GROMOV–WITTEN THEORY AND INTEGRABLE HIERARCHIES

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1. Introduction to Gromov–Witten theory

Let X be a projective manifold.

Definition 1.1. *A* stable map $(f, \Sigma, z_1, \dots, z_n)$ *consists of*

(1) *nodal Riemann surface* Σ*,*
(2) *marked points* $z_1, \dots, z_n$ *that are pairwise different and not nodal,*
(3) $f : \Sigma \to X$ *is a continuous map, holomorphic away from the nodal points,*

such that the automorphism group of the configuration $(f, \Sigma, z_1, \dots, z_n)$ *is finite.*

It is not hard to see that a map is stable iff the following holds. Let Σ_0 be an irreducible genus-g_0 component of Σ contracted by f and let n_0 be the total number of marked and nodal points on Σ_0, then $2g_0 - 2 + n_0 > 0$.

Definition 1.2. *Two stable maps* $(f, \Sigma, z_1, \dots, z_n)$ *and* $(f', \Sigma', z'_1, \dots, z'_n)$ *are called* equivalent *if there exists a diffeomorphism* $\varphi : \Sigma \to \Sigma'$*, such that:*

(1) $f = f' \circ \varphi$*,*
(2) $\varphi(z_i) = z'_i$ $(1 \leq i \leq n)$*,*
(3) $\varphi^* j' = j$*, where* j *and* j' *are the complex structures on* Σ *and* Σ'*.*

Given two non-negative numbers g and n and a homology class $d \in H_2(X; \mathbb{Z})$, we denote by $\overline{\mathcal{M}}_{g,n}(X; d)$ the space of equivalence classes of stable maps $(f, \Sigma, z_1, \dots, z_n)$ such that Σ has genus g and $f_*[\Sigma] = d$. Sometimes we will denote the space by $X_{g,n,d}$. We refer to it as the moduli space of stable maps. Using sequential convergence, one can introduce a topology and then it is a theorem of Gromov [7] that the moduli space is a compact topological space, i.e., every sequence has a convergent subsequence.

In general, $\overline{\mathcal{M}}_{g,n}(X; d)$ is not a manifold or an orbifold. The reason for this is that the infinitesimal deformations of a stable map might have obstructions, so we can not always extend them to actual deformations. Nevertheless, one can define a homology cycle, called virtual fundamental cycle, such that the integration theory on the moduli space is the same as if $\overline{\mathcal{M}}_{g,n}(X, d)$ were compact complex orbifolds.

1.1. **Deformations of stable maps.** We consider a simplified version of the deformation theory of a stable map. Namely, let (Σ, z, f), $z = (z_1, \dots, z_n)$, be a fixed stable map. We classify the infinitesimal deformations of the map f and their obstructions, keeping the Riemann surface and the marked points fixed. Choose an open covering $\{V_i\}$ of Σ by holomorphic disks and let U_i be coordinate charts of X such that $f(V_i) \subset U_i$. In each chart U_i we pick coordinates and so on each V_i the map f is represented by a collection of holomorphic functions $u_i = (u_i^1, \dots, u_i^D)$, $D = \dim_{\mathbb{C}} X$. Finally, let g_{ji} be the transition functions between the charts U_i and U_j, i.e., $u_j = g_{ji}(u_i)$.

Case 1: 1-st order deformations. Let $\overline{u}_i = u_i + \epsilon v_i$ be first order deformations. Compare the coefficient in front of ϵ in the gluing identity $\bar{u}_j^a = g_{ji}^a(\bar{u}_i^1, \dots, \bar{u}_i^D)$. We get:

$$v_j^a = \sum_{b=1}^{D} \frac{\partial g_{ji}^a}{\partial u_i^b} v_i^b \quad , \tag{1.1}$$

which implies that the vector fields $\sum_a v_j^a \frac{\partial}{\partial u_j^a} \in \Gamma(V_j, f^*TX)$ glue to give a global section of f^*TX, i.e., the infinitesimal deformations are classified by $H^0(\Sigma, f^*TX)$.

Case 2: 2-nd order deformations. Let $\bar{u}_i = u_i + \epsilon v_i + \epsilon^2 w_i$ be a second order deformation. Comparing the coefficients in front of ϵ^2 in the gluing identity $\bar{u}_j^a = g_{ji}^a(\bar{u}_i)$ we get:

$$w_j^a = \sum_b \frac{\partial g_{ji}^a}{\partial u_i^b} w_i^b + \frac{1}{2} \frac{\partial^2 g_{ji}^a}{\partial u_i^b \partial u_i^c} v_i^b v_i^c,$$

i.e.,

$$\sum_a w_j^a \frac{\partial}{\partial u_j^a} = \sum_b w_i^b \frac{\partial}{\partial u_i^b} + \frac{1}{2} \sum_{a,b,c} \frac{\partial^2 g_{ji}^a}{\partial u_i^b \partial u_i^c} v_i^b v_i^c \frac{\partial}{\partial u_j^a}, \tag{1.2}$$

The LHS and the first sum on the RHS are elements respectively of $H^0(V_i, f^*TX)$ and $H^0(V_j, f^*TX)$. We denote the second term on the RHS by w_{ji}. A direct computation (using also formula (1.1)) shows that $w_{ki} = w_{kj} + w_{ji}$, i.e., $w = (w_{ji})$ give rise to a Cech cocycle. Let $[w] \in H^1(\Sigma, f^*TX)$ be the corresponding cohomology class, then formula (1.2) means that $[w] = 0$, so the obstructions belong to the cohomology group $H^1(\Sigma, f^*TX)$.

Let $\mathcal{T}_\Sigma$ be the sheaf of holomorphic vector fields on Σ which vanish at the marked points and at the nodes. A similar argument shows that $H^1(\Sigma, \mathcal{T}_\Sigma)$ classifies the deformations of the complex structure on Σ, and $H^0(\Sigma, \mathcal{T}_\Sigma)$ are the automorphisms of (Σ, z). Finally, for $s \in \mathrm{Sing}(\Sigma)$ let T_s' and T_s'' be the tangent spaces at s to the two branches of Σ that meet at s. Then $T_s' \otimes T_s''$ can be identified with a space of infinitesimal deformations of (Σ, z, f) which come from resolving s. Namely, let x and y be coordinates on the two branches and let $\epsilon\partial_x \otimes \partial_y \in T'_s \otimes T''_s$. In a neighborhood of s the Riemann surface is given by the equation $xy = 0$ and we resolve the singularity by deforming the equation into $xy = \epsilon$.

The following space is called *virtual tangent space*

$$H^1(\Sigma, \mathcal{T}_\Sigma) - H^0(\Sigma, \mathcal{T}_\Sigma) + \bigoplus_{s\in \mathrm{Sing}\Sigma} T'_s \otimes T''_s + H^0(\Sigma, f^*TX) - H^1(\Sigma, f^*TX).$$

It should be understood as an element of the Grothendick group of vector spaces. Using the Riemann-Roch formula:

$$\dim_{\mathbb{C}} H^0(\Sigma, E) - \dim_{\mathbb{C}} H^1(\Sigma, E) = \mathrm{rk}(E)(1-g) + \int_\Sigma c_1(E),$$

we find that the dimension of the virtual tangent space is

$$3g - 3 + n + D(1-g) + \int_d c_1(TX).$$

Example. If X is a manifold whose tangent spaces are spanned by global vector fields $H^0(X, TX)$ (e.g. Grassmanians, flag manifolds) then $H^1(\Sigma, f^*TX) = 0$ for all genus-0 curves Σ. This implies that the obstructions vanish so the moduli space $\overline{\mathcal{M}}_{0,n}(X, d)$ is a compact complex orbifold.

Example. If the degree $d = 0$, i.e., the maps contracts the curve to a point. We have $\overline{\mathcal{M}}_{g,n}(X, 0) = \overline{\mathcal{M}}_{g,n} \times X$. On the other hand $H^0(\Sigma, f^*TX) = T_{f(\Sigma)}X$, $H^1(\Sigma, f^*TX) = H^0(\Sigma, \mathcal{O}_\Sigma) \otimes T_{f(\Sigma)}X$, so the tangent bundle is given by

$$\mathcal{T} = \mathcal{T}_{\overline{\mathcal{M}}_{g,n}} + TX - \mathbb{E} \otimes TX,$$

where $\mathbb{E}$ is the rank-g bundle on $\overline{\mathcal{M}}_{g,n}$ whose fiber at (Σ, p) is given by $H^1(\Sigma, \mathcal{O}_\Sigma)$ (the dual to this bundle is known as *the Hodge bundle*). Since the obstructions form a bundle we have that the virtual fundamental cycle is the Poincare dual to the Euler class, i.e.,

$$\int_{\overline{\mathcal{M}}_{g,n}(X,0)} \alpha = \int_{\overline{\mathcal{M}}_{g,n}\times X} \alpha \smile \mathrm{Euler}(\mathbb{E} \otimes TX).$$

1.2. **The Mori cone.** The space of all fundamental classes $f_*[\Sigma]$ of holomorphic maps $f : \Sigma \to X$ is called *the Mori cone* of X and it is denoted by $MC(X)$.

Proposition 1.3. *For every $d \in MC(X)$ there are only finitely many ways to decompose $d = d' + d''$, where $d', d'' \in MC(X)$.*

Proof. Recall that *a Cartier divisor* on X is an equivalence class of a collection $\{(U_i, f_i)\}$ of pairs, such that

(1) $\{U_i\}$ form an open covering of X,

(2) $f_i \neq 0$ are meromorphic functions on U_i, such that on the overlaps $U_i \cap U_j$ we have $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$.

Here $\mathcal{O}_X^*$ is the sheaf of holomorphic functions on X that take only non-zero values. Two collections $\{(U_i, f_i)\}$ and $\{(U_i', f_i')\}$ are equivalent if after passing to a common refinement $\{V_i\}$ of the two coverings we have: $f_i/f_i' \in \mathcal{O}_X^*(V_i)$. The set of Cartier divisors is naturally an abelian group:

$$D = \{(U_i, f_i)\},\ D' = \{(U_i, f_i')\}, \quad D + D' := \{(U_i, f_i f_i')\}.$$

Given a Cartier divisor, we can construct a line bundle $L(D)$ on X as follows. On the open covering $\{U_i\}$ we set $L(D)|_{U_i} = U_i \times \mathbb{C}$ and on the overlaps $U_i \cap U_j$ we glue the two copies of the line bundle via the isomorphism:

$$U_i \times \mathbb{C} \to U_j \times \mathbb{C}, \quad (x, \lambda) \mapsto (x, f_i^{-1} f_j \lambda).$$

Note that the sheaf $\mathcal{L}$ of holomorphic sections of L can be identified with a subsheaf of the (constant) sheaf of all meromorphic functions on X: $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i} f_i^{-1}$. It turns out that on a projective manifold all line bundles arise this way. Moreover, $L(D) = L(D')$ iff the functions f_i/f_i' glue together to produce a global meromorphic function on X. In this case the divisors D and D' are called linearly equivalent.

From a Cartier divisor $D = \{(U_i, f_i)\}$ one can construct a homology class as follows. Let V be a codimension-1 subvariety of X. Then the local representatives f_i of D have zeroes or poles of certain order along $V \cap U_i$, which if non-zero is the same for all of them and it is denoted by $\mathrm{ord}_V(D)$. We set

$$[D] = \sum_V \mathrm{ord}_V(D)\,[V] \ \in \ H_{2D-2}(X;\mathbb{Z}) \cong H^2(X;\mathbb{Z}),$$

where the sum is over all codimension-1 subvarieties of X and $[V]$ is the fundamental class of V. By definition, the first Chern class of L is $c_1(L) = [D]$. According to the Lefschetz $(1,1)$-theorem the cohomology classes of type $[D]$ span (over $\mathbb{R}$) the cohomology group $H^{1,1}(X;\mathbb{R})$.

A line bundle L is called *very ample* if there exists an imbedding $i : X \to \mathbb{C}P^N$ such that $L = i^*\mathcal{O}(1)$. The bundle is called *ample* if there exists an integer $m > 0$ such that $L^{\otimes m}$ is very ample. The same terminology applies to divisors via the correspondance between line bundles and divisors. Note that ample divisors have the following property:

$$\int_d c_1(L(D)) = [D] \cap d \geq 0, \quad \text{for all } \ d \in MC(X).$$

This is because $[mD]$, for m sufficiently large, is very ample, so we can imbed $i : X \to \mathbb{C}P^N$ and then the intersection number turns into the symplectic area of the holomorphic map: $i \circ f : \Sigma \to CP^N$ (note that the first Chern class $c_1(\mathcal{O}_{\mathbb{C}P^N}(1))$ is represented by a Kähler form known as *the Fubini-Studi form*).

The symplectic area of a holomorphic map with respect to a Kähler form is always ≥ 0.

Our manifold X is projective, so it admits a hyperplane section H which is a very ample divisor. It can be proved that if D is any divisor then $D + mH$ is ample for m sufficiently large. Therefore, we can choose an integral basis $\{p_a\}_{a=1}^{r}$in $H^2(X;\mathbb{R})$ such that $\langle p_a, d\rangle \geq 0$ for all $d \in MC(X)$.

Assume that there are infinitely many pairwise different decompositions: $d = d'_j + d''_j$. Then the number $\langle d, p_a\rangle$ is decomposed into a sum of two non-negative numbers $\langle d'_j, p_a\rangle + \langle d''_j, p_a\rangle$. So there are infinitely many j such that $\langle d'_j, p_a\rangle = d_a$ is a fixed constant. It follows that d'_j (and hence $d''_j = d - d'_j$ are the same for all j – contradiction. □

By definition the *Novikov ring* $\mathbb{C}[Q]$ of X is the vector space

$$\mathbb{C}[Q] = \Big\{ \sum_{d\in MC(X)} c_d Q^d \mid c_d \in \mathbb{C}\Big\}$$

equipped with the following multiplication:

$$\Big(\sum_{d'\in MC(X)} c_{d'} Q^{d'}\Big)\Big(\sum_{d''\in MC(X)} c_{d''} Q^{d''}\Big) = \sum_{d\in MC(X)} c_d Q^d, \quad c_d = \sum_{d'+d''=d} c_{d'} c_{d''}.$$

The multiplication is well defined thanks to Proposition 1.3.

1.3. **Gromov–Witten invariants.** First we explain the maps that appear in the following diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n+1}(X,d) & \xrightarrow{\mathrm{ev}_i} & X \\ s_i \uparrow \; \downarrow \pi & & \\ \overline{\mathcal{M}}_{g,n}(X,d) & & \end{array} \qquad \begin{array}{ccc} (f',\Sigma',z') & \mapsto & f(z'_i) \in X \\ \downarrow \pi & & (1 \leq i \leq n) \\ \sigma = (f,\Sigma,z) & & \end{array}$$

The map π forgets the last marked point and contracts all unstable components. The fiber $\pi^{-1}(\sigma)$ is canonically identified with Σ, i.e., π is the universal curve. Indeed, if $\pi(f',\Sigma',z') = (f,\Sigma,z)$ then an irreducible componenet Σ_0 of Σ' is contracted iff it is a copy of $\mathbb{C}P^1$ contracted by f and such that one of the following two cases hold:

(1) the only marke points on Σ_0 are z'_{n+1} and z'_i for some i $(1 \leq i \leq n)$ and Σ_0 has exactly one nodal point.
(2) the last marked point is sitting on Σ_0 and Σ_0 has exactly two nodal points,

The identification is given by

$$(\Sigma', z', f') \mapsto \begin{cases} z_{n+1}, & \text{if no contraction occures} \\ \mathrm{ct}(\Sigma_0) & \text{otherwise} \end{cases}$$

where $\mathrm{ct}(\Sigma_0)$ is the point obtained from the contraction of the irreducible component Σ_0.

The universal curve π has natural sections

$$s_i : \sigma = (\Sigma, z, f) \mapsto z_i \in \Sigma \cong \pi^{-1}(\sigma).$$

Introduce the divisor $S_i = [s_i(\overline{\mathcal{M}}_{g,n}(X,d))]$ and let $L_i = s_i^* N_{S_i}^\vee$ be the pullback of the conormal bundle to D_i. Intuitively L_i is the bundle formed by the cotangent lines $T_{z_i}^\vee \Sigma$.

From now on we will assume that the cohomology algebra $H^*(X;\mathbb{C})$ has only even degree cohomology classes. Let $\{\phi_a\}_{a=1}^N$ be a fixed basis. By definition the descendant GW invariants of X are the following correlators:

$$\langle \phi_{a_1}\psi^{k_1}, \ldots, \phi_{a_n}\psi^{k_n}\rangle_{g,n,d} := \int_{[X_{g,n,d}]} \psi_1^{k_1} \ldots \psi_n^{k_n} \mathrm{ev}^*(\phi_{a_1} \otimes \ldots \otimes \phi_{a_n}), \tag{1.3}$$

where $\psi_i = c_1(L_i)$.

Put

$$\mathbf{t}(z) = \sum_{k=0}^{\infty} t_k z^k, \quad t_k = \sum_{a=1}^{N} t_k^a \phi_a,$$

where t_k^a are formal variables. By definition the total descendant potential of X is the following generating series:

$$\mathcal{D}_X(\mathbf{t}) = \exp\Big(\sum_{g,n\geq 0} \sum_{d\in MC(X)} \frac{\hbar^{g-1}}{n!} Q^d \,\langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_1)\rangle_{g,n,d}\Big). \tag{1.4}$$

For reasons, which will become clear later, we change the variables according to the so called *dilaton shift*:

$$t_0 = q_0,\ t_1 = q_1 + 1,\ t_2 = q_2, \ldots, \quad \text{where } q_k = \sum_{a=1}^{N} q_k^a \phi_a.$$

We introduce the Fock space

$$\mathbb{C}_\hbar[Q][[q_0, q_1 + 1, q_2, \ldots]], \quad \mathbb{C}_\hbar[Q] := \mathbb{C}[Q]((\hbar)). \tag{1.5}$$

Note that if we set $\mathbf{t} = 0$ and $Q = 0$ in (1.4), then the correlators can be non-zero only if $g > 1$ (due to stability constraints). Therefore, the total descendant potential is a well defined element of the Fock space (1.5).

2. Frobenius structures in GW theory

2.1. **Definition of a Frobenius structure.** Let M be a small ball (with center at 0) in $\mathbb{C}^N$, equipped with the following structures:

(1) a non-degenerate bi-linear pairing g on TM,
(2) multiplication $\bullet_t$ in T_tM that depends holomorphically on $t \in M$,
(3) a vector field e, such that its restriction to T_tM is a unity with respect to $\bullet_t$,
(4) a vector field E.

The data $(g, \bullet_t, e, E)$ forms a Frobenius structure on M of *conformal dimension* $D \in \mathbb{C}$, if the following conditions are satisfied.

(i) g and $\bullet$ satisfy the Frobenius property:
$$g(X \bullet Y_1, Y_2) = g(Y_1, X \bullet Y_2),$$
(ii) The one-parameter group corresponding to E acts on M by conformal transformations of g, i.e., $\mathcal{L}_E g = (2 - D)g$,
(iii) e is a flat vector field: $\nabla^{\text{L.C.}} e = 0$, where $\nabla^{\text{L.C.}}$ is the Levi-Civitá connection of g,
(iv) The connection operator
$$\nabla = \nabla^{\text{L.C.}} - z^{-1} \sum_{i=1}^{N} \partial_{t_i} \bullet_t \, dt_i + \Big(z^{-2} \, E \bullet_t - z^{-1} \, \mu \Big) dz, \tag{2.1}$$
where
$$\mu := \nabla^{\text{L.C.}}(E) - (1 - \frac{D}{2})\text{Id} : TM \to TM$$

is *the Hodge grading operator,* is flat, i.e., $\nabla^2 = 0$.

Remark. The flatness of the family of connection operators implies that $\bullet_t$ is commutative and associative and that there exists a function $F(\tau)$, called potential of the Frobenius structure, such that the structure constants of the multiplication $\bullet_t$ are given by the third partial derivatives of F, i.e.,
$$g(\partial/\partial\tau^a \bullet_t \partial/\partial\tau^b, \partial/\partial\tau^c) = \partial^3 F/(\partial\tau^a \partial\tau^b \partial\tau^c),$$
where $\tau = (\tau^1, \ldots, \tau^N)$ is a flat coordinate system on M.

2.2. **Frobenius structures in GW theory.** Let $H := H^*(X; \mathbb{C}[Q])$. Using genus-0 GW invariants we will equip H with a Frobenius structure. Let $g = (\, , \,)$ be the Poincaré pairing. Note that if we set $\tau = \sum_{a=1}^N \tau^a \phi_a \in H$ then $(\tau^1, \ldots, \tau^N)$ are flat coordinates. *The qunatum cup product* is defined by
$$(\phi_a \bullet \phi_b, \phi_c) = \sum_{d,n} \frac{Q^d}{n!} \langle \phi_a, \phi_b, \phi_c, \tau, \ldots, \tau \rangle_{0,3+n,d}.$$

Assume that the basis $\{\phi_a\}_{a=1}^N$ is homogeneous and let $\deg_{\mathbb{C}} \phi_a := \deg \phi_a/2$. We introduce the following vector field on H:

$$E = \sum_{a=1}^{N} (1 - \deg_C \phi_a) \tau^a \frac{\partial}{\partial \tau^a} + c_1(TX).$$

Here

$$c_1(TX) = \sum_{a=2}^{r+1} \langle c_1(TX), \phi^a \rangle \frac{\partial}{\partial \tau^a},$$

where we arranged the basis $\{\phi_a\}_{a=1}^N$ in such a way that $\phi_1 = 1$, the next r cohomology classes $\phi_2, \dots, \phi_{r+1}$ form a basis of $H^2(X;\mathbb{C})$, and $\{\phi^a\}_{a=2}^r$ is a Poincaré dual basis of $H_2(X;\mathbb{C})$.

Theorem 2.1. *The data formed by the Poincaré pairing, the quantum cup product, the cohomology class* 1*, and the vector field* E *forms a Frobenius structure on* H *of conformal dimension* D*.*

The only non-obvious part in the proof of the above theorem is the flatness of the connection operators ∇. In other words we have to prove that

$$[\nabla_{\partial_a}, \nabla_{\partial_b}] = 0, \quad \text{and} \quad [\nabla_{\partial_a}, \nabla_{\partial/\partial z}] = 0, \tag{2.2}$$

where $\partial_a = \partial/\partial \tau^a$.

2.3. **The comparison lemma.** Let $\pi : X_{g,n+1,d} \to X_{g,n,d}$ be the universal curve. Denote by $\overline{L}_i$ the pullback via π of the line bundle $L_i \to X_{g,n,d}$. Note that L_i and $\overline{L}_i$ coincide everywhere, except for the points of the divisor D_i consisting of stable maps $(f, \Sigma, z_1, \dots, z_{n+1})$ such that Σ has an irreducible component which carries exactly two marked points: z_i and z_{n+1}.

Lemma 2.2. *The following relations hold:*

$$L_i = \overline{L}_i \otimes \mathcal{O}(D_i), \quad \pi_*(\psi_{n+1}) = 2g-2+n, \quad \pi_*(\mathrm{ev}_{n+1}^* p) = \int_d p, \quad p \in H^2(X;\mathbb{Z}).$$

Proof. [1] Let $(f, \Sigma, z_1, \dots, z_n)$ represent a point in $X_{g,n,d}$. Let Y be the blow up of $\Sigma \times \Sigma$ at the n points (z_i, z_i). Introduce also the set of divisors in $\Sigma \times \Sigma$:

$$S_i = \Sigma \times \{z_i\} \ (1 \le i \le n), \quad \Delta = \text{ the diagonal of } \Sigma \times \Sigma$$

Note that

$$\pi^{-1}(S_i) = \widetilde{S}_i + E_i \quad \text{and} \quad \pi^{-1}(\Delta) = \widetilde{\Delta} + \sum_{i=1}^{n} E_i,$$

where $\widetilde{S}_i$ and $\widetilde{\Delta}$ are smooth codimension-1 submanifolds of Y, E_i are the exceptional divisors, and π is the blow-down map. Note that Y is a family of

[1] I learned this proof from D. Oprea and he learned it from R. Pandharipande.

curves and that $\widetilde{S}_i$ $(1 \leq i \leq n)$ and $\widetilde{\Delta}$ determine $n+1$ sections. Therefore each fiber represents a point in $X_{g,n+1,d}$, i.e., we have an imbedding of Σ into $X_{g,n+1,d}$. In fact the image of this imbedding coinsides with the fiber of the universal curve at the point $(f, \Sigma, z_1, \dots, z_n)$. Moreover, Y is the pullback of the universal family $X_{g,n+2,d} \to X_{g,n+1,d}$.

Recall that if V is a codimension-1 submanifold of X then we have an exact sequence

$$0 \to TV \longrightarrow TX|_V \xrightarrow{\langle df,\ \rangle} \mathcal{O}([V])|_V \to 0, \quad \text{i.e.,} \quad \mathcal{O}([V])|_V \cong N_V,$$

where f is the section of $\mathcal{O}([V]$ glued by the local equations of the divisor V.

Now the relations are easy to prove. For the first one, note that $L_i = \overline{L}_i \otimes \mathcal{O}(nD_i)$ for some integer n, because the two line bundles are different only along the divisor D_i. By definition

$$L_i|_\Sigma = N^\vee_{\widetilde{S}_i}, \quad \overline{L}_i|_\Sigma = \mathcal{O}, \quad \mathcal{O}(D_i)|_\Sigma = \mathcal{O}_\Sigma(z_i).$$

Since $\pi^{-1}(S_i) = \widetilde{S}_i + E_i$, we get

$$\pi^*(\mathcal{O}(S_i)) = \mathcal{O}(\widetilde{S}_i) \otimes \mathcal{O}(E_i) \quad \Rightarrow \quad N^\vee_{\widetilde{S}_i} = \pi^* N^\vee_{S_i} \otimes \mathcal{O}(E_i)|_{\widetilde{S}_i}.$$

It remains only to notice that the bundle N_{S_i} is trivial and that $\mathcal{O}(E_i)|_{\widetilde{S}_i} = \mathcal{O}(z_i)$. It follows that the number $n = 1$.

A similar argument shows that $L_{n+1}|_\Sigma = T^*\Sigma(z_1 + \cdots + z_n)$ (you have to use here that $N_\Delta = T\Sigma$. The 2-nd relation then follows from the well known fact that the degree of the cotangent bundle $T^*\Sigma$ is $2g-2$.

For the last relation we just have to notice that $\mathrm{ev}^*_{n+1} p|_\Sigma = f^*p$. Lemma is proved. □

2.4. **Topological recursion relations.** We are going to prove the vanishing of the first commutator in (2.2) by using the so called *genus*-0 *topological recursion relations* (TRR).

Proposition 2.3. *The following identity holds:*

$$(2.3)\ \frac{1}{n!}\langle \phi_a\psi^{i+1}, \phi_b\psi^j, \phi_c\psi^k, \mathbf{t}(\psi)\dots, \mathbf{t}(\psi)\rangle_{0,n+3,d} = \sum_{\substack{n_1+n_2=n\\ d_1+d_2=d}} \sum_{\mu,\nu=1}^{N} \frac{g^{\mu\nu}}{n_1! n_2!} \times$$
$$\langle \phi_a\psi^i, \phi_\mu, \mathbf{t}(\psi), \dots, \mathbf{t}(\psi)\rangle_{0,n_1+2,d_1} \langle \phi_\nu, \phi_b\psi^j, \phi_c\psi^k, \mathbf{t}(\psi)\dots, \mathbf{t}(\psi)\rangle_{0,n_2+3,d_2} \quad ,$$

where $g_{\mu\nu} = (\phi_\mu\,,\,\phi_\nu)$ are the entries of the matrix of the Poincaré pairing and $g^{\mu\nu}$ are the entries of its inverse.

Proof. Let $\mathrm{ct} : \overline{\mathcal{M}}_{0,n+3}(X,d) \to \overline{\mathcal{M}}_{0,3}$ be the map forgetting the map, the last n marked points, and contracting all unstable components. Let $(f, \Sigma, z) \in \overline{\mathcal{M}}_{0,n+3}(X,d)$. Note that if we forget f and the last n marked points then only one of the irreducible components of Σ is stable (and hence is not contracted by ct). We call this distinguished component the central component of Σ. Let D be the divisor consisting of all stable maps such that the first marked point is not on the central component.

Using Lemma 2.2 we get $L_1 = \overline{L_1} \otimes \mathcal{O}(D) = \mathcal{O}(D)$, where $\overline{L_1}$ is the pullback via ct of the cotangent line bundle L_1 on $\overline{\mathcal{M}}_{0,3}$. The later is trivial, because $\overline{\mathcal{M}}_{0,3}$ is a point. It follows that the LHS of (2.3) can be written in the following form:

$$\frac{1}{n!}\int_{[D]} \phi_a\psi_1^i\phi_b\psi_2^j\phi_c\psi_3^k\mathbf{t}(\psi_4)\ldots\mathbf{t}(\psi_{n+3}). \tag{2.4}$$

On the other hand, given a point $(f,\Sigma,z)\in D$ we can split the curve into two parts Σ' and Σ'' such that Σ' is a tree of $\mathbb{C}P^1$s which carries the first marked point and such that under the contraction map it is contracted to a point on the central component. Σ'' is the complement of Σ'. Thus there is a natural map gl which to each stable map $(f,\Sigma,z)\in D$ assigns an element of the preimage of the diagonal of the following map:

$$\overline{\mathcal{M}}_{0,n_1+1+\circ}(X,d_1)\times\overline{\mathcal{M}}_{0,\bullet+2+n_2}(X,d_2)\xrightarrow{\mathrm{ev}_\circ\times\mathrm{ev}_\bullet} X\times X.$$

The map gl is a $\binom{n}{n_1}$-covering because if we split the last n marked points of Σ into two groups then there are exactly that many ways to re-number them so that the order of the marked points in each group does not change. Since the Poincaré dual to the diagonal in $X\times X$ has the form $\sum_{\mu,\nu} g^{\mu\nu}\phi_\mu\otimes\phi_\nu$ we see that (2.4) is transformed into:

$$\sum_{\substack{n_1+n_2=n\\ d_1+d_2=d}}\frac{1}{n_1!n_2!}\int_{\overline{\mathcal{M}}_{0,n_1+1+\circ}(X,d_1)\times\overline{\mathcal{M}}_{0,\bullet+2+n_2}(X,d_2)}\sum_{\mu,\nu} g^{\mu\nu}\mathrm{ev}_\circ^*\phi_\mu\mathrm{ev}_\bullet^*\phi_\nu(\ldots),$$

where the dots stand for the integrand in (2.4). Formula (2.3) follows. □

We introduce a series

$$S_\tau(z) = 1 + S_1(\tau)z^{-1} + S_2(\tau)z^{-2}+\ldots,\quad S_k\in\mathrm{End}(H),$$

defined by the following formula

$$(S_\tau\phi_a,\phi_b) = (\phi_a,\phi_b)+\sum_{k=0}^{\infty}\langle\phi_a\psi^k,\phi_b\rangle_{0,2}(\tau)z^{-k-1},$$

where we used the notation:

$$\langle\phi_{a_1}\psi^{k_1},\ldots,\phi_{a_n}\psi^{k_n}\rangle_{0,n}(\tau)=\sum_{d,l}\frac{Q^d}{l!}\langle\phi_{a_1}\psi^{k_1},\ldots,\phi_{a_n}\psi^{k_n},\tau,\ldots,\tau\rangle_{0,n+l,d}\ .$$

Proposition 2.4. *The series S_τ is a fundamental solution to the following system of differential equations:*

$$z\partial_a S_\tau(z) = (\phi_a\bullet_\tau)S_\tau(z),\quad 1\le a\le N.$$

Proof. We have to prove that

$$\sum_{k=0}^{\infty}\langle\phi_a,\phi_b,\phi_c\psi^k\rangle_{0,3}(\tau)z^{-k}=(S_\tau(z)\phi_c,\phi_a\bullet_\tau\phi_b)z^{-k}.$$

On the other hand, thanks to the TRR, the LHS in the above equality is equivalent to:

$$\langle\phi_a,\phi_b,\phi_c\rangle_{0,3}(\tau)+\sum_{k=1}^{\infty}\sum_{\mu,\nu}\langle\phi_a,\phi_b,\phi_\mu\rangle_{0,3}(\tau)g^{\mu\nu}\langle\phi_\nu,\phi_c\psi^{k-1}\rangle_{0,3}(\tau)$$

Using the definitions of the quantum cup product and the series $S_\tau(z)$, we get that the above expression equals

$$(\phi_a\bullet_\tau\phi_b,\phi_c)+\sum_{\mu,\nu}(\phi_a\bullet_\tau\phi_b,\phi_\mu)g^{\mu\nu}\left((S_\tau(z)-1)\phi_c,\phi_\nu\right).$$

The proposition follows. □

Since S_τ is a fundamental solution the corresponding system is compatible. We get the following corollary (see 1-st commutator in (2.2)).

Corollary 2.5. *The differential operators*

$$\nabla_{\partial_a}=\partial_a-z^{-1}(\phi_a\bullet_\tau)\quad\textit{and}\quad\nabla_{\partial_b}=\partial_b-z^{-1}(\phi_b\bullet_\tau)$$

commute.

2.5. **The divisor equation.** Now we turn to proving the vanishing of the second commutator in (2.2).

Proposition 2.6. *Assume that p is a cohomology class of degree ≤ 2. Then*

$$\langle\phi_{a_1}\psi^{k_1},\dots,\phi_{a_n}\psi^{k_n},p\rangle_{g,n+1,d}=\left(\int_d p\right)\langle\phi_{a_1}\psi^{k_1},\dots,\phi_{a_n}\psi^{k_n}\rangle_{g,n,d}+$$
$$\sum_{i=1}^{n}\langle\phi_{a_1}\psi^{k_1},\dots,p\cup\phi_{a_i}\psi^{k_i-1},\dots,\phi_{a_n}\psi^{k_n}\rangle_{g,n,d}$$

for all g,n,d such that $X_{g,n,d}$ is non-empty.

Proof. Let

$$\pi:X_{g,n+1,d}\to X_{g,n,d},\quad \overline{L}_i=\pi^*(L_i\to X_{g,n,d}),\quad \overline{\psi}_i=c_1(\overline{L}_i),$$

where π is the universal curve. According to Lemma 2.2, $L_i=\overline{L}_i\otimes\mathcal{O}(D_i)$, where the divisor D_i is the image of the gluing map:

$$\overline{\mathcal{M}}_{g,n}(X,d)\times\overline{\mathcal{M}}_{0,3}\longrightarrow\overline{\mathcal{M}}_{g,n+1}(X,d),$$

which attaches a sphere with 3 marked points by identifying the 1-st marked point on the sphere with the i-th one and then renumbering the 2-nd and the

3-rd marked points respectively by i and $n+1$. In particular, $L_i|_{D_i}$ is the cotangent line bundle $L_2 \to \overline{\mathcal{M}}_{0,3}$, so it is trivial.

Using Lemma 2.2, we get

$$\psi_i^k = (\overline{\psi}_i + [D_i])\psi_i^{k-1} = \overline{\psi}_i\psi_i^{k-1} = \cdots = \overline{\psi}_i^{k-1}(\overline{\psi}_i + [D_i]) = \overline{\psi}_i^k + [D_i]\overline{\psi}_i^{k-1}$$

Note also that $[D_i]\cdot[D_j] = 0$ for $i \neq j$, because the divisors do not intersect. Put $\alpha = \mathrm{ev}^*(\phi_{a_1}\otimes\ldots\otimes\phi_{a_n} \in H^*(X_{g,n,d};\mathbb{C})$. It follows that

$$\int_{X_{g,n+1,d}} (\pi^*\alpha)\wedge(\mathrm{ev}_{n+1}^* p)\bigwedge_{i=1}^n \psi_i^{k_i} = \int_{X_{g,n,d}} \alpha\wedge\pi_*(\mathrm{ev}_{n+1}^* p)\bigwedge_{i=1}^n \psi_i^{k_i} \quad +$$

$$\sum_{i=1}^n \int_{[D_i]} \mathrm{ev}^*(\phi_{a_1}\ldots\phi_{a_n})\wedge \mathrm{ev}_{n+1}^*(p)\wedge\overline{\psi}_1^{k_1}\ldots\overline{\psi}_i^{k_i-1}\ldots\overline{\psi}_n^{k_n}.$$

However $D_i \cong \overline{\mathcal{M}}_{g,n}(X,d)$ and under this identification ev_{n+1} on D_i corresponds to ev_i. Note that if p has degree < 2 then $\pi_*(p) = 0 = \int_d p$, while if the degree is 2 then $\pi_*(\mathrm{ev}_{n+1}^* p) = \int_d p$, according to Lemma 2.2. □

In case $p \in H^2(X;\mathbb{Z})$ the identity in Proposition 2.6 is called *the divisor equation* (DivE) and if $p = 1$ then it is called *the string equation* (SE). For completeness we mention one more identity, known as *the dilaton equation* (DE).

$$\langle\phi_{a_1}\psi^{k_1},\ldots,\phi_{a_n}\psi^{k_n},\psi\rangle_{g,n+1,d} = (2g-2+n)\ \langle\phi_{a_1}\psi^{k_1},\ldots,\phi_{a_n}\psi^{k_n}\rangle_{g,n,d}, \tag{2.5}$$

whenever the moduli space $X_{g,n,d}$ is non-empty. The proof of the dilaton equation is almost the same as of the divisor equation and it is left as an exercise to the reader.

Corollary 2.7. *The differential operators*

$$\nabla_{\partial_a} = \partial_a - z^{-1}(\phi_a\bullet_\tau) \quad \textit{and} \quad \nabla_{\partial/\partial z} = \partial_z + (z^{-2}E\bullet_\tau - z^{-1}\mu)$$

commute.

Proof. Note that in GW theory the Hodge grading operator μ is diagonal:

$$\mu(\phi_a) = (1 - d_a - (1 - D/2))\ \phi_a = (D/2 - d_a)\ \phi_a,$$

where $d_a = \deg_{\mathbb{C}}\phi_a = (\deg\phi_a)/2$. After a direct computation we find that the commutator of the differential operators is

$$z^{-2}\Big(\phi_a\bullet_\tau + [\mu,\phi_a\bullet_\tau] - \partial_a(E\bullet_\tau)\Big). \tag{2.6}$$

This expressian vanishes iff (apply (2.6) to ϕ_b and Poincaré pair the result with ϕ_c)

$$\partial_a\langle\phi_b,\phi_c,E\rangle_{0,3}(\tau) = (1 - D + d_b + d_c)\langle\phi_a,\phi_b,\phi_c\rangle_{0,3}(\tau).$$

Using the definition of the Euler vector field, we get

$$E\langle\phi_a,\phi_b,\phi_c\rangle_{0,3}(\tau)=(d_a+d_b+d_c-D)\langle\phi_a,\phi_b,\phi_c\rangle_{0,3}(\tau).$$

This identity follows esily from the dimension formula

$$\dim_{\mathbb{C}}\overline{\mathcal{M}}_{0,n}(X;d)=D-3+n+\int_d c_1(TX),$$

and the divisor equation. □

3. The Lagrangian cone of Givental

3.1. **Geometric interpretation of genus-0 GW theory.** Last time we proved that the correlators in GW theory satisfy SE (see Proposition 2.6 with $p = 1$), DE (formula (2.5)), and TRR. These identities can be written in the following form:

$$\frac{\partial \mathcal{F}^{(0)}}{\partial t_0^1} = \frac{1}{2}\,(t_0, t_0) \;+\; \sum_{k=0}^{\infty}\sum_{a=1}^{N} t_{k+1}^a \,\frac{\partial \mathcal{F}^{(0)}}{\partial t_k^a} \tag{3.1}$$

$$\frac{\partial \mathcal{F}^{(0)}}{\partial t_1^1} = \sum_{k=0}^{\infty}\sum_{a=1}^{N} t_k^a \frac{\partial \mathcal{F}^{(0)}}{\partial t_k^a} - 2\mathcal{F}^{(0)} \tag{3.2}$$

$$\frac{\partial^3 \mathcal{F}^{(0)}}{\partial t_{k+1}^a \partial t_l^b \partial t_m^c} = \sum_{\mu,\nu}^{N} \frac{\partial^2 \mathcal{F}^{(0)}}{\partial t_k^a \partial t_0^\mu}\, g^{\mu\nu}\, \frac{\partial^3 \mathcal{F}^{(0)}}{\partial t_0^\nu \partial t_l^b \partial t_m^c}, \tag{3.3}$$

where

$$\mathcal{F}^{(0)}(\mathbf{t}) = \sum_{n=0}^{\infty} \frac{1}{n!}\langle \mathbf{t}(\psi_1), \ldots, \mathbf{t}(\psi_n)\rangle_{0,n}$$

is the genus-0 descendant potential,

$$\mathbf{t}(z) = \sum_{k=0}^{\infty}\; t_k^a\, \phi_a\, z^k$$

$\{\phi_a\}_{a=1}^N$ is a basis of H such that $\phi_1 = 1$.

Let $\mathcal{H} = H((z^{-1}))$ be the vector space of all Laurent series in z^{-1}. We equip $\mathcal{H}$ with the symplectic structure:

$$\Omega(\mathbf{f}, \mathbf{g}) = \mathrm{Res}_{z=0}\,(\mathbf{f}(-z), \mathbf{g}(z))dz, \quad \mathbf{f}(z), \mathbf{g}(z) \in \mathcal{H}$$

and will refer to it as *the symplectic loop space.* There is a natural polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+ := H[z]$ and $\mathcal{H}_- := z^{-1}H[[z^{-1}]]$ are Lagrangian subspaces. Using the symplectic pairing we can identify $\mathcal{H}_-$ with with $\mathcal{H}_+^\vee$ and hence $\mathcal{H} \cong T^*\mathcal{H}_+$. It is not hard to see that if we set

$$\mathbf{f} = \sum_{k=0}^{\infty} p_{k,a}\; \phi^a\; (-z)^{-k-1} + q_k^a\; \phi_a\; z^k, \quad \phi^a = \sum_{\mu=1}^{N} g^{a\mu}\phi_\mu$$

then $\{p_{k,a}, q_k^a\}$ form a Darboux coordinate system on $\mathcal{H}$.

Exercise. Let

$$A(z) = \sum_{k=1}^{\infty} A_k\, z^k, \quad A_k \in \mathrm{End}(H).$$

a) Prove that $A(z)$ is an infinitesimal symplectic transformation iff $A(z) + A^T(-z) = 0$.

b) The map $\mathbf{f} \mapsto A(z)\mathbf{f}$ is a linear vector field X_A. Prove that X_A is Hamiltonian iff A is an infinitesimal symplectic transformation and that the corresponding Hamiltonian h_A (i.e. $dh_A + \iota_{X_A}\Omega = 0$) is $h_A = \frac{1}{2}\Omega(A(z)\mathbf{f}, \mathbf{f})$.

We change the variables via the so called *dilaton shift*:

$$t_0 = q_0,\ t_1 = q_1 + 1,\ t_2 = q_2,\ \dots \qquad q_k = \sum_{a=1}^{N} q_k^a \phi_a,$$

so that the potential becomes a function on $\mathcal{H}_+$, defined in the formal neighborhood of $-z$.

Definition 3.1. *We say that a cone $\mathcal{L} \subset \mathcal{H}$ with vertex at the origin is* over-ruled *if for every $\mathbf{f} \in \mathcal{L}$ the tangent space $L := T_{\mathbf{f}}\mathcal{L}$ has the following property*

$$\{g \in \mathcal{L} \mid T_g\mathcal{L} = L\} = zL.$$

Denote by $\mathcal{L} \subset T^*\mathcal{H} \cong \mathcal{H}$ the graph of the differential $d\mathcal{F}^{(0)}$, i.e.,

$$\mathcal{L} = \Big\{ \sum_{k=0}^{\infty} \widetilde{q}_k z^k + \widetilde{p}_k(-z)^{-k-1} \quad | \quad \widetilde{p}_{k,a} = \frac{\partial \mathcal{F}^{(0)}}{\partial q_k^a}\Big|_{q_k^a = \widetilde{q}_k^a} \Big\}.$$

Note that for a given $\mathbf{f} \in \mathcal{L}$, the tangent space $T_{\mathbf{f}}\mathcal{L}$ is given by the following formulas:

$$v(z) + \sum_{k,l=0}^{\infty} \sum_{a,b=1}^{N} \frac{\partial^2 \mathcal{F}^{(0)}}{\partial q_k^a \partial q_l^b}\, v_l^b \phi^a (-z)^{-k-1}$$

Finally, let us stress that $\mathcal{L}$ is interpreted in a formal sense, which means that the coefficients $\widetilde{q}_k$ are formal series in $q_0, q_1+1, q_2, \dots$, such that $\lim_{k\to\infty} \widetilde{q}_k = 0$ in the q-adic topology.

Theorem 3.2. *Let $\mathcal{F}^{(0)}$ be any function on $\mathcal{H}_+$ defined in a formal neighborhood of $-z$. Then $\mathcal{F}^{(0)}$ satisfies DE, SE and TRR iff the graph $\mathcal{L}$ is an over-ruled Lagrangian cone in $\mathcal{H}$.*

Proof. Assume that $\mathcal{L}$ is an over-ruled Lagrangian cone.

Step 1. If $(\mathbf{q}, \mathbf{p}) \in \mathcal{L}$ then $(t\mathbf{q}, t\mathbf{p}) \in \mathcal{L}$, because $\mathcal{L}$ is a cone. It follows that

$$\frac{\partial \mathcal{F}^{(0)}}{\partial q_k^a}\Big|_{\mathbf{q}\mapsto t\mathbf{q}} = t\, \frac{\partial \mathcal{F}^{(0)}}{\partial q_k^a}$$

Using the chain rule we get

$$\frac{\partial}{\partial t}\, \mathcal{F}^{(0)}(t\mathbf{q}) = \sum_{k=0}^{\infty} \sum_{a=1}^{N} \frac{\partial \mathcal{F}^{(0)}}{\partial q_k^a}\Big|_{\mathbf{q}\mapsto t\mathbf{q}} = t \sum_{k=0}^{\infty} \sum_{a=1}^{N} \frac{\partial \mathcal{F}^{(0)}}{\partial q_k^a}.$$

Integrating from 0 to 1 and recalling the dilaton shift we get that $\mathcal{F}^{(0)}$ satisfies the dilaton equation.

Step 2. Let $\mathbf{f}\in\mathcal{L}$ arbitrary and $L=T_{\mathbf{f}}\mathcal{L}$. Since $\mathcal{L}$ is overruled we get that $\mathbf{f}\in zL$, i.e., $z^{-1}\mathbf{f}\in L$. In other words the v.f. $\mathbf{f}\mapsto z^{-1}\mathbf{f}$ is tangent to the cone $\mathcal{L}$. This v.f. is Hamiltonian with Hamiltonian

$$\frac{1}{2}\Omega(z^{-1}\mathbf{f},\mathbf{f})=\frac{1}{2}(q_0,q_0)+\sum_{k=0}^{\infty}\sum_{a=1}^{N}q_{k+1}^a p_{k,a}.$$

It follows that $\mathcal{F}^{(0)}$ satisfies the string equation.

Step 3. We imbed H into $\mathcal{H}_+$ by $\tau\mapsto -z+\tau$. Put

$$\mathbf{f}=\sum_{k=0}^{\infty}\sum_{a=1}^{N}q_k^a\phi_a z^k+p_{k,a}\phi^a(-z)^{-k-1}\in\mathcal{L},\quad L=T_{\mathbf{f}}\mathcal{L}.$$

Denote by $(zL)_+$ the projection of zL along $\mathcal{H}_-$. Then

$$(zL)_+\cap H=\{-z+\tau\},\quad \text{where}\quad \tau^a(\mathbf{q})=\sum_{b=1}^{N}\frac{\partial^2\mathcal{F}^{(0)}}{\partial q_0^1\partial q_0^b}\,g^{ab}.$$

Using that $zL\subset\mathcal{L}$ we get

$$g:=\tau(q)-z+d_{\tau(\mathbf{q})-z}\mathcal{F}^{(0)}\in zL,$$

because we could not have two diffent elements of $\mathcal{L}$ whose projection along $\mathcal{H}_-$ is the same. Introduce the correlator notation:

$$\langle\phi_{a_1}\psi^{k_1},\dots,\phi_{a_n}\psi^{k_n}\rangle_{0,n}(\tau)=\left.\frac{\partial^n\mathcal{F}^{(0)}}{\partial t_{k_1}^{a_1}\dots\partial t_{k_n}^{a_n}}\right|_{t_0=\tau,t_1=t_2=\dots=0}.$$

We must have (since $T_g\mathcal{L}=L=T_{\mathbf{f}}\mathcal{L}$)

$$\frac{\partial^2\mathcal{F}^{(0)}}{\partial t_k^a\partial t_l^b}=\langle\phi_a\psi^k,\phi_b\psi^l\rangle_{0,2}(\tau(\mathbf{q})).$$

Differentiating with respect to t_0^a the string equation

$$\frac{\partial\mathcal{F}^{(0)}}{\partial t_0^1}=\frac{1}{2}\,(t_0,t_0)\;+\;\sum_{k=0}^{\infty}\sum_{a=1}^{N}t_{k+1}^a\,\frac{\partial\mathcal{F}^{(0)}}{\partial t_k^a}$$

we get that $\tau(\mathbf{q})$ is a solution to the following equation

$$G^a(\tau,\mathbf{t})=0,\quad G^a(\tau,\mathbf{t})=\tau^a-t_0^a-\sum_{k-0}^{\infty}\sum_{b,c=1}^{N}g^{ac}t_{k+1}^b\langle\phi_b\psi^k,\phi_c\rangle_{0,2}(\tau).$$

Using implicit differentiation it is easy to verify that the matrix with entries $\partial G^a/\partial t^b$ is the inverse to $\partial\tau^b/\partial t^c$. On the other hand comparing the derivatives

$\partial\tau^a/\partial t^a_{k+1}$ and $\partial G^a/\partial\tau^e$ we see that

$$\sum_{e=1}^{N} \frac{\partial\tau^e}{\partial t^b_{k+1}} \frac{\partial G^a}{\partial\tau^e} = \sum_{c=1}^{N} g^{ac}\langle\phi_b\psi^k, \phi_c\rangle_{0,2}(\tau).$$

In other words

$$\frac{\partial\tau^e}{\partial t^b_{k+1}} = \sum_{a,c=1}^{N} \frac{\partial\tau^e}{\partial t^a_0}\, g^{ac}\,\langle\phi_b\psi^k, \phi_c\rangle_{0,2}(\tau).$$

Now we can prove the TRR.

$$\frac{\partial^3\mathcal{F}^{(0)}}{\partial t^b_{k+1}\partial t^b_l\partial t^c_m} = \sum\langle\phi_b\psi^l, \phi_c\psi^m, \phi_d\rangle_{0,3}(\tau)\,\frac{\partial\tau^d}{\partial t^a_{k+1}} =$$

$$\sum\langle\phi_b\psi^l, \phi_c\psi^m, \phi_d\rangle_{0,3}(\tau)\,\frac{\partial\tau^d}{\partial t^e_0}\, g^{ef}\,\langle\phi_a\psi^k, \phi_f\rangle_{0,2}(\tau).$$

To finish the proof of TRR we just have to notice that

$$\sum_d\langle\phi_b\psi^l, \phi_c\psi^m, \phi_d\rangle_{0,3}(\tau)\,\frac{\partial\tau^d}{\partial t^e_0} = \frac{\partial}{\partial t^l_0}\,\langle\phi_b\psi^l, \phi_c\psi^m\rangle_{0,2}(\tau).$$

The opposite direction is left to the reader. The argument can be found in [6]. □

4. From descendant to ancestors

4.1. From two- to one-point descendants. Denote by

$$W_\tau(z,w)=\sum_{k,l} W_{kl}(\tau)z^{-k}w^{-l},\quad W_{kl}\in \mathrm{End}(H),$$

where the coefficients W_{kl} are defined by the following formulas

$$(\phi_a, W_\tau(z,w)\phi_b)=\sum_{k,l\geq 0}\langle \phi_a\psi^k,\phi_b\psi^l\rangle_{0,2}(\tau)z^{-k}w^{-l}.$$

Let $S_\tau(z)$ be the fundamental solution of the system of quantum differential equations (see Proposition 2.4.

Lemma 4.1. *The following formula holds:*

$$W_\tau(z,w)=\frac{{}^{\mathrm{t}}S_\tau(z)S_\tau(w)-1}{z^{-1}+w^{-1}},$$

where the transpose of S is with respect to the Poincaré pairing.

Proof. We need to verify that

$$(\phi_a, W_\tau(z,w)\phi_b)\left(z^{-1}+w^{-1}\right)+(\phi_a,\phi_b)=(S_\tau(z)\phi_a, S_\tau(w)\phi_b).$$

Using the (SE), it is easy to verify that the LHS of the above identity coincides with

$$\sum_{k,l\geq 0}\langle \phi_a\psi^k,\phi_b\psi^l,1\rangle_{0,3}(\tau)z^{-k}w^{-l}. \tag{4.1}$$

We split the summation range in the above sum into four groups. First if $k=l=0$ then the corresponding summand is just (ϕ_a,ϕ_b). The summands corresponding to $k,l\geq 1$ can be simplified first with TRR and then they add up to the following sum:

$$\sum_{\mu,\nu}\sum_{k,l\geq 1}\langle \phi_a\psi^{k-1},\phi_\mu\rangle_{0,2}(\tau)g^{\mu\nu}\langle\phi_\nu,\phi_b\psi^l,1\rangle_{0,3}(\tau)z^{-k}w^{-l}. \tag{4.2}$$

By definition we have

$$\sum_{k\geq 1}\langle\phi_a\psi^{k-1},\phi_\mu\rangle_{0,2}(\tau)z^{-k}=(\phi_\mu,(S_\tau(z)-1)\phi_a)$$

and

$$\sum_{l\geq 1}\langle\phi_\nu,\phi_b\psi^l,1\rangle_{0,3}(\tau)w^{-l}=\sum_{l\geq 1}\langle\phi_\nu,\phi_b\psi^{l-1}\rangle_{0,2}(\tau)w^{-l}=(\phi_\nu,(S_\tau(w)-1)\phi_b),$$

where for the first equality we used SE. Therefore the sum (4.2) equals

$$\sum_{\mu,\nu}(\phi_\mu,(S_\tau(z)-1)\phi_a)g^{\mu\nu}(\phi_\nu,(S_\tau(w)-1)\phi_b)=((S_\tau(z)-1)\phi_a,(S_\tau(w)-1)\phi_b).$$

Similarly, the summands in (4.1) corresponding to $k \geq 1$, $l = 0$ add up to $((S_\tau(z) - 1)\phi_a, \phi_b)$, and the ones corresponding to $k = 0$ and $l \geq 1$ to $(\phi_a, (S_\tau(w) - 1)\phi_b)$. The lemma follows. □

Corollary 4.2. *The series S_τ is a symplectic transformation of $\mathcal{H}$, i.e.,*

$$ {}^{\mathrm{T}}S_\tau(-z)S_\tau(z) = 1. $$

4.2. **Quantization formalism.** Given an infinitesimal symplectic transformation A we define a differential operator $\widehat{A}$ acting on the space of formal power series

$$ M := \mathbb{C}_{\sqrt{\hbar}}[Q][[q_0, q_1 + 1, q_2, \dots]], \quad \mathbb{C}_{\sqrt{\hbar}}[Q] = \mathbb{C}[Q]((\sqrt{\hbar})). $$

This space is called *Fock space.* We use the Weyl quantization rules:

$$ \widehat{q}_k{}^a = q_k^a/\sqrt{\hbar} \quad \text{and} \quad \widehat{p}_{k,a} = \sqrt{\hbar}\, \partial/\partial q_k^a. $$

Monomial expressions in p and q are quantized by representing each p (resp. q) by the corresponding differentiation (resp. multiplication) operator and moving all differentiation operators before the multiplication ones. We define $\widehat{A} := \widehat{h}_A$. Notice that the quantization of quadratic Hamiltonians is a projective representation of Lie algebras, i.e.,

$$ [\widehat{F}, \widehat{G}] = \{F, G\}\widehat{\ } + C(F, G), $$

where the cocycle is defined by:

$$ C(p_a p_b, q_a q_b) = -C(q_a q_b, p_a p_b) = \begin{cases} 1 & \text{if } a \neq b \\ 2 & \text{otherwise,} \end{cases} $$

and C vanishes for all other pairs of quadratic Darboux monomials.

By definition, *the twisted loop group* is defined as

$$ \mathcal{L}^{(2)}\mathrm{GL}(H) = \Big\{ M(z) = \sum_k M_k z^k \mid {}^{\mathrm{T}}M(-z)M(z) = 1 \Big\} $$

Given an element of $\mathcal{L}^{(2)}\mathrm{GL}(H)$ of the form $S(z) = 1 + S_1 z^{-1} + S_2 z^{-2} + \dots$, we define its quantization by $\widehat{S} = e^{\widehat{A}}$, where $A = \ln S$. We would like to describe the action of $\widehat{S}^{-1}$ on the Fock space. Introduce the quadratic form

$$ W(\mathbf{q}, \mathbf{q}) = \sum_{k,l} (W_{kl} q_l, q_k), \quad \text{where} \quad \sum_{k,l \geq 0} W_{kl} z^{-k} w^{-l} = \frac{{}^{\mathrm{T}}S(z)S(w) - 1}{z^{-1} + w^{-1}}. $$

Theorem 4.3. *The follwoing formula holds:*

$$ \widehat{S}^{-1}\mathcal{F} = e^{\frac{1}{2\hbar}W(\mathbf{q},\mathbf{q})}\mathcal{F}([S\mathbf{q}]_+), $$

where f_+ means the series obtained from f by truncating the terms with negative powers of z.

Proof. Write $A(z)=\sum_{k\geq 1} A_k z^{-k}$. Then it is not hard to see that the corresponding quadratic Hamiltonian is given by:

$$h_A = -\frac{1}{2}(A\mathbf{q},\mathbf{q}(-z)) - (A\mathbf{p},\mathbf{q}(-z)),$$

where

$$\mathbf{q}(z)=\sum_k q_k z^k = \sum_{k,a} q_k^a \phi_a z^k,$$

and

$$\mathbf{p}(z)=\sum_k p_k(-z)^{-k-1} = \sum_{k,a} p_{k,a}\phi^a(-z)^{-k-1}.$$

Put $\mathcal{G}(t,\mathbf{q}) = e^{-t\widehat{A}}\mathcal{F}$. We compute $\mathcal{G}$ for all t. In particular, the Theorem would follow from the case $t=1$.

Notice that $\mathcal{G}$ is a solution to the differential equation $\partial_t\mathcal{G} = -\widehat{A}\mathcal{G}$, which after the substitution $g=\log\mathcal{G}$, turns into:

$$\frac{\partial g}{\partial t} = \frac{1}{2\hbar}(A\mathbf{q},\mathbf{q}(-z)) + \sum_{k,a}(A\phi^a(-z)^{-k-1},\mathbf{q}(-z))\frac{\partial g}{\partial q_k^a}. \tag{4.3}$$

This is a 1-st order PDE which we solve by the method of the characteristics.

Step 1: first, we solve the homogeneus equation, i.e.,

$$\frac{\partial g}{\partial t} = \sum_{k,a}(A\phi^a(-z)^{-k-1},\mathbf{q}(-z))\frac{\partial g}{\partial q_k^a}.$$

The auxiliarly system of ODE's is

$$\frac{\partial q_k^a}{\partial t} = -(A\phi^a(-z)^{-k-1},\mathbf{q}(-z)) \quad\Leftrightarrow\quad \frac{\partial \mathbf{q}}{\partial t} = -[A\mathbf{q}]_+.$$

Notice that $[A[\ldots[A\mathbf{q}]_+]]_+ = [A^n\mathbf{q}]_+$, where on the LHS A is repeated n times. Therefore, the system of ODE's has the following solution: $\mathbf{q}(t) = [e^{-tA}\mathbf{c}]_+$, where $\mathbf{c}=\mathbf{q}(0)\in\mathcal{H}_+ = H[z]$ is an initial condition. The method of the characteristics is based on the fact that the solutions $g(t,\mathbf{q})$ of the PDE are constant along the curves $(t,\mathbf{q}(t))\in\mathbb{C}\times\mathcal{H}_+$. From here we find that if $(t,\mathbf{q})\in\mathbb{C}\times\mathcal{H}_+$ is any point then the curve $(s,\mathbf{q}(s))$ with initial condition $(0,[e^{tA}\mathbf{q}]_+)$ will pass through the point $(t,\mathbf{q})$. Therefore, the general solution of the PDE is given by: $g(t,\mathbf{q}) = f([e^{tA}\mathbf{q}]_+)$, where f is an arbitrary function on $\mathcal{H}_+$.

Step 2: a direct computation shows that the function

$$W_t(\mathbf{q},\mathbf{q}) = \frac{1}{2\hbar}\sum_{k,l}(W_{kl}(t)q_l,q_k),$$

defined by the formula:

$$\sum_{k,l\geq 0} W_{kl}(t)z^{-k}w^{-l} = \frac{e^{{}^{\mathrm{T}}A(z)t}e^{A(w)t}-1}{z^{-1}+w^{-1}}$$

is a solution to (4.3).

So the general solution to (4.3) is given by $g(t,\mathbf{q}) = W_t(\mathbf{q},\mathbf{q}) + f([e^{tA}\mathbf{q}]_+)$. Notice that for $t=0$ we have $\mathcal{G}=\mathcal{F}$, and $W_0(\mathbf{q},\mathbf{q})=0$, so $f=\log\mathcal{F}$. The theorem follows. □

4.3. **From descendants to ancestor GW invariants.** Let

$$\alpha_i(\psi,\overline{\psi}) = \sum_{k,m}\alpha_i^{k,m}\psi^k\overline{\psi}^m \in H[\psi,\overline{\psi}].$$

The correlator

$$\langle\langle \alpha_1(\psi,\overline{\psi}),\dots,\alpha_n(\psi,\overline{\psi})\rangle\rangle_{g,n}(\tau) \tag{4.4}$$

represents the following sum of integrals over the moduli spaces:

$$\sum_{d,l}\sum_{k.,m.}\frac{Q^d}{l!}\int_{\overline{\mathcal{M}}_{g,n+l}(X;d)}\psi_1^{k_1}\overline{\psi}_1^{m_1}\dots\psi_n^{k_n}\overline{\psi}_n^{m_n}\mathrm{ev}^*(\alpha_1^{k_1,m_1}\otimes\alpha_n^{k_n,m_n}\otimes\tau^{\otimes l}).$$

Here $\tau\in H$ is a formal parameter and $\overline{\psi}_i$ is the pullback of the ψ_i-class on $\overline{\mathcal{M}}_{g,n}$ via the (forgetfull) map $\pi:\overline{\mathcal{M}}_{g,n+l}(X,d)\to\overline{\mathcal{M}}_{g,n}$ which forgets the map, the last l marked points, and contracts all unstable components. By definition, the corelator (4.4) is 0 if $\overline{\mathcal{M}}_{g,n}$ is empty, i.e., for $(g,n)\in\{(0,0),(0,1),(0,2),(1,0)\}$.

Lemma 4.4. *Assume that $\alpha\in H^*(X)$ and (g,n) is a stable pair (i.e. $\overline{\mathcal{M}}_{g,n}$ is non-empty). Then the following formula holds:*

$$\begin{aligned}&\langle\alpha\psi^{k+1}\overline{\psi}^m,\alpha_2(\psi,\overline{\psi}),\dots,\alpha_n(\psi,\overline{\psi})\rangle_{g,n}(\tau) =\\ &= \langle\alpha\psi^k\overline{\psi}^{m+1} + S_{k+1}\alpha\overline{\psi}^m,\alpha_2(\psi,\overline{\psi}),\dots,\alpha_n(\psi,\overline{\psi})\rangle_{g,n}(\tau),\end{aligned}$$

where $S_\tau(z) = 1 + S_1(\tau)z^{-1}+\dots$ is the 1-point descendant series.

Proof. Let D_1 be the divisor in $\overline{\mathcal{M}}_{g,n+l}(X,d)$ of all points $(\Sigma,p.,f)$ such that the first marked point p_1 is not on the same irreducible component as any of the points p_i, $2\le i\le n$. Notice that $\psi_1=\overline{\psi}_1+[D_1]$ and that the divisor D_1 can be identified with the image of the gluing map:

$$\mathrm{gl}:\bigsqcup_{\substack{l'+l''=l\\ d'+d''=d}}\overline{\mathcal{M}}_{g,n-1+l'+\circ}(X,d')\times_X\overline{\mathcal{M}}_{0,1+l''+\bullet}(X,d'')\to\overline{\mathcal{M}}_{g,n+l}(X,d),$$

where in the fiber product the maps from the moduli spaces to X are given by the evaluations at the marked points $\circ$ and $\bullet$. Writing $\psi_1^{k+1}\overline{\psi}_1^m = \psi_1^k\overline{\psi}_1^{m+1} + [D_1]\psi_1^k\overline{\psi}^m$ we get that the integral

$$\int_{\overline{\mathcal{M}}_{g,n+l}(X,d)}\mathrm{ev}_1^*(\alpha)\psi_1^{k+1}\overline{\psi}_1^m\alpha_2\dots\alpha_n\tau^{\otimes l}$$

equals to

$$\int_{\overline{\mathcal{M}}_{g,n+l}(X,d)} \mathrm{ev}_1^*(\alpha)\psi_1^k\overline{\psi}_1^{m+1}\alpha_2\dots\alpha_n\tau^{\otimes l} + \sum_{\substack{l'+l''=l\\ d'+d''=d}} \frac{l!}{l'!l''!}\sum_{\mu,\nu} g^{\mu\nu}\times$$

$$\times\int_{\overline{\mathcal{M}}_{g,n-1+l'+\circ}(X,d')} \alpha_2\dots\alpha_n\tau^{\otimes l'}\mathrm{ev}_\circ^*(\phi_\mu)\overline{\psi}_\circ^m\int_{\overline{\mathcal{M}}_{0,1+l''+\bullet}(X,d'')}\mathrm{ev}_1^*(\alpha)\psi_1^k\tau^{\otimes l''}\mathrm{ev}_\bullet^*(\phi_\nu),$$

where the combinatorial factor $\binom{l}{l'}$ comes from the fact that in the gluing map gl the union of the l' marked points on the 1-st moduli space and the l'' marked points on the second one have to be renumbered with the numbers from $n+1$ to $n+l$. Notice that the expression $\sum_{\mu\nu} g^{\mu\nu}\phi_\mu\otimes\phi_\nu$ is the Poincaré dual to the diagonal in $X\times X$. The lemma follows. □

By definition *the total ancestor potential* is defined by the following formula:

$$\widetilde{\mathcal{A}}_\tau(\mathbf{t}) = \exp\ \Big(\sum_{g,n}\frac{1}{n!}\,\hbar^{g-1}\langle\langle\mathbf{t}(\overline{\psi}_1),\dots,\mathbf{t}(\overline{\psi}_n)\rangle\rangle_{g,n}(\tau)\Big).$$

Using the dilaton shift $\mathbf{t}(z)=\mathbf{q}(z)+z$, we identify $\widetilde{\mathcal{A}}_\tau$ with an element $\mathcal{A}_\tau(\mathbf{q})$ of the Fock space. Namely,

$$\mathcal{A}_\tau(\mathbf{q}) = \widetilde{A}_\tau(\mathbf{q}(z)+z)$$

The goal now is to express the total ancestor potential in terms of the total descendant potential.

Theorem 4.5. *The following formula holds*

$$\mathcal{D}(\mathbf{q}) = e^{F^{(1)}(\tau)}\ \widehat{S}_\tau^{-1}\ \mathcal{A}_\tau(\mathbf{q}),$$

where $F^{(1)}(\tau)=\mathcal{F}^{(0)}\big|_{t_0=\tau,t_1=t_2=\dots=0}$ *is the genus-1 GW potential.*

Proof. Recall that the total descendant potential is given by the formula

$$\widetilde{\mathcal{D}}(\mathbf{t}) = \exp\Big(\sum_{g,n}\frac{\epsilon^{2g-2}}{n!}\langle\mathbf{t}(\psi),\dots,\mathbf{t}(\psi)\rangle_{g,n}\Big).$$

It is identified with an element of the Fock space via the dilaton shift:

$$\mathcal{D}(\mathbf{q}) = \widetilde{\mathcal{D}}(\mathbf{q}(z)+z).$$

The above lemma implies the following identity:

$$\langle\langle\mathbf{t}(\psi),\dots,\mathbf{t}(\psi)\rangle\rangle_{g,n}(\tau) = \langle\langle[S_\tau\mathbf{t}]_+(\psi),\dots,[S_\tau\mathbf{t}]_+(\psi)\rangle\rangle_{g,n}(\tau),$$

where $\mathbf{t}(z)=\sum_{k,a}t_k^a\phi_a z^k\in\mathcal{H}_+$. Using the Taylor's formula we get

$$\widetilde{\mathcal{A}}_0(\mathbf{t}(z)+\tau) = \widetilde{\mathcal{A}}_\tau([S_\tau(z)\mathbf{t}(z)]_+$$

Note that

$$\widetilde{\mathcal{D}}(\mathbf{t}+\tau)/\widetilde{\mathcal{A}}_0(\mathbf{t}+\tau) = C(\mathbf{t}),$$

where

$$C_\tau(\mathbf{t}) = e^{F^{(1)}(\tau)} \exp\Big(\langle\ \rangle_{0,0}(\tau) + \langle\ \mathbf{t}(\psi)\rangle_{0,1}(\tau) + \frac{1}{2}\langle\ \mathbf{t}(\psi), \mathbf{t}(\psi)\rangle_{0,2}(\tau)\Big)\hbar^{-1}.$$

Therefore

$$\widetilde{\mathcal{D}}(\mathbf{t}+\tau) = C_\tau(\mathbf{t})\widetilde{\mathcal{A}}_\tau([S_\tau \mathbf{t}]_+).$$

Replacing in this formula $\mathbf{t}(z) \mapsto \mathbf{q}(z) + z - \tau$, we get

$$\mathcal{D}(\mathbf{q}) = C_\tau(\mathbf{q}(z) + z - \tau)\mathcal{A}_\tau(-z + [S_\tau(\mathbf{q}(z) + z - \tau)]_+).$$

First, let us simplify the argument in the ancestor potential:

$$-z + [S_\tau\, \mathbf{q}(z)]_+ + z + S_1 1 - \tau = [S_\tau\, \mathbf{q}(z)]_+.$$

Where we used that

$$(S_1 1, \phi_a) = \langle 1, \phi_a\rangle_{0,2}(\tau) = \langle 1, \phi_a, \tau\rangle_{0,3,0} = \int_X \phi_a \tau,$$

i.e., $S_1(\tau)1 = \tau$.

On the other hand, using the dilaton equation, it is not hard to verify that

$$\begin{aligned} \langle \psi - \tau, \mathbf{q}(\psi)\rangle_{0,2}(\tau) &= -\langle \mathbf{q}(\psi)\rangle_{0,1}(\tau) \\ \langle \psi - \tau, \psi - \tau\rangle_{0,2}(\tau) &= -\langle \psi - \tau\rangle_{0,1}(\tau) \\ \langle \psi - \tau\rangle_{0,1}(\tau) &= -2\langle\rangle_{0,0}(\tau). \end{aligned}$$

From this formulas we get

$$C_\tau(\mathbf{q}(z) + z - \tau) = e^{F^{(1)}(\tau)}\ e^{\frac{1}{2\hbar}\langle \mathbf{q}(\psi), \mathbf{q}(\psi)\rangle_{0,2}(\tau)}.$$

It remains only to recall Theorem 4.3 and the formula relating 1- to 2-point descendants. □

5. Semi-simple cohomological field theories I

In this lecture, following the work of C. Teleman (see [13]), we will see how Givental's quantization formalism arises naturally in the settings of the so called *Cohomological Field Theories* (CohFT).

5.1. **Definition of CohFT.** Let H be a vector space, equipped with a non-degenerate pairing, and a unit vector $1 \in H$. From now on we fix a basis $\{\phi_\mu\}_{\mu=1}^N$ of H, put $g_{\mu\nu} = (\phi_\mu, \phi_\nu)$ and denote by $(g^{\mu\nu})$ the matrix inverse to $(g_{\mu\nu})$.

A CohFT on H is a system of maps

$$\overline{Z}_{g,n} : H^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{C}), \quad 2g-2+n > 0,$$

satisfying the following axioms

(1) *Permutation invariance:* the expression $\overline{Z}_{g,n}(a_1, \dots, a_n)$ is symmetric in $a_1, \dots, a_n$.

(2) *Boundary axioms:* the boundary morphism

$$b : \overline{\mathcal{M}}_{g,n'+1} \times \overline{\mathcal{M}}_{g'',n''+1} \to \overline{\mathcal{M}}_{g,n}, \quad g' + g'' = g, n' + n'' = n$$

defined by gluing the last marked points satisfies

$$b^*\overline{Z}_{g,n}(a_1, \dots, a_n) = \sum_{\mu,\nu=1}^{N} g^{\mu\nu} \overline{Z}_{g',n'+1}(a_{i_1}, \dots, a_{i_{n'}}, \phi_\mu) \overline{Z}_{g,n''+1}(a_{j_1}, \dots, a_{j_{n''}}, \phi_\nu),$$

where

$$\{i_1, \dots, i_{n'}\} \sqcup \{j_1, \dots, j_{n''}\} = \{1, 2, \dots, n\}$$

is the partition imposed by b.

Similarly, the boundary morphism

$$b' : \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n},$$

consisting of gluing the last two marked points, must satisfy

$$(b')^*\overline{Z}_{g+1,n}(a_1, \dots, a_n) = \sum_{\mu,\nu=1}^{N} g^{\mu\nu} \overline{Z}_{g,n+2}(a_1, \dots, a_n, \phi_\mu, \phi_\nu).$$

(3) *Identity axiom:* $\overline{Z}_{0,3}(a, b, 1) = (a, b)$.

The CohFT comming from GW theory satisfy one additional axiom. Namely,

$$\pi^*\overline{Z}_{g,n}(a_1, \dots, a_n) = \overline{Z}_{g,n+1}(a_1, \dots, a_n, 1),$$

where $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the universal curve. We refer to this equation as *the flat identity axiom.*

5.2. **An important example.** Given a CohFT, we define a multiplication $\bullet$ on H as follows:

$$(a \bullet b, c) = \overline{Z}_{0,3}(a, b, c), \quad a, b, c \in H.$$

It is easy to verify that this multiplication and the pairing (,) turn H into a Frobenius algebra.

Assume now that H is a Frobenius algebra equipped with a unity. Then we can build a whole family of CohFT in the following way. The moduli space $\overline{\mathcal{M}}_{g,n}$ carries the so called κ-classes defined by $\kappa_i = \pi_*(\psi_{n+1}^{i+1})$, $i \geq 0$ (note that $\kappa_0 = 2g - 2 + n$). They satisfy the following crucial property:

$$b^*\kappa_i = \kappa_i \otimes 1 + 1 \otimes \kappa_i,$$

where b is the boundary morphism from the previous subsection. Let $s_i \in H$ ($i \geq 1$) be any sequence of vectors. It is easy to check that the following formulas:

$$\overline{Z}_{g,n}(a_1, \dots, a_n) = (P^g \bullet e^{\sum_{i=1}^{\infty} s_i\kappa_i}, a_1 \bullet \cdots \bullet a_n),$$

where

$$P = \sum_{\mu,\nu=1}^{N} g^{\mu\nu}\phi_\mu \bullet \phi_\nu,$$

is the so called *propagator*, form a CohFT. The propagator P is chosen so that this system of maps is compatible with the boundary morphisms of type b'. All multiplication in the above formula take place in the Frobenius algebra and in the cohomology $H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{C})$.

5.3. **Semi-simple CohFT.**

Definition 5.1. *A CohFT* $\{\overline{Z}_{g,n}\}$ *is called* semi-simple *if the Frobenius algebra* H *is semi-simple, i.e., there exists a basi* $\{e_i\}_{i=1}^N$ *such that*

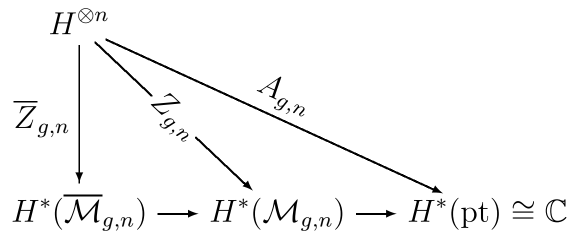
$$(e_i, e_j) = \delta_{ij}, \quad e_i \bullet e_j = \sqrt{\theta_i}\delta_{ij}e_j, \quad 1 \leq i, j \leq N,$$

where θ_i *(*$1 \leq i \leq N$*) are some* non-zero *complex numbers.*

Note that in a semi-simple Frobenius algebra, the propagator

$$P = \sum_{i=1}^{N} e_i \bullet e_i = \sum_{i=1}^{N} \sqrt{\theta_i}\; e_i$$

is *invertible.*



Given a CohFT, we denote by $Z_{g,n}$ and $A_{g,n}$ its restrictions respectively to $H^*(\overline{\mathcal{M}}_{g,n})$ and $H^*(\mathrm{pt}) = \mathbb{C}$ (see the above diagram). The set of all $A_{g,n}$ satisfy the axioms of the so called *topological field theories*. They can be computed explicitly (see [3]). The answer is the following

Theorem 5.2 (Dubrovin)**.** *The map $A_{g,n}$ is given by the following formula:*

$$A_{g,n}(a_1, \ldots, a_n) = (P^g, a_1 \bullet \cdots \bullet a_n), \quad a_1, \ldots, a_n \in H.$$

It turns out that for semi-simple CohFT the maps $Z_{g,n}$ can be computed explicitly as well. Let $\overline{\mathcal{M}}^1_{g,n}$ be the moduli space of Riemann surfaces equipped with n marked points and with 1 parametrized boundary. Forgetting the parametrization gives us a map $\overline{\mathcal{M}}^1_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$ which turns $\overline{\mathcal{M}}^1_{g,n}$ into a S^1-bundle. In fact, $\overline{\mathcal{M}}^1_{g,n}$ is the S^1-bundle associated with $L^\vee_{n+1}$. In particular its Euler class $e(\overline{\mathcal{M}}^1_{g,n}) = -\psi_{n+1}$. Similarly, one can define $\overline{\mathcal{M}}^r_{g,n}$ – the moduli space of Riemann surfaces with r parametrized boundaries, where the boundary circles are numbered with the integers from $n+1$ to $n+r$.

The moduli space $\overline{\mathcal{M}}^1_{g,n}$ admit the so called genus stabilization map $\overline{\mathcal{M}}^1_{g,n} \to \overline{\mathcal{M}}^1_{g+1,n}$ consisting of sewing a genus-1 Riemann surface with 2 boundary circles. It follows that the cohomology groups $H^*(\overline{\mathcal{M}}^1_{g,n})$ form an inverse system with respect to g. The Mumfords conjecture, proved by Madsen and Weiss ([12]) says

Theorem 5.3 (Madsen–Weiss)**.** *In the stable range $H^*(\mathcal{M}^1_{g,n})$ is a polynomial algebra in κ- and ψ-classes, i.e.,*

$$\lim_{g\to\infty} H^*(\overline{\mathcal{M}}^1_{g,n}; \mathbb{C}) = \mathbb{C}[\psi_1, \ldots, \psi_n, \kappa_1, \kappa_2, \ldots], \tag{5.1}$$

where the limit is the inverse limit of the inverse system $\{H^*(\overline{\mathcal{M}}_{g,n})\}_{g\geq 0}$.

Set

$$Z^r_{g,n}(a_1, \ldots, a_n, b_1, \ldots, b_r) = \pi^*\, Z_{g,n+r}(a_1, \ldots, a_n, b_1, \ldots, b_r),$$

where $\pi : \mathcal{M}^r_{g,n} \to \mathcal{M}_{g,n+r}$ is the map forgetting the parametrizations.

Proposition 5.4. *The class $Z^1_{g,n}(a_1,\ldots,a_n,b) \in H^*(\mathcal{M}^1_{g,n})$ is a polynomial expressions in κ- and ψ-classes.*

Proof. Consider the stabilization map $\mathrm{st} = b \circ i$

$$\mathcal{M}_{g,n} \xrightarrow{\ i\ } \mathcal{M}^2_G \times \mathcal{M}^1_{g,n} \xrightarrow{\ b\ } \mathcal{M}^1_{g+G,n},$$

where b is the map gluing two boundary cicrlces and i is the inclusion map. Using the gluing axioms, we find

$$\mathrm{st}^*(Z^1_{g+G,n}(a_1,\ldots,a_n,P^{-g-G}b) = i^*\Big(Z^2_G(\phi_\mu,P^{-g-G}b)g^{\mu\nu}Z^1_{g,n}(a_1,\ldots,a_n,\phi_\nu)\Big).$$

Note that

$$i^*(Z^2_G(\phi_\mu,P^{-g-G}b)) = A_{G,2}(\phi_\mu,P^{-g-G}b)) = (P^{-g}\bullet b,\phi_\mu).$$

It follows that

$$\mathrm{st}^*(Z^1_{g+G,n}(a_1,\ldots,a_n,P^{-g-G}b) = Z^1_{g,n}(a_1,\ldots,a_n,P^{-g}b).$$

Thanks to Mumfords conjecture, by taking G sufficiently large, we can arrange that the LHS is a polynomial in ψ- and κ-classes. □

In fact, using the gluing axioms, it is not hard to find all $Z^1_{g,n}$ explicitly. The answer is the following.

Proposition 5.5. *There are vectors $s_i \in H$, $i \geq 1$ and a series*

$$R(z) = 1 + R_1 z + R_2 z^2 + \ldots,\qquad R_k \in \mathrm{End}(H),$$

such that

$$Z^1_{g,n}(a_1,\ldots,a_n,b) = (P^g \ \bullet\ e^{\sum_{i=1}^\infty s_i\kappa_i}, (R(\psi_1)a_1)\bullet\cdots\bullet(R(\psi_n)a_n)\bullet b).$$

We leave the proof of this Proposition as an exercise. The only thing we have to use here are the boundary axioms.

Proposition 5.6. *The series $R(z)$ is a symplectic transformation, i.e.,*

$${}^{\mathrm{T}}R(-z)R(z) = 1.$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\ \widetilde{b}\ } & \mathcal{M}^2_{g+G} \\ \Big\downarrow p & & \Big\downarrow i \\ \mathcal{M}^1_{g,1}\times\mathcal{M}^1_{G,1} & \xrightarrow{\ b\ } & \overline{\mathcal{M}}_{g+G,2} \end{array}$$

where b is the map that glues the marked points and forgets the parametrization of the boundary circrcles. Note that $Im\, b \subset \overline{\mathcal{M}}_{g+G,2}$ consists of Riemann surfaces having exactly two irreducible components of topological type $(g,2)$

and $(G,2)$ respectively, glued along their 1-st marked points. A tubular neighborhood N of $Im\,b$ can be identified with a disk bundle of the normal bundle and then $E = b^*(\partial N)$ is the corresponding S^1-bundle. The bundle E is naturally imbedded in $\mathcal{M}^2_{g+G}$ because $\partial N \subset \mathcal{M}_{g+G,2}$.

Let $a, b \in H$ be arbitrary. By the definition of Z^2_{g+G} the following expression

$$\widetilde{b}^*\Big(Z^2_{g+G}(a,b) - i^*\,\overline{Z}_{g+G,2}(a,b)\Big) \tag{5.2}$$

is 0. On the other hand, using Proposition 5.5 we have

$$Z^2_{g+G}(a,b) = i^*\Big(P^{g+G}e^{\sum_{i=1}^\infty s_i\kappa_i}, a\bullet b\Big).$$

Using the commutative diagram and computing $b^*\overline{Z}_{g+G,2}$ via the boundary axiom, we get that (5.2) equals equals

$$p^*\Big(b^*\big(P^{g+G}e^{\sum_{i=1}^\infty s_i\kappa_i}, a\bullet b\big) - Z^1_{g,1}(\phi_\mu,a)g^{\mu\nu}Z^1_{G,1}(\phi_\nu,b)\Big),$$

where summation over the repeating indices μ and ν is assumed. Recalling Proposition 5.5 again and after some simplifications we get

$$Z^1_{g,1}(\phi_\mu,a)g^{\mu\nu}Z^1_{G,1}(\phi_\nu,b) = \Big({}^{\mathrm{T}}R(\psi_1')(P^g e^{\sum_{i=1}^\infty s_i\kappa_i'}a), {}^{\mathrm{T}}R(\psi_1'')(P^G e^{\sum_{i=1}^\infty s_i\kappa_i''}b)\Big).$$

It follows that

$$p^*\Big(\big(P^{g+G}e^{\sum_{i=1}^\infty s_i(\kappa_i'+\kappa_i'')}, a\bullet b\big) - \big(R(\psi_1'')^{\mathrm{T}}R(\psi_1')(P^g e^{\sum_{i=1}^\infty s_i\kappa_i'}a), P^G e^{\sum_{i=1}^\infty s_i\kappa_i''}b\big)\Big) = 0,$$

where $'$ (resp. $''$) indicate a cohomology class on the first (resp. second) factor of $\mathcal{M}^1_{g,1}\times\mathcal{M}^1_{G,1}$. Using the Gysin sequence, for the S^1-bundle E we get that the expression in the brackets is a multiple of the Euler class $e(E)$. Note that the normal bundle to Im b is $(L_1')^\vee\otimes(L_1'')^\vee$, so the Euler class $e(E) = -\psi_1' - \psi_1''$.

Replacing $a \mapsto P^{-g}a$ and $b \mapsto P^{-G}b$ and then taking the stable limit $g, G\to\infty$ we obtain some identity involving polynomial expressions in ψ_1', ψ_1'' and two coopies of the κ-classes. Thanks to Mumfords conjecture, these are independent variables, so we can set

$$\psi_1' = z,\quad \psi_1'' = -z,\quad \kappa_i' = \kappa_i'' = 0$$

and get

$$\Big(\big(R(-z)\ {}^{\mathrm{T}}R(z) - 1\big)a, b\Big) = 0.$$

The Proposition follows. □

It turns out that the symplectic condition is the only constraint that one has to impose in order to obtain a CohFT. More precisely,

Theorem 5.7 (Teleman)**.** *Let $s_i \in H,\ i \geq 1$ be a sequence of vectors and $R(z)$ is a symplectic transformation. Then there exists a unique CohFT $\overline{Z}_{g,n}$ such that*

$$Z_{g,n}(a_1,\dots,a_n) = \Big(P^g e^{\sum_{i=1}^\infty s_i\kappa_i}, R(\psi_1)a_1 \bullet \cdots \bullet R(\psi_n)a_n \Big).$$

For a proof and more conceptual description we refer to the article [13].

5.4. **Infinitesimal deformations.** Put

$$\overline{Z}_{g,n}(\mathbf{q},\dots,\mathbf{q}) = \sum \overline{Z}_{g,n}(\phi_{a_1},\dots,\phi_{a_n})\psi_1^{k_1}\dots\psi_n^{k_n} q_{k_1}^{a_1}\dots q_{k_n}^{a_n},$$

where $\mathbf{q}(z) = \sum_{k=0}^\infty \sum_{a=1}^N q_k^a\,\phi_a\,z^k$ and the summation is over all $k_1,\dots,k_n$ and $a_1,\dots,a_n$.

The total ancestor potential is defined by

$$\mathcal{A}(\mathbf{q}) = \exp\Big(\sum_{g,n} \frac{\hbar^{g-1}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \overline{Z}_{g,n}(\mathbf{q},\dots,\mathbf{q}) \Big).$$

In case, the CohFT is semi-simple, the potential will be denoted by $\mathcal{A}_{s,R}$, where the sequence $s=(s_1,s_2,\dots)$ and the symplectic transformation $R(z)$ are the parameters that (according to Proposition 5.7) determine the entire theory.

Given an infintesimal symplectic transformation

$$A(z) = A_1 z + A_2 z^2 + \dots,\quad A_k \in \mathrm{End}(H),\quad A(z) + {}^{\mathrm{T}}A(-z) = 0,$$

we define

$$\partial_A\ \mathcal{A}_{s,R} = \frac{d}{d\epsilon}\Big(\mathcal{A}_{s,Re^{\epsilon A}}\Big)\Big|_{\epsilon=0}.$$

Theorem 5.8. *The follwoing formula holds*

$$\partial_A\ \mathcal{A}_{s,R} = -\widehat{A}\ \mathcal{A}_{s,R}.$$

Proof. Let $\overline{\mathcal{M}}_{g,n}^{(k)},\ k\geq 0$ be the moduli space of Riemann surfaces with at least k nodes. We have a filtration

$$\cdots \subset \overline{\mathcal{M}}_{g,n}^{(k+1)} \subset \overline{\mathcal{M}}_{g,n}^{(k)} \subset \cdots \subset \overline{\mathcal{M}}_{g,n}^{(0)} = \overline{\mathcal{M}}_{g,n}.$$

For each $k\geq 0$, we introduce the open set in $\overline{\mathcal{M}}_{g,n}^{(k)}$ defined by

$$\mathcal{M}_{g,n}^{(k)} = \overline{\mathcal{M}}_{g,n}^{(k)} - \overline{\mathcal{M}}_{g,n}^{(k+1)}$$

We would like to find

$$\overline{Z}_{g,n} = Z_{g,n} + \overline{Z}_{g,n}^{(1)} + \overline{Z}_{g,n}^{(2)} + \dots\ ,$$

where the cohomology classes $\overline{Z}_{g,n}^{(k)} \in H^*(\overline{\mathcal{M}}_{g,n})$ are supported on $\overline{\mathcal{M}}_{g,n}^{(k)}$. First, we show how to find $\overline{Z}_{g,n}^{(1)}$. It will be clear that one can proceed inductively.

Consider the following commutative diagrams:

$$\begin{array}{ccc} E & \longrightarrow & \mathcal{M}_{g,n} \\ \Big\downarrow p & & \Big\downarrow i \\ \displaystyle\bigsqcup_{\substack{g'+g''=g\\ n'+n''=n}} \mathcal{M}_{g',n'+1}\times\mathcal{M}_{g'',n''+1} & \xrightarrow{\;b\;} & \overline{\mathcal{M}}_{g,n} \end{array}$$

and

$$\begin{array}{ccc} E' & \longrightarrow & \mathcal{M}_{g,n} \\ \Big\downarrow p & & \Big\downarrow i' \\ \mathcal{M}_{g-1,n+2} & \xrightarrow{\;b'\;} & \overline{\mathcal{M}}_{g,n} \end{array}$$

where b and b' are the boundary morphisms, and E (resp. E') is the S^1-bundle associated to the normal bundle of $Im\ b$ (resp. $Im\ b'$) in $\overline{\mathcal{M}}_{g,n}$. Note that both E and E' imbed naturally in $\mathcal{M}_{g,n}$, because they can be viewed as the boundary of a tubular neighborhood of $Im\ b$ and $Im\ b'$.

It follows from the explicit formula in Proposition 5.5 that $Z_{g,n}$ is a class on $\overline{\mathcal{M}}_{g,n}$. By definition, $i^*(\overline{Z}_{g,n} - Z_{g,n}) = 0$. Therefore, $p^*\ b^*\ (\overline{Z}_{g,n} - Z_{g,n}) = 0$, i.e.,

$$b^*\ (\overline{Z}_{g,n} - Z_{g,n}) = e(E)Z^{(1)}_{g,n} = -(\psi'_{n'+1} + \psi''_{n''+1})Z^{(1)}_{g,n},$$

for some cohomology class $Z^{(1)}_{g,n}$. It is convenient to introduce

$$F_{g,n}(a_1,\dots,a_n) = \Big(P^g e^{\sum_{i=1}^\infty s_i\kappa_i}, a_1\bullet\cdots\bullet a_n\Big).$$

This is a CohFT as it was explained in subsection 5.2 and we have

$$Z_{g,n}(a_1,\dots,a_n) = F_{g,n}(R(\psi_1)a_1,\dots,R(\psi_n)a_n).$$

According to the boundary axioms we have

$$b^*\,\overline{Z}_{g,n}(\mathbf{q},\dots,\mathbf{q}) = \sum_{\mu,\nu=1}^N F_{g',n'+1}(R\mathbf{q},\dots,R\mathbf{q},R\phi_\mu)g^{\mu\nu}F_{g'',n''+1}(R\mathbf{q},\dots,R\mathbf{q},R\phi_\nu).$$

and

$$b^*\,Z_{g,n}(\mathbf{q},\dots,\mathbf{q}) = \sum_{\mu,\nu=1}^N F_{g',n'+1}(R\mathbf{q},\dots,R\mathbf{q},\phi_\mu)g^{\mu\nu}F_{g'',n''+1}(R\mathbf{q},\dots,R\mathbf{q},\phi_\nu).$$

Using that

$$R\phi_\mu = \sum_{\mu'=1}^{N} (R\phi_\mu, \phi^{\mu'})\phi_{\mu'} \quad \text{and} \quad R\phi_\nu = \sum_{\nu'=1}^{N} (R\phi_\nu, \phi^{\nu'})\phi_{\nu'}$$

we get

$$Z^{(1)}_{g,n} = \sum_{\mu',\nu'=1}^{N} F_{g',n'+1}(R\mathbf{q}, \dots, R\mathbf{q}, \phi_{\mu'})(V\phi^{\mu'}, \phi^{\nu'})F_{g'',n''+1}(R\mathbf{q}, \dots, R\mathbf{q}, \phi_{\nu'})$$

where

$$V = V(\psi'_{n'+1}, \psi''_{n''+1}), \quad V(z,w) = \frac{1 - R(w)R^T(z)}{z+w}.$$

A similar argument shows that we have

$$(b')^*(\overline{Z}_{g,n} - Z_{g,n}) = e(E')(Z')^{(1)}_{g,n} = -(\psi_{n+1} + \psi_{n+2})(Z')^{(1)}_{g,n}$$

and therefore

$$(Z')^{(1)}_{g,n} = \sum_{\mu,\nu=1}^{N} F_{g,n+2}(R\mathbf{q}, \dots, R\mathbf{q}, \phi_\mu, \phi_\nu)(V\phi^\mu, \phi^\nu)$$

Set

$$\overline{Z}^{(1)}_{g,n} = (b')_*(Z')^{(1)}_{g,n} + \sum_b b_*(Z^{(1)}_{g,n}) \ \in \ H^*(\overline{\mathcal{M}}_{g,n}).$$

Then the restriction of the cohomology class $\overline{Z}_{g,n} - Z_{g,n} - \overline{Z}^{(1)}_{g,n}$ to $\mathcal{M}^{(1)}_{g,n}$ is 0 (recall that $b^*b_*(z) = e(E) \wedge z$), so we can proceed inductively. Now one has to check that

$$Z_{g,n} + \overline{Z}^{(1)}_{g,n} + \overline{Z}^{(2)}_{g,n} + \dots$$

defines a CohFT. Apriori, this theory might be different from $\overline{Z}_{g,n}$. The difference is given by a cohomology class $\Delta_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n})$ such that the restriction of $\Delta_{g,n}$ to $\mathcal{M}^{(i)}_{g,n}$ is 0 for all $i \geq 0$. There is no guarantee that $\Delta = 0$. However, according to C. Teleman, if two CohFT have the same restriction to $\mathcal{M}_{g,n}$ then they must coincide.

Let us apply the infinitesimal derivative ∂_A to each $\overline{Z}^{(k)}_{g,n}$. Using that

$$\partial_A R = RA \quad \text{and} \quad \partial_A V(z,w) = -R(w)\, \frac{A(w) + A^T(z)}{w+z}\, R^T(z)$$

we get

$$\partial_A Z_{g,n}(\mathbf{q}, \dots, \mathbf{q}) = nF_{g,n}(RA\mathbf{q}, R\mathbf{q}, \dots, R\mathbf{q}) = \sum (A_k q_l, \phi^a) \frac{\partial}{\partial q^a_{k+l}} Z_{g,n}(\mathbf{q}, \dots, \mathbf{q}).$$

The infinitesimal derivative $\partial_A\ Z^{(1)}_{g,n}(\mathbf{q},\dots,\mathbf{q})$ is the sum of

$$\sum (A_k q_l,\phi^a)\frac{\partial}{\partial q^a_{k+l}}\ Z^{(1)}_{g,n}(\mathbf{q},\dots,\mathbf{q}),$$

and infitesimal deformations corresponding two boundary morphisms. The later are devided into two types – that do not change genus and that do. Contributions from the first type look this way (the sum is over repeating indices)

$$-\sum b_*\Big(F_{g',n'+1}(R\mathbf{q},\dots,R\mathbf{q},\phi_\mu)\,F_{g'',n''+1}(R\mathbf{q},\dots,R\mathbf{q},\phi_\nu)(\psi''_{n''+1})^k(\psi'_{n'+1})^l\times$$
$$(R(\psi''_{n''+1})a_{kl}R^T(\psi'_{n'+1})\phi^\mu,\phi^\nu)\ \Big),$$

where

$$\frac{A(w)+A^T(z)}{w+z}=\sum_{k,l=0}^{\infty}a_{kl}w^kz^l\quad\Rightarrow\quad a_{kl}=(-1)^lA_{k+l+1}.$$

The above expression is simplified as follows. Combine the second line with the $F_{g'',n''+1}$-term and sum over ν. We get

$$Z_{g'',n''+1}(\mathbf{q},\dots,\mathbf{q},a_{kl}R^T(\psi'_{n'+1})\phi^\mu)=\sum_{\nu=1}^{N}(\phi^\mu,R(\psi'_{n'+1})\phi_\nu)Z_{g'',n''+1}(\mathbf{q},\dots,\mathbf{q},a_{kl}\phi^\nu)$$

It follows that the infinitesimal contribution is

$$-\sum b_*\Big(Z_{g',n'+1}(\mathbf{q},\dots,\mathbf{q},\phi_\nu)Z_{g'',n''+1}(\mathbf{q},\dots,\mathbf{q},\phi_\mu)(a_{kl}\phi^\nu,\phi^\mu)\ (\psi''_{n''+1})^k(\psi'_{n'+1})^l\Big).$$

A similar computation gives us that the infinitesimal contribution from the boundary terms of the second type is:

$$-(b')_*\Big(Z_{g-1,n+2}(\mathbf{q},\dots,\mathbf{q},\phi_\mu,\phi_\nu)(a_{kl}\phi^\nu,\phi^\mu)\ (\psi_{n+2})^k(\psi_{n+1})^l.\Big)$$

I have not analyzed the infinitesimal deformations of $\overline{Z}^{(k)}_{g,n}$ for $k\geq 2$ yet, but it should be clear that if we include their contributions as well, we would get that $\partial_A\overline{Z}_{g,n}$ equals

$$\sum_{k,l=0}^{\infty}\ \sum_{a=1}^{N}\ (A_kq_l,\phi^a)\frac{\partial}{\partial q^a_{k+l}}\ \overline{Z}_{g,n}(\mathbf{q},\dots,\mathbf{q})$$
$$-\frac{1}{2}\sum_b\binom{n}{n'}b_*\Big(\overline{Z}_{g',n'+1}(\mathbf{q},\dots,\mathbf{q},\phi_\nu)\overline{Z}_{g'',n''+1}(\mathbf{q},\dots,\mathbf{q},\phi_\mu)(a_{kl}\phi^\nu,\phi^\mu)\ (\psi''_{n''+1})^k(\psi'_{n'+1})^l\Big)$$
$$-\frac{1}{2}(b')_*\Big(\overline{Z}_{g-1,n+2}(\mathbf{q},\dots,\mathbf{q},\phi_\mu,\phi_\nu)(a_{kl}\phi^\nu,\phi^\mu)\ (\psi_{n+2})^k(\psi_{n+1})^l\Big),$$

where with respect to b the sum is over all boundary morphisms such that the marked points $\{1,2,\dots,n'\}$ and $\{1,2,\dots,n''\}$ correspond to $\{1,\dots,n',n'+$

$1,\ldots,n'+n''\}$ (and hence we need the combinatorial factor $\binom{n}{n'}$). Both factors of $1/2$ comes from the fact that switching

$$\mathcal{M}_{g',n'+1}\times\mathcal{M}_{g'',n''+1}\mapsto\mathcal{M}_{g'',n''+1}\times\mathcal{M}_{g',n'+1}$$

(resp. switching the last two marked points) does not change the image of b (resp. b'), i.e., b (resp. b') defines a 2-fold covering of the corresponding boundary stratum.

Now we are ready to prove the theorem. Let

$$\mathcal{F}=\sum_{g,n}\frac{\hbar^{g-1}}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\overline{Z}_{g,n}(\mathbf{q},\ldots,\mathbf{q}).$$

Then the formula for the infinitesimal deformation yields

$$\partial_A\mathcal{F}=\sum_{k,l=0}^{\infty}\sum_{a=1}^{N}(A_kq_l,\phi^a)\frac{\partial}{\partial q^a_{k+l}}\mathcal{F}-\frac{\hbar}{2}\sum_{k,l=0}^{\infty}\sum_{\mu,\nu=1}^{N}(a_{kl}\phi^\nu,\phi^\mu)\Big(\frac{\partial\mathcal{F}}{\partial q^\mu_k}\frac{\partial\mathcal{F}}{\partial q^\nu_l}+\frac{\partial^2\mathcal{F}}{\partial q^\mu_k\partial q^\nu_l}\Big).$$

It remains only to notice that

$$\widehat{A}=\frac{1}{2}\Omega(A\mathbf{f},\mathbf{f})=-\sum_{k,l=0}^{\infty}(A_kq_l,p_{k+l})\widehat{\ }+\sum_{k,l=0}^{\infty}(-1)^l(A_{k+l+1}p_l,p_k)\widehat{\ },$$

where

$$q_k=\sum_{a=1}^{N}q^a_k\phi_a\quad\text{and}\quad p_k=\sum_{a=1}^{N}p_{k,a}\phi^a.$$

Recall that $a_{k,l}=(-1)^lA_{k+l+1}$. The Theorem follows. □

Corollary 5.9. *Let $\{\overline{Z}_{g,n}\}$ be a semi-simple CohFT, whose restriction to $\mathcal{M}_{g,n}$ is described by the sequence $\{s_i\}_{i=1}^N$ and by the symplectic transformation $R(z)$. Then*

$$\mathcal{A}_{s,R}=\widehat{R}^{-1}\mathcal{A}_{s,\mathrm{Id}}.$$

Proof. Write $R(z)=e^{A(z)}$ and set $\mathcal{A}_t=\mathcal{A}_{s,e^{tA}}$. By the Theorem we have $\partial_t\mathcal{A}_t=-\widehat{A}\mathcal{A}_t$. Solving this equation for t and using the initial condition $\mathcal{A}_0=\mathcal{A}_{s,\mathrm{Id}}$ proves the Corollary. □

6. Semi-simple CohFT II

6.1. **The quantization operator of Givental.** Assume that H is a vector space equipped with a Frobenius structure. Let $\bullet_t$, $t \in H$ be the corresponding multiplication in T_tH, $(\tau^1, \dots, \tau^N)$ flat coordinate system on H. We denote the flat vector fields $\partial/\partial\tau^a$ by ∂_a. Finally, let E be the corresponding Euler vector field.

Definition 6.1. *The Frobenius structure is called* semi-simple *if there are local coordinates, called* canonical, $\{u^i\}_{i=1}^N$ *near some point* $t_0 \in H$ *such that*

$$\frac{\partial}{\partial u^i} \bullet_t \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^j}, \quad \textit{for all } t \textit{ near } t_0.$$

Note that in canonical coordinates, due to the Frobenius property, the flat pairing takes the form

$$\Big(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\Big) = \delta_{ij}\ \frac{1}{\Delta_i},$$

where Δ_i are some functions, defined in a neighborhood of t and taking only non-zero values.

The canonical coordinates determine a trivialization of the tangent bundle

$$\Psi(t) : \mathbb{C}^N \to T_tH, \quad e_i \mapsto \sqrt{\Delta_i}\partial/\partial u^i, \quad 1 \le i \le N.$$

Exercise 1. Put $A = \sum_{a=1}^N (\partial_a \bullet) d\tau^a$. Prove that $\Psi(t)^{-1} A \Psi(t) = dU(t)$, where $U(t)$ is the diagonal $N \times N$ matrix whose diagonal entries are $u^1(t), \dots, u^N(t)$.

Consider a formal series of the follwoing type

$$R(t,z) = 1 + R_1(t)z + R_2(t)z^2 + \dots, \quad R_k \in \mathrm{End}(\mathbb{C}^N).$$

It is easy to check that the following systems of differential equations

$$\partial_a\ R(t,z) = z^{-1}[\partial_a U(t), R(t,z)] - \Psi^{-1}\,(\partial_a \Psi)\,R, \quad 1 \le a \le N \tag{6.1}$$

$$z\partial z\, R(t,z) = z^{-1}[R(t,z), U(t)] + V(t)R(t,z), \quad V(t) = \Psi^{-1}(t)\,\mu\,\Psi(t), \tag{6.2}$$

and

$$\partial_a\ \Psi R e^{U/z} = (\partial_a \bullet_t)\Psi R e^{U/z}, \quad 1 \le a \le N$$
$$(z\partial_z + L_E)\Psi R e^{U/z} = \mu\ \Psi R e^{U/z},$$

where μ is the Hodge grading operator, are equivalent.

Theorem 6.2 ([5])**.** *There exists a unique series $R(t,z)$ such that $R(t,z)$ satisfies the differential equations (6.1) and (6.2). Moreover, the series R is a symplectic transformation, i.e., $R(t,-z)^T R(t,z) = 1$.*

It is easy to chek that the differential equations (6.1) are equivalent to

$$\partial_a\, \widetilde{R}(t,z) = z^{-1}[\partial_a \bullet_t, \widetilde{R}(t,z)] - \widetilde{R}(t,z)\,\partial_a \Psi\, \Psi^{-1}, \tag{6.3}$$

where $\widetilde{R} := \Psi^{-1} R \Psi$.

6.2. **Deformations of CohFT.** Let $\overline{Z}_{g,n}$ be a semi-simple CohFT. In the previous lecture we proved that the restriction of $\overline{Z}_{g,n}$ to $\mathcal{M}_{g,n}$ has the following form

$$Z_{g,n}(\phi_{a_1},\dots,\phi_{a_n}) = \Big(P^g e^{\sum_{k=1}^\infty s_k\kappa_k}, \widetilde{R}^{-1}(\psi_1)\phi_{a_1}\bullet\dots\bullet\widetilde{R}^{-1}(\psi_n)\phi_{a_n}\Big)$$

where $s_k\in H$ and $\widetilde{R}$ is a symplectic tranformation of $\mathcal{H}$. From now on we will assume that the flat identity axiom holds, i.e.,

$$\pi^*\overline{Z}_{g,n}(\phi_{a_1},\dots,\phi_{a_n}) = \overline{Z}_{g,n+1}(\phi_{a_1},\dots,\phi_{a_n},1)$$

where $\pi:\overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}$ is the universal curve. In particular, we have

$$\pi^*Z_{g,n}(\phi_{a_1},\dots,\phi_{a_n}) = Z_{g,n+1}(\phi_{a_1},\dots,\phi_{a_n},1).$$

Using that $\pi^*(\kappa_k)=\kappa_k-\psi_{n+1}^k$ we obtain the following relation:

$$\widetilde{R}^{-1}(t,z)\,1 = e^{-\sum_{k=1}^\infty s_kz^k}. \tag{6.4}$$

Given a formal parameter $\tau\in H$ we set

$$\overline{Z}_{\tau,g,n} = \sum_{l=0}^\infty \frac{1}{l!}\pi_*\overline{Z}_{g,n+l}(\phi_{a_1},\dots,\phi_{a_n},\tau,\dots,\tau),$$

where π is the morphism forgetting the last l marked points and contracting the unstable components. It is easy to check that $\overline{Z}_{\tau,g,n}$ is a CohFT and therefore we have a Frobenius multiplication $\bullet_\tau$ and a symplectic transformation $\widetilde{R}(t,z)$. Moreover, the family of Frobenius multiplications $\bullet_\tau$ forms a Frobenius structure. We are going to assume that this Frobenius structure is semi-simple and assume the same notations $\Psi(t)$, $u^1,\dots,u^N$ as in the previous subsection.

Proposition 6.3. *The operator $\widetilde{R}(t,z)$ coincides with Givental's quantization operator.*

Proof. We need to check that $R=\Psi^{-1}\widetilde{R}\Psi$ satisfies the differential equations (6.1) and (6.2). We will verify only (6.1) and leave (6.2) as an exercise.

So we need to prove (6.3) or equivalently

$$\partial_i(\widetilde{R}^{-1}) = z^{-1}[\phi_i\bullet_\tau,\widetilde{R}^{-1}] + [\phi_i\bullet,\widetilde{R}_1(\tau)]\widetilde{R}^{-1}.$$

Condsider the following diagram

$$\begin{array}{ccccc}
\mathcal{M}_{g,2}\times\mathcal{M}_{0,3} & \xrightarrow{b} & \iota^*\overline{\mathcal{M}}_{g,3} & \xrightarrow{\widetilde{\iota}} & \overline{\mathcal{M}}_{g,3} \\
 & & \Big\downarrow{\scriptstyle\widetilde{\pi}} & & \Big\downarrow{\scriptstyle\pi} \\
 & & \mathcal{M}_{g,2} & \xrightarrow[\iota]{} & \overline{\mathcal{M}}_{g,2}
\end{array}$$

where π is the universal curve and b is a boundary morphism. By definition

$$\partial_i \overline{Z}_{\tau,g,2}(\phi_a,\phi_b) = \pi_* \overline{Z}_{\tau,g,3}(\phi_a,\phi_b,\phi_i). \tag{6.5}$$

The goal now is to compute the restriction of this identity to $\mathcal{M}_{g,2}$. For the LHS we have

$$\iota^*\partial_i \overline{Z}_{\tau,g,2}(\phi_a,\phi_b) = \partial_i Z_{\tau,g,2}(\phi_a,\phi_b) = \partial_i \Big(E_\tau\, ,\, (\widetilde{R}_\tau^{-1}(\psi_1)\phi_a) \bullet_\tau (\widetilde{R}_\tau^{-1}(\psi_2)\phi_b)\Big),$$

where $E_\tau = P^g e^{\sum_{k=1}^\infty s_k\kappa_k} \in H^*(\mathcal{M}_{g,2})$.

In order to find the restriction of the RHS in (6.5) we set

$$\overline{Z}_{\tau,g,3}(\phi_a,\phi_b,\phi_i) = \alpha + Z_{\tau,g,3}(\phi_a,\phi_b,\phi_i)$$

where $\alpha := \overline{Z}_{\tau,g,3}(\phi_a,\phi_b,\phi_i) - Z_{\tau,g,3}(\phi_a,\phi_b,\phi_i)$ is a cohomology class on $\overline{\mathcal{M}}_{g,3}$ supported on the codimension-1 stratum consisting of Riemann surfaces with at least one nodal point. This implies that (see the formula for $\overline{Z}^{(1)}_{g,3}$ from the previous lecture) $\widetilde{\iota}^*\alpha$ is a sum of two boundary terms

$$b_*\left(\Big(E_\tau, \widetilde{R}^{-1}(\tau,\psi_1')\phi_a \bullet_\tau \phi_\mu\Big)\ \Big(\frac{1-\widetilde{R}^{-1}(\tau,\psi_2')^T\widetilde{R}^{-1}(\tau,\psi_1'')}{\psi_2'+\psi_1''}\ \phi^\mu,\phi^\nu\Big)\ (\phi_\nu,\phi_b\bullet_\tau\phi_i)\right)$$

and

$$b_*\left(\Big(E_\tau, \widetilde{R}^{-1}(\tau,\psi_1')\phi_b \bullet_\tau \phi_\mu\Big)\ \Big(\frac{1-\widetilde{R}^{-1}(\tau,\psi_2')^T\widetilde{R}^{-1}(\tau,\psi_1'')}{\psi_2'+\psi_1''}\ \phi^\mu,\phi^\nu\Big)\ (\phi_\nu,\phi_a\bullet_\tau\phi_i)\right).$$

Here both boundary morphisms glue the second marked point in $\mathcal{M}_{g,2}$ with the 1-st marked point in $\mathcal{M}_{0,3}$. The difference is only in the enumeration of the marked points after the gluing. Namely, in the first case we obtain nodal Riemann surfaces such that 2-nd and 3-rd marked points are on the genus-0 component, while in the second one the 1-st and the 3-rd marked points are on the genus-0 component.

Note that $\psi_1''=0$ because the moduli space $\mathcal{M}_{0,3}$ is a point. Using that $\iota^*\circ\pi_* = \widetilde{\pi}_*\circ\widetilde{\iota}^*$ we get

$$\iota^*(\pi_*\alpha) = \Big(E_\tau,\quad (\widetilde{R}^{-1}(\tau,\psi_1)\phi_a)\bullet_\tau\Big(\tfrac{1-\widetilde{R}^{-1}(\tau,\psi_2)}{\psi_2}\ (\phi_b\bullet_\tau\phi_i)\Big) + \\ +\Big(\tfrac{1-\widetilde{R}^{-1}(\tau,\psi_1)}{\psi_1}\ (\phi_a\bullet_\tau\phi_i)\Big)\bullet_\tau(\widetilde{R}^{-1}(\tau,\psi_2)\phi_b)\Big)$$

In order to compute the pushforward via π of

$$Z_{\tau,g,3}(\phi_a,\phi_b,\phi_i) = \Big(P^g e^{\sum_{k=1}^\infty s_k\kappa_k}, (\widetilde{R}^{-1}(\tau,\psi_1)\phi_a)\bullet_\tau(\widetilde{R}^{-1}(\tau,\psi_2)\phi_b)\bullet_\tau(\widetilde{R}^{-1}(\tau,\psi_3)\phi_i)\Big) \tag{6.6}$$

we have to use the following identities

$$\kappa_k - \pi^*\kappa_k = \psi_3^k$$

and

$$\widetilde{R}^{-1}(\tau,\psi_j)=\widetilde{R}^{-1}(\tau,\overline{\psi}_j)+\frac{\widetilde{R}^{-1}(\tau,\overline{\psi}_j)-1}{\overline{\psi}_j}\,[D_j],$$

where $\overline{\psi}_j=\pi^*\psi_j$, $j=1,2$ and D_j is the divisor in $\overline{\mathcal{M}}_{g,3}$ consisting of Riemann surfaces that have a genus-0 irreducible components that carries only the j-th and the 3-rd marked points. It is easy to see that the pushforward of (6.6) is

$$\Big(P^g e^{\sum_{k=1}^\infty s_k\kappa_k}\ ,\ \ (\widetilde{R}^{-1}(\psi_1)\phi_a)\bullet_\tau(\widetilde{R}^{-1}(\psi_2)\phi_b)\bullet_\tau\Big(\textstyle\sum_{l=1}^\infty A_l\phi_i\,\kappa_{i-1}\Big)+$$
$$+\Big(\tfrac{\widetilde{R}^{-1}(\psi_1)-1}{\psi_1}\,\phi_a\Big)\bullet_\tau(\widetilde{R}^{-1}(\psi_2)\phi_b)\bullet_\tau\phi_i+$$
$$+(\widetilde{R}^{-1}(\psi_1)\phi_a)\bullet_\tau\Big(\tfrac{\widetilde{R}^{-1}(\psi_1)-1}{\psi_1}\,\phi_b\Big)\bullet_\tau\phi_i\Big),$$

where

$$\sum_{l=0}^\infty A_l z^l=e^{\sum_{k=1}^\infty(s_k\bullet_\tau)z^k}\widetilde{R}^{-1}(z).$$

Combining this formula and the formula for $\iota^*\pi_*\alpha$ we get that the restriction of the differential equation (6.5) to $\mathcal{M}_{g,2}$ is

$$\partial_i\Big(E_\tau,(\widetilde{R}^{-1}(\psi_1)\phi_a)\bullet_\tau(\widetilde{R}^{-1}(\psi_2)\phi_b)\Big)=$$

$$\Big(E_\tau\ ,\ \ (\widetilde{R}^{-1}(\psi_1)\phi_a)\bullet_\tau(\widetilde{R}^{-1}(\psi_2)\phi_b)\bullet_\tau\Big(\textstyle\sum_{l=1}^\infty A_l\phi_i\,\kappa_{i-1}\Big)+$$
$$+\Big([\phi_i\bullet_\tau,\tfrac{\widetilde{R}^{-1}(\psi_1)}{\psi_1}]\,\phi_a\Big)\bullet_\tau(\widetilde{R}^{-1}(\psi_2)\phi_b)+$$
$$+(\widetilde{R}^{-1}(\psi_1)\phi_a)\bullet_\tau\Big([\phi_i\bullet_\tau,\tfrac{\widetilde{R}^{-1}(\psi_1)}{\psi_1}]\,\phi_b\Big)\Big),$$

Now the proposition follows easily. Namely, first set $\psi_1=\psi_2=0$. We get

$$\partial_i(E_\tau,\phi_a\bullet_\tau\phi_b)=\Big(E_\tau\ ,\ \ \phi_a\bullet_\tau\phi_b\bullet_\tau\Big(\textstyle\sum_{l=1}^\infty A_l\phi_i\,\kappa_{i-1}\Big)+$$
$$+([\phi_i\bullet_\tau,-\widetilde{R}_1]\,\phi_a)\bullet_\tau\phi_b)+$$
$$+\phi_a\bullet_\tau([\phi_i\bullet_\tau,-\widetilde{R}_1]\,\phi_b)\Big).$$

To finish the proof simply put $\psi_2=0$ and write the LHS in the following way:

$$\partial_i\Big(\sum_{c=1}^N(E_\tau,\phi_c\bullet_\tau\phi_b)(\widetilde{R}^{-1}(\psi_1)\phi_a,\phi^c)\Big).$$

It remains only to apply the product rule and to use the above formula with c instead of a. □

6.3. **Removing the κ-classes.** Recall that the total ancestor potential is by definition

$$\mathcal{A}_{s,\widetilde{R}^{-1}}(\mathbf{q}) = \exp \sum_{g,n} \frac{\hbar^{g-1}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \overline{Z}_{g,n}(\mathbf{q},\dots,\mathbf{q}).$$

According to Corollary 5.9 we have

$$\mathcal{A}_{s,\widetilde{R}^{-1}}(\mathbf{q}) = (\widetilde{R})\widehat{\ }\mathcal{A}_{s,Id}(\mathbf{q}).$$

Our goal now is to compute $\mathcal{A}_{s,Id}(\mathbf{q})$. When $R = Id$, the CohFT is given by the following formulas

$$\overline{Z}_{g,n} = (P^g e^{\sum_{k=1}^\infty s_k\kappa_k}, \mathbf{q}\bullet\cdots\bullet\mathbf{q}).$$

Put

$$\mathbf{q}(z) = \sum_{i=1}^N \mathbf{q}^i(z)\frac{\partial}{\partial u^i}, \quad s_k = \sum_{i=1}^N s_k^i \frac{\partial}{\partial u^i}.$$

Note that the propagator is

$$P = \sum_{i=1}^N \sqrt{\Delta_i}\partial/\partial u^i \quad \Rightarrow \quad P^g = \sum_{i=1}^N \Delta_i^g \partial/\partial u^i.$$

It follows that

$$\overline{Z}_{g,n}(\mathbf{q},\dots,\mathbf{q}) = \sum_{i=1}^N \Delta_i^{g-1} e^{\sum_{k=1}^\infty s_k^i\kappa_k}\mathbf{q}^i(\psi_1)\dots\mathbf{q}^i(\psi_n).$$

Proposition 6.4. *The following formula holds*

$$e^{\sum_{k=1}^\infty s_k^i\kappa_k} = 1 + \sum_{k=1}^\infty \frac{1}{k!}\pi_*\Big(\prod_{j=1}^k (1 - e^{-\sum_{a=1}^\infty s_a^i\psi_{n+j}^a})\psi_{n+j}\Big).$$

The proof of this proposition is a direct consequence from the results in [9]. Namely, the authors derived a formula expressing any polynomial expression in κ classes in terms of pushforward of a polynomial expression in ψ-classes.

Put

$$\mathbf{t}^i(z) = (1 - e^{-\sum_{a=1}^\infty s_a^i z^a})z \ \in \ z^2 H[z].$$

Then we have

$$\int_{\overline{\mathcal{M}}_{g,n}} \overline{Z}_{g,n}(\mathbf{q},\dots,\mathbf{q}) = \sum_{i=1}^N \Delta_i^{g-1}\frac{1}{k!}\int_{\overline{\mathcal{M}}_{g,n+k}} \mathbf{q}^i(\psi_1)\dots\mathbf{q}^i(\psi_n)\mathbf{t}^i(\psi_{n+1})\dots\mathbf{t}^i(\psi_{n+k}).$$

Note that in order to derive this formula we have to use that for each i, $1 \le i \le n$, the difference $\psi_i - \pi^*\psi_i$ is anihilated by ψ_{n+1}. Therefore, since $\mathbf{t}^i(\psi_{n+1})$ is divisible by ψ_{n+1}, we can replace $\pi^*(\psi_i)$ with ψ_i without changing the value of the integral.

From here we get that

$$\mathcal{A}_{s,Id}(\mathbf{q}) = \prod_{i=1}^{N} \mathcal{D}_{\rm pt}(\Delta_i\hbar, \mathbf{q}^i(z) + \mathbf{t}^i(z)),$$

where

$$\sum_{i=1}^{N} \mathbf{t}^i(z)\frac{\partial}{\partial u^i} = (1 - e^{-\sum_{k=1}^{\infty} s_k z^k})z = (1 - \widetilde{R}^{-1}(z)\,1)z = z - \widetilde{R}^{-1}(z)\,z.$$

Here the second equality follows from the flat identity axiom (see (6.4)).

Lemma 6.5. *Assume that $a(z) \in H[z]$ is an arbitrary vector. Let $T_a\mathcal{F}(\mathbf{q}) = \mathcal{F}(\mathbf{q} + a)$ be the translation operator. Then*

$$T_a\ \widehat{R} = \widehat{R}\ T_{R^{-1}a}.$$

The proof of this lemma is left as an exercise. From here we get

$$\mathcal{A}_{s,\widetilde{R}^{-1}}(\mathbf{q}(z) + z) = T_z\,\mathcal{A}_{s,\widetilde{R}^{-1}}(\mathbf{q}(z)) = T_z\ (\widetilde{R})^{\wedge}\mathcal{A}_{s,Id}(\mathbf{q}(z)) = (\widetilde{R})^{\wedge}T_{\widetilde{R}^{-1}z}\ \mathcal{A}_{s,Id}.$$

Note that

$$\widetilde{R}^{-1}z = \sum_{i=1}^{N}(z - \mathbf{t}^i(z))\,\frac{\partial}{\partial u^i}.$$

Therefore,

$$T_{\widetilde{R}^{-1}z}\ \mathcal{A}_{s,Id}(\mathbf{q}(z)) = \prod_{i=1}^{N} \mathcal{D}_{\rm pt}(\sqrt{\Delta_i}\hbar; \mathbf{q}^i(z) + z).$$

6.4. **Givental's formula.** Cohomological field theories arise in Gromov–Witten theory in the following way. Let X be a projective manifold. Then we define

$$\overline{Z}_{\tau,g,n}(a_1,\dots,a_n) = \sum_{l=0}^{\infty}\sum_{d\in MC(X)} \frac{1}{l!}Q^d\pi_*(\mathrm{ev}^*(a_1\otimes\ldots\otimes a_n\otimes\tau^{\otimes l}),$$

where $\pi : X_{g,n+l,d} \to \overline{\mathcal{M}}_{g,n}$ is the map that forgets the stable map and the last l marked points and conttracts the unstable components. It is straightforward to check that this is a CohFT. Moreover, the total ancestor potential of the CohFT coincides with the total ancestor potential of the manifold X. Therefore, we have the following formula, which was conjectured by Givental (see [5])

Theorem 6.6. *Assume that the quantum cohomology is semi-simple. Then*

$$\mathcal{D}_X(\mathbf{q}) = e^{F^{(1)}(\tau)}\widehat{S}(\tau,z)^{-1}\ (\widetilde{R}(\tau,z))^{\wedge}\prod_{i=1}^{N}\mathcal{D}_{\rm pt}(\sqrt{\Delta_i}\hbar; \mathbf{q}^i),$$

where the generating functions are identified with elements of the Fock space via the dilaton shift.

7. Singularity theory

Givental's formula makes sense for any semi-simple Frobenius manifold. It is known that this formula is always a highest weight vector for the Virasoro algebra. One of the main open questions is whether one can associate an integrable hierarchy with any semi-simple Frobenius manifold. If yes then is it true that Givental's formula is a tau-function.

In the remaining lectures, we will address this question in the settings of singularity theory. In particular, we will describe completely the case of simple singularities.

7.1. **Frobenius structures.** Let $f : (\mathbb{C}^{2l+1}, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function with an isolated critical point at 0.

Definition 7.1. *The family of holomorphic functions*

$$F : S \times \mathbb{C}^{2l+1} \to \mathbb{C},$$

where $S \subset \mathbb{C}^N$ is a small ball with center the origin, is called a miniversal deformation *of f if*

(1) $F(0, x) = f(x)$ *for all* $x \in \mathbb{C}^{2l+1}$.
(2) *The partial derivatives*

$$\frac{\partial F}{\partial t^i}(0, x), \quad 1 \le i \le N$$

represent a basis in the local algebra

$$H := \mathbb{C}[[x_0, \ldots, x_{2l}]]/\langle \partial_{x_0} f, \ldots, \partial_{x_{2l}} f\rangle.$$

A miniversal deformation always exists: it is enough to pick $F = f + t^1 g_1 + \cdots + t^N g_N$ where $\{g_i\}_{i=1}^N$ represents a basis in the local algebra H.

In what follows we denote by B_r^n the ball in $\mathbb{C}^n$ with center 0 and radius r. We pick $\rho > 0$ so small that the fiber $f^{-1}(0)$ interesects the boundary of B_r^{2l+1} transversely for every $0 < r \le \rho$. Given $t \in S$ we denote by $f_t = F|_{\{t\}\times\mathbb{C}^{2l+1}}$. Choose δ and S so small that $f_t^{-1}(\lambda)$ intersects transversely the boundary of B_ρ^{2l+1} for all $(t, \lambda) \in S \times B_\delta^1$.

Let

$$V = \left\{\ (t, x) \in S \times B_\rho^{2l+1} \mid F(t, x) \in B_\delta^1 \right\}.$$

The map

$$\partial/\partial t^i \quad \mapsto \quad \frac{\partial F}{\partial t^i} \ \operatorname{mod}\Big(\frac{\partial F}{\partial x^0}, \ldots, \frac{\partial F}{\partial x^{2l}}\Big)$$

gives an isomorphism between sheaves

$$\mathcal{T}_S \cong p_* \mathcal{O}_V / \Big\langle \frac{\partial F}{\partial x^0}, \ldots, \frac{\partial F}{\partial x^{2l}} \Big\rangle, \tag{7.1}$$

where $p : V \to S$ is induced from the projection $S \times B^{2l+1} \to S$. Using this isomorphism we equip each tangent space T_tS with a multiplication $\bullet_t$.

Given a holomorphic volume form

$$\omega = g(t, x)dx^0 \wedge \cdots \wedge dx^{2l}, \quad g(t, 0) \neq 0,$$

we introduce the following residue pairing

$$(\partial/\partial t^i, \partial/\partial t^j)_t = \frac{1}{(2\pi i)^{2l+1}} \int_{|\partial_x F|=\epsilon} \frac{\partial_{t^i} F g(t,x) \partial_{t^j} F g(t,x)}{\partial_{x^0} F \ldots \partial_{x^{2l}} F} dx^0 \wedge \cdots \wedge dx^{2l}.$$

It is independent of the choice of the coordinate system $(x^0, \ldots, x^{2l})$ (see [8]).

We introduce the oscillating integral

$$(J_{\mathcal{B}}(t, z), \partial/\partial t^i) = (-2\pi z)^{-l-\frac{1}{2}} \ (z\partial_{t^i}) \ \int_{\mathcal{B}} e^{F(t,x)/z} \omega \tag{7.2}$$

where the integration cycle $\mathcal{B}$ is an element of the homology group

$$\lim_{M\to\infty} \ H_{2l+1}(\mathbb{C}^{2l+1}, \mathrm{Re}(f_t/z) < -M; \mathbb{C}).$$

We view $J_{\mathcal{B}}$ as a section of the cotangent bundle which via the residue pairing is identified with the tangent bundle.

Theorem 7.2 (K. Saito, M. Saito)**.** *There exists a volume form ω such that the oscillating integral satisfies the following system of differential equations:*

$$z\nabla^{\mathrm{L.C.}}_{\partial/\partial t^i} J_{\mathcal{B}} = \partial_{t^i} \bullet_t J_{\mathcal{B}}, \quad 1 \le i \le N \tag{7.3}$$

$$(z\partial_z + \nabla^{\mathrm{L.C.}}_E) J_{\mathcal{B}} = \mu J_{\mathcal{B}} \tag{7.4}$$

Here E is the vector field which under the identification (7.1) corresponds to the function F. The last equation expresses homogeneity properties of the oscillating integral.

It follows that the residue pairing is flat. We denote by $(\tau^1, \ldots, \tau^N)$ a flat coordinate system on S and set $\partial_a := \partial/\partial\tau^a$. It can be proved that in an appropriately chosen flat coordinate system, the Euler vector field has the form

$$E = \sum_{a=1}^{N} (1 - d_a)\tau^a \partial_a + \sum_{a=1}^{N} r_a \partial_a,$$

where the degree spectrum d_a is in the interval $[0, D]$ (the minimal degree is 0 and the maximal one is D). In this case the Hodge grading operator is

$$\mu(\partial_a) = (D/2 - d_a)\partial_a, \quad 1 \le a \le N.$$

Theorem 7.3 (Hertling)**.** *The residue metric, the multiplication $\bullet_t$, and the Euler vector field form a Frobenius strcture on S of conformal dimension D.*

Proposition 7.4. *If $t \in S$ is a sufficiently generic point then the critical values $u^i(t), 1 \leq i \leq N$ form a canonical coordinate system, i.e.,*

$$\partial/\partial u^i \bullet \partial/\partial u^j = \delta_{ij}\partial/\partial u^j, \quad (\partial/\partial u^i, \partial/\partial u^j) = \delta_{ij}/\Delta_i.$$

Proof. Let $t \in S$ be such that f_t is a Morse function and its critical values $u^i(t)$ form a coordinate system. By definition

$$(\partial/\partial u^i, \partial/\partial u^j) = \frac{1}{(2\pi i)^{2l+1}} \int_{|\partial_x F|=\epsilon} \frac{\partial_{u_i} F g(t,x) \partial_{u_j} F g(t,x)}{F'_{x_0} \dots F'_{x_{2l}}} dx_0 \wedge \cdots \wedge dx_{2l}.$$

The residue on the RHS equals sum of the residues at the critical points ξ_k ($1 \leq k \leq N$) of f_t. Let $y_0, \dots, y_{2l}$ be a Morse coordinate system near $x = \xi_k$, i.e.,

$$f_t = u^k + \frac{1}{2}(y_0^2 + \cdots + y_{2l}^2).$$

We get

$$\partial_{u^i} f_t = \delta_{ik} + O(y) \quad \text{and} \quad \partial_{u^j} f_t = \delta_{jk} + O(y).$$

On the other hand the residue pairing is independent of the choice of coordinate system. Therefore, we can compute the residue at $x = \xi_k$ by switching to the Morse coordinates. It follows that the residue at $x = \xi_k$ equals

$$\frac{1}{(2\pi i)^{2l+1}} \int_{|y|=\epsilon} \frac{\delta_{ik}\delta_{jk} a_k^2 + O(y)}{y_0 \dots y_{2l}} dy_0 \wedge \cdots \wedge dy_{2l} = \delta_{ik}\delta_{jk} a_k^2,$$

where $a_k = g(t, \xi_k)$. This implies that

$$(\partial/\partial u^i, \partial/\partial u^j) = \sum_{k=1}^N \delta_{ik}\delta_{jk} a_k^2 = \delta_{ij} a_i^2.$$

By definition

$$(\partial/\partial u^i \bullet \partial/\partial u^j, \partial/\partial u^k) = \frac{1}{(2\pi i)^{2l+1}} \int_{|\partial_x F|=\epsilon} \frac{\partial_{u_i} F \partial_{u_j} F g(t,x) \partial_{u_k} F g(t,x)}{F'_{x_0} \dots F'_{x_{2l}}} dx_0 \wedge \cdots \wedge dx_{2l}.$$

Choosing Morse coordinates $y_0, \dots, y_{2l}$ near the critical point $x = \xi_m$ we get that the residue at ξ_m is

$$\delta_{im}\delta_{jm}\delta_{km} a_m^2,$$

so

$$(\partial/\partial u^i \bullet \partial/\partial u^j, \partial/\partial u^k) = \sum_{m=1}^N \delta_{im}\delta_{jm}\delta_{km} a_m^2 = \delta_{ij}(\partial/\partial u^i, \partial/\partial u^k).$$

□

Note that in particular we proved the following fact. Let t be a generic point such that the critical values $\{u^i\}_{i=1}^N$ form a coordinate system. Let ξ_i be the critical point of f_t corresponding to the critical value u^i, then

$$\Delta_i = (g(t, \xi_i))^2. \tag{7.5}$$

7.2. **The Milnor fibration.** Put $V_{t,\lambda} := f_t^{-1}(\lambda) \cap B_\rho^{2l+1}$. According to our choices of S, ρ, and δ, the boundaries of $V_{t,\lambda}$ are smooth manifolds. They form a smooth fibration over $S \times B_\delta^1$, which must be trivial, because $S \times B_\delta^1$ is contractible.

Let $\Sigma \subset S \times B_\delta^1$ be the set of all pairs (t, λ) such that the fiber $V_{t,\lambda}$ is singular, i.e., λ is a critical value of f_t. The collection of all fibers

$$\bigcup \Big\{ V_{t,\lambda} \quad | \quad (t,\lambda) \in S \times B_\delta^1 - \Sigma \Big\}$$

forms a smooth fibration over $S \times B_\delta^1 - \Sigma$ called *the Milnor fibration.*

Now we would like to describe the so called *vanishing cycles.* Let t be a generic point, such that the function f_t has N different Morse type critical points. Let

$$c : [0,1] \to S \times B_\delta^1 - \Sigma, \quad c(0) = (0,1), \quad c(1) = (t, u(t)) \in \Sigma$$

be a path. Here $u(t) = f_t(\xi)$ is a critical value of f_t. We assume that $c(s) = (t, \lambda(s))$ for s sufficiently close to 1. Near the point $x = \xi$ we pick a Morse coordinate system $(y_0, \dots, y_{2l})$, so that the function f_t takes the form:

$$f_t = u + \frac{1}{2}(y_0^2 + \cdots + y_{2l}^2).$$

Set $y_k = (q_k + \sqrt{-1}p_k)\sqrt{2(\lambda - u)}$. Then the equation

$$y_0^2 + \cdots + y_{2l}^2 = 2(\lambda - u^i)$$

is equivalent to

$$\sum_{k=0}^{2l} q_k^2 - p_k^2 = 1, \quad \text{and} \quad \sum_{k=0}^{2l} q_k p_k = 0. \tag{7.6}$$

On the other hand the map

$$(q,p) \mapsto \Big(\frac{q}{1 + \sum_k p_k^2} \; , \; p \Big)$$

identifies (7.6) with the tangent bundle TS^{2l} of the unit sphere. In other words for each s sufficiently close to 1, we obtain a map

$$b(s) : D(TS^{2l}) \to V_{c_i(s)},$$

where $D(TS^{2l})$ is a disk bundle associated to the tangent bundle. Note that $b(1)$ is a constant map – it contracts the disk bundle to a point. Using the homotopy lifting property, we obtain a map

$$b(s) : D(TS^{2l}) \to V_{c(s)}, \quad \text{for all} \;\; s \in [0,1].$$

The cycle $b(0)[S^{2l}] \in H_{2l}(V_{0,1}; \mathbb{Z})$ is called *vanishing cycle.*

Proposition 7.5. *Let $t \in S$ be a generic point and $c_i(s)$, $1 \le i \le N$ is a set of paths starting at $(0,1)$ and terminating at the points $(t, u^i(t))$, where u^i are the critical values of f_t. Then the homology group $H_{2l}(V_{0,1};\mathbb{Z})$ is spanned over $\mathbb{Z}$ by the corresponding vanishing cycles.*

Given a loop $\gamma \in \pi_1(S \times B^1_\delta - \Sigma)$ based at (t,λ) we obtain (using the homotopy lifting property) a map

$$h_\gamma : V_{t,\lambda} \to V_{t,\lambda}, \quad h_\gamma|_{\partial V_{t,\lambda}} = \mathrm{Id}.$$

The map h_γ is unique up to homotopy and it is called *geometric monodromy.* We have an induced map $h_{\gamma *}$ on homology and cohomology and the set of all such transformations forms a group called *the monodromy group* of the singularity.

Let γ be a path starting form $(0,1)$, avoiding the discriminant and terminating at a generic point on the discriminant. Let $\beta \in H_{2l}(V_{0,1};\mathbb{Z})$ be a corresponding vanishing cycle.

Lemma 7.6. *The self-intersection index $\beta \circ \beta$ is $(-1)^l 2$.*

Proof. Indeed β is the zero section of the tangent bundle TS^{2l} which is known to have self-intersection index equal to the Euler characteristic of the sphere S^{2l} which is 2. The sign $(-1)^l$ comes from the difference in the orientations. Namely, the local coordinates on $V_{t,\lambda}$ are given by $y_1, \dots, y_{2l}$, i.e., $q_1, p_1, \dots, q_{2l}, p_{2l}$, while the local coordinates on TS^{2l} are $q_1, \dots, q_{2l}, p_1, \dots, p_{2l}$. □

Let

$$(\alpha|\beta) = (-1)^l (\alpha \circ \beta), \quad \alpha, \beta \in H_{2l}(V_{0,1};\mathbb{C})$$

be the intersection form normalized by a sign, so that the self-intersection of a vanishing cycle is 2. Slightly abusing the notations, we denote by γ the path that coincides with γ except that at the end isntead of approaching a point on the discriminant, it makes a small loop around it.

Proposition 7.7 (Picard-Lefschetz formula)**.** *The following formula holds*

$$h_{\gamma *}(x) = x - (\alpha|x)\alpha, \quad x \in H_{2l}(V_{0,1};\mathbb{C}).$$

Definition 7.8. *We say that the singularity is simple of type X_N, $X = ADE$ if the vansihing cycles and the intersection form $(\ |\ )$ form a root system of type X_N.*

For more details and for the proves of the Propositions in this section we refer to the book [1].

7.3. **The Leray periods.** Let $\alpha \in H_{2l}(V_{0,1};\mathbb{C})$ be a middle homology cycle. We denote by $\alpha_{t,\lambda} \in H_{2l}(V_{t,\lambda};\mathbb{C})$ the cycle obtained from α via a parallel transport along some path connecting $(0,1)$ and (t,λ). Let $d^{-1}\omega$ be any holomorphic $2l$-form on $\mathbb{C}^{2l+1}$ (possibly depending on t) whose De Rham differential is the primitive form ω. For each $k\in\mathbb{Z}$ we associate the following *period vector*:

$$(I^{(k)}_{\alpha}(t,\lambda),\partial_a) = (2\pi)^{-l}\,(-\partial_a)\,(\partial_\lambda)^{k+l}\int_{\alpha_{t,\lambda}} d^{-1}\omega,\quad 1\le a\le N. \tag{7.7}$$

This definition is consistent with the operation of stabilization of the singularity. Namely, the following lemma holds

Lemma 7.9. *Let* $\widetilde{f} = f+\frac{1}{2}(y_1^2+y_2^2)$, $\quad\widetilde{\omega}=\omega\wedge dy_1\wedge dy_2$. *Then*

$$\int_{\alpha_{t,\lambda}} d^{-1}\omega = (2\pi)^{-1}\partial_\lambda\int_{\widetilde{\alpha}_{t,\lambda}} d^{-1}\widetilde{\omega}$$

Proof. Note that $\widetilde{f}_t := f_t+\frac{1}{2}(y_1^2+y_2^2)$ is a miniversal deformation of f_t. Let $U_\lambda = \{(y_1,y_2)\mid y_1^2+y_2^2=2\lambda\}$ be the fibers of the Milnor fibration for the A_1 singularity. It is known (see [1]) that the Milnor fiber

$$\widetilde{V}_{t,\lambda} := \widetilde{f}_t^{-1}(\lambda)\cap B^{2(l+1)+1}_{\widetilde{\rho}}$$

is homotopic to the joint

$$V_{t,\lambda}*U_\lambda = V_{t,\lambda}\times[0,1]\times U_\lambda/\sim,$$

where the equivalence relation is

$$(x,0,y)\sim(x',0,y),\quad (x,1,y)\sim(x,1,y'),\quad \text{for all } x,x'\in V_{t,\lambda},\ y,y'\in U_\lambda.$$

In fact a map $g: V_{t,\lambda}*U_\lambda\to\widetilde{V}_{t,\lambda}$ that induces a homotopy equivalence can be constructed as follows. First, since $V_{t,\lambda}\simeq V_{0,\lambda}$ we may assume that $t=0$. Fix a path $c:[0,1]\to B^1_\delta$ connecting 0 and λ. There exists a continuous family of continuous maps

$$h_s: V_{0,\lambda}\to V_{0,\,c(s)},\quad \text{s.t.,}\quad h_0(V_{0,\lambda})=0\in\mathbb{C}^{2l+1},\quad h_1=\mathrm{Id}.$$

Put

$$g(x,s,y) = (h_s(x),(2-2c(s)/\lambda)^{1/2}y).$$

By definition the vanishing cycle $\varphi\in H_1(U_\lambda;\mathbb{Z})$ is given by the following equations:

$$\varphi = \{(\sqrt{2\lambda}y_1,\sqrt{2\lambda}y_2)\mid y_1^2+y_2^2=1, y_1,y_2\in\mathbb{R}\}.$$

Therefore, the vanishing cycle $\widetilde{\alpha}=\alpha*\varphi$ is the union of

$$\alpha_{0,c(s)}\times(\sqrt{\lambda-c(s)}\,y_1,\sqrt{\lambda-c(s)}\,y_2)\ \ ,\quad 0\le s\le 1.$$

We have

$$\int_{\widetilde{\alpha}_{0,c(s)}} y_1 dy_2 \wedge \omega = \int_0^1 2(\lambda - c(s)) \int_{S^1} y_1 dy_2 \int_{\alpha_{0,c(s)}} \omega \ ds.$$

The integral $\int_{S^1} y_1 dy_2 = \pi$. Note that the union of all $\alpha_{0,c(s)},\ 0 \le s \le 1$ is a relative homology cycle $L \in H_{2l}(V, V_{0,\lambda}; \mathbb{Z})$. Therefore we get

$$2\int_0^1 (\lambda - c(s)) \int_{\alpha_{0,c(s)}} \omega \ ds = 2\int_L (\lambda - f(x))\omega = 2\int_{\alpha_{0,\lambda}} d^{-1}((\lambda - f(x))\omega),$$

where for the last equality we used the Stoke's theorem. The derivative with respect to λ of this integral is

$$2\int_{\alpha_{0,\lambda}} (\lambda - f(x))\frac{\omega}{df} + 2\int_{\alpha_{0,\lambda}} d^{-1}\omega = 2\int_{\alpha_{0,\lambda}} d^{-1}\omega.$$

The lemma follows. □

From this lemma we get that in the definition (7.7) of the period vectors we can take l as large as we wish. In particular, the period vectors can be defined unambiguously for all negative values of k.

7.4. **Stationary phase asymptotic.** Let $t \in S$ be a generic value such that f_t is a Morse function and its critical values $\{u^i(t)\}$ form a canonical coordinate system. Let $\mathcal{B}_i$ be the cycle in $\mathbb{C}^{2l+1}$ swept by the flat family of cycles $\beta_i(t,\lambda) \in H_{2l}(V_{t,\lambda}; \mathbb{Z})$ parametrized by the points λ of a semi-infinite path C in $\mathbb{C}$ starting at the critical value $u^i(t)$ and such that $\mathrm{Re}(\lambda/z) \to -\infty$ when $\lambda \to \infty$ along C. Assume also that when λ is close to $u^i(t)$ then the cycle β_i coincides with the vanishing cycle corresponding to the generic point $(t, u^i(t)) \in \Sigma$.

Lemma 7.10. *The oscillating integral (7.2) is a Laplace transform of the period vectors, i.e.,*

$$J_{\mathcal{B}_i}(t,z) = \frac{1}{\sqrt{-2\pi z}} \int_{u^i}^{\infty} e^{\lambda/z} I^{(0)}_{\beta_i}(t,\lambda)\ d\lambda.$$

The proof here is straightforward and it is left as an exercise.

Lemma 7.11. *Assume that λ is close to the critical value $u^i(t)$ then*

$$I^{(0)}_{\beta_i}(t,\lambda) = \frac{2}{\sqrt{2(\lambda - u^i)}}\Big(\sqrt{\Delta_i}\frac{\partial}{\partial u^i} + \sum_{k=1}^{\infty} A^i_k(t)(2(\lambda - u^i))^k\Big).$$

Proof. By definition

$$(I^{(0)}_{\beta_i}(t,\lambda), \partial_a) = (2\pi)^{-l}(-\partial_a)\partial_\lambda^l \int_{\beta_i} \frac{1}{\sqrt{\Delta_i}} y^0 dy^1 \wedge \cdots \wedge dy^{2l} + \ldots,$$

where $(y^0, \dots, y^{2l})$ is a Morse coordinate system for f_t and the dots stand for higher order terms in y. Here the leading term in the integrand on the RHS was determined in (7.5). Using that the vanishing cycle is

$$\beta_i = \{ \sqrt{2(\lambda - u^i)}(y_0, \dots, y_{2l}) \mid y_0^2 + \dots y_{2l}^2 = 1, y_i \in \mathbb{R} \}$$

we get (we ignore the higher order terms)

$$(2\pi)^{-l}(-\partial_a)\partial_\lambda^l \frac{1}{\sqrt{\Delta_i}} (2(\lambda - u^i))^{l+1/2} \int_{S^{2l}} y_0 dy_1 \dots dy_{2l}. \tag{7.8}$$

Using Stokes theorem we get that the above integral equals the volume of the unit ball, i.e.,

$$\int_{S^{2l}} y_0 dy_1 \dots dy_{2l} = \frac{\pi^l}{(l+1/2)\dots(1/2)}.$$

It follows that the lowest degree term in (7.8) is

$$(2(\lambda - u^i))^{-1/2}\ \partial_a u^i\ \frac{2}{\sqrt{\Delta_i}}.$$

The lemma follows because $du^i/\sqrt{\Delta_i} = \sqrt{\Delta_i}\,\partial/\partial u^i$. □

Recall that Givental's quantization operator

$$\widetilde{R}(t,z) = 1 + \widetilde{R}_1(t)z + \widetilde{R}_2(t)z^2 + \dots, \quad \widetilde{R}_k \in \mathrm{End}(H)$$

is defined as $\widetilde{R} = \Psi R \Psi^{-1}$, where to define R we have to take a formal asymptotical solution $\Psi R e^{U/z}$ that satisfies the differential equations (7.3) and (7.4). The differential equations uniquely determine R. Let us introduce the linear operators

$$A_k(t) : H \to H, \quad A_k(t)\sqrt{\Delta_i}\,\partial/\partial u^i = A_k^i(t),$$

where $A_k^i(t)$ are the vector coefficients that appear in the expansion in Lemma 7.11.

Proposition 7.12. *We have* $\widetilde{R}_k = (2k-1)!!\ (-1)^k\ A_k$.

Proof. Using the previous two lemmas we get

$$J_{\mathcal{B}_i}(t,z) \sim \frac{2}{\sqrt{-2\pi z}} \sum_{k=0}^{\infty} \int_{u^i}^{\infty} e^{\lambda/z} A_k (2(\lambda - u^i))^{k-1/2}\ \sqrt{\Delta_i}\,\partial/\partial u^i$$

Changing the variables

$$(\lambda - u^i)/z = -t^2/z, \quad d\lambda = -ztdt,$$

we get that $J_{\mathcal{B}_i}$ is asymptotic to

$$\frac{2}{\sqrt{-2\pi z}}\, e^{u^i/z}\ \sum_{k=0}^{\infty} (-z)^{k+1/2} \int_0^{\infty} e^{-t^2/2} t^{2k} dt\ A_k\ \sqrt{\Delta_i}\,\partial/\partial u^i =$$

$$e^{u^i/z}\ \sum_{k=0}^{\infty} (2k-1)!!(-z)^k A_k\ \sqrt{\Delta_i}\,\partial/\partial u^i.$$

By definition $\Psi(t)e_i = \sqrt{\Delta_i}\,\partial/\partial u^i$. It follows that

$$R_k = (2k-1)!!(-1)^k\, \Psi^{-1} A_k \Psi.$$

□

8. Vertex operators

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