0$.

<u>Thm</u> [Levelt] The regular singular point $z=0$ is Fuchsian ~~if and only if the matrix $U(0)$ is invertible.~~

<u>Pf.</u> Substitute (2) in (1); then we get

Put $B(z) = \frac{B_0(z)}{z}$ and

$$(3) \qquad B_0(z)\, U(z) = z \frac{dU}{dz} + U(z) L(z), \quad \text{where } L(z) = A + z^A E z^{-A}$$

↖ holomorphic at 0!

$\Rightarrow$) <u>Assume $z=0$ is Fuchsian</u> $\Rightarrow$ $B_0(z)$ is holomorphic at 0

$$B_0(0) \cdot U(0) = U(0)\, L(0)$$

$\Rightarrow$ ~~$L(0) : \mathrm{Ker}\, U(0) \longrightarrow \mathrm{Ker}\, U(0)$~~. Assume $\mathrm{Ker}(U(0)) \neq 0$ and

Let $\underset{\not= 0}{c} \in \mathrm{Ker}\, U(0)$ be an eigen-vector of $L(0)$. Put

$$y_c(z) = Y(z) \cdot c$$

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We compute $\varphi(y_c(z))$ in two different ways.

1-st way:

$$L(z) = \begin{bmatrix} L_1(z) & & 0 \\ & \ddots & \\ 0 & & L_s(z) \end{bmatrix}, \quad L_i(z) = A_i + z^{A_i} E_i z^{-A_i}$$

$$= \begin{bmatrix} \psi_i^1 I + E_i^{11} & E_i^{1,2} z^{\psi_i^1 - \psi_i^2} & \cdots & E_i^{1,m_i} z^{\psi_i^1 - \psi_i^{m_i}} \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \psi_i^{m_i} I + E_i^{m_i m_i} \end{bmatrix}$$

$$L_i(0) = \begin{bmatrix} (\rho^i + \psi_i^1) I + N_i^{11} & 0 & 0 \\ \vdots & \ddots & \\ 0 & \cdots & (\rho^i + \psi_i^{m_i}) I + N_i^{m_i m_i} \end{bmatrix}$$

$$L(0) = \begin{bmatrix} L_1(0) & & 0 \\ & \ddots & \\ 0 & & L_s(0) \end{bmatrix} \Rightarrow c = \begin{bmatrix} c_1 \\ \vdots \\ c_s \end{bmatrix}, \quad c_i = \begin{bmatrix} c_i^1 \\ \vdots \\ c_i^{m_i} \end{bmatrix}$$

c_i^l is a vector of size $\dim X_i^l / X_i^{l-1}$

c is an eigenvector of $L(0)$ only if $c_i^l \neq 0$ for precisely one pair (i, l), $1 \leq i \leq s$, $1 \leq l \leq m_i$ and the eigen-value of c is $\rho^i + \psi_i^l$

Note that $y_c = Y(z) . c$ is a linear combin. of the solutions belonging to $Y_{i,l}$ (← they project to a basis X_i^l / X_i^{l-1})

$$\Rightarrow \boxed{\varphi(y_c) = \psi_i^l}$$

2-nd way:

$$y_c(z) = U(z)\, z^A z^E \cdot c$$

Note that $E = R + N$, where $R = \begin{bmatrix} R_1 & & 0 \\ & \ddots & \\ 0 & & R_s \end{bmatrix}$

$R_i = \rho_i I$, We get ($[R,N] = 0$!)

$$y_c(z) = U(z)\, \underbrace{z^A z^N z^{-A}}_{1+\sum_{k=1}^{\infty} \frac{(\ln z)^k}{k!} N^k}\ \underbrace{z^A z^R \cdot c}_{= z^{\rho_i+\varphi_i^l} c} =$$

$$= z^{\rho_i+\varphi_i^l}\, U(z) \left(1 + \sum_{k=1}^{\rho-1} \frac{(\ln z)^k}{k!} \left(z^A N z^{-A}\right)^k \right) \cdot c$$

We have:

$$z^A N z^{-A} = z^A (E-R) z^{-A} = z^A E z^{-A} - R = L(z) - A - R =$$

$$= L(0) - A - R + O(z)$$

and $(L(0) - A - R) \cdot c = 0$ (since c is an eigenvector of $L(0)$ w/ eigenvalue $\rho_i + \varphi_i^l$)

$$\Rightarrow y_c(z) = z^{\rho_i+\varphi_i^l} \Big(\underbrace{U(z) \cdot c}_{= \underbrace{U(0)\cdot c}_{0} + O(z)} + O(z (\ln z)^{\rho-1}) \Big) = z^{\rho_i+\varphi_i^l+1}\, O((\ln z)^{\rho-1})$$

if $\lambda < \varphi_i^l + 1$ then $\lim_{z\to 0} \frac{y_c(z)}{|y|^\lambda} = 0 \Rightarrow \varphi(y_c(z)) \geq \varphi_i^l + 1$

contradiction. $\boxtimes$

$\Leftarrow$) From formula (3) we have

$$B_0(z) = z \frac{dU}{dz} \cdot U^{-1}(z) + U(z) L(z) U^{-1}(z)$$

if $U(0)$ is invertible then the RHS is holomorphic. $\boxtimes$

Corollary. The Levelt's thm holds for Levelt's basis as well.

Pf. e' - Levelt's basis $\quad 0 \subset X^1 \subset X^2 \subset \dots \subset X^m = X$

$$Y_{e'}(z) = U'(z)\, z^{A'} z^{E'} = U(z)\, z^{A} z^{E} \cdot S$$

const. matrix (invertible)

Note that $\det(z^{E'}) = z^{\operatorname{tr}(E')} = z^{\operatorname{tr}(\sigma^*)\frac{1}{2\pi\sqrt{-1}}} = \det(z^{E})$

$$\Rightarrow \det(U'(z)) \cdot z^{\operatorname{tr} A'} = \det(U(z))\, z^{\operatorname{tr} A} \cdot \det(S)$$

$$0 \subset X^1 \subset X^2 \subset \dots \subset X^m = X$$

$$\underbrace{X^1 \cap X_i = \dots = X^{k_i^0} \cap X_i}_{0} \subset \underbrace{X^{k_i^0+1} \cap X_i = \dots = X^{k_i^1} \cap X_i}_{X_i^1} \subset \dots \subset \underbrace{X^{k_i^{m-1}+1} \cap X_i = \dots = X^{k_i^m} \cap X_i}_{X_i^{m_i}}$$

$$\Rightarrow \operatorname{tr} A_i = \sum_{l=1}^{m_i} \psi_i^l \dim(X_i^l / X_i^{l-1}) = \sum_{l=1}^{m} \psi^l \cdot \dim(X^l \cap X_i / X^{l-1} \cap X_i)$$

$$\Rightarrow \operatorname{tr} A = \sum_{i=1}^{s} \operatorname{tr} A_i = \sum_{l=1}^{m} \psi^l \cdot \dim(X^l / X^{l-1}) = \operatorname{tr} A'$$

$$\Rightarrow \det(U'(z)) = \det(U(z)) \det(S) \quad \square$$

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Lecture 5: The global theory (Lecture 7 in Bolibrukh)

1. Exercises.

Def: If $\frac{dy}{dz} = \frac{B_0(z)}{z^k} \cdot y$, $B_0(0) \neq 0$ has a regular singular point; then k is called Poincaré rank of the singularity.

$b := \varphi(\det U(z))$.

Claim 1: $b \geq r$.

Claim 2 [Sauvage]: If $U(z)$ is holomorphic around $z=0$, invertible outside 0 (i.e. for $z \neq 0$); then $\exists$ $\Gamma(z)$ is holomorphic at $z=0$ and $c_1 \geq c_2 \geq \dots \geq c_p = 0$ s.t.

$$\Gamma(z) U(z) = z^C V(z)$$

where $C = \begin{bmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_p \end{bmatrix}$, $V(z)$ is holom. and $V(0)$ is invertible.

Claim 3. Prove that $b \leq \frac{p(p-1)}{2} r$.

Hint: Use Claim 2. and prove $c_i - c_{i+1} \leq r$ $\forall i$.

2. Fuchsian systems on $\mathbb{P}^1$.

F - holomorphic v.b. on $\mathbb{P}^1$ w/ merom. connection ∇

$a_1, \dots, a_n \in \mathbb{P}^1$ the set of sing. points ($\infty \notin \{a_1, \dots, a_n\}$)

O_i : small neighborhood of a_i $\Rightarrow$ horiz. sections of ∇ are given by

$$\frac{dy}{d\xi_i} = B_i(\xi_i)\cdot y \quad , \quad \xi_i = z - a_i$$

we have local invariants $Y_i(z) = U_i(z)\, z^{A_i} z^{E_i}$

ρ_i^j, $1 \le j \le m$, φ_i^ℓ, $1 \le \ell \le m$, $\beta_i^j = \rho_i^j + \varphi_i^j$ Levelt's exponents

Questions 1:

1) What is ∇ for trivial v.b. F

2) What are the relations between Levelt's filtrations and exponents in different points.

3) Conditions on (β_i^j) for ∇ to be Fuchsian.

$\boxed{\text{Assume } F \text{ is trivial and } \nabla \text{ is Fuchsian}}$. Then the system looks:

$$dy = \omega \cdot y$$

where ω is a 1-form on $\mathbb{P}^1$

Define $B_i = \operatorname{res}_{z=a_i} \omega \implies \omega - \sum_{i=1}^n \frac{B_i}{z-a_i}dz \in \Gamma(\mathbb{P}^1, \overset{\mathcal{O}(-2)}{\overset{\shortparallel}{\Omega_{\mathbb{P}^1}}}) = 0$

$$\implies \omega = \left(\sum_{i=1}^n \frac{B_i}{z-a_i}\right)dz \quad , \quad \sum_{i=1}^n B_i = 0$$

Thm 7.1. If ∇ is a connection w/ regular singular points on a trivial bundle (on $\mathbb{P}^1$); then

(a) $\Sigma := \sum_{i=1}^n \sum_{j=1}^p \beta_i^j \le 0$ and $\Sigma \in \mathbb{Z}$

(b) ∇ is Fuchsian $\iff$ $\Sigma = 0$.

Pf:

$$\det Y_i(z) = c_0 \exp\left(\int \operatorname{tr} B_i(z)\, dz \right)$$

$$\| $$

$$\det U_i(z) \cdot (z-a_i)^{\operatorname{tr} A_i + \operatorname{tr} E_i} = h(z)\,(z-a_i)^{b_i + \sum_{j=1}^{p} \beta_i^j}$$

$$b_i = \varphi_{z=a_i}\left(\det\left(U_i(z)\right)\right)$$

$$\Rightarrow \operatorname{tr} B_i(z)\, dz = d\log\left(\det Y_i(z)\right)$$

$$\Rightarrow \operatorname{res}_{z=a_i} \operatorname{tr} B_i(z)\, dz = b_i + \sum_{j=1}^{p} \beta_i^j$$

$$\Rightarrow 0 = \sum_{i=1}^{m} \operatorname{res}_{z=a_i} \operatorname{tr} B(z)\, dz = \Sigma b_i + \Sigma$$

$$\Rightarrow \Sigma = -\sum_{i=1}^{n} b_i \leq 0 \quad . \quad \boxed{\text{M}}$$

3. Fuchsian equations.

$$u^{(p)} + q_1(z)\, u^{(p-1)} + \cdots + q_p(z)\cdot u = 0$$

$a_1, a_2, \ldots, a_n = \infty$ singular points

In coordinate $\zeta = z^{-1}$

$$(\partial_z)^j = (-\zeta^2 \partial_\zeta)^j = \sum_{i=1}^{j} c_i^j\, \zeta^{i+j}\, \partial_\zeta^i$$

$\Rightarrow$ we get

$$(7.8) \quad \left(\partial_\zeta^p + \tilde{q}_1(\zeta)\, \partial_\zeta^{p-1} + \cdots + \tilde{q}_p(\zeta) \right)\cdot u = 0$$

(7.8) is Fuchsian at $\zeta = 0$ iff $R_i(\zeta) = \zeta^{-i} q_i(\zeta^{-1})$ is holom. at $\zeta = 0$ $\quad 1 \leq i \leq p$.

$$q_i(z) = \frac{r_i(z)}{[(z-a_1)\cdots(z-a_n)]^i}$$ where $r_i(z)$ is holom. in $\mathbb{C} \subset \mathbb{P}^1$

at $z=\infty$, $r_i(z)$ has a polynomial growth z^{k_i}, $k_i \le (n-2)i$

$r_i(z) = q_i\,[(z-a_1)\cdots(z-a_n)]^i$ has degree $(n-2)i$

$\Rightarrow$ r_i is a polynomial of degree k_i+1

$\Rightarrow$ # of parameters is

$$\boxed{N = \sum_{i=1}^{p} (k_i+1) = (n-2)\frac{p(p+1)}{2} + p}$$

Thm 2. For Fuchsian equations we have:

$$\sum_{i=1}^{n}\sum_{j=1}^{p} \beta_i^j = (n-2)\frac{p(p-1)}{2}$$

Pf. Assume $z=\infty$ is not a singular point. Switch to a system:

$$y^\ell = \prod_{i=1}^{n}(z-a_i)^{\ell-1}\,\partial_z^{\ell-1} u, \quad 1 \le \ell \le p$$

$\Rightarrow$ new system is Fuchsian w/ same exponents at $a_1, \dots, a_n$

Choose a basis $\{e_1, \dots, e_p\}$ for the equation $\Rightarrow$

$$Y(z) = \Gamma(z)\cdot W(z), \quad W = \begin{bmatrix} e_1 & \cdots & e_p \\ e_1' & \cdots & e_p' \\ \vdots & & \vdots \\ e_1^{(p-1)} & \cdots & e_p^{(p-1)} \end{bmatrix}$$

$$\Gamma(z) = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & \left(\prod_{i=1}^{n}(z-a_i)\right)^{p-1} \end{bmatrix}, \quad W(z) = \Gamma_1(z)\,\Gamma_2(z)\,V(\zeta)$$

$V(\zeta)$: Wronskian w.r.t. ζ, invertible near $\zeta=\infty$ for all ζ.

$$\Gamma_1 = \begin{bmatrix} 1 & & 0 \\ & z^{-2} & \\ & & \ddots \\ 0 & & z^{-2(p-1)} \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ * & & & -1 \\ & & & & \ddots \end{bmatrix}$$

$$\Rightarrow \varphi_{\zeta=\infty}(\det(W(z))) = p(p-1), \quad \varphi_{\zeta=\infty}(\det\Gamma) = -n\frac{p(p-1)}{2}$$

Def. Degree of a vector bundle F on $\mathbb{P}^1$ is

$$c_1(F) = \sum_{i:\text{sing. of the connection}} \operatorname{res}_i \det \nabla$$

(independent of the choice of a meromorphic conn. ∇ on $\mathbb{P}^1$)

We have similar results for non-trivial bundle F.

Thm. If ∇ is a connection on F w/ regular singularities then (a) $\Sigma := \sum_{i=1}^{n} \sum_{j=1}^{p} \beta_i^j \le c_1(F)$;

~~(b)~~ ∇ is logarithmic iff $\Sigma = c_1(F)$. ∎

Exercises.

1) Exponents does not change under $\mathrm{Aut}(\mathbb{P}^1)$

2) Use Claim 1 and 3 ~~that~~ to prove that

$$-\frac{p(p-1)}{2} \sum_{i=1}^{n} r_i \le \sum_{i=1}^{n} \sum_{j=1}^{p} \beta_i^j \le -\sum_{i=1}^{n} r_i$$

r_i - Poincaré rank at a_i

3) Hypergeometric equation is a Fuchsian equation w/ $n=3$ $\{0,1,\infty\}$, $p=2$ and exponents $\beta_0^1 = \beta_1^1 = 0$

$\beta_0^2 = 1-\gamma$, $\beta_1^2 = \gamma-\alpha-\beta$, $\beta_\infty^1 = \alpha$, $\beta_\infty^2 = \beta$

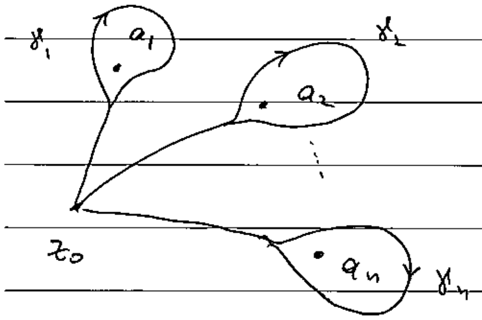
Lecture 6: The Riemann-Hilbert problem (Ch. 8 from Bolibrukh)

1. The 21-st Hilbert problem

$a_1, \dots, a_n \in \overline{\mathbb{C}} = \mathbb{P}^1$

$\chi : \pi_1(\mathbb{C} \setminus \{a_1, \dots, a_n\}) \to GL(p; \mathbb{C})$ representation

Can χ be realized as monodromy repres. of a Fuchsian system.



$G_j = \chi(\gamma_j)$, $1 \le j \le n$

Since $\gamma_n \dots \gamma_1 = 1$, we must have

$G_n G_{n-1} \dots G_1 = 1$

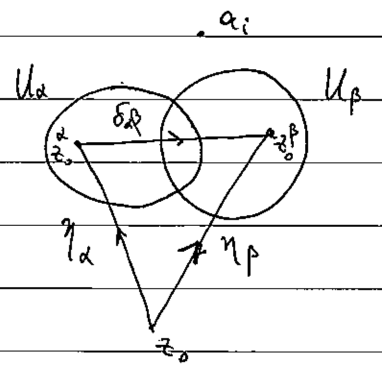
Fix χ. Try to find a merom. conn. (F, ∇)

2. Extensions

Step 1. Find a conn. (F^0, ∇) on $B = \mathbb{P}^1 \setminus \{a_1, \dots, a_n\}$ that realizes χ. Cover B by $\{U_i\}$ s.t.

(1) U_i are connected, simply connected

(2) $U_i \cap U_j$ — " —



$g_{\alpha\beta} = \chi(\eta_\alpha \circ \delta_{\alpha\beta} \circ \eta_\beta^{-1})$

Consider $\{U_\alpha \times \mathbb{C}^p\}$ w/ $g_{\alpha\beta}$ gluing cocycles

$\nabla_\alpha = d + 0$

$\Rightarrow$ get a bundle w/ flat connection.

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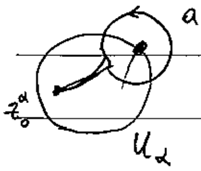
Step 2. Continue (F°, ∇) to (F, ∇) F holom. on $\bar{\mathbb{C}}$ s.t.
∇ merom. at $a_1, \dots, a_n$

Fix a_i, $\mathcal{U}_\alpha$ s.t. $a_i \in \overline{\mathcal{U}_\alpha}$

δ_i small loop around a_i



$(e_1^\alpha, \dots, e_p^\alpha)$ basis of horiz. sections over $\mathcal{U}_\alpha$

analytical continuation along δ_i gives

$$(e_1^\alpha, \dots, e_p^\alpha) \cdot G_i$$

Put $E_i = \frac{1}{2\pi\sqrt{-1}} \ln G_i$ s.t. eigenv. ρ of E_i satisfy $0 \le \operatorname{Re}\rho < 1$

Fix a branch of $(z-a_i)^{-E_i}$ in $\mathcal{U}_\alpha$.

$\mathcal{O}_i$ - open neighb. of a_i.

$s = (s_1, \dots, s_p)$ sections of $\mathcal{O}_i \times \mathbb{C}^p$ s.t. $[s_1 \ \dots \ s_p] = I_p$ identity matrix

Put $\xi^\alpha = e^\alpha \cdot (z-a_i)^{-E_i}$ basis of $F^\circ|_{\mathcal{U}_\alpha}$

it induces a trivializ. of $F^\circ|_{\mathring{\mathcal{O}}_i}$.

Define: $g_{\alpha 0} : \mathring{\mathcal{O}}_i \cap \mathcal{U}_\alpha \to GL_p(\mathbb{C})$ s.t. $s_i \equiv \cdot\, \xi_i^\alpha$

Note that in the trivializ. given by ξ^α the conn. takes the form:

$$\omega = \frac{E_i\, dz}{z - a_i}.$$

Let $\{\tilde{e}^\alpha\}$ be another basis of horiz. sections in U_α

$(\tilde{e}^\alpha) = (e^\alpha)\cdot S$ then the monodromy becomes $\tilde{G} = S^{-1} G S$

$\Rightarrow$ we can assume the monodromy $\tilde{G}$ and $\tilde{E} = \frac{1}{2\pi\sqrt{-1}} \ln \tilde{G}$

are upper triangular.

Choose $\Lambda = \mathrm{diag}[\lambda_1, \ldots, \lambda_p]$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \in \mathbb{Z}$

admissible matrix. Take a basis

$$\xi^{\Lambda, \tilde{\alpha}} = \tilde{e}^\alpha \, (z-a_i)^{-\tilde{E}_i} \, (z-a_i)^{-\Lambda_i}$$

single valued near $z = a_i$ $\Rightarrow$ give a trivializ. of $F^{(0)}\big|_{\mathring{D}_i}$

$\Rightarrow$ we get an extension of $F^{(0)}$ that depends on Λ_i.

The connection matrix in the new trivializ. becomes

$$\omega^{-\Lambda_i} = \left(\Lambda_i + (z-a_i)^{\Lambda_i} \tilde{E}_i \, (z-a_i)^{-\Lambda_i}\right) \frac{dz}{z-a_i}.$$

$\mathcal{F} = \{$the set of all these extensions (F, ∇) for different $\Lambda_i, S_i\}$

<u>Thm.</u> $\mathcal{F}$ contains <u>all</u> extensions of (F°, ∇) to (F, ∇) so that ∇ has Fuchsian singularities at $\{a_1, \ldots, a_n\}$, and monodromy repr. χ.

<u>Pf.</u> Let (F', ∇') be any bundle w/ a logarithmic connection and a monodromy repr. χ.

$$(F', \nabla')|_B = (F^\circ, \nabla)$$

Let (ξ) be a basis of local holom. sections of F' over $O_i \ni a_i$

$(F', \nabla')|_{O_i}$ is a Fuchsian system near $z = a_i$ $\Rightarrow$

we can choose a Levelt fundamental matrix [w/ respect to some trivializ. $\xi^\alpha = (\xi_1^\alpha, \dots, \xi_p^\alpha)$ of F' in $O_i \cap U_\alpha$]

$$Y(z) = U(z)\underbrace{(z-a_i)^{A_i}}_{\text{admissible}}(z-a_i)^{E_i}$$

$U(z)$ is holomorphically invertible.

The basis (s^α) of horiz. sections of ∇' over $U_\alpha \cap O_i$ w/ matrix w/ coords. $Y(z)$ (we have to fix a branch of $(z-a_i)^{E_i}$ in $U_\alpha \cap O_i$)

Define $\xi^\alpha = (\xi_1^\alpha \dots \xi_p^\alpha)$, $\xi_i^\alpha : O_i \cap U_\alpha \to \mathbb{C}^p$ by

$$(s^\alpha) = (\xi^\alpha) . Y(z) = (\underbrace{\xi^\alpha . U(z)}) (z-a_i)^{A_i} (z-a_i)^{\tilde{E}_i}$$

$$(\xi')^\alpha = (s^\alpha) \cdot (z-a_i)^{-\tilde{E}_i} (z-a_i)^{-A_i}$$

$\Rightarrow (F', \nabla')$ is isomorphic to a bundle from the class $\mathcal{F}$ w/ $\Lambda_i = A_i$ and $(s_1^\alpha, \dots, s_p^\alpha)$ as $\tilde{e}^\alpha$.

Lecture 7: The Birkhoff–Grothendieck thm

1. Vector bundles on $\mathbb{P}^1$.

Thm 1. [B.–G.] Every holomorphic vector bundle E (of rank p) on $\mathbb{P}^1$ has the form:

$$E \cong \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_p)$$

for some integers $k_1 \geq \cdots \geq k_p$. ☑

Def 1: The integers $k_1 \geq \cdots \geq k_p$ are called splitting numbers (integers).

Corollary 1 [Lemma 10.1 in Bolibrukh].

$E := \bigoplus_{i=1}^{p} \mathcal{O}(k_i) \cong \bigoplus_{i=1}^{p} \mathcal{O}(k_i') =: E'$ iff $k_i = k_i'$ for all $i = 1, 2, \cdots, p$

Pf. Cover $\mathbb{P}^1$ by $U_0 = \mathbb{C}$ and $U_\infty = \mathbb{P}^1 \setminus \{0\}$

put $K = \mathrm{diag}[k_1, \cdots, k_p]$, $K' = \mathrm{diag}[k_1', \cdots, k_p']$

then E is glued from $U_0 \times \mathbb{C}^p$ and $U_\infty \times \mathbb{C}^p$ via

$$g_{0\infty}(z) = z^K : \underset{\cap\; U_\infty}{(U_0 \cap U_\infty)} \times \mathbb{C}^p \to \underset{\cap\; U_0}{(U_0 \cap U_\infty)} \times \mathbb{C}^p$$

Similarly E' is glued via $g'_{0\infty}(z) = z^{K'}$.

$\exists$ holom. invertible h_α in U_α $(\alpha \in \{0, \infty\})$ s.t.

$$h_0(z) \cdot \underbrace{g'_{0\infty}(z)}_{z^{K'}} = \underbrace{g_{0\infty}(z)}_{z^K} h_\infty(z), \text{ i.e. } z^{k_j' - k_i} h_0^{ij}(z) = h_\infty^{ij}(z).$$

Assume $k_i = k_i'$ for $i = 1, 2, \dots, l-1$ and $k_l' > k_l \geq k_{l+1} \geq \dots \geq k_p$

$$h_\infty(z) = \begin{array}{c|ccccccc|} & 1 & 2 \dots & l-1 & l & l+1 & \dots & p \\ \hline 1 & & & & * & & & * \\ \vdots & & & & \vdots & & & \vdots \\ l-1 & & & & * & & & * \\ l & & & & * & - & - - & * \\ l+1 & & & & & & & \\ \vdots & & & & & & & \\ p & & & & & & & \end{array}$$

these entries are holomorphic in $\mathbb{P}^1$ and vanish at $z=0$ $\Rightarrow$ must be 0.

$\Rightarrow \det\left(h_\infty(z)\right) = 0$. $\boxtimes$

Corollary 2 [Prop. 10.2 in Bolibrukh]. E: holomorphic bundle on $\mathbb{P}^1$ and $b \in \mathbb{P}^1$ $\Rightarrow$ we can find a basis of meromorphic sections that are holomorphic on $\mathbb{P}^1 \setminus \{a\}$.

Pf. Assume $E = \mathcal{O}(k_1) \oplus \dots \oplus \mathcal{O}(k_p)$; $(U_0 \setminus \{b\}) \times \mathbb{C}^p$, $(e_1^0, \dots, e_p^0)$ $\downarrow z^{k}$

$s_i^\infty = \left(\frac{z-b}{z}\right)^{k_i} e_i^\infty$, $s_i^0 = (z-b)^{k_i} e_i^0$ $\quad (U_\infty \setminus \{b\}) \times \mathbb{C}^p$, $(e_1^\infty, \dots, e_p^\infty)$

$(S^\infty) = (s_1^\infty, \dots, s_p^\infty)$ sections over $\mathbb{P}^1 \setminus \{\infty, b\}$, $\quad (S^0) = (s_1^0, \dots, s_p^0)$ sections over $\mathbb{C} \setminus \{b\}$

Since $(S^0) = (S^\infty)$, we get the desired sections by gluing (S^0) and (S^∞). $\boxtimes$

Remark about trivializ. of v.b.:

E

$\downarrow$

X

$U_i \quad U_j$

$(s) = (s_1, \dots, s_p)$ sections over U_i $\rightsquigarrow$ trivial. $E|_{U_i} \cong U_i \times \mathbb{C}^p$

$(s') = (s_1', \dots, s_p')$: $E|_{U_j} \cong U_j \times \mathbb{C}^p$

$y = \begin{pmatrix} y_1(z) \\ \vdots \\ y_p(z) \end{pmatrix}$ gives section over U_i : $y = \sum_{a=1}^{p} y_a s_a$

$y' = \begin{pmatrix} y_1' \\ \vdots \\ y_p' \end{pmatrix}$ — " — $y' = \sum y_b' s_b'$; then

section over U_j

$$\boxed{\begin{array}{c} y' = g_{ji} \cdot y \\ \Updownarrow \\ (s) = (s') \cdot g_{ji} \end{array}}$$

$s_a = \sum_{b=1}^{p} (g_{ji})_{ba} s_b'$

connection matrices transform as:

$$\boxed{B' = \partial_z g_{ji}\, g_{ji}^{-1} + g_{ji} \underset{\substack{\parallel \\ B_i}}{B}\, g_{ji}^{-1}} \quad \text{with } B' = B_j$$

$\leftarrow$ connection matrices on U_i and U_j

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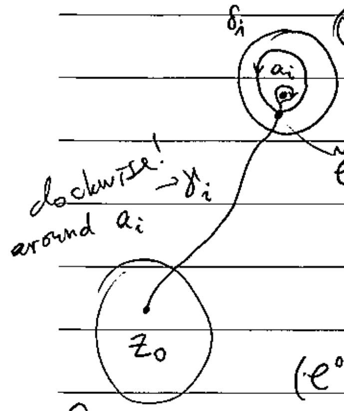
2. Applications to the RH problem.

(E, ∇)-bundle in the class $\mathcal{F}$.

Assume we are given $\chi : \pi_1(\mathbb{P}^1 \setminus \{a_1, \ldots, a_n\}) \longrightarrow GL(p; \mathbb{C})$



$\tilde{e}^i = (\tilde{e}^i_1, \ldots, \tilde{e}^i_p)$

analyt. contin.

$(e^0) = (e^0_1, \ldots, e^0_p)$

O_0 basis of flat sections over O_0

Given a path $[\gamma]$ ~~eta~~ we get by analytical continuoction along γ a new basis:

$$\gamma \cdot (e^0) = (e^0) \cdot M_\gamma$$

Since $\gamma_1 \cdot \gamma_2$ means 1-st γ_1 then γ_2 we must define

$$\chi(\gamma) := M_{\gamma^{-1}}$$

Pick a matrix S_i s.t. in $(e^i) = (\tilde{e}^i) \cdot S_i$ the monodromy along δ_i: $(e^i) \longmapsto (e^i) \cdot G_i$, G_i - upper triang.

$$\left[\ \tilde{e}^i \xrightarrow{\gamma_i^{-1}} \tilde{e}^i M_{\gamma_i^{-1}} = \tilde{e}^i \chi(\gamma_i) \right.$$

$$(e^i) \cdot G_i \cdot S_i^{-1} = (\tilde{e}^i) S_i G_i S_i^{-1} \quad \Rightarrow \quad \boxed{\chi(\gamma_i) = S_i G_i S_i^{-1}}$$

Let $\Lambda_i = \{\lambda_i^1 \geq \lambda_i^2 \geq \ldots \geq \lambda_i^p\} \in \mathbb{Z}$, $1 \leq i \leq n$; then

$$\xi^{\Lambda_i} = (\xi^{\Lambda_i}_1, \ldots, \xi^{\Lambda_i}_p) = (e^i) \cdot \underbrace{(z - a_i)^{-\tilde{E}_i} (z - a_i)^{-\Lambda_i}}_{g_{\alpha i}}$$

where $\tilde{E}_i = \frac{1}{2\pi\sqrt{-1}} \ln G_i$

give a trivialization of $E|_{\mathring{O}_i}$ and ξ^{Λ_i}, by defin. to extend holomorphic~~ally~~ ~~...~~ sections of E on O_i!

the connection matrix in the frame ξ^{Λ_i} becomes:

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$$\left(\Lambda_i + (z-a_i)^{\Lambda_i} \tilde{E}_i \, (z-a_i)^{-\Lambda_i} \right) \frac{dz}{z-a_i} .$$

<u>Theorem 2</u>. [Plemel] $\exists$ a connection ∇ on the trivial bundle on $\mathbb{P}^1$, s.t.,

1) ~~∇ has a regular singular point at 1 of the points (say a_i);~~

2) ~~∇ has logarithmic (Fuchsian) singularities at the other points~~

3) ~~The monodromy representation of ∇ coincides w/ the given one χ.~~

<u>Pf</u>. $(E, \nabla) \in \mathcal{F}$; pick $b = a_i$ $\Rightarrow$ we can identify $E \cong \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_n)$ $\Rightarrow$ choose sections

$e = (e_1, \ldots, e_p)$: holomorphic basis (trivializ.) of E over $\mathbb{P}^1 \setminus \{a_i\}$ meromorphic at $z = a_i$

Let $(e^{a_i}) = (e_1^{a_i} \cdots e_p^{a_i})$ be trivializ. of $\oplus \mathcal{O}(k_i)$ at $z = a_i$ as in <u>Corollary 2</u>.

$(\xi^{\Lambda_i}) = (e^{a_i}) \cdot \bar{V}(z)$ $\qquad$ $\bar{V}$ - invertible holom. at $z = a_i$

Since $(e) := (e^{a_i}) \cdot (z-a_i)^K$ extends to global sections of E

$\Rightarrow (\xi^{\Lambda_i}) = (e) \, (z-a_i)^{-K} \, \bar{V}(z)$

In the frame (e) the connection becomes:

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$$-\frac{K}{z-a_i} + (z-a_i)^{-K}\,\omega_i\,(z-a_i)^{K}$$

where $\omega_i = \partial_z V \cdot V^{-1} + V\left(\Lambda_i + (z-a_i)^{\Lambda_i}\tilde{E}_i\,(z-a_i)^{-\Lambda_i}\right)V^{-1}$

and the $\overbrace{\text{fundamental}}^{\text{corresp.}}$ matrix is

$$\boxed{Y_e(z) = (z-a_i)^{-K}\,V(z)\,(z-a_i)^{\Lambda_i}\,(z-a_i)^{\tilde{E}_i}}$$

$\Rightarrow$ (1) a_i is a regular singular point;

(2) in the other points, the connection remains Fuchsian;

(3) the monodromy repres. remains the same as for $\mathcal{E}$, i.e., χ ☐

Corollary 3: The degree $c_1(E)$ (as introduced by Sergey)

is $k_1 + \dots + k_p$. ☐

Thm 3. [Plemel] Given χ, s.t. one of the matrices is diagonalizable then the RH problem has a solution.

The proof is easy if we use the above ~~#~~ discussion and the following Lemma:

Lemma 1. [10.2 in Bolibrukh]. Assume;

$U(z)$: holomorphically invertible at $z=0$

$K = \mathrm{diag}(k_1 I^{m_1}, \dots, k_t I^{m_t})$, $k_1 > \dots > k_t \in \mathbb{Z}$

Then we have

$$z^K U(z) = \Gamma^{-1}(1/z)\cdot\tilde{U}(z)\cdot z^D,$$

for some matrices: $\Gamma(1/z)$ - polynomial in $1/z$ and invertible in $\mathbb{P}^1 \setminus 0$

$D = \mathrm{diag}[d_1, \dots, d_p]$ $\leftarrow$ perm. of $[k_1 I^{m_1}, \dots, k_t I^{m_t}]$, and $\tilde{U}(z)$ holom. invert. at 0.

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Assume all principal minors of $U(0)$ are invertible.

<u>Pf.</u> Induction on t. For $t=1$: trivial.

$$z^K U(z) = z^{K'} U'(z) z^{K''}, \quad K = K' + K'', \quad U' = z^{K''} U z^{-K''}$$

$$K' = \begin{bmatrix} (k_1 - k_{t-1}) I^{m_1} & & & & 0 \\ & \ddots & & & \\ & & (k_{t-2} - k_{t-1}) I^{m_{t-2}} & & \\ & & & 0 & \\ 0 & & & & 0 \end{bmatrix}, \quad K'' = \begin{bmatrix} k_{t-1} I^{n_1} & 0 \\ 0 & k_t I^{m_t} \end{bmatrix}, \quad n_1 = p - m_t$$

$$U' = \begin{array}{c|cc} & n_1 & m_t \\ \hline n_1 & V & T \\ m_t & W & * \end{array}$$

V is a principal minor of U

T is holom. at $z=0$ w/ order of vanishing $\geq m := k_{t-1} - k_t$

$*$: same as corresp. block in U, so holom. at $z=0$

~~W : has a pole of order $\leq m$~~

$$\Gamma_t\left(\tfrac{1}{z}\right) = \begin{bmatrix} I_{n_1} & 0 \\ z^{-m} \underbrace{R(z)}_{R_0 + R_1 z + \cdots + R_m z^m} & I_{m_t} \end{bmatrix}, \quad \Gamma_t \cdot U' = \begin{bmatrix} V & T \\ z^{-m} R(z) V + W & z^{-m} R(z) \cdot T + * \end{bmatrix} =: U''(z)$$

$V_0 + V_1 z + \cdots + V_m z^m + O(z^{m+1})$

choose R_i, $(0 \leq i \leq m)$ so that $\quad z^{-m}(W_0 + W_1 z + \cdots + W_m z^m) + O(z)$

$z^{-m} R(z) V(z) + W(z) = O(z)$ (Note V_0 is an invertible matrix!)

all principal minors of $U''(0)$ are invertible $\xrightarrow{\text{by induction}}$

$\Gamma'(\frac{1}{z}) \, z^{K'} U''(z) = \tilde{U}(z) \, z^{K'}$; Put $\Gamma(\frac{1}{z}) = \Gamma'(\frac{1}{z}) (z^{K'} \Gamma_t(\frac{1}{z}) z^{-K'})$

$$\Gamma(\tfrac{1}{z}) \cdot z^K U(z) = \Gamma(\tfrac{1}{z}) z^{K'} U'(z) z^{K''} = \Gamma'(\tfrac{1}{z}) z^{K'} \underbrace{\Gamma_t(\tfrac{1}{z}) U'(z)}_{U''} z^{K''} = \tilde{U}(z) z^{K' + K'' = K}.$$

General case is reduced to this one by conjugating $U(z)$ w/ a const. matrix. ☐

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Irreducible monodromy

Lecture 8: Another solutions to the RH problem

Prop 1 [11.1 in Bol.] The degree of any subbundle $F \subset \underline{\mathbb{C}}^p$ (trivial bundle)

$\Rightarrow c_1(F) \leq 0$.

Corollary. Every subbundle of deg $= 0$ of a trivial v.b. $/ \overline{\mathbb{C}} = \mathbb{P}^1$ is trivial.

Def1: If $E \to \mathbb{P}^1$ is a v.b.; then $k(E) = \dfrac{c_1(E)}{rk(E)}$ (slope of E)

Def2: E is called stable if $k(F) \lneq k(E)$ $\forall F \subset E$

~~semi-stable if $k(F) < k(E)$~~

Claim. There are no stable bundles on $\mathbb{P}^1$ of $rk > 1$.

$E = \mathcal{O}(k_1) \oplus \dots \oplus \mathcal{O}(k_p)$, $k(E) = \dfrac{k_1 + \dots + k_p}{p} > k_i$ $\forall i$

$\Rightarrow k_1 + \dots + k_p > \sum k_i$ contr.

Prop 2. [11.2 in Bol.] E is semi-stable if $k_1 = k_2 = \dots = k_p = k$

(E, ∇) holom. v.b. $/ \mathbb{P}^1$ w/ a logarithmic connection.

Def3: A subbundle $F \subset E$ is stabilized by ∇ if

$$\nabla(\Gamma(F)) \subset \Gamma(T^*_B \otimes F)$$ (i.e. $\mathcal{F}$ is ∇-invariant)

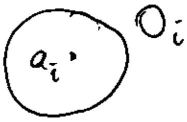
Def4: (E, ∇) is stable if: $k(F) < k(E)$ $\forall F \subset E$ s.t. $\nabla F \subset F$

semi-stable if: $k(F) \leq k(E)$

Thm 1. [II.1 in Bol.] Let $(E, \nabla) \in \mathcal{F}$ be semi-stable; then $k_i - k_{i+1} \le n-2$, where $(k_1, \dots, k_p)$ are the splitting numbers of E and $n = \#$ of singular pts of ∇.

Pf:

$$\Lambda_i, S_i \quad , 1 \le i \le p \quad \text{just like before}$$



(ξ^{Λ_i}) holomorphic frame of E over O_i s.t. the connection matrix of ∇ is

$$\omega^{\Lambda_i} = \left(\Lambda_i + (z-a_i)^{\Lambda_i} \underset{\substack{\uparrow \\ \text{upper triangular}}}{E_i} (z-a_i)^{-\Lambda_i} \right) \frac{dz}{z-a_i} .$$

$\dot{z}_0$

If $(\xi^{\Lambda_i}) = (e^i) \cdot V(z)$, (e^i) some other frame on O_i

then the connection matrix is:

$$\omega_i = dV \cdot V^{-1} + V \cdot \omega^{\Lambda_i} \cdot V^{-1} .$$

Fix i; then E admits a chart $\left(\mathbb{C}, \overline{\mathbb{C}} \setminus \{0, a_i\}, (z-a_i)^K\right)$

gluing $\mathbb{C} \times \mathbb{C}^p \ni (x, v) \sim (x, (z-a_i)^K \cdot v) \in (\overline{\mathbb{C}} \setminus \{0, a_i\}) \times \mathbb{C}^p$

$\Rightarrow$ in the global basis $(e^i)(z-a_i)^K$ holom. on $\mathbb{C} \setminus \{a_i\}$

in the frame (e^i) the connection matrix

is $\omega' = -\dfrac{K}{z-a_i} dz + (z-a_i)^{-K} \omega_i (z-a_i)^K$,

Suppose (E, ∇) is semistable but $\exists\, l$, s.t.

$$k_l - k_{l+1} > n-2$$

$\omega' = (\omega_{mj})_{1 \le m, j \le p}$; $\omega := \omega_i = (u_{mj})_{1 \le m, j \le p}$

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$\boxed{k_j - k_m \geq k_\ell - k_{\ell+1} > n-2}$

$\left(\begin{array}{l} k_j - k_m > n-2 \\ \ell \geq j \ , m > \ell \end{array} \right)$

$\omega_{mj} = u_{mj}(z)\,(z-a_i)^{-k_m+k_j}$ for $m \neq j$

$\Rightarrow$ multipl. of the zero at $z=a_i$ of $\omega_{mj} \geq n-3$.

$\Rightarrow$ ω' at $z=a_i$ has a zero of order $> n-3$

ω' at $z \neq a_j$ has at most pole of order ≤ 1

$\Rightarrow$ ω' has zero $\geq n-4$ and poles of order $\leq n-1$

$$\Rightarrow \quad \omega' = \begin{array}{c} \\ \ell \\ p-\ell \\ \end{array} \left[\begin{array}{c|c} \overbrace{\omega^1}^{\ell} & \overbrace{*}^{p-\ell} \\ \hline 0 & \omega^2 \\ \underbrace{}_{\ell} & \end{array} \right] \qquad \omega' \in T^*_{\mathbb{P}^1} \otimes \mathcal{O}(-1) \\ \qquad\qquad\qquad\qquad \cong \mathcal{O}(-3)$$

conn. matrix ω^1

$\Rightarrow$ we can choose a rank ℓ subbundle (F^1, ∇^1)

~~Note~~ Note that $F^1 \cong \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_\ell)$, $k_1 \geq \cdots \geq k_\ell$

$$\Rightarrow \frac{k_1 + \cdots + k_\ell}{\ell} > \frac{k_1 + \cdots + k_p}{p} \quad \text{- contradiction.} \ \square$$

because $k_\ell > k_{\ell+1}$

Thm 2. [10.4 in Bol.] Every irreducible! repr. $\chi : \pi_1(\mathbb{P}^1 \setminus \{a_1, \ldots, a_n\}) \to GL(p; \mathbb{C})$ can be constr. as a monodr. repr. of a Fuchsian system.

Pf. Take $E \in \mathcal{F}$ using $\Lambda_1 = \mathrm{diag}\{\lambda_1, \ldots, \lambda_p\}$, $\Lambda_2 = \cdots = \Lambda_n = 0$

λ_j any integers s.t. $\lambda_j - \lambda_{j+1} > (n-2)(p-1)$ $\forall j$.

Choose a meromorphic frame of E which is holom. outside of a_1 and s.t. ~~in the~~ the system

$$dy = \omega \cdot y$$

has a fundam. matrix $Y_1(z)$ whose expansion near $z=a_1$ has the form:

$$Y_1(z) = (z-a_1)^{-K}\, V(z)\, (z-a_1)^{\Lambda_1}\, (z-a_1)^{E_1}$$

$V(z)$: holom. invertible at a_1

Using Lemma 10.2, $\exists\ \Gamma(z)$ holom. invertible on $\mathbb{P}^1 \setminus \{a_1\}$

$U(z)$: holom. invertible at $z=a_1$, and

D: diagonal w/ entries permut. of $\{-k_1, \dots, -k_p\}$

$$\Gamma(z)\,(z-a_1)^{-K}\, V(z) = U(z)\,(z-a_1)^{D}$$

χ is irreducible $\Rightarrow$ $k_\ell - k_{\ell+1} \le n-2$ $\Rightarrow |k_i - k_j| \le (n-2)(j-i)$

$\Rightarrow |d_i - d_j| \le (n-2)(p-1)$

$\Rightarrow$ $H_1 = D + \Lambda_1$ is still an admissible matrix.

$\tilde{Y}_1(z) = \Gamma(z)\, Y_1(z)$ gauge transformation (global one) holom. on $\mathbb{P}^1 \setminus \{a_1\}$ merom. on O_1.

$$\tilde{Y}_1(z) = U(z)\,(z-a_1)^{H_1}\,(z-a_1)^{E_1}$$

$$\Rightarrow \partial_z \tilde{Y}_1 = \left(\partial_z U \cdot U^{-1} + \frac{U \cdot H_1\, U^{-1}}{z-a_1} + \frac{U(z)\,(z-a_1)^{H_1} E_1 (z-a_1)^{-H_1}\, U^{-1}}{z-a_1}\right)\tilde{Y}_1$$

holomorphic at $z=a_1$

$\Rightarrow$ $\tilde{Y}_1$ is the fund. matrix of a Fuchsian system. ☑

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Example 11.1

$$G_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 3 & 1 & 1 & -1 \\ -4 & -1 & 1 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -4 & -1 \end{bmatrix}, \quad G_3 = \begin{bmatrix} -1 & 0 & 2 & -1 \\ 4 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & -1 \end{bmatrix}$$

$$G_1 \cdot G_2 \cdot G_3 = 1 \qquad S_2 = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 3 & -3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 3 \end{bmatrix}, \quad S_3 = \frac{1}{64}\begin{bmatrix} 0 & 16 & 4 & 3 \\ 64 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & -16 & -12 \end{bmatrix}$$

$$S_2^{-1} G_2 S_2 = G_1, \quad S_3^{-1} G_3 S_3 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The repr. $\chi : \underset{\mathbb{Z} * \mathbb{Z}}{\underset{\shortparallel}{\pi_1(\mathbb{P}^1 \setminus \{a_1, a_2, a_3\})}} \to GL(4; \mathbb{C})$ is reducible

Def. A repres. χ is a B-representation if
(1) all G_i have 1 Jordan block
(2) it is reducible

Thm 11.2. B-repr. ~~of~~ is the monodromy of some Fuchsian system iff $\underset{\Delta_i = 0 \ \forall i}{F^0}$ is semi-~~simple~~ stable.

In particular $c_1(F^{(0)}) = k \cdot p$ for some integer k.

For the ~~ex~~ example from above $c_1(F^{(0)}) = 2$

Pf of Thm 11.2. $\Rightarrow$) If $F^{(0)}$ is semi-stable

Assume all sing. are Fuchsian except for 1 of them

$$Y_i(z) = (z-a_i)^{-K}\, \overline{V}(z)\, (z-a_i)^{\Lambda_i = 0} (z-a_i)^{E_i}$$

fund. solution at the remaining one point a_i.

if $F^{(0)}$ is semi-stable, then $K = k \cdot I^p \Rightarrow$ commutes w/ everything $\Rightarrow$ $Y_i^{a_i}$ is Fuchsian.

$\Leftarrow$) If the system is Fuchsian

$$\begin{array}{l} E, \nabla \\ \downarrow \\ B \end{array} \quad \text{trivial v.b. w/ Fuchsian connection.}$$

χ' subrepr. of χ, dim $\chi' = l$

$X_l \subset X$: monodromy invariant subspace of dim l.

(X ← solutions)

near a_i we have fund. matrix (from Levelt (=Jordan) filtr. for E_i)

$$Y_i(z) = U_i(z)(z-a_i)^{\Lambda_i}(z-a_i)^{E_i} \cdot S_i$$

$$\Lambda_i = \begin{bmatrix} \lambda_i^1 \geq & & 0 \\ & \ddots & \\ 0 & & \geq \lambda_i^p \end{bmatrix} \Rightarrow \exists \text{ subbundle } F' \subset E$$

$\operatorname{rk} F' = l$

$$\varkappa(E) = \quad \frac{\lambda_i^1 + \dots + \lambda_i^l}{l} \geq \frac{\lambda_i^1 + \dots + \lambda_i^p}{p}$$

$$\Rightarrow \varkappa(F') = \sum_{i=1}^{n} \left(\frac{\lambda_i^1 + \dots + \lambda_i^l}{l} + f_i \right) \geq \varkappa(E) = 0$$

but we know $\varkappa(F') \leq 0 \Rightarrow c_1(F') = 0$

$\Rightarrow \Lambda_i = c_i \cdot I$ are scalar matrices $\quad c = \sum_{i=2}^{m} c_i, \quad Y_1(z) = z-$

$$Y' = \frac{(z-a_1)^c}{\prod (z-a_i)^{c_i}}\, Y(z),$$

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Birkhoff normal form

(1) $z \frac{dy}{dz} = C(z) \cdot y$ near $z = \infty$

$C(z) = z^r \sum_{n=0}^{\infty} C_n z^{-n}$, $C_0 \neq 0$, $r \geq 0$ Poincaré rank

converges in $\mathcal{O}_\infty = \{ z \in \mathbb{P}^1 \mid |z| > R \}$

$x = \Gamma(z) y$

anal. invertible in $\mathcal{O}_\infty$ or merom.

$z \frac{dx}{dz} = \tilde{C}(z) x$

$\tilde{C}(z) = d\Gamma(z) \, \Gamma(z)^{-1} + \Gamma(z) C(z) \Gamma(z)^{-1}$

Thm [Birkhoff] There is some anal. transf. $\Gamma(z)$ s.t.

$$\tilde{C} = (\tilde{C}_0 + \tilde{C}_1 \cdot z + \dots + \tilde{C}_r \cdot z^r)$$

provided the system (1) satisfies some conditions. e.g. irreducible monodromy

Remark: such $\tilde{C}$ is called Birkhoff standard form. (BSF)

Example 12.1 [Gantmacher 50's]

$z \frac{dy}{dz} = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + z^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) y$, $r = 0$

BSF should be $\tilde{C} = \tilde{C}_0$ — does not exist!

Define a vector bundle $F \to \mathbb{P}^1$ from (1):

$\mathcal{O}_\infty \subset \mathbb{P}^1 \setminus \{0\}$

$Y(z) = T(z) \, z^E$ $\quad (\mathbb{C}, \cancel{E}, g_{0\infty}(z) = T(z))$

∇ given by $\frac{C(z)}{z} dz = \omega_\infty$, $\omega_0 = \frac{E}{z} dz$

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$\Rightarrow$ we can reduce the connection (by choosing a holom. trivializ. over $\mathbb{P}^1 \setminus \{0, \infty\}$) to

$$\tilde{C} = \sum_{i=-k}^{r} c_i z^i .$$

We can similarly construct bundles F^Λ, $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_p \end{bmatrix}$

and get connection ∇^Λ. We get a class of v.b. $\mathcal{E}$

Thm 12.2. The system (1) admits a (BSF) iff $\mathcal{E}$ contains a trivial vector bundle.

Thm 12.3. Assume the system is irreducible (i.e. gauge $\cdot\, C \neq \begin{bmatrix} C_1 & * \\ 0 & C_2 \end{bmatrix}$) and $E \in \mathcal{E}$ then

$$k_i - k_{i+1} \leq r , \qquad i = 1, \dots, p-1$$

Thm 12.4. If C is irreducible, then BSF exists.