

Primitive forms and vertex operators

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ABSTRACT. The notion of a primitive form was introduced in the 80's in order to define a period map in singularity theory similar to the classical period map for Riemann surfaces. A decade later, it was discovered that the primitive form plays an important role in the formulation of mirror symmetry. Moreover, the period integrals of the primitive forms found several interesting applications for constructing integrable hierarchies in the form of Hirota Bilinear Equations. This book is an introduction to the theory of primitive forms and its applications to integrable hierarchies. The first part of the book is written more abstractly using the general framework of semi-simple Frobenius manifolds. We introduce a certain set of vertex operators and propagators and establish their general properties. The main result of the book is in chapter 3 where the analytic properties of the propagators are established. The second part of the book is an introduction to the theory of primitive forms. Strictly speaking, we have extended the original framework having in mind the applications to mirror symmetry. Semi-simple Frobenius structures appear quite naturally in the theory of primitive forms and the vertex operators have a clear geometric significance, i.e., their coefficients are period integrals of vanishing cycles. The last chapter is an example of how the constructions from the previous chapters work in the case of simple singularities.

The book could be of interest to people working in Gromov–Witten theory, deformations of complex structures and singularity theory, vertex operator algebras and integrable hierarchies. We have tried to keep the exposition as elementary as possible and to make the text accessible to graduate students and young researchers.

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Preface

The notion of a primitive form was introduced by one of the authors in the early 80's (see [53]) in order to define a period map in singularity theory similar to the classical period map for Riemann surfaces. To give some idea of the definition, let us consider a triple $(\pi : Y \rightarrow B, f, \omega)$ consisting of a family $\pi : Y \rightarrow B$ of complex manifolds, a holomorphic functions $f : Y \rightarrow \mathbb{C}$ and a holomorphic volume form ω on Y relative to B . If ω is a primitive form then the oscillatory integrals $\int e^{f/z} \omega$ provide a solution to a certain family of flat connections on the tangent bundle TB . The flat connection can be described in terms of the residue pairing and the local algebra of f . In particular, the existence of a primitive form implies that the base B of the family carries a special structure which was called by Saito *flat structure*. A decade later, in the early 90's, the theory proposed by Saito had an important application to the so-called mirror symmetry phenomenon. The later has its origin in string theory as a duality between certain classes of Quantum Field Theories. Mathematically, the predictions coming from mirror symmetry are quite striking. Namely, there is a duality between symplectic and complex geometry. Giving a precise mathematical statement of mirror symmetry is not easy. There are several proposals which motivated many developments in both symplectic and complex geometry. It was Givental who noticed that the flat structure of Saito plays an important role in the formulation of mirror symmetry. Let us briefly explain the connection. Suppose that we have a compact Kähler manifold X . Let $\mathcal{M}_{g,n}(X, d)$ be the moduli space of stable maps from a genus- g nodal Riemann surface equipped with n marked points to X , such that, the image of the fundamental class of the Riemann surface has a fixed homology class $d \in H_2(X, \mathbb{Z})$. Such moduli spaces were proposed by Witten [64] in order to give a mathematical definition of the correlation functions in the so-called *topological string model*. Mathematically, the correlation functions are symplectic invariants of X that are now known as the Gromov–Witten (GW) invariants of X . The genus-0 GW invariants of X can be used to define a deformation of the classical cup product known as the *quantum cup product*. The cohomology vector space $H^*(X, \mathbb{C})$ equipped with the quantum cup product is known as the *quantum cohomology of X* . It was Dubrovin who was able to give a geometric formulation of the properties of quantum cohomology. He introduced the notion of a Frobenius manifold which turns out to be an important object not only in GW theory but also in complex geometry and in the theory of integrable systems. It turns out that the flat structure of Saito is an example of a Frobenius manifold. Givental's proposal for mirror symmetry is that the Frobenius structure underlying quantum cohomology is isomorphic to the flat structure of Saito associated to an appropriate family of functions and an appropriate primitive form. Making Givental's proposal a precise mathematical conjecture requires a little bit more work. Namely, the original definition of a primitive form is in the settings of singularity theory where the manifold Y is a germ of a complex

manifold, while in most examples of mirror symmetry Y is a global complex manifold. Therefore, the question is what are the most general settings in which the theory of primitive forms makes sense. This question is partially addressed in chapter 4 of this book. We gave a precise definition for a family of functions, introduced Gauss-Manin connection and higher residue pairing. Although we can define a primitive form, proving its existence in such a generality is quite difficult.

The applications to mirror symmetry will not be pursued in this book. We would like to focus instead on the applications to integrable systems. Let us point out that at the time when the primitive form was invented M. Sato and his students were investigating an approach to the Kadomtsev–Petviashvili (KP) equation based on ideas from quantum field theory. Sato was able to prove that the solutions of the KP hierarchy are naturally parameterized by the points of an infinite Grassmannian which is now known as the *Sato Grassmannian* (see [56]). The Plücker relations describing the embedding of the Grassmannian in an infinite dimensional Fock space can be identified with an infinite system of PDEs known as the Hirota Bilinear Equations (HBEs). The HBEs are given by certain quadratic polynomials in the partial derivatives and for that reason they are also sometimes called Hirota Quadratic Equations (HQEs). On the other hand, the definition of the primitive form involves an infinite system of quadratic relations. Already in the early 80's there was a question whether the primitive form and the HQEs are closely related. The first evidence for such a relation is the conjecture of Witten proved by Kontsevich which says that the generating function of the GW invariants of a point is a tau-function (a solution to the HQEs) of the KdV hierarchy. The mirror partner of the point is the A_1 -singularity, so this is the first example for which the answer is yes. The next important evidence comes from the work of Givental. Let us point out that the Frobenius manifolds that come from the flat structure of Saito are always generically semi-simple because the critical values of f provide local coordinates in which the Frobenius multiplication is semi-simple. Motivated by several computations with fixed-point localization Givental conjectured (see [19]) and Teleman proved (see [60]) that if the quantum cohomology of X is semi-simple, then the higher genus GW invariants are uniquely determined from the Frobenius structure. More precisely, Givental has invented a certain quantization formalism that allows us to construct a formal function for any semi-simple Frobenius manifold. This formal function is called the *total descendant potential* of the Frobenius manifold. It is obtained from the product of several copies of the total descendant potential of a point by the action of a quantized symplectic transformation. Therefore, there is a natural project to construct HQEs for the total descendant potential of any semi-simple Frobenius manifold by using several copies of the HQEs for the KdV hierarchy and the quantized symplectic transformation of Givental. Let us try to give several evidence in favor of this project. First of all, if we conjugate the vertex operators of KdV with Givental's quantized symplectic transformation we obtain a vertex operator that has a very elegant interpretation (see [22]): the coefficients of the vertex operator are period integrals. In the case of simple singularities (see [24]) and Fano orbifold lines (see [47]) the HQEs of the total descendant potential exist and they can be identified either with the HQEs of a Kac–Wakimoto hierarchy or an extended Kac–Wakimoto hierarchy. The main feature of the works [24, 47] is that there was a natural candidate for the HQEs. There was no need to construct HQEs but only to check that they hold. The main difficulty in the proof comes from the so-called propagators. The latter are certain multi-valued functions arising when we consider the product of two vertex operators. These

propagators appear naturally when we start conjugating vertex operators by quantized symplectic transformations. The proofs in both [24] and [47] amount to establishing certain identities between the propagators. More precisely, one has to understand how the propagators behave under the analytic continuation along a closed loop and along a path approaching the discriminant. It turns out that these two kinds of analytic continuations of the propagators can be described in general in the settings of all semi-simple Frobenius manifolds. This is exactly one of the main goals of this book. The corresponding results are contained in chapter 3. Our expectation is that if one has a conjecture saying that the total descendant potential of a semi-simple Frobenius manifold satisfies HQEs, then the techniques developed in chapter 3 should be sufficient to prove this conjecture. Let us point out that the work [24] did not make use of propagators. The importance of the latter was discovered in [16]. In chapter 6 we will give an alternative proof of the main result in [24] by demonstrating how the techniques from chapter 3 work.

The main question that remains to be addressed is whether the HQEs exist. There is a very interesting observation which might provide an answer to this question. The idea comes from the work of Bakalov–Milanov [8]. It can be explained using the language of lattice vertex algebras (see [38] for some background). For simplicity, let us consider the case when the Frobenius structure comes from Saito’s flat structure. Let us consider the lattice vertex algebra V_Λ corresponding to the Milnor lattice Λ . Following Bakalov–Milanov we can construct a twisted representation of V_Λ on the Fock space to which the total descendant potential belongs, such that, the state $e^\alpha \in V_\Lambda$, where $\alpha \in \Lambda$, is mapped to the vertex operator corresponding to the periods of the cycle α . This representation induces a representation of $V_\Lambda \otimes V_\Lambda$ on the tensor square of the Fock space. In all known examples the HQEs represent a state $\text{Hir} \in V_\Lambda \otimes V_\Lambda$ satisfying the screening equations $(e_{(0)}^\alpha \otimes 1 + 1 \otimes e_{(0)}^\alpha) \text{Hir} = 0$ for all vanishing cycles α . Let us point out that most probably we have to complete $V_\Lambda \otimes V_\Lambda$ appropriately in order to have a chance that such a state Hir exist. Therefore, the existence of the HQEs for the total descendant potential of a semi-simple Frobenius manifold seems to be a problem in the theory of lattice vertex algebras, that is, it is a problem independent from the theory of semi-simple Frobenius manifolds and its solution requires new ideas.

Let us comment on the structure of this book. We have tried to write a self-contained pedagogical text accessible to graduate students. In general, we omitted the proofs only of results which can be found in standard textbooks, such as, [7, 45] in commutative algebra, [62] in homological algebra, [25, 28, 63] in complex geometry. Chapter 1 is an introduction to Givental’s higher genus reconstruction. We introduce the necessary background of Frobenius manifolds and semi-simple Frobenius manifolds. Only Section 1.2 requires a deeper understanding of complex geometry. The rest of the chapter should be easy to follow. Chapter 2 can be viewed as an introduction to the analytic theory of Ordinary Differential Equations (ODEs). The first goal in this chapter is to prove the Painleve property for the Schlesinger equations. This result is due to Malgrange and one of the pedagogical goals of this books is to make the result accessible to a wider audience. The original proof requires specialized knowledge of functional analysis. On the other hand, Bolibruch outlined an elementary proof of Malgrange’s result in his book [10]. We filled in the missing details and hence obtained an elementary self-contained proof of Malgrange’s result. This allows us to prove the Painleve property for all semi-simple Frobenius manifolds which will be used later on in an essential way in chapter 3. The content of chapter 3 was already mentioned above. We introduce vertex operators and

their propagators and establish their properties. This chapter contains the main tools that are needed in order to prove that the total descendant potential satisfies HQEs. In some sense the results of this chapter are one of the main motivations for writing this book. The next two chapters can be viewed as an introduction to the theory of primitive forms. In chapter 4 we developed the theory in quite general settings. Following ideas of Iritani, we also introduced certain families of functions for which one can apply Morse theory. We have to admit that proving the existence of a primitive form in such general settings is quite challenging. In chapter 5 we specialize the theory from chapter 4 to the case of weighted homogeneous singularities. We prove the existence of a primitive form and classify all Frobenius (or flat) structures that arise from opposite subspaces. Many of the concepts introduced in chapters 3 and 4 have a very elegant interpretation in the settings of singularity theory. Chapter 5 provides a huge class of examples of semi-simple Frobenius manifolds that are expected to be very important in the applications to mirror symmetry and integrable systems. The last chapter is an example of how the techniques developed in chapter 3 work in the settings of simple singularities.

Semi-simple Frobenius manifolds

1.1. Frobenius manifolds

The main goal of this section is to recall the notion of a semi-simple Frobenius manifold and to prove that semi-simple Frobenius manifolds can be classified by solutions of a certain system of PDEs. The general references for more details are [12, 31, 44].

1.1.1. Definition. There are several ways to introduce the notion of a Frobenius manifold. Our definition is equivalent to (Definition 1.2 in [12]). Let M be a complex manifold and \mathcal{T}_M denotes the sheaf of holomorphic vector fields on M . Let us assume that M is equipped with the following structures

- (a) Each tangent space $T_t M$, $t \in M$, is equipped with the structure of a *Frobenius algebra* depending holomorphically on t . In other words, we have a commutative associative multiplication \bullet_t and symmetric non-degenerate bi-linear pairing $(,)_t$ satisfying the Frobenius property

$$(v_1 \bullet_t w, v_2) = (v_1, w \bullet_t v_2), \quad v_1, v_2, w \in T_t M.$$

The pointwise multiplication \bullet_t defines a multiplication \bullet in \mathcal{T}_M , i.e., a \mathcal{O}_M -bilinear map

$$\mathcal{T}_M \otimes \mathcal{T}_M \rightarrow \mathcal{T}_M, \quad v_1 \otimes v_2 \mapsto v_1 \bullet v_2.$$

The pairing $(,)_t$ determine a \mathcal{O}_M -bilinear pairing

$$(,) : \mathcal{T}_M \otimes \mathcal{T}_M \rightarrow \mathcal{O}_M.$$

- (b) There exists a global vector field $e \in \mathcal{T}_M$, called *unit vector field*, such that

$$\nabla_v^{\text{L.C.}} e = 0, \quad e \bullet v = v, \quad \forall v \in \mathcal{T}_M,$$

where $\nabla^{\text{L.C.}}$ is the Levi-Civita connection on \mathcal{T}_M corresponding to the bi-linear pairing $(,)$.

- (c) There exists a global vector field $E \in \mathcal{T}_M$, called *Euler vector field*, such that

$$E(v_1, v_2) - ([E, v_1], v_2) - (v_1, [E, v_2]) = (2 - D)(v_1, v_2),$$

for all $v_1, v_2 \in \mathcal{T}_M$ and for some constant $D \in \mathbb{C}$.

The above data allows us to define the so-called *structure connection* or *Dubrovin's connection* ∇ on the vector bundle $TM \times \mathbb{C}^* \rightarrow M \times \mathbb{C}^*$. Namely,

$$\begin{aligned} \nabla_v &:= \nabla_v^{\text{L.C.}} - z^{-1}v \bullet, \quad v \in \mathcal{T}_M \\ \nabla_{\partial/\partial z} &:= \frac{\partial}{\partial z} - z^{-1}\theta + z^{-2}E \bullet, \end{aligned}$$

$v\bullet$ and $E\bullet$ are \mathcal{O}_M -linear maps $\mathcal{T}_M \rightarrow \mathcal{T}_M$ corresponding to the Frobenius multiplication by respectively v and E . The \mathcal{O}_M -linear map $\theta : \mathcal{T}_M \rightarrow \mathcal{T}_M$ is defined by

$$(1.1) \quad \theta(v) := \nabla_v^{\text{L.C.}} E - (1 - D/2)v.$$

The operator θ is sometimes called *grading operator*. Let us point out that the term $(1 - D/2)v$ in the definition of $\theta(v)$ is inserted so that θ becomes skew-symmetric with respect to the Frobenius pairing

$$(\theta(v_1), v_2) + (v_1, \theta(v_2)) = 0, \quad v_1, v_2 \in \mathcal{T}_M.$$

DEFINITION 1.1. The data $((,), \bullet, e, E)$ satisfying conditions (a), (b), and (c) from above is said to be a *Frobenius structure* on M of *conformal dimension* D if the structure connection ∇ is flat.

1.1.2. Properties. The following proposition is a direct consequence of the definition.

PROPOSITION 1.2. *Suppose that $(M, ((,), \bullet, e, E))$ is a Frobenius structure. Then*

a) The Levi-Civita connection $\nabla^{\text{L.C.}}$ is flat.

b) Let $t = (t_1, \dots, t_N)$ be $\nabla^{\text{L.C.}}$ -flat coordinates defined on a contractible open subset $U \subset M$. There exists a holomorphic function $F \in \mathcal{O}_M(U)$, such that

$$(\partial/\partial t_a \bullet \partial/\partial t_b, \partial/\partial t_c) = \frac{\partial^3 F}{\partial t_a \partial t_b \partial t_c}$$

and

$$EF = (3 - D)F + H,$$

where H is a polynomial in t_1, \dots, t_N of degree at most 2.

c) The grading operator is covariantly constant: $\nabla_v^{\text{L.C.}} \theta = 0$. Equivalently, in flat coordinates $t = (t_1, \dots, t_N)$, the matrix $(\theta_{ab})_{a,b=1}^N$ of θ defined by

$$\theta(\partial/\partial t_b) = \sum_{a=1}^N \theta_{ab} \partial/\partial t_a$$

is constant.

d) The following identity holds

$$[E, v \bullet w] - [E, v] \bullet w - v \bullet [E, w] = v \bullet w, \quad v, w \in \mathcal{T}_M.$$

PROOF. Parts a) and b) are straightforward. We will prove c) and d) simultaneously. To begin with note that both c) and d) are \mathcal{O}_M -linear in v and w . Therefore, we may assume v and w are flat with respect to $\nabla^{\text{L.C.}}$. The flatness of ∇ implies that

$$\nabla_{z\partial_z + E} \nabla_v w - \nabla_v \nabla_{z\partial_z + E} w - \nabla_{[E, v]} w = 0.$$

By definition

$$\nabla_v = \nabla_v^{\text{L.C.}} - z^{-1}v\bullet$$

and

$$\nabla_{z\partial_z + E} = z\partial_z + \nabla_E^{\text{L.C.}} - \theta.$$

Substituting these operators in the 0-curvature equation and using that v and w are flat we get a polynomial expression in z^{-1} of degree 1 for which the coefficient in front of z^0 is

$$\nabla_v^{\text{L.C.}} \cdot \theta(w)$$

and the coefficient in front of z^{-1} is

$$(1.2) \quad v \bullet w + [E, v] \bullet w - v \bullet \theta(w) + \theta(v \bullet w) - \nabla_E^{\text{L.C.}}(v \bullet w).$$

Therefore both expressions must vanish. The vanishing of $\nabla_v^{\text{L.C.}} \cdot \theta(w)$ for all flat vector fields v and w is equivalent to the statement in c). Using the definition (1.1) of θ and that $\nabla^{\text{L.C.}}$ is torsion free we get

$$-v \bullet \theta(w) + \theta(v \bullet w) = v \bullet [E, w] - [E, v \bullet w] + \nabla_E^{\text{L.C.}}(v \bullet w).$$

Substituting this identity in (1.2) we get the identity of part d). \square

REMARK 1.3. Locally the Frobenius structure is completely determined by the Euler vector field E and the holomorphic function F . It is possible to give a definition of a Frobenius manifold in terms of F (see [12]). This leads to the so-called WDVV equations for F . In many applications the Frobenius structures arise as solutions of the WDVV equations. \square

1.1.3. Semi-simple Frobenius manifolds.

DEFINITION 1.4. A Frobenius manifold $(M, (\cdot, \cdot), \bullet, e, E)$ is said to be *semi-simple* if there are local coordinates $u = (u_1, \dots, u_N)$ defined in a neighborhood of some point on M such that

$$\partial/\partial u_i \bullet \partial/\partial u_j = \delta_{ij} \partial/\partial u_j, \quad 1 \leq i, j \leq N.$$

The coordinates u_i are called *canonical coordinates*.

As we will see now, canonical coordinates are unique up to permutation and constant shifts. To avoid cumbersome notation we put $\partial_{u_i} := \partial/\partial u_i$.

PROPOSITION 1.5. *Let $u = (u_1, \dots, u_N)$ be canonical coordinates defined on some open subset $U \subset M$. Then*

a) *The Frobenius pairing takes the form*

$$(\partial/\partial u_i, \partial/\partial u_j) = \delta_{ij} \eta_j(u), \quad 1 \leq i, j \leq N,$$

where $\eta_j \in \mathcal{O}_M(U)$ and $\eta_j(u) \neq 0$ for all $u \in U$.

b) *The unit vector field takes the form $e = \sum_{i=1}^N \partial/\partial u_i$.*

c) *The 1-form $\sum_{i=1}^N \eta_i(u) du_i$ is closed.*

d) *There are constants c_i ($1 \leq i \leq N$) such that*

$$E = \sum_{i=1}^N (u_i + c_i) \frac{\partial}{\partial u_i}.$$

PROOF. a) If $i \neq j$ then we have

$$(\partial_{u_i}, \partial_{u_j}) = (e \bullet \partial_{u_i}, \partial_{u_j}) = (e, \partial_{u_i} \bullet \partial_{u_j}) = 0.$$

The fact that $\eta_i(u) := (\partial_{u_i}, \partial_{u_i}) \neq 0$ follows from the non-degeneracy of the Frobenius pairing.

b) Let $e = \sum_{i=1}^N e_i(u)\partial_{u_i}$. Then

$$\partial_{u_j} = \partial_{u_j} \bullet e = e_j(u)\partial_{u_j}.$$

Therefore $e_j(u) = 1$ for all j .

c) We have to check that $\partial_{u_j}\eta_i = \partial_{u_i}\eta_j$. On the other hand

$$\partial_{u_j}\eta_i = \partial_{u_j}(\partial_{u_i}, e) = (\nabla_{\partial_{u_j}}^{\text{L.C.}}\partial_{u_i}, e),$$

where we used the Leibnitz rule and the fact the e is a flat vector field. It remains only to recall that the Levi-Civita connection is torsion free, so

$$\nabla_{\partial_{u_j}}^{\text{L.C.}}\partial_{u_i} = \nabla_{\partial_{u_i}}^{\text{L.C.}}\partial_{u_j}.$$

d) Put $E = \sum_{i=1}^N E_i(u)\partial_{u_i}$. Let us recall Proposition 1.2, part d) with $v = \partial_{u_i}$ and $w = \partial_{u_j}$. For $i \neq j$ we get

$$(\partial_{u_i}E_j)\partial_{u_j} + (\partial_{u_j}E_i)\partial_{u_i} = 0.$$

Hence $\partial_{u_i}E_j = 0$ for $i \neq j$. If $i = j$ then we get $\partial_{u_i}E_i = 1$. Therefore $E_i(u) = u_i + c_i$ for some constant c_i . \square

Part d) of the above proposition shows that in every canonical coordinate system up to some constant shifts the canonical coordinates coincide with the eigenvalues of the operator $E\bullet$. Therefore, up to constant shifts and permutations the canonical coordinates are uniquely determined. From now on we will work only with canonical coordinates such that

$$E = \sum_{i=1}^N u_i\partial_{u_i}.$$

The question that we would like to answer now is the following: Suppose that U is an open contractible subset of \mathbb{C}^N and that $\sum_{i=1}^N \eta_i(u)du_i$ is a closed 1-form on U . Using the 1-form we define a pairing

$$(\partial_{u_i}, \partial_{u_j}) = \delta_{ij} \eta_j(u).$$

Let us also define multiplication

$$\partial_{u_i} \bullet \partial_{u_j} = \delta_{ij} \partial_{u_j}$$

and vector fields

$$e = \sum_{i=1}^N \partial_{u_i}, \quad E = \sum_{i=1}^N u_i \partial_{u_i}.$$

The problem then is to classify all 1-forms $\sum_{i=1}^N \eta_i(u)du_i$ such that the above data determines a Frobenius structure on U with canonical coordinates (u_1, \dots, u_N) . The answer is given by the following theorem:

THEOREM 1.6. *The closed 1-form $\sum_{i=1}^N \eta_i(u)du_i$ determines a Frobenius structure on U of conformal dimension D if and only if the following conditions are satisfied:*

- (1) $\eta_i(u) \neq 0$ for all i and for all $u \in U$.
- (2) $e\eta_i(u) = 0$ for all i .
- (3) $E\eta_i(u) = -D\eta_i(u)$.

(4) For all $k \neq i \neq j \neq k$ we have

$$\frac{\partial \eta_{ij}}{\partial u_k} = \frac{1}{2} \left(\frac{\eta_{ij} \eta_{kj}}{\eta_j} + \frac{\eta_{jk} \eta_{ik}}{\eta_k} + \frac{\eta_{ki} \eta_{ji}}{\eta_i} \right),$$

where $\eta_{ab}(u) := \partial_{u_a} \eta_b(u)$.

PROOF. Step 1. Determine when does the 1-form $\sum_i \eta_i(u) du_i$ defines data satisfying conditions (a), (b), and (c) in the definition of a Frobenius manifold.

In part (a), we would like the multiplication and the pairing to give a holomorphic family of Frobenius algebras. This is clearly satisfied for any choice of the 1-form. The requirement that the pairing is non-degenerate yields that $\eta_i(u) \neq 0$ for all i and for all $u \in U$.

For condition (b), we would like to know when is e a flat vector field. Let Γ_{ij}^k be the Christoffel's symbols of the pairing $g_{ij}(u) = \delta_{ij} \eta_j$. A straightforward computation yields

$$\Gamma_{ij}^j = \Gamma_{ji}^j = \frac{\eta_{ij}}{2\eta_j}, \quad 1 \leq i, j \leq N,$$

$$\Gamma_{ii}^j = -\frac{\eta_{ij}}{2\eta_j}, \quad 1 \leq i \neq j \leq N,$$

and

$$\Gamma_{ij}^k = 0, \quad k \neq i \neq j \neq k.$$

Using the above formulas we compute directly that

$$\nabla_{\partial_{u_i}}^{\text{L.C.}} e = \frac{e \eta_i}{2\eta_i} \partial_{u_i}.$$

Therefore e is a flat vector field if and only if $e \eta_i = 0$ for all i .

Finally, for condition (c) to hold we must have $E \eta_i = -D \eta_i$ for all i . Therefore, the 1-form will define a data satisfying conditions (a), (b), and (c) if and only if the functions $\eta_i(u)$ satisfy conditions (1), (2), and (3) in Theorem 1.6.

Step 2. When is the Levi-Civita connection flat?

The flatness of $\nabla^{\text{L.C.}}$ is equivalent to: the expression

$$2(\nabla_{\partial_{u_i}}^{\text{L.C.}} \nabla_{\partial_{u_j}}^{\text{L.C.}} \partial_{u_k}, \partial_{u_\ell})$$

is symmetric in i and j . Using the Leibnitz rule we transform this expression into

$$(1.3) \quad \partial_{u_i} \left(2\Gamma_{jk}^\ell \eta_\ell \right) - \sum_{a=1}^N 2\Gamma_{jk}^a \Gamma_{i\ell}^a \eta_a.$$

Let us assume first that i, j , and k are pairwise distinct. Then we get

$$\delta_{\ell j} \left(\frac{\partial \eta_{jk}}{\partial u_i} - \frac{\eta_{ij} \eta_{kj}}{2\eta_j} \right) + \delta_{\ell i} \left(\frac{\eta_{kj} \eta_{ij}}{2\eta_j} + \frac{\eta_{jk} \eta_{ik}}{2\eta_k} \right) + \delta_{\ell k} \left(\frac{\partial \eta_{jk}}{\partial u_i} - \frac{\eta_{ik} \eta_{jk}}{2\eta_k} \right).$$

The last term is symmetric in i and j , so a non-trivial condition will be obtained either if $\ell = i$ or $\ell = j$. Due to the symmetry between i and j we may assume that $\ell = j$. Then we get

$$\frac{\partial \eta_{jk}}{\partial u_i} - \frac{\eta_{ij} \eta_{kj}}{2\eta_j} = \frac{\eta_{ki} \eta_{ji}}{2\eta_i} + \frac{\eta_{ik} \eta_{jk}}{2\eta_k}.$$

This is exactly the PDE given in condition (4).

There are 3 more cases to analyze. Indeed, since we may assume that $i \neq j$ we get that $k = i$ or $k = j$. Again exchanging the role of LHS and RHS provides a symmetry between i and j , which allows us to assume that $k = i$. Therefore the remaining cases are: $(k, \ell) = (i, i), (i, j)$, or $k = i$ and $\ell \neq i, j$. The first case yields $\partial_{u_i} \eta_{ij} = \partial_{u_j} \eta_{ii}$, which is always satisfied because the 1-form $\sum_i \eta_i du_i$ is closed. The 2nd case $(k, \ell) = (i, j)$ yields

$$\partial_{u_i} \eta_{ij} + \partial_{u_j} \eta_{ij} = \frac{\eta_{ij} \eta_{ij}}{2\eta_i} + \frac{\eta_{ij} \eta_{ij}}{2\eta_j} + \frac{\eta_{ij} \eta_{ii}}{2\eta_i} + \frac{\eta_{ij} \eta_{jj}}{2\eta_j} - \sum_{a: a \neq i, j} \frac{\eta_{ia} \eta_{ja}}{2\eta_a}.$$

It is easy to see that this identity is a consequence of (2) and (4). In the last case if $k = i$ and $\ell \neq i, j$ we get

$$\partial_{u_j} \eta_{i\ell} = \frac{\eta_{ij} \eta_{i\ell}}{2\eta_i} + \frac{\eta_{i\ell} \eta_{j\ell}}{2\eta_\ell} + \frac{\eta_{ij} \eta_{j\ell}}{2\eta_j}$$

which is equivalent to (4).

Step 3. It remains only to verify that under the conditions (1)–(4) the structure connection ∇ is flat. The argument is similar to the argument in Step 2, so it will be left as an exercise. \square

REMARK 1.7. The system of PDEs in condition (4) of Theorem 1.6 is equivalent to the so-called Darboux–Egoroff equation – see [44], Chapter 2, Section 1.2.

1.2. Analytic spectrum of a Frobenius manifold

Let M be a Frobenius manifold. We say that a point $t^\circ \in M$ is semi-simple if there exists an open coordinate chart of M containing t° , such that, the corresponding coordinates are canonical coordinates. Clearly, if a point t° is semi-simple, then the Frobenius algebra $T_{t^\circ} M$ is semi-simple, i.e., a direct sum of fields. We would like to prove that the converse is also true. Moreover, we would like to establish several other criteria for semi-simplicity. Also, the problem of analytic continuation of the canonical coordinates comes up in many applications. It turns out that all these problems can be reformulated in terms of the so-called analytic spectrum of the Frobenius manifold (see [31], Section 2.2) which allows us to use the methods of complex geometry.

1.2.1. Analytic spectrum. The tangent sheaf \mathcal{T}_M has a structure of a sheaf of \mathcal{O}_M -algebras induced from the Frobenius multiplication. Let us define a complex subspace L of T^*M . Suppose that $U \subset M$ is a coordinate chart with coordinates $t = (t_1, \dots, t_N)$. Note that every $\omega \in T_t^*U$ can be written as $\omega = \sum_{i=1}^N p_i dt_i$. We can interpret $(t_1, \dots, t_N, p_1, \dots, p_N)$ as coordinate functions which turn T^*U into a coordinate chart of T^*M . Let $c_{ij}^k(t)$ be the structure constants of the Frobenius multiplication, that is, $\frac{\partial}{\partial t_i} \bullet \frac{\partial}{\partial t_j} = \sum_{k=1}^N c_{ij}^k(t) \frac{\partial}{\partial t_k}$. Let $\mathcal{I}_U \subset \mathcal{O}_{T^*U}$ be the ideal sheaf generated by the functions

$$f_{ij}^U(t, p) := p_i p_j - \sum_{k=1}^N c_{ij}^k(t) p_k \quad (1 \leq i, j \leq N), \quad p_1 - 1,$$

where we assume that $\frac{\partial}{\partial t_1}$ is the unit vector field. It is easy to check that the ideal sheaves \mathcal{I}_U can be glued to an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{T^*M}$. Let $L \subset T^*M$ be the complex space corresponding to the ideal sheaf \mathcal{I} and $\pi : L \rightarrow M$ be the map induced from the natural projection $T^*M \rightarrow M$.

There is a natural way to construct holomorphic functions on T^*M from vector fields on M . Namely, suppose that $v \in \Gamma(U, TM)$ is a holomorphic vector field, then we define $f_v : T^*U \rightarrow \mathbb{C}$ by $f_v(t, \xi) = \langle \xi, v(t) \rangle$, where $t \in U$, $\xi \in T_t^*M$, and $v(t) \in T_tM$. This construction yields the following important map

$$(1.4) \quad \alpha : \mathcal{T}_M \rightarrow \pi_*\mathcal{O}_L, \quad v \mapsto f_v|_L.$$

We claim that $\pi : L \rightarrow M$ is a proper finite map and that α is an isomorphism identifying the Frobenius multiplication on \mathcal{T}_M with the natural multiplication in $\pi_*\mathcal{O}_L$. We are not going to use this fact in the sequel, but one can easily check that L is the *analytic spectrum* $L := \text{Specan}(\mathcal{T}_M)$ of \mathcal{T}_M . Namely, if $\varphi : X \rightarrow M$ is a proper finite map, then the natural map

$$\text{Hol}_M(X, L) \rightarrow \text{Hom}_{\mathcal{O}_M}(\mathcal{T}_M, \varphi_*\mathcal{O}_X), \quad (f, f^\sharp) \mapsto \pi_*(f^\sharp) \circ \alpha,$$

is a bijection. Here Hol_M denotes the set of morphisms of complex spaces over M , $\text{Hom}_{\mathcal{O}_M}$ denotes the set of morphisms of sheaves of \mathcal{O}_M -modules, and the pair (f, f^\sharp) consisting of a continuous map $f : X \rightarrow L$ and a sheaf morphism $f^\sharp : \mathcal{O}_L \rightarrow f_*\mathcal{O}_X$ is a morphism of complex spaces $(X, \mathcal{O}_X) \rightarrow (L, \mathcal{O}_L)$. We refer to [15], Section 1.14 for some introduction to analytic spectrum.

Recall that a commutative ring A is said to be *Artinian* if it satisfies the *descending chain condition*: every descending sequence of ideals $I_1 \supseteq I_2 \supseteq \dots$ stabilizes, that is, there exists n , such that, $I_{n+i} = I_n$ for all $i \geq 0$. In our case A will be a \mathbb{C} -algebra which is a finite dimensional vector space over \mathbb{C} . Note that since $\mathbb{C} \subset A$ every ideal is a vector subspace of A , so the descending chain condition is obvious. According to the structure theorem for Artin rings (see [7], Theorem 8.7), we have a direct sum decomposition $A = \bigoplus_{i=1}^r A_i$, where A_i is an Artin local ring. Note that the maximal ideals of A are precisely $\mathfrak{m}_i := \mathfrak{m}_{A_i} \oplus \bigoplus_{j:j \neq i} A_j$, where \mathfrak{m}_{A_i} is the maximal ideal of A_i . In particular, A has finitely many maximal ideals. It is easy to check that under the localization map $A \rightarrow A_{\mathfrak{m}_i}$ all subrings A_j ($j \neq i$) are mapped to 0, while A_i is mapped isomorphically to $A_{\mathfrak{m}_i}$.

LEMMA 1.8. *The map $\pi : L \rightarrow M$ is a proper finite map.*

PROOF. Suppose that $t \in M$, then $\pi^{-1}(t)$ coincides with the points $p \in \mathbb{C}^N$, such that,

$$p_i p_j = \sum_{k=1}^N c_{ij}^k(t) p_k \quad (1 \leq i, j \leq N), \quad p_1 = 1.$$

However, the quotient ring

$$A := \mathbb{C}[p_1, \dots, p_N] / (p_i p_j - \sum_{k=1}^N c_{ij}^k(t) p_k \quad (1 \leq i, j \leq N), \quad p_1 - 1) \cong T_t M$$

is Artinian, so it has finitely many maximal ideals. Since the points in $\pi^{-1}(t)$ are in bijection with the maximal ideals of A , we get that $\pi^{-1}(t)$ consists of finitely many points. In order to prove that $\pi : L \rightarrow M$ is a proper map, it is sufficient to prove that the map $\pi_U := \pi|_{L \cap T^*U} : L \cap T^*U \rightarrow U$ is proper for all coordinate charts $U \subset M$. Suppose that $K \subset U$ is a compact subset. Clearly $\pi_U^{-1}(K)$ is a closed subset of $L \cap T^*U$, so we need only to prove that $\pi_U^{-1}(K)$ is a bounded subset of $T^*U = U \times \mathbb{C}^N$. Let C be the maximal possible value of $|c_{ij}^k(t)|$ for $t \in K$ and $1 \leq i, j, k \leq N$. Since K is compact,

$0 < C < \infty$. Given a point $(t, p) \in T^*U \cap L$, such that $t \in K$, let us denote by i the index for which $|p_i|$ is maximal, that is, $p = (p_1, \dots, p_N)$ and $|p_j| \leq |p_i|$ for all j . Using the equation $p_i^2 = \sum_{k=1}^N c_{ii}^k(t)p_k$ and the triangle inequality we get $|p_i|^2 \leq CN|p_i|$, that is, $|p_i| \leq CN$, which implies that the set $\pi_U^{-1}(K)$ is bounded as claimed. \square

In the proof of next lemma we will make use of the GAGA principle (see [57]) in the case of a point. Recall that a commutative ring B with a unity $1 \in B$ is said to be *faithfully flat* over its subring A if the following conditions are satisfied:

- (i) $1 \in A$.
- (ii) B is a flat A -module.
- (iii) For every non-zero A -module X , the B -module $B \otimes_A X$ is non-zero.

It is well known (see [59], Theorem 13.3.5) that the analytic local ring $\mathcal{O}_{\mathbb{C}^N, 0} := \mathbb{C}\{p_1, \dots, p_N\}$ consisting of convergent power series in p_1, \dots, p_N is faithfully flat over the algebraic local ring $\mathbb{C}[p_1, \dots, p_N]_{(0)}$ consisting of rational functions $h(p)/g(p)$ regular at 0 (i.e. $g(0) \neq 0$). We will need the following characterization of faithful flatness (see [59], Lemma 13.1.2): B is faithfully flat over A if and only if B is a flat A -module and $BI \cap A = I$ for every ideal I of A .

LEMMA 1.9. *The sheaf $\pi_*\mathcal{O}_L$ is locally free of rank N .*

PROOF. Since π is a proper map, the sheaf $\pi_*\mathcal{O}_L$ is coherent. Since M is a non-singular variety, it is sufficient to prove that the fibers of $\pi_*\mathcal{O}_L$ are vector spaces of the same dimension N . Suppose that $t \in M$ is an arbitrary point. Let $\mathfrak{m}_{M,t} \subset \mathcal{O}_{M,t}$ be the ideal of the point t , and $I \subset \mathbb{C}[p_1, \dots, p_N]$ be the ideal generated by $(p_i p_j - \sum_{k=1}^N c_{ij}^k(t)p_k \ (1 \leq i, j \leq N), p_1 - 1)$. Using the direct image theorem for finite maps we get that the fiber of $\pi_*\mathcal{O}_L$ at t is

$$(\pi_*\mathcal{O}_L)_t/\mathfrak{m}_{M,t} \cong \bigoplus_{\omega \in \pi^{-1}(t)} \mathcal{O}_{L,\omega}/\mathfrak{m}_{M,t} = \bigoplus_{\omega \in \pi^{-1}(t)} \mathcal{O}_{\mathbb{C}^N,\omega}/I\mathcal{O}_{\mathbb{C}^N,\omega}.$$

In order to complete the proof of the lemma it is sufficient to construct an isomorphism

$$(1.5) \quad \bigoplus_{\omega \in \pi^{-1}(t)} \mathcal{O}_{\mathbb{C}^N,\omega}/I\mathcal{O}_{\mathbb{C}^N,\omega} \cong T_t M.$$

As we already pointed out above, the Frobenius multiplication turns $T_t M$ into an Artin ring isomorphic to $A := \mathbb{C}[p_1, \dots, p_N]/I$. According to the structure theorem for Artin rings, if $\mathfrak{m}_i \ (1 \leq i \leq r)$ are the maximal ideals of A , then $T_t M \cong A \cong \bigoplus_{i=1}^r A_{\mathfrak{m}_i}$. Every maximal ideal \mathfrak{m} of A corresponds via the quotient map $\mathbb{C}[p_1, \dots, p_N] \rightarrow A$ to a uniquely determined maximal ideal $(p_1 - \omega_1, \dots, p_N - \omega_N)$ of $\mathbb{C}[p_1, \dots, p_N]$ containing I . Put $\omega = (\omega_1, \dots, \omega_N)$, then the condition $(p_1 - \omega_1, \dots, p_N - \omega_N) \supset I$ is equivalent to the system of algebraic equations

$$\omega_i \omega_j = \sum_{k=1}^N c_{ij}^k(t) \omega_k \quad (1 \leq i, j \leq N), \quad \omega_1 = 1$$

which are the defining equations of the fiber $\pi^{-1}(t)$. In other words we have a one-to-one correspondence between the maximal ideals of $\mathfrak{m} \subset A$ and the points $\omega \in \pi^{-1}(t)$. We need only to prove that there is an isomorphism of local rings $A_{\mathfrak{m}} \cong \mathcal{O}_{L,\omega}/\mathfrak{m}_{M,t}$. By definition $\mathcal{O}_{L,\omega}/\mathfrak{m}_{M,t} = \mathcal{O}_{\mathbb{C}^N,\omega}/I$, where $\mathcal{O}_{\mathbb{C}^N,\omega} := \mathbb{C}\{p_1 - \omega_1, \dots, p_N - \omega_N\}$ is the ring of convergent power series. The natural inclusion $\mathbb{C}[p_1, \dots, p_N] \subset \mathcal{O}_{\mathbb{C}^N,\omega}$ induces an injective

map $A_{\mathfrak{m}} \rightarrow \mathcal{O}_{L,\omega}/\mathfrak{m}_{M,t}$, because $\mathcal{O}_{\mathbb{C}^N,\omega}$ is a faithfully flat $\mathbb{C}[p_1, \dots, p_N]_{(p_1-\omega_1, \dots, p_N-\omega_N)}$ -module and hence

$$I\mathcal{O}_{\mathbb{C}^N,\omega} \cap \mathbb{C}[p_1, \dots, p_N]_{(p_1-\omega_1, \dots, p_N-\omega_N)} = I\mathbb{C}[p_1, \dots, p_N]_{(p_1-\omega_1, \dots, p_N-\omega_N)}.$$

Finally, to prove that $A_{\mathfrak{m}} \rightarrow \mathcal{O}_{L,\omega}/\mathfrak{m}_{M,t}$ is surjective, note that the zero locus of the ideal I in a neighborhood of $\omega \in \mathbb{C}^N$ is $\{\omega\}$, so by Rückert Nullstellensatz there exists an integer $k > 0$ such that $(p_i - \omega_i)^k \in I\mathcal{O}_{\mathbb{C}^N,\omega}$ for all $1 \leq i \leq N$. Therefore, every element of $\mathcal{O}_{L,\omega}/\mathfrak{m}_{M,t}$ can be represented by a polynomial, i.e., the map $A \rightarrow \mathcal{O}_{L,\omega}/\mathfrak{m}_{M,t}$ is surjective, so the induced map $A_{\mathfrak{m}} \rightarrow \mathcal{O}_{L,\omega}/\mathfrak{m}_{M,t}$ is also surjective. \square

The map (1.4) induces a map $\alpha(t)$ between the fibers at t , that is, $\alpha(t) : T_t M \rightarrow (\pi_* \mathcal{O}_L)_t / \mathfrak{m}_{M,t} (\pi_* \mathcal{O}_L)_t$. Comparing the definitions we get that $\alpha(t)$ coincides with the isomorphism (1.5). Since both \mathcal{T}_M and $\pi_* \mathcal{O}_L$ are locally free \mathcal{O}_M -modules we get that α must be an isomorphism. Moreover, since (1.5) is an isomorphism of algebras, we get that α identifies the Frobenius multiplication on \mathcal{T}_M with the natural multiplication in $\pi_* \mathcal{O}_L$.

1.2.2. Semi-simplicity criteria. Using the results from the previous section we will give several equivalent characterizations of when is a given point $t^\circ \in M$ semi-simple.

LEMMA 1.10. *The map $\pi : L \rightarrow M$ is flat and hence it is an open finite surjection.*

PROOF. The local freeness of $\pi_* \mathcal{O}_L$ implies that $\pi : L \rightarrow M$ is a flat map. Indeed, suppose that $\omega \in L$ and $t = \pi(\omega)$. Recalling the direct image theorem for finite maps, we get that the local ring $\mathcal{O}_{L,\omega}$ is a direct summand of $(\pi_* \mathcal{O}_L)_t$. Since $(\pi_* \mathcal{O}_L)_t$ is a free $\mathcal{O}_{M,t}$ -module, we get that $\mathcal{O}_{L,\omega}$ is a projective module. However for local Noetherian rings the notion of a projective module, free module, and flat module are equivalent. Therefore $\mathcal{O}_{L,\omega}$ is a flat $\mathcal{O}_{M,t}$ -module. Since flat maps are known to be open, we get that $\pi : L \rightarrow M$ is open. On the other hand π is also a closed map, because proper maps are closed. Therefore $\pi(L)$ is both an open and a closed subset of M . Since M is a connected topological space we must have $\pi(L) = M$. \square

LEMMA 1.11. *A point $t \in M$ is semi-simple if and only if the algebra $T_t M$ has no nilpotents.*

PROOF. If the point t is semi-simple then clearly $T_t M \cong \mathbb{C}^{\oplus N}$ has no nilpotents. The difficult part is to prove the inverse. Suppose that $T_{t^\circ} M$ has no nilpotents. Let $T_{t^\circ} M = \bigoplus_{i=1}^r A_i$ be the decomposition of the Artin ring $T_{t^\circ} M$ into direct sum of Artin local rings. The maximal ideal \mathfrak{m}_{A_i} of A_i is nilpotent, so $\mathfrak{m}_{A_i} = 0$ (since we assumed that $T_{t^\circ} M$ has no nilpotents), $A_i \cong \mathbb{C}$, and $r = N$. Let $\omega \in \pi^{-1}(t^\circ)$. According to Lemma 1.10 $(\pi_* \mathcal{O}_L)_{t^\circ} = \bigoplus_{\omega \in \pi^{-1}(t^\circ)} \mathcal{O}_{L,\omega}$ is a free \mathcal{O}_{M,t° -module, so $\mathcal{O}_{L,\omega}$ is a free \mathcal{O}_{M,t° -module. On the other hand, $\mathcal{O}_{L,\omega}/\mathfrak{m}_{t^\circ} \cong A_i \cong \mathbb{C}$ for some i , where $\mathfrak{m}_{t^\circ} \subset \mathcal{O}_{M,t^\circ}$ is the ideal of the point $t^\circ \in M$. We get that the rank of $\mathcal{O}_{L,\omega}$ is 1, that is, $\mathcal{O}_{L,\omega} = \mathcal{O}_{M,t^\circ} \cdot 1_\omega$, where 1_ω is the unit of the ring $\mathcal{O}_{L,\omega}$. Let us fix a coordinate chart U of M containing the point t° and let $(t, p) = (t_1, \dots, t_N, p_1, \dots, p_N)$ be the coordinates on T^*U . Each p_i represents the germ of a holomorphic function in $\mathcal{O}_{L,\omega}$. Therefore, there exists $h_{i\omega} \in \mathcal{O}_{M,t^\circ}$ such that $p_i = h_{i\omega} \pmod{I_U}$, where $I_U = (f_{ij}^U(t, p) \mid 1 \leq i, j \leq N, p_1 - 1) \subset \mathcal{O}_{T^*U}$ is the ideal sheaf of $L \cap T^*U$. Since $\pi : L \rightarrow M$ is a proper finite map, we can choose U to be so small that $\pi^{-1}(U) = L \cap T^*U$ is a disjoint union of N open subsets L_ω ($\omega \in \pi^{-1}(t^\circ)$).

Decreasing U if necessary we may assume that all the germs $h_{i\omega} \in \mathcal{O}_{M,t^\circ}$ are represented by holomorphic functions on U . We claim that

$$(1.6) \quad (p_1 - h_{1\omega}, \dots, p_N - h_{N\omega}) \mathcal{O}_{T^*U,\omega} = I_U \mathcal{O}_{T^*U,\omega}.$$

We already know that $p_i - h_{i\omega} \in I_U \mathcal{O}_{T^*U,\omega}$. The rest of our claim follows easily if we knew that $\mathcal{O}_{M,t^\circ} \cap I_U \mathcal{O}_{T^*U,\omega} = \{0\}$, where we view $\mathcal{O}_{M,t^\circ} \subset \mathcal{O}_{T^*U,\omega}$ via the natural pullback map. If Δ is a holomorphic function representing a non-zero germ in $\mathcal{O}_{M,t^\circ} \cap I_U \mathcal{O}_{T^*U,\omega}$, then we have

$$\Delta(t) = \sum_{i,j=1}^N g_{ij}(t,p) f_{ij}^U(t,p) + g_1(t,p)(p_1 - 1),$$

for some $g_{ij}, g_1 \in \mathcal{O}_{T^*U,\omega}$. If $(t,p) \in L$, then $\Delta(t) = 0$, so the image via the projection π of the germ of L at the point ω is contained in the closed hypersurface $\{\Delta(t) = 0\} \subset M$ – contradicting the fact that the map π is open (see Lemma 1.10). This proves (1.6) and we get

$$(1.7) \quad L_\omega = \{(t,p) \in T^*U \mid p_i = h_{i\omega}(t) \forall 1 \leq i \leq N\}.$$

Suppose that $t \in U$ and let us consider the Artin ring

$$T_t M \cong \mathbb{C}[p_1, \dots, p_N] / (p_i p_j - \sum_{k=1}^N c_{ij}^k(t) p_k \ (1 \leq i, j \leq N), p_1 - 1).$$

Since this is the coordinate ring of $\pi^{-1}(t) = L \cap T_t^*U = \mathbb{C}^N$, using (1.7), we get that $T_t M$ contains precisely N maximal ideals, i.e., the maximal ideals of $T_t M$ are $(p_1 - h_{1\omega}(t), \dots, p_N - h_{N\omega}(t))$ ($\omega \in \pi^{-1}(t^\circ)$). Note that if we decompose $A := T_t M$ as a direct sum of local Artin rings, then we have a decomposition of the form $A \cong \bigoplus_{\mathfrak{m}} A/\mathfrak{m} \cong \mathbb{C}^{\oplus N}$, where the sum is over the maximal ideals of A and $A \rightarrow A/\mathfrak{m}$ is the natural quotient map. In other words, we get that for all $t \in U$ the map

$$T_t M \xrightarrow{\cong} \mathbb{C}^N, \quad \frac{\partial}{\partial t_i} \mapsto (h_{i\omega}(t))_{\omega \in \pi^{-1}(t^\circ)}$$

is an isomorphism of algebras, where the multiplications in $T_t M$ and \mathbb{C}^N are respectively the Frobenius multiplication and the componentwise multiplication. We get that the $N \times N$ matrix with entries $h_{i\omega}$ is invertible for all $t \in U$ and hence the inverse matrix exists. Let us denote the entries of the inverse matrix by $h^{\omega i}(t)$. Then $e_\omega := \sum_{i=1}^N h^{\omega i}(t) \frac{\partial}{\partial t_i}$ are holomorphic vector fields on U that are idempotents for the Frobenius multiplication, that is, $e_\omega \bullet e_\eta = \delta_{\omega,\eta} e_\eta$. It is an easy exercise to prove that there are holomorphic coordinates $u_\omega(t)$ on U , such that, $e_\omega = \frac{\partial}{\partial u_\omega}$ (see [12], Lecture 3, Main Lemma). This implies that the open chart U has canonical coordinates, so t° is a semi-simple point. \square

Following Abrams [1] let us define the *characteristic element* $\Delta(t) := \sum_{i=1}^N e_i \bullet e^i$, where $\{e_i\}_{i=1}^N$ and $\{e^i\}_{i=1}^N$ is a pair of bases of $T_t M$ dual with respect to the Frobenius pairing. The definition is independent of the choice of bases and $t \mapsto \Delta(t)$ is a holomorphic vector field on M .

PROPOSITION 1.12. *Let $t \in M$ be an arbitrary point. The following conditions are equivalent:*

- (i) *The point t is semi-simple.*
- (ii) *The Frobenius algebra $T_t M$ has no nilpotent elements.*
- (iii) *The Frobenius algebra $T_t M \cong \mathbb{C} \oplus \dots \oplus \mathbb{C}$.*

(iv) *The characteristic element $\Delta(t)$ is invertible in T_tM .*

PROOF. The equivalence of (i) and (ii) follows from Lemma 1.11. The equivalence of (ii) and (iii) follows from the structure theorem for Artin algebras. The equivalence of (iii) and (iv) is Theorem 3.4 in [1]. For the sake of completeness let us reproduce the argument (with small modifications). The implication (iii) \Rightarrow (iv) is obvious. Suppose that $\Delta(t)$ is invertible in $A := T_tM$. By the structure theorem for Artin rings $A = \bigoplus_{i=1}^r A_i$, where A_i are local Artin rings. Since $A_i A_j = 0$ and $(A_i, A_j) = 0$ the characteristic element of A is a sum of the characteristic elements $\Delta_i(t)$ of A_i . Moreover, the invertibility of $\Delta(t)$ implies the invertibility of each $\Delta_i(t)$. Therefore, it is sufficient to prove that if A is a local Artin algebra whose characteristic element is invertible, then $A = \mathbb{C}$, that is, the maximal ideal \mathfrak{m} of A is 0. It is known that \mathfrak{m} is a nilpotent ideal. We have a filtration

$$0 \subset \mathfrak{m}^k \subset \mathfrak{m}^{k-1} \subset \dots \subset \mathfrak{m} \subset A =: \mathfrak{m}^0,$$

where k is the smallest integer such that $\mathfrak{m}^{k+1} = 0$. Note that $\mathfrak{m}^i \neq \mathfrak{m}^{i+1}$ for $i < k+1$, otherwise $\mathfrak{m}^i = 0$ by Nakayama's lemma. Let us first choose a basis of \mathfrak{m}^k , extend it to a basis of \mathfrak{m}^{k-1} , etc. We get a basis e_1, \dots, e_N of A that has the following property: for every i there exists an l , such that, $e_i \in \mathfrak{m}^l$ and $e_i \notin \mathfrak{m}^{l+1}$. If $a \in \mathfrak{m}$, then $e_i a \in \mathfrak{m}^{l+1}$ is a linear combination of elements e_j different from e_i . Therefore, $(e_i a, e^i) = 0 \Rightarrow (e_i e^i, a) = 0$ for all $a \in \mathfrak{m}$. This implies that $(e_i e^i a, x) = (e_i e^i, ax) = 0$ for all $x \in A$. However, since the Frobenius pairing is non-degenerate, we get $e_i e^i a = 0$ for all $a \in \mathfrak{m}$. Summing over i we get $\Delta(t)a = 0$ and since $\Delta(t)$ is invertible, we get $a = 0$, that is, $\mathfrak{m} = 0$. \square

1.2.3. The caustic of a semi-simple Frobenius manifold. Suppose now that M is a semi-simple Frobenius manifold. The subset \mathcal{K} of all points in M that are not semi-simple is called the *caustic* of M . Proposition 1.12 gives us various conditions that characterize the points of \mathcal{K} . In particular, condition (iv) implies the following proposition.

PROPOSITION 1.13. *If M is a semi-simple Frobenius manifold, then the caustic \mathcal{K} is either the empty set or it is an analytic hypersurface of M .*

PROOF. Let $f(t)$ be the determinant of the linear operator in T_tM defined by Frobenius multiplication by the characteristic element $\Delta(t)$ of T_tM . Clearly $f(t)$ is a holomorphic function on M and according to Proposition 1.12, (iv) $\mathcal{K} = \{t \in M \mid f(t) = 0\}$. \square

Let us point out that in the proof of Lemma 1.11 (see formula (1.7)) we also proved that π induces a regular covering $L \setminus \pi^{-1}(\mathcal{K}) \rightarrow M \setminus \mathcal{K}$, while Proposition 1.13 implies that $M \setminus \mathcal{K}$ is dense in M .

LEMMA 1.14. *The complex space L is reduced.*

PROOF. We already know that the points in $L \setminus \pi^{-1}(\mathcal{K})$ are smooth and hence reduced. Suppose that $t^\circ \in \mathcal{K}$ is an arbitrary point and $\omega^\circ \in \pi^{-1}(t^\circ)$. We have to prove that the local ring $\mathcal{O}_{L, \omega^\circ}$ has no nilpotents. Since $\pi : L \rightarrow M$ is an open finite surjection, we can find a sufficiently small open neighborhood $U \subset M$, $t^\circ \in U$, such that, $\pi^{-1}(U)$ is a disjoint union of open subsets L_i ($1 \leq i \leq k$) and $\pi_i := \pi|_{L_i} : L_i \rightarrow U$ is an open finite surjection. Without any loss of generality we may assume that $\omega^\circ \in L_1$. By shrinking U we can make L_1 as small as we wish. In particular, given a germ in $\mathcal{O}_{L_1, \omega^\circ}$ we may assume that it is represented by a function in $\Gamma(L_1, \mathcal{O}_{L_1})$. Since L_i for $i \neq 1$ are disjoint from L_1 we can extend by 0 any holomorphic function on L_1 to a holomorphic function on L_i . Shrinking U even further if necessary we may assume that $\pi_* \mathcal{O}_L|_U \cong \mathcal{O}_U^N$. We have

to prove that if $f \in \Gamma(\pi^{-1}U, \mathcal{O}_L)$ and the germs $f_\omega \forall \omega \in L$ are nilpotent, then $f = 0$. Let $\phi_i \in \Gamma(\pi^{-1}U, \mathcal{O}_L)$ ($1 \leq i \leq N$) be an \mathcal{O}_U -basis, that is, $\pi_*\mathcal{O}_L|_U = \mathcal{O}_U \phi_1 \oplus \cdots \oplus \mathcal{O}_U \phi_N$. We have

$$f = \sum_{i=1}^N \alpha_i \phi_i, \quad \alpha_i \in \Gamma(U, \mathcal{O}_M).$$

Let $\tilde{\phi}_i \in \mathcal{O}_{T^*U}$ be a holomorphic function defined in a neighborhood of $\pi^{-1}(U) = L \cap T^*U$ that represents $\phi_i \in \mathcal{O}_L = \mathcal{O}_{T^*U}/I_U \mathcal{O}_{T^*U}$ where I_U is the ideal sheaf of L . Clearly $\sum_{i=1}^N \alpha_i \tilde{\phi}_i \in \mathcal{O}_{T^*U}$ represents the class of f and since the germs of f are nilpotent we have $\sum_{i=1}^N \alpha_i(t) \tilde{\phi}_i(t, p) = 0$ for all $(t, p) \in L$. On the other hand, if $t \in U \setminus \mathcal{K}$ and $\omega \in \pi^{-1}(t)$, that is, $\omega = (t, p) \in L$, then the local algebras $\mathcal{O}_{L, \omega}/\mathfrak{m}_{M, t} \cong \mathbb{C}$ and therefore using that $\{\phi_i\}_{i=1}^N$ is a \mathcal{O}_U -basis of $(\pi_*\mathcal{O}_L)_t = \bigoplus_{\omega \in \pi^{-1}(t)} \mathcal{O}_{L, \omega}$, we get that the columns of the $N \times N$ matrix, whose (i, ω) -entry ($1 \leq i \leq N, \omega \in \pi^{-1}(t)$) is $\tilde{\phi}_i(\omega)$, are linearly independent. Therefore, the rows are also linearly independent, so from $\sum_{i=1}^N \alpha_i(t) \tilde{\phi}_i(t, p) = 0$ for all $(t, p) =: \omega \in \pi^{-1}(t)$ we get that $\alpha_i(t) = 0$ for all i . We proved that $\alpha_i(t) = 0$ for all $t \in U \setminus \mathcal{K}$. Since \mathcal{K} is a thin set and α_i are continuous functions on U , we must have $\alpha_i(t) = 0$ for all $1 \leq i \leq N$ and $t \in U$. \square

The main result of this section can be stated as follows.

THEOREM 1.15. *A Frobenius manifold M is semi-simple if and only if its analytic spectrum L is an analytic variety, i.e., reduced complex space. If M is semi-simple, then the projection $\pi : L \rightarrow M$ is a branched covering with branching locus the caustic \mathcal{K} of M .*

1.3. Genus-0 gravitational descendants

Motivated by Witten's formulation of topological gravity, Givental has proposed in [21] a geometric interpretation of the so-called *genus 0 gravitational descendants*. The latter can be organized into a single generating formal power series \mathcal{F} satisfying three axioms: an infinite set of Topological Recursion Relations (TRR), Dilaton Equation (DE), and String Equation (SE). The three axioms can be reformulated in terms of the geometry of the graph \mathcal{L} of the differential of \mathcal{F} . It turns out that \mathcal{L} is a Lagrangian cone with vertex at the origin and that its tangent spaces vary in a finite dimensional family generating a *Variation of Semi-infinite Hodge Structures* (VSHS) in the sense of Barannikov [9]. In particular, the base of the family has a natural formal germ of a Frobenius structure. In this section we would like to prove that the above construction of a Frobenius structure is general, i.e., the formal germ of a Frobenius structure at any point $t^\circ \in M$ can be obtained by Givental's Lagrangian cone and Barannikov's VSHS.

1.3.1. Calibration at a point. Suppose that M is a Frobenius manifold and that $t^\circ \in M$ is an arbitrary fixed point. Let $H := T_{t^\circ}M$ be the holomorphic tangent space. The restriction of the Dubrovin's connection at $t = t^\circ$ is a connection on the trivial bundle $H \times \mathbb{C}^* \rightarrow \mathbb{C}^*$ that has an irregular singular point at $z = 0$ and a regular singular point at $z = \infty$. The general theory of Fuchsian systems implies that near the regular singular point $z = \infty$, there exists a fundamental solution

$$(1.8) \quad Y^\circ(z) = S^\circ(z) z^\delta z^\nu,$$

where $S^\circ(z) = \sum_{k=0}^{\infty} S_k^\circ z^{-k}$ is an operator-valued series, that is, $S_k^\circ \in \text{End}(H)$, δ is a diagonalizable operator, and ν is a nilpotent operator. We would like however to prove a slightly stronger result. Let us introduce some more notation. Let $\theta = \theta_s + \theta_n$ be the Jordan–Chevalley decomposition of the grading operator, where θ_s is diagonalizable, θ_n is nilpotent, and $[\theta_s, \theta_n] = 0$. If δ is a diagonalizable operator, then

$$\text{ad}_\delta : \mathfrak{gl}(H) \rightarrow \mathfrak{gl}(H), \quad X \mapsto [\delta, X]$$

is also diagonalizable, where $\mathfrak{gl}(H)$ is the Lie algebra of all linear transformations of H . Let $\text{spec}(\delta)$ be the set of eigenvalues of ad_δ . We have a direct sum decomposition into eigensubspaces

$$\mathfrak{gl}(H) = \bigoplus_{a \in \text{spec}(\delta)} \mathfrak{gl}_a(H), \quad \mathfrak{gl}_a(H) = \{X \in \mathfrak{gl}(H) \mid [\delta, X] = aX\}.$$

If $X \in \mathfrak{gl}(H)$, then we denote by $X_{[a]}$ the projection of X onto $\mathfrak{gl}_a(H)$ for $a \in \text{spec}(\delta)$.

PROPOSITION 1.16. *Let $t^\circ \in M$ be any point. The restriction of the Dubrovin’s connection to $t = t^\circ$ has a fundamental solution of the form (1.8), such that,*

- (i) *The operator δ coincides with the semi-simple part θ_s of θ .*
- (ii) *The nilpotent operator decomposes as $\nu = \nu_0 + \sum_{i=1}^{\infty} \nu_i$, where $\nu_0 = \theta_n$ and $[\delta, \nu_i] = -i\nu_i$, that is, $\nu_i \in \mathfrak{gl}_{-i}(H)$.*
- (iii) *$S_0^\circ = \text{id}_H$ and $S^\circ(-z)^T S^\circ(z) = 1$, where T is transposition with respect to the Frobenius pairing on the tangent space H .*

PROOF. The Proposition is proved in the master’s thesis of Chenghan Zha [66]. For the reader’s convenience we reproduce Zha’s argument. Substituting $Y^\circ(z)$ in $\nabla_{\partial_z}(Y) = 0$ and comparing the coefficients in front of the powers of z , we get first that the projection $\nu_{[a]} = 0$ if a is not an integer ≤ 0 and the following system of recursion equations:

$$(1.9) \quad \begin{aligned} \theta &= \delta + \nu_{[0]} \\ kS_k^\circ + [\theta, S_k^\circ] &= E \bullet S_{k-1}^\circ + \sum_{l=1}^k S_{k-l}^\circ \nu_{[-l]} \quad (k > 0). \end{aligned}$$

Since $[\delta, \nu_{[0]}] = 0$, the uniqueness in the Jordan–Chevalley decomposition implies that $\delta = \theta_s$ and $\nu_{[0]} = \theta_n$. We argue by induction on k that the recursion equations (1.9) have a solution, such that $S_0^\circ := 1$ and

$$(1.10) \quad \sum_{i=0}^k (-1)^i (S_{k-i}^\circ)^T S_i^\circ = 0, \quad \forall k > 0.$$

Before proving our claim, let us point out the following two properties of the transposition T with respect to the Frobenius pairing:

$$(X_{[a]})^T = (X^T)_{[a]}, \quad X \in \mathfrak{gl}(H), \quad a \in \text{spec}(\delta)$$

and $(\text{ad}_{\nu_{[0]}}(X))^T = \text{ad}_{\nu_{[0]}}(X^T)$. Both properties follow easily from the identities $\delta^T = -\delta$ and $(\nu_{[0]})^T = -\nu_{[0]}$ which can be proved as follows. We have $\theta = -\theta^T = -\delta^T - (\nu_{[0]})^T$. Since δ^T is a diagonalizable operator and $(\nu_{[0]})^T$ is a nilpotent operator the uniqueness of the Jordan–Chevalley decomposition implies that $\delta^T = -\delta$ and $\nu_{[0]}^T = -\nu_{[0]}$.

For $k = 0$ our claim is trivially satisfied. Suppose that we proved the existence of S_i° and $\nu_{[-i]}$ for $1 \leq i \leq k-1$. Decomposing each $S_k^\circ = \sum_{a \in \text{spec}(\delta)} (S_k^\circ)_{[a]}$ into eigenvectors, we get that (1.9) is equivalent to

$$(1.11) \quad (k + a + \text{ad}_{\nu_{[0]}})(S_k^\circ)_{[a]} = (E \bullet S_{k-1}^\circ)_{[a]} + \sum_{l=1}^k (S_{k-l}^\circ)_{[a+l]} \nu_{[-l]},$$

where we used that $\text{ad}_{\nu_{[0]}}$ preserves the eigensubspace $\mathfrak{gl}_a(H)$. If $k + a \neq 0$, then we can uniquely solve for $(S_k^\circ)_{[a]}$, because the operator $k + a + \text{ad}_{\nu_{[0]}}$ is invertible. Hence, $(S_k^\circ)_{[a]}$ is uniquely determined by S_{k-l}° and $\nu_{[-l]}$ for $1 \leq l \leq k-1$. Let us prove that if $a + k \neq 0$, then

$$(1.12) \quad \left(\sum_{i=0}^k (-1)^i (S_{k-i}^\circ)^T S_i^\circ \right)_{[a]} = 0,$$

where note that although S_k° is still not defined completely, only the component $(S_k^\circ)_{[a]}$ is needed and since we have already defined it, our claim makes sense.

Using the recursion formula (1.11), we get that $(k + a + \text{ad}_{\nu_{[0]}}) \left((S_k^\circ)^T + (-1)^k S_k^\circ \right)_{[a]}$ is equal to

$$(1.13) \quad \left((E \bullet S_{k-1}^\circ)_{[a]}^T + \sum_{j=1}^{k-1} \nu_{[-j]}^T (S_{k-j}^\circ)_{[a+j]}^T \right) + (-1)^k \left((E \bullet S_{k-1}^\circ)_{[a]} + \sum_{j=1}^{k-1} (S_{k-j}^\circ)_{[a+j]} \nu_{[-j]} \right).$$

Note that

$$(k + a + \text{ad}_{\nu_{[0]}}) \left((S_{k-i}^\circ)^T S_i^\circ \right)_{[a]} = (k + a + \text{ad}_{\nu_{[0]}}) \sum_{b \in \text{spec}(\delta)} \left((S_{k-i}^\circ)^T \right)_{[b]} (S_i^\circ)_{[a-b]}$$

is equal to

$$\sum_{b \in \text{spec}(\delta)} (k - i + b + \text{ad}_{\nu_{[0]}}) \left((S_{k-i}^\circ)^T \right)_{[b]} (S_i^\circ)_{[a-b]} + (S_{k-i}^\circ)^T_{[b]} (i + a - b + \text{ad}_{\nu_{[0]}}) (S_i^\circ)_{[a-b]}$$

Using the recursion relation (1.11) with (k, a) replaced by $(k-i, b)$ and $(i, a-b)$ we get that $(-1)^i (k + a + \text{ad}_{\nu_{[0]}}) \left((S_{k-i}^\circ)^T S_i^\circ \right)_{[a]}$ is equal to

$$(1.14) \quad (-1)^i \sum_{b \in \text{spec}(\delta)} \left(\left((E \bullet S_{k-i-1}^\circ)_{[b]}^T + \sum_{j=1}^{k-i} \nu_{[-j]}^T (S_{k-i-j}^\circ)_{[b+j]}^T \right) (S_i^\circ)_{[a-b]} + (S_{k-i}^\circ)^T_{[b]} \left((E \bullet S_{i-1}^\circ)_{[a-b]} + \sum_{j=1}^i (S_{i-j}^\circ)_{[a-b+j]} \nu_{[-j]} \right) \right).$$

Since the operator $(k + a + \text{ad}_{\nu_{[0]}}$ for $k \neq -a$ is invertible, formula (1.12) will be proved if we prove that the sum of (1.14) for $1 \leq i \leq k-1$ and (1.13) is 0.

The sum of the terms involving $E \bullet$ cancels out. Indeed, first note that since \bullet is a Frobenius multiplication the operator $E \bullet$ is self-adjoint, that is, $(E \bullet)^T = E \bullet$. This implies that the terms (involving $E \bullet$) in (1.13) cancel out with term on the first line of

(1.14) for $i = k - 1$ and the term on the second line of (1.14) corresponding to $i = 1$. The remaining terms involving $E\bullet$ are

$$\left(\sum_{i=1}^{k-2} (-1)^i (S_{k-i-1}^\circ)^T E\bullet S_i^\circ + \sum_{i=2}^{k-1} (-1)^i (S_{k-i}^\circ)^T E\bullet S_{i-1}^\circ \right)_{[a]},$$

where again we used that $(E\bullet)^T = E\bullet$. Shifting the summation index in the second sum $i \mapsto i + 1$, we see that the above two sums cancel out.

Let us fix j ($1 \leq j \leq k - 1$). We claim that the sum of the terms involving $\nu_{[-j]}^T$ cancels out. Indeed, the sum is equal to

$$\nu_{[-j]}^T \left(\sum_{i=0}^{k-j} (-1)^i \sum_{b \in \text{spec}(\delta)} (S_{k-i-j}^\circ)^T_{[b+j]} (S_i^\circ)_{[a-b]} \right) = \nu_{[-j]}^T \sum_{i=0}^{k-j} (-1)^i \left((S_{k-i-j}^\circ)^T S_i^\circ \right)_{[a+j]} = 0,$$

where we used the inductive assumption (1.10). Similarly, the sum of the terms involving $\nu_{[-j]}$ is

$$\sum_{i=j}^k (-1)^i \left((S_{k-i}^\circ)^T S_{i-j}^\circ \right)_{[a+j]}$$

and it vanishes according to the inductive assumption (1.10).

In order to define S_k° , we need only to specify the component $(S_k^\circ)_{[-k]}$. Let us choose $B_k \in \mathfrak{gl}_{-k}(H)$ arbitrary, such that, $B_k^T = -(-1)^k B_k$, for example pick an arbitrary $B \in \mathfrak{gl}_{-k}(H)$ and let $B_k := B - (-1)^k B^T$. Let us define

$$(S_k^\circ)_{[-k]} := B_k - \frac{1}{2} \sum_{i=1}^{k-1} (-1)^{k-i} \left((S_{k-i}^\circ)^T S_i^\circ \right)_{[-k]}.$$

Note that the symplectic condition (1.12) for $a = -k$ is trivially satisfied due to the above choice of $(S_k^\circ)_{[-k]}$. In other words, the operators S_i° ($1 \leq i \leq k$) satisfy the symplectic condition (1.10) and the recursion (1.11) for all $a \neq -k$. On the other hand, if $a = -k$, then the LHS of (1.11) depends only on operators S_i° and $\nu_{[-i]}$ that are already defined, while the RHS has the form $\nu_{[-k]} + \dots$, where the dots stand for some term in $\mathfrak{gl}_{-k}(H)$ that depends only on S_i° and $\nu_{[-i]}$ for $1 \leq i \leq k - 1$. Therefore, we can define $\nu_{[-k]}$ uniquely, so that (1.11) holds for $a = -k$ too. This completes the induction step and the proof of the proposition. \square

DEFINITION 1.17. Suppose that $t^\circ \in M$ is a given point. An operator series

$$S^\circ(z) = 1 + \sum_{k=1}^{\infty} S_k^\circ z^{-k}, \quad S_k^\circ \in \text{End}(T_{t^\circ} M)$$

is said to be a *calibration* of the Frobenius manifold at the point t° if there exists an operator ν , such that $Y^\circ(z) = S^\circ(z) z^\delta z^\nu$ is a fundamental solution of the restriction of the Dubrovin's connection to $t = t^\circ$ satisfying conditions (i)–(iii) of Proposition 1.16. \square

As we will see now, for a given operator series $S^\circ(z)$, if ν exists, then it is unique. Moreover, we are going to prove that the set \mathcal{S}° of all pairs $(S^\circ(z), \nu)$ satisfying conditions

(i)–(iii) of Proposition 1.16 form a single orbit for the following unipotent group

$$(1.15) \quad G = \left\{ C(z) = 1 + \sum_{m=1}^{\infty} C_m z^{-m} \mid C_m \in \mathfrak{gl}_{-m}(H), C(-z)^T C(z) = 1 \right\}.$$

The group G acts on \mathcal{S}° as follows:

$$C(z) \cdot (S^\circ(z), \nu) = (S^\circ(z)C(z), C(1)^{-1}\nu C(1)), \quad C(z) \in G, \quad (S^\circ(z), \nu) \in \mathcal{S}^\circ.$$

The result is again due to Chenghan Zha (see [66]) and we will follow his argument.

PROPOSITION 1.18. *The action of G on \mathcal{S}° is faithful and transitive.*

PROOF. The fact that the action is faithful is obvious. We need only to prove that it is transitive. Suppose that $(T^\circ(z), \mu) \in \mathcal{S}^\circ$ is an arbitrary element. Since $Y^\circ(z)$ and $\tilde{Y}^\circ(z) = T^\circ(z)z^\delta z^\mu$ are fundamental solutions of the same connection, there exists a constant invertible operator $C \in \mathrm{GL}(H)$ such that

$$T^\circ(z)z^\delta z^\mu = S^\circ(z)z^\delta z^\nu C.$$

We have

$$S^\circ(z)^{-1}T^\circ(z) = z^\delta z^\nu C z^{-\mu} z^{-\delta} = z^{N(z)}C(z)z^{-M(z)},$$

where $N(z) = z^\delta \nu z^{-\delta}$, $M(z) = z^\delta \mu z^{-\delta}$, and $C(z) = z^\delta C z^{-\delta}$.

The main difficulty is to prove that $C(z) \in 1 + \mathfrak{gl}(H)[z^{-1}]z^{-1}$. Suppose that this fact was established. Then

$$(1.16) \quad z^{N(z)}C(z)z^{-M(z)} = e^{N(z)\log z}C(z)e^{-M(z)\log z} = C(z) + \sum_{i=1}^{\infty} P_i(z)(\log z)^i,$$

where $P_i(z) \in \mathfrak{gl}(H)[z^{-1}]$ ($i \geq 1$) and only finitely many $P_i(z)$ are non-zero, because $N(z)$ and $M(z)$ are nilpotent operators. Since (1.16) coincides with $S^\circ(z)^{-1}T^\circ(z)$ which is a single-valued analytic function in a neighborhood of $z = \infty$, we get that $P_i(z) = 0$ for all $i \geq 1$. In particular, $C(z) = S^\circ(z)^{-1}T^\circ(z)$ satisfies $C(-z)^T C(z) = 1$, that is $C(z) \in G$ and $0 = P_1(z) = N(z)C(z) - C(z)M(z)$, that is, $\nu C = C\mu$. Therefore, $C(z) \in G$ and $(T^\circ(z), \mu) = C(z) \cdot (S^\circ(z), \nu)$, which is exactly what we have to prove.

Let us prove that $C(z) \in 1 + \mathfrak{gl}(H)[z^{-1}]z^{-1}$. We will need the following simple lemma:

LEMMA 1.19. *If $\beta_1, \dots, \beta_m \in \mathbb{R} \setminus \{0\}$ are pairwise different real numbers and $C_1, \dots, C_m \in \mathfrak{gl}(H)$ are such that $\lim_{t \rightarrow +\infty} \sum_{i=1}^m C_i e^{\beta_i t \sqrt{-1}}$ exists, then $C_1 = \dots = C_m = 0$.*

PROOF. Put $L(t) = \sum_{i=1}^m C_i e^{\beta_i t \sqrt{-1}}$ and let us choose a real number ϵ , such that, $\frac{\epsilon}{2\pi}(\beta_i - \beta_j) \notin \mathbb{Z}$ for all $1 \leq i < j \leq m$. We have

$$\sum_{i=1}^m C_i e^{\beta_i(t+j\epsilon)\sqrt{-1}} = L(t+j\epsilon), \quad 0 \leq j \leq m-1.$$

This is a system of m linear equations for $C_i e^{\beta_i t \sqrt{-1}}$ whose coefficient matrix is given by λ_i^{j-1} ($1 \leq i, j \leq m$), where $\lambda_i = e^{\beta_i \epsilon}$. The determinant is a Vandermonde determinant, that is, $\prod_{1 \leq i < j \leq m} (\lambda_j - \lambda_i) \neq 0$, according to our choice of ϵ . Therefore, we can write each $C_i e^{\beta_i t \sqrt{-1}}$ as a linear combination of $L(t+j\epsilon)$ ($0 \leq j \leq m-1$) with coefficients independent of t . Since $\lim_{t \rightarrow +\infty} L(t+j\epsilon)$ exists for all j , we get that $\lim_{t \rightarrow +\infty} C_i e^{\beta_i t \sqrt{-1}}$ exists for all i . However $\beta_i \neq 0$, so the limit will exist only if $C_i = 0$. \square

By definition,

$$C(z) = z^\delta C z^{-\delta} = C_{[0]} + \sum_{a \in \text{spec}(\delta) \setminus \{0\}} C_{[a]} z^a.$$

We claim that if $C_{[a]} \neq 0$, then $\text{Re}(a) < 0$. Let m be the maximal number in the set $\{\text{Re}(a) \mid a \in \text{spec}(\delta) \setminus \{0\}, C_{[a]} \neq 0\}$. Let us restrict z to be a positive real number. The limit

$$(1.17) \quad \lim_{z \rightarrow +\infty} C(z) = \lim_{z \rightarrow +\infty} e^{-N(z) \log z} S^\circ(z)^{-1} T^\circ(z) e^{M(z) \log z} = 1.$$

If $m > 0$, then $\lim_{z \rightarrow +\infty} C(z) z^{-m} = 0$. Let $\{a_i\}_{i=1}^s$ be the set of all $a \in \text{spec}(\delta)$, such that $\text{Re}(a) = m$. Then $a_i = m + \sqrt{-1} \beta_i$, where β_i ($1 \leq i \leq s$) are pairwise distinct. Let us write $z = e^t$, then we have

$$0 = \lim_{z \rightarrow +\infty} C(z) z^{-m} = \lim_{t \rightarrow +\infty} \sum_{i=1}^s C_{[a_i]} e^{\sqrt{-1} \beta_i t}.$$

If all β_i are non-zero, then Lemma 1.19 would imply that $C_{[a_i]} = 0$ – contradiction with the fact that $m = \text{Re}(a)$ for some a with $C_{[a]} \neq 0$. If $\beta_i = 0$ for some i , then we get

$$0 = C_{[a_i]} + \lim_{t \rightarrow +\infty} \sum_{j: j \neq i} C_{[a_j]} e^{\sqrt{-1} \beta_j t}.$$

However, we can recall Lemma 1.19 again and we get that $C_{[a_j]} = 0$ for all $j \neq i$ and the above identity will be reduced to $C_{[a_i]} = 0$, that is, we get the same contradiction as above. This proves that $m \leq 0$. If $m = 0$, then we have

$$1 = \lim_{z \rightarrow +\infty} C(z) = C_{[0]} + \lim_{z \rightarrow +\infty} \left(\sum_{a: \text{Re}(a)=0, a \neq 0} C_{[a]} z^a \right).$$

Recalling Lemma 1.19, we get $C_{[a]} = 0$ for all a with $\text{Re}(a) = 0$. This is again contradiction, so our claim that $\text{Re}(a) < 0$ for all a with $C_{[a]} \neq 0$ is proved.

The condition $\lim_{z \rightarrow +\infty} C(z) = 1$ implies that $C_{[0]} = 1$. Note that

$$e^{N(z) \log z} C(z) e^{-M(z) \log z} = S^\circ(z)^{-1} T^\circ(z)$$

is invariant under the analytic continuation around $z = \infty$. Therefore,

$$(1.18) \quad \sum_a e^{2\pi\sqrt{-1}N(z)} C_{[a]} z^a e^{2\pi\sqrt{-1}a} e^{-2\pi\sqrt{-1}M(z)} = \sum_a C_{[a]} z^a.$$

Suppose that there exists some $a \notin \mathbb{Z}$, such that, $C_{[a]} \neq 0$. Put

$$m := \max \{\text{Re}(a) \mid a \notin \mathbb{Z}, C_{[a]} \neq 0\}.$$

Suppose that $a \notin \mathbb{Z}$, $C_{[a]} \neq 0$, and $\text{Re}(a) = m$. Let us compare the coefficients in front of z^a in (1.18). Since $N(z) = \nu_{[0]} + \sum_{l \geq 1} \nu_{[-l]} z^{-l}$ and $M(z) = \mu_{[0]} + \sum_{l \geq 1} \mu_{[-l]} z^{-l}$, the coefficients $\nu_{[-l]}$ and $\mu_{[-l]}$ with $l \geq 1$ do not contribute. Note also that $\mu_{[0]} = \nu_{[0]} = \theta_n$, we get

$$e^{2\pi\sqrt{-1}\nu_{[0]}} C_{[a]} e^{-2\pi\sqrt{-1}\nu_{[0]}} e^{2\pi\sqrt{-1}a} = C_{[a]}.$$

This formula implies that

$$(1.19) \quad e^{2\pi\sqrt{-1}\nu_{[0]}k} C_{[a]} e^{-2\pi\sqrt{-1}\nu_{[0]}k} e^{2\pi\sqrt{-1}ka} = C_{[a]}, \quad \forall k \in \mathbb{Z}.$$

Let us denote by $f_{ij}(x)$ the (i, j) -entry of the matrix $e^{2\pi\sqrt{-1}\nu_{[0]}x}C_{[a]}e^{-2\pi\sqrt{-1}\nu_{[0]}x}$. Note that $f_{ij}(x) \in \mathbb{C}[x]$ is a polynomial because $\nu_{[0]}$ is a nilpotent operator. Formula (1.19) is equivalent to $f_{ij}(k) = e^{-2\pi\sqrt{-1}ak}f_{ij}(0)$ for all $k \in \mathbb{Z}$ and $1 \leq i, j \leq N$. Let us pick (i, j) , such that $f_{ij}(0) \neq 0$. This is possible because $f_{ij}(0)$ is the (i, j) -entry of the non-zero matrix $C_{[a]}$. First observe that $\text{Im}(a) = 0$. Otherwise, if $\text{Im}(a) > 0$ (resp. < 0), then $e^{-2\pi\sqrt{-1}ak}f_{ij}(0)$ has an exponential growth for $k \rightarrow -\infty$ (resp. $k \rightarrow +\infty$), while $f_{ij}(k)$ has at most polynomial growth. This proves that a is a real number. In particular, the set $e^{-2\pi\sqrt{-1}ak}f_{ij}(0)$ ($k \in \mathbb{Z}$) is bounded. The only way that $f_{ij}(k)$ is bounded for $k \in \mathbb{Z}$ is if $f_{ij}(x) = f_{ij}(0)$ is the constant polynomial (\because if the polynomial has positive degree $d > 0$, then $f_{ij}(k) = O(k^d)$ as $k \rightarrow \infty$, so in particular $f_{ij}(k)$ is not bounded). But if $f_{ij}(x)$ is the constant polynomial then clearly $e^{2\pi\sqrt{-1}a} = 1$, which means that $a \in \mathbb{Z}$ – contradiction. This completes the proof of our claim that $C(z) \in 1 + \mathfrak{gl}(H)[z^{-1}]z^{-1}$ and hence the proof of the proposition too. \square

PROPOSITION 1.20. *Suppose that $(S(z), \nu) \in \mathcal{S}^\circ$ is a calibration at a point $t^\circ \in M$. Let us decompose $\nu = \sum_{l=0}^{\infty} \nu_l$ as in Proposition 1.16. Then $\nu_l^T = (-1)^{l+1}\nu_l$.*

PROOF. The fact that the transposition operation T commutes with the projection to $\mathfrak{gl}_{-l}(H)$ is already established in the proof of Proposition 1.16. By definition, $\nu_l = \nu_{[-l]}$. Therefore, $(\nu_l)^T = (\nu^T)_{[-l]}$. It remains only to prove that $(\nu^T)_{[-l]} = (-1)^{l+1}\nu_l$.

By definition $Y(z) = S(z)z^\delta z^\nu$ is a solution to the differential equation

$$z\partial_z Y(z) = \left(-z^{-1}E \bullet + \theta\right)Y(z),$$

where $E \bullet$ is the operator in $H = T_{t^\circ}M$ of Frobenius multiplication by E . Let us assume that z is a positive real number and let us denote by $Y(-z)$ the analytic continuation of $Y(z)$ along the arc $s \mapsto e^{\pi i s}z$, $0 \leq s \leq 1$. Using the above differential equation and the fact that $(E \bullet)^T = E \bullet$ and $\theta^T = -\theta$, we get that $Y(z)^T Y(-z)$ is a constant C independent of z . Using that $S(z)^T S(-z) = 1$, we get

$$C = z^{\nu^T} z^{\delta^T} (-z)^\delta (-z)^\nu.$$

Since $\delta^T = -\delta$ and $[\delta, \nu_l] = -l\nu_l$, the above identity can be written as

$$z^{\nu^T} z^{\sum_{l=0}^{\infty} (-1)^l \nu_l} = C e^{-\pi i \nu} e^{-\pi i \delta}.$$

Setting $z = 1$ we get that the RHS must be 1, that is, $C = e^{\pi i \delta} e^{\pi i \nu}$. Differentiating by $z\partial_z$ and setting $z = 1$, we get $\nu^T + \sum_{l=0}^{\infty} (-1)^l \nu_l = 0$. Projecting this identity to $\mathfrak{gl}_{-l}(H)$ we get $(\nu^T)_{[-l]} + (-1)^l \nu_l = 0$ which completes the proof. \square

1.3.2. Gravitational descendants. The notion of calibration was introduced in the settings of quantum cohomology by Givental [20]. One of its main applications is that it allows us to reconstruct all genus 0 Gromov–Witten invariants. Since we proved that the calibration exists, we can use Givental’s formula (see [20], Corollary 5.4) to define gravitational descendants for any Frobenius manifold.

Let us fix a point $t^\circ \in M$ and denote by $H := T_{t^\circ}M$ the holomorphic tangent space. Suppose that $S^\circ(z)$ is a calibration at the point t° . Put $a = S^\circ(z)e$ and for technical convenience let us assume that $\phi_1 = e$ coincides with the unit of the Frobenius algebra H . The genus 0 total descendant potential of the Frobenius manifold, will be an element of the following ring of formal power series:

$$\mathbb{C}[[q_0 - a, q_1 + e, q_2, q_3, \dots]],$$

where $q_k := \sum_{i=1}^N q_{k,i} \phi_i$ are formal vector variables. The formal power series of the vector variables should be understood as formal power series of the components of the vectors, that is, let us write $a = \sum_{i=1}^N a_i \phi_i$ and recall that $e = \phi_1$, then the above ring is defined to be the ring of formal power series in

$$q_{0,i} - a_i \quad (1 \leq i \leq N), \quad q_{1,i} + \delta_{i,1} \quad (1 \leq i \leq N), \quad q_{k,i} \quad (k \geq 2, 1 \leq i \leq N).$$

Sometimes, it is convenient to substitute $t_0 := q_0 - a$, $t_1 = q_1 + e$, and $t_k = q_k$ for $k \geq 2$, where $t_k = \sum_{i=1}^N t_{k,i} \phi_i$. This gives an identification

$$\mathbb{C}[[q_0 - a, q_1 + e, q_2, q_3, \dots]] = \mathbb{C}[[t_0, t_1, \dots]].$$

Let $\tau_i \in \mathcal{O}_{M, t^\circ}$ ($1 \leq i \leq n$) be the set of holomorphic germs uniquely determined by the following conditions:

- (i) (τ_1, \dots, τ_N) define a holomorphic flat coordinate system on some open neighborhood V of the point t° .
- (ii) $\tau_i(t^\circ) = 0$ ($1 \leq i \leq N$).
- (iii) $\frac{\partial}{\partial \tau_i} = \phi_i$ ($1 \leq i \leq N$) in $T_{t^\circ} M = H$.

Let us trivialize the tangent bundle $TM|_V \cong V \times H$ using the frame of the flat vector fields $\partial_i := \frac{\partial}{\partial \tau_i}$ ($1 \leq i \leq N$). Using the Dubrovin's connection we extend the calibration to a calibration

$$S(t, z) = 1 + \sum_{k=1}^{\infty} S_k(t) z^{-k}, \quad S_k(t) \in \text{End}(H), \quad t \in V,$$

that is, $S(t, z)$ is the solution to the following Cauchy problem:

$$(1.20) \quad \begin{aligned} z \partial_i S(t, z) &= \phi_i \bullet S(t, z) \quad (1 \leq i \leq N) \\ S(t^\circ, z) &= S^\circ(z). \end{aligned}$$

We need also the following operator series:

$$(1.21) \quad \sum_{k, l=0}^{\infty} W_{kl}(t) z^{-k} w^{-l} = \frac{S(t, z)^T S(t, w) - 1}{z^{-1} + w^{-1}},$$

where the RHS is a power series in z^{-1} and w^{-1} thanks to the condition $S(t, -w)^T S(t, w) = 1$. Using the basis $\{\phi_i\}_{i=1}^N$ and the flat coordinates τ_i we represent all linear operators $S_k(t)$ and $W_{kl}(t)$ ($k, l \geq 0$) by matrices whose entries are in $\mathbb{C}\{\tau_1, \dots, \tau_N\}$, that is, convergent power series in τ_1, \dots, τ_N . Finally, we have a canonical embedding $V \subset H$, $t \mapsto \sum_{i=1}^N \tau_i(t) \phi_i$, which is independent of the choice of a basis $\{\phi_i\}_{i=1}^N$ of H . In particular, we have a canonical identification $\mathcal{O}_{H,0} \cong \mathcal{O}_{M, t^\circ} = \mathbb{C}\{\tau_1, \dots, \tau_N\}$.

LEMMA 1.21. *There exists a unique formal power series $f \in H[[q_0 - a, q_1 + e, q_2, \dots]]$, such that*

$$(1.22) \quad q_0 + \sum_{k=1}^{\infty} S_k(f) q_k = 0.$$

PROOF. Since $\partial_i S_1(t) e = \phi_i$ ($1 \leq i \leq N$), we have $S_1(\tau) e = a + \tau$, where $\tau := \sum_{i=1}^N \tau_i(t) \phi_i \in V \subset H$. Therefore the equation (1.22) can be written as

$$q_0 - a + S_1(f)(q_1 + e) + \sum_{k=2}^{\infty} S_k(f) q_k = f.$$

The LHS can be viewed as an operator on $M := H[[q_0 - a, q_1 + e, q_2, \dots]]$, so the problem is to prove that the operator has a fixed point. This however is easy because starting with any initial approximation, say $f^{(0)} := 0$, the iterations

$$f^{(n+1)} := q_0 - a + S_1(f^{(n)})(q_1 + e) + \sum_{k=2}^{\infty} S_k(f^{(n)})q_k, \quad n \geq 0,$$

define a sequence, which is convergent in the formal topology. Clearly, the limit $f := \lim_{n \rightarrow \infty} f^{(n)}$ is a fixed point.

Let $\mathfrak{m} := (q_0 - a, q_1 + e, q_2, \dots)$ be the maximal ideal of $\mathbb{C}[[q_0 - a, q_1 + e, q_2, \dots]]$. Note that any solution f to our problem belongs to $\mathfrak{m}M$. Suppose that f and g are two different solutions. If $f - g \in \mathfrak{m}^n M$, then $S_k(f) - S_k(g) \in \mathfrak{m}^n M$ and since

$$f - g = (S_1(f) - S_1(g))(q_1 + e) + \sum_{k=2}^{\infty} (S_k(f) - S_k(g))q_k$$

we get that $f - g \in \mathfrak{m}^{n+1}M$. Arguing by induction, we get that $f - g \in \bigcap_{n \geq 1} \mathfrak{m}^n M = 0$. \square

Let $f(\mathbf{q}) \in H[[q_0 - a, q_1 + e, q_2, \dots]]$ be a solution to (1.22), then the *genus-0 descendant potential* of the Frobenius manifold is defined by

$$(1.23) \quad \mathcal{F}(\mathbf{q}) := \frac{1}{2} \sum_{k,l=0}^{\infty} (W_{kl}(f(\mathbf{q}))q_l, q_k),$$

where the argument of \mathcal{F} and f is the sequence of all formal variables $\mathbf{q} = (q_{k,i})$. The series \mathcal{F} can be written also in terms of the formal variables $\mathbf{t} = (t_{k,i})$. If $\kappa = ((k_1, i_1), \dots, (k_s, i_s))$ is a sequence of pairs, then we denote by $|\text{Aut}(\kappa)|$ the number of permutations that leave the sequence invariant. The coefficient in front of the monomial

$$\frac{1}{|\text{Aut}(\kappa)|} t_{k_1, i_1} \cdots t_{k_s, i_s}$$

will be denoted by the correlator

$$\langle \phi_{i_1} \psi^{k_1}, \dots, \phi_{i_s} \psi^{k_s} \rangle_{0,s}.$$

We refer to such correlators as *genus-0 gravitational descendants*.

PROPOSITION 1.22. *The genus-0 descendant potential \mathcal{F} satisfies the following three relations:*

a) *String Equation (SE):*

$$\frac{1}{2}(q_0, q_0) + \sum_{k=0}^{\infty} \sum_{i=1}^N q_{k+1,i} \frac{\partial \mathcal{F}}{\partial q_{k,i}} = 0.$$

b) *Dilaton Equation (DE):*

$$\sum_{k=0}^{\infty} \sum_{i=1}^N q_{k,i} \frac{\partial \mathcal{F}}{\partial q_{k,i}} = 2\mathcal{F}.$$

c) *Topological Recursion Relations (TRR):*

$$\frac{\partial^3 \mathcal{F}}{\partial q_{k+1,a} \partial q_{l,b} \partial q_{m,c}} = \sum_{i,j=1}^N \frac{\partial^2 \mathcal{F}}{\partial q_{k,a} \partial q_{0,i}} g^{ij} \frac{\partial^3 \mathcal{F}}{\partial q_{0,j} \partial q_{l,b} \partial q_{m,c}},$$

where $g^{ij} = (\phi^i, \phi^j)$ is the matrix of the Frobenius pairing with respect to the dual basis.

PROOF. Parts a) and b) follow easily from the following formula:

$$\frac{\partial \mathcal{F}}{\partial q_{k,a}} = \frac{1}{2} \sum_{l=0}^{\infty} \left((W_{kl}(f)q_l, \phi_a) + (W_{lk}(f)\phi_a, q_l) \right).$$

For the proof, note that the partial derivative is supposed to have one more term, that is,

$$(1.24) \quad \sum_{l,m=0}^{\infty} (\partial_{q_{k,a}}(W_{lm}(f))q_m, q_l).$$

Recalling the definition of W_{lm} , we get

$$\partial_i W_{lm}(t) = S_l(t)^T (\phi_i \bullet) S_m(t).$$

Therefore,

$$(1.25) \quad \partial_{q_{k,a}}(W_{lm}(f)) = \sum_{i=1}^N \frac{\partial f_i}{\partial q_{k,a}} S_l(f)^T (\phi_i \bullet) S_m(f).$$

Substituting this formula in (1.24), we get a factor that involves $\sum_{l=0}^{\infty} S_l(f)q_l = 0$. The rest of the details in the proof of a) and b) are left as an exercise.

Let us prove c). Similar argument proves that

$$(1.26) \quad \frac{\partial^2 \mathcal{F}}{\partial q_{k,a} \partial q_{l,b}} = (W_{kl}(f)\phi_b, \phi_a).$$

Let us prove first the following formula:

$$(1.27) \quad \frac{\partial f}{\partial q_{m,i}} = \frac{\partial f}{\partial q_{0,1}} \bullet S_m(f)\phi_i.$$

Using formula (1.26), we get

$$a + f = S_1(f)e = \sum_{j=1}^N (W_{00}\phi_1, \phi_j)\phi^j = \sum_{j=1}^N \frac{\partial^2 \mathcal{F}}{\partial q_{0,j} \partial q_{0,1}} \phi^j,$$

where we used that $W_{00} = S_1$. Differentiating the above formula with respect to $q_{m,i}$, we get

$$\frac{\partial f}{\partial q_{m,i}} = \sum_{j=1}^N \partial_{q_{0,1}} \left((W_{0m}\phi_i, \phi_j)\phi^j \right) = \partial_{q_{0,1}} S_{m+1}(f)\phi_i = \frac{\partial f}{\partial q_{0,1}} \bullet S_m(f)\phi_i,$$

where we used that $W_{0m} = S_{m+1}$ and $\partial_a S_{m+1}(t) = \phi_a \bullet S_m(t)$.

The 3-jet derivative becomes

$$(1.28) \quad \frac{\partial^3 \mathcal{F}}{\partial q_{k+1,a} \partial q_{l,b} \partial q_{m,c}} = \left(\frac{\partial f}{\partial q_{k+1,a}}, S_l(f)\phi_b \bullet S_m(f)\phi_c \right),$$

where we used formulas (1.26) and (1.25). Recalling formula (1.27) and the Frobenius property of the pairing, we get

$$\left(\frac{\partial f}{\partial q_{0,1}} \bullet S_l(f)\phi_b \bullet S_m(f)\phi_c, S_{k+1}\phi_a \right) = \sum_{i,j=1}^N \left(\frac{\partial f}{\partial q_{0,1}} \bullet S_l(f)\phi_b \bullet S_m(f)\phi_c, \phi_j \right) g^{ij} (\phi_i, S_{k+1}\phi_a).$$

Formula (1.27) implies that $\frac{\partial f}{\partial q_{0,1}} \bullet \phi_j = \frac{\partial f}{\partial q_{0,j}}$, which in combination with (1.28) gives

$$\left(\frac{\partial f}{\partial q_{0,1}} \bullet S_l(f)\phi_b \bullet S_m(f)\phi_c, \phi_j \right) = \frac{\partial^3 \mathcal{F}}{\partial q_{l,b} \partial q_{m,c} \partial q_{0,j}}.$$

Formula (1.26) and $W_{0k} = S_{k+1}$ imply that

$$(\phi_i, S_{k+1}\phi_a) = \frac{\partial^2 \mathcal{F}}{\partial q_{0,i} \partial q_{k,a}}. \quad \square$$

1.4. Givental's quantization formalism

Recall that a *symplectic pairing* on a complex vector space \mathcal{H} is a complex bi-linear pairing $\Omega : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$, satisfying the following two conditions:

- (i) It is perfect: if $\Omega(v, w) = 0$ for all $w \in \mathcal{H}$, then $v = 0$.
- (ii) It is skew-symmetric: $\Omega(v, w) = -\Omega(w, v)$ for all $v, w \in \mathcal{H}$.

The pair (\mathcal{H}, Ω) of a complex vector space \mathcal{H} and a symplectic pairing Ω is called *symplectic vector space* over \mathbb{C} . In this book we will work only with coefficients in \mathbb{C} and if the symplectic pairing Ω is understood from the context, then we will simply say that \mathcal{H} is a symplectic vector space. Furthermore, a subspace $L \subset \mathcal{H}$ is said to be *Lagrangian* if it is maximally isotropic with respect to Ω , that is, $\Omega(v, w) = 0$ for all $v, w \in L$ and if $w \in \mathcal{H}$ is such that $\Omega(v, w) = 0$ for all $v \in L$, then $w \in L$. Finally, a *polarization* of a symplectic vector space \mathcal{H} is a direct sum vector spaces decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, such that, both subspaces \mathcal{H}_+ and \mathcal{H}_- are Lagrangian.

If \mathcal{H} is a finite dimensional polarized symplectic vector space, then there is a natural way to assign to every polynomial function h on \mathcal{H} a differential operator \hat{h} . The correspondence $h \mapsto \hat{h}$ is very similar to the canonical quantization procedure in quantum mechanics. It defines a projective representation of the Lie algebra of at most quadratic polynomials h – see Section 1.4.1. Givental discovered that the quantization can be extended to a certain class of infinite dimensional symplectic vector spaces and that the resulting tool for constructing differential operators has very important applications to both Gromov–Witten theory (see [20]) and the theory of integrable systems (see [22]). The main goal of this section is to recall Givental's symplectic loop space and the corresponding quantization formalism.

1.4.1. Finite dimensional symplectic vector spaces. We would like to explain the idea of Givental's quantization first in the case of a finite dimensional symplectic vector space. The advantage is that now we can use the notion of holomorphic functions and holomorphic vector fields, which will help us to understand the formal definitions in the infinite dimensional case.

1.4.1.1. *Holomorphic symplectic manifolds.* Symplectic geometry is usually embedded in the category of real smooth manifolds (see [5] for some background). The definitions however make sense also in the category of complex manifolds. Let us recall the basic terminology. A complex manifold \mathcal{H} is said to be holomorphic symplectic if it is equipped with a global holomorphic 2-form ω which is closed and non-degenerate, that is, $d\omega = 0$ and if $v \in T_t\mathcal{H}$ satisfies $\omega(v, w) = 0$ for all $w \in T_t\mathcal{H}$, then $v = 0$. Let us recall the operation ι_X of contraction a holomorphic form by a holomorphic vector field X . If ω is a 2-form, then $\iota_X\omega = \langle \omega, X \rangle$ is just the natural pairing between sections of $T\mathcal{H}$ and

its dual $T^*\mathcal{H}$. For higher-degree forms ι_X is extended uniquely in such a way that the graded Leibnitz rule holds

$$\iota_X(\omega_1 \wedge \omega_2) = \iota_X(\omega_1) \wedge \omega_2 + (-1)^{p_1} \omega_1 \wedge \iota_X(\omega_2),$$

where ω_1 is a holomorphic p_1 -form. If ω is a p -form and X_1, \dots, X_p are holomorphic vector fields, then we put

$$\omega(X_1, \dots, X_p) := \iota_{X_p} \cdots \iota_{X_1} \omega.$$

Every holomorphic function $h \in \mathcal{O}(\mathcal{H})$ determines a holomorphic vector field X , such that, $dh = -\iota_X \omega$. The vector field X is called *Hamiltonian* and the corresponding holomorphic function h is called the *Hamiltonian* of X . Let us denote by X_h the Hamiltonian vector field corresponding to h . The algebra of holomorphic functions $\mathcal{O}(\mathcal{H})$ is equipped with a *Poisson bracket*

$$\{f, g\} := \omega(X_f, X_g) = \iota_{X_g} \iota_{X_f} \omega, \quad f, g \in \mathcal{O}(\mathcal{H}).$$

In other words, the bracket $\{, \}$ is a Lie bracket satisfying the Leibniz rule

$$\{f, g_1 g_2\} = \{f, g_1\} g_2 + g_1 \{f, g_2\}, \quad f, g_1, g_2 \in \mathcal{O}(\mathcal{H}).$$

The standard example of a holomorphic symplectic manifold is \mathbb{C}^{2n} with symplectic form

$$(1.29) \quad \omega := dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n,$$

where $(q_1, \dots, q_n, p_1, \dots, p_n)$ is the standard coordinates systems on \mathbb{C}^{2n} . The Poisson bracket takes the following form:

$$(1.30) \quad \{f, g\} := \sum_{i=1}^n \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right), \quad f, g \in \mathcal{O}(\mathbb{C}^{2n}).$$

The holomorphic version of the Darboux theorem holds – the proof in the real case works in the holomorphic case too. In other words, every point of a holomorphic symplectic manifold admits a local holomorphic coordinate system $(q_1, \dots, q_n, p_1, \dots, p_n)$, such that, the symplectic form and the Poisson bracket take the form respectively (1.29) and (1.30). The coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ are called *Darboux coordinates*. Note that the Poisson bracket of the Darboux coordinates takes the form

$$\{p_i, q_j\} = \delta_{i,j}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0, \quad 1 \leq i, j \leq n.$$

We will refer to these relations as the *canonical Poisson bracket relations*. Note that a holomorphic coordinate system consists of Darboux coordinates if and only if the coordinate functions satisfy the canonical Poisson bracket relations.

1.4.1.2. *Symplectic vector spaces*. Suppose that (\mathcal{H}, Ω) is a finite dimensional symplectic vector space \mathcal{H} . The tangent bundle $T\mathcal{H}$ has a natural trivialization defined via the linear structure on \mathcal{H}

$$\mathcal{H} \times \mathcal{H} \cong T\mathcal{H}, \quad (v, w) \mapsto X_w(v),$$

where $X_w(v) \in T_v \mathcal{H} = \text{Der}(\mathcal{O}_{\mathcal{H},v}, \mathbb{C})$ is the directional derivative

$$X_w(v)f = \left. \frac{d}{d\epsilon} f(v + \epsilon w) \right|_{\epsilon=0}, \quad f \in \mathcal{O}_{\mathcal{H},v}.$$

Using the trivialization we can extend the symplectic pairing Ω to a symplectic form ω on \mathcal{H} . The form ω is uniquely determined by the following relation:

$$\omega(X_{w'}, X_{w''}) = \Omega(w', w''), \quad w', w'' \in \mathcal{H}.$$

Given $w \in \mathcal{H}$, let us denote by $h_w \in \mathcal{O}(\mathcal{H})$ the linear function defined by $h_w(v) := \Omega(v, w)$.

PROPOSITION 1.23. a) For every $w \in \mathcal{H}$, the vector field X_w is Hamiltonian with Hamiltonian h_w .

b) The Poisson bracket $\{h_{w'}, h_{w''}\} = \Omega(w', w'')$.

PROOF. a) We have to prove that $dh_w = -\iota_{X_w}\omega$. Since every holomorphic vector field on \mathcal{H} is a $\mathcal{O}_{\mathcal{H}}$ -linear combination of vector fields of the form $X_{w'}$, it is sufficient to check that $\iota_{X_{w'}}dh_w = -\iota_{X_{w'}}\iota_{X_w}\omega$. The LHS is

$$X_{w'}h_w(v) = \left. \frac{d}{d\epsilon} h_w(v + \epsilon w') \right|_{\epsilon=0} = \Omega(w', w) = \omega(X_{w'}, X_w)$$

which is exactly what we have to prove. Part b) follows from a) and the definition of the Poisson bracket. \square

Suppose that $A : \mathcal{H} \rightarrow \mathcal{H}$ is a linear transformation. Let X_A be the vector field on \mathcal{H} which under the trivialization isomorphism $T\mathcal{H} \cong \mathcal{H} \times \mathcal{H}$ takes the form $X_A(v) := -Av \in \mathcal{H} \cong T_v\mathcal{H}$. A linear transformation A is said to be an *infinitesimal symplectic transformation* if

$$\Omega(Av, w) + \Omega(v, Aw) = 0, \quad \forall v, w \in \mathcal{H}.$$

The set of all infinitesimal symplectic transformations is a complex Lie algebra with bracket the commutator $[A, B] := AB - BA$. Let us denote by $\mathfrak{sp}(\mathcal{H}, \Omega)$ the Lie algebra of all infinitesimal symplectic transformations. If the symplectic form Ω is understood from the context we will simply write $\mathfrak{sp}(\mathcal{H})$ instead of $\mathfrak{sp}(\mathcal{H}, \Omega)$. Given $A \in \mathfrak{sp}(\mathcal{H})$, let us denote by $h_A \in \mathcal{O}(\mathcal{H})$ the quadratic function defined by $h_A(v) := \frac{1}{2}\Omega(Av, v)$.

PROPOSITION 1.24. a) The vector field X_A is Hamiltonian if and only if $A \in \mathfrak{sp}(\mathcal{H})$. In addition, if X_A is Hamiltonian, then the corresponding Hamiltonian is h_A .

b) The following commutation relations hold:

$$\{h_A, h_B\} = h_{[A, B]}, \quad A, B \in \mathfrak{sp}(\mathcal{H})$$

and

$$\{h_A, h_w\} = h_{Aw}, \quad A \in \mathfrak{sp}(\mathcal{H}), \quad w \in \mathcal{H}.$$

PROOF. a) Just like in the proof of Proposition 1.23, it is sufficient to prove that if there exists an $h \in \mathcal{O}(\mathcal{H})$, such that,

$$\iota_{X_w}dh = -\iota_{X_w}\iota_{X_A}\omega = \omega(X_w, X_A),$$

then $A \in \mathfrak{sp}(\mathcal{H})$ and $h = h_A + \text{const}$. Since the symplectic pairing Ω is non-degenerate, there exists a pair of bases $\{\phi_i\}_{i=1}^{2n}$ and $\{\phi^i\}_{i=1}^{2n}$, such that, $\Omega(\phi^i, \phi_j) = \delta_{ij}$ for all $1 \leq i, j \leq 2n$. Note that every $w \in \mathcal{H}$ can be written in the form $w = \sum_{i=1}^{2n} \Omega(\phi^i, w)\phi_i$. In particular,

$$Av = \sum_{i=1}^{2n} \Omega(\phi^i, Av)\phi_i \quad \Rightarrow \quad X_A = - \sum_{i=1}^{2n} \Omega(\phi^i, Av)X_{\phi_i}.$$

Therefore,

$$\omega(X_w, X_A)(v) = - \sum_{i=1}^{2n} \Omega(\phi^i, Av)\omega(X_w, X_{\phi_i}) = - \sum_{i=1}^{2n} \Omega(\phi^i, Av)\Omega(w, \phi_i) = \Omega(Av, w).$$

On the other hand, the function $(\iota_{X_w} dh)(v) = (X_w h)(v)$ is linear in v for all $w \in \mathcal{H}$ if and only if, up to a constant, h is quadratic in v . Note that every quadratic function on \mathcal{H} can be written in the form $h(v) = \frac{1}{2}\Omega(Bv, v)$ for some linear transformation B . Since Ω is non-degenerate, every linear transformation B has a transpose B^T defined by $\Omega(B^T v, w) := \Omega(v, Bw)$. We have $(B^T)^T = B$ and $B \in \mathfrak{sp}(\mathcal{H})$ if and only if $B^T = -B$. Replacing B by $\frac{1}{2}(B - B^T)$ if necessary, we may assume that $B \in \mathfrak{sp}(\mathcal{H})$. We have

$$(\iota_{X_w} dh)(v) = \left. \frac{d}{d\epsilon} \frac{1}{2}\Omega(B(v + \epsilon w), v + \epsilon w) \right|_{\epsilon=0} = \frac{1}{2}(\Omega(Bw, v) + \Omega(Bv, w)) = \Omega(Bv, w).$$

Therefore, X_A will be Hamiltonian if and only if $A = B$ which completes the proof of a).

Let us prove b). Just like in a) we have

$$X_A = - \sum_{i=1}^{2n} \Omega(\phi^i, Av) X_{\phi_i},$$

a similar formula for X_B , and $\omega(X_{\phi_i}, X_{\phi_j}) = \Omega(\phi_i, \phi_j)$. We get

$$\{h_A, h_B\}(v) = \omega(X_A, X_B)(v) = \sum_{i,j=1}^{2n} \Omega(\phi^i, Av) \Omega(\phi^j, Bv) \Omega(\phi_i, \phi_j) = \Omega(Av, Bv).$$

On the other hand

$$h_{[A,B]}(v) = \frac{1}{2}\Omega([A, B]v, v) = \frac{1}{2}(\Omega(ABv, v) - \Omega(BAv, v)) = \Omega(Av, Bv),$$

where we used that A and B are infinitesimal symplectic transformations to transform $\Omega(BAv, v) = -\Omega(Av, Bv)$ and

$$\Omega(ABv, v) = -\Omega(Bv, Av) = \Omega(Av, Bv).$$

For the 2nd commutation relation we use a similar argument. Namely,

$$\{h_A, h_w\}(v) = - \sum_{i=1}^{2n} \Omega(\phi^i, Av) \Omega(\phi_i, w) = -\Omega(Av, w) = \Omega(v, Aw) = h_{Aw}(v). \quad \square$$

1.4.1.3. Quantization of at most quadratic Hamiltonians. Suppose that (\mathcal{H}, Ω) is a finite dimensional symplectic vector space equipped with a polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Note that the subspaces \mathcal{H}_+ and \mathcal{H}_- are dual to each other, that is, using the symplectic form Ω we can identify \mathcal{H}_- (resp. \mathcal{H}_+) with the dual of \mathcal{H}_+ (resp. \mathcal{H}_-). There is a natural way to construct Darboux coordinates from the polarization. Let us fix a basis ϕ_i ($1 \leq i \leq n$) of \mathcal{H}_+ . The Lagrangian subspace \mathcal{H}_- has a uniquely determined basis ϕ^i ($1 \leq i \leq n$), such that, $\Omega(\phi^i, \phi_j) = \delta_{ij}$ for all $1 \leq i, j \leq n$. Let us define the linear functions

$$p_i(v) := \Omega(v, \phi_i), \quad q_i(v) := \Omega(\phi^i, v), \quad 1 \leq i \leq n.$$

We have an isomorphism of complex vector spaces

$$\varphi : \mathcal{H} \rightarrow \mathbb{C}^{2n}, \quad v \mapsto (q_1(v), \dots, q_n(v), p_1(v), \dots, p_n(v)).$$

In other words, the functions $(q_1, \dots, q_n, p_1, \dots, p_n)$ are global holomorphic coordinates on the complex manifold \mathcal{H} . Note that in the notation from the previous section $p_i = h_{\phi_i}$ and $q_i = -h_{\phi^i}$. Recalling Proposition 1.23 we get

$$\{p_i, q_j\} = -\Omega(\phi_i, \phi^j) = \delta_{i,j}, \quad 1 \leq i, j \leq n.$$

Similarly, $\{p_i, p_j\} = \{q_i, q_j\} = 0$. Therefore, the coordinate functions $(q_1, \dots, q_n, p_1, \dots, p_n)$ satisfy the canonical Poisson bracket relations, that is, $(q_1, \dots, q_n, p_1, \dots, p_n)$ is a Darboux coordinate system on \mathcal{H} .

Let \hbar be a positive real number. Given a polynomial function $h \in \mathbb{C}[q_1, \dots, q_n, p_1, \dots, p_n]$, we denote by \widehat{h} the differential operator obtained from h by the following two operations:

- (i) Normal ordering: for every monomial of h , move each p -variable to the right of all q -variables.
- (ii) Substitution:

$$q_i \mapsto q_i/\sqrt{\hbar}, \quad p_i \mapsto \sqrt{\hbar} \frac{\partial}{\partial q_i}, \quad 1 \leq i \leq n.$$

For example, $(p_1^3 q_1 q_2)^\wedge = \sqrt{\hbar} q_1 q_2 \frac{\partial^3}{\partial q_1^3}$. Let us denote by \mathfrak{g} the vector space of all polynomials h of degree ≤ 2 . Note that if $h_1, h_2 \in \mathfrak{g}$, then $\{h_1, h_2\} \in \mathfrak{g}$. Therefore, \mathfrak{g} is a Lie algebra. Let us define a *cocycle* C on \mathfrak{g} . Recall that a 2-cocycle on \mathfrak{g} with coefficients in \mathbb{C} is a bi-linear form $C : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ satisfying the following two conditions:

- (i) Skew-symmetry: $C(h_1, h_2) = -C(h_2, h_1)$ for all $h_1, h_2 \in \mathfrak{g}$.
- (ii) Jacobi identity:

$$C(h_1, \{h_2, h_3\}) = C(\{h_1, h_2\}, h_3) + C(h_2, \{h_1, h_3\}), \quad h_1, h_2, h_3 \in \mathfrak{g}.$$

Every 2-cocycle with coefficients in \mathbb{C} defines a central extension, i.e., the bracket

$$[(h_1, c_1), (h_2, c_2)] := (\{h_1, h_2\}, C(h_1, h_2)), \quad h_1, h_2 \in \mathfrak{g}, \quad c_1, c_2 \in \mathbb{C},$$

is a Lie bracket on $\mathfrak{g} \oplus \mathbb{C}$. Let us define a skew-symmetric bilinear pairing C on \mathfrak{g} , such that,

- (i) If h_1 or h_2 have degree ≤ 1 , then $C(h_1, h_2) = 0$.
- (ii) If h_1 and h_2 are quadratic Darboux monomials, then $C(h_1, h_2)$ is non-zero only in the following two cases:

$$C(p_i p_j, q_i q_j) = 1 \quad (1 \leq i \neq j \leq n), \quad C(p_i^2, q_i^2) = 2 \quad (1 \leq i \leq n).$$

PROPOSITION 1.25. *If $h_1, h_2 \in \mathfrak{g}$, then*

$$[\widehat{h}_1, \widehat{h}_2] = \{h_1, h_2\}^\wedge + C(h_1, h_2).$$

The proof of Proposition 1.25 is straightforward and it will be left as an exercise. In other words, the map $h \mapsto \widehat{h}$ defines a *projective* representation of \mathfrak{g} on the space $\mathbb{C}[q_1, \dots, q_n]$ of polynomial functions on \mathcal{H}_+ . We will say that \widehat{h} is the *quantization* of h .

Let us finish this subsection by giving a closed formula for the co-cycle C . Note that according to Propositions 1.23 and 1.24, we have the following isomorphism of vector spaces:

$$\mathbb{C} \times \mathcal{H} \times \mathfrak{sp}(\mathcal{H}) \cong \mathfrak{g}, \quad (c, w, A) \mapsto c + h_w + h_A,$$

and the Poisson bracket on \mathfrak{g} becomes

$$\{(c_1, w_1, A_1), (c_2, w_2, A_2)\} = (\Omega(w_1, w_2), A_1 w_2 - A_2 w_1, [A_1, A_2]).$$

Let $J : \mathcal{H} \rightarrow \mathcal{H}$ be the linear operator defined by the following two conditions: $J(v) = v$ for all $v \in \mathcal{H}_+$ and $J(v) = -v$ for all $v \in \mathcal{H}_-$. Note that J is an infinitesimal symplectic transformation. We have

$$(1.31) \quad C((c_1, w_1, A_1), (c_2, w_2, A_2)) = \frac{1}{2} \operatorname{Tr}(A_1 J A_2).$$

1.4.2. The symplectic loop space. Let H be a finite-dimensional complex vector space of complex dimension N . Suppose that H is equipped with a distinguished vector $\mathbf{1}$ and a non-degenerate, symmetric bi-linear pairing (\cdot, \cdot) . Let $H[z, z^{-1}]$ be the vector space of Laurent polynomials in z with coefficients in H . Following Givental, we introduce the symplectic form

$$(1.32) \quad \Omega(f, g) = \text{Res}_{z=0}(f(-z), g(z))dz, \quad f, g \in H[z, z^{-1}],$$

where the residue is understood formally as the coefficient in front of dz/z . The vector space $H[z, z^{-1}]$ is equipped with a topology in which a basis of open neighborhoods of 0 is given by the vector subspaces $z^{-n}H[z^{-1}]$ ($n = 1, 2, \dots$). The completion of $H[z, z^{-1}]$ with respect to the above topology is naturally identified with the vector space $\mathcal{H} := H((z^{-1}))$ consisting of formal Laurent series in z^{-1} with coefficients in H . Moreover, if we equip \mathbb{C} with the discrete topology, then the symplectic pairing Ω defines a continuous map $H[z, z^{-1}] \times H[z, z^{-1}] \rightarrow \mathbb{C}$. Therefore, Ω extends by continuity to a skew-symmetric bilinear pairing on \mathcal{H} . It is easy to check that (\mathcal{H}, Ω) is a symplectic vector space. We will refer to it as the *symplectic loop space*. The symplectic vector space \mathcal{H} has a standard polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+ := H[z]$ and $\mathcal{H}_- := H[[z^{-1}]]z^{-1}$ are Lagrangian subspaces.

Let us fix a pair of bases $\{\phi_i\}_{i=1}^N$ and $\{\phi^i\}_{i=1}^N$ of H dual with respect to the pairing: $(\phi_i, \phi^j) = \delta_{i,j}$. Note that $\phi_i z^k$ ($1 \leq i \leq N, k \geq 0$) is a basis of \mathcal{H}_+ and that

$$\Omega(\phi^i(-z)^{-k-1}, \phi_j z^l) = \delta_{ij} \delta_{kl}, \quad 1 \leq i, j \leq N, \quad k, l \geq 0.$$

Motivated by our discussion in the finite dimensional case (see Section 1.4.1.3) we introduce the linear functions on \mathcal{H} defined by

$$p_{k,i}(v) := \Omega(v, \phi_i z^k), \quad q_{k,i}(v) := \Omega(\phi^i(-z)^{-k-1}, v), \quad v \in \mathcal{H}$$

and we will refer to them as *Darboux coordinates on \mathcal{H}* . Note that

$$v = \sum_{k=0}^{\infty} \sum_{i=1}^N p_{k,i}(v) \phi^i(-z)^{-k-1} + \sum_{k=0}^{\infty} \sum_{i=1}^N q_{k,i}(v) \phi_i z^k, \quad v \in \mathcal{H},$$

so the values of $p_{k,i}$ and $q_{k,i}$ on a given vector $v \in \mathcal{H}$ uniquely determine v .

We will be interested in linear and quadratic Hamiltonians on \mathcal{H} . The former are defined just like in the finite dimensional case, that is,

$$h_f(v) := \Omega(v, f), \quad f \in \mathcal{H}.$$

In order to define quadratic Hamiltonians however, we have to be a little bit more careful with the choice of an infinitesimal symplectic transformation. By definition, the vector space \mathcal{H} has a topology in which a basis of open neighborhoods of $0 \in \mathcal{H}$ is given by the subspaces $z^{-n}\mathcal{H}_-$ ($n \in \mathbb{Z}$). Some subset $U \subset \mathcal{H}$ is open if for every $f \in U$, there exists an integer n , such that, $f + z^{-n}\mathcal{H}_- \subset U$. Let us denote by $\mathfrak{sp}(\mathcal{H})$ the set of all *continuous* infinitesimal symplectic transformations, that is, linear transformations $A : \mathcal{H} \rightarrow \mathcal{H}$, such that,

$$\Omega(Av, w) + \Omega(v, Aw) = 0, \quad \forall v, w \in \mathcal{H}$$

and for every $n \in \mathbb{Z}$, there exists $k_0 \in \mathbb{Z}$, such that, $A(z^{-k}\mathcal{H}_-) \subset z^{-n}\mathcal{H}_-$ for all $k \geq k_0$. Let us characterize the matrices of the linear operators in $\mathfrak{sp}(\mathcal{H})$. Suppose that $A : \mathcal{H} \rightarrow \mathcal{H}$

is a linear transformation. Let us define the numbers $A_{ki,lj}^{ab}$ ($a, b \in \{+, -\}$, $1 \leq i, j \leq N$, $k, l \geq 0$) by the following decompositions

$$\begin{aligned} A\phi_j z^l &=: \sum_{k=0}^{\infty} \sum_{i=1}^N A_{ki,lj}^{++} \phi_i z^k + \sum_{k=0}^{\infty} \sum_{i=1}^N A_{ki,lj}^{-+} \phi_i (-z)^{-k-1}, \\ A\phi^j (-z)^{-l-1} &=: \sum_{k=0}^{\infty} \sum_{i=1}^N A_{ki,lj}^{+-} \phi_i z^k + \sum_{k=0}^{\infty} \sum_{i=1}^N A_{ki,lj}^{--} \phi_i (-z)^{-k-1}. \end{aligned}$$

Let us consider matrices whose rows and columns are enumerated by pairs (k, i) where $k \geq 0$ and $1 \leq i \leq N$ are integers. Let A^{ab} be the matrix with entries $A_{ki,lj}^{ab}$. We will refer to the block matrix $\begin{bmatrix} A^{++} & A^{+-} \\ A^{-+} & A^{--} \end{bmatrix}$ as the matrix of the linear operator A with respect to the basis $\phi_i z^k, \phi^i (-z)^{-k-1}$ ($1 \leq i \leq N, k \geq 0$).

PROPOSITION 1.26. *Suppose that A^{ab} ($a, b \in \{+, -\}$) are matrices with entries $A_{ki,lj}^{ab}$ given by some complex numbers. The block matrix $\begin{bmatrix} A^{++} & A^{+-} \\ A^{-+} & A^{--} \end{bmatrix}$ is a matrix of some operator $A \in \mathfrak{sp}(\mathcal{H})$ if and only if the following conditions are satisfied:*

- (i) *The matrices A^{ab} with $(a, b) = (+, -)$ or $(-, +)$ are symmetric, that is, $A_{ki,lj}^{ab} = A_{lj,ki}^{ab}$ for all (k, i) and (l, j) .*
- (ii) *$(A^{++})^T = -A^{--}$, that is, $A_{ki,lj}^{++} = -A_{lj,ki}^{--}$ for all (k, i) and (l, j) .*
- (iii) *The matrix A^{+-} has only finitely many non-zero entries.*
- (iv) *Each column (resp. row) of the matrix A^{++} (resp. A^{--}) has only finitely many non-zero entries.*

The proof is straightforward and it will be left as an exercise. The quadratic Hamiltonian

$$h_A(v) := \frac{1}{2} \Omega(Av, v), \quad A \in \mathfrak{sp}(\mathcal{H}),$$

takes the form

$$(1.33) \quad h_A = \frac{1}{2} \sum_{k,l=0}^{\infty} \sum_{i,j=1}^N \left(-A_{ki,lj}^{++} p_{k,i} q_{l,j} - A_{ki,lj}^{+-} p_{k,i} p_{l,j} + A_{ki,lj}^{-+} q_{k,i} q_{l,j} + A_{ki,lj}^{--} q_{k,i} p_{l,j} \right),$$

while the linear Hamiltonian becomes

$$(1.34) \quad h_f = \sum_{k=0}^{\infty} \sum_{i=1}^N \left(p_{k,i} \Omega(\phi^i (-z)^{-k-1}, f) + q_{k,i} \Omega(\phi_i z^k, f) \right).$$

Let \mathfrak{g} be the vector subspace of $\mathbb{C}[\mathbf{p}][[\mathbf{q}]]$ consisting of at most quadratic elements. More precisely, here $\mathbf{p} := (p_{k,i})$, $\mathbf{q} := (q_{k,i})$ ($1 \leq i \leq N, k \geq 0$), and the elements of \mathfrak{g} are formal power series in \mathbf{q} whose coefficients are polynomials in \mathbf{p} and such that the degree of each Darboux monomial is ≤ 2 . Note that $h_f, h_A \in \mathfrak{g}$ and that just like in the finite dimensional case we have an isomorphism

$$(1.35) \quad \mathbb{C} \times \mathcal{H} \times \mathfrak{sp}(\mathcal{H}) \cong \mathfrak{g}, \quad (c, f, A) \mapsto c + h_f + h_A.$$

LEMMA 1.27. *The vector space $\mathbb{C}[\mathbf{p}][\mathbf{q}]$ is a Poisson algebra with Poisson bracket*

$$\{f, g\} := \sum_{k=0}^{\infty} \sum_{i=1}^N \left(\frac{\partial f}{\partial p_{k,i}} \frac{\partial g}{\partial q_{k,i}} - \frac{\partial f}{\partial q_{k,i}} \frac{\partial g}{\partial p_{k,i}} \right)$$

PROOF. We have to check that the above formula for the Poisson bracket makes sense and that the value $\{f, g\} \in \mathbb{C}[\mathbf{p}][\mathbf{q}]$. Let us introduce the following notation. Given a sequence $\alpha = (\alpha_{ki})$ ($k \geq 0$, $1 \leq i \leq N$) of non-negative integers, such that, $\alpha_{k,i} \neq 0$ for only finitely many (k, i) , let us define $\mathbf{q}^\alpha := \prod_{k,i} q_{k,i}^{\alpha_{k,i}}$ and similarly $\mathbf{p}^\alpha = \prod_{k,i} p_{k,i}^{\alpha_{k,i}}$. Every element $f \in \mathbb{C}[\mathbf{p}][\mathbf{q}]$ can be written in the form

$$f = \sum_{\alpha, \beta} f_{\alpha, \beta} \mathbf{q}^\alpha \mathbf{p}^\beta, \quad f_{\alpha, \beta} \in \mathbb{C},$$

where the sum is over all sequences $\alpha = (\alpha_{k,i})$ and $\beta = (\beta_{k,i})$ of the above form. Note that for each α , the coefficients $f_{\alpha, \beta} \neq 0$ only for finitely many β . With this notation we have

$$(1.36) \quad \frac{\partial f}{\partial p_{k,i}} \frac{\partial g}{\partial q_{k,i}} = \sum_{\alpha, \beta} \left(\sum_{\alpha' \leq \alpha, \beta' \leq \beta} (\beta'_{ki} + 1)(\alpha''_{ki} + 1) f_{\alpha', \beta' + 1_{ki}} g_{\alpha'' + 1_{ki}, \beta''} \right) \mathbf{q}^\alpha \mathbf{p}^\beta,$$

where the 1st sum on the RHS is over all sequences α and β , the 2nd sum is over all sequences α' and β' , such that, $\alpha'_{lj} \leq \alpha_{lj}$ and $\beta'_{lj} \leq \beta_{lj}$ for all (l, j) , 1_{ki} is the sequence which has only one non-zero entry which is 1 and it is in position (k, i) , and $\alpha'' := \alpha - \alpha'$ and $\beta'' := \beta - \beta'$. If we fix α , then there are only finitely many possibilities for α' and hence the coefficient $f_{\alpha', \beta' + 1_{ki}}$ is non-zero only for finitely many β' and finitely many pairs (k, i) . Furthermore, since $\alpha'' \leq \alpha$ we have finitely many choices for α'' . Recall that we have already concluded that there are finitely many choices for (k, i) , therefore, the range of $\alpha'' + 1_{ki}$ is finite. Therefore, the range for β'' is also finite. In particular, $\beta = \beta' + \beta''$ will take only finitely many values. We get that the sum over all (k, i) in (1.36) makes sense and that the resulting formal power series is an element of $\mathbb{C}[\mathbf{p}][\mathbf{q}]$. \square

The Poisson bracket $\{h_1, h_2\} \in \mathfrak{g}$ for all $h_1, h_2 \in \mathfrak{g}$. Therefore, \mathfrak{g} is a Lie algebra with Lie bracket $\{, \}$. The commutation relations remain the same as in the finite dimensional case.

PROPOSITION 1.28. *The following formulas hold:*

- a) $\{h_f, h_g\} = \Omega(f, g)$ for all $f, g \in \mathcal{H}$.
- b) $\{h_A, h_f\} = h_{Af}$ for all $A \in \mathfrak{sp}(\mathcal{H})$ and $f \in \mathcal{H}$.
- c) $\{h_A, h_B\} = h_{[A, B]}$ for all $A, B \in \mathfrak{sp}(\mathcal{H})$.

The proof of Proposition 1.28 is a straightforward computation using the explicit formulas (1.33) and (1.34). Note that Proposition 1.28 can be stated equivalently as follows: under the isomorphism (1.35), the Lie bracket on \mathfrak{g} takes the form

$$\{(c_1, f_1, A_1), (c_2, f_2, A_2)\} = (\Omega(f_1, f_2), A_1 f_2 - A_2 f_1, [A_1, A_2]),$$

1.4.3. Quantization. Given an element $h \in \underline{\mathfrak{g}}$, just like in the finite dimensional case, we can construct a formal differential operator \widehat{h} . Namely, for each monomial in the formal power series h , we move each \mathbf{p} -variable to the right of all \mathbf{q} -variables and then substitute

$$q_{k,i} \mapsto q_{k,i}/\sqrt{\hbar}, \quad p_{k,i} \mapsto \sqrt{\hbar} \frac{\partial}{\partial q_{k,i}}, \quad k \geq 0, \quad 1 \leq i \leq N.$$

Recalling formulas (1.34) and (1.33), we get

$$\widehat{h}_A = \frac{1}{2} \sum_{k,l=0}^{\infty} \sum_{i,j=1}^N \left(-A_{ki,lj}^{++} q_{l,j} \frac{\partial}{\partial q_{k,i}} - A_{ki,lj}^{+-} \hbar \frac{\partial^2}{\partial q_{k,i} \partial q_{l,j}} + A_{ki,lj}^{-+} \hbar^{-1} q_{k,i} q_{l,j} + A_{ki,lj}^{--} q_{ki} \frac{\partial}{\partial q_{l,j}} \right)$$

and

$$\widehat{h}_f = \sum_{k=0}^{\infty} \sum_{i=1}^N \left(\Omega(\phi^i(-z)^{-k-1}, f) \sqrt{\hbar} \frac{\partial}{\partial q_{k,i}} + \Omega(\phi_i z^k, f) \frac{1}{\sqrt{\hbar}} q_{k,i} \right).$$

Just like in the finite dimensional case, let us define the 2-cocycle $C : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$, such that, $C(h_1, h_2) = 0$ if h_1 or h_2 is at most linear, while on a pair of quadratic Darboux monomials, the cocycle is non-zero only in the following cases:

$$C(p_{k,i} p_{l,j}, q_{k,i} q_{l,j}) = 1 \quad \text{if } (k, i) \neq (l, j), \quad C(p_{k,i}^2, q_{k,i}^2) = 2.$$

Under the isomorphism (1.35), the cocycle takes the form

$$C((c_1, f_1, A_1), (c_2, f_2, A_2)) = \frac{1}{2} \text{Tr}(A_1^{-+} A_2^{+-} - A_1^{+-} A_2^{-+}),$$

where A_i^{ab} ($1 \leq a, b \leq 2$) are the blocks of the matrix of the operator A_i with respect to the basis $\phi_i z^k, \phi^i(-z)^{-k-1}$ ($k \geq 0, 1 \leq i \leq N$). Note that the above trace exists because the matrices A_i^{+-} have finitely many non-zero entries.

REMARK 1.29. The formula for the 2-cocycle C in the finite dimensional settings (see (1.31)) can not be applied directly. Namely, one has to rewrite formula (1.31) in terms of the blocks of the matrices A_1 and A_2 . Indeed, we have

$$A_1 J A_2 = \begin{bmatrix} A_1^{++} A_2^{++} - A_1^{+-} A_2^{-+} & A_1^{++} A_2^{+-} - A_1^{+-} A_2^{--} \\ A_1^{-+} A_2^{++} - A_1^{-+} A_2^{-+} & A_1^{-+} A_2^{+-} - A_1^{-+} A_2^{--} \end{bmatrix}$$

Recalling Proposition 1.26, we get $A_1^{-+} A_2^{-+} = (A_2^{++} A_1^{++})^T$. In the finite dimensional case, we would have $\text{Tr}(A_1^{++} A_2^{++}) = \text{Tr}(A_1^{-+} A_2^{-+})$, so the trace of $A_1 J A_2$ would coincide with the trace of $A_1^{-+} A_2^{+-} - A_1^{+-} A_2^{-+}$. Note that in the infinite dimensional case, the trace $A_1^{++} A_2^{++}$ might fail to exist. \square

Let us denote by \mathfrak{d} the following Lie algebra of differential operators:

$$\mathbb{C} \oplus \bigoplus_{k,i} \left(\mathbb{C} \hbar^{-1/2} q_{k,i} + \mathbb{C} \hbar^{1/2} \frac{\partial}{\partial q_{k,i}} \right) \oplus \bigoplus_{k,i} \bigoplus_{l,j} \left(\mathbb{C} \hbar^{-1} q_{k,i} q_{l,j} + \mathbb{C} q_{k,i} \frac{\partial}{\partial q_{l,j}} + \mathbb{C} \hbar \frac{\partial^2}{\partial q_{k,i} \partial q_{l,j}} \right)$$

where the Lie bracket is the usual commutator of differential operators. Let us equip the vector space \mathfrak{d} with a topology, such that, a basis of neighborhoods of 0 is given by the vector subspaces $\mathbb{V}_n(\mathfrak{d})$ ($n \in \mathbb{Z}_{\geq 0}$) spanned by elements of the form

$$\hbar^{-1/2} q_{k,i}, \quad \hbar^{-1} q_{k,i} q_{l,j}, \quad q_{k,i} \frac{\partial}{\partial q_{l,j}},$$

where l, j, i are arbitrary and $k \geq n$. The topological space \mathfrak{d} is Hausdorff and the two operations $(P, Q) \mapsto P - Q$ and $(P, Q) \mapsto [P, Q]$ define continuous maps $\mathfrak{d} \times \mathfrak{d} \rightarrow \mathfrak{d}$.

REMARK 1.30. Note that \mathfrak{d} is not a topological vector space in the sense of functional analysis, because the scalar multiplication operation $\mathbb{C} \times \mathfrak{d} \rightarrow \mathfrak{d}$, $(\lambda, P) \mapsto \lambda P$ is not continuous (see [61]). However, if we equip \mathbb{C} with the discrete topology, then scalar multiplication does define a continuous map. \square

The quantized Hamiltonians \widehat{h} for $h \in \mathfrak{g}$ belong to the completion \overline{d} of d with respect to the above topology. Moreover, one can check that the commutator in d extends continuously to the completion \overline{d} , that is, \overline{d} is a Lie algebra whose Lie bracket defines a continuous map $\overline{d} \times \overline{d} \rightarrow \overline{d}$.

PROPOSITION 1.31. *The following formula holds:*

$$[\widehat{h}_1, \widehat{h}_2] = \{h_1, h_2\}^\wedge + C(h_1, h_2), \quad \forall h_1, h_2 \in \mathfrak{g}.$$

PROOF. Let us outline the key steps in the proof leaving the details as an exercise. The map $h \mapsto \widehat{h}$ defines a linear isomorphism $\mathfrak{g} \rightarrow \overline{d}$, so the topology on \overline{d} induces a topology on \mathfrak{g} , such that, the quantization map becomes a homeomorphism of topological vector spaces. Now one has to check that the Poisson bracket $\{, \} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is a continuous map. Both sides of the identity that we want to prove are continuous bilinear maps $\mathfrak{g} \times \mathfrak{g} \rightarrow \overline{d}$. Therefore, it is sufficient to check the identity when h_1 and h_2 are Darboux monomials which is straightforward. \square

1.4.4. Completed Fock spaces. Our goal now is to introduce the vector spaces on which the quantized Hamiltonians will act. Let \hbar be a formal variable. Given a commutative associative \mathbb{C} -algebra K , let us denote by $K_\hbar := K((\hbar^{1/2}))$ the algebra of all formal Laurent series in $\hbar^{1/2}$ with coefficients in K . We will be interested also in the set $\mathbb{C}[K_\hbar]$ of all formal finite sums

$$c_1 e^{a_1} + \cdots + c_r e^{a_r}, \quad c_i \in \mathbb{C}, \quad a_i \in K_\hbar.$$

Let us define a multiplication

$$\left(\sum_{i=1}^r c'_i e^{a'_i} \right) \cdot \left(\sum_{j=1}^r c''_j e^{a''_j} \right) := \sum_{i,j=1}^r c'_i c''_j e^{a'_i + a''_j}.$$

Then $\mathbb{C}[K_\hbar]$ is a commutative associative algebra known as the *group algebra* of K_\hbar .

REMARK 1.32. Our motivation to introduce $\mathbb{C}[K_\hbar]$ comes from the generating functions in Gromov–Witten theory. They usually have the form $e^{\sum_{g=0}^{\infty} \mathcal{F}^{(g)}(\mathbf{q}) \hbar^{g-1}}$, where each $\mathcal{F}^{(g)}(\mathbf{q})$ is a formal power series in \mathbf{q} . We would like to consider shifts $\mathbf{q} \mapsto \mathbf{q} + \sqrt{\hbar} f$ and think of the resulting series as a formal power series in \mathbf{q} with coefficients in \mathbb{C}_\hbar . However, sometimes the series $\mathcal{F}^{(g)}$ have constant terms and the exponentiation does not make sense. A natural solution of this complication is to interpret the exponential of the constant term as an element in the group algebra $\mathbb{C}[\mathbb{C}_\hbar]$. \square

Suppose that $a \in \mathcal{H}_+$. Let $a_k = \sum_{i=1}^N a_{k,i} \phi_i \in H$ be the coefficients in the expansion $a =: \sum_{k=0}^{\infty} a_k z^k$. In this section we will be interested in sequences $m = (m_{k,i})$, $k \geq 0$, $1 \leq i \leq N$, such that, $m_{k,i}$ are non-negative integers and only finitely many of them are non-zero. We will refer to such sequences as *sequences of finite energy* and the number $\|m\| := \sum_{k,i} (k+1) m_{k,i}$ will be called the *energy* of m . Let us introduce the following multi-index notation: $(\mathbf{q} - a)^m := \prod_{k,i} (q_{k,i} - a_{k,i})^{m_{k,i}}$. Our main interest is in vector spaces of the following form:

$$\widehat{\mathcal{O}}_{\mathcal{H}_+, a} := \mathbb{C}_\hbar[q_0 - a_0, q_1 - a_1, \dots] \otimes \mathbb{C}[\mathbb{C}_\hbar],$$

that is, the ring of formal power series in the shifted variables $q_{k,i} - a_{k,i}$. Elements of $\widehat{\mathcal{O}}_{\mathcal{H}_+,a}$ are infinite sums of the form

$$f = \sum_{\phi} \sum_m f_{m,\phi}(\mathbf{q} - a)^m e^{\phi}, \quad f_{m,\phi} = \sum_{g \geq g_0} f_{m,\phi}^{(g)} \hbar^{g-1} \in \mathbb{C}_{\hbar},$$

where the first sum is over a finite set of $\phi \in \mathbb{C}_{\hbar}$ and the second sum is over all sequences $m = (m_{k,i})$ of finite energy. The complex numbers $f_{m,\phi}^{(g)}$ will be called the *coefficients* of the formal series $f \in \widehat{\mathcal{O}}_{\mathcal{H}_+,0}$. Let us equip $\widehat{\mathcal{O}}_{\mathcal{H}_+,a}$ with a topology, such that, a basis of neighborhoods of 0 is given by the following subspaces:

$$\mathbb{V}_n(\widehat{\mathcal{O}}_{\mathcal{H}_+,a}) := \left\{ \sum_{\phi} \sum_{m: \|m\| \geq n} f_{m,\phi}(\mathbf{q} - a)^m e^{\phi} \mid f_{m,\phi} \in \mathbb{C}_{\hbar} \right\}, \quad n \in \mathbb{Z}.$$

Informally, we can think of $\widehat{\mathcal{O}}_{\mathcal{H}_+,a}$ as the completion of the local ring of \mathbb{C}_{\hbar} -valued functions on \mathcal{H}_+ holomorphic at a .

Before we continue further, it is convenient to introduce the following notation. Given $f \in \mathcal{H}$ and $A \in \mathfrak{sp}(\mathcal{H})$ let us define $\widehat{f} := \widehat{h}_f$ and $\widehat{A} := \widehat{h}_A$. In other words, we have a map

$$\widehat{\cdot} : \mathbb{C} \times \mathcal{H} \times \mathfrak{sp}(\mathcal{H}) \rightarrow \overline{d}, \quad (c, f, A) \mapsto c + \widehat{f} + \widehat{A},$$

where \overline{d} is the completed Lie algebra of differential operators in \mathbf{q} of order ≤ 2 . Recalling Propositions 1.28 and 1.31 we get the following commutation relations.

PROPOSITION 1.33. *The following identities hold.*

- a) $[\widehat{f}, \widehat{g}] = \Omega(f, g)$ for all $f, g \in \mathcal{H}$.
- b) $[\widehat{A}, \widehat{f}] = (Af)^{\widehat{\cdot}}$ for all $f \in \mathcal{H}$ and $A \in \mathfrak{sp}(\mathcal{H})$.
- c) $[\widehat{A}, \widehat{B}] = [A, B]^{\widehat{\cdot}} + C(A, B)$, where $A, B \in \mathfrak{sp}(\mathcal{H})$ and $C(A, B) := C(h_A, h_B)$ is the 2-cocycle defined above (see Proposition 1.31). \square

The vector space $\widehat{\mathcal{O}}_{\mathcal{H}_+,a}$ has a natural structure of a d -module. It is straightforward to check that the action of d extends by continuity to the completion \overline{d} . In other words, all vector spaces $\widehat{\mathcal{O}}_{\mathcal{H}_+,a}$ are \overline{d} -modules and the map $\overline{d} \times \widehat{\mathcal{O}}_{\mathcal{H}_+,a} \rightarrow \widehat{\mathcal{O}}_{\mathcal{H}_+,a}$ defined by the action is continuous.

The differential operators in d of order ≤ 1 form a Lie subalgebra, which is a central extension of an abelian Lie algebra. Such central extensions are usually called *Heisenberg Lie algebras* and their irreducible representations are called *Fock spaces*. Slightly abusing the terminology we will refer to $\widehat{\mathcal{O}}_{\mathcal{H}_+,a}$ as a *Fock space*, although it is a completion of a Fock space.

1.4.5. Lower-triangular symplectic transformations. Suppose that

$$S(z) = 1 + S_1 z^{-1} + S_2 z^{-2} + \dots \in \text{End}(H)[[z^{-1}]]$$

is a formal operator series. The series $S(z)$ defines naturally a linear transformation of \mathcal{H} :

$$S(z)f := \sum_{n \in \mathbb{Z}} \left(f_n + \sum_{k=1}^{\infty} S_k(f_{n+k}) \right) z^n, \quad f = \sum_{n \in \mathbb{Z}} f_n z^n \in \mathcal{H},$$

where note that for each $n \in \mathbb{Z}$ the sum over k must be finite because $f_{n+k} = 0$ for $k \gg 0$. Similarly, since for n sufficiently big $f_{n+k} = 0$ for all $k \geq 0$, we get that the above formula takes values in \mathcal{H} , that is, the powers of z are bounded from above. We will

refer to linear transformations of the above form as *lower-triangular*. Suppose now that $S(z)$ is a lower triangular symplectic transformation. There exists a unique infinitesimal symplectic transformation

$$A(z) = A_1 z^{-1} + A_2 z^{-2} + \dots \in \text{End}(H)[[z^{-1}]]z^{-1},$$

such that, $S(z) = e^{A(z)}$. The exponential of $A(z)$ is convergent in the topology of point-wise convergence in $\text{End}(\mathcal{H})$, that is, for every $f \in \mathcal{H}$ the series $\sum_{n=0}^{\infty} \frac{1}{n!} A(z)^n f$ converges in the topology of \mathcal{H} introduced above. The linear transformation $A(z)$ is continuous because $A(z)z^{-n}\mathcal{H}_- \subset z^{-n-1}\mathcal{H}_- \Rightarrow A(z) \in \mathfrak{sp}(\mathcal{H})$. Let us denote by \widehat{A} the quantization of $A(z)$. It is easy to see from the explicit formula for $h_{A(z)}$ (see formula (1.40) below), that the operator \widehat{A} increases the energy of every monomial \mathbf{q}^m (see Section 1.4.4 for the definition of energy). Therefore, the action of the operator \widehat{A} on the Fock space $\widehat{\mathcal{O}}_{\mathcal{H}_+,0}$ can be exponentiated. The operator $\widehat{S} := e^{\widehat{A}}$ will be called the *quantization* of $S(z)$. It turns out that there is a simple formula for the action of \widehat{S} on the Fock space. In order to state the formula we have to introduce the following quadratic form W on \mathcal{H}_+ :

$$(1.37) \quad W(f, g) = \sum_{k,l=0}^{\infty} (f_k, W_{kl}g_l),$$

where $f = \sum_{k=0}^{\infty} f_k z^k$, $g = \sum_{l=0}^{\infty} g_l z^l \in \mathcal{H}_+$, and the linear operators $W_{kl} \in \text{End}(H)$ are defined by the following identity:

$$(1.38) \quad \sum_{k,l=0}^{\infty} W_{kl} z^{-k} w^{-l} := \frac{S(z)^T S(w) - 1}{z^{-1} + w^{-1}}.$$

PROPOSITION 1.34. *The following properties hold:*

- a) *We have: $W_{kl} = \sum_{i=1}^{k+1} (-1)^{i+1} S_{k+1-i}^T S_{l+i}$.*
- b) *We have: $(W_{kl})^T = W_{lk}$, that is, $W(f, g) = W(g, f)$ is a symmetric form.*
- c) *We have: $W(f, g) = \Omega((Sf)_+, Sg)$.* □

The proof of Proposition 1.34 is straightforward and it is left as an exercise.

LEMMA 1.35. *Suppose that $S(z) = 1 + S_1 z^{-1} + \dots$ is a symplectic transformation in \mathcal{H} and that $\mathcal{F} \in \widehat{\mathcal{O}}_{\mathcal{H}_+,0}$. Then*

$$(1.39) \quad \widehat{S}^{-1} \mathcal{F}(\mathbf{q}) = e^{\frac{1}{2\hbar} W(\mathbf{q}, \mathbf{q})} \mathcal{F}([S\mathbf{q}]_+),$$

where f_+ means the projection of $f \in \mathcal{H}$ on \mathcal{H}_+ along \mathcal{H}_- and

$$W(\mathbf{q}, \mathbf{q}) := \sum_{k,l=0}^{\infty} (q_k, W_{kl}q_l) = \sum_{k,l=0}^{\infty} \sum_{i,j=1}^N (\phi_i, W_{kl}\phi_j) q_{k,i} q_{l,j}.$$

PROOF. Put $S(z) = e^{A(z)}$, where $A(z) = \sum_{k \geq 1} A_k z^{-k}$. It is convenient to introduce the following notation

$$\mathbf{q}(z) = \sum_{k=0}^{\infty} q_k z^k = \sum_{k=0}^{\infty} \sum_{a=1}^N q_{k,a} \phi_a z^k,$$

$$\mathbf{p}(z) = \sum_{k=0}^{\infty} p_k (-z)^{-k-1} = \sum_{k=0}^{\infty} \sum_{a=1}^N p_{k,a} \phi_a^a (-z)^{-k-1},$$

and to introduce the residue pairing on \mathcal{H} :

$$(\mathbf{f}(z), \mathbf{g}(z)) := \text{Res}_{z=0}(\mathbf{f}(z), \mathbf{g}(z)).$$

The quadratic Hamiltonian corresponding to A is $-\frac{1}{2}(A\mathbf{q}, \mathbf{q}(-z)) - (A\mathbf{p}, \mathbf{q}(-z))$, that is,

$$(1.40) \quad h_A = \frac{1}{2} \sum_{m,l} (-1)^{m+1} (A_{m+l+1} q_l, q_m) + \sum_{k,l \geq 0} (-1)^k (A_k p_l, q_{k+l}).$$

Put $\mathcal{G}(t, \mathbf{q}) = e^{-t\hat{A}}\mathcal{F}$ and let us compute \mathcal{G} for all t . The theorem would follow from the case $t = 1$. Note that \mathcal{G} is a solution to the differential equation $\partial_t \mathcal{G} = -\hat{A}\mathcal{G}$, which after the substitution $g = \log \mathcal{G}$, turns into:

$$(1.41) \quad \frac{\partial g}{\partial t} = \frac{1}{2\hbar} (A\mathbf{q}, \mathbf{q}(-z)) + \sum_{k=0}^{\infty} \sum_{a=1}^N (A\phi^a(-z)^{-k-1}, \mathbf{q}(-z)) \frac{\partial g}{\partial q_{k,a}}.$$

This is a 1-st order PDE which we solve by the method of the characteristics.

Step 1. First, we solve the homogeneous equation, that is,

$$\frac{\partial g}{\partial t} = \sum_{k,a} (A\phi^a(-z)^{-k-1}, \mathbf{q}(-z)) \frac{\partial g}{\partial q_{k,a}},$$

where $g(t, \mathbf{q})$ is a formal power series in $\widehat{\mathcal{O}}_{\mathcal{H}_+,0}$ whose coefficients depend smoothly on $t \in \mathbb{C}$. Let us apply the method of the characteristics as if the solutions $g(t, \mathbf{q})$ were smooth functions depending on finitely many variables. The auxiliary system of ODEs is

$$\frac{\partial q_{k,a}}{\partial t} = -(A\phi^a(-z)^{-k-1}, \mathbf{q}(-z)) \quad \Leftrightarrow \quad \frac{\partial \mathbf{q}}{\partial t} = -[A\mathbf{q}]_+.$$

Notice that $[A[\dots [A\mathbf{q}]_+]_+]_+ = [A^n \mathbf{q}]_+$, where on the LHS A is repeated n times. Therefore, the above system of ODEs has the following solution: $\mathbf{q}(t) = [e^{-tA} \mathbf{c}]_+$, where $\mathbf{c} = \mathbf{q}(0) \in \mathcal{H}_+ = H[z]$ is an initial condition. The method of the characteristics is based on the fact that the solutions $g(t, \mathbf{q})$ of the PDE are constant along the curves $(t, \mathbf{q}(t)) \in \mathbb{C} \times \mathcal{H}_+$. From here we find that if $(t, \mathbf{q}) \in \mathbb{C} \times \mathcal{H}_+$ is any point then the curve $(s, \mathbf{q}(s))$ with initial condition $(0, [e^{tA} \mathbf{q}]_+)$ will pass through the point (t, \mathbf{q}) . Therefore, the general solution of the PDE is given by: $g(t, \mathbf{q}) = f([e^{tA} \mathbf{q}]_+)$, where f is an arbitrary function on \mathcal{H}_+ .

The key ingredients of the above argument make sense in our infinite dimensional settings too. Namely, put $f(\mathbf{q}) := g(0, \mathbf{q}) \in \widehat{\mathcal{O}}_{\mathcal{H}_+,0}$. We claim that $g(t, \mathbf{q}) = f([e^{tA} \mathbf{q}]_+)$. Note that this is an identity between formal power series in $\widehat{\mathcal{O}}_{\mathcal{H}_+,0}$ whose coefficients are smooth functions in t . Let us fix $t \in \mathbb{C}$ and consider the following family $G(s, \mathbf{q}) := g(s, [e^{(t-s)A} \mathbf{q}]_+) \in \widehat{\mathcal{O}}_{\mathcal{H}_+,0}$, where $s \in \mathbb{C}$. Using the chain rule and the partial differential equations for g , it is easy to check that $\partial_s G(s, \mathbf{q}) = 0$, that is, $G(s, \mathbf{q})$ is independent of s . Therefore, $g(t, \mathbf{q}) = G(t, \mathbf{q}) = G(0, \mathbf{q}) = f([e^{tA} \mathbf{q}]_+)$.

Step 2: a direct computation shows that the function

$$W_t(\mathbf{q}, \mathbf{q}) = \frac{1}{2\hbar} \sum_{k,l} (W_{kl}(t) q_l, q_k),$$

defined by the formula:

$$\sum_{k,l \geq 0} W_{kl}(t) z^{-k} w^{-l} = \frac{e^{A^T(z)t} e^{A(w)t} - 1}{z^{-1} + w^{-1}}$$

is a solution to (1.41).

We get that the general solution to (1.41) is given by $g(t, \mathbf{q}) = W_t(\mathbf{q}, \mathbf{q}) + f([e^{tA}\mathbf{q}]_+)$. Note that for $t = 0$ we have $\mathcal{G} = \mathcal{F}$, and $W_0(\mathbf{q}, \mathbf{q}) = 0$, so $f = \log \mathcal{F}$. The lemma follows. \square

Suppose now that $a \in \mathcal{H}_+$. We would like to exponentiate the action of \widehat{A} on $\widehat{\mathcal{O}}_{\mathcal{H}_+, a}$. There is a slight complication, because if we expand the quantized Hamiltonian \widehat{h}_A in the powers of $q_0 - a_0, q_1 - a_1, \dots$, where $a_k \in H$ are the coefficients in the expansion $a = \sum_{k=0}^{\infty} a_k z^k$, then we will have a translation term of the form $\sum_{k,l} (\phi^i, A_k a_{k+l}) \frac{\partial}{\partial q_{l,i}}$. Therefore, in order to exponentiate the action we have to interpret the resulting operator as an operator between two different Fock spaces. Let us give the formal definition. Suppose that $S(z) = e^{A(z)}$ is a lower-triangular symplectic transformation. The operator

$$(1.42) \quad \widehat{S}_a^{-1} : \widehat{\mathcal{O}}_{\mathcal{H}_+, a} \rightarrow \widehat{\mathcal{O}}_{\mathcal{H}_+, [S^{-1}a]_+}$$

is defined as follows. Given $\mathcal{F} \in \widehat{\mathcal{O}}_{\mathcal{H}_+, a}$, we can write uniquely \mathcal{F} in the form $\mathcal{G}(\mathbf{q} + a)$, where $\mathcal{G} \in \mathcal{O}_{\mathcal{H}_+, 0}$. In other words, \mathcal{F} is obtained from \mathcal{G} via the substitutions $q_k \mapsto q_k + a_k$ ($k \geq 0$). Then we define

$$(1.43) \quad \widehat{S}_a^{-1} \mathcal{F} := e^{\frac{1}{2\hbar} W(\mathbf{q}, \mathbf{q})} \mathcal{G}([S\mathbf{q}]_+ + a).$$

The above definition is motivated by the formula in Lemma 1.35. The exponential $e^{\frac{1}{2\hbar} W(\mathbf{q}, \mathbf{q})}$ is interpreted in the following way: we write $W(\mathbf{q}, \mathbf{q}) = W(b, b) + (W(\mathbf{q}, \mathbf{q}) - W(b, b))$, where $b := [S^{-1}a]_+$. Since $b \in H[z]$, the quadratic form W can be evaluated at b , while the term $W(\mathbf{q}, \mathbf{q}) - W(b, b)$ is a formal power series in $q_0 - b_0, q_1 - b_1, \dots$ which can be exponentiated in $\widehat{\mathcal{O}}_{\mathcal{H}_+, b}$. We define

$$e^{\frac{1}{2\hbar} W(\mathbf{q}, \mathbf{q})} := e^{\frac{1}{2\hbar} W(b, b)} e^{\frac{1}{2\hbar} (W(\mathbf{q}, \mathbf{q}) - W(b, b))},$$

where the first exponential is interpreted formally as an element of $\mathbb{C}[\mathbb{C}\hbar]$, while the second one takes value in the topological ring $\widehat{\mathcal{O}}_{\mathcal{H}_+, b}$. Note that

$$[S(\mathbf{q} + b)]_+ = [S\mathbf{q} + S[S^{-1}a]_+]_+ = [S\mathbf{q}]_+ + [S[S^{-1}a]_+]_+ = [S\mathbf{q}]_+ + a,$$

where in the third equality we used that $S(\mathcal{H}_-) \subseteq \mathcal{H}_-$. In other words, the function $\mathcal{G}([S\mathbf{q}]_+ + a)$ is obtained from $\mathcal{G}([S\mathbf{q}]_+) \in \widehat{\mathcal{O}}_{\mathcal{H}_+, 0}$ via the substitution $q_k \mapsto q_k + b_k$ ($k \geq 0$), where $b_k \in H$ are the coefficients of b in the expansion $b = \sum_{k=0}^{\infty} b_k z^k$.

1.4.6. Heisenberg group. The Fock space $\widehat{\mathcal{O}}_{\mathcal{H}_+, 0}$ is isomorphic to all other Fock spaces $\widehat{\mathcal{O}}_{\mathcal{H}_+, a}$ via the natural isomorphism $\mathcal{G}(\mathbf{q}) \mapsto \mathcal{G}(\mathbf{q} + a)$. We would like to interpret these isomorphisms in terms of the quantized action of linear Hamiltonians. If the function $\mathcal{G}(\mathbf{q})$ is a polynomial in \mathbf{q} , then by using Taylor's formula we get

$$\mathcal{G}(\mathbf{q} + a) = e^{\sum_{k=0}^{\infty} \sum_{i=1}^N a_{k,i} \frac{\partial}{\partial q_{k,i}}} \mathcal{G}(\mathbf{q}),$$

where $a = \sum_{k=0}^{\infty} a_k z^k$ and $a_k = \sum_{i=1}^n a_{k,i} \phi_i$. Note that the differential operator in the exponent coincides with the quantized linear Hamiltonian $\frac{1}{\sqrt{\hbar}} \widehat{a}$. On the other hand, recalling the Baker–Campbell–Hausdorff formula and the commutation relations from Proposition 1.33, a), we get

$$e^{\frac{1}{\sqrt{\hbar}} \widehat{f}} e^{\frac{1}{\sqrt{\hbar}} \widehat{g}} = e^{\frac{1}{2\hbar} \Omega(f, g)} e^{\frac{1}{\sqrt{\hbar}} (f+g)^{\widehat{}}}, \quad f, g \in \mathcal{H}.$$

Let us introduce the set of formal symbols

$$\text{Heis}(\mathcal{H}) := \{e^{c/\hbar}e^f \mid c \in \mathbb{C}, f \in \mathcal{H}\} \cong \mathbb{C} \times \mathcal{H}$$

and define multiplication

$$e^{c_1/\hbar}e^{f_1} \cdot e^{c_2/\hbar}e^{f_2} := e^{\frac{1}{\hbar}(c_1+c_2+\frac{1}{2}\Omega(f_1,f_2))}e^{f_1+f_2}, \quad e^{c_i/\hbar}e^{f_i} \in \text{Heis}(\mathcal{H}).$$

It is straightforward to check that $\text{Heis}(\mathcal{H})$ is a group. We will refer to it as the *Heisenberg group* of the symplectic vector space \mathcal{H} . Every element of the Heisenberg group can be written uniquely in the form $e^{c/\hbar}e^{f_+}e^{f_-}$, where $f_+ \in \mathcal{H}_+$ and $f_- \in \mathcal{H}_-$. Let us quantize the elements of the Heisenberg group by the following formula:

$$(1.44) \quad \left(e^{c/\hbar}e^{f_+}e^{f_-}\right)^\wedge := e^{c/\hbar}e^{\frac{1}{\sqrt{\hbar}}\widehat{f}_+}e^{\frac{1}{\sqrt{\hbar}}\widehat{f}_-},$$

where for every $a \in \mathcal{H}_+$, the above expression is interpreted as an operator $\widehat{\mathcal{O}}_{\mathcal{H}_+,a} \rightarrow \widehat{\mathcal{O}}_{\mathcal{H}_+,a+f_+}$ as follows. We have

$$\frac{1}{\sqrt{\hbar}}\widehat{f}_- = \frac{1}{\hbar}\Omega(\mathbf{q}, f_-) = -\frac{1}{\hbar}\Omega(a, f_-) + \frac{1}{\hbar}\Omega(\mathbf{q} + a, f_-)$$

and we define

$$e^{\frac{1}{\sqrt{\hbar}}\widehat{f}_-} = e^{-\frac{1}{\hbar}\Omega(a, f_-)}e^{\frac{1}{\hbar}\Omega(\mathbf{q}+a, f_-)},$$

where the first exponential on the RHS is interpreted as an element of $\mathbb{C}[\mathbb{C}_\hbar]$ and the second one takes values in $\widehat{\mathcal{O}}_{\mathcal{H}_+,a}$. The operator $e^{\frac{1}{\sqrt{\hbar}}\widehat{f}_+}$, as we discussed above, is defined to be the translation $\mathbf{q} \mapsto \mathbf{q} + f_+$. In particular, it is a map $\widehat{\mathcal{O}}_{\mathcal{H}_+,a} \rightarrow \widehat{\mathcal{O}}_{\mathcal{H}_+,a+f_+}$. The remaining expression $e^{c/\hbar}$ is interpreted as an element in the ring $\mathbb{C}[\mathbb{C}_\hbar]$.

Put $\widehat{\mathcal{O}}_{\mathcal{H}_+} := \bigsqcup_{a \in \mathcal{H}_+} \widehat{\mathcal{O}}_{\mathcal{H}_+,a}$. Using Lemma 1.35 and the commutation relations from Proposition 1.33, we get the following proposition.

PROPOSITION 1.36. *a) Formula (1.43) defines an action of the group of lower-triangular symplectic transformations on $\widehat{\mathcal{O}}_{\mathcal{H}_+}$.*

b) The operators (1.44) define a representation of the Heisenberg group on $\widehat{\mathcal{O}}_{\mathcal{H}_+}$.

c) Suppose that S is a lower-triangular symplectic transformation and $f \in \mathcal{H}$, then the following formula holds:

$$\widehat{S} \circ (e^f)^\wedge \circ \widehat{S}^{-1} = (e^{Sf})^\wedge,$$

where the identity should be viewed as an identity between operators acting on $\widehat{\mathcal{O}}_{\mathcal{H}_+}$.

Let us apply the formula from Proposition 1.36, c) to $f = -S^{-1}a$ for $a \in \mathcal{H}_+$ and act with both sides of the identity on some $\mathcal{F} \in \widehat{\mathcal{O}}_{\mathcal{H}_+,a}$. We get

$$\widehat{S}_0 \circ \left(e^{-S^{-1}a}\right)^\wedge \circ \widehat{S}_a^{-1} \mathcal{F} = (e^{-a})^\wedge \mathcal{F}.$$

Solving for $\widehat{S}_a^{-1} \mathcal{F}$, we get the following formula:

$$\widehat{S}_a^{-1} = \left(e^{S^{-1}a}\right)^\wedge \widehat{S}_0^{-1} (e^{-a})^\wedge.$$

REMARK 1.37. It is possible to invert the logic here. Namely, we could have used the above formula to define the operators \widehat{S}_a^{-1} , prove Proposition 1.36, and finally use Lemma 1.35 to prove that (1.43) holds. \square

1.4.7. Upper-triangular symplectic transformations. Let $H((z))$ be the vector space of formal Laurent series in z with coefficients in H . Note that $H((z))$ is the completion of $H[z, z^{-1}]$ with respect to the topology in which the sets $z^n H[z]$ ($n = 1, 2, \dots$) form a basis of neighborhood of 0. The symplectic form Ω (see (1.32)) on $H[z, z^{-1}]$ extends by continuity to $H((z))$. Note that $p_{k,i}(v) = \Omega(v, \phi_i z^k)$ and $q_{k,i}(v) = \Omega(\phi^i(-z)^{-k-1}, v)$ are linear functions in $v \in H((z))$. Just like before (see Section 1.4.2) we introduce the Poisson Lie algebra of at most quadratic Hamiltonians:

$$h_f(v) := \Omega(v, f), \quad h_A(v) = \frac{1}{2} \Omega(Av, v), \quad v, f \in H((z))$$

where A is an infinitesimal symplectic transformation of $H((z))$. The only difference now is that both h_f and h_A take values in $\mathbb{C}[\mathbf{q}][\mathbf{p}]$. An upper-triangular symplectic transformation $R(z)$ is a symplectic transformation of $H((z))$ of the form $R(z) = 1 + R_1 z + R_2 z^2 + \dots \in \text{End}(H)[[z]]$. There exists a unique infinitesimal symplectic transformation $A(z) = \sum_{k=1}^{\infty} A_k z^k \in \text{End}(H)[[z]]$, such that, $R(z) = e^{A(z)}$. Note that the corresponding quadratic Hamiltonian takes the form

$$(1.45) \quad h_A = - \sum_{k,l=0}^{\infty} (A_{k+1} q_l, p_{k+l+1}) + \frac{1}{2} \sum_{k,l=0}^{\infty} (-1)^l (A_{k+l+1} p_l, p_k),$$

where $q_k = \sum_{i=1}^N q_{k,i} \phi_i$ and $p_l = \sum_{i=1}^N p_{l,i} \phi^i$. We would like to quantize $R(z)$, that is, define $\hat{A} := \hat{h}_A$ and $\hat{R} := e^{\hat{A}}$, where quadratic Hamiltonians are quantized by the same rules as before: for each Darboux monomial in h_A put the \mathbf{p} -variables on the right of all \mathbf{q} -variables and substitute $q_{k,i} \mapsto \hbar^{-1/2} q_{k,i}$ and $p_{k,i} \mapsto \hbar^{1/2} \partial / \partial q_{k,i}$. Although the action of \hat{A} on the entire Fock space $\hat{\mathcal{O}}_{\mathcal{H}_+, a}$ does not make sense, there is a certain subspace $\hat{\mathcal{O}}_{\mathcal{H}_+, a}^{\text{tame}} \subset \hat{\mathcal{O}}_{\mathcal{H}_+, a}$ on which the actions of both \hat{A} and \hat{R} are defined. For our purposes it is sufficient to consider only the Fock spaces for which $a = -\mathbf{1} z \in \mathcal{H}_+$, where $\mathbf{1} \in H$ is a non-zero vector. Let us assume that $\phi_1 := \mathbf{1}$. To avoid cumbersome notation we will write $\hat{\mathcal{O}}_{\mathcal{H}_+, -z}$ for $\hat{\mathcal{O}}_{\mathcal{H}_+, -\mathbf{1} z}$.

REMARK 1.38. In the case of a Frobenius manifold, the above formalism is applied to $H :=$ the space of flat vector fields and $\mathbf{1} :=$ the unit vector field. \square

Let us introduce first another sequence of formal vector variables $\mathbf{t} = (t_0, t_1, t_2, \dots)$, so that $t_k = q_k + \delta_{k,1} \mathbf{1}$. Recall that if $m = (m_{k,i})$, $k \geq 0, 1 \leq i \leq N$, is a sequence of finite energy (i.e., finitely many non-zero elements), then we denoted by $\mathbf{t}^m = \prod_{k,i} t_{k,i}^{m_{k,i}}$. An element in the Fock space $\hat{\mathcal{O}}_{\mathcal{H}_+, -z}$ written in the form

$$(1.46) \quad F = \sum_{g \in \frac{1}{2} \mathbb{Z}} \sum_{m=(m_{k,i})} c_m^{(g)} \hbar^{g-1} \frac{\mathbf{t}^m}{m!},$$

where $m! := \prod_{k,i} m_{k,i}!$, is said to be *tame* if $c_m^{(g)} = 0$ for all (g, m) , such that, $\sum_{k,i} k m_{k,i} > 3g - 3 + \sum_{k,i} m_{k,i}$. Here the coefficients $c_m^{(g)}$ take value in the group ring $\mathbb{C}[\mathbb{C}_\hbar]$. It is convenient to introduce the following terminology. Given a monomial $\hbar^{g-1} \mathbf{t}^m$ we define its *codimension* by

$$\text{codim}(\hbar^{g-1} \mathbf{t}^m) := \text{codim}(g, m) := 3g - 3 + \sum_{k,i} (1 - k) m_{k,i}.$$

Note that the tame elements of the Fock space are arbitrary, possibly infinite, linear combinations of monomials with non-negative codimension. Clearly, the product of tame elements is again tame, so the set of all tame elements is a subalgebra $\widehat{\mathcal{O}}_{\mathcal{H}_+,-z}^{\text{tame}}$ of $\widehat{\mathcal{O}}_{\mathcal{H}_+,-z}$. If $F \in \widehat{\mathcal{O}}_{\mathcal{H}_+,-z}$, then we denote by $\text{supp}(F)$ the set of all pairs (g, m) , such that, the coefficient $c_m^{(g)}$ in the expansion (1.46) is not 0.

PROPOSITION 1.39. *Suppose that $A(z) = \sum_{k=1}^{\infty} A_k z^k$ is an infinitesimal symplectic transformation and that $F \in \widehat{\mathcal{O}}_{\mathcal{H}_+,-z}^{\text{tame}}$ is tame.*

- a) *If F is a tame monomial, then $\widehat{A}(F)$ is a tame polynomial consisting of monomials of codimension $\geq \text{codim}(F) + 1$.*
- b) *The action of \widehat{A} on the space of tame polynomials extends to an action on $\widehat{\mathcal{O}}_{\mathcal{H}_+,-z}^{\text{tame}}$.*
- c) *The infinite series $\sum_{n=0}^{\infty} \frac{\widehat{A}^n}{n!} F$ is convergent in the formal topology of $\widehat{\mathcal{O}}_{\mathcal{H}_+,-z}$ and it defines an element in $\widehat{\mathcal{O}}_{\mathcal{H}_+,-z}^{\text{tame}}$.*

PROOF. a) We have to prove that if $\hbar^{g-1} \mathbf{t}^m$ is an arbitrary monomial, then $\widehat{A}(\hbar^{g-1} \mathbf{t}^m)$ is a finite linear combination of monomials whose codimensions are strictly larger. Recalling formula 1.45 we get

$$(1.47) \quad \widehat{A} = \sum_{k=0}^{\infty} \sum_{b=1}^N (A_{k+1} \phi_1, \phi^b) \frac{\partial}{\partial t_{k+2,b}} - \sum_{k,l=0}^{\infty} \sum_{a,b=1}^N (A_{k+1} \phi_a, \phi^b p_{k+l+1}) t_{l,a} \frac{\partial}{\partial t_{k+l+1,b}} + \frac{\hbar}{2} \sum_{k,l=0}^{\infty} \sum_{a,b=1}^N (-1)^l (A_{k+l+1} \phi^a p_l, \phi^b p_k) \frac{\partial^2}{\partial t_{k,b} \partial t_{l,a}},$$

where we changed the Darboux variables via the dilaton shift. Note that multiplication by $t_{l,a}$ changes the codimension by $1-l$, the differentiation $\frac{\partial}{\partial t_{k,a}}$ changes the codimension by $k-1$, and multiplication by \hbar increases the codimension by 3. This implies that \widehat{A} increases the codimension at least by 1.

Note that c) is an easy consequence from a) and b). Indeed, every monomial in the support of $\widehat{A}^n(F)$ has codimension at least n , so the exponential series $\sum_{n=0}^{\infty} \frac{\widehat{A}^n}{n!} F$ is convergent in the formal topology.

Let us prove b). Suppose that $F = \sum_{g,m} c_m^{(g)} \hbar^{g-1} \mathbf{t}^m / m!$ is a tame series. We have to prove that the infinite series

$$(1.48) \quad \widehat{A}(F) = \sum_{g,m} c_m^{(g)} \widehat{A}(\hbar^{g-1} \frac{\mathbf{t}^m}{m!})$$

is convergent in the formal topology. A monomial $\hbar^{g-1} \mathbf{t}^m$ is uniquely determined by its genus g and its finite energy sequence $m = (m_{k,i})$. Let us call such a monomial a (g, m) -monomial and let us define $a(m) := \sum_{i=1}^N m_{0,i}$ and $b(m) := \sum_{i=1}^N m_{1,i}$. Let us fix a tame monomial $\hbar^{G-1} \mathbf{t}^\lambda$, where $\lambda = (\lambda_{k,i})$ is a finite energy sequence. We have to prove that there are finitely many tame (g, m) -monomials, such that, (G, λ) is in the support of the series $\widehat{A}(\hbar^{g-1} \mathbf{t}^m)$. Let us fix a tame (g, m) -monomial and analyze the action of \widehat{A} on it. There are 3 cases corresponding to the 3 double sums in (1.47). Put $\alpha := a(\lambda)$, $\beta := b(\lambda)$, and $\gamma := \text{codim}(G, \lambda)$. If the action of the operator $\frac{\partial}{\partial t_{k+2,b}}$ gives the (G, λ) -monomial, then we have $g = G$, $a(m) = \alpha$, $b(m) = \beta$, and $\text{codim}(g, m) = \gamma - k - 1$. The last relation implies that $k \leq \gamma - 1$, that is, only finitely many summands in the first double sum in

(1.47) could produce a (G, λ) -monomial from a tame (g, m) -monomial. Furthermore, we have

$$\begin{aligned} 0 \leq \text{codim}(g, m) &= 3g - 3 + \sum_{k,i} (1-k)m_{k,i} \leq \\ &\leq 3g - 3 + a(m) - \sum_{k=2}^{\infty} \sum_{i=1}^N m_{k,i} = \\ &= 3G - 3 + 2\alpha + \beta - \sum_{k=0}^{\infty} \sum_{i=1}^N m_{k,i}. \end{aligned}$$

Therefore, $\sum_{k,i} m_{k,i} \leq 3G - 3 + 2\alpha + \beta$ and

$$\sum_{k,i} km_{k,i} \leq 3G - 3 + \sum_{k,i} m_{k,i} \leq 6G - 6 + 2\alpha + \beta$$

are bounded by constants that depend only on (G, λ) . The conclusion is that the set of tame (g, m) -monomials for which the action of the first double sum in (1.47) produces a (G, λ) -monomial is finite. For the 2nd case, suppose that the operator $t_{l,a} \frac{\partial}{\partial t_{k+l+1,b}}$ produces a (G, λ) -monomial, then $g = G$, $a(m) \in \{\alpha - 1, \alpha\}$, $b(m) \in \{\beta - 1, \beta, \beta + 1\}$, and $\text{codim}(g, m) = \gamma - k - 1$. Note that since $\lambda_{l,a} = m_{l,a} + 1 > 0$, there are only finitely many possibilities for l , while by tameness $\text{codim}(g, m) \geq 0 \Rightarrow k \leq \gamma - 1$. Therefore, there are only finitely many summands in the 2nd double sum in (1.47) that could produce a (G, λ) -monomial from a tame (g, m) -monomial. Furthermore, we have

$$\begin{aligned} 0 \leq \text{codim}(g, m) &= 3g - 3 + \sum_{k,i} (1-k)m_{k,i} \leq \\ &\leq 3g - 3 + 2a(m) + b(m) - \sum_{k,i} m_{k,i} \leq \\ &\leq 3G - 3 + 2\alpha + \beta + 1 - \sum_{k,i} m_{k,i}. \end{aligned}$$

Therefore, $\sum_{k,i} m_{k,i} \leq 3G - 2 + 2\alpha + \beta$ and $\sum_{k,i} km_{k,i} \leq 6G - 5 + 2\alpha + \beta$ are bounded, so there are only finitely many tame (g, κ) -monomials from which the 2nd double sum in (1.47) could produce a (G, λ) -monomial. Finally, in the 3rd case, if the operator $\hbar \frac{\partial^2}{\partial t_{k,b} \partial t_{l,a}}$ produces a (G, λ) -monomial from a (g, m) -monomial, then $g = G - 1$, $a(m) \in \{\alpha, \alpha + 1, \alpha + 2\}$, $b(m) \in \{\beta, \beta + 1, \beta + 2\}$, and $\text{codim}(g, m) = \gamma - k - l - 1$. Again the tameness of the (g, m) monomial implies $k + l \leq \gamma - 1$, so only finitely many summands in the 3rd double sum in (1.47) could produce a (G, λ) -monomial. Furthermore, we have

$$\begin{aligned} 0 \leq \text{codim}(g, m) &= 3(G - 1) - 3 + \sum_{k,i} (1-k)m_{k,i} \leq \\ &\leq 3G - 6 + 2a(m) + b(m) - \sum_{k,i} m_{k,i} \leq \\ &\leq 3G - 6 + 2(\alpha + 2) + \beta + 2 - \sum_{k,i} m_{k,i}. \end{aligned}$$

Therefore, $\sum_{k,i} m_{k,i} \leq 3G + 2\alpha + \beta$ and $\sum_{k,i} km_{k,i} \leq 6G - 6 + 2\alpha + \beta$ are bounded, so there are only finitely many tame (g, m) -monomials from which the 3rd double sum in (1.47) could produce a (G, λ) -monomial. \square

Similarly to Lemma 1.35, there is a simple formula for the action of \widehat{R} on the tame Fock space $\widehat{\mathcal{O}}_{\mathcal{H}_+, -z}$. In order to state the formula, we have to introduce first the following quadratic form on $H[z^{-1}]z^{-1}$:

$$(1.49) \quad V(f, g) = \sum_{k,l=0}^{\infty} (-1)^{k+l} (f_{-1-k}, V_{kl}g_{-1-l}),$$

where $f = \sum_{k=0}^{\infty} f_{-1-k}z^{-1-k}$, $g = \sum_{l=0}^{\infty} g_{-1-l}z^{-1-l}$, and the linear operators $V_{kl} \in \text{End}(H)$ are defined by

$$(1.50) \quad \sum_{k,l=0}^{\infty} V_{kl}z^k w^l := \frac{1 - R(z)R^T(w)}{z + w}.$$

PROPOSITION 1.40. *The following properties hold.*

- a) We have: $V_{kl} = \sum_{i=0}^l (-1)^{i+1} R_{k+1+i} R_{l-i}^T$.
- b) We have: $(V_{kl})^T = V_{lk}$, that is, $V(f, g) = V(g, f)$ is a symmetric form.
- c) We have: $V(f, g) = \Omega((R^{-1}f)_+, (R^{-1}g)_-)$. \square

The proof of Proposition 1.40 is straightforward and it is left as an exercise.

LEMMA 1.41. *Suppose that $R(z) = 1 + R_1z + R_2z^2 + \dots$, $R_k \in \text{End}(H)$ is an operator series satisfying the symplectic condition $R(z)R(-z)^T = 1$ and that $\mathcal{F}(\mathbf{q})$ is a tame element of $\widehat{\mathcal{O}}_{\mathcal{H}_+, -z}$. Then*

$$(1.51) \quad (\widehat{R}^{-1}\mathcal{F})(\mathbf{q}) = \left(e^{\frac{\hbar}{2}V(\partial_{\mathbf{q}}, \partial_{\mathbf{q}})} \mathcal{F} \right)(R\mathbf{q}),$$

where

$$(1.52) \quad V(\partial_{\mathbf{q}}, \partial_{\mathbf{q}}) = \sum_{k,l=0}^{\infty} \sum_{a,b=1}^N (\phi^a, V_{kl}\phi^b) \frac{\partial^2}{\partial q_{k,a} \partial q_{l,b}}.$$

PROOF. The idea is to use a certain formal Fourier transform and to obtain the formula from the formula in Lemma 1.35. If $L(\mathbf{p}, \partial_{\mathbf{p}})$ is a differential operator in $\mathbf{p} = (p_{k,a})$, then we define the vacuum expectation value

$$\left\langle L(\mathbf{p}, \partial_{\mathbf{p}}) e^{\frac{1}{\hbar}\Omega(\mathbf{p}, \mathbf{q})} \right\rangle := L(\mathbf{p}, \partial_{\mathbf{p}}) e^{\frac{1}{\hbar}\Omega(\mathbf{p}, \mathbf{q})} \Big|_{\mathbf{p}=0},$$

where just like in the proof of Lemma 1.35 we put

$$\mathbf{p} = \mathbf{p}(z) = \sum_{k,a} p_{k,a} \phi^a (-z)^{-k-1} \quad \text{and} \quad \mathbf{q} = \mathbf{q}(z) = \sum_{k,a} q_{k,a} \phi_a z^k.$$

Since the formula that we want to prove is linear in \mathcal{F} , we may assume that \mathcal{F} is a polynomial in \mathbf{q} . Note that we can write $\mathcal{F}(\mathbf{q})$ as a vacuum expectation value

$$(1.53) \quad \mathcal{F}(\mathbf{q}) = \left\langle \mathcal{F}(\hbar\partial_{\mathbf{p}}) e^{\frac{1}{\hbar}\Omega(\mathbf{p}, \mathbf{q})} \right\rangle,$$

where $\mathcal{F}(\hbar\partial_{\mathbf{p}})$ is the differential operator obtained from $\mathcal{F}(\mathbf{q})$ by the substitution $q_{k,a} \mapsto \hbar\frac{\partial}{\partial p_{k,a}}$. The following two properties hold

$$\begin{aligned}\hbar\partial_{q_{k,a}}\mathcal{F}(\mathbf{q}) &= \left\langle \mathcal{F}(\hbar\partial_{\mathbf{p}}) p_{k,a} e^{\frac{1}{\hbar}\Omega(\mathbf{p},\mathbf{q})} \right\rangle, \\ q_{k,a}\mathcal{F}(\mathbf{q}) &= \left\langle \mathcal{F}(\hbar\partial_{\mathbf{p}}) \hbar\partial_{p_{k,a}} e^{\frac{1}{\hbar}\Omega(\mathbf{p},\mathbf{q})} \right\rangle.\end{aligned}$$

Put $A(z) := \log R(z) = \sum_{m=1}^{\infty} A_m z^m$. This is an infinitesimal symplectic transformation and the corresponding Hamiltonian is

$$h_A(\mathbf{p}, \mathbf{q}) = \frac{1}{2}\Omega(A\mathbf{f}, \mathbf{f}) = \sum_{k,l=0}^{\infty} \left(\frac{(-1)^k}{2} (A_{k+l+1} p_k, p_l) - (A_k q_l, p_{k+l}) \right),$$

where as usual $p_k = \sum_a p_{k,a} \phi^a$ and $q_l = \sum_a q_{l,a} \phi_a$. Using formula (1.53) we get

$$\widehat{R}^{-1}\mathcal{F}(\mathbf{q}) = \left\langle \mathcal{F}(\hbar\partial_{\mathbf{p}}) e^{-:h_A(\mathbf{p}/\sqrt{\hbar}, \sqrt{\hbar}\partial_{\mathbf{p}}):} e^{\frac{1}{\hbar}\Omega(\mathbf{p},\mathbf{q})} \right\rangle.$$

The operator

$$(1.54) \quad - : h_A(\mathbf{p}/\sqrt{\hbar}, \sqrt{\hbar}\partial_{\mathbf{p}}) := \sum_{k,l=0}^{\infty} \left(\frac{(-1)^{k+1}}{2\hbar} (A_{k+l+1} p_k, p_l) + (A_k \phi_a, p_{k+l}) \frac{\partial}{\partial p_{l,a}} \right)$$

can be interpreted as a quantization of a quadratic Hamiltonian in the following way. Let $\mathcal{H}^{\vee} = H((w^{-1}))$ be another copy of Givental's symplectic loop space in which we choose Darboux coordinates $P = (P_{k,a})$ and $Q = (Q_{k,a})$, such that, the vector in \mathcal{H}^{\vee} with coordinates (P, Q) is given by

$$\sum_{k=0}^{\infty} \sum_{a=1}^N \left(Q_{k,a} \phi^a w^k + P_{k,a} \phi_a (-w)^{-k-1} \right).$$

Let $B(w) := A(-w^{-1})$, then (see formula (1.40)) we have

$$\widehat{h}_B = \sum_{k,l=0}^{\infty} \left(\frac{(-1)^{l+1}}{2\hbar} (B_{k+l+1} Q_k, Q_l) + (-1)^k (B_k \phi_a, Q_{k+l}) \frac{\partial}{\partial Q_{l,a}} \right).$$

Since $B_k = (-1)^k A_k$, we get that \widehat{h}_B coincides with (1.54) under the substitution $p_k = \mathbf{i}Q_k$. Therefore,

$$\widehat{R}^{-1}\mathcal{F}(\mathbf{q}) = \left\langle \mathcal{F}(-\mathbf{i}\hbar\partial_Q) e^{\widehat{h}_B} e^{\frac{\mathbf{i}}{\hbar}\Omega(-w^{-1}\mathbf{q}(-w^{-1}), Q(w))} \right\rangle,$$

where we used that

$$\frac{1}{\hbar}\Omega(\mathbf{p}, \mathbf{q}) = \frac{1}{\hbar} \sum_{k,a} p_{k,a} q_{k,a} = \frac{\mathbf{i}}{\hbar} \sum_{k,a} Q_{k,a} q_{k,a} = \frac{\mathbf{i}}{\hbar} \Omega(-w^{-1}\mathbf{q}(-w^{-1}), Q(w)).$$

Recalling Lemma 1.35, since $e^{\widehat{h}_B}$ is the quantization of $R(-w^{-1})$, we get

$$e^{\widehat{h}_B} e^{\frac{\mathbf{i}}{\hbar}\Omega(-w^{-1}\mathbf{q}(-w^{-1}), Q(w))} = e^{\frac{\mathbf{i}}{2\hbar}W(Q,Q)} e^{\frac{\mathbf{i}}{\hbar}\Omega(-w^{-1}\mathbf{q}(-w^{-1}), [R(-w^{-1})^{-1}Q(w)]_+)},$$

where the quadratic form $W(Q, Q) = \sum_{k,l} (Q_k, W_{kl} Q_l)$ is defined by

$$\sum_{k,l=0}^{\infty} W_{kl} w_1^{-k} w_2^{-l} = \frac{R(-w_1^{-1})^{-T} R(-w_2^{-1})^{-1} - 1}{w_1^{-1} + w_2^{-1}} = \frac{R(w_1^{-1}) R(w_2^{-1})^T - 1}{w_1^{-1} + w_2^{-1}},$$

where for a linear operator A we put $A^{-T} := (A^{-1})^T$. Comparing with (1.50) we get $W_{kl} = -V_{kl}$, so

$$(1.55) \quad W(Q, Q) = - \sum_{k,l=0}^{\infty} \sum_{a,b=1}^N (\phi^a, V_{kl}\phi^b) Q_{k,a} Q_{l,b}.$$

Furthermore, we have

$$\begin{aligned} \Omega(-w^{-1}\mathbf{q}(-w^{-1}), [R(-w^{-1})^{-1}Q(w)]_+) &= \Omega(-w^{-1}R(-w^{-1})\mathbf{q}(-w^{-1}), Q(w)) = \\ &= \sum_{k=0}^{\infty} \sum_{a=1}^N [R(z)^{-1}\mathbf{q}(z)]_{k,a} Q_{k,a}, \end{aligned}$$

where $[\]_{k,a}$ denotes the coefficient in front of $\phi_a z^k$. Combining our observations we get

$$\widehat{R}^{-1}\mathcal{F}(\mathbf{q}) = \left\langle \mathcal{F}(-\mathbf{i}\hbar\partial_Q) e^{\frac{1}{2\hbar}W(Q,Q)} e^{\frac{\mathbf{i}}{\hbar}\sum_{k,a}[R(z)\mathbf{q}(z)]_{k,a}Q_{k,a}} \right\rangle.$$

Commuting the exponential of the linear terms in $Q_{k,a}$ through the differential operator $\mathcal{F}(-\mathbf{i}\hbar\partial_Q)$ we get

$$(1.56) \quad \left\langle \mathcal{F}(-\mathbf{i}\hbar\partial_Q + R(z)\mathbf{q}(z)) e^{\frac{1}{2\hbar}W(Q,Q)} \right\rangle,$$

where the substitution in \mathcal{F} is given by $q_{k,a} \mapsto -\mathbf{i}\hbar\frac{\partial}{\partial Q_{k,a}} + [R(z)\mathbf{q}(z)]_{k,a}$. Let us first examine the vacuum expectation value in the case of one variable, i.e., let us compute

$$\langle f(-\mathbf{i}\hbar\partial_x + c)x^a \rangle,$$

where c is a constant independent of x and the brackets $\langle \ \rangle$ mean that we apply the differential operator to x^a and then we substitute $x = 0$. Note that since we set $x = 0$ at the end, we can replace multiplication by x operator by commutator, i.e., the above expectation value is

$$\langle [\dots [f(-\mathbf{i}\hbar\partial_x + c), x] \dots x] \rangle.$$

Since only the commutation relations matter here, we can replace $-\mathbf{i}\hbar\partial_x \mapsto y$ and $x \mapsto \mathbf{i}\hbar\partial_y \Rightarrow$ the above expectation value becomes

$$\langle (-\mathbf{i}\hbar\partial_y)^a f(y+c) \rangle = \left((-\mathbf{i}\hbar\partial_y)^a f(y) \right) \Big|_{y=c}.$$

so we get the following formula:

$$\langle f(-\mathbf{i}\hbar\partial_x + c)x^a \rangle = \left((-\mathbf{i}\hbar\partial_y)^a f(y) \right) \Big|_{y=c}.$$

The generalization of the above formula to many variables is straightforward. Namely, in order to compute (1.56), we need to switch $Q_{k,a}$ with $-\mathbf{i}\hbar\frac{\partial}{\partial Q_{k,a}}$, that is, (1.56) coincides with

$$\left(e^{-\frac{\hbar}{2}W(\partial_Q, \partial_Q)} \mathcal{F}(Q) \right) \Big|_{Q_{k,a}=[R(z)\mathbf{q}(z)]_{k,a}}.$$

Using (1.55) we get that $-W(\partial_Q, \partial_Q)$ coincides with the differential operator (1.52). We get the formula stated in the lemma. \square

1.5. Vertex operators

Let K be a commutative associative \mathbb{C} -algebra with a unit. Given an infinite series $f = \sum_{n \in \mathbb{Z}} I^{(n)}(-z)^n$ with coefficients $I^{(n)} \in K \otimes H$, the formal expression

$$e^{\widehat{f}^-} e^{\widehat{f}^+} := \exp \left(- \sum_{k=0}^{\infty} \sum_{a=1}^N (I^{(-k-1)}, \phi_a) \frac{q_{k,a}}{\sqrt{\hbar}} \right) \exp \left(\sum_{k=0}^{\infty} \sum_{a=1}^N (I^{(k)}, \phi^a) (-1)^k \sqrt{\hbar} \frac{\partial}{\partial q_{k,a}} \right)$$

is called a *vertex operator* with coefficients in K . Here we denoted by f_+ (resp. f_-) the series obtained from f by truncating all negative (resp. non-negative) powers of z . The differential operators in the exponents on the RHS are obtained by quantizing each term $I^{(n)}(-z)^n \in K \otimes \mathcal{H}$, where the quantization rules are extended K -linearly to $K \otimes \mathcal{H}$:

$$(I \otimes f)^{\widehat{\cdot}} := I \otimes \widehat{f}, \quad I \in K, \quad f \in \mathcal{H}.$$

Let us give an example of K which includes all coefficients rings that will be considered in this book.

1.5.1. Example. Suppose that $U, T \in \text{End}(H)$ are given linear operators. We will assume that U is diagonalizable and T arbitrary. Let us consider the following family of Fuchsian ODEs

$$(1.57) \quad (\lambda - U) \partial_{\lambda} I^{(m)}(\lambda) = (T - m) I^{(m)}(\lambda),$$

where $m \in \mathbb{Z}$ is an integer parameter and $I^{(m)}(\lambda)$ is an H -valued function in λ . The singularities of this system are at the eigenvalues of U and $\lambda = \infty$.

The differential equation (1.57) has the following symmetry: if $I^{(m)}(\lambda)$ is a solution, then $\partial_{\lambda} I^{(m)}$ is a solution to (1.57) with m replaced by $m+1$. Let us organize the solutions to (1.57) in such a way that $I^{(m+1)}(\lambda) = \partial_{\lambda} I^{(m)}(\lambda)$ for all $m \in \mathbb{Z}$. This can be achieved as follows. Suppose that $m < 0$ is smaller than all eigenvalues of T . Then we can define $I^{(m-1)} := (T - m + 1)^{-1} (\lambda - U) I^{(m)}$ and check that $\partial_{\lambda} I^{(m-1)} = I^{(m)}$. Similarly, we can define $I^{(m-2)}$ in terms of $I^{(m-1)}$ etc., so $I^{(m-k)}$ is uniquely determined from $I^{(m)}$ for all $k \geq 0$. For the remaining solutions we must have $I^{(m+k)} = \partial_{\lambda}^k I^{(m)}$, so they are also uniquely determined from $I^{(m)}$.

Let us fix a basis ϕ_i ($1 \leq i \leq N$) of H and a base point $\lambda^{\circ} \in \mathbb{C}$ which is not an eigenvalue of U . Let $I_a^{(m)}(\lambda)$ be the solution $I^{(m)}(\lambda)$ of (1.57) satisfying the initial condition $I^{(m)}(\lambda^{\circ}) = \phi_a$. Let $I_{ai}^{(m)}(\lambda)$ be the coordinates of $I_a^{(m)}(\lambda)$ with respect to the basis ϕ_i . We define K to be the subring of $\mathcal{O}_{\mathbb{C}, \lambda^{\circ}}$ generated by the set of all $I_{ai}^{(m)}(\lambda)$ with $m \in \mathbb{Z}$, $1 \leq a, i \leq N$.

We will be interested also in the completions of K defined in the following way. Let u be a singular point of (1.57), that is, $u = \infty$ or u is an eigenvalue of U . Let us fix a reference path from λ° to u avoiding the singularities of (1.57). Using analytic continuation along the reference path and taking the Laurent series expansion at $\lambda = u$ gives an embedding of K into the ring $\mathbb{C}\{(\lambda - u)^{1/h}\}[\log(\lambda - u)]$, where $h > 0$ is an integer number depending on the local monodromy of (1.57) near the singular point $\lambda = u$. More precisely, near the singular point, the elements f of K can be expanded into infinite series of the following form:

$$(1.58) \quad f(\lambda) = f_0(\lambda) + f_1(\lambda) \log \lambda + \cdots + f_r(\lambda) (\log \lambda)^r,$$

where $f_j(\lambda)$ is a convergent Laurent series in $(\lambda - u)^{1/h}$. Let \mathfrak{m} be the subalgebra of K consisting of those $f(\lambda) \in K$ for which $\lim_{\lambda \rightarrow u} f(\lambda) = 0$, where the limit is taken along

the reference path. The elements in the \mathfrak{m} -adic completion \overline{K} of K (see Section 1.5.3) can be identified with series of the form (1.58) for which $f_j(\lambda)$ is a formal Laurent series in $\lambda - u$.

1.5.2. Fock space with coefficients in K . Slightly abusing the notation we define

$$K \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a} := K_{\hbar}[[q_0 - a_0, q_1 - a_1, \dots]] \otimes \mathbb{C}[K_{\hbar}],$$

where $a \in \mathcal{H}_+$ and $\mathbb{C}[K_{\hbar}]$ is the group algebra of K_{\hbar} (see Section 1.4.4). Since $\mathbb{C} \subset K$, we can view $K \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a}$ as an extension of $\widehat{\mathcal{O}}_{\mathcal{H}_+, a}$. Every element of the Fock space $K \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a}$ can be written as an infinite sum

$$(1.59) \quad \sum_{g \in \frac{1}{2}\mathbb{Z}} \sum_m c_m^{(g)} \hbar^{g-1} \frac{(\mathbf{q} - a)^m}{m!},$$

where the second sum is over all sequences $m = (m_{k,i})$ of finite energy and there are finitely many $b_i \in K_{\hbar}$ ($1 \leq i \leq r$), such that, all coefficients $c_m^{(g)}$ can be written in the form $c_m^{(g)} = \sum_{i=1}^r c_{m,i}^{(g)} e^{b_i}$ for some $c_{m,i}^{(g)} \in K$. Every vertex operator with coefficients in K determines a map

$$e^{\widehat{f}^-} e^{\widehat{f}^+} : \widehat{\mathcal{O}}_{\mathcal{H}_+, a}^{\text{tame}} \rightarrow K \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a}$$

defined by

$$(1.60) \quad e^{\widehat{f}^-} e^{\widehat{f}^+} \mathcal{F} := e^{\frac{1}{\sqrt{\hbar}} \Omega(\mathbf{q}, f^-)} \mathcal{F}(\mathbf{q} + \sqrt{\hbar} f_+(z)),$$

where the translation by $\sqrt{\hbar} f_+(z)$ is defined formally by applying the Taylor's formula and the exponential factor, just like before, should be split into product of two

$$e^{\frac{1}{\sqrt{\hbar}} \Omega(\mathbf{q}, f^-)} = e^{\frac{1}{\sqrt{\hbar}} \Omega(a, f^-)} e^{\frac{1}{\sqrt{\hbar}} \Omega(\mathbf{q} - a, f^-)}.$$

The first factor is interpreted as an element of $\mathbb{C}[K_{\hbar}]$, while the second one can be exponentiated in the usual way yielding a formal power series in $K_{\hbar}[[q_0 - a_0, q_1 - a_1, \dots]]$. Let us check that $\mathcal{F}(\mathbf{q} + \sqrt{\hbar} f_+(z))$ makes sense. We have

$$\mathcal{F}(\mathbf{q} + \sqrt{\hbar} f_+(z)) = \sum_{g, m, n} c_{m+n}^{(g)} \hbar^{g + \frac{|n|}{2} - 1} \frac{(\mathbf{q} - a)^m}{m!} \frac{(f_+(z))^n}{n!},$$

where the notation is as follows. The sum is over all $g \in \frac{1}{2}\mathbb{Z}$ and all sequences m and n of finite energy. We put $|n| := \sum_{k,i} n_{k,i}$. Finally, let us write $f_+(z) = \sum_{k=0}^{\infty} \sum_{a=1}^N I_a^{(k)} \phi_a z^k$, where $I_a^{(k)} = (-1)^k (I^{(k)}, \phi^a)$, then $(f_+(z))^n := \prod_{k,a} (I_a^{(k)})^{n_{k,a}}$. Let us fix $G := g + \frac{|n|}{2}$ and m . We need to check that the sum

$$(1.61) \quad \sum_{n=(n_{k,a})} c_{m+n}^{(G-|n|/2)} \prod_{k=0}^{\infty} \prod_{a=1}^N \frac{(I_a^{(k)})^{n_{k,a}}}{n_{k,a}!}$$

has only finitely many non-zero terms. Since \mathcal{F} is tame, the coefficients $c_n^{(g)}$ are non-zero only if $3g - 3 + \sum_{k,i} (1-k)n_{k,i} \geq 0$. Therefore, for the non-zero terms in (1.61) we have

$$3(G - |n|/2) - 3 + \sum_{k,a} (1-k)(m_{k,a} + n_{k,a}) \geq 0.$$

We get that $\sum_{k,a} (k + \frac{1}{2})n_{k,a} \leq 3G - 3 + \sum_{k,a} (1-k)m_{k,a}$ which implies that for fixed G and m there are only finitely many choices for n .

1.5.3. \mathfrak{m} -adic Completion. Let $\mathfrak{m} \subset K$ be a \mathbb{C} -subalgebra. The set \mathfrak{m}^n consisting of all finite sums of elements of \mathfrak{m} of the form $a_1 \cdots a_n$ with $a_i \in \mathfrak{m}$, is also a \mathbb{C} -subalgebra and we have $\mathfrak{m}^{n+1} \subset \mathfrak{m}^n$. We will require that the pair (K, \mathfrak{m}) satisfies the following two conditions:

- (i) Hausdorff condition: $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = \{0\}$.
- (ii) Continuity condition: for every $a \in K$ there exists $k \in \mathbb{Z}_{>0}$, such that, $a\mathfrak{m}^k \subseteq \mathfrak{m}$.

The ring K has a unique structure of a topological ring, such that, the subsets \mathfrak{m}^n ($n = 1, 2, \dots$) form a basis for the neighborhoods of 0 of K . We will refer to this topology as the \mathfrak{m} -adic topology of K . Condition (i) implies that K is a Hausdorff topological space and condition (ii) is necessary in order for the multiplication operation $K \times K \rightarrow K$, $(a, b) \mapsto ab$ to be continuous. The continuity of the additive operation $K \times K \rightarrow K$, $(a, b) \mapsto a - b$ follows from the requirement that \mathfrak{m} and hence \mathfrak{m}^n for all n are subalgebras. Let \overline{K} be the completion of K , that is, elements of \overline{K} are infinite sums $\sum_{i=1}^{\infty} a_i$ with $a_i \in K$, such that, for every $n \in \mathbb{Z}_{\geq 0}$ there are only finitely many i , such that, $a_i \notin \mathfrak{m}^n$. Let $\overline{\mathfrak{m}} \subset \overline{K}$ be the closure of \mathfrak{m} in \overline{K} .

The K -module $K \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a}$ has a natural topology with respect to which it becomes a topological K -module. This topology can be described conveniently in terms of the *order function* $\text{ord} : K \rightarrow \mathbb{Z} \cup \{\infty\}$ defined by

$$\text{ord}(a) := \begin{cases} 0 & \text{if } a \notin \mathfrak{m}, \\ n & \text{if } a \in \mathfrak{m}^n \text{ and } a \notin \mathfrak{m}^{n+1}, \\ \infty & \text{if } a = 0. \end{cases}$$

Let us define $\text{ord}(c_m^{(g)}) := \min_i \text{ord}(c_{m,i}^{(g)})$. By definition for each fixed m , the coefficients $c_m^{(g)} = 0$ for all sufficiently negative g . We equip $K \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a}$ with a topology, such that, a basis for the neighborhoods of 0 is given by the subsets $\mathbb{V}_n(K \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a})$ ($n \geq 0$) consisting of all elements of the form (1.59), such that,

$$\text{ord}(c_m^{(g)}) + \sum_{k,i} (k+1)m_{k,i} \geq n, \quad \forall g, m.$$

Similarly, let us introduce also the Fock space

$$\overline{K} \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a} := \overline{K}_{\hbar}[[q_0 - a_0, q_1 - a_1, \dots]] \otimes \mathbb{C}[\overline{K}_{\hbar}].$$

Suppose that $S(z) = 1 + S_1 z^{-1} + S_2 z^{-2} + \dots$, $S_k \in \text{End}(H)$ is a lower-triangular symplectic transformation of \mathcal{H} . The quantization \widehat{S}^{-1} defines a continuous linear operator $\widehat{\mathcal{O}}_{\mathcal{H}_+, a} \rightarrow \widehat{\mathcal{O}}_{\mathcal{H}_+, [S^{-1}a]_+}$. Elements of the form $I \otimes \mathcal{F}$ with $I \in K$ and $\mathcal{F} \in \widehat{\mathcal{O}}_{\mathcal{H}_+, a}$ are dense in $K \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a}$. Therefore, we can extend uniquely \widehat{S}^{-1} to a continuous K -linear transformation

$$\widehat{S}^{-1} : K \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a} \rightarrow K \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, [S^{-1}a]_+}.$$

For the same reason, the quantization extends also to the completions

$$\widehat{S}^{-1} : \overline{K} \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a} \rightarrow \overline{K} \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, [S^{-1}a]_+}.$$

The formula for the action of the quantization remains the same – see (1.43). Similarly, we can extend the action of the Heisenberg group $\text{Heis}(\mathcal{H})$ to $K \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a}$ and $\overline{K} \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a}$ and the statements of Proposition 1.36 remain the same.

DEFINITION 1.42. A vertex operator

$$e^{\widehat{f}^-} e^{\widehat{f}^+}, \quad f = \sum_{n \in \mathbb{Z}} I^{(n)} (-z)^n, \quad I^{(n)} \in K \otimes H$$

is said to be *regular* if for every $l \in \mathbb{Z}_{>0}$ there exists $k_0 \in \mathbb{Z}$, such that, $I^{(k)} \in \mathfrak{m}^l$ for all $k \geq k_0$.

If a vertex operator $e^{\widehat{f}^-} e^{\widehat{f}^+}$ is regular, then the coefficient of the series

$$S(z)f = \sum_{n \in \mathbb{Z}} \left(\sum_{k=0}^{\infty} (-1)^k S_k I^{(n+k)} \right) (-z)^n$$

in front of $(-z)^n$ ($\forall n \in \mathbb{Z}$) is convergent in \overline{K} . Therefore, $e^{\widehat{S}f^-} e^{\widehat{S}f^+}$ is a vertex operator with coefficients in \overline{K} . In particular, it defines a linear map $\widehat{\mathcal{O}}_{\mathcal{H}_+, a}^{\text{tame}} \rightarrow \overline{K} \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a}$. Note that in general the quantized operator \widehat{S}^{-1} does not preserve the tameness property. Nevertheless, we have the following lemma.

LEMMA 1.43. *Suppose that $\mathcal{F} = \widehat{S}^{-1}\mathcal{G}$, $\mathcal{G} \in \widehat{\mathcal{O}}_{\mathcal{H}_+, a}^{\text{tame}}$, and that $e^{\widehat{f}^-} e^{\widehat{f}^+}$ is a regular vertex operator with coefficients in K . Then*

$$e^{\widehat{f}^-} e^{\widehat{f}^+} \mathcal{F} := e^{\frac{1}{\sqrt{\hbar}} \Omega(\mathbf{q}, f^-)} \mathcal{F}(\mathbf{q} + \sqrt{\hbar} f_+(z))$$

is an element of $\overline{K} \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, b}$, where $b := [S^{-1}a]_+$.

PROOF. For the exponential term we have

$$e^{\frac{1}{\sqrt{\hbar}} \Omega(\mathbf{q}, f^-)} = e^{\frac{1}{\sqrt{\hbar}} \Omega(b, f^-)} e^{\frac{1}{\sqrt{\hbar}} \Omega(\mathbf{q} - b, f^-)}.$$

The first exponential on the RHS is interpreted as an element of the group algebra $\mathbb{C}[K_{\hbar}]$ while the second one can be exponentiated in $K \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, b} \subset \overline{K} \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, b}$. We are given that $\mathcal{F}(\mathbf{q}) = e^{\frac{1}{2\hbar} W(\mathbf{q}, \mathbf{q})} \mathcal{G}([S\mathbf{q}]_+)$. The exponential term $e^{\frac{1}{2\hbar} W(\mathbf{q} + \sqrt{\hbar} f_+, \mathbf{q} + \sqrt{\hbar} f_+)}$ is interpreted as an element of $\overline{K} \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, b}$. Finally, let us examine

$$\mathcal{G}([S(\mathbf{q} + \sqrt{\hbar} f_+)]_+) = \mathcal{G}([S\mathbf{q}]_+ + \sqrt{\hbar} [Sf]_+).$$

We have

$$[Sf]_+ = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k S_k I^{(k+n)} (-z)^n,$$

where the coefficients in front of the powers of $-z$ are convergent in \overline{K} thanks to the regularity of the vertex operator. Since $\mathcal{G} \in \widehat{\mathcal{O}}_{\mathcal{H}_+, a}^{\text{tame}}$ is tame, as we already argued in Section 1.5.2, the translation $\mathcal{G}(\overline{\mathbf{q}} + \sqrt{\hbar} [Sf]_+)$ is still a formal power series in $\overline{\mathbf{q}} - a$ with coefficients in \overline{K} . Since S is invertible, the substitution $\overline{\mathbf{q}} = [S\mathbf{q}]_+$ will transform the formal power series into a formal power series in $\mathbf{q} - b$, where $b = [S^{-1}a]_+$. \square

PROPOSITION 1.44. *Let S be a lower-triangular symplectic transformation of \mathcal{H} and $e^{\widehat{f}^-} e^{\widehat{f}^+}$ a regular vertex operator. Then the following formula holds:*

$$(1.62) \quad e^{\widehat{f}^-} e^{\widehat{f}^+} \widehat{S}^{-1} = e^{\frac{1}{2} W(f_+, f_+)} \widehat{S}^{-1} e^{(Sf)^-} e^{(Sf)^+},$$

where $e^{\frac{1}{2} W(f_+, f_+)}$ is interpreted as an element of $\mathbb{C}[\overline{K}_{\hbar}]$ and the identity is viewed as an equality between linear operators $\widehat{\mathcal{O}}_{\mathcal{H}_+, a}^{\text{tame}} \rightarrow \overline{K} \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, b}$ where $b := [S^{-1}a]_+$.

The proof is straightforward, using the definition of the action of the vertex operator (see (1.60)) and the symplectic transformation (see (1.43)), and the properties of the quadratic form W (see Proposition 1.34).

1.5.4. Tame vertex operators. Suppose that the pair (K, \mathfrak{m}) is the same as in the previous subsection. Let us introduce the *tame* subspace $(K \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a})^{\text{tame}}$ of $K \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a}$ consisting of all series of the form (1.59), such that,

$$\text{ord}(c_m^{(g)}) + 3g - 3 + \sum_{k,i} (1-k)m_{k,i} \geq 0$$

for all g and m for which $c_m^{(g)} \neq 0$. Similarly, we can define the tame subspace $(\overline{K} \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a})^{\text{tame}}$.

DEFINITION 1.45. A vertex operator with coefficients in K is said to be

a) *tame* if $I^{(-k-1)} \in \mathfrak{m}^{k+1} \otimes H$ for all $k \geq 0$.

b) *tame composable* if $\mathfrak{m}I^{(k)} \in \mathfrak{m} \otimes H$ for all $k \geq 0$. \square

Let us point out that in the applications that we have in mind, tame vertex operators are never regular. The name tame composable is justified by part c) of the following proposition.

PROPOSITION 1.46. Let $e^{\widehat{f}^-} e^{\widehat{f}^+}$ be a vertex operator with coefficients in K .

a) If $e^{\widehat{f}^-} e^{\widehat{f}^+}$ is a tame composable vertex operator, then formula (1.60) defines a map

$$e^{\widehat{f}^-} e^{\widehat{f}^+} : (K \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a})^{\text{tame}} \rightarrow \overline{K} \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a}.$$

b) If $e^{\widehat{f}^-} e^{\widehat{f}^+}$ is a tame vertex operator, then formula (1.60) defines a map

$$e^{\widehat{f}^-} e^{\widehat{f}^+} : \widehat{\mathcal{O}}_{\mathcal{H}_+, a}^{\text{tame}} \rightarrow (K \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a})^{\text{tame}}.$$

c) If the vertex operators $e^{\widehat{f}^-} e^{\widehat{f}^+}$ and $e^{\widehat{g}^-} e^{\widehat{g}^+}$ are respectively tame composable and tame, then

$$(e^{\widehat{f}^-} e^{\widehat{f}^+}) \circ (e^{\widehat{g}^-} e^{\widehat{g}^+}) = e^{\Omega(f_+, g_-)} \left(e^{(f+g)_-} e^{(f+g)_+} \right),$$

where both sides are viewed as operators $\widehat{\mathcal{O}}_{\mathcal{H}_+, a}^{\text{tame}} \rightarrow \overline{K} \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a}$.

PROOF. Part c) follows immediately from the definition (1.60). The arguments for a) and b) are very similar, so let us just prove b). First of all note that the assumption that the vertex operator is tame implies that multiplication by the exponential $e^{\frac{1}{\hbar}\Omega(\mathbf{q}, f_-)}$ leaves the tame Fock space $(K \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a})^{\text{tame}}$ invariant. We have to check that if $\mathcal{F} \in \widehat{\mathcal{O}}_{\mathcal{H}_+, a}^{\text{tame}}$, then the sum (1.61) is finite and that $\hbar^{G-1}(\mathbf{q} - a)^m$ is a tame monomial. By the tameness of \mathcal{F} , if $c_{m+n}^{(G-|n|/2)} \neq 0$, then we must have

$$3(G - |n|/2) - 3 + \sum_{k,i} (1-k)(m_{k,i} + n_{k,i}) \geq 0.$$

We get $\sum_{k,i} (k + 1/2)n_{k,i} \leq 3G - 3 + \sum_{k,i} (1-k)m_{k,i}$. Since G and m are fixed, this inequality proves that there are only finitely many possible choices for the sequence n . Clearly, $3G - 3 + \sum_{k,i} (1-k)m_{k,i} \geq 0$, so the monomial $\hbar^{G-1}(\mathbf{q} - a)^m$ is tame. \square

PROPOSITION 1.47. *Let $R(z) = 1 + R_1z + R_2z^2 + \dots$, $R_k \in \text{End}(H)$ be an upper-triangular symplectic transformation. Then*

- a) *The quantization \widehat{R} extends to a linear operator $(\overline{K} \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a})^{\text{tame}} \rightarrow (\overline{K} \otimes \widehat{\mathcal{O}}_{\mathcal{H}_+, a})^{\text{tame}}$.*
b) *If $e^{\widehat{f}^-} e^{\widehat{f}^+}$ is a tame vertex operator, then the following formula holds:*

$$(1.63) \quad \widehat{R}^{-1} e^{\widehat{f}^-} e^{\widehat{f}^+} \widehat{R} = e^{\frac{1}{2}V(f_-, f_-)} e^{(R^{-1}f_-)\widehat{-}} e^{(R^{-1}f_+)\widehat{+}}.$$

PROOF. The proof of part a) is a straightforward generalization of the proof of Proposition 1.39. Let us prove b). Since both operators $e^{\widehat{f}^-}$ and $\widehat{R} = e^{\widehat{A}}$ are defined by exponentiating respectively the operators \widehat{f}_- and \widehat{A} we have

$$\widehat{R}^{-1} e^{\widehat{f}^-} \widehat{R} = \exp\left(e^{-\widehat{A}} \widehat{f}_- e^{\widehat{A}}\right) = \exp\left(e^{-\text{ad}_{\widehat{A}}}(\widehat{f}_-)\right) = e^{(R^{-1}f_-)\widehat{-}},$$

where for the last equality we used Proposition 1.33, b). Note that

$$e^{(R^{-1}f_-)\widehat{-}} e^{(R^{-1}f_+)\widehat{+}} = e^{\frac{1}{2}[(R^{-1}f_-)\widehat{-}, (R^{-1}f_+)\widehat{+}]} e^{(R^{-1}f_-)\widehat{-}} = e^{\frac{1}{2}\Omega((R^{-1}f_-)\widehat{-}, (R^{-1}f_+)\widehat{+})} e^{(R^{-1}f_-)\widehat{-}}$$

where in the last equality we used Proposition 1.33, a). Recall that $\Omega((R^{-1}f_-)\widehat{-}, (R^{-1}f_+)\widehat{+}) = -V(f_-, f_-)$ (see Proposition 1.40, c). We get

$$\widehat{R}^{-1} e^{\widehat{f}^-} \widehat{R} = e^{\frac{1}{2}V(f_-, f_-)} e^{(R^{-1}f_-)\widehat{-}} e^{(R^{-1}f_+)\widehat{+}}.$$

Similarly,

$$\widehat{R}^{-1} e^{\widehat{f}^+} \widehat{R} = e^{(R^{-1}f_+)\widehat{+}}.$$

Therefore,

$$\widehat{R}^{-1} e^{\widehat{f}^-} e^{\widehat{f}^+} \widehat{R} = e^{\frac{1}{2}V(f_-, f_-)} e^{(R^{-1}f_-)\widehat{-}} e^{(R^{-1}f_+)\widehat{+}} e^{(R^{-1}f_+)\widehat{+}} = e^{\frac{1}{2}V(f_-, f_-)} e^{(R^{-1}f_-)\widehat{-}} e^{(R^{-1}f_+)\widehat{+}}$$

which is what we had to prove. \square

1.6. Higher genus reconstruction

The goal of this section is to define the total descendant and ancestor potentials of a semi-simple Frobenius manifold. The main motivation for this definition comes from quantum cohomology. Namely, it was conjectured by Givental [19] and proved by Teleman [60] that if the quantum cohomology of a smooth projective variety X is semi-simple, then the total descendant (resp. ancestor) potential coincides with the generating function of descendant (resp. ancestor) Gromov–Witten invariants of X .

1.6.1. The Witten–Kontsevich tau-function. Recall that a *nodal Riemann surface* is a connected projective variety Σ of dimension 1 with at most nodal singularities, that is, if $x_0 \in \Sigma$ is a singular point, then there is an open neighborhood of x_0 biholomorphic to $\{(x, y) \in \mathbb{C}^2 \mid xy = 0\}$. The *genus* of a nodal Riemann surface is defined to be the genus of its normalization. A marked nodal Riemann surface $(\Sigma, x_1, \dots, x_n)$ is a nodal Riemann surface Σ together with a sequence of non-singular pairwise distinct points x_1, \dots, x_n , called *marked points*. Two marked nodal Riemann surfaces are called equivalent if there is a biholomorphism between them that induces a bijection between the marked points. The group of automorphisms of $(\Sigma, x_1, \dots, x_n)$ is known to be finite if and only if $2g - 2 + n > 0$, where g is the genus of Σ . Let $\overline{M}_{g,n}$ be the set of equivalence classes of marked nodal Riemann surfaces with fixed genus g and fixed number of marked points n . If $2g - 2 + n > 0$, then the set $\overline{M}_{g,n}$ is known to have a structure of a projective variety of dimension $3g - 3 + n$. Moreover, the singularities of $\overline{M}_{g,n}$ are at most

of quotient type, i.e., there exists an orbifold groupoid $\overline{\mathcal{M}}_{g,n}$, whose coarse moduli space is $\overline{M}_{g,n}$. The set $L_i := \bigcup T_{x_i}^* \Sigma / \text{Aut}(\Sigma, x_1, \dots, x_n)$, where the union is over all points $(\Sigma, x_1, \dots, x_n) \in \overline{M}_{g,n}$ has a structure of an orbifold line bundle. The constructions and the proofs of the above facts are quite involved. We refer to [4] for further details.

Let us define the following intersection numbers:

$$(1.64) \quad \langle \psi_1^{k_1}, \dots, \psi_n^{k_n} \rangle_{g,n} := \int_{[\overline{M}_{g,n}]} \psi_1^{k_1} \cup \dots \cup \psi_n^{k_n}, \quad k_1, \dots, k_n \geq 0$$

where $\psi_i = c_1(L_i) \in H^2(\overline{M}_{g,n}; \mathbb{Q})$ and the intersection number is defined to be 0 if $2g - 2 + n \leq 0$. The intersection numbers are assembled into the following generating function:

$$(1.65) \quad \mathcal{D}_{\text{pt}}(\hbar, \mathbf{t}) := \exp \left(\sum_{g,n \geq 0} \frac{\hbar^{g-1}}{n!} \langle \mathbf{t}(\psi_1), \dots, \mathbf{t}(\psi_n) \rangle_{g,n} \right),$$

where $\mathbf{t} = (t_0, t_1, \dots)$ is an infinite sequence of formal variables, $\mathbf{t}(\psi) := \sum_{k=0}^{\infty} t_k \psi^k$, and each correlator is expanded multilinearly into a formal series in \mathbf{t} whose coefficients are the intersection numbers (1.64).

It was conjectured by Witten [64] and proved by Kontsevich [41] that under the substitutions $t_k = \sqrt{\hbar}(2k+1)!! T_{2k+1}$ ($k \geq 0$), the generating function $\mathcal{D}_{\text{pt}}(\hbar, \mathbf{t})$ is a tau-function of the KdV hierarchy. That is why $\mathcal{D}_{\text{pt}}(\hbar, \mathbf{t})$ is also known as the *Witten-Kontsevich tau-function*. The KdV hierarchy will be defined later on. Let us postpone the precise formulations for now and outline instead how to obtain a combinatorial recursion that allows us to compute all intersection numbers. To begin with, Witten's conjecture can be reformulated equivalently in terms of *Virasoro constraints* (see [11, 18, 39]). The latter, according to Givental [20], can be stated in terms of the quantization formalism from the previous section as follows. Let $H = \mathbb{C}$ and $(\ , \)$ be the standard pairing $(1, 1) = 1$. Put $D = z \frac{\partial}{\partial z}$. Then $\ell_m = z^{-1/2} D^{m+1} z^{-1/2}$ ($m \geq -1$) are infinitesimal symplectic transformations of $\mathcal{H} = \mathbb{C}((z^{-1}))$. Recalling the quantization formalism, we define the operators $L_m := \widehat{\ell}_m + \frac{1}{16} \delta_{m,0}$. The first few operators take the form

$$\begin{aligned} L_{-1} &= \frac{1}{2\hbar} q_0^2 + \sum_{m=0}^{\infty} q_{m+1} \partial_m, \\ L_0 &= \frac{1}{16} + \sum_{m=0}^{\infty} \left(m + \frac{1}{2}\right) q_m \partial_m, \\ L_1 &= \frac{\hbar}{8} \partial_0^2 + \sum_{m=0}^{\infty} \left(m + \frac{1}{2}\right) \left(m + \frac{3}{2}\right) q_m \partial_{m+1}, \end{aligned}$$

where $\partial_m := \frac{\partial}{\partial q_m}$. It is straightforward to check that the above operators satisfy the commutation relation $[L_m, L_n] = (m-n)L_{m+n}$ and hence they determine a representation of the Lie algebra of vector fields on the circle. It turns out that in order to obtain a representation relevant for the intersection numbers, we have to introduce the following *Fock space*:

$$(1.66) \quad \mathbb{C}_{\hbar}[\widehat{q}_0, q_1 + 1, q_2, \dots], \quad \mathbb{C}_{\hbar} := \mathbb{C}((\hbar)).$$

The Witten–Kontsevich tau-function is identified with an element in the Fock space via the so-called *dilaton shift*: $t_k = q_k + \delta_{k,1}$. Givental’s reformulation of Virasoro constraints can be stated as follows: $L_m \mathcal{D}_{\text{pt}} = 0$ for all $m \geq -1$. The Virasoro constraints $L_{-1} \mathcal{D}_{\text{pt}} = 0$ and $L_0 \mathcal{D}_{\text{pt}} = 0$ are known respectively as the *string equation* and the *dilaton equation*. They can be proved geometrically (see [64]).

The Virasoro constraints are equivalent to a recursion for the intersection numbers which was discovered in the settings of matrix models. The abstract formulation of the recursion and its general properties were studied by Eynard and Orantin [14]. For the case at hands, the recursion produces a set of symmetric meromorphic differentials $\omega_{g,n}(x_1, \dots, x_n)$ on $\mathbb{C} \times \dots \times \mathbb{C}$ with finite order poles at the divisors $x_i = 0$ ($1 \leq i \leq n$), except for the case $(g, n) = (0, 2)$. All forms $\omega_{g,n}$ with $2g - 2 + n \leq 0$ are 0, except for

$$\omega_{0,2}(x_1, x_2) := \frac{dx_1 dx_2}{(x_1 - x_2)^2},$$

and

$$\begin{aligned} \omega_{g,n+1}(x_0, x_1, \dots, x_n) = \\ \text{Res}_{y=0} K(x_0, y) \left(\omega_{g-1, n+2}(y, -y, x_1, \dots, x_n) + \right. \\ \left. \sum_{g'+g''=g} \sum_{i'_1, \dots, i'_{n'}} \omega_{g', n'+1}(y, x_{i'_1}, \dots, x_{i'_{n'}}) \omega_{g'', n''+1}(-y, x_{i''_1}, \dots, x_{i''_{n''}}) \right), \end{aligned}$$

where the first sum is over all pairs (g', g'') of non-negative integers, such that, $g' + g'' = g$, the second sum is over all subsets $\{i'_1, \dots, i'_{n'}\} \subseteq \{1, 2, \dots, n\}$ and

$$\{i''_1, \dots, i''_{n''}\} := \{1, 2, \dots, n\} \setminus \{i'_1, \dots, i'_{n'}\},$$

and

$$K(x_0, y) = \frac{1}{2y(y-x_0)(y+x_0)} \frac{dx_0}{dy}$$

is the *recursion kernel*. It is proved in [13], formula (5.11), that the solution of the above recursion is given by

$$\omega_{g,n}(y_1, \dots, y_n) = (-1)^n \sum_{d_1, \dots, d_n \geq 0} \langle \psi_1^{d_1}, \dots, \psi_n^{d_n} \rangle_{g,n} \prod_{i=1}^n (2d_i + 1)!! y_i^{-2d_i - 2} dy_i,$$

where the forms $W_{g,n}^K(t_1, \dots, t_n)$ in [13], formula (5.11), coincide with $\frac{1}{2^{2g-2+n}} \omega_{g,n}(y_1, \dots, y_n)$ under the substitution $y_i = -\frac{2}{t_i}$. Note that $\omega_{g,n}$ are polynomials in y_i^{-1} , because the intersection number in the above sum is non-zero only if $d_1 + \dots + d_n = 3g - 3 + n$. In particular, we can take the inverse Laplace transform. We get the following answer. Suppose that z_1, \dots, z_n are negative real numbers, then the correlator

$$\left\langle \frac{1}{\psi_1 - z_1}, \dots, \frac{1}{\psi_n - z_n} \right\rangle_{g,n},$$

where the i th slot is expanded into a gemetric series near $z_i = \infty$, coincides with the oscillatory integral

$$\prod_{i=1}^n (-2\pi z_i)^{-1/2} \int_{x_1 \in \Gamma} \dots \int_{x_n \in \Gamma} e^{\frac{f(x_1)}{z_1} + \dots + \frac{f(x_n)}{z_n}} \omega_{g,n}(x_1, \dots, x_n),$$

where $f(x) = \frac{x^2}{2}$ and the integration contour is $\Gamma = \sqrt{-1}\epsilon + \mathbb{R}$.

1.6.2. The asymptotic operator series. Let us fix a semi-simple point $s \in M \setminus \mathcal{K}$. By definition, there exists a coordinate system $\{u_i\}_{i=1}^N$ defined locally on some open neighborhood of s in which both the Frobenius multiplication and the flat metric are diagonal, that is,

$$\partial_{u_i} \bullet \partial_{u_j} = \delta_{i,j} \partial_{u_j}, \quad (\partial_{u_i}, \partial_{u_j}) = \frac{\delta_{i,j}}{\Delta_j},$$

where $\partial_{u_i} := \frac{\partial}{\partial u_i}$ and $\{\Delta_j\}_{j=1}^N$ are analytic functions with no zeros, defined in a neighborhood of s . Furthermore, let us fix a flat coordinate systems $t = (t_1, \dots, t_N)$, defined in a neighborhood of s . For brevity put $\partial_a := \frac{\partial}{\partial t_a}$ ($1 \leq a \leq N$) for the corresponding flat vector fields. Given an endomorphism $A \in \text{End}(TM)$, then we denote by A^{ft} the matrix with entries $A_{ab}^{\text{ft}} \in \mathcal{O}_M$ ($1 \leq a, b \leq N$) defined by $A\partial_b = \sum_{a=1}^N A_{ab}^{\text{ft}} \partial_a$. Finally, let us define the $N \times N$ matrix Ψ with entries

$$(1.67) \quad \Psi_{ai} = \sqrt{\Delta_i} \frac{\partial t_a}{\partial u_i} \in \mathcal{O}_{M,s}, \quad 1 \leq a, i \leq N.$$

Some basic properties of Ψ are summarized in the following proposition, whose proof is left as an exercise.

PROPOSITION 1.48. *Let U be the diagonal matrix of size $N \times N$ whose diagonal entries are $U_{ii} = u_i$ ($1 \leq i \leq N$). The matrix Ψ has the following properties:*

- (1) *If $g = (g_{ab})$, $g_{ab} = (\partial_a, \partial_b)$, is the matrix of the Frobenius pairing, then*

$$\Psi \Psi^T = g^{-1},$$

where T is the usual transposition of matrices.

- (2) *Let $A = \sum_{a=1}^N A_a dt_a$ be the connection 1-form on M where A_a is the endomorphism of TM defined by the Frobenius multiplication by ∂_a . Then*

$$\Psi^{-1} A^{\text{ft}} \Psi = dU.$$

- (3) *The Euler vector field has the form $E = \sum_{i=1}^N u_i \partial_{u_i}$. In particular,*

$$\Psi^{-1} (E \bullet)^{\text{ft}} \Psi = U,$$

where $E \bullet$ is the endomorphism of TM defined by Frobenius multiplication by E . \square

The Dubrovin's connection ∇ in flat coordinates takes the form

$$\nabla = d - A^{\text{ft}} z^{-1} + \left(-\theta^{\text{ft}} z^{-1} + (E \bullet)^{\text{ft}} z^{-2} \right) dz,$$

where A is the same as in Proposition 1.48, (2).

LEMMA 1.49. *The Euler vector field in flat coordinates has the following form*

$$E = \sum_{a=1}^N \left(r_a + \sum_{b=1}^N \left(\theta_{ab}^{\text{ft}} + (1 - D/2) \delta_{a,b} \right) t_b \right) \partial_a,$$

where r_a are some constants.

PROOF. The formula follows immediately from the definition of the grading operator θ , that is, we have

$$[\partial_b, E] = \theta(\partial_b) + (1 - D/2) \partial_b. \quad \square$$

PROPOSITION 1.50. *The connection ∇ has a unique formal asymptotical solution of the form*

$$(1.68) \quad \Psi(1 + R_1^{\text{can}}z + R_2^{\text{can}}z^2 + \dots)e^{U/z},$$

where R_k^{can} are $N \times N$ matrices whose entries are functions holomorphic in a neighborhood of s .

PROOF. Using Proposition 1.48 we get

$$\Psi^{-1}\nabla\Psi = d + \Psi^{-1}d\Psi - dUz^{-1} + (Vz^{-1} + Uz^{-2})dz,$$

where $V := -\Psi^{-1}\theta^{\text{ft}}\Psi$. The asymptotical series (1.68) is a solution to the Dubrovin's connection if and only if $\{R_k^{\text{can}}\}_{k=0}^{\infty}$ (we set $R_0^{\text{can}} = 1$) satisfies the following system of differential equations:

$$(1.69) \quad dR_k^{\text{can}} + (\Psi^{-1}d\Psi)R_k^{\text{can}} = [dU, R_{k+1}^{\text{can}}], \quad \forall k \geq 0$$

and

$$(1.70) \quad kR_k^{\text{can}} + [U, R_{k+1}^{\text{can}}] = -VR_k^{\text{can}}, \quad \forall k \geq 0.$$

We have to prove that the above system has a unique solution. In order to avoid cumbersome notation, let us drop the superscript and simply write $R_i := R_i^{\text{can}}$. Arguing by induction on ℓ we will prove that there is a unique sequence R_1, \dots, R_ℓ satisfying (1.69) and (1.70) for all $k \leq \ell - 1$, the diagonal part of (1.70) for $k = \ell$, and $E(R_k) = -kR_k$ for all $k \leq \ell$.

Let us first prove the statement for $\ell = 1$. Using (1.69) with $k = 0$ and comparing the (i, j) -th entries of the matrices with $i \neq j$ we get

$$(\Psi^{-1}d\Psi)_{ij} = (du_i - du_j)(R_1)_{ij}.$$

The flatness of ∇ implies that $[dU, \Psi^{-1}d\Psi] = 0$. In particular, $(du_i - du_j) \wedge (\Psi^{-1}d\Psi)_{ij} = 0$, which by the de Rham lemma implies that $(\Psi^{-1}d\Psi)_{ij} = \alpha_{ij}(du_i - du_j)$ for some function α_{ij} analytic in a neighborhood of s . Hence $(R_1)_{ij} = \alpha_{ij}$, that is,

$$(R_1)_{ij} = (\Psi^{-1}\partial_{u_i}\Psi)_{ij} = -(\Psi^{-1}\partial_{u_j}\Psi)_{ij}.$$

As a byproduct, our argument here also yields

$$(\Psi^{-1}\partial_{u_p}\Psi)_{ij} = 0, \quad p \neq i, j.$$

Comparing the diagonal entries in (1.70) for $k = 1$ we get

$$(R_1)_{ii} = -\sum_{p \neq i} V_{ip}(R_1)_{pi},$$

so R_1 is uniquely determined. Let us check that R_1 satisfies (1.70) with $k = 0$. We need only to compare the off-diagonal entries. Fix $i \neq j$, then

$$[U, R_1]_{ij} = (u_i - u_j)(R_1)_{ij} = (\Psi^{-1}E(\Psi))_{ij},$$

where $E = \sum_{i=1}^N u_i \partial_{u_i}$ is the Euler vector field. Since by definition $\text{Lie}_E(\cdot, \cdot) = (2-D)(\cdot, \cdot)$ we get that $E(\Delta_i) = D \Delta_i \Rightarrow [E, \sqrt{\Delta_i} \partial_{u_i}] = \left(\frac{D}{2} - 1\right) \sqrt{\Delta_i} \partial_{u_i}$. Therefore,

$$E(\Psi_{ai}) = \left(\frac{D}{2} - 1\right) \Psi_{ai} + \sqrt{\Delta_i} \partial_{u_i}(E(t_a)) = \sum_{b=1}^N \theta_{ab} \Psi_{bi},$$

where in the second equality we used Lemma 1.49. In other words $\Psi^{-1}E(\Psi) = \Psi^{-1}\theta^{\text{ft}}\Psi = -V$. Finally, note that $E(U) = U$ and $E(V) = 0$, so the identity $[U, R_1] = -V$ implies that $E(R_1) = -R_1$.

Assume that we have constructed R_1, \dots, R_ℓ . We would like to construct $R_{\ell+1}$ so that the inductive assumption holds. Note that since ∇ is flat we have

$$\left(d + \Psi^{-1}d\Psi\right)^2 = \Psi^{-1}d^2\Psi = 0, \quad [dU, d + \Psi^{-1}d\Psi] = 0,$$

so

$$[dU, dR_\ell + \Psi^{-1}d\Psi R_\ell] = (d + \Psi^{-1}d\Psi)[dU, R_\ell] = (d + \Psi^{-1}d\Psi)^2 R_{\ell-1} = 0.$$

Now the same argument that we used to construct R_1 can be used to construct $R_{\ell+1}$. The details are straightforward and will be left as an exercise. \square

1.6.3. The total ancestor potential. Suppose that $s \in M \setminus \mathcal{K}$ is a semi-simple point. Let us fix both a flat coordinate system $t = (t_1, \dots, t_N)$ and a canonical coordinate system $u = (u_1, \dots, u_N)$ on a sufficiently small contractible open neighborhood \mathcal{U} of s . We are going to use Givental's quantization formalism with $H = T_s M$ and $(\ , \)$ – the Frobenius pairing. Using parallel transport with respect to the Levi-Civita connection we identify H with the space of flat vector fields on \mathcal{U} , that is, $H = \mathbb{C} \frac{\partial}{\partial t_1} \oplus \dots \oplus \mathbb{C} \frac{\partial}{\partial t_N}$. Note that $\phi_a := \frac{\partial}{\partial t_a}$ ($1 \leq a \leq N$) and $\phi^a := dt_a$ ($1 \leq a \leq N$) are dual bases of H , where we identified $T^*M \cong TM$ via the Frobenius pairing. Let us assume that $\phi_1 = \frac{\partial}{\partial t_1} = e$ is the unit vector field.

Let us introduce the vector fields $e_i := \sqrt{\Delta_i} \frac{\partial}{\partial u_i}$ ($1 \leq i \leq N$). They form a frame for $\mathcal{T}\mathcal{U}$. Therefore, every endomorphism $A \in \text{End}(\mathcal{T}\mathcal{U})$ can be represented by a matrix A^{can} with entries $A_{ij}^{\text{can}} \in \mathcal{O}_M(\mathcal{U})$ defined by $Ae_j = \sum_{i=1}^N A_{ij}^{\text{can}} e_i$. Recalling the definition of the matrix Ψ , we get $A^{\text{ft}} = \Psi A^{\text{can}} \Psi^{-1}$, where A^{ft} is the matrix of the endomorphism A with respect to the flat frame ∂_a ($1 \leq a \leq N$) of $\mathcal{T}\mathcal{U}$. Finally, let us define the following evaluation maps. Using the flat Levi-Civita connection $\nabla^{\text{L.C.}}$ we identify all tangent spaces $T_t \mathcal{U} \cong H \ \forall t \in \mathcal{U}$. Given an endomorphism $A \in \text{End}(\mathcal{T}\mathcal{U})$, (resp. a vector field $v \in \mathcal{T}\mathcal{U}$) and a point $t \in \mathcal{U}$, we denote by $A(t) \in \text{End}(H)$ (resp. $v(t) \in H$) the restriction $A|_t \in \text{End}(T_t \mathcal{U}) \cong \text{End}(H)$ (resp. $v|_t \in T_t \mathcal{U} \cong H$). Note that both values $A(t)$ and $v(t)$ depend on the Levi-Civita connection, but not on the choice of the flat frame ∂_a ($1 \leq a \leq N$).

Let us recall the asymptotic series solution from Proposition 1.50. Let $R_k \in \text{End}(\mathcal{T}\mathcal{U})$ be the endomorphism whose matrix with respect to the frame e_i ($1 \leq i \leq N$) is the coefficient R_k^{can} of the asymptotic solution. Put

$$R(t, z) := \sum_{k=0}^{\infty} R_k(t) z^k \in \text{End}(H)[[z]],$$

where the evaluation map $t \mapsto R_k(t)$ is defined as explained above.

LEMMA 1.51. *The operator series $R(t, z)$ is a symplectic transformation of $H((z))$.*

PROOF. We have to check that $R(t, z)R(t, -z)^T = 1$, where T is transposition with respect to the Frobenius pairing. Let us denote by g the matrix of the Frobenius pairing $g_{ab} = (\partial_a, \partial_b)$ and by ${}^{\text{tr}}$ the usual transposition of matrices. Let us identify $\text{End}(H)$ with the space of $N \times N$ -matrices via the flat basis ϕ_a ($1 \leq a \leq N$), that is, via the map $A \mapsto A^{\text{ft}}$. To avoid cumbersome notation we drop the superscript ${}^{\text{ft}}$. Note that

since $A^T = g^{-1}A^{\text{tr}}g$, it is sufficient to prove that $R(t, z)g^{-1}R(t, -z)^{\text{tr}}g = 1$. Recalling that by definition $R(t, z) = \Psi_t R^{\text{can}}(t, z)\Psi_t^{-1}$, where $\Psi_t := \Psi(t)$ and $R^{\text{can}}(t, z) := 1 + \sum_{k \geq 1} R^{\text{can}}(t)z^k$, we get

$$\Psi_t R^{\text{can}}(t, z)\Psi_t^{-1}g^{-1}(\Psi_t^{-1})^{\text{tr}}R^{\text{can}}(t, -z)^{\text{tr}}\Psi_t^{\text{tr}}g = 1.$$

Recalling Proposition 1.48, (1), we get that the above identity is equivalent to

$$R^{\text{can}}(t, z)R^{\text{can}}(t, -z)^{\text{tr}} = 1.$$

Let us consider the formal expression

$$C := e^{-U/z}R^{\text{can}}(t, -z)^{\text{tr}}\Psi^{\text{tr}}g\Psi R^{\text{can}}(t, z)e^{U/z}.$$

Using that $\Psi R^{\text{can}}(t, z)e^{U/z}$ is a solution to ∇ , it is straightforward to check that C is independent of t and z . Recalling again Proposition 1.48, since $\Psi^{\text{tr}}g\Psi = 1$, we get that $A := R^{\text{can}}(t, -z)^{\text{tr}}R^{\text{can}}(t, z)$ satisfies the same differential equations in u_1, \dots, u_N , and z as $e^{U/z}Ce^{-U/z}$. In other words,

$$\begin{aligned} \partial_{u_i} A &= z^{-1}[E_{ii}, A] \\ z\partial_z A &= -z^{-1}[U, A] \end{aligned}$$

where E_{ii} is the matrix with only one non-zero entry in position (i, i) and that entry is 1. The series $A = 1 + \sum_{k=1}^{\infty} A_k z^k$, where A_k are holomorphic functions on \mathcal{U} with values $N \times N$ matrices. Suppose k is the minimal index > 1 for which $A_k \neq 0$. The differential equation in z implies that $[U, A_k] = 0 \Rightarrow A_k$ is a diagonal matrix. Again the differential equation in z implies that $kA_k = -[U, A_{k+1}] \Rightarrow$ the diagonal entries of A_k must be 0 $\Rightarrow A_k = 0$ – contradiction. This proves that $A = 1$. \square

Let us denote by \widehat{R}_t the quantization of the upper-triangular symplectic transformation $R(t, z)$. Let us recall the Witten–Kontsevich tau-function $\mathcal{D}_{\text{pt}}(\hbar; \mathbf{t})$ and let us identify $\mathbf{t} = (t_0, t_1, t_2, \dots)$ with $\mathbf{t}(z) := t_0 + t_1 z + t_2 z^2 + \dots$. Similarly, the sequence of formal vector variables $\mathbf{q} = (q_{k,a})_{1 \leq a \leq N, k \geq 0}$ is identified with the series $\mathbf{q}(z) := \sum_{k,i} q_{k,a} \partial_a z^k$. Note that the following product is an element of $\widehat{\mathcal{O}}_{\mathcal{H}_+, -z}^{\text{tame}}$:

$$(1.71) \quad \prod_{i=1}^N \mathcal{D}_{\text{pt}}(\hbar \Delta_i, \mathbf{q}(u_i) + z),$$

where $\mathbf{q}(u_i) := \sum_{k=0}^{\infty} \sum_{a=1}^N q_{k,a} \frac{\partial u_i}{\partial t_a} z^k$, that is, the i th term in the above product is obtained from the Witten–Kontsevich tau-function via the substitutions $\hbar \mapsto \hbar \Delta_i$ and $t_k \mapsto q_k(u_i) + \delta_{k,1}$. Proposition 1.39 implies that the quantized symplectic transformation \widehat{R}_t acts on the tame Fock space $\widehat{\mathcal{O}}_{\mathcal{H}_+, -z}^{\text{tame}}$. The tame formal function

$$(1.72) \quad \mathcal{A}_t(\hbar, \mathbf{q}) := \widehat{R}_t \prod_{i=1}^N \mathcal{D}_{\text{pt}}(\hbar \Delta_i, \mathbf{q}(u_i) + z)$$

is called the *total ancestor potential* of the Frobenius manifold M . Let us give a coordinate independent interpretation of the coefficients of the total ancestor potential. In order to do this, let us fix the formal variables $q_{k,a}$ once and for all and think of $\mathbb{C}_{\hbar}[[q_0, q_1 + \mathbf{1}, q_2, + \dots]]$ as an universal ring in which all ancestor potentials of all semi-simple Frobenius manifolds of dimension N take values. Recalling the definition of $\mathcal{A}_t(\hbar; \mathbf{q})$, we get that, a priori the total ancestor potential depends on the choice of canonical coordinates and the choice of a flat basis $\partial := (\partial_1, \dots, \partial_N)$, such that, $\partial_1 = e$ is the unit vector field. In fact, it

looks that we have also a dependence on the choice of a sign for $\sqrt{\Delta_i}$. However, as we will argue below, the total ancestor potential depends only on the choice of the flat basis $\partial = (\partial_1, \dots, \partial_N)$ satisfying $\partial_1 = e$.

PROPOSITION 1.52. *Suppose that*

$$\mathcal{A}_t(\hbar; \mathbf{q}) := \sum_{g \in \mathbb{Z}} \sum_{\kappa = \{(k_1, a_1), \dots, (k_r, a_r)\}} c_{\kappa}^{(g)}(t) \hbar^{g-1} t_{k_1, a_1} \cdots t_{k_r, a_r},$$

where the variables \mathbf{q} and \mathbf{t} are related via the dilaton shift. Then

- a) The coefficients $c_{\kappa}^{(g)}(t)$ can be extended analytically in t along any path in $M \setminus \mathcal{K}$.
- b) For every fixed sequence of integers $k_1, \dots, k_r \geq 0$, the tensor

$$(1.73) \quad \sum_{a_1, \dots, a_r=1}^N c_{(k_1, a_1), \dots, (k_r, a_r)}^{(g)}(t) dt_{a_1} \otimes \cdots \otimes dt_{a_r}$$

is a symmetric r -form on $M \setminus \mathcal{K}$, that is, the above tensor is a section of $\text{Sym}^r(T^*M)$ on $M \setminus \mathcal{K}$. Moreover, the form (1.73) is independent of the choice of flat basis in the definition of $\mathcal{A}_t(\hbar; \mathbf{q})$.

PROOF. a) In order to prove that the coefficients $c_{\kappa}^{(g)}$ extend analytically along any path in $M \setminus \mathcal{K}$, it is enough to prove that the canonical coordinates u_i have this property. Let $L \subset T^*M$ be the analytic spectrum (see Section 1.2). If s is a semi-simple point, then we can choose canonical coordinates (u_1, \dots, u_N) around s and fiberwise linear coordinates x_1, \dots, x_N on T^*M corresponding to the basis $\{du_i\}_{i=1}^N$. In the local coordinates $(u_1, \dots, u_N, x_1, \dots, x_N)$ the analytic spectrum L is given by the equations

$$x_i x_j - \delta_{i,j} x_j = 0, \quad 1 \leq i, j \leq N.$$

It follows that over a neighborhood of s the subvariety L is a N -sheet covering and the N sections of T^*M that define L are precisely the 1-forms du_i ($1 \leq i \leq N$). Recall that the projection $T^*M \rightarrow M$ induces a map $\pi : L \rightarrow M$ which is a branched covering of degree N and moreover the set \mathcal{K} of non-semi-simple points coincides with the branching locus. Since L induces a regular covering on $M \setminus \mathcal{K}$ the differential forms du_i extend along any path in $M \setminus \mathcal{K}$, which proves that u_i also extends.

b) We have to check that the form (1.73) is invariant under the analytic continuation along a closed loop around the caustic \mathcal{K} . Let us look more carefully at the definition of the total ancestor potential. Let $R^{\text{ft}}(t, z) := \sum_{k \geq 0} R_k^{\text{ft}}(t) z^k$, where $R_k^{\text{ft}}(t)$ is the matrix of $R_k(t)$ with respect to the flat basis. Since $R^{\text{ft}}(t, z) = \Psi_t R^{\text{can}}(t, z) \Psi_t^{-1}$, the system (1.69)–(1.70) is equivalent to the following system of differential equations:

$$(1.74) \quad dR^{\text{ft}} = -R^{\text{ft}}(d\Psi \Psi^{-1}) + [z^{-1} A^{\text{ft}}, R^{\text{ft}}] + (\theta^{\text{ft}} R^{\text{ft}} - [z^{-2} (E \bullet)^{\text{ft}}, R^{\text{ft}}]) dz,$$

where $A = \sum_{a=1}^N dt_a \partial_a \bullet$. Therefore, $R^{\text{ft}}(t, z)$ is uniquely determined from the matrices $d\Psi \Psi^{-1}$, A , θ , and $E \bullet$, where the last three matrices are the matrices of the corresponding operators with respect to the flat basis $\partial := (\partial_1, \dots, \partial_N)$. In fact, $d\Psi \Psi^{-1}$ has also an intrinsic interpretation, i.e., it is the matrix of the linear operator:

$$\Gamma : \mathcal{T}_M|_{M \setminus \mathcal{K}} \rightarrow \mathcal{T}_M \otimes \Omega_M^1|_{M \setminus \mathcal{K}},$$

defined by

$$(1.75) \quad \Gamma(v) = \frac{1}{2} \left(\sum_{i=1}^N \frac{d\Delta_i}{\Delta_i} \right) v + \sum_{i=1}^N v(u_i) \nabla^{\text{L.C.}}(\partial_{u_i}).$$

Suppose that under the analytic continuation along a closed loop around the caustic, the flat basis ∂ is transformed into the flat basis $\partial' = \partial C$, that is, $\partial'_b = \sum_{a=1}^N C_{ab} \partial_a$, where C is some constant matrix satisfying the symplectic condition $CC^T = 1$. The matrix $R^{\text{ft}}(t, z)$ is transformed into $C^{-1}R^{\text{ft}}(t, z)C$. It is an easy exercise to check that if $CC^T = 1$, then

$$\left(C^{-1}R^{\text{ft}}(t, z)C \right)^\wedge = \widehat{R}_t|_{q_k \mapsto Cq_k},$$

where q_k should be viewed as a vector column with entries $(q_{k,1}, \dots, q_{k,N})$, that is, in components the substitution takes the form $q_{k,a} \mapsto \sum_{b=1}^N C_{ab} q_{k,b}$. The argument in the i th factor in (1.71) is transformed into

$$q_k(u_i) = \sum_{a=1}^N q_{k,a} \partial'_a(u_i) = \sum_{a,b=1}^N q_{k,a} C_{ba} \partial_b.$$

Therefore, the operation of analytic continuation of the product (1.71) is equivalent to the substitution $q_k \mapsto Cq_k$. The conclusion is that under the analytic continuation the ancestor potential $\mathcal{A}_t(\hbar; \mathbf{q})$ is transformed into $\mathcal{A}_t(\hbar; C\mathbf{q})$. Equivalently the coefficient $c_{(k_1, b_1), \dots, (k_r, b_r)}^{(g)}(t)$ is transformed into

$$\sum_{a_1, \dots, a_r=1}^N c_{(k_1, a_1), \dots, (k_r, a_r)}^{(g)}(t) C_{a_1 b_1} \cdots C_{a_r b_r}$$

Note that the dual bases are related by $dt_a = \sum_{b=1}^N C_{ab} dt'_b$, so the form (1.73) is invariant.

It remains only to prove that (1.73) is independent of the choice of the flat basis. Note that if we have two flat bases $\partial = (\partial_1, \dots, \partial_N)$ and $\partial' = \partial C$, where C is a constant matrix (possibly non-symplectic), then the quadratic Hamiltonian h_A with respect to ∂' is obtained from the quadratic Hamiltonian with respect to ∂ via the substitutions $q_k \mapsto Cq_k$ and $p_k \mapsto p_k C^{-1}$, where q_k is a vector column (just like above) and p_k is a vector row with entries $(p_{k,1}, \dots, p_{k,N})$. We get that the change of the flat basis changes the quantized Hamiltonian \widehat{h}_A via the substitution $q_k \mapsto Cq_k$. The conclusion is that

$$\mathcal{A}_{t, \partial C}(\hbar; \mathbf{q}) = \mathcal{A}_{t, \partial}(\hbar; C\mathbf{q}),$$

where $\partial = (\partial_1, \dots, \partial_N)$ is a flat basis, C is any invertible constant matrix, and we added ∂ to the notation in order to keep track of the dependence of $\mathcal{A}_t(\hbar; \mathbf{q})$ on the choice of the flat basis. Just like above, this relation implies that the form (1.73) is independent of the choice of flat basis. \square

Finally, we would like to prove that the total ancestor potential satisfies the following differential equations:

$$(1.76) \quad \partial_a \mathcal{A}_t(\hbar; \mathbf{q}) = \left((z^{-1} A_a)^\wedge - \partial_a F^{(1)}(t) \right) \mathcal{A}_t(\hbar; \mathbf{q}),$$

where $A_a = \partial_a \bullet$ is the operator of Frobenius multiplication by ∂_a and

$$(1.77) \quad F^{(1)}(t) = \int^t \left(\frac{1}{2} \text{Tr}(R_1(t)dU) + \frac{1}{48} \sum_{i=1}^N \frac{d\Delta_i}{\Delta_i} \right)$$

is the so-called *primary genus-1 potential* of the Frobenius structure.

To begin with, let us recall the differential equations for $R(t, z)$ with respect to t (see formula (1.74))

$$\partial_a R(t, z) = -R(t, z)(\partial_a \Psi)\Psi^{-1} + [z^{-1}A_a, R(t, z)].$$

In order to avoid cumbersome notation, let us denote by $\alpha := z^{-1}A_a$, $\beta := \log R(t, z)$, and $\gamma := (\partial_a \Psi)\Psi^{-1}$. Note that these 3 operators are infinitesimal symplectic transformations. We have

$$R(t + \epsilon \partial_a, z) = e^{\beta + \epsilon[\alpha, \beta] + O(\epsilon^2)} e^{-\epsilon\gamma + O(\epsilon^2)},$$

where $t + \epsilon \partial_a = (t_1, \dots, t_a + \epsilon, \dots, t_N)$. Recalling Proposition 1.31, we get

$$\widehat{h}_{[\alpha, \beta]} = \{h_\alpha, h_\beta\}^\wedge = [\widehat{h}_\alpha, \widehat{h}_\beta] - C(\alpha, \beta),$$

where $C(\alpha, \beta) := C(h_\alpha, h_\beta)$. Note that the Hamiltonian corresponding to γ is

$$(1.78) \quad h_\gamma = - \sum_{k=0}^{\infty} \sum_{i, j=1}^N \gamma_{ij} q_{k, j} p_{k, i}.$$

The Hamiltonian h_β is given by formula (1.45). Clearly, the cocycle $C(h_\beta, h_\gamma) = 0$. We get

$$\widehat{R}_{t+\epsilon \partial_a} = e^{-\epsilon C(\alpha, \beta)} e^{\widehat{\beta} + \epsilon[\widehat{\alpha}, \widehat{\beta}] + O(\epsilon^2)} e^{-\epsilon \widehat{\gamma} + O(\epsilon^2)}.$$

Differentiating with respect to ϵ and setting $\epsilon = 0$ we get

$$(1.79) \quad \partial_a \widehat{R}_t = -C(\alpha, \beta) \widehat{R}_t - \widehat{R}_t \widehat{\gamma} + [\widehat{\alpha}, \widehat{R}_t].$$

Let us compute the cocycle pairing $C(\alpha, \beta)$. After a straightforward computation we get

$$h_\alpha = -\frac{1}{2} \sum_{i, j=1}^N (A_a \phi_i, \phi_j) q_{0, i} q_{0, j} - \sum_{k=0}^{\infty} \sum_{i, j=1}^N (A_a \phi_i, \phi^j) q_{k+1, i} p_{k, j}.$$

Comparing with the formula (1.45), we get that only the first term in h_α contributes to the cocycle and hence

$$C(\alpha, \beta) = \frac{1}{2} \sum_{i, j=1}^N (\phi_i, A_a \phi_j) (\beta_1 \phi^i, \phi^j) = \frac{1}{2} \text{Tr}(\beta_1 A_a),$$

where $\beta =: \sum_{k=1}^{\infty} \beta_k z^k$, we used that $C(q_{0, i} q_{0, j}, p_{0, i} p_{0, j}) = -1$ (resp. $= -2$) for $i \neq j$ (resp. $i = j$). Note that $R(t, z) = e^\beta \Rightarrow \beta_1^{\text{can}} = R_1^{\text{can}}$ and $A^{\text{can}} = dU \Rightarrow A_a^{\text{can}} = \iota_{\partial_a} A^{\text{can}} = \partial_a(U)$. Therefore, we can write the cocycle as

$$(1.80) \quad C(\alpha, \beta) = \frac{1}{2} \text{Tr}(R_1^{\text{can}} \partial_a(U)) = \frac{1}{2} \sum_{i=1}^N R_{1ii}^{\text{can}} \partial_a(u_i),$$

where R_{1ii}^{can} is the (i, i) -entry of the matrix R_1^{can} . Let us compute the derivative ∂_a of the product of Witten–Kontsevich tau functions (1.71). We will have to use the dilaton equation (see [64]), i.e., if $2g - 2 + n > 0$, then

$$\langle \psi_1^{k_1}, \dots, \psi_n^{k_n}, \psi_{n+1} \rangle_{g, n+1} = (2g - 2 + n) \langle \psi_1^{k_1}, \dots, \psi_n^{k_n} \rangle_{g, n}$$

and $\langle \psi_1 \rangle_{1,1} = \frac{1}{24}$. In terms of the generating function, the dilaton equation becomes

$$\left(2\hbar \partial_{\hbar} + \sum_{k=0}^{\infty} (t_k - \delta_{k,1}) \frac{\partial}{\partial t_k} \right) \mathcal{D}_{\text{pt}}(\hbar, \mathbf{t}) = -\frac{1}{24} \mathcal{D}_{\text{pt}}(\hbar, \mathbf{t}),$$

where slightly abusing the notation, here and until the end of this section, we will denote by $\mathbf{t} = (t_0, t_1, \dots)$ the variables of the Witten–Kontsevich tau-function, not the variables related to \mathbf{q} via the dilaton shift. Using the chain rule we compute

$$\partial_a \mathcal{D}_{\text{pt}}(\hbar \Delta_i; \mathbf{q}(u_i) + z) = \frac{\partial_a \Delta_i}{\Delta_i} (\Delta_i \hbar \partial_{\hbar} \mathcal{D}_{\text{pt}}) + \sum_{k=0}^{\infty} q_k (\partial_a(u_i)) \partial_{t_k} \mathcal{D}_{\text{pt}},$$

where on the RHS we first differentiate \mathcal{D}_{pt} with respect to \hbar and t_k and then substitute $\hbar \mapsto \hbar \Delta_i$, $\mathbf{t} \mapsto \mathbf{q}(u_i) + z$. Using the dilaton equation to compute the \hbar -derivative, we get

$$(1.81) \quad \partial_a \mathcal{D}_{\text{pt}}(\hbar \Delta_i; \mathbf{q}(u_i) + z) = -\frac{1}{48} \frac{\partial_a \Delta_i}{\Delta_i} \mathcal{D}_{\text{pt}} + \sum_{k=0}^{\infty} \left(q_k (\partial_a(u_i)) - \frac{1}{2} \frac{\partial_a \Delta_i}{\Delta_i} q_k(u_i) \right) \partial_{t_k} \mathcal{D}_{\text{pt}}.$$

On the other hand, let us compute also the action of

$$-\hat{\gamma} = \sum_{k=0}^{\infty} \sum_{l,b=1}^N \gamma_{lb} q_{k,b} \frac{\partial}{\partial q_{k,l}}$$

on the i th factor of (1.71). Using the chain rule we get

$$-\hat{\gamma} \mathcal{D}_{\text{pt}}(\hbar \Delta_i; \mathbf{q}(u_i) + z) = \sum_{k=0}^{\infty} \sum_{l,b=1}^N \gamma_{lb} q_{k,b} \partial_l(u_i) \partial_{t_k} \mathcal{D}_{\text{pt}}.$$

Recalling the definition of $\gamma = (\partial_a \Psi) \Psi^{-1}$ and Ψ , we get

$$\gamma_{lb} = \sum_{j=1}^N \left(\frac{1}{2} \frac{\partial_a \Delta_j}{\Delta_j} \frac{\partial t_l}{\partial u_j} \frac{\partial u_j}{\partial t_b} - \frac{\partial t_l}{\partial u_j} \frac{\partial^2 u_j}{\partial t_a \partial t_b} \right),$$

where we used that $\sum_{j=1}^N \partial_a(\partial_{u_j}(t_l)) \partial_b(u_j) = -\sum_{j=1}^N \partial_{u_j}(t_l) \partial_a(\partial_b(u_j))$. Therefore,

$$\sum_{l,b=1}^N \gamma_{lb} q_{k,b} \partial_l(u_i) = \frac{1}{2} \frac{\partial_a \Delta_i}{\Delta_i} q_k(u_i) - q_k(\partial_a(u_i)).$$

Comparing with formula (1.81) we get

$$\partial_a \mathcal{D}_{\text{pt}}(\hbar \Delta_i; \mathbf{q}(u_i) + z) = \left(-\frac{1}{48} \frac{\partial_a \Delta_i}{\Delta_i} + \hat{\gamma} \right) \mathcal{D}_{\text{pt}}.$$

Combining the above formula with (1.79) and (1.80) we get that in order to prove formula (1.76) we need only to prove that

$$\hat{\alpha} \prod_{i=1}^N \mathcal{D}_{\text{pt}}(\hbar \Delta_i; \mathbf{q}(u_i) + z) = 0.$$

The above constraint is a consequence of the string equation for $\mathcal{D}_{\text{pt}}(\hbar; \mathbf{t})$, that is, the Virasoro constraint (see [64])

$$\left(\frac{t_0^2}{2\hbar} + \sum_{k=0}^{\infty} (t_{k+1} - \delta_{k,0}) \frac{\partial}{\partial t_k} \right) \mathcal{D}_{\text{pt}}(\hbar; \mathbf{t}) = 0.$$

We have

$$\hat{\alpha} = -\frac{1}{2\hbar} (A_a q_0, q_0) - \sum_{k=0}^{\infty} \sum_{b,l=1}^N (A_a \phi_b, \phi^l) q_{k+1,b} \frac{\partial}{\partial q_{k,l}}.$$

Using the chain rule we get

$$\sum_{b,l=1}^N (A_a \phi_b, \phi^l) q_{k+1,b} \frac{\partial}{\partial q_{k,l}} \mathcal{D}_{\text{pt}}(\hbar \Delta_i; \mathbf{q}(u_i) + z) = (\partial_a \bullet q_{k+1}, du_i) \partial_{t_k} \mathcal{D}_{\text{pt}} = \partial_a(u_i) q_{k+1}(u_i) \partial_{t_k} \mathcal{D}_{\text{pt}}.$$

Summing the above formula over all $k \geq 0$ and recalling the string equation we get

$$\sum_{k=0}^{\infty} \sum_{b,l=1}^N (A_a \phi_b, \phi^l) q_{k+1,b} \frac{\partial}{\partial q_{k,l}} \mathcal{D}_{\text{pt}}(\hbar \Delta_i; \mathbf{q}(u_i) + z) = -\frac{\partial_a(u_i)}{2\hbar \Delta_i} (q_0(u_i))^2 \mathcal{D}_{\text{pt}}(\hbar \Delta_i; \mathbf{q}(u_i) + z).$$

It remain only to prove that

$$(A_a q_0, q_0) = (\partial_a \bullet q_0, q_0) = \sum_{i=1}^N \frac{\partial_a(u_i) q_0(u_i)^2}{\Delta_i}.$$

This however is obvious because $v = \sum_{i=1}^N v(u_i) \partial_{u_i}$ for any vector field v and u_i are canonical coordinates, that is, $\partial_{u_i} \bullet \partial_{u_j} = \delta_{i,j} \partial_{u_j}$ and $(\partial_{u_i}, \partial_{u_j}) = \delta_{i,j} / \Delta_j$.

1.6.4. The total descendant potential. Suppose that M is a semi-simple Frobenius manifold. The total descendant potential depends on the choice of a calibration. As we had already discussed calibrations are sections of a certain principal G -bundle on M , where G is the unipotent group (1.15). Let us define the total descendant potential locally on a coordinate chart of M . Suppose that $t^\circ \in M$ is an arbitrary point and that $S^\circ(z)$ is a calibration at the point t° . Let \mathcal{U} be an open chart of M and H be the space of flat vector fields on \mathcal{U} . Let us fix a basis $\{\phi_a\}_{a=1}^N$ of H such that $\phi_1 = e$ is the unit vector field. For a fixed calibration at a point and a basis of flat vector fields, there is a canonical way to construct a flat coordinate system $\tau = (\tau_1, \dots, \tau_N)$ on \mathcal{U} . Namely, let us extend the calibration at t° to a calibration $S(t, z) = 1 + S_1(t)z^{-1} + \dots$ for all $t \in \mathcal{U}$ by using parallel transport with respect to Dubrovin's connection

$$\begin{aligned} \partial_a S(t, z) &= \phi_a \bullet_t S(t, z), \quad 1 \leq a \leq N, \\ S(t^\circ, z) &= S^\circ(z), \end{aligned}$$

where ∂_a is the derivation corresponding to the flat vector field ϕ_a . If we decompose $S_1(t)e =: \sum_{a=1}^N \tau_a(t) \phi_a$, then the coefficients $\tau_a(t)$ ($1 \leq a \leq N$) form a flat coordinate system. Indeed, the derivative $\partial_a S_1(t)e = \phi_a$, so $\partial_a \tau_b(t) = \delta_{a,b} \Rightarrow \partial_a = \frac{\partial}{\partial \tau_a}$.

Suppose now that $t \in \mathcal{U}$ is a semi-simple point. Note that the calibration $S(t, z)$ is a symplectic transformation of $\mathcal{H} = H((z^{-1}))$, because by definition it satisfies the

symplectic condition $S(t, z)S(t, -z)^T = 1$. Let us denote by \widehat{S}_t the quantization of $S(t, z)$. The total descendant potentia is defined by the following formula:

$$(1.82) \quad \mathcal{D}(\hbar; \mathbf{q}) := e^{F^{(1)}(t)} \widehat{S}_t^{-1} \mathcal{A}_t(\hbar; \mathbf{q}),$$

where $\mathcal{A}_t(\hbar; \mathbf{q})$ is the total ancestor potential. Since the total ancestor potential is an element of the Fock space $\widehat{\mathcal{O}}_{\mathcal{H}_+, -z}$ and $[S^{-1}(-z)]_+ = S_1 e = \tau(t)$, recalling the definition for the quantization of a lower-triangular symplectic transformation, we get that the total descendant potential is an element of the Fock space

$$\widehat{\mathcal{O}}_{\mathcal{H}_+, \tau-z} = \mathbb{C}_{\hbar}[[q_0 - \tau, q_1 + \mathbf{1}, q_2, \dots]] \otimes \mathbb{C}[\mathbb{C}_{\hbar}].$$

Let us determine the component of the total descendant potential that belongs to the group algebra $\mathbb{C}[\mathbb{C}_{\hbar}]$.

LEMMA 1.53. *Let $S(t, z)$ be a calibration of the Frobenius manifold and W be the corresponding quadratic form on \mathcal{H}_+ – see (1.38). Then*

$$W(\tau(t) - ze, \tau(t) - ze) = ((S_2(t)S_1(t) - S_3(t))e, e),$$

where $\tau(t) := S_1(t)e$.

PROOF. Recalling Proposition 1.34 we get

$$\begin{aligned} W(\tau(t) - ze, \tau(t) - ze) &= \Omega([S(t, z)(\tau(t) - ze)]_+, S(t, z)(\tau(t) - ze)) = \\ &= \Omega(-ze, S(t, z)(\tau(t) - ze)), \end{aligned}$$

where for the 2nd equality we used that $[S(t, z)(\tau(t) - ze)]_+ = \tau(t) - ze - S_1(t)e = -ze$. Recalling the definition of the symplectic pairing we get

$$\Omega(-ze, S(t, z)(\tau(t) - ze)) = \text{Res}_{z=0}(ze, S(t, z)(\tau(t) - ze)) = (e, S_2(t)\tau(t) - S_3(t)e)$$

which coincides with the RHS of the formula that we had to prove. \square

In the case when the Frobenius structure comes from the quantum cohomology of a smooth projective variety X , the function

$$F^{(0)}(t) := \frac{1}{2}((S_2(t)S_1(t) - S_3(t))e, e)$$

coincides with the generating function of genus-0 Gromov–Witten invariants with no descendants (i.e. ψ -classes) involved. The generating function is also known as the *genus-0 primary potential* of X . Therefore, we will refer to $F^{(0)}(t)$ as the *primary genus-0 potential* of the Frobenius manifold. Using the differential equations for the calibration (see formula (1.20))

$$\partial_a S_k(t) = \partial_a \bullet S_{k-1}, \quad \forall k \geq 1, \quad 1 \leq a \leq N$$

and (see formula (1.9))

$$E(S_k) = kS_k + [\theta, S_k] - \sum_{l=1}^k S_{k-l}\nu_l$$

we get that $F^{(0)}(t)$ satisfies the following equations (see Proposition 1.2, b):

$$\frac{\partial^3 F^{(0)}}{\partial t_a \partial t_b \partial t_c} = (\partial_a \bullet \partial_b, \partial_c)$$

and

$$E(F^{(0)}) = (3 - D)F^{(0)} - \frac{1}{2}(\nu_1\tau, \tau) - (\nu_2\tau, e) + \frac{1}{2}(\nu_3e, e),$$

where D is the conformal dimension of the Frobenius manifold. Note that the total ancestor potential belongs to $\mathbb{C}_\hbar[[q_0, q_1 + \mathbf{1}, q_2, \dots]]$. Recalling the formula for the action of \widehat{S}_t (see (1.39)) we get that

$$\mathcal{D}(\hbar; \mathbf{q}) \in \mathbb{C}_\hbar[[q_0 - \tau(t), q_1 + \mathbf{1}, q_2, \dots]] \otimes e^{F^{(0)}(t)/\hbar + F^{(1)}(t)}.$$

Our notation for the total descendant potential seems to be a bit misleading because we did not include the semi-simple point t in it. This can be justified to some extent by the following lemma.

LEMMA 1.54. *The derivatives $\partial_a \mathcal{D}(\hbar; \mathbf{q}) = 0$ for all $1 \leq a \leq N$.*

PROOF. By definition $\partial_a S(t, z) = z^{-1} \phi_a \bullet S(t, z)$. Note that $z^{-1} \phi_a \bullet$ is an infinitesimal symplectic transformation and that the corresponding Hamiltonian does not involve p^2 -terms. The Hamiltonian for the infinitesimal symplectic transformation $\log S(t, z)$ also does not involve p^2 -terms. Therefore, using an argument similar to the proof of (1.76) we get that

$$\partial_a (S(t, z))^\wedge = (z^{-1} \phi_a \bullet)^\wedge (S(t, z))^\wedge.$$

Therefore,

$$\partial_a (S(t, z)^{-1})^\wedge = -(S(t, z))^\wedge (z^{-1} \phi_a \bullet)^\wedge.$$

The lemma follows easily from (1.76). \square

Using Lemma 1.54 we can give a slightly different interpretation of the total descendant potential. Namely, it makes sense to substitute $\tau = q_0$ in the RHS of (1.82) and think of $q_0 = (q_{0,1}, \dots, q_{0,N})$ as the coordinates of a point in \mathcal{U} . Then we get that the total descendant potential is identified with an element in the space

$$(1.83) \quad \mathcal{O}_\hbar(\mathcal{U} \setminus \mathcal{K})[[q_1 + \mathbf{1}, q_2, q_3, \dots]] \otimes \mathbb{C}[\mathcal{O}_\hbar(\mathcal{U} \setminus \mathcal{K})],$$

where \mathcal{K} is the caustic of the Frobenius manifold and for a sheaf \mathcal{F} on M , we define the sheaf $\mathcal{F}_\hbar(V) := \mathcal{F}(V)((\hbar))$, where $V \subset M$ is an open subset. In other words, the coefficient in front of a given monomial in $\hbar, q_1 + \mathbf{1}, q_2, \dots$ in $\mathcal{D}(\hbar; \mathbf{q})$ is a holomorphic function in $q_0 \in \mathcal{U} \setminus \mathcal{K}$. Therefore, in order to obtain an element in the Fock space $\widehat{\mathcal{O}}_{\mathcal{H}_+, \tau - z}$ we simply have to expand each coefficient into a Taylor series about the point $q_0 = \tau$.

1.6.5. Genus expansion. We would like to prove that the total descendant and ancestor potentials have the following form:

$$(1.84) \quad \mathcal{D}(\hbar; \mathbf{q}) = \exp \left(\sum_{g=0}^{\infty} \hbar^{g-1} \mathcal{F}^{(g)}(\mathbf{q}) \right)$$

and

$$(1.85) \quad \mathcal{A}_t(\hbar; \mathbf{q}) = \left(\sum_{g=0}^{\infty} \hbar^{g-1} \overline{\mathcal{F}}_t^{(g)}(\mathbf{q}) \right),$$

where $\mathcal{F}^{(g)}$ and $\overline{\mathcal{F}}_t^{(g)}$ are called respectively the *genus- g descendant potential* and the *genus- g ancestor potential*. To begin with, note that the expansion (1.84) can be obtained from the expansion (1.85) by using Lemma 1.35. More precisely, we get

$$(1.86) \quad \mathcal{F}^{(0)}(\mathbf{q}) = \frac{1}{2} W_t(\mathbf{q}, \mathbf{q}) + \overline{\mathcal{F}}_t^{(0)}([S(t, z)\mathbf{q}(z)]_+),$$

where W_t is the quadratic form corresponding to the calibration $S(t, z)$ (see also (1.21)),

$$\mathcal{F}^{(1)}(\mathbf{q}) = F^{(1)}(t) + \overline{\mathcal{F}}_t^{(1)}([S(t, z)\mathbf{q}(z)]_+),$$

where $F^{(1)}$ is the primary genus-1 potential (1.77) of the Frobenius structure, and

$$\mathcal{F}^{(g)}(\mathbf{q}) = \overline{\mathcal{F}}_t^{(g)}([S(t, z)\mathbf{q}(z)]_+), \quad \text{for } g > 1.$$

Therefore, we need only to prove that the total ancestor potential has an expansion of the form (1.85). However, let us make one more remark before we go into the details of the proof. Namely, let us point out that our earlier definition (1.23) of genus-0 descendant potential agrees with $\mathcal{F}^{(0)}(\mathbf{q})$. Indeed, due to tameness, the genus-0 ancestor potential $\overline{\mathcal{F}}_t(\mathbf{q})$ vanishes under the substitution $q_0 = 0$. Using the flat coordinates $\tau(t) = S_1(t)e$ and letting $a = (a_1, \dots, a_N) := \tau(t^\circ)$, we get that the local ring of holomorphic functions $\mathcal{O}_{M, t^\circ} = \mathbb{C}\{\tau_1 - a_1, \dots, \tau_N - a_N\}$. According to Lemma 1.21 there exists a formal power series $f \in H[[q_0 - a, q_1 + \mathbf{1}, q_2, \dots]]$, such that, under the substitution $\tau = a + f(q_0 - a, q_1 + \mathbf{1}, q_2, \dots)$ we have $q_0 + \sum_{k=1}^{\infty} S_k(\tau)q_k = 0$. Therefore, under the substitution $\tau = a + f(q_0 - a, q_1 + \mathbf{1}, q_2, \dots)$ formula (1.86) takes the form

$$\mathcal{F}^{(0)}(\mathbf{q}) = \frac{1}{2} W_\tau(\mathbf{q}, \mathbf{q}),$$

which is exactly the formula that we used before to define the genus-0 descendant potential.

REMARK 1.55. Note that the flat coordinates used here are such that the coordinates of the reference point t° of our coordinate neighborhood are $a = (a_1, \dots, a_N)$, while in Section 1.3.2 the flat coordinates were chosen to be 0 at t° .

REMARK 1.56. Since the genus-0 descendant potential $\mathcal{F}^{(0)}(\mathbf{q})$ can be reconstructed from the calibration $S(t, z)$ and the latter depends holomorphically on t , we get that $\mathcal{F}^{(0)}(\mathbf{q})$ extends holomorphically through the caustic, that is, it is an element in $\mathcal{O}(\mathcal{U})[[q_1 + \mathbf{1}, q_2, \dots]]$. The higher-genus descendant potentials $\mathcal{F}^{(g)}(\mathbf{q})$ ($g > 0$) however, do not extend across the caustic in general. Their definition depends on the asymptotic operator $R_t(z)$, which has singularities with respect to t along the caustic. Nevertheless, for many important classes of semi-simple Frobenius manifolds, that have applications to geometry, $\mathcal{F}^{(g)}(\mathbf{q})$ does extend holomorphically across the caustic.

Let us proceed with the proof of the expansion (1.85). According to Lemma 1.41, in order to prove that the total ancestor potential has an expansion of the type (1.85), it is sufficient to solve the following problem. Suppose that $V = \{V_{kl}\}, k, l \geq 0$ is an infinite symmetric matrix and that $\mathcal{F}(\mathbf{q}) = \sum_{g=0}^{\infty} \hbar^{g-1} \mathcal{F}^{(g)}(\mathbf{q})$ is an arbitrary tame formal series in $\mathbf{q} = (q_k)_{k=0}^{\infty}$, where q_k are formal scalar variables. Then

$$e^{\frac{\hbar}{2} \sum_{k, l=0}^{\infty} V_{kl} \frac{\partial^2}{\partial q_k \partial q_l}} e^{\mathcal{F}(\mathbf{q})} = \exp \left(\sum_{g=0}^{\infty} \hbar^{g-1} \mathcal{F}_V^{(g)}(\mathbf{q}) \right),$$

where $\mathcal{F}_V^{(g)}(\mathbf{q})$ are some tame formal power series in \mathbf{q} . Such identities are well known to physicists. Namely, the series $\mathcal{F}_V(\mathbf{q}) := \sum_{g=0}^{\infty} \hbar^{g-1} \mathcal{F}_V^{(g)}(\mathbf{q})$ can be written explicitly in terms of V and the partial derivatives of \mathcal{F} as an infinite sum over graphs that resembles the summation over Feynman diagrams in Quantum Field Theory. Let us give a precise statement. To begin with, let us clarify the notion of a graph that we need. Secretely, we think of a graph as a 1-dimensional finite CW-complex. The 0-cells are called *vertices* and the 1-cells are called *edges*. In order to obtain a more formal/combinatorial definition, let us cut each edge in the middle and denote by F the set of all half-edges. Note that there is an involution $\iota : F \rightarrow F$ which to each half-edge $f \in F$ associates the remaining half of the edge to which f belongs. Note that ι has no fixed points. Also, if V is the set of all vertices, then there is a natural map of sets $\pi : F \rightarrow V$ that to each half-edge $f \in F$ associates the vertex that belongs to f .

DEFINITION 1.57. A graph is a quadruple (V, F, π, ι) consisting of finite sets V and F , called respectively *vertices* and *flags* (or half-edges), a set-theoretic map $\pi : F \rightarrow V$, and a set-theoretic involution $\iota : F \rightarrow F$, which has no fixed points, that is, $\iota^2(f) := f$ and $\iota(f) \neq f$ for all $f \in F$.

Note that our definition of a graph allows multiple edges between two vertices and loops. Given a graph (V, F, π, ι) , an edge is defined as an ι -orbit $\{f, \iota(f)\}$ for some $f \in F$. Furthermore, we will be interested in decorated graphs $\Gamma = (V, F, \pi, \iota, \kappa)$, where in addition we have a function $\kappa : F \rightarrow \mathbb{Z}_{\geq 0}$, that is, each flag is assigned a non-negative integer. An automorphism of Γ is a pair $\sigma = (\sigma_V, \sigma_F)$ of bijections $\sigma_V : V \rightarrow V$ and $\sigma_F : F \rightarrow F$, such that,

- (i) Compatibility with $\pi : \sigma_V \circ \pi = \pi \circ \sigma_F$.
- (ii) Compatibility with $\iota : \sigma_F \circ \iota = \iota \circ \sigma_F$.
- (iii) Compatibility with $\kappa : \kappa \circ \sigma_F = \kappa$.

The group of all automorphisms is denoted by $\text{Aut}(\Gamma)$. It is a finite group and the number of its elements will be denoted by $|\text{Aut}(\Gamma)|$.

Suppose now that $\Gamma = (V, F, \pi, \iota, \kappa)$ is a decorated graph and let $E(\Gamma)$ denotes the set of all edges of Γ . For each edge $e = \{f, \iota(f)\}$ ($f \in F$) of Γ put $V_e = \hbar V_{ij}$, where $i = \kappa(f)$ and $j = \kappa(\iota(f))$. Note that there is no ambiguity in the definition of V_e , because the matrix (V_{ij}) is required to be symmetric. Also, for each vertex $v \in V(\Gamma)$ we define the differential operator $\partial_v = \partial_{q_{i_1}} \dots \partial_{q_{i_r}}$ where i_1, \dots, i_r are the labels of the flags incident with v . In the degenerate case $r = 0$, that is, if there are no flags incident with v , we set ∂_v to be the identity operator.

LEMMA 1.58. *a) The following formula holds:*

$$(1.87) \quad e^{\frac{\hbar}{2} \sum_{i,j \geq 0} V_{ij} \frac{\partial^2}{\partial q_i \partial q_j}} e^{\mathcal{F}} = \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} \prod_{e \in E(\Gamma)} V_e \prod_{v \in V(\Gamma)} \partial_v \mathcal{F},$$

where the sum is taken over all, possibly disconnected, isomorphism classes of graphs.

b) The logarithm of the RHS is given by the same formula except that the summation is over all connected graphs.

PROOF. Part b) is a standard easy to prove fact, so we leave it as an exercise. Let us prove a). Let us denote the infinite graph sum on the RHS of (1.87) by $\mathcal{A}(\hbar; \mathbf{q})$. It is

sufficient to prove that $\mathcal{A}(\hbar; \mathbf{q})$ satisfies the following differential equation

$$(1.88) \quad \frac{\partial}{\partial \hbar} \mathcal{A}(\hbar; \mathbf{q}) = \frac{1}{2} \left(\sum_{i,j=0}^{\infty} V_{ij} \frac{\partial^2}{\partial q_i \partial q_j} \right) \mathcal{A}(\hbar; \mathbf{q}).$$

Indeed, the LHS clearly satisfies the above differential equation and for $\hbar = 0$ formula (1.87) holds. Both sides of formula (1.87) are formal power series in \hbar with coefficients in $\mathbb{C}[[q_0, q_1, \dots]]$. The differential equation (1.88) is equivalent to a recursion about the coefficients in front of the powers of \hbar , which uniquely determines a solution from its initial value at $\hbar = 0$, i.e., the coefficient in front of \hbar^0 . Therefore, the LHS and the RHS must coincide.

It remains to prove (1.88). If $\Gamma = (V, F, \pi, \iota, \kappa)$ is a decorated graph and $i, j \in \mathbb{Z}_{\geq 0}$, then we denote by $E_{ij}(\Gamma)$ the set of all edges $e = \{f, \iota(f)\}$ of Γ , such that, $\{\kappa(f), \kappa(\iota(f))\} = \{i, j\}$. The derivative with respect to \hbar is given by

$$(1.89) \quad \sum_{0 \leq i \leq j} \sum_{\Gamma} \sum_{e \in E_{ij}(\Gamma)} \frac{V_{ij}}{|\text{Aut}(\Gamma)|} \prod_{e' \in E(\Gamma) \setminus \{e\}} V_{e'} \prod_{v \in V(\Gamma)} \partial_v \mathcal{F},$$

where the first sum is over all sequences (i, j) , $i, j \in \mathbb{Z}$, satisfying $0 \leq i \leq j$. Note that $E(\Gamma) \setminus \{e\}$ coincides with the set of edges of the graph $\Gamma' := \Gamma \setminus e$ obtained from Γ by removing the edge e . Let us fix a pair of graphs Γ' and Γ and denote by $n_{ij}(\Gamma, \Gamma')$ the number of edges $e \in E_{ij}(\Gamma)$, such that, $\Gamma \setminus e \cong \Gamma'$. In other words if we list all graphs $\Gamma \setminus e$ ($e \in E_{ij}(\Gamma)$), then $n_{ij}(\Gamma, \Gamma')$ is the number of times Γ' will appear in our list. The sum (1.89) takes the form

$$(1.90) \quad \sum_{0 \leq i \leq j} \sum_{\Gamma', \Gamma} \frac{n_{ij}(\Gamma, \Gamma')}{|\text{Aut}(\Gamma)|} V_{ij} \prod_{e' \in E(\Gamma')} V_{e'} \prod_{v \in V(\Gamma)} \partial_v \mathcal{F}.$$

In order to compute the derivative with respect to q_i and q_j , note that

$$\frac{\partial^2}{\partial q_i \partial q_j} \prod_{v \in V(\Gamma)} \partial_v \mathcal{F} = \sum_{v, w \in V(\Gamma)} \prod_{v' \in V(\Gamma \cup \{(v, i), (w, j)\})} \partial_{v'} \mathcal{F},$$

where $\Gamma \cup \{(v, i), (w, j)\}$ denotes the graph obtained from Γ by adding an edge between the vertices v and w , that is, adding two flags f_1 and f_2 to Γ and extending the structure maps π, ι, κ of Γ by

$$\pi(f_1) := v, \quad \pi(f_2) = w, \quad \iota(f_1) = f_2, \quad \iota(f_2) = f_1, \quad \kappa(f_1) = i, \quad \kappa(f_2) = j.$$

The RHS of (1.88) takes the form

$$(1.91) \quad \frac{1}{2} \sum_{i,j=0}^{\infty} \sum_{\Gamma'} \sum_{v', w' \in V(\Gamma')} \frac{1}{|\text{Aut}(\Gamma')|} V_{ij} \prod_{e' \in E(\Gamma')} V_{e'} \prod_{v \in V(\Gamma' \cup \{(v, i), (w, j)\})} \partial_v \mathcal{F}.$$

For a given pair (Γ, Γ') of decorated graphs, let us denote by $m_{ij}(\Gamma, \Gamma')$ the number of elements in the set

$$(1.92) \quad M_{ij}(\Gamma, \Gamma') := \{(v', w') \in V(\Gamma') \times V(\Gamma') \mid \Gamma' \cup \{(v, i), (w, j)\} \cong \Gamma\}.$$

Note also that the summand in (1.91) is symmetric with respect to i and j . Formula (1.91) takes the form

$$(1.93) \quad \sum_{0 \leq i \leq j} \sum_{\Gamma', \Gamma} \frac{m_{ij}(\Gamma, \Gamma')}{|\text{Aut}(\Gamma')|} \frac{V_{ij}}{|\text{Aut}(i, j)|} \prod_{e' \in E(\Gamma')} V_{e'} \prod_{v \in V(\Gamma)} \partial_v \mathcal{F},$$

where $|\text{Aut}(i, j)|$ is the number of permutations leaving the sequence (i, j) invariant, that is,

$$|\text{Aut}(i, j)| = \begin{cases} 1, & \text{if } i \neq j, \\ 2, & \text{otherwise.} \end{cases}$$

Comparing formulas (1.93) and (1.90) we get that in order to prove that they agree we need to prove that

$$(1.94) \quad m_{ij}(\Gamma, \Gamma') |\text{Aut}(\Gamma)| = n_{ij}(\Gamma, \Gamma') |\text{Aut}(\Gamma')| |\text{Aut}(i, j)|,$$

for all $i, j \in \mathbb{Z}_{\geq 0}$ and for all pairs of decorated graphs Γ' and Γ . Let us define the set

$$N_{ij}(\Gamma, \Gamma') := \{(e, \phi) \mid e \in E_{ij}(\Gamma), \phi : \Gamma \setminus e \xrightarrow{\cong} \Gamma'\},$$

where ϕ is an isomorphism between graphs, and let

$$\Pi : N_{ij}(\Gamma, \Gamma') \rightarrow M_{ij}(\Gamma, \Gamma')$$

be the map defined as follows: if $e = \{f_1, f_2\}$ with the flags f_1 and f_2 such that $\kappa(f_1) = i$ and $\kappa(f_2) = j$, then

$$\Pi(e, \phi) := (\phi \circ \pi(f_1), \phi \circ \pi(f_2)).$$

Note that the number of elements in $N_{ij}(\Gamma, \Gamma')$ is $n_{ij}(\Gamma, \Gamma') |\text{Aut}(\Gamma')|$, because if we choose $e \in E_{ij}(\Gamma)$, such that, $\Gamma \setminus e \cong \Gamma'$ then the ambiguity in choosing a graph isomorphism $\phi : \Gamma \setminus e \rightarrow \Gamma'$ is given by an automorphism in $\text{Aut}(\Gamma')$.

Let us consider first the case when $i \neq j$. Let $(v', w') \in M_{ij}(\Gamma, \Gamma')$ be an arbitrary element. We claim that the group $\text{Aut}(\Gamma)$ acts faithfully and transitively on the fiber $\Pi^{-1}(v', w')$. This fact would imply that the number of elements in $|\Pi^{-1}(v', w')|$ is $|\text{Aut}(\Gamma)| \Rightarrow$ the number of elements in $N_{ij}(\Gamma, \Gamma')$ is $m_{ij}(\Gamma, \Gamma') |\text{Aut}(\Gamma)|$. On the other hand, since we already know that the number of elements in $N_{ij}(\Gamma, \Gamma')$ is $n_{ij}(\Gamma, \Gamma') |\text{Aut}(\Gamma')|$, formula (1.94) follows. Let us prove our claim. If $\psi \in \text{Aut}(\Gamma)$ and $(e, \phi) \in N_{ij}(\Gamma, \Gamma')$, then we define $\psi(e, \phi) := (\psi(e), \phi \circ \psi^{-1})$. Note that $\phi \circ \psi^{-1}$ induces an isomorphism $\Gamma \setminus \psi(e) \rightarrow \Gamma'$, that is, we have an action of $\text{Aut}(\Gamma)$ on $N_{ij}(\Gamma, \Gamma')$. Clearly, $\Pi(\psi(e, \phi)) = \Pi(e, \phi) \Rightarrow$ the action preserves the fibers of Π . If (e_1, ϕ_1) and (e_2, ϕ_2) are two elements in the fiber $\Pi^{-1}(v', w')$, then $\phi_2^{-1} \circ \phi_1 : \Gamma \setminus e_1 \rightarrow \Gamma \setminus e_2$ is an isomorphism of graphs. Moreover, e_i is an edge between the vertices $\phi_i^{-1}(v')$ and $\phi_i^{-1}(w')$ and the κ -labels of e_1 and e_2 agree. Therefore, we can extend $\phi_2^{-1} \circ \phi_1$ to an automorphism $\psi \in \text{Aut}(\Gamma)$ such that $\psi(e_1) = e_2 \Rightarrow \psi(e_1, \phi_1) = (e_2, \phi_2)$. This proves that the action is transitive. Suppose now that $\psi(e, \phi) = (e, \phi)$, that is, $\psi(e) = e$ and $\phi \circ \psi^{-1} = \phi$ on $\Gamma \setminus e$. Let $e = \{f_1, f_2\}$ where f_1 is the flag with label i and f_2 – the flag with label j . Since $\psi(e) = e$ and ψ should be compatible with the κ -labels, we must have $\psi(f_1) = f_1$ and $\psi(f_2) = f_2$. Note that this is the place where we used that $i \neq j$ and that in the case $i = j$ we have also the option $\psi(f_1) = f_2$ and $\psi(f_2) = f_1$. However, on $\Gamma \setminus e$ we have $\phi \circ \psi^{-1} = \phi$ and since ϕ is an isomorphism, we must have that the restriction of ψ to $\Gamma \setminus e$ is the identity. Therefore, ψ must be the identity. This proves that the action is faithful.

In the case when $i = j$, the argument is absolutely the same. Namely, we can prove that the action of $\text{Aut}(\Gamma)$ on $\Pi^{-1}(v', w')$ is transitive and that if $\psi \in \text{Aut}(\Gamma)$ is in the stabilizer of a point (e, ϕ) , then $\psi|_{\Gamma \setminus e}$ is the identity. However, the identity $\psi(e) = e$, as we already explained above, has two solutions, that is, $\text{Aut}(\Gamma)$ acts transitively on $\Pi^{-1}(v', w')$ with a stabilizer of order 2. Therefore, in this case the number of elements

in $N_{ii}(\Gamma, \Gamma')$ is $m_{ii}(\Gamma, \Gamma') |\text{Aut}(\Gamma)|/2$. Just like before, this number must coincide with $n_{ii}(\Gamma, \Gamma') |\text{Aut}(\Gamma')|$, which yields formula (1.94) for $i = j$. \square

Cobining Lemma 1.41 and Lemma 1.58 we get that in order to prove that the total ancestor potential has genus expansion of the form (1.85) we have to check the following fact. Let us substitute in the RHS of formula (1.87) $\mathcal{F} = \sum_{g=0}^{\infty} \hbar^{g-1} \mathcal{F}^{(g)}$ and suppose that we sum over connected graphs Γ only, then the power of \hbar must be ≥ -1 . Indeed, the smallest possible power of \hbar in the summand corresponding to a fixed graph Γ is $|E(\Gamma)| - |V(\Gamma)|$. On the other hand, let X_Γ be the CW-complex corresponding to the graph Γ . In other words, X_Γ is a 1-dimensional complex with one 0-cell for each vertex in $V(\Gamma)$ and one 1-cell for each edge in $E(\Gamma)$. The attaching maps of X_Γ are determined by the flags, i.e., the 1-cell corresponding to the edge $e = \{f_1, f_2\}$ is attached to the 0-cells corresponding to $\pi(f_1)$ and $\pi(f_2)$. The Euler characteristic of a CW-complex is an alternating sum of the number of cells in each dimension. In the case of X_Γ , we get

$$\chi(X_\Gamma) = \dim H_0(X_\Gamma; \mathbb{C}) - \dim H_1(X_\Gamma; \mathbb{C}) = |V(\Gamma)| - |E(\Gamma)|.$$

By definition, the graph Γ is connected if X_Γ is a connected topological space. Therefore, $\dim H_0(X_\Gamma; \mathbb{C}) = 1$. Since $\dim H_1(X_\Gamma; \mathbb{C}) \geq 0$, the above identity implies that $|E(\Gamma)| - |V(\Gamma)| \geq -1$. Moreover, equality is achieved if and only if the graph Γ has no loops, i.e., it is a tree.

Analytic continuation

2.1. Levelt's theory for Fuchsian connections

The main goal in this section is to prove the existence of weak Levelt solutions for Fuchsian systems. We follow closely [10].

2.1.1. Fuchsian systems. Let $D = \{\lambda : |\lambda| < R\}$ be the open disk of radius R and $B_0(\lambda) \in \mathfrak{gl}(\mathbb{C}^p)$ be a $p \times p$ -matrix whose entries depend holomorphically on $\lambda \in D$. We will be interested in the system of ODEs defined by

$$\frac{\partial y}{\partial \lambda}(\lambda) = B(\lambda)y(\lambda), \quad B(\lambda) := B_0(\lambda)/\lambda.$$

Systems of this type are said to be *Fuchsian in a neighborhood of 0*.

Let us fix a small sector S in D containing the open interval $(0, R)$, e.g.,

$$S = \{\lambda \in D - \{0\} : -\epsilon < \text{Arg}(\lambda) < \epsilon\}$$

where $0 < \epsilon < 2\pi$ is fixed arbitrary. Furthermore, let us fix a reference point $\lambda_0 \in (0, R) \subset S$ and denote by X the space of holomorphic functions $y : S \rightarrow \mathbb{C}^p$ that solve the above system. The general theory of ODEs implies that X is a finite dimensional vector space of dimension p . More precisely

$$X \cong \mathbb{C}^p, \quad y \mapsto y(\lambda_0).$$

Since the coefficients of the linear system are holomorphic in $D - \{0\}$, every solution $y \in X$ can be extended analytically along any path in $D - \{0\}$. In particular, we have a linear map

$$\sigma : X \rightarrow X$$

corresponding to analytic continuation along a loop based at λ_0 that goes once around $\lambda = 0$ in counter-clockwise direction.

2.1.2. Fuchsian singularities are regular. The following result is well known in the theory of ODEs. Nevertheless, we give our own proof as well. For a different argument, which is shorter but yields a slightly weaker result, see [10], Theorem 4.1 and Lemma 4.1.

PROPOSITION 2.1. *Every solution $y \in X$ has the form*

$$y(\lambda) = \sum_{\rho} \sum_{k=0}^{p-1} y_{\rho,k}(\lambda) \lambda^{\rho} (\log \lambda)^k,$$

where the first sum is over all eigenvalues ρ of $B_0(0)$ and $y_{\rho,k}(\lambda)$ are \mathbb{C}^p -valued functions analytic for all $\lambda \in D$.

PROOF. We will prove that the system has a fundamental matrix whose columns have the above form. Using a constant gauge transformation $y(\lambda) \mapsto Cy(\lambda)$ we can reduce the general case to the case when $B_0(0)$ is in Jordan normal form. Moreover, we may assume that the Jordan blocks are ordered in such a way that $B_0(0) = D + N^{(0)}$, where D and $N^{(0)}$ have the following properties. Both

$$D = \text{diag}(D_1, \dots, D_s) \quad \text{and} \quad N^{(0)} = \text{diag}(N_1^{(0)}, \dots, N_s^{(0)})$$

are block-diagonal. The block $D_i = \rho_i I_i$, where I_i is an identity matrix of size the multiplicity of ρ_i as an eigenvalue of $B_0(0)$ and

$$\text{Re}(\rho_1) > \dots > \text{Re}(\rho_s).$$

The block $N_i^{(0)}$ ($1 \leq i \leq s$) is an upper-triangular nilpotent matrix whose size is the same as the size of I_i . Note that the commutator $[D, N^{(0)}] = 0$.

LEMMA 2.2. *There exists a formal solution*

$$Y(\lambda) = U(\lambda)\lambda^D\lambda^N,$$

where

$$U(\lambda) = 1 + U_1\lambda + U_2\lambda^2 + \dots, \quad U_k \in \mathfrak{gl}(\mathbb{C}^p)$$

and N is upper-triangular nilpotent matrix of the form

$$N = N^{(0)} + N^{(1)} + \dots, \quad [D, N^{(k)}] = kN^{(k)}.$$

PROOF. Put $B_0(\lambda) = B_{0,0} + B_{0,1}\lambda + \dots$, substitute $Y(\lambda)$ in the differential equation, and compare the coefficients in front of the powers of λ^k . For $k = 0$ we get $D + N^{(0)} = B_0(0) = B_{0,0}$, which is true by definition. For $k > 0$ we get

$$kU_k + [U_k, D + N^{(0)}] + N^{(k)} = B_{0,k} + \sum_{i=1}^{k-1} (B_{0,k-i}U_i - U_iN^{(k-i)}).$$

The linear operator

$$\text{ad}_D : \mathfrak{gl}(\mathbb{C}^p) \rightarrow \mathfrak{gl}(\mathbb{C}^p), \quad x \mapsto [D, x]$$

is diagonalizable, i.e., we have a decomposition

$$\mathfrak{gl}(\mathbb{C}^p) = \bigoplus_{a \in \text{spec}(D)} \mathfrak{gl}_a(\mathbb{C}^p),$$

where $\text{spec}(D)$ denotes the set of eigenvalues of ad_D and for $a \in \text{spec}(D)$

$$\mathfrak{gl}_a(\mathbb{C}^p) = \{x : [D, x] = ax\}$$

is the corresponding eigen-subspace. Let us denote by

$$\pi_a : \mathfrak{gl}(\mathbb{C}^p) \rightarrow \mathfrak{gl}_a(\mathbb{C}^p)$$

the projection map defined via the above decomposition. Let us assume that we have determined U_1, \dots, U_{k-1} and $N^{(1)}, \dots, N^{(k-1)}$. Then $U_k = \sum_{a \in \text{spec}(D)} \pi_a(U_k)$ and $N^{(k)} \in \mathfrak{gl}_k(\mathbb{C}^p)$ are defined by projecting via π_a the above recursion relation and solving for $\pi_a(U_k)$ and $\pi_a(N^{(k)})$. There are two cases. First, if $a = k$, then we set $\pi_k(U_k) = 0$. Note that since $N^{(0)}$ commutes with D , we have $\pi_k([U_k, N^{(0)}]) = [\pi_k(U_k), N^{(0)}] = 0$ and

$\pi_k(N^{(k)}) = N^{(k)}$. Therefore, we can uniquely solve for $N^{(k)}$. The second case is if $a \neq k$, then $\pi_a(N^{(k)}) = 0$ and

$$\pi_a(kU_k + [U_k, D + N^{(0)}]) = (k - a - \text{ad}_{N^{(0)}})\pi_a(U_k).$$

Since $N^{(0)}$ is nilpotent, the linear operator $\text{ad}_{N^{(0)}}$ is also nilpotent. Therefore the linear operator $k - a - \text{ad}_{N^{(0)}}$ is invertible, so we can uniquely solve for $\pi_a(U_k)$. \square

It remains to prove that the formal series $U(\lambda)$ is convergent. Note that $U(\lambda)$ satisfies the following differential equation

$$(2.1) \quad (\lambda\partial_\lambda - \text{ad}_{D+N^{(0)}})U = (\alpha(\lambda)U + U\beta(\lambda)),$$

where

$$\alpha(\lambda) = -\sum_{i=1}^{\infty} N^{(i)}\lambda^i \quad \beta(\lambda) = \sum_{i=1}^{\infty} B_{0,i}\lambda^i.$$

Let us fix an integer $k > 0$, such that the set $\text{spec}(D)$ does not contain any integers $\ell > k$. Note that $N^{(\ell)} = 0$ for all $\ell > k$, so $\alpha(\lambda)$ is polynomial in λ . Let us write the formal series in the form

$$U(\lambda) = U_{\leq k}(\lambda) + \lambda^k V(\lambda), \quad U_{\leq k}(\lambda) = 1 + \sum_{i=1}^k U_i \lambda^i,$$

where $V(\lambda) = \sum_{j=1}^{\infty} U_{j+k} \lambda^j$. Then $V(\lambda)$ satisfies the following differential equation

$$(2.2) \quad (\lambda\partial_\lambda + k - \text{ad}_{D+N^{(0)}})V = \alpha(\lambda)V + V\beta(\lambda) + \gamma(\lambda),$$

where

$$\gamma(\lambda) = \lambda^{-k} \left(\alpha(\lambda)U_{\leq k}(\lambda) + U_{\leq k}(\lambda)\beta(\lambda) - (\lambda\partial_\lambda - \text{ad}_{D+N^{(0)}})U_{\leq k}(\lambda) \right)$$

By definition $U_{\leq k}(\lambda)$ satisfies the differential equation (2.1) up to terms of order $O(\lambda^{k+1})$. Therefore, $\gamma(\lambda)$ is analytic at $\lambda = 0$ and $\gamma(0) = 0$. It is enough to prove that the differential equation (2.2) has a solution $V_{hol}(\lambda)$ analytic at $\lambda = 0$. Indeed, the linear operator $k - \text{ad}_{D+N^{(0)}}$ is invertible, so after substituting the Taylor series of $V_{hol}(\lambda)$ in the differential equation we get that the Taylor series must coincide with the formal series $V(\lambda)$.

In order to construct a holomorphic solution, we use the standard idea to identify V_{hol} with the fixed point of a certain integral operator. Let us fix a closed disk $D_\rho = \{\lambda : |\lambda| \leq \rho\}$ with radius $\rho < R$. Let us define a sequence of holomorphic $\mathfrak{gl}(\mathbb{C}^p)$ -valued functions

$$V_n : D_\rho \rightarrow \mathfrak{gl}(\mathbb{C}^p), \quad n = 0, 1, 2, \dots$$

as follows. Put $V_0(\lambda) = 0$ and let $V_{n+1}(\lambda)$ be such that

$$(\lambda\partial_\lambda + k - \text{ad}_{D+N^{(0)}})V_{n+1} = V_n\alpha(\lambda) + \beta(\lambda)V_n + \gamma(\lambda).$$

Note that

$$V_{n+1}(\lambda) = \int_0^1 t^{k - \text{ad}_{D+N^{(0)}}} \left(V_n(t\lambda)\alpha(t\lambda) + \beta(t\lambda)V_n(t\lambda) + \gamma(t\lambda) \right) \frac{dt}{t}.$$

The convergence of the integral follows from the fact that if we choose k sufficiently large the real part of the eigenvalues of $k - \text{ad}_{D+N^{(0)}}$ will be positive. Therefore $V_{n+1}(\lambda)$ is an analytic function for all $\lambda \in D_\rho$.

In order to prove that the sequence V_n is convergent we introduce the following norm. Let $|\cdot| : \mathfrak{gl}(\mathbb{C}^p) \rightarrow \mathbb{R}_{\geq 0}$ be the standard matrix norm

$$|A| = \sup_{v \neq 0} \frac{|Av|}{|v|},$$

where $|v| = \sqrt{|v_1|^2 + \cdots + |v_p|^2}$ is the standard Euclidean norm of $v \in \mathbb{C}^p$. If $A : D_\rho \rightarrow \mathfrak{gl}(\mathbb{C}^p)$ is holomorphic, then we define

$$\|A\|_\rho = \sum_{i=0}^{\infty} |A_i| \rho^i,$$

where $A(\lambda) = \sum_{i=0}^{\infty} A_i \lambda^i$ is the Taylor series expansion. Let B_ρ be the space of those holomorphic maps A for which $\|A\|_\rho < \infty$. It is known (see [25]) that B_ρ is a Banach algebra. Using the Cauchy inequality it is easy to prove that if $A(\lambda)$ is holomorphic for all $\lambda \in D$ then $A \in B_\rho$.

LEMMA 2.3. *Suppose $k > |\mathrm{ad}_{D+N(0)}|$. Then the map*

$$F : B_\rho \rightarrow B_\rho, \quad F(A)(\lambda) := \int_0^1 t^{k-\mathrm{ad}_{D+N(0)}} A(t\lambda) \frac{dt}{t}$$

is a bounded linear operator of norm less or equal to 1, i.e., $\|F(A)\|_\rho \leq \|A\|_\rho$.

PROOF. Put $A(\lambda) = \sum_{i=0}^{\infty} A_i \lambda^i$. Then the coefficient in front of λ^i in $F(A)$ is

$$F(A)_i = \int_0^1 t^{k+i-1-\mathrm{ad}_{D+N(0)}} A_i dt.$$

Using that

$$|t^{k+i-1-\mathrm{ad}_{D+N(0)}}| = t^{k+i-1} |t^{-\mathrm{ad}_{D+N(0)}}| \leq t^{k+i-1} t^{-|\mathrm{ad}_{D+N(0)}|}, \quad 0 \leq t \leq 1$$

we get

$$|F(A)_i| \leq \frac{|A_i|}{k+i-|\mathrm{ad}_{D+N(0)}|} \leq |A_i|. \quad \square$$

Note that $V_{n+1} = F(V_n \alpha + \beta V_n + \gamma)$. Therefore

$$\|V_{n+1} - V_n\|_\rho \leq (\|\alpha\|_\rho + \|\beta\|_\rho) \|V_n - V_{n-1}\|_\rho.$$

Since $\alpha(0) = \beta(0) = 0$ we can always choose ρ so small that $\|\alpha\|_\rho + \|\beta\|_\rho < 1$. Then the above inequality shows that $\{V_n\}$ is a Cauchy sequence in B_ρ , so the limit $V_{hol} = \lim_{n \rightarrow \infty} V_n$ exists and it gives a solution to the differential equation (2.2).

Finally, note that the series $U(\lambda)$ must be analytic for all $\lambda \in D$, because the fundamental matrix $Y(\lambda) = U(\lambda) \lambda^D \lambda^N$ extends analytically along any path inside $D - \{0\}$. \square

COROLLARY 2.4. *If $B_0(0)$ is nilpotent, then the matrix of the monodromy of the Fuchsian system with respect to a basis of X given by the columns of the fundamental matrix $Y(\lambda)$ satisfying the initial condition $Y(\lambda_0) = 1$ is $e^{2\pi\sqrt{-1}B_0(0)}$.*

2.1.3. The Levelt evaluation. Let us denote by $\mathcal{O}[S]$ the space of holomorphic maps $y : S \rightarrow \mathbb{C}^p$, such that

$$\lim_{\substack{\lambda \rightarrow 0 \\ \lambda \in S}} \frac{y(\lambda)}{|\lambda|^m} = 0$$

for some integer m . Such functions are also sometimes said to be of *moderate growth* at $\lambda = 0$. The key to Levelt's theory is the map

$$\varphi : \mathcal{O}[S] \rightarrow \mathbb{Z} \cup \{\infty\}$$

defined by

$$\varphi(y) := \max \left\{ m \in \mathbb{Z} \mid \lim_{\substack{\lambda \rightarrow 0 \\ \lambda \in S}} \frac{y(\lambda)}{|\lambda|^\ell} = 0 \text{ for all } \ell < m \right\}$$

for all $y \in \mathcal{O}[S] \setminus \{0\}$ and $\varphi(0) = \infty$. Note that according to Proposition 2.1 the space of solutions $X \subset \mathcal{O}[S]$.

LEMMA 2.5. *The map φ satisfies the following properties.*

- a) *If $y_1, y_2 \in \mathcal{O}[S]$, then $\varphi(y_1 + y_2) \geq \min(\varphi(y_1), \varphi(y_2))$. If $\varphi(y_1) \neq \varphi(y_2)$, then the equality in the above inequality holds.*
- b) *If $c \in \mathbb{C} \setminus \{0\}$, then $\varphi(cy) = \varphi(y)$ for all $y \in \mathcal{O}[S]$.*

The proof is an elementary consequence from the definitions, so it will be omitted.

LEMMA 2.6. *Let θ_i , $1 \leq i \leq n$ be real numbers such that $\theta_i \neq \theta_j$ for all $i \neq j$. Suppose that*

$$f(x) = \sum_{i=1}^n a_i(x) e^{\sqrt{-1}\theta_i x}, \quad a_i \in \mathbb{C}[x]$$

and that there is a real number $\epsilon > 0$, such that

$$\lim_{x \rightarrow +\infty} f(x) e^{\ell x} = 0, \quad \forall \ell < \epsilon.$$

Then $a_i = 0$ for all i .

PROOF. It is enough to prove the lemma in the case when the polynomials are constants. Indeed, let m be the maximal degree among the degrees of a_i , i.e., $a_i(x) = \sum_{\mu=0}^m a_{i,\mu} x^\mu$ and $a_{i,m} \neq 0$ for at least one i . There exists a constant C , s.t.,

$$\left| \sum_{i=1}^n a_{i,m} e^{\sqrt{-1}\theta_i x} \right| x^m e^{\lambda x} \leq C \left| \sum_{i=1}^n a_i(x) e^{\sqrt{-1}\theta_i x} \right| e^{\lambda x}, \quad \forall x \geq 0.$$

Therefore we must have

$$\lim_{x \rightarrow +\infty} \left(\sum_{i=1}^n a_{i,m} e^{\sqrt{-1}\theta_i x} \right) e^{\lambda x} = 0 \quad \forall \lambda < \epsilon.$$

Therefore, if we knew that the lemma holds for constant polynomials, then we would get $a_{i,m} = 0$ for all i – contradiction with the definition of m .

Let us assume that $a_i \in \mathbb{C}$ are constants. Using induction on m it is easy to prove that if $\lambda_m < \dots < \lambda_1 < \lambda_0 := \epsilon$ is any sequence of real numbers, then

$$\lim_{x \rightarrow +\infty} \left(\sum_{i=1}^n \frac{a_i e^{\sqrt{-1}\theta_i x}}{(\sqrt{-1}\theta_i + \lambda_1) \cdots (\sqrt{-1}\theta_i + \lambda_m)} \right) e^{\lambda x} = 0, \quad \forall \lambda < \lambda_m.$$

Indeed, the starting point of the induction is $m = 0$ and the statement is true by definition. Suppose the statement is true for m and that $\lambda_{m+1} < \lambda_m$. Let us pick λ' in the open interval $(\lambda_{m+1}, \lambda_m)$. Using the inductive assumption we get

$$\left| \sum_{i=1}^n \frac{a_i e^{(\sqrt{-1}\theta_i + \lambda_{m+1})y}}{(\sqrt{-1}\theta_i + \lambda_1) \cdots (\sqrt{-1}\theta_i + \lambda_m)} \right| \leq C' e^{(\lambda_{m+1} - \lambda')y}, \quad \forall y \geq 0$$

for some constant C' depending on the choice of λ' . Integrating the function inside the absolute value on the LHS for y from 0 to x and using the above inequality to estimate the absolute value of the integral, we get the following inequality

$$\left| \sum_{i=1}^n \frac{a_i (e^{(\sqrt{-1}\theta_i + \lambda_{m+1})x} - 1)}{(\sqrt{-1}\theta_i + \lambda_1) \cdots (\sqrt{-1}\theta_i + \lambda_m)(\sqrt{-1}\theta_i + \lambda_{m+1})} \right| \leq C' \frac{e^{(\lambda_{m+1} - \lambda')x} - 1}{\lambda_{m+1} - \lambda'}.$$

If $\lambda < \lambda_{m+1}$ is any given number, then we multiply the above inequality by $e^{(\lambda - \lambda_{m+1})x}$, and let $x \rightarrow +\infty$.

To complete the proof of the lemma we proceed as follows. Let us choose a sequence of n numbers $0 < \lambda_n < \cdots < \lambda_1 < \epsilon$ and define the matrix C with entries

$$C_{im} := \frac{1}{(\sqrt{-1}\theta_i + \lambda_1) \cdots (\sqrt{-1}\theta_i + \lambda_m)}, \quad 1 \leq i, m \leq n.$$

Note that for $\lambda_1 = \cdots = \lambda_n$ the determinant of C turns into a Vandermonde determinant, which is not 0 according to the assumption $\theta_i \neq \theta_j$ for $i \neq j$. Therefore choosing λ_1 sufficiently close to λ_n we may guarantee that C is invertible. On the other hand if we define

$$g_m(x) = \sum_{i=1}^n a_i e^{\sqrt{-1}\theta_i x} C_{im}, \quad 1 \leq m \leq n,$$

then according to the above fact $\lim_{x \rightarrow +\infty} g_m(x) = 0$. However, since C is invertible, we can solve the above equations and express each $a_i e^{\sqrt{-1}\theta_i x}$ as a linear combination of $g_m(x)$ with constant coefficients. Therefore $\lim_{x \rightarrow +\infty} a_i e^{\sqrt{-1}\theta_i x} = 0$. This however is possible only if $a_i = 0$. \square

PROPOSITION 2.7. *If $y \in X$, then $\varphi(\sigma y) = \varphi(y)$.*

PROOF. Recalling Proposition 2.1 we write the solution as

$$y(\lambda) = \sum_{i=1}^n y_i(\lambda) \lambda^{\rho_i}, \quad y_i(\lambda) = \sum_{k=0}^{p-1} y_{i,k}(\lambda) (\log \lambda)^k$$

where $y_{i,k}(\lambda)$ are analytic at $\lambda = 0$. We may further assume that $\operatorname{Re}(\rho_1) \leq \cdots \leq \operatorname{Re}(\rho_n)$. Let us write the solution as

$$y(\lambda) = \lambda^{\rho_1} \left(f(\lambda) + \sum_j y_j(\lambda) \lambda^{\rho_j - \rho_1} \right),$$

where the sum is over all j , s.t., that $\operatorname{Re}(\rho_j) > \operatorname{Re}(\rho_1)$ and

$$f(\lambda) = \sum_i y_i(\lambda) \lambda^{\rho_i - \rho_1},$$

where the sum is over all i , s.t., $\operatorname{Re}(\rho_i) = \operatorname{Re}(\rho_1)$. Let us assume that $y_{1,k}(0) \neq 0$ for at least one k . Otherwise, we can replace ρ_1 with an exponent with a larger real part. Note that $\varphi(y) = \lfloor \operatorname{Re}(\rho_1) \rfloor$, where $\lfloor x \rfloor$ is the largest integer that does not exceed x . Indeed,

by definition we have that if $\ell < \lfloor \operatorname{Re}(\rho_1) \rfloor$, then $\lim y(\lambda)/|\lambda|^\ell = 0$, so $\varphi(y) \geq \lfloor \operatorname{Re}(\rho_1) \rfloor$. If the inequality is strict then we can find $\epsilon > 0$, such that

$$\lfloor \operatorname{Re}(\rho_1) \rfloor \leq \operatorname{Re}(\rho_1) < \operatorname{Re}(\rho_1) + 2\epsilon < \varphi(y)$$

and $\epsilon < \operatorname{Re}(\rho_j) - \operatorname{Re}(\rho_1)$ for all j for which $\operatorname{Re}(\rho_j) \neq \operatorname{Re}(\rho_1)$. We have

$$\frac{y(\lambda)}{|\lambda|^{\operatorname{Re}(\rho_1)+\epsilon+\ell}} = \frac{\lambda^{\rho_1}}{|\lambda|^{\operatorname{Re}(\rho_1)+\epsilon}} \left(\frac{f(\lambda)}{|\lambda|^\ell} + \sum_j y_j(\lambda) \frac{\lambda^{\rho_j - \rho_1}}{|\lambda|^\ell} \right).$$

If $\ell < \epsilon$, then the LHS has limit 0 as $\lambda \rightarrow 0$, while the limit of the first factor on the RHS is ∞ and the limit of the sum over j is 0. Therefore, we must have

$$\lim_{\lambda \rightarrow 0} \frac{f(\lambda)}{|\lambda|^\ell} = 0, \quad \forall \ell < \epsilon.$$

If we put $\lambda = e^{-x}$, $x \in \mathbb{R}_{>0}$ and let $x \rightarrow +\infty$ we get that

$$\sum_i \sum_{k=0}^{p-1} y_{i,k}(0) x^k e^{\sqrt{-1}\theta_i x}, \quad \sqrt{-1}\theta_i = \rho_i - \rho_1$$

satisfies the condition of Lemma 2.6, so it must be 0, which contradicts the choice of ρ_1 .

Note also that we have

$$\sigma y(\lambda) = \sum_{i=1}^n \sum_{k=0}^{p-1} y_{i,k}(\lambda) e^{2\pi\sqrt{-1}\rho_i} \lambda^{\rho_i} (\log \lambda + 2\pi\sqrt{-1})^k.$$

Therefore, choosing k to be the largest integer such that $y_{1,k}(0) \neq 0$ we get

$$\varphi(y) = \varphi(y_{1,k}(\lambda) \lambda^{\rho_1} (\log \lambda)^k) = \varphi(y_{1,k}(\lambda) e^{2\pi\sqrt{-1}\rho_1} \lambda^{\rho_1} (\log \lambda + 2\pi\sqrt{-1})^k) \leq \varphi(\sigma y),$$

where in the last equality we used Lemma 2.5, Part a). Similarly $\varphi(y) \leq \varphi(\sigma^{-1}y)$ for all $y \in X$. Finally we get

$$\varphi(y) \leq \varphi(\sigma y) \leq \varphi(\sigma^{-1}(\sigma y)) = \varphi(y). \quad \square$$

2.1.4. Weak Levelt solutions. The eigenvalues of σ can be written uniquely as

$$e^{2\pi\sqrt{-1}\rho_i}, \quad 0 \leq \operatorname{Re}(\rho_i) < 1, \quad 1 \leq i \leq s.$$

Let

$$X = X_1 \oplus \cdots \oplus X_s, \quad X_i := \{y \in X : (\sigma - e^{2\pi\sqrt{-1}\rho_i})^n y = 0 \text{ for all } n \gg 0\}$$

be the decomposition of X into generalized eigensubspaces.

Using Lemma 2.5 we get that $\varphi(X)$ is a finite set. Let us define the set

$$\{\infty, \psi_i^1, \dots, \psi_i^{m_i}\} := \varphi(X_i), \quad 1 \leq i \leq s,$$

where in addition we assume that $\psi_i^1 > \cdots > \psi_i^{m_i}$. Put

$$X_i^\ell = \{y \in X \mid \varphi(y) \geq \psi_i^\ell\}, \quad 1 \leq i \leq s, \quad 1 \leq \ell \leq m_i.$$

According to Lemma 2.5 the sets X_i^ℓ are vector subspaces of X_i , so we have a strictly increasing filtration (in particular we see that φ could take only finitely many values on X_i)

$$X_i^1 \subset X_i^2 \subset \cdots \subset X_i^{m_i} = X_i$$

Using Proposition 2.7 we get that the above filtration is σ -invariant.

A *weak Levelt* solution $Y(\lambda)$ is by definition a fundamental matrix whose columns are splitted into s groups

$$Y(\lambda) = [Y_1(\lambda) \cdots Y_s(\lambda)],$$

where the columns in $Y_i(\lambda)$ represent a basis of X_i with the following property. We can split $Y_i(\lambda)$ into m_i groups

$$Y_i(\lambda) = [Y_{i,1}(\lambda) \cdots Y_{i,m_i}(\lambda)]$$

such that

- (i) The columns in $Y_{i,\ell}(\lambda)$ represent a basis of the quotient subspace $X_i^\ell/X_i^{\ell-1}$.
- (ii) The matrix of the linear operator in $X_i^\ell/X_i^{\ell-1}$ induced by σ in the basis represented by the columns of $Y_{i,\ell}(\lambda)$ is upper-triangular.

Let G be the matrix of σ with respect to the basis of X given by the columns of a weak Levelt solution $Y(\lambda)$. Note that the matrix G is block-diagonal

$$G = \text{diag}(G_1, \dots, G_s)$$

where each block is a square matrix of size $\dim_{\mathbb{C}}(X_i)$. Each block G_i has a natural block-matrix form corresponding to the filtration $X_i^1 \subset \cdots \subset X_i^{m_i}$

$$G_i = \begin{bmatrix} G_i^{11} & G_i^{12} & \cdots & G_i^{1m_i} \\ 0 & G_i^{22} & \cdots & G_i^{2m_i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_i^{m_i m_i} \end{bmatrix},$$

where the size of the block G_i^{ab} is $\dim_{\mathbb{C}}(X_i^a/X_i^{a-1}) \times \dim_{\mathbb{C}}(X_i^b/X_i^{b-1})$. The definition of a weak Levelt solution implies that $G_i^{ab} = 0$ for $a > b$ (\because the filtration is σ -invariant) and that the block $G_i^{\ell\ell}$ has the form of an upper-triangular matrix with all diagonal entries being equal to $e^{2\pi\sqrt{-1}\rho_i}$ (\because the matrix of the linear map in $X_i^\ell/X_i^{\ell-1}$ induced by σ is upper-triangular).

2.1.5. Levelt's theorem. Let $Y(\lambda)$ be a weak Levelt solution. Let us write the monodromy matrix $G = e^{2\pi\sqrt{-1}E}$, where E has the same block-matrix structure as G . Namely,

$$E = \text{diag}(E_1, \dots, E_s)$$

is block-diagonal and each block E_i has the form

$$E_i = \begin{bmatrix} E_i^{11} & E_i^{12} & \cdots & E_i^{1m_i} \\ 0 & E_i^{22} & \cdots & E_i^{2m_i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_i^{m_i m_i} \end{bmatrix},$$

where $E_i^{\ell\ell} = \rho_i I_i^\ell + N_i^{\ell\ell}$ is upper-triangular matrix whose diagonal entries are all equal to ρ_i . We have denoted by I_i^ℓ the identity matrix of size $\dim_{\mathbb{C}}(X_i^\ell/X_i^{\ell-1})$ while by $N_i^{\ell\ell}$ we have denoted the strictly upper-triangular part of $E_i^{\ell\ell}$.

Let us define also the matrix A with the same block-diagonal structure as G and E , i.e.,

$$A = \text{diag}(A_1, \dots, A_s)$$

where the block A_i is given by the diagonal matrix

$$A_i = \text{diag}(\psi_i^1 I_i^1, \dots, \psi_i^{m_i} I_i^{m_i}).$$

THEOREM 2.8 (Levelt). *Suppose that $Y(\lambda)$ is a weak Levelt solution and that A and E are the matrices defined as above. Then*

$$Y(\lambda) = U(\lambda)\lambda^A\lambda^E,$$

where $U(\lambda)$ is holomorphic and invertible at $\lambda = 0$.

PROOF. Our argument follows [10]. Note that the analytic continuation of $Y(\lambda)$ and λ^E around $\lambda = 0$ are respectively $Y(\lambda)G$ and $\lambda^E G$. Therefore, the holomorphic branch of

$$U(\lambda) := Y(\lambda)\lambda^{-E}\lambda^{-A}$$

defined in the sector $S \subset D - \{0\}$ extends analytically to the entire punctured disc $D - \{0\}$. Using Proposition 2.1 we get that $U(\lambda)$ has at most a finite order pole at $\lambda = 0$.

Let us prove that $U(\lambda)$ is holomorphic at $\lambda = 0$. Let us denote by $r = \max_{1 \leq j \leq s} \text{Re}(\rho_j)$. Since $r < 1$ we can find a real number $\epsilon > 0$, such that $r + 2\epsilon < 1$. We claim that $\lim_{\lambda \rightarrow 0} U(\lambda)\lambda^{r+2\epsilon} = 0$. This clearly implies that $U(\lambda)$ does not have a pole at $\lambda = 0$. To prove that the limit is 0 we write

$$U(\lambda)\lambda^{r+2\epsilon} = Y(\lambda)\lambda^{-A+\epsilon} \exp\left((r - \lambda^A E \lambda^{-A}) \log \lambda\right) \lambda^\epsilon.$$

Note that the first two factors on the RHS give a matrix obtained from $Y(\lambda)$ by multiplying each column in $Y_{i,\ell}$ by $\lambda^{-\psi_i^\ell + \epsilon}$. Since the Levelt evaluation of every column in $Y_{i,\ell}$ is at least ψ_i^ℓ we get that the limit of $Y(\lambda)\lambda^{-A+\epsilon}$ is 0. Since A and E have the same block-diagonal structure we get that the 3rd and the 4th factor give a matrix which is also block-diagonal and the i -th block is

$$(2.3) \quad \lambda^{\epsilon+r-\rho_i} e^{\lambda^{A_i} N_i \lambda^{-A_i} \log \lambda},$$

where N_i is strictly upper triangular. Since A_i is diagonal with decreasing diagonal entries the matrix $\lambda^{A_i} N_i \lambda^{-A_i}$ is holomorphic at $\lambda = 0$. Therefore the limit of (2.3) is 0.

It remains only to prove that $U(0)$ is invertible. Substituting $Y(\lambda) = U(\lambda)\lambda^A\lambda^E$ in the differential equation we get

$$\lambda U'(\lambda) + U(\lambda)L(\lambda) = B_0(\lambda)U(\lambda),$$

where $L(\lambda) = A + \lambda^A E \lambda^{-A}$. As we discussed above the matrix $\lambda^A E \lambda^{-A}$ is holomorphic at $\lambda = 0$. Note that $L(0)$ is block-diagonal and that the i th block is

$$\begin{bmatrix} (\psi_i^1 + \rho_i)I_i^1 + N_i^{11} & 0 & \cdots & 0 \\ 0 & (\psi_i^2 + \rho_i)I_i^2 + N_i^{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & (\psi_i^{m_i} + \rho_i)I_i^{m_i} + N_i^{m_i m_i} \end{bmatrix}.$$

Since $U(0)L(0) = B_0(0)U(0)$, we get that $L(0)$ is a linear operator in $\text{Ker}(U(0))$. If we assume that $U(0)$ is not invertible, then $L(0)$ has a non-zero eigenvector $c \in \text{Ker}(U(0))$. Let us denote by $y_c(\lambda) = Y(\lambda)c$.

Let us split the vector-column c in the following way

$$c = \begin{bmatrix} c_1 \\ \vdots \\ c_s \end{bmatrix}, \quad c_i = \begin{bmatrix} c_{i,1} \\ \vdots \\ c_{i,m_i} \end{bmatrix},$$

where the length of the subcolumn $c_{i,\ell}$ is the same as the dimension of $X_i^\ell/X_i^{\ell-1}$. Since $L(0)$ is block-diagonal and upper triangular, we get that there exists a unique pair (i, ℓ) for which $c_{i,\ell} \neq 0$ and that the eigenvalue of c is $\rho_i + \psi_i^\ell$. Note that

$$y_c(\lambda) = Y_{i,\ell} c_{i,\ell}$$

is a linear combination of elements in X_i^ℓ that project to a basis in $X_i^\ell/X_i^{\ell-1}$. Therefore $\varphi(y_c) = \psi_i^\ell$.

On the other hand, let us denote by R the diagonal part of E and write $E = R + N$. Note that $[R, N] = 0$. Therefore

$$y_c(\lambda) = U(\lambda) \lambda^A \lambda^N \lambda^{-A} c \lambda^{\rho_i + \psi_i^\ell},$$

where we used that $A+R$ is the diagonal part of $L(0)$. Furthermore, using that $\lambda^A N \lambda^{-A} = L(\lambda) - A - R$ is a holomorphic nilpotent matrix we get

$$U(\lambda) \lambda^A \lambda^N \lambda^{-A} = U(\lambda) e^{(L(\lambda) - A - R) \log \lambda}.$$

Expanding near $\lambda = 0$ we get

$$U(0) + \sum_{k=1}^m \frac{1}{k!} (L(0) - A - R)^k (\log \lambda)^k + O(\lambda (\log \lambda)^m),$$

where m is an integer such that $N^m = 0$. However $(L(0) - A - R)c = U(0)c = 0$, so we get that $\varphi(y_c) \geq 1 + \psi_i^\ell$ - contradiction. \square

2.2. Vector bundles on \mathbb{P}^1

Suppose that $E \rightarrow \mathbb{P}^1 \times \tilde{\Pi}$ is a holomorphic vector bundle of rank p , where

$$\tilde{\Pi} = \{u = (u_1, \dots, u_N) \in \mathbb{C}^N \mid |u_i - u_i^\circ| < \tilde{\delta}_i, 1 \leq i \leq N\},$$

is the polydisc with center $u^\circ := (u_1^\circ, \dots, u_N^\circ)$ and polyradius $\tilde{\delta} = (\tilde{\delta}_1, \dots, \tilde{\delta}_N)$. The first goal in this section is to prove the existence of Birkhoff factorization for the transition matrix of E . This result is also known as the Birkhoff–Grothendieck theorem with parameters. Our second goal is to prove the following theorem due to Malgrange:

THEOREM 2.9 ([43]). *Suppose that T is a connected complex manifold and that E is a holomorphic vector bundle on $\mathbb{P}^1 \times T$, such that, $E_{\mathbb{P}^1 \times \{t_0\}}$ and $E|_{\{b_0\} \times T}$ are trivial for some $(b_0, t_0) \in \mathbb{P}^1 \times T$. Then*

a) *The subset*

$$\Theta = \{t \in T : E_{\mathbb{P}^1 \times \{t\}} \text{ is not trivial}\}$$

is either empty or it is an analytic hypersurface of T .

b) *$E|_{\mathbb{P}^1 \times (T - \Theta)}$ is trivial and meromorphic along $\mathbb{P}^1 \times \Theta$.*

Let us clarify the meaning of being meromorphic in Theorem 2.9. It means that we can find a trivializing frame $\{e_i\}_{i=1}^p$ for $E|_{\mathbb{P}^1 \times (T-\Theta)}$, such that, if $\{e_i^U\}_{i=1}^p$ is a local frame for E in a neighborhood U of some point on $\mathbb{P}^1 \times \Theta$, then the transition function between the two frames is a $p \times p$ matrix whose entries are meromorphic functions on U with poles along $U \cap (\mathbb{P}^1 \times \Theta)$.

The arguments presented here are due to Bolibruch [10].

2.2.1. Transition function. We will be interested in transition functions of E of the following type. Let us fix a point $b \in \mathbb{C} \subset \mathbb{P}^1$, real numbers $0 < r < R$, and a polydisc

$$\Pi = \{u \in \mathbb{C}^N \mid |u_i - u_i^o| < \delta_i, 1 \leq i \leq N\},$$

where $0 < \delta_i < \tilde{\delta}_i$ for all i . The discs

$$D_b = \{\lambda \in \mathbb{C} \mid |\lambda - b| < R\}, \quad D_\infty = \{\lambda \in \mathbb{P}^1 \mid |\lambda - b| > r\}$$

give an open cover of \mathbb{P}^1 . The open subsets $D_\nu \times \Pi$, $\nu = b, \infty$, are Stein and contractible, so according to the Grauert–Oka principle $E|_{D_\nu \times \Pi}$ is trivial. Let us define row vectors

$$e_\nu = (e_{\nu,1}, \dots, e_{\nu,p}), \quad e_{\nu,i} \in \Gamma(D_\nu \times \Pi, E),$$

such that, $\{e_{\nu,i}\}_{i=1}^p$ is a trivializing frame for $E|_{D_\nu \times \Pi}$. On the intersection the two frames are related by a holomorphic invertible matrix

$$e_\infty(\lambda, u) = e_b(\lambda, u)M(\lambda, u), \quad (\lambda, u) \in D_{b\infty} \times \Pi,$$

where $D_{b\infty} = D_b \cap D_\infty$ and

$$M : D_{b\infty} \times \Pi \rightarrow \mathrm{GL}(\mathbb{C}^p)$$

is a holomorphic map. Choosing different trivialization frames $\tilde{e}_b = e_b U$ and $\tilde{e}_\infty = e_\infty W$, where

$$U : D_b \rightarrow \mathrm{GL}(\mathbb{C}^p) \quad \text{and} \quad W : D_\infty \rightarrow \mathrm{GL}(\mathbb{C}^p)$$

are holomorphic maps, yields a new transition matrix $\tilde{e}_\infty = \tilde{e}_b \tilde{M}$, where

$$\tilde{M}(\lambda, u) = U(\lambda, u)^{-1} M(\lambda, u) W(\lambda, u), \quad (\lambda, u) \in D_{b\infty} \times \Pi.$$

Our main goal can be stated as follows. We would like to prove that after decreasing Π if necessary and removing an analytic hypersurface from Π we can always arrange that

$$\tilde{M} = \mathrm{diag}((\lambda - b)^{k_1}, \dots, (\lambda - b)^{k_p}),$$

where $k_1 \geq \dots \geq k_p$ is a decreasing sequence of integers.

2.2.2. GAGA reduction.

DEFINITION 2.10. We say that a map $M : D_{b\infty} \times \Pi \rightarrow \mathfrak{gl}(\mathbb{C}^p)$ is Π -rational if the entries of $M(\lambda, u)$ are quotients of polynomials in $\mathcal{O}(\Pi)[\lambda]$, where $\mathcal{O}(\Pi)$ is the ring of holomorphic functions on Π .

We would like to reduce the general analytic problem to an algebraic one. More precisely we would like to prove the following proposition

PROPOSITION 2.11. *Decreasing the size of Π if necessary, we can find a transition matrix*

$$M : D_{b\infty} \times \Pi \rightarrow \mathrm{GL}(\mathbb{C}^p),$$

such that,

- (i) M is Π -rational.
- (ii) The zeroes of $\det(M(\lambda, u))$ and the poles of $M(\lambda, u)$ for $(\lambda, u) \in \mathbb{P}^1 \times \Pi$ are independent of u .

Let us introduce the following notation. If Π is an open polydisc, then we denote by $\bar{\Pi}$ the corresponding closed polydisc. If $X \subset \mathbb{P}^1 \times \bar{\Pi}$ is an open subset, then we define

$$H(X) := \{\phi : \bar{X} \rightarrow \mathfrak{gl}(\mathbb{C}^p) \mid \phi \text{ is continuous in } \bar{X} \text{ and holomorphic in } X\},$$

and

$$H^0(X) := \{\phi \in H(X) \mid \phi(x) \text{ is invertible for all } x \in X\}.$$

Recall that $H(X)$ is a Banach algebra with norm

$$\|A\| = \sup_{(\lambda, u) \in \bar{X}} |A(\lambda, u)|,$$

where $|\cdot| : \mathfrak{gl}(\mathbb{C}^p) \rightarrow \mathbb{R}_{\geq 0}$ is the matrix norm

$$|A| := \sup_{v \in \mathbb{C}^p - \{0\}} |Av|/|v|,$$

where for $w \in \mathbb{C}^p$ we denote by $|w| = (|w_1|^2 + \cdots + |w_p|^2)^{1/2}$ the Euclidean norm of w .

LEMMA 2.12. *There exists an $\epsilon > 0$, depending on r , and R , such that, if $\|B\| < \epsilon$, $B \in H(D_{b\infty} \times \Pi)$, then $1 + B \in H^0(D_{b\infty} \times \Pi)$ and we have a factorization*

$$1 + B = UW, \quad U \in H^0(D_b \times \Pi), \quad W \in H^0(D_\infty \times \Pi).$$

PROOF. The Laurent series expansion gives a decomposition

$$H(D_{b\infty} \times \Pi) = H(D_b \times \Pi) \bigoplus H(D_\infty \times \Pi)(\lambda - b)^{-1}, \quad B = B_+ + B_-.$$

Let pr_\pm be the corresponding projection maps $B \mapsto B_\pm$. We have

$$\text{pr}_+(B)(\lambda, u) = \frac{1}{2\pi\sqrt{-1}} \int_{|\zeta-b|=R} \frac{B(\zeta, u)d\zeta}{\zeta - \lambda}$$

and

$$\text{pr}_-(B)(\lambda, u) = -\frac{1}{2\pi\sqrt{-1}} \int_{|\zeta-b|=r} \frac{B(\zeta, u)d\zeta}{\zeta - \lambda}.$$

It is easy to check that $\|\text{pr}_\pm(B)\| \leq C\|B\|$ for some constant C that depends on r and R . Using these estimates and choosing ϵ sufficiently small we can prove that the series

$$w = \sum_{n=1}^{\infty} (-\text{pr}_- \circ B)^n \mathbf{1} = -B_- + (BB_-)_- - (B(BB_-)_-)_- + \cdots$$

is convergent. Note that $(Bw)_- + w + B_- = 0$ therefore, $(1 + B)(1 + w) = 1 + u$, with $u = B_+ + (Bw)_+$. Decreasing ϵ if necessary we can arrange that $1 + u$ and $1 + w$ are invertible, so the lemma follows with $U = 1 + u$ and $W = (1 + w)^{-1}$. \square

Proof of Proposition 2.11. Let us fix positive numbers $0 < r' < r'' < r < R < R'' < R'$ and polydiscs $\Pi \subset \Pi'' \subset \tilde{\Pi} \subset \bar{\Pi}$ with center at u° . Here Π' is chosen arbitrary, while the sizes of Π'' and Π will be specified later on.

Let us pick an arbitrary transition matrix

$$M' : D'_{b\infty} \times \Pi' \rightarrow \text{GL}(\mathbb{C}^p), \quad D'_{b\infty} := \{r' < |\lambda - b| < R'\}.$$

Note that $M' \in H^0(D''_{b\infty} \times \Pi'')$ where $D''_{b\infty} = \{r'' < |\lambda - b| < R''\}$. The Laurent series expansion of $M'(\lambda, u_0)^{-1}$ at $\lambda = b$ is uniformly convergent for $r'' \leq |\lambda - b| \leq R''$, while

$M'(\lambda, u)^{-1}$ is uniformly continuous for $(\lambda, u) \in \overline{D''_{b\infty}} \times \overline{\Pi''}$. Therefore by truncating the Laurent series expansion of $M'(\lambda, u_0)^{-1}$ appropriately and choosing Π'' sufficiently small, we can find a Laurent polynomial $P \in \mathfrak{gl}(\mathbb{C}^p)[(\lambda-b)^{\pm 1}]$, such that, $\|M'P - 1\|_{r'', R'', \Pi''} < \epsilon$, where the norm is in the space $H(D''_{b\infty} \times \Pi'')$. Recalling Lemma 2.12, we find $U_1 \in H^0(D''_b \times \Pi)$ and $W_1 \in H^0(D''_\infty \times \Pi)$, such that, $M'P = U_1W_1$, that is,

$$M' = U_1W_1P^{-1}.$$

Similarly, we can find $Q \in \mathfrak{gl}(\mathbb{C}^p)[(\lambda-b)^{\pm 1}]$, such that, $QW_1P^{-1} = U_2W_2$. Therefore,

$$M' = U_1Q^{-1}U_2W_2.$$

We claim that the matrix $M := Q^{-1}U_2$ has the required properties. Condition (ii) is easy to verify. Let us prove that U_2 is Π -rational. We have

$$U_2 = QW_1P^{-1}W_2^{-1}.$$

Let $g(\lambda) \in \mathbb{C}[\lambda]$ be a common denominator for the entries of Q and P^{-1} . The matrix $g^2U_2 = (gQ)W_1(gP^{-1})W_2^{-1}$ is holomorphic for all $\lambda \in \mathbb{C}$, because U_2 is holomorphic in $D_b \times \Pi$ while W_1 and W_2 are holomorphic in $D_\infty \times \Pi$. Moreover, since W_1 and W_2 are holomorphic at $\lambda = \infty$, the matrix g^2U_2 has at most a finite order pole at $\lambda = \infty$, so it must be polynomial in λ . \square

2.2.3. Birkhoff factorization with parameters. Let Π be a polydisc with center u° and $\Theta_0 \subset \Pi$ be an analytic hypersurface with finitely many irreducible components. Since Π is Stein and contractible, there exists a holomorphic function $f_0 \in \mathcal{O}(\Pi)$ such that Θ_0 is the zero locus of f_0 . Suppose that we have a transition matrix

$$M : D_{b\infty} \times (\Pi - \Theta_0) \rightarrow \mathrm{GL}(\mathbb{C}^p),$$

such that,

- (i) M is Π -rational.
- (ii) The zeroes of $\det(M(\lambda, u))$ and the poles of $M(\lambda, u)$ for $(\lambda, u) \in D_b \times (\Pi - \Theta_0)$ are independent of u .

According to the previous section such a transition matrix exists provided we choose Π sufficiently small and $\Theta_0 = \emptyset$.

Note that condition (i) implies that the points $(\lambda, u) \in D_b \times \Pi$ for which $M(\lambda, u)$ is not holomorphic form an analytic hypersurface $Z_\infty(M)$. More precisely $Z_\infty(M)$ is the union of all irreducible hypersurfaces $V \subset D_b \times \Pi$ such that there exists an entry $m = g/f$ ($g, f \in \mathcal{O}(D_b \times \Pi)$) of M for which $\mathrm{ord}_V(f) > \mathrm{ord}_V(g)$, where $\mathrm{ord}_V(h)$ denotes the order of vanishing of the holomorphic function $h \in \mathcal{O}(D_b \times \Pi)$ along V .

LEMMA 2.13. *Every irreducible component of $Z_\infty(M)$ has either the form $\{b'\} \times \Pi$ for some $b' \in D_b - D_{b\infty}$ or $D_b \times \Theta'_0$, where Θ'_0 is an irreducible component of Θ_0 .*

PROOF. Let V be an irreducible component. Condition (ii) implies that

$$V \cap D_b \times (\Pi - \Theta_0) = \bigcup_{i=1}^s \{b_i\} \times (\Pi - \Theta_0),$$

for some $b_i \in D_b$. Since $M(\lambda, u)$ is holomorphic and invertible for $(\lambda, u) \in D_{b\infty} \times (\Pi - \Theta_0)$ we have $b_i \in D_b - D_{b\infty}$ and

$$V \subset \left(\bigcup_{i=1}^s \{b_i\} \times \Pi \right) \cup D_b \times \Theta_0.$$

The RHS of the above inclusion relation is an analytic hypersurface, so V must be one of its irreducible components. \square

PROPOSITION 2.14. *a) There exists an analytic hypersurface $\Theta \subset \Pi$ that contains Θ_0 and has finitely many irreducible components, such that,*

$$M(\lambda, u) = U(\lambda, u)(\lambda - b)^K W(\lambda, u),$$

where

- (i) U and W are Π -rational.
- (ii) $U(\lambda, u)$ (resp. $W(\lambda, u)$) is holomorphic and invertible for all $(\lambda, u) \in D_b \times (\Pi - \Theta)$ (resp. $D_\infty \times (\Pi - \Theta)$).
- (iii) $K = \text{diag}(k_1, \dots, k_p)$, where $k_1 \geq \dots \geq k_p$ are integers.

b) Suppose that $u \in \Pi$ is fixed and that

$$M(\lambda, u) = U_i(\lambda, u)(\lambda - b)^{K^{(i)}} W_i(\lambda, u), \quad i = 1, 2,$$

are two factorizations, such that, $U_i(\lambda, u)$ ($i = 1, 2$) is holomorphic invertible for all $\lambda \in D_b$ and $W_i(\lambda, u)$ ($i = 1, 2$) is holomorphic invertible for all $\lambda \in D_\infty$, then $K^{(1)} = K^{(2)}$.

PROOF. a) We split the proof into two cases.

Case 1: If $\det(M(\lambda, u)) \neq 0$ for all $(\lambda, u) \in (D_b - \{b\}) \times (\Pi - \Theta_0)$. We may assume that $M(\lambda, u) = L(\lambda, u)(\lambda - b)^K$, where $L(\lambda, u)$ is holomorphic for $(\lambda, u) \in D_b \times (\Pi - \Theta_0)$ and $K = \text{Diag}(k_1, \dots, k_p)$, where $k_1 \geq k_2 \geq \dots \geq k_p$ are integers. This can be always achieved by first multiplying M from the right by matrices of the type $(\lambda - b_0)^K(\lambda - b)^{-K}$, so that we clear all the poles of $M(\lambda, u)$ from $D_b \times (\Pi - \Theta_0)$, and finally multiply by a constant permutation matrix to arrange that the entries of K are decreasing. Moreover, according to Lemma 2.13 there exists an integer n , such that $L(\lambda, u)f_0(u)^n$ is holomorphic for all $(\lambda, u) \in D_b \times \Pi$.

The Taylor's series expansion of L has the form

$$L(\lambda, u) = L_0(u) + L_1(u)(\lambda - b) + L_2(u)(\lambda - b)^2 + \dots.$$

Let us denote by $m_i(u)$, $1 \leq i \leq p$, the columns of the matrix $L_0(u)$. We may assume that $m_1 \neq 0$, otherwise we can factor out $(\lambda - b)$ from the first column of $L(\lambda, u)$ and increase k_1 by one. We can also assume that $\det(L_0(u)) = 0$, otherwise the matrix $L(\lambda, u)$ is invertible for all $(\lambda, u) \in D_b \times (\Pi - \Theta)$, where $\Theta \subset \Pi$ is the union of Θ_0 and the zero locus of $\det(L_0(u))$, and this is already a Birkhoff factorization, so there is nothing to prove.

Let us denote by i the maximal index, such that, some $i \times i$ minor of $L_0(u)$ contained in the first i -columns is not identically 0 for $u \in \Pi - \Theta_0$. If there are several such minors, then we choose one of them, write it in the form $g(u)/f_0(u)^n$ for some $g \in \mathcal{O}(\Pi)$ and denote by $\Theta \subset \Pi$ the analytic hypersurface defined by the zero locus of $g(u)f_0(u)$. There are functions $s_1(u), \dots, s_i(u)$, holomorphic for $u \in \Pi - \Theta$ and meromorphic along Θ , such that,

$$m_{i+1}(u) = s_1(u)m_1(u) + \dots + s_i(u)m_i(u).$$

Let

$$W(\lambda, u) = 1 - \sum_{a=1}^i s_a(u)(\lambda - b)^{-k_a + k_{i+1}} E_{a, i+1},$$

where $E_{a,i+1}$ is the matrix with only one non-zero entry, which is equal to 1 and it is in row a and column $i+1$. Note that $k_a \geq k_{i+1}$, so $W(\lambda, u)$ is holomorphic and invertible for $(\lambda, u) \in D_\infty \times (\Pi - \Theta)$ and meromorphic along $D_\infty \times \Theta$. It is easy to check that $M(\lambda, u)W(\lambda, u) = \tilde{L}(\lambda, u)(\lambda - b)^{\tilde{K}}$, where $\tilde{K} = \text{Diag}(\tilde{k}_1, \dots, \tilde{k}_p)$ satisfies $\tilde{k}_j = k_j$ for $j \neq i+1$ and $\tilde{k}_{i+1} > k_{i+1}$. Multiplying if necessary W by a constant permutation matrix from the right we can arrange that $\tilde{k}_1 \geq \dots \geq \tilde{k}_p$. Note that

$$\det(L(\lambda, u)) = \det(\tilde{L}(\lambda, u)) (\lambda - b)^{\sum_{i=1}^p (\tilde{k}_i - k_i)},$$

so the order of vanishing of $\det(L(\lambda, u))$ at $\lambda = b$ decreases strictly. Repeating the above procedure finitely many times we will eventually get a matrix $L(\lambda, u)$, such that, $\det(L(\lambda, u)) \neq 0$ at $\lambda = b$, which as explained above would give a Birkhoff factorization.

Case 2: general case. Just like in Case 1, multiplying $M(\lambda, u)$ from the right by an appropriate holomorphic invertible matrix defined for all $(\lambda, u) \in D_\infty \times \Pi$ and by $(\lambda - b)^m \text{Id}$ with $m \gg 0$, we may reduce the proof to the case when $M(\lambda, u)$ is holomorphic for all $(\lambda, u) \in D_b \times (\Pi - \Theta_0)$ and meromorphic along $D_b \times \Theta_0$. We argue by induction on the number of zeroes of $\det(M(\lambda, u))$ in $D_b \times (\Pi - \Theta_0)$. If there are no zeroes, then $M(\lambda, u)$ is holomorphic and invertible for all $(\lambda, u) \in D_b \times (\Pi - \Theta_0)$ and there is nothing to prove.

Let $b_1 \in D_b$ be a zero of $\det(M(\lambda, u))$. Since $M(\lambda, u)$ is invertible for $r \leq |\lambda - b| \leq R$ we have $|b_1 - b| < r$. Let us choose a disc $D_1 = \{|\lambda - b_1| < R_1\}$ with center b_1 and radius R_1 so small that $D_1 \subset \{\lambda \in \mathbb{C} \mid |\lambda - b| < r\}$ and D_1 does not contain other zeroes of $\det(M(\lambda, u))$. Let us recall Case 1 for M and the covering of \mathbb{P}^1 given by the discs D_1 and $D_1^\infty := \{|\lambda - b_1| > r_1\}$, where $0 < r_1 < R_1$. We get a Birkhoff factorization

$$M(\lambda, u) = M_1(\lambda, u)(\lambda - b_1)^{K_1} W_1(\lambda, u),$$

where

- (i) M_1 and W_1 are Π -rational.
- (ii) $M_1(\lambda, u)$ (resp. $W_1(\lambda, u)$) is holomorphic and invertible for all $(\lambda, u) \in D_1 \times (\Pi - \Theta_1)$ (resp. $D_1^\infty \times (\Pi - \Theta_1)$) for some analytic hypersurface $\Theta_1 \subset \Pi$ with $\Theta_0 \subset \Theta_1$.
- (iii) K_1 is a diagonal matrix with decreasing integer entries.

By comparing the domains of analyticity and invertibility of the LHS and the RHS in the following identity we get that

$$(2.4) \quad M_1(\lambda, u) = M(\lambda, u)W_1(\lambda, u)^{-1}(\lambda - b_1)^{-K_1}$$

is holomorphic for $(\lambda, u) \in D_b \times (\Pi - \Theta_1)$ and invertible for $(\lambda, u) \in D_{b_\infty} \times (\Pi - \Theta_1)$. The zeroes of $\det(M_1(\lambda, u))$ for $(\lambda, u) \in D_b \times (\Pi - \Theta_1)$ are first of all in $(D_b - D_1) \times (\Pi - \Theta_1)$ and then by expecting the RHS of (2.4), we get that they are contained in the set of zeroes of $\det(M(\lambda, u))$. Note that if $\lambda = b_1$ is the only zero of $\det(M(\lambda, u))$ for $(\lambda, u) \in D_b \times (\Pi - \Theta_1)$, then we are done, because $M_1(\lambda, u)$ will be holomorphic and invertible for $(\lambda, u) \in D_b \times (\Pi - \Theta_1)$ and we have the following Birkhoff factorization:

$$M(\lambda, u) = M_1(\lambda, u) (\lambda - b)^{K_1} \left(\frac{\lambda - b_1}{\lambda - b} \right)^{K_1} W_1(\lambda, u),$$

where note that $\frac{\lambda - b_1}{\lambda - b}$ is holomorphic invertible for all $\lambda \in D_\infty$ because $b_1 \notin D_\infty$. Otherwise, let $b_2 \in D_b$ be a 2nd zero of $\det(M(\lambda, u))$ and let $m > 0$ be an integer, such that, the diagonal entries of K_1 are greater than $-m$. We get that the number of zeroes of

$\det \left(M_1(\lambda, u)(\lambda - b_2)^{K_1+m} \right)$ in $D_b \times (\Pi - \Theta_1)$ is at least 1 less than the number of zeroes of $\det(M(\lambda, u))$. Using the inductive assumption we get a Birkhoff factorization

$$M_1(\lambda, u)(\lambda - b_2)^{K_1+m} = U(\lambda, u)(\lambda - b)^K W'(\lambda, u),$$

where

- (i) U and W' are Π -rational.
- (ii) $U(\lambda, u)$ (resp. $W'(\lambda, u)$) is holomorphic and invertible for all $(\lambda, u) \in D_b \times (\Pi - \Theta)$ (resp. $D_\infty \times (\Pi - \Theta)$) for some analytic hypersurface $\Theta \subset \Pi$ with $\Theta_1 \subset \Theta$.
- (iii) K is a diagonal matrix with decreasing integer entries.

Therefore,

$$M(\lambda, u) = U(\lambda, u)(\lambda - b)^{K-m} W'(\lambda, u) \left(\frac{\lambda - b_1}{\lambda - b_2} \right)^{K_1} \left(\frac{\lambda - b}{\lambda - b_2} \right)^m W_1(\lambda, u)$$

which provides a Birkhoff factorization for all $u \in \Pi - \Theta$.

b) Put $K^{(i)} = \text{Diag}(k_1^{(i)}, \dots, k_p^{(i)})$. We argue by induction on i that $k_i^{(1)} = k_i^{(2)}$ for all i . Assume that $k_a^{(1)} = k_a^{(2)}$ for $a = 1, 2, \dots, i-1$ and $k_i^{(1)} > k_i^{(2)}$. Comparing the two Birkhoff factorization, we get

$$(U_2^{-1}U_1)_{a\ell} = (\lambda - b)^{k_a^{(2)} - k_\ell^{(1)}} (W_2W_1^{-1})_{a\ell},$$

where $A_{a\ell}$ denotes the (a, ℓ) -entry of the matrix A . The LHS is analytic for $\lambda \in D_b$. If $k_a^{(2)} < k_\ell^{(1)}$, then the RHS is analytic in D_∞ and vanishes for $\lambda = \infty$, so by Liouville's theorem both sides must vanish. We get that $(U_2^{-1}U_1)_{a\ell} = 0$ for $1 \leq \ell \leq i$ and $a \geq i$, because according to our assumptions

$$k_a^{(2)} \leq k_i^{(2)} < k_i^{(1)} \leq k_\ell^{(1)}.$$

The first i -columns of $U_2^{-1}U_1$ have non-zero entries only in the first $(i-1)$ places, therefore they must be linearly dependent. This however contradicts the fact that $U_2^{-1}U_1$ is invertible for $\lambda \in D_b$. Similarly, the assumption $k_i^{(1)} < k_i^{(2)}$ would contradict the invertibility of $W_2W_1^{-1}$, so $k_i^{(1)} = k_i^{(2)}$. \square

2.2.4. Proof of Theorem 2.9. a) We argue by induction on the dimension of T . If T is 0-dimensional, then there is nothing to prove. Let us define the set

$$N = \{t \in T : E|_{\mathbb{P}^1 \times \{t\}} \text{ is trivial}\}.$$

LEMMA 2.15. *If $t' \in N$, then there exists an open neighborhood V of t' in T such that $E|_{\mathbb{P}^1 \times V}$ is trivial.*

PROOF. Let V be an open polydisc neighborhood of t' . We can find trivializations of $E|_{D_b \times V}$, such that, the transition function $M(\lambda, t') = 1$. Indeed, using that $E|_{\mathbb{P}^1 \times \{t'\}}$ is trivial we get that $M(\lambda, t') = U'(\lambda)W'(\lambda)$, where $U'(\lambda)$ (resp. $W'(\lambda)$) is holomorphic and invertible for $\lambda \in D_b$ (resp. D_∞). Changing the trivialization frames of $E|_{D_b \times V}$ and $E|_{D_\infty \times V}$ via U' and W' we can transform the transition matrix into $U'(\lambda)^{-1}M(\lambda, t)W'(\lambda)^{-1}$, which turns into 1 at $t = t'$.

Let us assume now that the transition matrix is such that $M(\lambda, t') = 1$. Decreasing V if necessary, we can make $M(\lambda, t)$ sufficiently close to $M(\lambda, t')$. Recalling Lemma 2.12, we get a Birkhoff factorization $M(\lambda, t) = U(\lambda, t)W(\lambda, t)$, which implies that $E|_{\mathbb{P}^1 \times V}$ is trivial. \square

The above lemma shows that N is an open subset.

LEMMA 2.16. *The vector bundle $E|_{\mathbb{P}^1 \times N}$ is trivial.*

PROOF. Let Σ be the set of open subsets $V \subset T$ such that $E|_{\mathbb{P}^1 \times V}$ is trivial. By definition $t_0 \in N$, so according to Lemma 2.15 the set Σ is non-empty. Using the inclusion of open subset we can define a partial ordering on Σ . Clearly every increasing chain $V_1 \subset V_2 \subset \dots$ in Σ is bounded by $\cup_i V_i \in \Sigma$. Therefore, recalling the Zorn's lemma the set Σ has a maximal element, say V . If $V \neq N$, then let $t' \in N$ be a boundary point of V . According to Lemma 2.15 we can find an open neighborhood $V' \in \Sigma$ that contains t' . Let e' and e be row vectors whose entries give trivialisations of respectively $E|_{\mathbb{P}^1 \times V'}$ and $E|_{\mathbb{P}^1 \times V}$. Then $e' = eU$, where $U : \mathbb{P}^1 \times (V' \cap V) \rightarrow \text{GL}(\mathbb{C}^p)$ is a transition matrix. Since the entries of $U(\lambda, u)$ are holomorphic for all $\lambda \in \mathbb{P}^1$, by Louiville's theorem they must be constants independent of λ , i.e., $U(\lambda, u) = U(b_0, u)$. On the other hand by definition $E|_{\{b_0\} \times T}$ is a trivial bundle, so we can factorize $U(b_0, u) = A(u)A'(u)^{-1}$. Therefore $eA(u) = e'A'(u)$ for $u \in V \cap V'$, so we get that $E|_{\mathbb{P}^1 \times (V \cup V')}$ is trivial. Since V is maximal we get $V' \subseteq V$, which however contradicts the fact that $t' \in V'$ is a boundary point of V . \square

If $N = T$, then we are done. Let us assume that $N \neq T$. We have to show that $T - N$ is an analytic subvariety of codimension 1. Let $u^0 \in T$ be a boundary point of N and Π be a polydisc with center u^0 . Let $M(\lambda, u)$ be the transition matrix for some trivialisations $E|_{D_b \times \Pi}$, $\nu = b, \infty$. According to Proposition 2.11 we may assume that M is Π -rational. Decreasing Π if necessary, we get a Birkhoff factorization (see Proposition 2.14, part a)) $M(\lambda, u) = U(\lambda, u)(\lambda - b)^K W(\lambda, u)$, where $U(\lambda, u)$ (resp. $W(\lambda, u)$) is holomorphic and invertible for $(\lambda, u) \in D_b \times (\Pi - \Theta)$ (resp. $D_\infty \times (\Pi - \Theta)$). On the other hand, if $V \subset (\Pi - \Theta) \cap N \subset N$ is an open subset, then $E|_{\mathbb{P}^1 \times V}$ is trivial. Therefore, the transition function $M(\lambda, u) = U'(\lambda, u)W'(\lambda, u)$. Comparing the two Birkhoff factorizations of M and recalling Proposition 2.14, part b) we get that $K = 0$, which implies that $E|_{\mathbb{P}^1 \times (\Pi - \Theta)}$ is trivial, that is, $\Pi - \Theta \subset N$.

Let $\Theta = \Theta_1 \cup \dots \cup \Theta_s$ be the decomposition of Θ into irreducible components. By decreasing Π if necessary we may assume that $u^0 \in \Theta_i$ for all $i = 1, \dots, s$ and that Θ_i ($1 \leq i \leq s$) represent the germs of the irreducible components of the germ of Θ at u^0 . Let Θ' be the union of the irreducible components Θ_i that are disjoint from N and Θ'' is the union of the remaining irreducible components of Θ . We claim that $\Theta' = \Pi - N$. Equivalently, since $\Theta' \cap N = \emptyset$ we have to prove that $\Pi - \Theta' \subset N$. Let us split $\Theta'' = \Theta''_{\text{sing}} \cup \Theta''_{\text{reg}}$ into singular and regular points. Θ''_{sing} is at least complex codimension 2 in Π . Since $N \cap \Theta''_{\text{reg}} \neq \emptyset$, recalling the inductive assumption we get that there exists an analytic hypersurface $\Theta''_0 \subset \Theta''_{\text{reg}}$, such that, $N \cap \Theta''_{\text{reg}} = \Theta''_{\text{reg}} - \Theta''_0$. Therefore,

$$\Pi - (\Theta' \cup \Theta''_{\text{sing}} \cup \Theta''_0) = (\Pi - \Theta) \cup N \cap \Theta''_{\text{reg}} \subset N.$$

We get that the restriction of the vector bundle E to $\mathbb{P}^1 \times (\Pi - (\Theta' \cup \Theta''_{\text{sing}} \cup \Theta''_0))$ is trivial. Therefore, the transition matrix has a factorization $M(\lambda, u) = U(\lambda, u)W(\lambda, u)$, where $U(\lambda, u)$ (resp. $W(\lambda, u)$) is holomorphic and invertible in $D_b \times ((\Pi - \Theta') - (\Theta''_{\text{sing}} \cup \Theta''_0))$ (resp. $D_\infty \times ((\Pi - \Theta') - (\Theta''_{\text{sing}} \cup \Theta''_0))$). On the other hand, since $\Theta''_{\text{sing}} \cup \Theta''_0$ is located in complex codimension at least 2 in $\Pi - \Theta'$, the Hartogues extension theorem implies that $U(\lambda, u)$ (resp. $W(\lambda, u)$) extends analytically for all $(\lambda, u) \in D_b \times (\Pi - \Theta')$ (resp. $D_\infty \times (\Pi - \Theta')$). Furthermore, the zero locus of $\det(U(\lambda, u))$ in $D_b \times (\Pi - \Theta')$ is contained in the subset $D_b \times (\Theta''_{\text{sing}} \cup \Theta''_0)$ which is contained in an analytic subset of codimension at least 2. Since the zero locus of a holomorphic function is either empty or it has complex

codimension 1, we get that $U(\lambda, u)$ must be invertible for all $(\lambda, u) \in D_b \times (\Pi - \Theta')$. Similarly, $W(\lambda, u)$ is invertible for all $(\lambda, u) \in D_\infty \times (\Pi - \Theta')$. We get that $E|_{\mathbb{P}^1 \times (\Pi - \Theta')}$ is trivial, that is, $\Pi - \Theta' \subset N$ which is what we had to prove.

b) Let $e = (e_1, \dots, e_p)$, $e_i \in \Gamma(\mathbb{P}^1 \times (T - \Theta), E)$ be a trivializing frame. Every other trivializing frame has the form eC , where $C : T - \Theta \rightarrow \text{GL}(\mathbb{C}^p)$. On the other hand, since $E|_{\{b_0\} \times T}$ is trivial we get that we can always choose C in such a way that eC extends to a trivializing frame of $E|_{\{b_0\} \times T}$. Therefore there exists a frame e such that $e|_{\{b_0\} \times (T - \Theta)}$ extends to a trivializing frame of $E|_{\{b_0\} \times T}$. We claim that such a frame e is meromorphic.

Let us choose the covering $\{D_b, D_\infty\}$ of \mathbb{P}^1 in such a way that $b_0 \in D_{b_\infty} = D_b \cap D_\infty$. Let $u_0 \in \Theta$ be an arbitrary point and let $\Pi \subset T$ be an open polydisc neighborhood of u_0 . Let $e_\nu^\Pi = (e_{\nu 1}^\Pi, \dots, e_{\nu p}^\Pi)$, $e_{\nu i}^\Pi \in \Gamma(D_\nu \times \Pi, E)$, $\nu = b, \infty$ be trivializing frames of E and $M : D_{b_\infty} \rightarrow \text{GL}(\mathbb{C}^p)$ be the corresponding transition matrix, that is, $e_\infty^\Pi = e_b^\Pi M$. There exists a Birkhoff factorization

$$M(\lambda, u) = U_1(\lambda, u) (\lambda - b)^K W_1(\lambda, u),$$

such that, U_1 and W_1 are Π -rational, $U_1(\lambda, u)$ (resp. $W_1(\lambda, u)$) is holomorphic and invertible for all $(\lambda, u) \in D_b \times (\Pi - \Theta_1)$ (resp. $D_b \times (\Pi - \Theta_1)$), where $\Theta_1 \subset \Pi$ is an analytic hypersurface. Since $E|_{\mathbb{P}^1 \times (\Pi - \Theta)}$ is trivial, we have $K_1 = 0$ and $\Pi - \Theta_1 \subseteq \Pi - \Theta$, that is, $\Theta \subseteq \Theta_1$. Let U and W be the transition matrices between the frame e and respectively e_b^Π and e_∞^Π , that is, $e = e_b^\Pi U$ and $e = e_\infty^\Pi W^{-1}$, where $U = U(\lambda, u)$ is holomorphic and invertible for all $(\lambda, u) \in D_b \times (\Pi - \Theta)$ and $W = W(\lambda, u)$ is holomorphic and invertible for all $(\lambda, u) \in D_\infty \times (\Pi - \Theta)$. Note that

$$M(\lambda, u) = U_1(\lambda, u) W_1(\lambda, u) = U(\lambda, u) W(\lambda, u).$$

Therefore, $U_1(\lambda, u)^{-1} U(\lambda, u) = W_1(\lambda, u) W(\lambda, u)^{-1}$ and by comparing the domain of analyticity of the LHS and RHS we get that for every fixed $u \in \Pi - \Theta_1$ both the LHS and the RHS are analytic for all $\lambda \in \mathbb{P}^1$. Recalling Liouville's theorem, we get that there exists a matrix $Q(u)$ independent of λ and holomorphic for all $u \in \Pi - \Theta_1$, such that, $U(\lambda, u) = U_1(\lambda, u) Q(u)$ and $W_1(\lambda, u) = Q(u) W(\lambda, u)$. Let us specialize $\lambda = b_0$. Since both e and e_b^Π are holomorphic frames for E over $\{b_0\} \times \Pi$, the matrix $U(b_0, u)$ is holomorphic for all $u \in \Pi$. By construction, U_1 is Π -rational which in particular implies that the entries of $U_1(b_0, u)$ are meromorphic. Therefore, $Q(u)$ is also meromorphic in Π . Recalling again that $U_1(\lambda, u)$ is Π -rational, we get that $U(\lambda, u) = U_1(\lambda, u) Q(u)$ is meromorphic in $D_b \times \Pi$. Finally, $U(\lambda, u)$ is holomorphic and invertible for $(\lambda, u) \in D_b \times (\Pi - \Theta) \Rightarrow$ the poles of $U(\lambda, u)$ are along the hypersurface $D_b \times \Theta$. The proof that $W(\lambda, u)$ is meromorphic in $D_\infty \times \Pi$ with possible poles along $D_\infty \times \Theta$ is similar. \square

2.3. Painleve property for the Schlesinger equations

Let ∇° be a Fuchsian connection on the trivial vector bundle $\mathbb{P}^1 \times \mathbb{C}^p$. Written in coordinates

$$\nabla^\circ = d - A^\circ(\lambda) d\lambda,$$

where

$$A^\circ(\lambda) = \frac{A_1^\circ}{\lambda - u_1^\circ} + \dots + \frac{A_N^\circ}{\lambda - u_N^\circ},$$

where A_i° are $p \times p$ matrices and u_i° are the finite poles of ∇° . Let us also assume that $\sum_{i=1}^N A_i^\circ \neq 0$, so that the connection has a Fuchsian singularity at $\lambda = \infty$.

The Schlesinger equations are the following system of differential equations

$$(2.5) \quad \frac{\partial A_i}{\partial u_j} = \frac{[A_j, A_i]}{u_j - u_i}, \quad 1 \leq i \neq j \leq N,$$

$$(2.6) \quad \frac{\partial A_i}{\partial u_1} + \cdots + \frac{\partial A_i}{\partial u_N} = 0, \quad 1 \leq i \leq N,$$

$$(2.7) \quad A_i(u^\circ) = A_i^\circ \quad 1 \leq i \leq N,$$

where $u^\circ = (u_1^\circ, \dots, u_N^\circ)$. Here

$$A_i(u_1, \dots, u_N) \in \mathfrak{gl}(\mathbb{C}^p), \quad 1 \leq i \leq N,$$

is a set of matrix-valued functions which should be viewed as deformations of the coefficients of the Fuchsian connection ∇° . It is easy to check that the Schlesinger equations are compatible (integrable). Therefore the solution exists for all $u = (u_1, \dots, u_N)$ sufficiently close to u° .

Let us denote by $U \subset \mathbb{C}^N$ a sufficiently small neighborhood of u° and let $(U \times \mathbb{C})'$ be the set of all $(u, \lambda) \in U \times \mathbb{C}$, such that

$$(\lambda - u_1) \cdots (\lambda - u_N) \neq 0.$$

The Schlesinger equations are equivalent to the 0-curvature equations for the following connection on the trivial bundle on $(U \times \mathbb{C})'$ of rank p :

$$\begin{aligned} \nabla_{\partial_{u_i}}^{Schl} &= \partial_{u_i} + \frac{A_i(u)}{\lambda - u_i} \quad (1 \leq i \leq N), \\ \nabla_{\partial_\lambda}^{Schl} &= \partial_\lambda - \sum_{i=1}^N \frac{A_i(u)}{\lambda - u_i}. \end{aligned}$$

REMARK 2.17. The Schlesinger equations provide a solution to the isomonodromy problem for the Fuchsian connection ∇° : find a deformation of ∇° , such that the monodromy of the deformed connection is independent of the deformation parameters. We refer to [10] for more details on the theory of isomonodromic deformations. \square

Let us identify the configuration space of N points on \mathbb{C} with

$$Z_N = \{u \in (\mathbb{P}^1)^{N+1} : u_i \neq u_j \text{ for } i \neq j \text{ and } u_{N+1} = \infty\}$$

Every point $u \in Z_N$ corresponds to a punctured sphere

$$\mathbb{P}^1 - \{u_1, \dots, u_{N+1}\} = \mathbb{C} - \{u_1, \dots, u_N\}.$$

Let us denote by T the universal cover of Z_N . The point $u^\circ \in Z_N$ will be fixed as a base point and we identify T as the set of pairs $(u, [c])$ such that $u \in Z_N$ and $[c]$ is the homotopy class of a path c in Z_N from u° to u . A small neighborhood of u° in Z_N has a natural lift to a small neighborhood of $t^\circ := (u^\circ, [1]) \in T$, where $[1]$ is the trivial path from u° to u° . The solution to the Schlesinger equations (2.5)–(2.7) exists locally in a neighborhood of $t^\circ \in T$. The main goal of this chapter is to prove the following theorem due to Malgrange:

THEOREM 2.18 ([43]). *If $\{A_i(u)\}_{i=1}^N$ is a solution to the Schlesinger equations, then each A_i extends to a meromorphic function on T .*

2.3.1. Malgrange's vector bundle. The Fuchsian connection ∇° determines a monodromy representation

$$\mu : \pi_1(\mathbb{C} - \{u_1^\circ, \dots, u_N^\circ\}, b^\circ) \rightarrow \mathrm{GL}(\mathbb{C}^p),$$

where b° is a reference point. The representation is defined as follows. Let $Y^\circ(\lambda)$ be the fundamental solution of ∇° defined in a neighborhood of b° such that $Y^\circ(b^\circ) = 1$. If γ is a closed path in $\mathbb{C} - \{u_1^\circ, \dots, u_N^\circ\}$ then the analytic continuation of $Y^\circ(\lambda)$ along γ has the form $Y^\circ(\lambda)\mu(\gamma)$.

Let us denote by $D_i \subset \mathbb{P}^1 \times T$, $1 \leq i \leq N+1$, the hypersurface consisting of points $(\lambda, u, [c])$ such that $\lambda = u_i$ ($u_{N+1} := \infty$). Let $\mathcal{C} := \mathbb{P}^1 \times T - \cup_{i=1}^{N+1} D_i$. The projection map

$$\pi : \mathcal{C} \rightarrow T, \quad (\lambda, u, [c]) \mapsto (u, [c])$$

is a smooth fibration with fiber diffeomorphic to $\pi^{-1}(t^\circ) = \mathbb{C} - \{u_1^\circ, \dots, u_N^\circ\}$. Since $\pi_k(T) = \pi_k(Z_N) = 0$ for $k > 1$ and $\pi_1(T) = \{1\}$, we get that T is a contractible space. Using the long exact sequence of homotopy groups we get that the natural inclusion

$$(\mathbb{C} - \{u_1^\circ, \dots, u_N^\circ\}, b^\circ) \rightarrow (\mathcal{C}, (b^\circ, t^\circ))$$

induces an isomorphism between the fundamental groups. Therefore the monodromy representation μ° of ∇° induces a representation

$$(2.8) \quad \mu : \pi_1(\mathcal{C}, (b^\circ, t^\circ)) \rightarrow \mathrm{GL}(\mathbb{C}^p).$$

There exists a unique vector bundle $E \rightarrow \mathcal{C}$ of rank p equipped with a flat connection ∇ such that the monodromy representation of ∇ is equivalent to the given representation (2.8). We will refer to $E \rightarrow \mathcal{C}$ as the *Malgrange's vector bundle*. The equivalence between the monodromy representation of ∇ and (2.8) means that there exists a row vector $f^\circ = (f_1^\circ, \dots, f_p^\circ)$ whose entries form a basis of the fiber E_{b°, t° such that the parallel transport with respect to ∇ along a closed loop γ based at (b°, t°) transforms f° into $f^\circ \mu(\gamma)$.

For the reader's convenience let us recall the construction of E . We choose a covering of \mathcal{C} by open balls $\{B_i\}_{i \in \mathcal{C}}$ that have contractible connected intersections. This can be achieved by choosing a Riemannian metric on \mathcal{C} and letting B_i be the ball with center $i \in \mathcal{C}$ of radius r_i , where r_i is the injectivity radius of \mathcal{C} at the point i . It is known that if $x', x'' \in B_i$, then there exists a unique geodesic in \mathcal{C} from x' to x'' whose length is the distance between x' and x'' . Moreover, such a geodesic is entirely in B_i . If $B_i \cap B_j \neq \emptyset$, then we choose a smooth path γ_{ij} in $B_i \cup B_j$ between the centers of B_i and B_j . Let us also fix B_0 to be the ball with center the base point (b°, t°) . Let us also fix a path γ_i from B_0 to B_i consisting of paths γ_{ab} . Then we define $E|_{B_i} := B_i \times \mathbb{C}^p$ and let $e^i = (e_1^i, \dots, e_p^i)$ be the trivializing frame corresponding to the standard basis of \mathbb{C}^p . On the overlaps $B_i \cap B_j \neq \emptyset$ the bundles are glued via

$$e^j = e^i g_{ij}, \quad g_{ij} = \mu(\gamma_i^{-1} \circ \gamma_{ji} \circ \gamma_j),$$

where μ is the given monodromy representation (2.8). Since g_{ij} are constants, the standard flat connections given by the de Rham differential on $B_i \times \mathbb{C}^p$ glue together, so the bundle E is naturally equipped with a flat connection.

2.3.2. Extension of E . Recall that $Y^\circ(\lambda)$ is the fundamental solution of ∇° defined in a neighborhood of a fixed reference point $\lambda = b^\circ$. $Y^\circ(\lambda)$ is uniquely determined by requiring that it satisfies the initial condition $Y^\circ(b^\circ) = 1$. For every singular point u_i° ($1 \leq i \leq N$) of ∇° let us fix a sector with vertex at u_i° of the following form

$$\{\lambda \in \mathbb{C} : 0 < |\lambda - u_i^\circ| < R_i^\circ, -\epsilon < \text{Arg}(\lambda - u_i^\circ) < \epsilon\},$$

where R_i is sufficiently small so that the disc with center u_i° and radius R_i° does not contain other singular points u_j° and $0 < \epsilon < 2\pi$. Let us fix a path γ_i° ($1 \leq i \leq N+1$) in $\mathbb{C} - \{u_1^\circ, \dots, u_N^\circ\}$ from b° to a point $u_i^\circ + \lambda_i^\circ$ in the above sector, e.g., $\lambda_i^\circ := R_i^\circ/2$. Let us extend analytically $Y^\circ(\lambda)$ along γ_i° . We get an analytic solution of ∇° defined in the above sector. Finally, let us choose an invertible matrix $S_i^\circ \in \text{GL}(\mathbb{C}^p)$, such that $Y^\circ(\lambda)S_i^\circ$ is a weak Levelt solution for the Fuchsian singularity of ∇° at $\lambda = u_i^\circ$. We have

$$(2.9) \quad Y^\circ(\lambda)S_i^\circ = U_i^\circ(\lambda)(\lambda - u_i^\circ)^{K_i}(\lambda - u_i^\circ)^{E_i},$$

where the matrix

$$E_i = \text{diag}(E_i^1, \dots, E_i^{p_i})$$

is block diagonal with each block corresponding to an eigenvalue of E_i , the block $E_i^j = \rho_i^j I + N_i^j$, where N_i^j is an upper-triangular nilpotent matrix and the eigenvalue ρ_i^j satisfies

$$0 \leq \text{Re}(\rho_i^j) < 1,$$

$K_i = \text{diag}(K_i^1, \dots, K_i^{p_i})$ has the same block diagonal structure as E_i with each block K_i^j being a diagonal matrix with decreasing integer entries, and $U_i^\circ(\lambda)$ is holomorphically invertible in a neighborhood of $\lambda = u_i^\circ$.

It is convenient to extend our notation for the singular points of ∇° in order to include also the singularity at $\lambda = u_{N+1}^\circ = \infty$. The above statements remain the same except that we have to replace everywhere $\lambda - u_i^\circ$ with λ^{-1} . In particular, the fundamental solution takes the forms

$$(2.10) \quad Y^\circ(\lambda)S_{N+1}^\circ = U_{N+1}^\circ(\lambda)\lambda^{-K_{N+1}}\lambda^{-E_{N+1}}.$$

The vector bundle E can be extended across the divisors D_i , $1 \leq i \leq N+1$ as follows. Let us take a tubular neighborhood

$$T_i = \{(\lambda, u, [c]) : |\lambda - u_i| < R_i(u)\} \subset \mathbb{P}^1 \times T,$$

where $R_i : Z_N \rightarrow \mathbb{R}_{>0}$ is a smooth function satisfying

$$R_i(u) < |u_j - u_i|, \quad \text{for all } 1 \leq i \neq j \leq N,$$

and

$$R_{N+1}(u) > |u_j|, \quad \text{for all } 1 \leq j \leq N.$$

Using parallel transport with respect to the flat connection ∇ we construct a multivalued flat frame $f = (f_1, \dots, f_p)$ of E whose value at a point $(\lambda, t) \in \mathcal{C}$

$$f(\lambda, t) = (f_1(\lambda, t), \dots, f_p(\lambda, t)), \quad f_i(\lambda, t) \in E_{\lambda, t}$$

depends on the choice of a reference path in \mathcal{C} from (b°, t°) to (λ, t) : the component $f_i(\lambda, t)$ is obtained from $f_i^\circ \in E_{b^\circ, t^\circ}$ via a parallel transport along the reference path. Let us trivialize $E|_{T_i - D_i}$ via the frame

$$(2.11) \quad f(\lambda, t)S_i^\circ(\lambda - u_i)^{-E_i}(\lambda - u_i)^{-K_i}, \quad (\lambda, t) \in T_i - D_i,$$

where $t = (u, [c]) \in T$ and the path specifying the value of $f(\lambda, t)$ is chosen as follows. We identify $\mathbb{C} - \{u_1^\circ, \dots, u_N^\circ\}$ with the fiber $\mathbb{C}_{t^\circ} := \pi^{-1}(t^\circ)$. Note that the path $\gamma_i^\circ \subset \mathbb{C}_{t^\circ}$ connects the reference point (b°, t°) with the point $(u_i^\circ + R_i^\circ/2, t^\circ) \in T_i$ (provided we define $R_i^\circ := R_i(u^\circ)$). The path that we would like to select consists of two pieces the path γ_i° and any path in $T_i - D_i$ connecting the end point of γ_i° and (λ, t) . The analytic continuation of $f(\lambda, t^\circ)$ and $Y^\circ(\lambda)$ along a closed loop around $\lambda = u_i^\circ$ are respectively $f(\lambda, t^\circ)M_i$ and $Y^\circ(\lambda)M_i$. Note that $M_i S_i^\circ = S_i^\circ e^{2\pi\sqrt{-1}E_i}$, therefore the monodromy of $f(\lambda, t)S_i$ around D_i cancels out the monodromy of $(\lambda - u_i)^{-E_i}(\lambda - u_i)^{-K_i}$ around D_i . Hence the frame (2.11) provides a holomorphic trivialization of $E|_{T_i - D_i}$. We extend E across D_i in the obvious way: on the overlap of T_i and $T_i - D_i$ we identify the standard frame of $T_i \times \mathbb{C}^p$ with the frame (2.11) of $E|_{U_i - D_i}$.

2.3.3. Proof of Theorem 2.18. We are going to construct a multivalued analytic function $Y(\lambda, t)$ with values in $\text{GL}(\mathbb{C}^p)$ defined for all $(\lambda, t) \in \mathcal{C}$ such that

- (1) $Y(\lambda, t^\circ) = Y^\circ(\lambda)$.
- (2) The 1-form $\omega := dY(\lambda, t)Y(\lambda, t)^{-1}$ is a meromorphic 1-form on $\mathbb{P}^1 \times T$ of the form

$$\sum_{i=1}^N \frac{A_i(t)}{\lambda - u_i} (d\lambda - du_i),$$

where A_i is a $\mathfrak{gl}(\mathbb{C}^p)$ -valued meromorphic function on T and $u_i : T \rightarrow \mathbb{C}$ is the i th component of the projection map $T \rightarrow Z_N$.

If we manage to do this then Theorem 2.18 follows immediately. Indeed, the 1st condition implies that $A_i(t^\circ) = A_i^\circ$. While the fact that $A_i(t)$ satisfy the Schlesinger equations follows from the fact that ω is a 1-form satisfying

$$d\omega + \omega \wedge \omega = d(dY Y^{-1}) + dY Y^{-1} \wedge dY Y^{-1} = 0.$$

The matrix-valued function $Y(\lambda, t)$ is constructed by comparing two trivializing frames of E . The first one is the multivalued flat frame

$$f(\lambda, t) = (f_1(\lambda, t), \dots, f_p(\lambda, t)), \quad f_i(\lambda, t) \in E_{\lambda, t},$$

defined by the parallel transport with respect to ∇ with initial value $f(b^\circ, t^\circ) := f^\circ$. Recall that f° is the frame of E_{b°, t° that we fixed so that the monodromy representation of ∇ coincides with the monodromy representation (2.8).

The 2nd frame will be constructed by using Theorem 2.9, which guarantees the existence of a meromorphic trivialization of E . Let us check that the conditions of Theorem 2.9 are satisfied. By definition, $D_{N+1} = \{\infty\} \times T$ and $E|_{D_{N+1}}$ is trivial.

LEMMA 2.19. *The restriction $E|_{\mathbb{P}^1 \times t^\circ}$ is trivial.*

PROOF. We will prove that $f(\lambda, t^\circ)Y^\circ(\lambda)^{-1}$ is a trivializing frame. By definition the monodromy of the frame $f(\lambda, t^\circ)$ and the monodromy of the matrix $Y^\circ(\lambda)^{-1}$ cancel each other. Therefore the above frame provides a trivialization of $E|_{\mathbb{P}^1 \times t^\circ}$ on $\mathbb{C} - \{u_1^\circ, \dots, u_N^\circ\}$. Let us check that the trivialization extends analytically in a neighborhood of $\lambda = u_i^\circ$ for all $1 \leq i \leq N + 1$. Let us assume that $1 \leq i \leq N$. The case $i = N + 1$ is the same but one has to use slightly different notation. By definition the trivializing frame of $E|_{\mathbb{P}^1 \times t^\circ}$ in a neighborhood of $\lambda = u_i^\circ$ is given by

$$f(\lambda, t^\circ)S_i^\circ(\lambda - u_i^\circ)^{-E_i}(\lambda - u_i^\circ)^{-K_i}.$$

However, recalling the definition of S_i° we get that the above frame coincides with

$$f(\lambda, t^\circ)Y^\circ(\lambda)^{-1}U_i^\circ(\lambda).$$

According to Levelt's theorem $U_i^\circ(\lambda)$ is holomorphically invertible at $\lambda = u_i^\circ$. Therefore the frame $f(\lambda, t^\circ)Y^\circ(\lambda)^{-1}$ extends holomorphically and it remains a frame at the point $\lambda = u_i^\circ$. \square

According to Theorem 2.9, there exists an analytic hypersurface $\Theta \subset T$, such that $E|_{\mathbb{P}^1 \times (T - \Theta)}$ is a trivial vector bundle. Let

$$\tilde{e} = (\tilde{e}_1, \dots, \tilde{e}_p), \quad \tilde{e}_i \in \Gamma(\mathbb{P}^1 \times (T - \Theta), E)$$

be a trivializing frame. We may further assume that $\tilde{e}(\lambda, t^\circ) = f(\lambda, t^\circ)Y^\circ(\lambda)^{-1}$. The frame that we need in order to define $Y(\lambda, u)$ is slightly different. The necessary modification is constructed as follows. In the tubular neighborhood T_{N+1} we have

$$f(\lambda, t)S_{N+1}^\circ \lambda^{E_{N+1}} \lambda^{K_{N+1}} = \tilde{e}(\lambda, t)\tilde{U}(\lambda, t), \quad \forall (\lambda, t) \in T_{N+1} - T_{N+1} \cap (\mathbb{P}^1 \times \Theta),$$

where $\tilde{U}(\lambda, t)$ is holomorphic and invertible for all $(\lambda, t) \in T_{N+1} - T_{N+1} \cap (\mathbb{P}^1 \times \Theta)$ and meromorphic along $T_{N+1} \cap (\mathbb{P}^1 \times \Theta)$. The Taylor series expansion at $\lambda = \infty$ yields

$$\tilde{U}(\lambda, t) = \tilde{U}_0(t) + \tilde{U}_1(t)\lambda^{-1} + \tilde{U}_2(t)\lambda^{-2} + \dots,$$

where $\tilde{U}_0(t)$ is holomorphic and invertible for all $t \in T - \Theta$ and meromorphic along Θ . The frame that we need is

$$e(\lambda, t) = \tilde{e}(\lambda, t)\tilde{U}_0(t)^{-1}\tilde{U}_0(t^\circ).$$

Note that the above frame is holomorphic for all $t \in T - \Theta$ and meromorphic along Θ .

Let us define $Y(\lambda, t) \in \text{GL}(\mathbb{C}^p)$ as the transition matrix

$$f(\lambda, t) = e(\lambda, t)Y(\lambda, t), \quad (\lambda, t) \in \mathcal{C} - (\mathcal{C} \cap (\mathbb{P}^1 \times \Theta)).$$

Note that at $t = t^\circ$ we have $Y(\lambda, t^\circ) = Y^\circ(\lambda)$. Therefore, we need to check that the 1-form $\omega = dY Y^{-1}$ has the required properties.

To begin with, note that ω is single valued and analytic. Indeed, the monodromy of $Y(\lambda, t)$ is the same as the monodromy of $f(\lambda, t)$, i.e., under the analytic continuation along a closed loop γ the value of $Y(\lambda, t)$ changes into $Y(\lambda, t)\mu(\gamma)$. However, $\mu(\gamma)$ is independent of λ and t , so the value of ω remains the same. Since being analytic is a local property and locally $Y(\lambda, t)$ is analytic the same is true for ω .

Let us analyze the singularities of ω as a 1-form on $\mathbb{P}^1 \times T$. The possible singular locus is along the following divisors

$$D_i \ (1 \leq i \leq N + 1), \quad \mathbb{P}^1 \times \Theta.$$

Let us fix $t \notin \Theta$ and look in a neighborhood of $\lambda = u_i$ for $1 \leq i \leq N$. We have

$$f(\lambda, t)S_i^\circ(\lambda - u_i)^{-E_i}(\lambda - u_i)^{-K_i} = e(\lambda, u)U_i(\lambda, t),$$

where U_i is holomorphic and invertible for all $(\lambda, t) \in T_i - T_i \cap (\mathbb{P}^1 \times T)$ and meromorphic along $T_i \cap (\mathbb{P}^1 \times T)$. In particular, the Taylor series expansion at $\lambda = u_i$ takes the form

$$U_i(\lambda, t) = U_{i,0}(t) + U_{i,1}(t)(\lambda - u_i) + \dots,$$

where $U_{i,0}(t)$ is holomorphic and invertible for $t \in T - \Theta$ and meromorphic along Θ . Recalling the definition of $Y(\lambda, t)$ we get

$$Y(\lambda, t)S_i^\circ = U_i(\lambda, t)(\lambda - u_i)^{K_i}(\lambda - u_i)^{E_i},$$

where the branch of $Y(\lambda, t)$ is determined by an appropriate reference path (see Section 2.3.2 and the definition of the frame (2.11)). Similarly, at $\lambda = \infty$ we get

$$Y(\lambda, t) = \tilde{U}_0(t^\circ) \tilde{U}_0(t)^{-1} \tilde{U}(t, \lambda) \lambda^{-K_{N+1}} \lambda^{-E_{N+1}}.$$

Put

$$A_i(t, \lambda) := -(\lambda - u_i)(\partial_{u_i} Y(\lambda, t)) Y(\lambda, t)^{-1}.$$

If $t \notin \Theta$ is fixed then A_i is an analytic matrix-valued function on $\mathbb{C} - \{u_1, \dots, u_N\}$. Near $\lambda = u_j$ with $1 \leq j \neq i \leq N$ we get

$$A_i(t, \lambda) = (u_i - u_j)(\partial_{u_i} U_{j,0}(t)) U_{j,0}(t)^{-1} + O(\lambda - u_j),$$

which is analytic in a neighborhood of $\lambda = u_j$. Near $\lambda = u_i$ we get

$$\begin{aligned} A_i(t, \lambda) &= -(\lambda - u_i)(\partial_{u_i} U_i(\lambda, t)) U_i(\lambda, t)^{-1} + \\ &U_i(\lambda, t) \left(K_i + (\lambda - u_i)^{K_i} E_i (\lambda - u_i)^{-K_i} \right) U_i(\lambda, t)^{-1}. \end{aligned}$$

Using the special form of the matrices E_i and K_i we get that the above expression is analytic at $\lambda = u_i$. Finally at $\lambda = \infty$ we have

$$A_i(t, \lambda) = -(\lambda - u_i) \tilde{U}_0(t^\circ) \partial_{u_i} (\tilde{U}_0(t)^{-1} \tilde{U}(t, \lambda)) \tilde{U}(t, \lambda)^{-1} \tilde{U}_0(t) \tilde{U}_0(t^\circ)^{-1},$$

and this again is analytic at $\lambda = \infty$. According to Liouville's theorem $A_i(t, \lambda)$ is independent of λ . Setting $\lambda = u_i$ we get that

$$A_i(t) := A_i(t, \lambda) = U_{i,0}(t) C_i U_{i,0}(t)^{-1},$$

where C_i is a constant upper triangular matrix. Moreover, we get that A_i is meromorphic along Θ .

Similar argument shows that the matrix

$$A(\lambda, t) := (\partial_\lambda Y(\lambda, t)) Y(\lambda, t)^{-1}$$

is holomorphic at $\lambda = \infty$ and equal to 0 at $\lambda = \infty$. While at $\lambda = u_i$ we have

$$A(\lambda, t) = \frac{A_i(t)}{\lambda - u_i} + \dots,$$

where the dots stand for terms analytic at $\lambda = u_i$. This implies that

$$A(\lambda, t) - \sum_{i=1}^N \frac{A_i(t)}{\lambda - u_i}$$

is analytic for all $\lambda \in \mathbb{P}^1$ and vanishing at $\lambda = \infty$. Recalling again Liouville's theorem we get that

$$A(\lambda, t) = \sum_{i=1}^N \frac{A_i(t)}{\lambda - u_i}.$$

Summarizing, we get that

$$\omega = dY Y^{-1} = \sum_{i=1}^N \frac{A_i(t)}{\lambda - u_i} (d\lambda - du_i),$$

where A_i are meromorphic functions on T . This completes the proof of Theorem 2.18. \square

2.4. Painleve property for semi-simple Frobenius manifolds

Let us assume that M is a semi-simple Frobenius manifold. Let $t^\circ \in M$ be a semi-simple point, such that, the canonical coordinates $(u_1(t^\circ), \dots, u_N(t^\circ))$ of t° are pairwise distinct. In particular, $u^\circ := (u_1(t^\circ), \dots, u_N(t^\circ))$ is a point in the configuration space Z_N and the germ of the Frobenius structure at t° determines a Frobenius structure defined in a neighborhood of u° in Z_N . The goal in this section is to prove the following theorem.

THEOREM 2.20. *Let U be a contractible open neighborhood of u° . There exists an analytic hypersurface K , such that, the germ of the Frobenius structure at u° extends to a Frobenius structure on $U \setminus K$.*

The proof of the above theorem is based on the fact that a semi-simple Frobenius manifold can be viewed as a solution to the Schlesinger equations satisfying certain special initial conditions. This result is due to Manin [44]. A precise statement with a proof will be given below in Theorem 2.28. Once this fact is established, Theorem 2.20 becomes an easy consequence of Theorem 2.18.

REMARK 2.21. The main application of Theorem 2.20 will be for the case when U is an open neighborhood of a fixed path in Z_N .

REMARK 2.22. In fact we can say a little bit more. The extended Frobenius structure in Theorem 2.20 is meromorphic in the following sense: in a flat trivialization of the tangent bundle the operators of Frobenius multiplication by ∂_{u_i} are represented by matrices whose entries are *meromorphic functions* on U . In that sense, the extended Frobenius structure is meromorphic and hence it is appropriate to say that semi-simple Frobenius manifolds have the Painleve property.

2.4.1. Second structure connection. Suppose that M is a Frobenius manifold. The hypersurface in $M \times \mathbb{C}$ defined by

$$\{(t, \lambda) \in M \times \mathbb{C} \mid \det(\lambda - E_{\bullet t}) = 0\}$$

is called the *discriminant* of the Frobenius manifold M . The complement of the discriminant will be denoted by $(M \times \mathbb{C})'$. Let $\text{pr}_M : (M \times \mathbb{C})' \rightarrow M$ be the projection $(t, \lambda) \mapsto t$. For each complex number n we define a connection $\nabla^{(n)}$ on the vector bundle $\text{pr}_M^* T_M$, by the following formulas:

$$\begin{aligned} \nabla_v^{(n)} &:= \nabla_v^{L.C.} + (\lambda - E_{\bullet})^{-1} v \bullet \left(\theta - n + \frac{1}{2}\right), \quad v \in \mathcal{T}_M \\ \nabla_{\partial_\lambda}^{(n)} &:= \partial_\lambda - (\lambda - E_{\bullet})^{-1} \left(\theta - n + \frac{1}{2}\right), \end{aligned}$$

where $\partial_\lambda := \frac{\partial}{\partial \lambda}$. The connection $\nabla^{(n)}$ is usually called the *second structure connection*.

REMARK 2.23. Informally, the second structure connection should be thought off as the Laplace transform of the Dubrovin's connection in the following sense: If $I^{(n)}(t, \lambda)$ is a horizontal section of $\nabla^{(n)}$, then the integral

$$J(t, z) := z^{-n-\frac{1}{2}} \int_\Gamma e^{\lambda/z} I^{(n)}(t, \lambda) d\lambda$$

is a horizontal section for the Dubrovin's connection, provided that the path Γ can be chosen such that the integral is convergent and integration by parts works. \square

PROPOSITION 2.24. *The second structure connection $\nabla^{(n)}$ is flat.*

PROOF. This is a straightforward computation, so let us just sketch the main steps leaving the details as an exercise. The problem is local, so let us choose a local flat coordinate system (t_1, \dots, t_N) for $\nabla^{\text{L.C.}}$. The Dubrovin's connection takes the form

$$\nabla_{\partial_{t_i}} = \partial_{t_i} - z^{-1}A_i(t) \quad (1 \leq i \leq N), \quad \nabla_{\partial_z} = \partial_z - z^{-1}\theta + z^{-2}\mathcal{E}(t),$$

where the connection matrices $A_i(t)$ ($1 \leq i \leq N$) and $\mathcal{E}(t)$ pairwise commute, because they represent respectively the operators of Frobenius multiplication by ∂_{t_i} and E and the Frobenius multiplication is commutative. The connection matrix θ is a constant matrix because it represents the grading operator θ . The flatness of the Dubrovin's connection is equivalent to the following relations:

$$(2.12) \quad \partial_{t_i}(A_j) = \partial_{t_j}(A_i) \quad (1 \leq i, j \leq N), \quad \partial_{t_i}(\mathcal{E}) = A_i + [\theta, A_i] \quad (1 \leq i \leq N).$$

The second structure connection takes the form

$$\nabla_{\partial_{t_i}}^{(n)} = \partial_{t_i} - z^{-1}B_i(t, \lambda) \quad (1 \leq i \leq N), \quad B_i(t, \lambda) := -(\lambda - \mathcal{E}(t))^{-1}A_i(t)(\theta - n - 1/2)$$

and

$$\nabla_{\partial_\lambda}^{(n)} = \partial_\lambda - B_0(t, \lambda), \quad B_0(t, \lambda) := -(\lambda - \mathcal{E}(t))^{-1}(\theta - n - 1/2).$$

Using (2.12) we get

$$\partial_{t_j}(B_i) = (\lambda - \mathcal{E})^{-1}(A_j + [\theta, A_j])B_i - (\lambda - \mathcal{E})^{-1}\partial_{t_j}(A_i)(\theta - n - 1/2).$$

Using $\partial_{t_i}(A_j) = \partial_{t_j}(A_i)$ and $A_jB_i = A_iB_j$ ($\because A_i$ and $(\lambda - \mathcal{E})^{-1}$ commute), we get

$$\partial_{t_j}(B_i) - \partial_{t_i}(B_j) = [B_j, B_i],$$

which is precisely the 0-curvature equation $[\nabla_{\partial_{t_i}}^{(n)}, \nabla_{\partial_{t_j}}^{(n)}] = 0$. The remaining 0-curvature equations $[\nabla_{\partial_{t_i}}^{(n)}, \nabla_{\partial_\lambda}^{(n)}] = 0$ are proved similarly. \square

2.4.2. Schlesinger equations. We would like to prove that the second structure connection is a solution to the Schlesinger equations. In particular, it is an isomonodromic family of Fuchsian connections. Suppose that M is a semi-simple Frobenius manifold. Let t° be a generic semi-simple point, such that, the corresponding canonical coordinates $(u_1^\circ, \dots, u_N^\circ)$ are pairwise distinct. Let us fix an open neighborhood U of t° , such that U admits a canonical coordinate system (u_1, \dots, u_N) and $u_i(t) \neq u_j(t)$ for all $t \in U$ and $i \neq j$. Put $H = T_{t^\circ}U$ and let us trivialize the tangent bundle

$$(2.13) \quad TU \cong U \times H \cong U \times \mathbb{C}^N$$

using the Levi-Civita connection. In other words, we fix a basis $\{\phi_a\}_{a=1}^N$ of H and let $\partial_{t_a} \in \mathcal{T}_U$ be the flat vector field on U obtained by parallel transport with respect to the Levi-Civita connection. Then the isomorphisms (2.13) are given by the map

$$(u, v) \in TU \mapsto (u, v_1\phi_1 + \dots + v_N\phi_N) \in U \times H \mapsto (u, v_1, \dots, v_N) \in U \times \mathbb{C}^N,$$

where $v \in T_uU$ and $v =: v_1\partial_{t_1} + \dots + v_N\partial_{t_N}$. The isomorphism (2.13) identifies the structure connection of the Frobenius structure with the flat connection on the trivial bundle

$$(U \times \mathbb{C}^*) \times \mathbb{C}^N \rightarrow U \times \mathbb{C}^*$$

defined by

$$\begin{aligned}\nabla_{\partial_{u_i}} &= \partial_{u_i} - z^{-1}P_i(u), \quad 1 \leq i \leq N, \\ \nabla_{\partial_z} &= \partial_z - z^{-1}\theta + z^{-2}\mathcal{E}(u),\end{aligned}$$

where $P_i : U \rightarrow \mathfrak{gl}(\mathbb{C}^N)$ is a holomorphic map whose (a, b) -entry $P_{iab}(u)$ is defined by the identity

$$\partial_{u_i} \bullet \partial_{t_b} = \sum_{a=1}^N P_{iab}(u) \partial_{t_a},$$

$\mathcal{E} = \sum_{i=1}^N u_i P_i(u)$, and θ is a constant matrix whose (a, b) -entry θ_{ab} is defined by

$$\theta(\partial_{t_b}) = [\partial_{t_b}, E] - (1 - D/2)\partial_{t_b} =: \sum_{a=1}^N \theta_{ab} \partial_{t_a}.$$

The second structure connection takes the following form:

$$\begin{aligned}\nabla_{\partial_{u_i}}^{(n)} &= \partial_{u_i} + (\lambda - \mathcal{E})^{-1}P_i(u)(\theta - n - 1/2), \quad 1 \leq i \leq N, \\ \nabla_{\partial_\lambda}^{(n)} &= \partial_\lambda - (\lambda - \mathcal{E})^{-1}(\theta - n - 1/2).\end{aligned}$$

This is a connection on

$$(U \times \mathbb{C})' \times \mathbb{C}^N \rightarrow (U \times \mathbb{C})',$$

where

$$(U \times \mathbb{C})' = \{(u, \lambda) \in U \times \mathbb{C} \mid (\lambda - u_1) \cdots (\lambda - u_N) \neq 0\}.$$

LEMMA 2.25. *Let $\tilde{\Psi}$ be the matrix whose (a, i) -entry is given by $\tilde{\Psi}_{ai} = \partial t_a / \partial u_i$. Then*

$$\tilde{\Psi}^{-1}P_i\tilde{\Psi} = E_{ii}, \quad \tilde{\Psi}^{-1}\mathcal{E}\tilde{\Psi} = \text{diag}(u_1, \dots, u_N),$$

where E_{ii} is the matrix whose entry in position (i, i) is 1 and all other entries are 0.

PROOF. We have

$$\partial_{u_i} \bullet \partial_{t_b} = \partial_{u_i} \bullet \sum_{j=1}^N \frac{\partial u_j}{\partial t_b} \partial_{u_j} = \frac{\partial u_i}{\partial t_b} \partial_{u_i} = \sum_{a=1}^N \frac{\partial u_i}{\partial t_b} \frac{\partial t_a}{\partial u_i} \partial_{t_a}.$$

Therefore

$$P_{iab} = \frac{\partial u_i}{\partial t_b} \frac{\partial t_a}{\partial u_i}.$$

Using this formula we find that the (a, j) -entry of $P_i\tilde{\Psi}$ is

$$\sum_{b=1}^N \frac{\partial u_i}{\partial t_b} \frac{\partial t_a}{\partial u_i} \tilde{\Psi}_{bj} = \sum_{b=1}^N \frac{\partial u_i}{\partial t_b} \frac{\partial t_a}{\partial u_i} \frac{\partial t_b}{\partial u_j} = \delta_{ij} \frac{\partial t_a}{\partial u_i} = \delta_{ij} \tilde{\Psi}_{aj}.$$

The latter is precisely the (a, j) -entry of $\tilde{\Psi}E_{ii}$. Therefore $P_i\tilde{\Psi} = \tilde{\Psi}E_{ii}$. \square

PROPOSITION 2.26. *Let $n \in \mathbb{C}$ be arbitrary. Then the matrix-valued functions*

$$A_i^{(n)}(u) := P_i(u)(\theta - n - 1/2), \quad 1 \leq i \leq N,$$

satisfy the Schlesinger equations.

PROOF. We have to prove that the connection

$$\begin{aligned}\nabla_{\partial_{u_i}}^{\text{Schl}} &= \partial_{u_i} + \frac{A_i^{(n)}(u)}{\lambda - u_i}, \quad 1 \leq i \leq N \\ \nabla_{\partial_\lambda}^{\text{Schl}} &= \partial_\lambda - \sum_{i=1}^N \frac{A_i^{(n)}(u)}{\lambda - u_i}\end{aligned}$$

is flat. However, using Lemma 2.25 we get

$$(\lambda - \mathcal{E})^{-1} P_i(\theta - n - 1/2) = \frac{A_i^{(n)}(u)}{\lambda - u_i}.$$

Therefore $\nabla^{\text{Schl}} = \nabla^{(n)}$, so it remains only to recall Proposition 2.24. \square

2.4.3. Special initial conditions. In this section we prove a result due to Manin [44], which answers the following question: what kind of initial conditions for the Schlesinger equations determine semi-simple Frobenius structures. Following Manin, we give the following definition.

DEFINITION 2.27. Let H be a vector space equipped with a non-degenerate symmetric bi-linear pairing $(\ , \)$ and a distinguished vector $e \in H$. Suppose also that we have a set of linear operators $\theta, \{P_i^\circ\}_{i=1}^N \in \mathfrak{gl}(H)$. The data $(H, (\ , \), e, \theta, \{P_i^\circ\}_{i=1}^N)$ is said to be a *special initial condition* if the following conditions are satisfied:

- (1) θ is skew-symmetric: $(\theta(a), b) + (a, \theta(b)) = 0$ for all $a, b \in H$.
- (2) e is an eigenvector of θ with eigenvalue $D/2$.
- (3) The set $\{P_i^\circ\}_{i=1}^N$ is a complete set of orthogonal projectors of H , that is,
 - (a) $P_i^\circ P_j^\circ = \delta_{ij} P_j^\circ$ for all $1 \leq i, j \leq N$.
 - (b) $P_1^\circ + \cdots + P_N^\circ = 1$.
 - (c) $(P_i^\circ(a), b) = (a, P_i^\circ(b))$ for all $1 \leq i \leq N$ and for all $a, b \in H$.
 - (d) $P_i^\circ e \neq 0$ for all $1 \leq i \leq N$. \square

Suppose that $((\ , \), \bullet, e, E)$ is a semi-simple Frobenius structure on some complex manifold M and that $t^\circ \in M$ is a semi-simple point, such that the canonical coordinates $u_i \neq u_j$ for $i \neq j$ in a sufficiently small neighborhood of t° . The data

$$H := T_{t^\circ} M, (\ , \), e, \theta := \nabla^{\text{L.C.}} E - (1 - D/2), P_i^\circ = P_i(u^\circ), \quad 1 \leq i \leq N,$$

is a special initial condition. In fact, the only property that we did not check yet is that e is an eigenvector of θ . However

$$\theta(e) = [e, E] - (1 - D/2)e = e - (1 - D/2)e = (D/2)e,$$

where in the first equality we used that e is flat and in the second equality we used that $e = \sum_i \partial_{u_i}$ and $E = \sum_i u_i \partial_{u_i}$. Let

$$Z_N = \{u \in \mathbb{C}^N \mid u_i \neq u_j \text{ for } i \neq j\}$$

be the configuration space of N points on the complex line \mathbb{C} .

THEOREM 2.28. *Suppose that $(H, (\ , \), e, \theta, \{P_i^\circ\}_{i=1}^N)$ is a special initial condition and that $u^\circ \in Z_N$ is any point. Let $U \subset Z_N$ be a contractible open neighborhood of u° , such that, the Schlesinger equations admit a solution in U satisfying the special initial condition at u° . Then there exists an analytic hypersurface $K \subset U \setminus \{u^\circ\}$ and an isomorphism*

$T_{u^\circ}U \cong H$, such that, the special initial condition is obtained from a uniquely determined semi-simple Frobenius structure on $U \setminus K$.

PROOF. Let $A_i^{(n)}(u)$, $1 \leq i \leq N$ be solutions to the Schlesinger equations such that

$$A_i^{(n)}(u^\circ) = P_i^\circ(\theta - n - 1/2).$$

If $n + \frac{1}{2}$ is not an eigenvalue of θ , then we define

$$P_i^{(n)}(u) = A_i^{(n)}(u)(\theta - n - 1/2)^{-1}.$$

LEMMA 2.29. *The set $\{P_i^{(n)}(u)\}_{i=1}^N$ is a complete set of orthogonal projections for all u sufficiently close to u° .*

PROOF. Let us fix a basis $\{\phi_i\}_{i=1}^N$ of H and identify $\mathfrak{gl}(H)$ with the space of $p \times p$ -matrices. Let \mathcal{A} be the polynomial ring

$$\mathcal{A} = \mathbb{C}[(u_i - u_j)^{\pm 1} : 1 \leq i < j \leq N] \otimes \mathbb{C}[A_1, \dots, A_N],$$

where $A_i = (A_{iab})_{a,b=1}^N$ are matrix variables. We define derivations $\partial_{u_1}, \dots, \partial_{u_N}$ of \mathcal{A} such that

$$\begin{aligned} \partial_{u_i} A_j &:= \frac{[A_i, A_j]}{u_i - u_j}, \quad 1 \leq i \neq j \leq N, \\ (\partial_{u_1} + \dots + \partial_{u_N}) A_j &:= 0, \end{aligned}$$

and if $f \in \mathcal{A}$ depends only on u_1, \dots, u_N then ∂_{u_i} is defined to be the usual derivative. It is easy to check that these differentiations pairwise commute, so \mathcal{A} becomes a \mathcal{D} -module for the ring \mathcal{D} of differential operators on Z_N .

Let us define $\mathcal{I} \subset \mathcal{A}$ to be the ideal generated by the relations corresponding to conditions (a)–(c) in Definition 2.27. More precisely, we replace P_i° by $A_i(\theta - n - 1/2)^{-1}$ and take the entries of the corresponding matrix identities as generators of \mathcal{I} . Condition (a) yields generators given by the entries of

$$R_{ij}(A_1, \dots, A_N) = A_i(\theta - n - 1/2)^{-1} A_j - \delta_{ij} A_j, \quad 1 \leq i, j \leq N.$$

Condition (b) gives the entries of

$$R(A_1, \dots, A_N) = A_1 + \dots + A_N - \theta + n + \frac{1}{2}.$$

Finally, condition (c) gives the entries of

$$R_i(A_1, \dots, A_N) = A_i(\theta - n - 1/2)^{-1} + (\theta + n + 1/2)^{-1} A_i^T, \quad 1 \leq i \leq N,$$

where T is the transposition operation in $\mathfrak{gl}(H)$ with respect to the pairing (\cdot, \cdot) .

We claim that in order to prove the lemma it is enough to check that \mathcal{I} is \mathcal{D} -invariant. Indeed, condition (a) in Definition 2.27 will be satisfied if

$$R_{ij}(A_1^{(n)}(u), \dots, A_N^{(n)}(u)) = 0.$$

On the other hand, the Taylor series expansion of $R_{ij}(A_1^{(n)}(u), \dots, A_N^{(n)}(u))$ at $u = u^\circ$ has the form

$$\sum_{m_1, \dots, m_N=0}^{\infty} \left(\frac{\partial_{u_1}^{m_1}}{m_1!} \dots \frac{\partial_{u_N}^{m_N}}{m_N!} R_{ij} \right) (A_1^{(n)}(u^\circ), \dots, A_N^{(n)}(u^\circ)) (u_1 - u_1^\circ)^{m_1} \dots (u_N - u_N^\circ)^{m_N},$$

where we used that $A_i^{(n)}(u)$ solve the Schlesinger equations, so the evaluation maps $A_i \mapsto A_i^{(n)}(u)$ are \mathcal{D} -equivariant. It remains only to notice that all Taylor's coefficients must vanish, because $P_i^{(n)}(u^\circ) = P_i^\circ$ form a complete system of orthogonal projections, so the evaluation $R(A_1^{(n)}(u^\circ), \dots, A_N^{(n)}(u^\circ)) = 0$ for all generators R of \mathcal{I} and hence for all $R \in \mathcal{I}$.

Let us check that \mathcal{I} is \mathcal{D} -invariant. We will prove only that $\partial_{u_k} R_{ij} \in \mathcal{I}$ because the remaining cases can be dealt in the same way. It is more convenient to prove that

$$dR_{ij} := \sum_{k=1}^N \partial_{u_k} R_{ij} \otimes du_k \in \mathcal{I} \otimes \Omega^1(Z_N),$$

where $\Omega^1(Z_N)$ denotes the ring of holomorphic 1-forms on Z_N . By definition dR_{ij} is

$$\begin{aligned} & \sum_{k:k \neq i} \frac{[A_k, A_i]}{u_k - u_i} (\theta - n - 1/2)^{-1} A_j \otimes (du_k - du_i) + \\ & + \sum_{k:k \neq j} A_i (\theta - n - 1/2)^{-1} \frac{[A_k, A_j]}{u_k - u_j} \otimes (du_k - du_j) + \\ & - \delta_{ij} \sum_{k:k \neq j} \frac{[A_k, A_j]}{u_k - u_j} \otimes (du_k - du_j) \end{aligned}$$

On the other hand

$$[A_k, A_i] (\theta - n - 1/2)^{-1} A_j = \delta_{ij} A_k A_j - \delta_{kj} A_i A_j \pmod{\mathcal{I}}$$

and

$$A_i (\theta - n - 1/2)^{-1} [A_k, A_j] = \delta_{ik} A_k A_j - \delta_{ij} A_j A_k \pmod{\mathcal{I}}$$

Therefore modulo terms in \mathcal{I} the differential dR_{ij} coincides with the sum of the following 2 terms

$$\delta_{ij} \left(A_k A_j \otimes \frac{du_k - du_i}{u_k - u_i} - A_j A_k \otimes \frac{du_k - du_j}{u_k - u_j} - [A_k, A_j] \otimes \frac{du_k - du_j}{u_k - u_j} \right)$$

and

$$-\delta_{kj} A_i A_j \otimes \frac{du_k - du_i}{u_k - u_i} + \delta_{ik} A_k A_j \otimes \frac{du_k - du_j}{u_k - u_j}.$$

Both terms vanish, which proves that the entries of dR_{ij} are in $\mathcal{I} \otimes \Omega^1(Z_N)$.

This completes the proof that the set $\{P_i^{(n)}(u)\}_{i=1}^N$ satisfies conditions (a)–(c) in Definition 2.27. The last condition (d) will be satisfied for all u sufficiently close to u° , because $P_i^{(n)}(u)$ is continuous and $P_i^{(n)}(u^\circ)e = P_i^\circ e \neq 0$. \square

LEMMA 2.30. *If $n + \frac{1}{2}$ and $m + \frac{1}{2}$ are not eigenvalues of θ , then $P_i^{(m)}(u) = P_i^{(n)}(u)$.*

PROOF. According to Lemma 2.29 the matrices $P_i^{(n)}(u)$ pairwise commute. Using that $A_i^{(n)}(u)$ satisfy the Schlesinger equations we get

$$dP_i^{(n)}(u) = \sum_{j:j \neq i} \frac{du_j - du_i}{u_j - u_i} \left(P_j^{(n)}(u) \theta P_i^{(n)}(u) - P_i^{(n)}(u) \theta P_j^{(n)}(u) \right).$$

Using these equations and the fact that $P_i^{(n)}(u)$ pairwise commute we get that the matrix-valued functions $\tilde{A}_i^{(n)}(u) := P_i^{(m)}(u)\left(\theta - n - \frac{1}{2}\right)$ ($1 \leq i \leq N$) satisfy the Schlesinger equations. However the initial condition $\tilde{A}_i^{(n)}(u^\circ) = A_i^{(n)}(u^\circ)$. Therefore $\tilde{A}_i^{(n)}(u) = A_i^{(n)}(u)$. \square

According to Lemma 2.30 the matrices $P_i(u) := P_i^{(n)}(u)$ are independent of n , while Lemma 2.29 implies that they form a complete system of orthogonal projections.

LEMMA 2.31. *The 1-form*

$$\sum_{i=1}^N \eta_i(u) du_i, \quad \eta_i(u) := (P_i(u)e, e), \quad 1 \leq i \leq N,$$

defines a Frobenius structure on $U \setminus K$, where K is the analytic hypersurface in U defined by $\eta_1 \cdots \eta_N = 0$. Moreover, we have $u^\circ \notin K$.

PROOF. Let us first check that the above 1-form is closed. We have

$$\eta_{ij}(u) := \partial_{u_j} \eta_i = \partial_{u_j} (P_i(u)e, e) = \frac{2}{D-1-2n} (\partial_{u_j} A_i^{(n)}(u)e, e),$$

where we used that $P_i(u) = A_i^{(n)}(u) (\theta - n - 1/2)^{-1}$ and that $\theta(e) = (D/2)e$. We have to prove that $\eta_{ij}(u) = \eta_{ji}(u)$. Let us assume that $i \neq j$. Since $A_i^{(n)}(u)$ ($1 \leq i \leq N$) satisfy the Schlesinger equations we get

$$\partial_{u_j} A_i^{(n)} = \frac{[A_j, A_i]}{u_j - u_i} = \partial_{u_i} A_j^{(n)},$$

which implies that $\eta_{ij} = \eta_{ji}$, so the 1-form is closed. To complete the proof we have to check that the 4 conditions of Theorem 1.6 are satisfied.

The first condition is satisfied by definition. We need only to check that $u^\circ \notin K$. Note that the vectors $P_i^\circ e$ ($1 \leq i \leq N$) form a basis of H . Indeed, if $\sum_i \alpha_i P_i^\circ e = 0$, then applying to both sides P_i° we get $\alpha_i P_i^\circ e = 0$. By assumption $P_i^\circ e \neq 0$, so $\alpha_i = 0$. The matrix of the form $(\ , \)$ is diagonal in the basis $P_i^\circ e$ with diagonal entries $\eta_i(u^\circ)$. Therefore $\eta_i(u^\circ) \neq 0$ for all i , that is, $u^\circ \notin K$.

The second condition that we have to check is $e\eta_i = 0$. This follows from the fact that

$$\sum_{j=1}^N \eta_j(u) = \left(\sum_{j=1}^N P_j(u)e, e \right) = (e, e)$$

is a constant independent of u .

The third condition that we have to check is $E\eta_i = -D\eta_i$. We have (see above)

$$\eta_i(u) = \frac{2}{D-1-2n} (A_i^{(n)}(u)e, e).$$

Note that

$$EA_i^{(n)}(u) = \iota_E dA_i^{(n)}(u) = \iota_E \sum_{j:j \neq i} \frac{du_j - du_i}{u_j - u_i} [A_j^{(n)}(u), A_i^{(n)}(u)] = [\theta, A_i^{(n)}(u)],$$

where in the second equality we used the Schlesinger equations and in the third one we used that

$$\sum_{j=1}^N A_j^{(n)}(u) = \sum_{j=1}^N P_j(u)(\theta - n - 1/2) = \theta - n - 1/2.$$

Therefore

$$E\eta_i = \frac{2}{D-1-2n}([\theta, A_i^{(n)}(u)]e, e).$$

It remains only to use that $\theta(e) = (D/2)e$ and that θ is skew-symmetric with respect to the pairing.

Finally, the last condition that we have to check is

$$(2.14) \quad \frac{\partial \eta_{ij}}{\partial u_k} = \frac{1}{2} \left(\frac{\eta_{ik}\eta_{jk}}{\eta_k} + \frac{\eta_{ji}\eta_{ki}}{\eta_i} + \frac{\eta_{kj}\eta_{ij}}{\eta_j} \right), \quad k \neq i \neq j \neq k.$$

Let us explain how to express the LHS as a quadratic expression in the functions η_{ab} . Recall that we have the following differential equation

$$\partial_{u_j} P_i = \frac{1}{u_j - u_i} (P_j \theta P_i - P_i \theta P_j).$$

Using the above differential equations and the fact that the operators P_a are self-adjoint and θ is skew symmetric with respect to (\cdot, \cdot) we get

$$(2.15) \quad \eta_{ij} = (\partial_{u_j} P_i(u)e, e) = \frac{2}{u_i - u_j} (P_i(u)e, \theta P_j(u)e).$$

The derivative $\partial_{u_k} \eta_{ij}$ becomes

$$\frac{2}{u_i - u_j} \left(\frac{(P_k \theta P_i e, \theta P_j e)}{u_k - u_i} - \frac{(P_k \theta P_j e, \theta P_i e)}{u_k - u_j} + \frac{(P_i \theta P_k e, \theta P_j e)}{u_i - u_k} - \frac{(P_j \theta P_k e, \theta P_i e)}{u_j - u_k} \right).$$

Using the projection formula $P_i x = (x, P_i e) \frac{P_i e}{\eta_i}$ we get

$$(2.16) \quad \frac{(P_k \theta P_i e, \theta P_j e)}{u_k - u_i} = (\theta P_i e, P_k e)(P_k e, \theta P_j e) \frac{1}{\eta_k} = \frac{\eta_{ik}\eta_{jk}}{4\eta_k} (u_k - u_j).$$

Similar formulas hold for the remaining 3 terms above, so for the derivative $\partial_{u_k} \eta_{ij}$ we get

$$\frac{2}{u_i - u_j} \left(\frac{\eta_{ik}\eta_{jk}}{4\eta_k} (u_k - u_j) - \frac{\eta_{ik}\eta_{jk}}{4\eta_k} (u_k - u_i) + \frac{\eta_{ki}\eta_{ji}}{4\eta_i} (u_i - u_j) - \frac{\eta_{kj}\eta_{ij}}{4\eta_j} (u_j - u_i) \right).$$

The above expression is precisely the RHS of (2.14). \square

The proof of the theorem can be completed as follows. Let us define the isomorphisms

$$T_{u^\circ} U \cong H, \quad \partial_{u_i} \mapsto P_i^\circ e,$$

where slightly abusing the notation we have denoted by ∂_{u_i} the tangent vector in $T_{t^\circ} U$ representing the value of the coordinate vector field ∂_{u_i} at u° . We claim that the special initial condition corresponding to the Frobenius structure defined by Lemma 2.31 coincides with the given special initial condition. The easiest way to see this is if we fix the basis of H to be $\phi_i = P_i^\circ e$. Then for the given special initial condition we have: the matrix P_j° is E_{jj} (the matrix with 1 on place (j, j) and 0 elsewhere), the matrix of

the pairing $(\ , \)$ is diagonal with diagonal entries $(P_i^\circ e, e) = \eta_i(u^\circ)$, the vector e has coordinates $(1, \dots, 1)$, and the matrix of θ becomes (see formula (2.15))

$$\theta_{ij} = (u_i^\circ - u_j^\circ) \frac{\eta_{ij}(u^\circ)}{2\eta_i(u^\circ)}, \quad 1 \leq i, j \leq N.$$

Comparing with the special initial condition corresponding to the Frobenius structure we see that the only thing left to prove is that the grading operator $\tilde{\theta}|_{T_{u^\circ}U}$ coincides with θ . Let us compute the matrix of $\tilde{\theta}$ in canonical coordinates. Note that $\tilde{\theta}_{ij} = 0$ for $i = j$ due to skew-symmetry. Let us assume that $i \neq j$. Then

$$\tilde{\theta}_{ij}(u)\eta_i(u) = (\partial_{u_i}, \nabla_{\partial_{u_j}}^{L.C.} E) = \partial_{u_j}(\partial_{u_i}, E) - \sum_{k=1}^N \Gamma_{ij}^k(u)(\partial_{u_k}, E),$$

where Γ_{ij}^k are the Christoffel's symbols of the Frobenius pairing. Recalling the formulas for the Christoffel's symbols (see Step 1 in the proof of Theorem 1.6) we get

$$(2.17) \quad \tilde{\theta}_{ij}(u)\eta_i(u) = (u_i - u_j) \frac{\eta_{ij}(u)}{2} \quad \Rightarrow \quad \tilde{\theta}_{ij}(u) = (u_i - u_j) \frac{\eta_{ij}(u)}{2\eta_i(u)}.$$

Restricting to $u = u^\circ$ we get that $\tilde{\theta}(u^\circ) = \theta$.

Finally, in order to prove that the Frobenius structure on $U \setminus K$ is uniquely determined, we need only to use that the solution of Schlesinger equation satisfying the given (special) initial condition is unique. \square

Vertex operators

The goal of this chapter is to define a certain set of vertex operators associated with any semi-simple Frobenius manifold and to establish some of their properties. The construction was initiated by Givental in [22] and was developed further in the sequence of works [24, 16, 8, 46]. The main application that we have in mind is the construction of Hirota Quadratic Equations (HQEs) for the total descendent potential. Although such an application is known in some very special cases, the methods developed so far should be sufficient for the most general case too. The problem of whether HQEs exist in general can be formulated as a problem in the settings of lattice VertexOperator Algebras (VOAs).

3.1. HQEs for the KdV hierarchy

Let us start with the computation of Givental in [22] that motivated the definition and study of vertex operators for semi-simple Frobenius manifolds.

Recall that a formal power series $\tau \in \mathbb{C}[[y_1, y_2, \dots]]$ is said to be a tau-function of the Kadomtsev–Petviashvili (KP) hierarchy if the following condition is satisfied:

$$(3.1) \quad \text{Res}_{\zeta=\infty} \left(\Gamma^+(\zeta) \otimes \Gamma^-(\zeta) \right) \tau \otimes \tau = 0,$$

where

$$\Gamma^\pm(\zeta) = \exp \left(\pm \sum_{n=1}^{\infty} y_n \zeta^n \right) \exp \left(\mp \sum_{n=1}^{\infty} \frac{\zeta^{-n}}{n} \partial_{y_n} \right)$$

are vertex operators and the residue is interpreted formally as the coefficient in front of $-z^{-1}$. The standard interpretation of the above equation is to make a substitution

$$(3.2) \quad x_n = \frac{1}{2}(y'_n - y''_n), \quad t_n = \frac{1}{2}(y'_n + y''_n),$$

where $y'_n := y_n \otimes 1$ and $y''_n := 1 \otimes y_n$ are two copies of the variables of the tau-function. Then (3.1) can be expanded into a Taylor series in x_1, x_2, \dots whose coefficients are quadratic expressions in the partial derivatives of $\tau(t_1, t_2, \dots)$. In other words, (3.1) is equivalent to an infinite system of Partial differential equations, which are usually called *Hirota Bilinear Equations* (HBEs).

We will interpret (3.1) in a slightly different way. Namely, let us expand the expression $\Gamma^+(\zeta) \otimes \Gamma^-(\zeta) \tau(y') \tau(y'')$ as a formal power series in $y' = (y'_1, y'_2, \dots)$ and $y'' = (y''_1, y''_2, \dots)$ with coefficients formal Laurent series in ζ^{-1} , whose coefficients are quadratic polynomials in the Taylor coefficients of $\tau(y_1, y_2, \dots)$. Then (3.1) is equivalent to an infinite system of algebraic equations, which we will refer to as *Hirota Quadratic Equations* (HQEs).

The KP hierarchy was studied quite extensively by many people. We will not give a historical overview here, since this is quite a difficult task. Nevertheless, let us point out that the above approach to the KP hierarchy is due to M. Sato (see [56]). He proved that

the solutions to the KP hierarchy can be parametrized by the points of an infinite Grassmannian, which is now called the *Sato Grassmannian*. The HQEs of the KP hierarchy defined above coincide with the Plücker relations defining the Plücker embedding of the Sato Grassmannian in a projective space (see [56]). Sato's result motivated a series of works investigating the role of affine Lie algebras in the theory of integrable hierarchies. In particular, the most general construction based on Sato's idea was obtained by Kac and Wakimoto (see [40]).

There are many integrable hierarchies that can be obtained as a reduction of the KP hierarchy. The example that we would like to discuss is the *Korteweg-de Vries* (KdV) hierarchy. The tau-functions τ of the KdV hierarchy are tau-functions of the KP hierarchy independent of the even variables y_2, y_4, \dots , that is, $\tau = \tau(y_1, y_3, \dots)$. Let us decompose the vertex operators $\Gamma^\pm(\zeta) = \Gamma_{\text{odd}}^\pm(\zeta)\Gamma_{\text{ev}}^\pm(\zeta)$, where

$$\Gamma_{\text{odd}}^\pm(\zeta) = \exp\left(\pm \sum_{k=0}^{\infty} y_{2k+1} \zeta^{2k+1}\right) \exp\left(\mp \sum_{k=0}^{\infty} \frac{\zeta^{-2k-1}}{2k+1} \partial_{y_{2k+1}}\right)$$

and $\Gamma_{\text{ev}}^\pm(\zeta)$ is the remaining part of the vertex operator involving only the even variables y_{2k} ($k \geq 1$). If τ is a tau-function of the KdV hierarchy, then (3.1) takes the following form

$$(3.3) \quad \text{Res}_{\zeta=\infty} e^{\sum_{k=1}^{\infty} (y'_{2k} - y''_{2k}) \zeta^{2k}} \left(\Gamma_{\text{odd}}^+(\zeta) \otimes \Gamma_{\text{odd}}^-(\zeta) \right) \tau(y'_1, y'_3, \dots) \tau(y''_1, y''_3, \dots) = 0.$$

Note that (3.3) is equivalent to

$$(3.4) \quad \text{Res}_{\zeta=\infty} \zeta^{2k} \left(\Gamma_{\text{odd}}^+(\zeta) \otimes \Gamma_{\text{odd}}^-(\zeta) \right) \tau(y'_1, y'_3, \dots) \tau(y''_1, y''_3, \dots) = 0$$

for all $k \geq 0$. In other words, the coefficients in front of ζ^{-2k-1} ($k \geq 0$) in $\Gamma_{\text{odd}}^+(\zeta) \otimes \Gamma_{\text{odd}}^-(\zeta) \tau \otimes \tau$ vanish. On the other hand, there is a natural way to extract the odd coefficients of a Laurent series $f(\zeta)$, that is, $\frac{1}{2}(f(\zeta) - f(-\zeta))$ is the series obtained from $f(\zeta)$ by truncating all terms involving even powers of ζ . Therefore, (3.4) is equivalent to saying that the following expression does not have negative powers of ζ :

$$(3.5) \quad \frac{1}{\zeta} \left(\Gamma_{\text{odd}}^+(\zeta) \otimes \Gamma_{\text{odd}}^-(\zeta) - \Gamma_{\text{odd}}^-(\zeta) \otimes \Gamma_{\text{odd}}^+(\zeta) \right) \tau(y'_1, y'_3, \dots) \tau(y''_1, y''_3, \dots),$$

where we used that $\Gamma_{\text{odd}}^\pm(-\zeta) = \Gamma_{\text{odd}}^\mp(\zeta)$. Note that if τ is an arbitrary power series in y_1, y_3, \dots , then (3.5) takes values in $\mathbb{C}((\zeta^{-2}))[[y', y'']]$. The HQEs of KdV are equivalent to the condition that (3.5) belongs to $\mathbb{C}[\zeta^2][[y', y'']]$.

Let us transform (3.5) having in mind the applications to the Witten–Kontsevich tau-function (see Section 1.6.1). Namely, put $q_k = \sqrt{\hbar}(2k+1)!! y_{2k+1}$ and $\lambda = \zeta^2/2$.

Then the vertex operators $\Gamma_{\text{odd}}^\pm(\zeta) = \Gamma_{\text{pt}}^{\pm \frac{1}{2}}(\lambda)$, where

$$(3.6) \quad \Gamma_{\text{pt}}^c(\lambda) = \exp\left(2c \sum_{k=0}^{\infty} \frac{(2\lambda)^{k+\frac{1}{2}}}{(2k+1)!!} \frac{q_k}{\sqrt{\hbar}}\right) \exp\left(-2c \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2\lambda)^{k+\frac{1}{2}}} \sqrt{\hbar} \frac{\partial}{\partial q_k}\right),$$

where $c \in \mathbb{C}$ is a complex number. Therefore, we can reformulate Witten's conjecture proved by Kontsevich as follows. The total descendent potential of a point $\mathcal{D}_{\text{pt}}(\hbar, \mathbf{q})$ satisfies the HQEs of KdV, that is, the expression

$$(3.7) \quad \frac{1}{\sqrt{\lambda}} \left(\Gamma_{\text{pt}}^{\frac{1}{2}}(\lambda) \otimes \Gamma_{\text{pt}}^{-\frac{1}{2}}(\lambda) - \Gamma_{\text{pt}}^{-\frac{1}{2}}(\lambda) \otimes \Gamma_{\text{pt}}^{\frac{1}{2}}(\lambda) \right) \mathcal{D}_{\text{pt}}(\hbar, \mathbf{q}') \mathcal{D}_{\text{pt}}(\hbar, \mathbf{q}'')$$

takes values in $\mathbb{C}[\lambda][[\mathbf{q}', \mathbf{q}'']]$.

The vertex operators (3.6) have a very nice interpretation in terms of Givental's quantization formalism (see Section 1.4) applied to the symplectic loop space $\mathcal{H} := \mathbb{C}((z^{-1}))$. The differential operator in the exponent is a quantization of the linear Hamiltonian

$$2c \sum_{k=0}^{\infty} \frac{(2\lambda)^{k+\frac{1}{2}}}{(2k+1)!!} q_k - 2c \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2\lambda)^{k+\frac{1}{2}}} p_k.$$

Recalling the trivialization of the tangent bundle $T\mathcal{H} \cong \mathcal{H} \times \mathcal{H}$ in which $\frac{\partial}{\partial q_k} \mapsto z^k$ and $\frac{\partial}{\partial p_k} \mapsto (-z)^{-k-1}$, we get that the corresponding Hamiltonian vector field is

$$(3.8) \quad \mathbf{f}^c(\lambda, z) := - \sum_{n \in \mathbb{Z}} I_c^{(n)}(\lambda) (-z)^n,$$

where $I_c^{(n+1)}(\lambda) = \partial_\lambda I_c^{(n)}(\lambda)$ and $I_c^{(0)}(\lambda) = 2c(2\lambda)^{-1/2}$. The function $I_c^{(0)}(\lambda)$ is the period of A_1 -singularity! Namely, let $f(x) := \frac{x^2}{2}$ and $\omega = dx$ be the standard holomorphic volume form on $Z := \mathbb{C}$. The fiber $Z_\lambda = f^{-1}(\lambda)$ consists of two points $x_\pm(\lambda) = \pm\sqrt{2\lambda}$. The 0-dimensional cycle $\varphi_\lambda := [x_+(\lambda)] - [x_-(\lambda)] \in \widetilde{H}^0(X_\lambda; \mathbb{Z})$ is a vanishing cycle and we have the following 0-dimensional period integral corresponding to the cycle $c\varphi$:

$$\int_{c\varphi_\lambda} \frac{\omega}{df} = \int_{c\varphi_\lambda} \frac{1}{x} = \frac{c}{x_+(\lambda)} - \frac{c}{x_-(\lambda)} = \frac{2c}{\sqrt{2\lambda}} = I_c^{(0)}(\lambda).$$

It turns out that the above interpretation of the coefficients of the vertex operators as periods generalizes to any singularity and even more generally to any semi-simple Frobenius manifold. The resulting vertex operators have very interesting applications to representation theory of vertex algebras and to integrable hierarchies.

3.2. Periods of a semi-simple Frobenius manifold

We would like to define the periods of a semi-simple Frobenius manifold as solutions to the second structure connection having a certain normal form in a neighborhood of the singularity at infinity. Before going into the details, let us specify the settings in which we would like to work. Let $(M, \bullet, (\ , \), e, E)$ be a semi-simple Frobenius manifold (see Definitions 1.1 and 1.4). For simplicity, let us assume that M is simply connected. Let $t^\circ \in M \setminus \mathcal{K}$ be a semi-simple point and $t = (t_1, \dots, t_N)$ be flat coordinates defined in a neighborhood of t° . Since M is simply connected, the flat vector fields $\phi_a := \frac{\partial}{\partial t_a}$ ($1 \leq a \leq N$) extend to global vector fields on M and they provide a trivialization of the tangent bundle: $TM \cong M \times H$, where $H = \bigoplus_{1 \leq a \leq N} \mathbb{C} \partial_a$ denotes the space of flat vector fields on M . Using the Frobenius pairing $(\ , \)$ we identify the cotangent and the tangent bundles, that is, the 1-forms $\phi^a = dt^a$ ($1 \leq a \leq N$) correspond to a basis of H dual to ϕ_a with respect to the Frobenius pairing.

We will make a further assumption that the grading operator θ is diagonalizable with rational eigenvalues and that the Frobenius manifold M has a calibration $S(t, z) = 1 + \sum_{k=1}^{\infty} S_k(t) z^{-k}$ satisfying the following conditions:

- (i) $S_k : M \rightarrow \text{End}(H)$ is a holomorphic map.
- (ii) Symplectic condition holds: $S(t, z)S(t, -z)^T = 1$.
- (iii) The Dubrovin connection admits a fundamental solution of the form $Y(t, z) = S(t, z)z^\theta z^{-\rho}$, where $\rho : H \rightarrow H$ is a nilpotent operator, satisfying $[\theta, \rho] = -\rho$.

Recalling the recursive formula (1.11) for the calibration, we get that $\rho = -\nu_{[-1]} = (E\bullet)_{[-1]}$, that is, the operator ρ is uniquely determined from the operator of Frobenius multiplication by the Euler vector field. If the grading operator θ satisfies the above conditions, then we will say that θ is a *Hodge grading operator*.

REMARK 3.1. The construction that will follow can be generalized to semi-simple Frobenius manifolds for which the grading operator is not necessarily diagonalizable and $[\theta, \rho] \neq -\rho$. However, the formulas become more cumbersome and so far we do not have any examples for which such a generality is needed.

3.2.1. Calibrated periods. Let $u = (u_1, \dots, u_N)$ be the canonical coordinates defined in a neighborhood of t° . Let us fix a positive real number λ° , $\lambda^\circ > |u_i(t^\circ)|$ for all $1 \leq i \leq N$. We will use (t°, λ°) as a reference point in $M \times \mathbb{C}$ in order to keep track of the branches of the solutions to the second structure connection. Using the calibration $S(t, z)$ we would like to describe the space of solutions of the second structure connection $\nabla^{(m)}$ in a neighborhood of $\lambda = \infty$. To begin with, put

$$(3.9) \quad \tilde{I}_a^{(m)}(\lambda) := e^{-\rho \partial_\lambda \partial_m} \left(\frac{\lambda^{\theta-m-\frac{1}{2}}}{\Gamma(\theta-m+\frac{1}{2})} \right) a,$$

where m is a complex number. Note that ρ is a nilpotent operator, so the exponential produces a finite order differential operator. Formula (3.9) defines a multivalued function on $\mathbb{C} \setminus \{0\}$ with values in $\text{End}(H)$. Its value in a neighborhood of $\lambda = \lambda^\circ$ is determined by the principal branch of the logarithm, while the values at the remaining points depend on the choice of a reference path.

LEMMA 3.2. *The functions (3.9) satisfy the following differential equation:*

$$(\lambda - \rho) \partial_\lambda \tilde{I}^{(m)} = \left(\theta - m - \frac{1}{2} \right) \tilde{I}^{(m)}.$$

PROOF. Let $L := (\lambda - \rho) \partial_\lambda - \theta + m + \frac{1}{2}$. We need to prove that $L \tilde{I}^{(m)} = 0$. Note that

$$[\rho \partial_\lambda \partial_m, L] = \rho \partial_\lambda.$$

Using the formula $e^A B e^{-A} = B + \sum_{k=1}^{\infty} \frac{1}{k!} \text{ad}_A^k(B)$ with $A = \rho \partial_\lambda \partial_m$ and $B = L$, we get

$$e^{\rho \partial_\lambda \partial_m} L e^{-\rho \partial_\lambda \partial_m} = L + \rho \partial_\lambda = \lambda \partial_\lambda - \theta + m + \frac{1}{2}.$$

Therefore, the lemma follows from the fact that

$$\lambda \partial_\lambda \lambda^{\theta-m-\frac{1}{2}} = \left(\theta - m - \frac{1}{2} \right) \lambda^{\theta-m-\frac{1}{2}}. \quad \square$$

The functions (3.9) will be called *calibrated periods*. Note that since $\frac{1}{\Gamma(z)}$ is an entire function in z , definition (3.9) makes sense for all $m \in \mathbb{C}$. We will be interested however, only in the case when $m \in \mathbb{Z}$. Let us point out the following important property:

$$(3.10) \quad \partial_\lambda \tilde{I}_a^{(m)}(\lambda) = \tilde{I}_a^{(m+1)}(\lambda).$$

Put

$$(3.11) \quad I_a^{(m)}(t, \lambda) := \sum_{k=0}^{\infty} (-1)^k S_k(t) \tilde{I}_a^{(m+k)}(\lambda),$$

where $a \in H$, $m \in \mathbb{C}$, and the RHS is interpreted as a formal Laurent series near $\lambda = \infty$. We claim that $I_a^{(m)}(t, \lambda)$ is a solution to the second structure connection (see Section 2.4). More precisely, we have the following proposition.

PROPOSITION 3.3. *The series $I_a^{(m)}(t, \lambda)$ is a solution to the following system of differential equations:*

$$\begin{aligned} (\lambda - E \bullet) \partial_\lambda I_a^{(m)}(t, \lambda) &= \left(\theta - m - \frac{1}{2} \right) I_a^{(m)}(t, \lambda) \\ \partial_{t_i} I_a^{(m)}(t, \lambda) &= -\phi_i \bullet \partial_\lambda I_a^{(m)}(t, \lambda) \quad (1 \leq i \leq N). \end{aligned}$$

PROOF. The proof is straightforward verification using Lemma 3.2, formula (3.10), and that the coefficients $S_k(t)$ of the calibration series satisfy the following recursion equations:

$$\begin{aligned} kS_k(t) + [\theta, S_k(t)] &= E \bullet S_{k-1}(t) - S_{k-1}(t)\rho, \\ \partial_{t_i} S_k(t) &= -\phi_i \bullet S_{k-1}(t). \end{aligned}$$

We leave the details as an exercise. \square

Let us list several key properties of the period vectors, which will be used quite frequently.

1) As we already discussed in great details in Section 2.4, the system in Proposition 3.3 is an isomonodromic family of Fuchsian equations (in λ). Therefore, the formal series (3.11) is convergent for all (t, λ) sufficiently close to the reference point (t°, λ°) . Using the differential equations from Proposition 3.3, we get that the periods $I_a^{(m)}(t, \lambda)$ extend analytically along any path avoiding the discriminant.

2) We have a *monodromy representation*

$$(3.12) \quad r : \pi_1((M \times \mathbb{C})', (t^\circ, \lambda^\circ)) \rightarrow \text{End}(H),$$

where $(M \times \mathbb{C})'$ is the complement of the discriminant, defined as follows: if C is a closed loop in $(M \times \mathbb{C})'$ based at (t°, λ°) , then the analytic continuation of $I_a^{(m)}(t, \lambda)$ along C is equal to $I_{r_C(a)}^{(m)}(t, \lambda)$ for some $r_C(a) \in H$. Clearly, $r_C \in \text{End}(H)$ depends only on the homotopy class of the map C and hence the map $C \mapsto r_C$ induces a representation of the fundamental group.

3) We have $\partial_\lambda I_a^{(m)}(t, \lambda) = I_a^{(m+1)}(t, \lambda)$. This follows immediately from (3.10).

4) Since $\partial_{t_1} I_a^{(m)}(t, \lambda) = -\partial_\lambda I_a^{(m)}(t, \lambda)$ the periods have the following translation invariance property:

$$I_a^{(m)}(t, \lambda) = I_a^{(m)}(t - \lambda \mathbf{1}, 0),$$

where $t - \lambda \mathbf{1} \in M$ is the time $-\lambda$ -flow of the unit vector field, that is, in flat coordinates $t = (t_1, \dots, t_N)$ such that $\frac{\partial}{\partial t_1}$ is the unit vector field of the Frobenius manifold, we have $t - \lambda \mathbf{1} = (t_1 - \lambda, t_2, \dots, t_N)$.

5) There is a canonical way to change the reference point. Suppose that we choose a 2nd reference point $(t^\#, \lambda^\#) \in (M \times \mathbb{C})'$, such that, $\lambda^\#$ is a positive real number and $|\lambda^\#| > |u^\#|$ for all eigenvalues $u^\#$ of the operator of Frobenius multiplication by $E_{\bullet t^\#}$. Now we have two ways to define the periods $I_a^{(m)}(t^\#, \lambda^\#)$. First, by using formula (3.11) and second, by fixing a path A in $(M \times \mathbb{C})'$ connecting (t°, λ°) and $(t^\#, \lambda^\#)$ and analytically continuing $I_a^{(m)}(t^\circ, \lambda^\circ)$ along A . We claim that we can always choose the path A in such a way that the two definitions will agree. Such a path A will be called

space of eigenvectors with eigenvalue 0 and a 1-dimensional space of eigenvectors with eigenvalue $m + \frac{1}{2}$.

Let us construct the solutions to (3.13)–(3.14) corresponding to $\alpha = 0$, that is, the solutions that are holomorphic at $\lambda = u_i$. To begin with, let us substitute the ansatz (3.15) in (3.14) and compare the coefficients in front of $(\lambda - u_i)^{k-1}$ for $k \geq 1$. We get a recursion relation of the following form:

$$(3.16) \quad \left(k + P_i(u)(-\theta + m + \frac{1}{2})\right)a_k = \sum_{j:j \neq i} \sum_{s=1}^k \frac{P_j(u)}{(u_j - u_i)^s} (-\theta + m + \frac{1}{2})a_{k-s}.$$

We get that the solution, if it exists, is uniquely determined from a_0 . To determine a_0 , let us substitute (3.15) in (3.13)–(3.14) and specialize $\lambda = u_i$. We get that the coefficient $a_0(u)$ is a solution to the following system of ODEs:

$$(3.17) \quad \partial_{u_j} a_0 = \frac{P_j(u)}{u_j - u_i} \left(\theta - m - \frac{1}{2}\right) a_0, \quad j \neq i,$$

$$(3.18) \quad \partial_{u_i} a_0 = - \sum_{j:j \neq i} \frac{P_j(u)}{u_j - u_i} \left(\theta - m - \frac{1}{2}\right) a_0.$$

The compatibility of the above system follows from the fact that the operator valued functions $A_j(u) = P_j(u)(-\theta + m + \frac{1}{2})$ ($1 \leq j \leq N$) are solutions to the Schlesinger equations (see Proposition (2.26)). Let us choose $a_0^\circ \in H$, such that, $P_i(u^\circ)(\theta - m - \frac{1}{2})a_0^\circ = 0$, where $u^\circ = (u_1(t^\circ), \dots, u_N(t^\circ))$ are the canonical coordinates of the reference point t° . Let us define $a_0(u)$ to be the solution to the system (3.17)–(3.18), satisfying the initial condition $a_0(u^\circ) = a_0^\circ$. Furthermore, we define $a_k(u)$ for $k \geq 1$ by the recursion (3.16). We leave it as an exercise for the reader to check that the series (3.15) with coefficients defined as above is indeed a solution to (3.13)–(3.14).

Let us construct the solution to (3.13)–(3.14) of the form (3.15) with $\alpha = m + \frac{1}{2}$. Although one can follow the same method as above, it turns out that the solution in this case can be expressed in terms of the asymptotic series $R(t, z)$ – see Proposition 1.68. Namely, using the differential equations (1.69)–(1.70) defining the asymptotic series $R(t, z)$ we get that the Laurent series

$$(3.19) \quad I_i^{(m)}(t, \lambda) = \sqrt{2\pi} \sum_{k=0}^{\infty} (-1)^k \Psi(t) R_k(t) e_i \frac{(\lambda - u_i)^{k-m-\frac{1}{2}}}{\Gamma(k - m + \frac{1}{2})}$$

is a solution to (3.13)–(3.14). Since (3.14) has a regular singular point at $\lambda = u_i$, the above series must be convergent. Here e_i is a vector column with 1 on the i th position and 0 elsewhere and the matrix $\Psi(t)$ is viewed as an isomorphism $\mathbb{C}^N \cong H$, defined by

$$\Psi(t)e_i = \sqrt{\Delta_i} \frac{\partial}{\partial u_i} = \sum_{a=1}^N \Psi_{ai} \phi_a.$$

In other words, we proved the following proposition.

PROPOSITION 3.4. *The space of solutions to the second structure connection (3.13)–(3.14) in a neighborhood of a generic point (t°, u°) on the discriminant has the following properties.*

- a) *The subspace of solutions analytic at $(t, \lambda) = (t^\circ, u^\circ)$ is $(N - 1)$ -dimensional.*

b) A solution $I^{(m)}(t, \lambda)$ is analytic at $(t, \lambda) = (t^\circ, u^\circ)$ if and only if it is locally monodromy invariant, i.e., it is invariant under the analytic continuation along a closed loop around the discriminant branch $\lambda = u_i(t)$, where i is such that $u^\circ = u_i(t^\circ)$.

c) The Laurent series (3.19), up to a constant factor, is the unique solution which is anti-invariant with respect to the analytic continuation along a closed loop around the discriminant branch $\lambda = u_i(t)$ – same convention as in part b). \square

3.2.3. Vertex operators and phase factors. Our main interest is in the following vertex operators

$$(3.20) \quad \Gamma^\alpha(t, \lambda) = e^{\mathbf{f}^\alpha(t, \lambda, z)} \widehat{e^{\tilde{\mathbf{f}}_+^\alpha(t, \lambda, z)}}$$

and

$$(3.21) \quad \tilde{\Gamma}^\alpha(\lambda) = e^{\tilde{\mathbf{f}}^\alpha(\lambda, z)} \widehat{e^{\tilde{\mathbf{f}}_+^\alpha(\lambda, z)}}$$

where $\alpha \in H$,

$$\mathbf{f}^\alpha(t, \lambda, z) = \sum_{m \in \mathbb{Z}} I_\alpha^{(m)}(t, \lambda) (-z)^m,$$

and

$$\tilde{\mathbf{f}}^\alpha(\lambda, z) = \sum_{m \in \mathbb{Z}} \tilde{I}_\alpha^{(m)}(\lambda) (-z)^m.$$

The second structure connection has the form (1.57). In particular, the construction in the example of Section 1.5.1 applies in the current settings. In other words, the coefficients of the above vertex operators take values in the ring generated by the components of all period vectors. Note that the definition (3.11) of the periods can be written equivalently as

$$\mathbf{f}^\alpha(t, \lambda, z) = S(t, z) \tilde{\mathbf{f}}^\alpha(\lambda, z).$$

Recalling formula (1.62) we get

$$(3.22) \quad \tilde{\Gamma}^\alpha(\lambda) \widehat{S}^{-1} = \widehat{S}^{-1} e^{W(\tilde{\mathbf{f}}_+^\alpha, \tilde{\mathbf{f}}_+^\alpha)/2} \Gamma^\alpha(t, \lambda).$$

Let us fix a path C avoiding the discriminant from (t°, λ°) to a generic point on the discriminant $(t, u_i(t))$. Then the anti-invariant solution $I_i^{(m)}(t, \lambda) = I_\alpha^{(m)}(t, \lambda)$ for some $\alpha \in H$, where the value of the period vector $I_\alpha^{(m)}(t, \lambda)$ is specified by the path C . Such a vector α will be called *reflection vector*. The set of all reflection vectors will be denoted by \mathcal{R} . Our interest in the reflection vectors comes from the fact that the corresponding vertex operators $\Gamma^\alpha(t, \lambda)$ are conjugated to the vertex operators used in the definition of the HQEs of the KdV hierarchy. More precisely, put

$$\mathbf{f}_{\text{pt}}(u, \lambda, z) = \sum_{m \in \mathbb{Z}} (-z \partial_\lambda)^m \frac{2}{\sqrt{2(\lambda - u)}} = \sqrt{2\pi} \sum_{m \in \mathbb{Z}} (-z)^m \frac{(\lambda - u)^{-m - \frac{1}{2}}}{\Gamma(-m + \frac{1}{2})}.$$

The vertex operator

$$\Gamma_{\text{pt}}^c(u_i, \lambda) := e^{c(\mathbf{f}_{\text{pt}}^-(u_i, \lambda, z) e_i)} \widehat{e^{c(\mathbf{f}_{\text{pt}}^+(u_i, \lambda, z) e_i)}},$$

where $c \in \mathbb{C}$ is a complex constant and $e_i := \sqrt{\Delta_i} \frac{\partial}{\partial u_i}$, takes the form

$$\exp\left(c \sum_{k=0}^{\infty} \sum_{a=1}^N I_{\text{pt}}^{(-k-1)}(u_i, \lambda) \frac{\partial u_i}{\partial t_a} \frac{q_{k,a}}{\sqrt{\hbar \Delta_i}}\right) \exp\left(c \sum_{k=0}^{\infty} \sum_{a=1}^N (-1)^{k+1} I_{\text{pt}}^{(k)}(u_i, \lambda) \frac{\partial t_a}{\partial u_i} \sqrt{\hbar \Delta_i} \frac{\partial}{\partial q_{k,a}}\right),$$

where

$$I_{\text{pt}}^{(m)}(u_i, \lambda) = \sqrt{2\pi} \frac{(\lambda - u_i)^{-m - \frac{1}{2}}}{\Gamma(-m + \frac{1}{2})} = \partial \lambda^m \left(\frac{2}{\sqrt{2(\lambda - u_i)}} \right).$$

Note that up to some linear change of the variables, the operator $\Gamma_{\text{pt}}^1(u_i, \lambda)$ coincides with the vertex operator (3.6). On the other hand, the definition (3.19) is equivalent to

$$\mathbf{f}_i(t, \lambda, z) := \sum_{m \in \mathbb{Z}} I_i^{(m)}(t, \lambda) (-z)^m = (\Psi(t)R(t, z)\Psi(t)^{-1}) e_i \mathbf{f}_{\text{pt}}(u_i, \lambda, z).$$

Recalling the conjugation formula (1.63) we get that if $\alpha \in \mathcal{R}$ then

$$(3.23) \quad \Gamma^{\alpha}(t, \lambda) (\Psi R \Psi^{-1})^{\wedge} = e^{\frac{c^2}{2} V(\mathbf{f}_{\pm}^{\alpha}; \mathbf{f}_{\pm}^{\alpha})} (\Psi R \Psi^{-1})^{\wedge} \Gamma_{\text{pt}}^c(u_i, \lambda),$$

where λ is sufficiently close to u_i and the reference path specifying the value of $\Gamma^{\alpha}(t, \lambda)$ is the same as the path used in the definition of the reflection vector α , that is, $I_{\alpha}^{(m)}(t, \lambda) = I_i^{(m)}(t, \lambda)$. Formula (3.23) makes sense because if α is a reflection vector, then $\Gamma^{\alpha}(t, \lambda)$ is a tame vertex operator and hence the conjugation by upper-triangular symplectic transformations is defined (see Section 1.5.4).

Formulas (3.22) and (3.23) involve the quadratic expressions $W(\tilde{\mathbf{f}}_{\pm}^{\alpha}(\lambda, z), \tilde{\mathbf{f}}_{\pm}^{\alpha}(\lambda, z))$ and $V(\mathbf{f}_{\pm}^{\alpha}(t, \lambda, z), \mathbf{f}_{\pm}^{\alpha}(t, \lambda, z))$, which we will refer to as *phase factors*, where the quadratic forms W and V are defined in terms of the symplectic transformations $S(t, z)$ and $\Psi(t)R(t, z)\Psi(t)^{-1}$ according to respectively formulas (1.37)–(1.38) and (1.49)–(1.50). Our goal now is to express the phase factors as integrals along the path of a certain multivalued 1-form on $(M \times \mathbb{C})'$ (see also [22], Section 7).

Let us begin with the phase factor corresponding to W . More generally, let us compute

$$(3.24) \quad W_{\alpha, \beta}(t, \lambda, \mu) = W(\tilde{\mathbf{f}}_{\pm}^{\alpha}(\lambda, z), \tilde{\mathbf{f}}_{\pm}^{\beta}(\mu, z)) = \sum_{k, l=0}^{\infty} (-1)^{k+l} (W_{kl}(t) \tilde{I}_{\alpha}^{(l)}(\lambda), \tilde{I}_{\beta}^{(k)}(\mu)).$$

Recalling formula (1.38) and the differential equation for the calibration $d^M S(t, z) = z^{-1} A(t) S(t, z)$, where d^M is the de Rham differential on M and $A(t) = \sum_{a=1}^N (\phi_a \bullet) dt_a$, we get $d^M W_{kl}(t) = S_k(t)^T A(t) S_l(t)$ and hence

$$\begin{aligned} d^M W_{\alpha, \beta}(t, \lambda, \mu) &= \sum_{k, l=0}^{\infty} (-1)^{k+l} (A(t) S_l(t) \tilde{I}_{\alpha}^{(l)}(\lambda), S_k(t) \tilde{I}_{\beta}^{(k)}(\mu)) = \\ &= (A(t) I_{\alpha}^{(0)}(t, \lambda), I_{\beta}^{(0)}(t, \mu)) = \sum_{a=1}^N (\phi_a, I_{\alpha}^{(0)}(t, \lambda) \bullet I_{\beta}^{(0)}(t, \mu)) dt_a = \\ &= I_{\alpha}^{(0)}(t, \lambda) \bullet I_{\beta}^{(0)}(t, \mu) \in T_t^* M, \end{aligned}$$

where in the last equality we used the identification $TM \cong T^*M$ defined via the Frobenius pairing.

Let us compute the phase factor corresponding to V . Put

$$\tilde{R}(t, z) := \Psi(t)R(t, z)\Psi(t)^{-1} =: \sum_{k=0}^{\infty} \tilde{R}_k(t)z^k.$$

Recalling the differential equations (1.69) for the asymptotic series $R(t, z)$ we get

$$d^M \tilde{R}_k = -\tilde{R}_k B + [A, \tilde{R}_{k+1}],$$

where $B := d^M \Psi \Psi^{-1}$ and we suppressed the dependence on t . Note that $B^T = -B$ and $A^T = A$. On the other hand, according to Proposition 1.40, part a), we have

$$V_{kl}(t) = \sum_{i=0}^l (-1)^{i+1} \tilde{R}_{k+1+i}(t) \tilde{R}_{l-i}(t)^T.$$

Therefore,

$$d^M V_{kl}(t) = A(t)V_{k+1,l}(t) + V_{k,l+1}(t)A(t) + \tilde{R}_{k+1}(t)A(t)\tilde{R}_{l+1}(t)^T,$$

for all $k, l \geq 0$. By definition (see formula (1.49))

$$V(\mathbf{f}_-^\alpha(t, \lambda, z), \mathbf{f}_-^\alpha(t, \lambda, z)) = \sum_{k,l=0}^{\infty} (I_\alpha^{(-1-k)}(t, \lambda), V_{kl}(t)I_\alpha^{(-1-l)}(t, \lambda))$$

and $d^M I_\alpha^{(-1-k)}(t, \lambda) = -A(t)I_\alpha^{(-k)}(t, \lambda)$. Therefore, after a straightforward computation, we get

$$\begin{aligned} d^M V(\mathbf{f}_-^\alpha(t, \lambda, z), \mathbf{f}_-^\alpha(t, \lambda, z)) = \\ - (A(t)I_\alpha^{(0)}(t, \lambda), I_\alpha^{(0)}(t, \lambda)) + \sum_{k,l=0}^{\infty} (A(t)\tilde{R}_l(t)^T I_\alpha^{(-l)}(t, \lambda), \tilde{R}_k(t)^T I_\alpha^{(-k)}(t, \lambda)). \end{aligned}$$

The above sums can be simplified in the case when $\alpha \in \mathcal{R}$ is a reflection vector, (t, λ) is sufficiently close to a generic point $(t^\circ, u^\circ) = (t^\circ, u_i(t^\circ))$ on the discriminant, and the periods $I_\alpha^{(\cdot)}(t, \lambda)$ coincide with the corresponding anti-invariant solution of the second structure connection. Recall that in that case we have

$$\mathbf{f}^\alpha(t, \lambda, z) = \tilde{R}(t, z)e_i \left(\sum_{n \in \mathbb{Z}} (-z\partial_\lambda)^n \left(\frac{2}{\sqrt{2(\lambda - u_i)}} \right) \right).$$

The symplectic condition implies that $\tilde{R}(t, z)^{-1} = \tilde{R}(t, -z)^T$, so applying to both sides of the above formula the operator $\tilde{R}(t, z)^{-1}$ and comparing the coefficients in front of z^0 we get

$$\sum_{k=0}^{\infty} \tilde{R}_k(t)^T I_\alpha^{(-k)}(t, \lambda) = \frac{2e_i}{\sqrt{2(\lambda - u_i)}}.$$

The formula for the differential of the phase factor takes the form

$$d^M V(\mathbf{f}_-^\alpha(t, \lambda, z), \mathbf{f}_-^\alpha(t, \lambda, z)) = \frac{2du_i}{\lambda - u_i} - I_\alpha^{(0)}(t, \lambda) \bullet I_\alpha^{(0)}(t, \lambda).$$

Since $\mathbf{f}_-^\alpha(t, \lambda, z)$ vanishes at $\lambda = u_i$, we get

$$(3.25) \quad V(\mathbf{f}_-^\alpha(t, \lambda, z), \mathbf{f}_-^\alpha(t, \lambda, z)) = \int_{t+(\lambda-u_i)\mathbf{1}}^t \left(\frac{2du_i(s)}{\lambda - u_i(s)} - I_\alpha^{(0)}(s, \lambda) \bullet I_\alpha^{(0)}(s, \lambda) \right),$$

where s is the integration variable.

We proved the following proposition.

PROPOSITION 3.5. *a) We have*

$$d^M W_{\alpha,\beta}(t, \lambda, \mu) = I_{\alpha}^{(0)}(t, \lambda) \bullet I_{\beta}^{(0)}(t, \mu),$$

where $W_{\alpha,\beta}(t, \lambda, \mu)$ is defined by formula (3.24).

b) If $\alpha \in \mathcal{R}$ and (t, λ) is sufficiently close to the corresponding generic point $(t^\circ, u^\circ) = (t^\circ, u_i(t^\circ))$ on the discriminant, then

$$V(\mathbf{f}_-^\alpha(t, \lambda, z), \mathbf{f}_-^\alpha(t, \lambda, z)) = \lim_{\epsilon \rightarrow 0} \int_t^{t+(\lambda-u_i+\epsilon)\mathbf{1}} \left(I_{\alpha}^{(0)}(s, \lambda) \bullet I_{\alpha}^{(0)}(s, \lambda) + \frac{2du_i(s)}{u_i(s) - \lambda} \right),$$

where the branch of the period vectors is specified by the same reference path as the one used in the definition of the reflection vector α . \square

The formula in part b) of Proposition 3.5 is obtained from (3.25) by the substitution $x = s - \lambda\mathbf{1}$. Note that $u_i(x) = u_i(s) - \lambda$, because $\frac{\partial}{\partial t_1} = \partial_{u_1} + \dots + \partial_{u_N}$, so $\frac{\partial u_i}{\partial t_1} = 1$, that is, $u_i(t)$ is the sum of t_1 and a function that depends only on t_2, \dots, t_N .

3.3. Propagators

The main tool for constructing Hirota quadratic equations for the total descendent potential is the formalism of vertex operators that we introduced in Section 3.2. Recall that conjugating vertex operators by quantized symplectic transformations produces the so-called phase factors (3.24). The latter are multivalued analytic functions on the complement of the discriminant. Our next goal is to understand the dependence of the phase factors on the choice of a reference path, i.e., how do the phase factors transform under the analytic continuation along a closed loop? The answer to this question will be very important for the applications to integrable systems.

3.3.1. Product of vertex operators. Let us consider a product of two vertex operators

$$\Gamma^\alpha(t, \lambda_1) \Gamma^\beta(t, \lambda_2) = e^{\Omega_{\alpha,\beta}(t, \lambda_1, \lambda_2)} : \Gamma^\alpha(t, \lambda_1) \Gamma^\beta(t, \lambda_2) :,$$

where $: :$ denotes normal ordering – all differentiation operators should be moved to the right, and

$$\begin{aligned} \Omega_{\alpha,\beta}(t, \lambda_1, \lambda_2) &= \Omega(\mathbf{f}_+^\alpha(t, \lambda_1, z), \mathbf{f}^\beta(t, \lambda_2, z)) = \\ (3.26) \quad &= \sum_{k=0}^{\infty} (-1)^{k+1} (I_{\alpha}^{(k)}(t, \lambda_1), I_{\beta}^{(-1-k)}(t, \lambda_2)). \end{aligned}$$

Following physicists terminology, we refer to $\Omega_{\alpha,\beta}(t, \lambda_1, \lambda_2)$ as *propagators*. We will always assume that λ_2 is sufficiently close to λ_1 , so that the disk in $\{t\} \times \mathbb{C}$ with center (t, λ_1) and radius $|\lambda_2 - \lambda_1|$ does not intersect the discriminant. In particular, the composition of the reference path for $\Gamma^\alpha(t, \lambda_1)$ and the line segment in $\{t\} \times \mathbb{C}$ between (t, λ_1) and (t, λ_2) qualifies as a reference path for the vertex operator $\Gamma^\beta(t, \lambda_2)$. The propagator (3.26) is interpreted as a formal Laurent series in λ_1^{-1} . We will see later on that the formal Laurent series is in fact convergent.

REMARK 3.6. One might ask whether expanding (3.26) as a Laurent series in λ_1^{-1} and choosing $(t, \lambda_1, \lambda_2)$, such that, the resulting Laurent series is convergent is equivalent to requiring that the partial sums of (3.26) are convergent, that is, the limit

$$\lim_{K \rightarrow \infty} \sum_{k=0}^K (-1)^{k+1} (I_\alpha^{(k)}(t, \lambda_1), I_\beta^{(-1-k)}(t, \lambda_2))$$

exists. The answer is no. Let us consider the following series

$$\sum_{k=0}^{\infty} (\lambda_1 - t)^{-k-1} (\lambda_2 - t)^k,$$

where $t \in \mathbb{C}$ is a complex number. The partial sums of the above series are convergent iff $|\lambda_2 - t| < |\lambda_1 - t|$. On the other, if we expand in the powers of λ_1^{-1} , then we get

$$\sum_{l=1}^{\infty} \left(\sum_{k=1}^l \binom{-k}{-k+l} \right) (-t)^{-k+l} (\lambda_2 - t)^{k-1} \lambda_1^{-l}.$$

Differentiating the coefficient in front of λ_1^{-l} with respect to t , it is easy to check that they are independent of t . Setting $t = 0$ we get that the above sum is $\sum_{l=1}^{\infty} \lambda_2^{l-1} \lambda_1^{-l}$. The domain of convergence is given by $|\lambda_2| < |\lambda_1|$. \square

Similarly, consider the product

$$\tilde{\Gamma}^\alpha(\lambda_1) \tilde{\Gamma}^\beta(\lambda_2) = e^{\tilde{\Omega}_{\alpha,\beta}(\lambda_1, \lambda_2)} : \tilde{\Gamma}^\alpha(\lambda_1) \tilde{\Gamma}^\beta(\lambda_2) : ,$$

where the expression

$$\tilde{\Omega}_{\alpha,\beta}(\lambda_1, \lambda_2) = \Omega(\tilde{\mathbf{f}}_+^\alpha(\lambda_1, z), \tilde{\mathbf{f}}_+^\beta(\lambda_2, z)) = \sum_{k=0}^{\infty} (-1)^{k+1} (\tilde{I}_\alpha^{(k)}(\lambda_1), \tilde{I}_\beta^{(-k-1)}(\lambda_2))$$

will be referred to as *calibrated propagator*.

LEMMA 3.7. *The following identity holds:*

$$\Omega_{\alpha,\beta}(t, \lambda_1, \lambda_2) = \tilde{\Omega}_{\alpha,\beta}(\lambda_1, \lambda_2) + W_{\alpha,\beta}(t, \lambda_1, \lambda_2),$$

where $W_{\alpha,\beta}(t, \lambda_1, \lambda_2)$ is the phase factor (3.24).

PROOF. Put \mathbf{f}^α and $\tilde{\mathbf{f}}^\alpha$ for respectively $\mathbf{f}^\alpha(t, \lambda_1, z)$ and $\tilde{\mathbf{f}}^\alpha(\lambda_1, z)$, and \mathbf{f}^β and $\tilde{\mathbf{f}}^\beta$ for respectively $\mathbf{f}^\beta(t, \lambda_2, z)$ and $\tilde{\mathbf{f}}^\beta(\lambda_2, z)$. We have

$$\Omega_{\alpha,\beta}(t, \lambda_1, \lambda_2) = \Omega(\mathbf{f}_+^\alpha, \mathbf{f}_+^\beta) = \Omega((S\tilde{\mathbf{f}}^\alpha)_+, S\tilde{\mathbf{f}}^\beta) = \Omega((S\tilde{\mathbf{f}}^\alpha)_+, S\tilde{\mathbf{f}}^\beta) + \Omega((S\tilde{\mathbf{f}}^\alpha)_+, S\tilde{\mathbf{f}}^\beta).$$

According to part c) of Proposition 1.34, the symplectic pairing $\Omega((S\tilde{\mathbf{f}}^\alpha)_+, S\tilde{\mathbf{f}}^\beta) = W_{\alpha,\beta}(t, \lambda_1, \lambda_2)$. Using that S is a symplectic transformation, $H[z]$ and $H[[z^{-1}]]z^{-1}$ are Lagrangian subspaces, and $S\tilde{\mathbf{f}}^\beta \in H[[z^{-1}]]z^{-1}$, we get

$$\Omega((S\tilde{\mathbf{f}}^\alpha)_+, S\tilde{\mathbf{f}}^\beta) = \Omega(S\tilde{\mathbf{f}}^\alpha, S\tilde{\mathbf{f}}^\beta) = \Omega(\tilde{\mathbf{f}}^\alpha, \tilde{\mathbf{f}}^\beta) = \Omega(\tilde{\mathbf{f}}_+^\alpha, \tilde{\mathbf{f}}^\beta),$$

which by definition is the calibrated propagator $\tilde{\Omega}_{\alpha,\beta}(\lambda_1, \lambda_2)$. \square

3.3.2. Energy propagators. According to Lemma 3.7 the propagator decomposes as a calibrated propagator and a phase factor. Our next goal is to find an explicit formula for the calibrated propagator. Note that the phase factor is analytic at $\lambda_1 = \lambda_2$. Therefore, our explicit formula will describe the singularity of the propagator at $\lambda_1 = \lambda_2$.

It turns out, that it is easier to compute the propagator with respect to a slightly different polarization of the symplectic vector space $\mathcal{H} = H((z^{-1}))$. Let us consider the following operator

$$\ell := w\partial_w + \frac{1}{2} - \theta : \mathcal{H} \rightarrow \mathcal{H}.$$

The origin of the above operator is in Givental's formulation of Virasoro conjecture in Gromov–Witten theory (see [20]). More precisely, ℓ is the semi-simple part of an infinitesimal symplectic transformation whose quantization is the Virasoro operator L_0 .

The operator ℓ is diagonalizable – an eigenbasis is given by $\phi_i w^k$, where ϕ_i ($1 \leq i \leq N$) is an eigenbasis for the grading operator θ , that is, $\theta(\phi_i) = \theta_i \phi_i$ for some rational numbers $\theta_i \in \mathbb{Q}$. Note that $\ell(\phi_i w^k) = \left(k + \frac{1}{2} - \theta_i\right) \phi_i w^k$. Let us decompose $\mathcal{H} = \mathcal{H}_{<0} \oplus \mathcal{H}_0 \oplus \mathcal{H}_{>0}$, where $\mathcal{H}_0 = \ker(\ell)$ and $\mathcal{H}_{<0}$ (resp. $\mathcal{H}_{>0}$) is the vector subspace spanned by eigenvectors with negative (resp.) positive eigenvalues. If $\mathbf{f} \in \mathcal{H}$, then we denote by $\mathbf{f}_{<0}$, \mathbf{f}_0 , and $\mathbf{f}_{>0}$ the corresponding projections of \mathbf{f} on $\mathcal{H}_{<0}$, \mathcal{H}_0 , and $\mathcal{H}_{>0}$. Our goal is to compute the symplectic pairings $\Omega(\tilde{\mathbf{f}}_\alpha(\lambda_1, w)_{>0}, \tilde{\mathbf{f}}_\beta(\lambda_2, w))$ and $\Omega(\tilde{\mathbf{f}}_\alpha(\lambda_1, w)_0, \tilde{\mathbf{f}}_\beta(\lambda_2, w))$. In order to state the answer we need some further notation. Put

$$(3.27) \quad \sigma := e^{2\pi i(\theta + \frac{1}{2})} e^{2\pi i\rho} : H \rightarrow H.$$

Using formula (3.9) we get that σ is the endomorphism representing via (3.12) a big loop in $(M \times \mathbb{C})'$ around the discriminant. We will refer to σ as the *classical monodromy operator*. Let us fix a logarithm of the classical monodromy operator

$$(3.28) \quad \sigma = e^{-2\pi i\mathcal{N}}, \quad \mathcal{N} = \mathcal{N}_s + \mathcal{N}_n, \quad [\mathcal{N}_s, \mathcal{N}_n] = 0.$$

where $\mathcal{N}_n = -\rho$ and \mathcal{N}_s is defined as follows. Let us write the eigenvalues of $\theta + \frac{1}{2}$ in the form

$$\theta_i + \frac{1}{2} =: -p_i - \nu_i, \quad p_i \in \mathbb{Z}, \quad -1 < \nu_i \leq 0.$$

In other words,

$$\nu_i = [\theta_i + \frac{1}{2}] - \theta_i - \frac{1}{2},$$

where $[x]$ denotes the integral part of x . Then $\mathcal{N}_s : H \rightarrow H$ is defined by $\mathcal{N}_s(\phi_i) = \nu_i \phi_i$. Note that $e^{-2\pi i\mathcal{N}_s} = e^{2\pi i(\theta + \frac{1}{2})}$ and that $[\mathcal{N}_s, \mathcal{N}_n] = 0 \Rightarrow$ formula (3.28) holds. Furthermore, let us introduce the following power series

$$\text{Li}_\sigma(x) = \sum_{m=1}^{\infty} \frac{x^{m+\mathcal{N}}}{m + \mathcal{N}},$$

where $x^A := e^{A \log x}$. The series is convergent for $|x| < 1$ and therefore it defines a multivalued analytic function on the unit disk with values in $\text{End}(H)$. Finally, let us define the (non-symmetric in general) bilinear pairing

$$(3.29) \quad \langle , \rangle : H \otimes H \rightarrow \mathbb{C}, \quad \langle a, b \rangle := \frac{1}{2\pi} (a, e^{\pi i\theta} e^{\pi i\rho} b).$$

In the case of a Frobenius structure coming from quantum cohomology the pairing (3.29) coincides with the Euler pairing. For that reason we will refer to (3.29) as the *Euler pairing* of the Frobenius manifold. The symmetrization of the Euler pairing

$$(3.30) \quad (|) : H \otimes H \rightarrow \mathbb{C}, \quad (a|b) := \langle a, b \rangle + \langle b, a \rangle$$

will be called *intersection pairing*. Our motivation for such a name comes from mirror symmetry. Namely, in all known examples in which the quantum cohomology is isomorphic to the Frobenius structure associated with a primitive form of a mirror family of functions, the pairing (|) coincides with the intersection pairing in vanishing homology. Using that $\theta^T = -\theta$ and $\rho^T = \rho$, it is easy to prove that

$$(3.31) \quad (a|b) = \langle (1 - \sigma)a, b \rangle, \quad a, b \in H.$$

In particular, since the Euler pairing is non-degenerate, the kernel of the intersection pairing coincides with $\text{Ker}(1 - \sigma)$. The following spectral decomposition will play an important role: $H = H_0 \oplus H_{\neq 0}$, where

$$H_0 := \text{Ker}(\mathcal{N}_s) \quad \text{and} \quad H_{\neq 0} := \bigoplus_{\lambda \in \mathbb{Q} \setminus \{0\}} \text{Ker}(\mathcal{N}_s - \lambda).$$

LEMMA 3.8. *Let \mathcal{N}_s^T be the transpose of \mathcal{N}_s with respect to the Frobenius pairing.*

- a) *The subspaces H_0 and $H_{\neq 0}$ are orthogonal with respect to the Frobenius pairing.*
- b) *We have $\mathcal{N}_s^T(x) = -x - \mathcal{N}_s(x)$ for all $x \in H_{\neq 0}$.*
- c) *We have $\mathcal{N}_s^T(x) = -\mathcal{N}_s(x)$ for all $x \in H_0$.*

PROOF. Recall that $-\theta_i - \frac{1}{2} = \nu_i + p_i$. The subspace H_0 has a basis given by the set of all ϕ_i with $\nu_i = 0$, while $H_{\neq 0}$ has a basis given by the set of all ϕ_i with $\nu_i \neq 0$.

a) We have to prove that if $\nu_i = 0$ and $\nu_j \neq 0$, then $(\phi_i, \phi_j) = 0$. If this is not the case, then since $\theta^T = -\theta$, we must have $\theta_i + \theta_j = 0$. Therefore $\nu_i + \nu_j$ must be an integer – contradiction.

b) We have to prove that

$$(\mathcal{N}_s \phi_i, \phi_j) + (\phi_i, \mathcal{N}_s \phi_j) = -(\phi_i, \phi_j),$$

for all i, j , such that, $\nu_i \neq 0$ and $\nu_j \neq 0$. If $(\phi_i, \phi_j) = 0$, then the identity is obviously true. Otherwise, $\theta_i + \theta_j = 0 \Rightarrow \nu_i + \nu_j$ must be an integer. However, $-1 < \nu_i, \nu_j < 0$, so the only option is that $\nu_i + \nu_j = -1$ so the identity holds again.

c) The proof is similar to the proof of part b), so we will leave it as an exercise. \square

THEOREM 3.9. *The following formulas hold:*

$$(3.32) \quad \Omega(\tilde{\mathbf{f}}_\alpha(\lambda_1, w)_{>0}, \tilde{\mathbf{f}}_\beta(\lambda_2, w)) = -(\text{Li}_\sigma(\lambda_2/\lambda_1)\alpha|\beta)$$

and

$$(3.33) \quad \Omega(\tilde{\mathbf{f}}_\alpha(\lambda_1, w)_0, \tilde{\mathbf{f}}_\beta(\lambda_2, w)) = -\left\langle \frac{e^{2\pi i \rho} - 1}{\rho} (\lambda_2/\lambda_1)^{-\rho} \alpha_0, \beta_0 \right\rangle,$$

where α_0 and β_0 are the projections of α and β on H_0 .

PROOF. In order to avoid cumbersome notation we introduce the following notation: $\lambda^{(x)} = \frac{\lambda^x}{\Gamma(x+1)}$, where x is a rational number which is not a negative integer. By definition

$$(3.34) \quad \tilde{\mathbf{f}}_\alpha(\lambda, w) = \sum_{m \in \mathbb{Z}} (-w)^m \partial_\lambda^{m+1} e^{\rho \partial_\lambda \partial_i} \left(\lambda^{(\theta+l-\frac{1}{2})} \right) \alpha,$$

where first l is allowed to be a real number, so that ∂_l makes sense and then, after all derivations in l are computed, we specialize $l = -m$. Note that since $\partial_\lambda \lambda^{(x)} = \lambda^{(x-1)}$, we can choose any $l \geq -m$. We have

$$e^{\rho \partial_\lambda \partial_l} \left(\lambda^{(\theta+l-\frac{1}{2})} \right) = \sum_{k=0}^{\infty} \frac{1}{k!} \rho^k \partial_l^k \lambda^{(\theta+l-k-\frac{1}{2})} = \sum_{k=0}^{\infty} \frac{1}{k!} \partial_l^k \lambda^{(\theta+l-\frac{1}{2})} \rho^k,$$

where in the second equality we used that $\rho\theta = (\theta+1)\rho$. Substituting the above formula in (3.34) we get

$$(3.35) \quad \tilde{\mathbf{f}}_\alpha(\lambda, w) = \sum_{m \in \mathbb{Z}} \sum_{k=0}^{\infty} (-w)^m \partial_\lambda^{m+l} \frac{1}{k!} \partial_l^k \left(\lambda^{(\theta+l-\frac{1}{2})} \rho^k \alpha \right).$$

Let us transform the following expression:

$$\lambda^{(\theta+l-\frac{1}{2})} \rho^k \alpha = \sum_{i=1}^N (\lambda^{(\theta+l-\frac{1}{2})} \rho^k \alpha, \phi_i) \phi^i = \sum_{i=1}^N (\alpha, \rho^k \lambda^{(-\theta+l-\frac{1}{2})} \phi_i) \phi^i,$$

where we used that $\theta^T = -\theta$. Recall that $\{\phi_i\}_{i=1}^N$ is an eigenbasis of θ and that θ_i denotes the eigenvalue corresponding to ϕ_i . Note that the dual basis $\{\phi^i\}_{i=1}^N$ is also an eigenbasis and that the eigenvalue corresponding to ϕ^i is $-\theta_i$. Finally, recall also that $-\theta_i - \frac{1}{2} = \nu_i + p_i$, where $-1 < \nu_i \leq 0$ and $p_i \in \mathbb{Z}$. Using these remarks we get

$$\sum_{k=0}^{\infty} \frac{1}{k!} \partial_l^k \left(\lambda^{(\theta+l-\frac{1}{2})} \rho^k \alpha \right) = \sum_{i=1}^N \sum_{k=0}^{\infty} \frac{1}{k!} \partial_l^k (\alpha, \rho^k \lambda^{(l+\nu_i+p_i)} \phi_i) \phi^i = \sum_{i=1}^N (\alpha, \lambda^{(l+\nu_i+p_i+\rho)} \phi_i) \phi^i.$$

Substituting the above formula in (3.35) we get

$$\tilde{\mathbf{f}}_\alpha(\lambda, w) = \sum_{i=1}^N \sum_{m \in \mathbb{Z}} (-w)^m (\alpha, \lambda^{(\nu_i+p_i-m+\rho)} \phi_i) \phi^i.$$

Changing the summation index $m = k + p_i$ and using that $\mathcal{N}_s(\phi_i) = \nu_i \phi_i$ we get

$$\tilde{\mathbf{f}}_\alpha(\lambda, w) = \sum_{i=1}^N \sum_{k \in \mathbb{Z}} (-w)^{k+p_i} (\alpha, \lambda^{(\mathcal{N}_s+\rho-k)} \phi_i) \phi^i.$$

Let us decompose $\alpha = \alpha_{\neq 0} + \alpha_0$ and recall Lemma 3.8. We get

$$(3.36) \quad \tilde{\mathbf{f}}_\alpha(\lambda, w) = \sum_{i=1}^N \sum_{k \in \mathbb{Z}} (-w)^{k+p_i} \phi^i \left((\lambda^{(-\mathcal{N}_s+\rho-k-1)} \alpha_{\neq 0}, \phi_i) + (\lambda^{(-\mathcal{N}_s+\rho-k)} \alpha_0, \phi_i) \right).$$

Formula (3.36) allows us to compute easily the projection on $\mathcal{H}_{>0}$. Indeed, the energy of the monomial $w^{k+p_i} \phi^i$, that is, the eigenvalue of the Virasoro operator ℓ is $k+p_i+\frac{1}{2}+\theta_i = k-\nu_i$. If $\nu_i = 0$, then the energy is positive iff $k \geq 1$. If $\nu_i \neq 0$, then the energy is positive iff $k \geq 0$. We get

$$(3.37) \quad \tilde{\mathbf{f}}_\alpha(\lambda, w)_{>0} = \sum_{i=1}^N \phi^i \left(\sum_{k=0}^{\infty} (-w)^{k+p_i} (\lambda^{(-\mathcal{N}_s+\rho-k-1)} \alpha_{\neq 0}, \phi_i) + \sum_{k=1}^{\infty} (-w)^{k+p_i} (\lambda^{(-\mathcal{N}_s+\rho-k)} \alpha_0, \phi_i) \right).$$

Using formulas (3.36) and (3.37) we get the following formula:

$$\begin{aligned} \Omega(\tilde{\mathbf{f}}_\alpha(\lambda_1, w)_{>0}, \tilde{\mathbf{f}}_\beta(\lambda_2, w)) = \\ \sum_{i,j=1}^N \left(\sum_{k=0}^{\infty} \left((\phi^i, e^{\pi i(l'+p_j)} \phi^j) (\lambda_1^{(-\mathcal{N}_s+\rho-k-1)} \alpha_{\neq 0}, \phi_i) (\lambda_2^{(-\mathcal{N}_s+\rho-l'-1)} \beta_{\neq 0}, \phi_j) \right) + \right. \\ \left. \sum_{k=1}^{\infty} \left((\phi^i, e^{\pi i(l''+p_j)} \phi^j) (\lambda_1^{(-\mathcal{N}_s+\rho-k)} \alpha_0, \phi_i) (\lambda_2^{(-\mathcal{N}_s+\rho-l'')} \beta_0, \phi_j) \right) \right), \end{aligned}$$

where l' and l'' are determined from the conditions $k + l' + p_i + p_j = -1$ and $k + l'' + p_i + p_j = -1$. Note that in the first sum over k , since $\nu_i \neq 0$ we have $p_i + p_j = 0$ ($\because -1 = -\theta_i - \frac{1}{2} - \theta_j - \frac{1}{2} = \nu_i + \nu_j + p_i + p_j$ and $\nu_i + \nu_j = -1$) $\Rightarrow l' = -k - 1$. Similar reasoning yields $l'' = -k$.

In order to simplify the above formula let us rewrite

$$e^{\pi i(l'+p_j)} \phi^j = e^{\pi i(-k-1-\nu_j-\theta_j-\frac{1}{2})} \phi^j = e^{\pi i(-k+\mathcal{N}_s+\theta-\frac{1}{2})} \phi^j,$$

where we used that

$$\mathcal{N}_s(\phi^j) = \begin{cases} -(\nu_j + 1)\phi^j & \text{if } \nu_j \neq 0, \\ 0 & \text{if } \nu_j = 0. \end{cases}$$

The proof here is as follows. Suppose that $\nu_j \neq 0$, then using Lemma 3.8 we get

$$(\mathcal{N}_s \phi^j, \phi_i) = (\phi^j, (-\mathcal{N}_s - 1)\phi_i) = -(\nu_j + 1)(\phi^j, \phi_i) = -(\nu_j + 1)(\phi^j, \phi_i), \quad \forall i,$$

where in the last equality we used that $(\phi^j, \phi_i) = \delta_{ij}$. Since the pairing is non-degenerate we get $\mathcal{N}_s \phi^j = -(\nu_j + 1)\phi^j$ as claimed. The case $\mathcal{N}_s \phi^j = 0$ for $\nu_j = 0$ is proved similarly. Similar computation yields

$$e^{\pi i(l''+p_j)} \phi^j = e^{\pi i(-k-\nu_j-\theta_j-\frac{1}{2})} \phi^j = e^{\pi i(-k+\mathcal{N}_s+\theta-\frac{1}{2})} \phi^j$$

The formula for the propagator takes the form

$$\begin{aligned} \Omega(\tilde{\mathbf{f}}_\alpha(\lambda_1, w)_{>0}, \tilde{\mathbf{f}}_\beta(\lambda_2, w)) = \\ \sum_{k=0}^{\infty} (\lambda_1^{(-\mathcal{N}_s+\rho-k-1)} \alpha_{\neq 0}, e^{\pi i(-k+\mathcal{N}_s+\theta-\frac{1}{2})} \lambda_2^{(-\mathcal{N}_s+\rho+k)} \beta_{\neq 0}) + \\ \sum_{k=1}^{\infty} (\lambda_1^{(-\mathcal{N}_s+\rho-k)} \alpha_0, e^{\pi i(-k+\mathcal{N}_s+\theta-\frac{1}{2})} \lambda_2^{(-\mathcal{N}_s+\rho+k)} \beta_0). \end{aligned}$$

Let us look at the first sum over $k \geq 0$. If we move the term $\lambda_2^{(-\mathcal{N}_s+\rho+k)}$ to the left through $e^{\pi i\theta}$, the operator ρ will change to $-\rho$, that is we will get $\lambda_2^{(-\mathcal{N}_s-\rho+k)}$. If we move the latter to the left slot of the Frobenius pairing, then using Lemma 3.8, part b), we will get $\lambda_2^{(\mathcal{N}_s+1-\rho+k)}$. Finally, after moving the exponential term $e^{\pi i(-k+\mathcal{N}_s+\theta-\frac{1}{2})}$ on the left slot of the Frobenius pairing, the summand of the 1st sum takes the form

$$(e^{\pi i(-k-1-\mathcal{N}_s-\theta-\frac{1}{2})} \lambda_2^{(\mathcal{N}_s-\rho+k+1)} \lambda_1^{(-\mathcal{N}_s+\rho-k-1)} \alpha_{\neq 0}, \beta_{\neq 0})$$

Rewriting the summand of the 2nd sum in a similar way we get

$$(e^{\pi i(-k-\mathcal{N}_s-\theta-\frac{1}{2})} \lambda_2^{(\mathcal{N}_s-\rho+k)} \lambda_1^{(-\mathcal{N}_s+\rho-k)} \alpha_0, \beta_0).$$

Note that after shifting the summation index in the 1st sum $k \mapsto k - 1$ the two sums become identical except for the following difference: the terms $\alpha_{\neq 0}$ and $\beta_{\neq 0}$ in the 1st sum and the terms α_0 and β_0 in the 2nd sum are the only places where the two sums do not coincide. In other words, the two sums add up to

$$(3.38) \quad \sum_{k=1}^{\infty} (e^{\pi i(-k-\mathcal{N}_s-\theta-\frac{1}{2})} \lambda_2^{(\mathcal{N}_s+k-\rho)} \lambda_1^{(-\mathcal{N}_s-k+\rho)} \alpha, \beta).$$

Let us simplify

$$\lambda_2^{(\mathcal{N}_s+k-\rho)} \lambda_1^{(-\mathcal{N}_s-k+\rho)} = \frac{(\lambda_2/\lambda_1)^{\mathcal{N}_s+k-\rho}}{\Gamma(-\mathcal{N}_s-k+\rho+1)\Gamma(\mathcal{N}_s+k-\rho+1)}.$$

Using that $\Gamma(1+x)\Gamma(1-x) = \frac{2\pi i x}{e^{\pi i x} - e^{-\pi i x}}$, we get

$$\frac{1}{\Gamma(-\mathcal{N}_s-k+\rho+1)\Gamma(\mathcal{N}_s+k-\rho+1)} = \frac{1}{2\pi i} \frac{1}{\mathcal{N}_s+k-\rho} \left(e^{\pi i(\mathcal{N}_s+k-\rho)} - e^{-\pi i(\mathcal{N}_s+k-\rho)} \right).$$

Note that

$$\frac{1}{2\pi i} e^{\pi i(-k-\mathcal{N}_s-\theta-\frac{1}{2})} \left(e^{\pi i(\mathcal{N}_s+k-\rho)} - e^{-\pi i(\mathcal{N}_s+k-\rho)} \right) = -\frac{1}{2\pi} e^{-\pi i\theta} \left(e^{-\pi i\rho} - e^{-\pi i(2\mathcal{N}_s-\rho)} \right).$$

Recalling that $\sigma = e^{-2\pi i(\mathcal{N}_s-\rho)}$, we get that the above expression is

$$-\frac{1}{2\pi} e^{-\pi i\theta} e^{-\pi i\rho} \left(1 - e^{-2\pi i(\mathcal{N}_s-\rho)} \right) = -\frac{1}{2\pi} e^{\pi i\rho} e^{-\pi i\theta} (1 - \sigma).$$

Formula (3.38) takes the form

$$-\frac{1}{2\pi} \sum_{k=1}^{\infty} \left(e^{\pi i\rho} e^{-\pi i\theta} (1 - \sigma) \frac{(\lambda_2/\lambda_1)^{\mathcal{N}_s+k}}{\mathcal{N}_s+k} \alpha, \beta \right) = -\frac{1}{2\pi} \left((1 - \sigma) \text{Li}_\sigma(\lambda_2/\lambda_1) \alpha, e^{\pi i\theta} e^{\pi i\rho} \beta \right).$$

Using (3.31) and the definition (3.29) of the Euler pairing, we get that the above formula is precisely the RHS of (3.32).

The proof of formula (3.33) is completely analogous, so let us just sketch the main steps leaving the details as an exercise. Using formula (3.36), we get that the projection of $\tilde{\mathbf{f}}_\alpha(\lambda_1, w)$ on \mathcal{H}_0 is

$$\tilde{\mathbf{f}}_\alpha(\lambda_1, w)_0 = \sum_{i=1}^N \phi^i(-w)^{p_i} (\lambda_1^{(\rho)} \alpha_0, \phi_i).$$

The symplectic pairing

$$\Omega(\tilde{\mathbf{f}}_\alpha(\lambda_1, w)_0, \tilde{\mathbf{f}}_\beta(\lambda_2, w)) = \sum_{i,j=1}^N (\phi^i, e^{\pi i p_j} \phi^j) (\lambda_1^{(\rho)} \alpha_0, \phi_i) (\lambda_2^{(\rho)} \beta_0, \phi_j).$$

Note that in the above sum the non-trivial contributions come only from i and j , such that, $\nu_i = \nu_j = 0$. In particular, $p_j = p_j + \nu_j = -\theta_j - \frac{1}{2}$. Therefore, $e^{\pi i p_j} \phi^j = e^{\pi i(\theta - \frac{1}{2})} \phi^j$ and the symplectic pairing becomes

$$\Omega(\tilde{\mathbf{f}}_\alpha(\lambda_1, w)_0, \tilde{\mathbf{f}}_\beta(\lambda_2, w)) = (\lambda_1^{(\rho)} \alpha_0, e^{\pi i(\theta - \frac{1}{2})} \lambda_2^{(\rho)} \beta_0) = \left(e^{-\pi i(\theta + \frac{1}{2})} \lambda_2^{(-\rho)} \lambda_1^{(\rho)} \alpha_0, \beta_0 \right).$$

Just like before, using $\Gamma(1+x)\Gamma(1-x) = \frac{2\pi i x}{e^{\pi i x} - e^{-\pi i x}}$ we get

$$e^{-\pi i(\theta + \frac{1}{2})} \lambda_2^{(-\rho)} \lambda_1^{(\rho)} = -\frac{1}{2\pi} e^{-\pi i\theta} e^{-\pi i\rho} (1 - e^{2\pi i\rho}) \frac{(\lambda_2/\lambda_1)^{-\rho}}{-\rho}.$$

It remains only to notice that $(e^{-\pi i \theta} e^{-\pi i \rho})^T = (e^{\pi i \rho} e^{-\pi i \theta})^T = e^{\pi i \theta} e^{\pi i \rho}$ and to recall the definition of the Euler pairing (3.29). \square

3.3.3. Polylogarithms. The polylogarithm function is defined by the following series:

$$\text{Li}_p(x) := \sum_{k=1}^{\infty} \frac{x^k}{k^p},$$

where $p \geq 1$. The series is convergent for $|x| < 1$, so $\text{Li}_p(x)$ is an analytic function inside the unit disk. Note that

$$\text{Li}_p(x) = \int_0^x \text{Li}_{p-1}(t) \frac{dt}{t},$$

and that

$$\text{Li}_1(x) = -\text{Log}(1-x),$$

where

$$\text{Log}(y) := \ln |y| + \mathbf{i} \text{Arg}(y), \quad -\pi < \text{Arg}(y) < \pi,$$

is the principal branch of the logarithm function. Arguing by induction on p , we get that $\text{Li}_p(x)$ extends analytically along any path avoiding $x = 0$ and $x = 1$. Our main interest in the polylogarithm function is due to the following Lemma.

LEMMA 3.10. *The following formula holds:*

$$\text{Li}_\sigma(x) = x^{-\rho} \sum_{p=1}^{\infty} \sum_{r=1}^{|\sigma|} (|\sigma| \rho)^{p-1} \text{Li}_p(\eta^r x^{1/|\sigma|}) \sigma_s^r,$$

where $\sigma_s := e^{2\pi i(\theta + \frac{1}{2})}$ is the semi-simple part in the Jordan decomposition of σ , $|\sigma|$ is the order of the automorphism σ_s , and $\eta = e^{2\pi i/|\sigma|}$.

PROOF. We have

$$\text{Li}_\sigma(x) = \sum_{m=1}^{\infty} \frac{x^{m+\mathcal{N}_s-\rho}}{m+\mathcal{N}_s-\rho} = \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} \frac{x^{m+\mathcal{N}_s-\rho}}{(m+\mathcal{N}_s)^p} \rho^{p-1}.$$

By definition the eigenvalues of \mathcal{N}_s are rational numbers ν satisfying $-1 < \nu \leq 0$. Since the order of σ_s is $|\sigma|$ the number $|\sigma|\nu$ must be an integer. The identity that we have to prove is equivalent to the following identity:

$$(3.39) \quad \frac{1}{|\sigma|^p} \sum_{m=1}^{\infty} \frac{x^{m+\mathcal{N}_s}}{(m+\mathcal{N}_s)^p} = \frac{1}{|\sigma|} \sum_{r=1}^{|\sigma|} \sum_{k=1}^{\infty} \frac{(\eta^r x^{1/|\sigma|})^k}{k^p} \sigma_s^r.$$

This is an identity between two operator valued series. It is sufficient to prove that applying both sides of (3.39) to an eigenvector ϕ of \mathcal{N}_s yields the same vector. Let ν be the eigenvalue of ϕ . The RHS takes the form

$$\frac{1}{|\sigma|} \sum_{r=1}^{|\sigma|} \sum_{k=1}^{\infty} \frac{(\eta^r x^{1/|\sigma|})^k}{k^p} \eta^{-\nu|\sigma|r} \phi.$$

Exchanging the order of the summation and using that

$$\frac{1}{|\sigma|} \sum_{r=1}^{|\sigma|} \eta^{r(k-\nu|\sigma|)} = \begin{cases} 1 & \text{if } k = (\nu + m)|\sigma| \text{ for some } m \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

we get that in the summation over k only the terms for which $k = (\nu + m)|\sigma|$ with $m \in \mathbb{Z}$ and $m \geq 1$ could have non-trivial contribution. Therefore, it remains only to check that after substituting $k = (\nu + m)|\sigma|$ and summing over all $m \geq 1$ we get precisely the LHS of (3.39) applied to the eigenvector ϕ . \square

Lemma 3.10 implies that $\text{Li}_\sigma(x)$ extends analytically along any path avoiding 0 and the $|\sigma|$ -roots of 1. In particular, using analytic continuation we can define

$$(3.40) \quad \tilde{\Omega}_{\alpha,\beta}(\lambda_1, \lambda_2) - \tilde{\Omega}_{\beta,\alpha}(\lambda_2, \lambda_1).$$

This difference measures the failure of the vertex operators $\tilde{\Gamma}^\alpha(\lambda_1)$ and $\tilde{\Gamma}^\beta(\lambda_2)$ to commute and as we will see later on it will play a key role in understanding the analytic continuation of the phase factors. It turns out that (3.40) can be expressed in terms of the Euler pairing. In order to derive the precise formula we need to recall the so-called Jonquière's inversion formula ([37]). Let $B_p(x)$ ($p \geq 0$) be the *Bernoulli polynomials*. They are defined by the following formula:

$$\frac{te^{tx}}{e^t - 1} =: \sum_{p=0}^{\infty} B_p(x) \frac{t^p}{p!}.$$

LEMMA 3.11. *If $0 < |x| < 1$, then*

$$(3.41) \quad \text{Li}_p(1/x) = (-1)^{p+1} \text{Li}_p(x) + (-1)^{p+1} \frac{(2\pi\mathbf{i})^p}{p!} B_p\left(\frac{1}{2\pi\mathbf{i}} \log x\right),$$

where the value of $\text{Li}_p(1/x)$ is determined from the value of $\text{Li}_p(x)$ via analytic continuation along a path that does not wind around $x = 1$ and the value of $\log x$ is chosen in such a way that the formula holds for $p = 1$.

PROOF. By continuity, it is sufficient to prove the lemma for x such that $\text{Im}(x) \neq 0$. Let C be the path from x to $1/x$ along which $\text{Li}(x)$ is extended analytically. Since C does not wind around $x = 1$, we may assume that C intersect the real axis once. There are four cases depending on whether $\text{Im}(x)$ is > 0 or < 0 and C intersects or it does not intersect the interval $(1, \infty)$. Let us consider the case when $\text{Im}(x) < 0$ and C does not intersect $(1, \infty)$. The argument in the remaining cases is similar. Put

$$f_p(x) = \text{Li}_p(x) + (-1)^p \text{Li}_p(1/x).$$

Note that since $x\partial_x f_p(x) = f_{p-1}(x)$, the function $f_p(x)$ must be a polynomial in $\log x$, that is,

$$f_p(x) = -\frac{(2\pi\mathbf{i})^p}{p!} h_p\left(\frac{\log x}{2\pi\mathbf{i}}\right),$$

for some polynomial h_p . In order to determine h_p , we will find the analytic continuation of $f(x)$ when x varies along a circle γ with center at 0. To begin with, let us determine the branch of $\log x$. Formula (3.41) for $p = 1$ takes the form

$$-\text{Log}(1 - 1/x) = -\text{Log}(1 - x) + 2\pi\mathbf{i}\left(\frac{1}{2\pi\mathbf{i}} \log x - \frac{1}{2}\right),$$

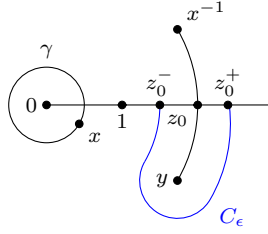


FIGURE 1. Contour deformation.

where we used that the path $1 - C := \{1 - y \mid y \in C\}$ connecting $1 - x$ and $1 - 1/x$ is in the domain of analyticity for the principal branch of the logarithm, i.e., the path $1 - C$ does not intersect the negative real axis. We get

$$\log x = \ln |x| + \mathbf{i} \left(\text{Arg}(1 - x) - \text{Arg}(1 - 1/x) + \pi \right) = \ln |x| + \mathbf{i} \left(\text{Arg}(x) + 2\pi \right) = \text{Log}(x) + 2\pi\mathbf{i},$$

where we used that $1 - x = -x(1 - \frac{1}{x})$ and hence

$$\text{Arg}(1 - x) = \text{Arg}(-x) + \text{Arg}(1 - 1/x) = \pi + \text{Arg}(x) + \text{Arg}(1 - 1/x).$$

We have the following integral representation:

$$(3.42) \quad \text{Li}_p(y) := \frac{y}{\Gamma(p)} \int_1^\infty \frac{(\ln z)^{p-1}}{z - y} \frac{dz}{z}, \quad |y| < 1,$$

where the integration path is the real interval $[1, \infty)$. Indeed, for $|y| < 1$ we can expand $\frac{1}{z-y}$ into a geometric series in the powers of y/z . Changing the integration variable $z = e^t$ and integrating termwise the geometric series we get precisely the polylogarithmic series $\text{Li}_p(y)$. The RHS of formula (3.42) is an analytic function in $y \in \mathbb{C} \setminus [1, \infty)$, that is, formula (3.42) provides the analytic continuation of $\text{Li}_p(y)$ in the complex plane cut along the interval $[1, \infty)$. If we vary x anti-clockwise along the circle γ , then $y = x^{-1}$ varies clockwise along a circle γ' that intersects the interval $[1, \infty)$. If $y \in \gamma'$ is a point in the lower half-plane, then let us compare the value of $\text{Li}_p(y)$ and the value $\tilde{\text{Li}}_p(y)$ obtained from $\text{Li}_p(y')$ when y' approaches y along an arc that crosses the interval $(1, \infty)$. The value $\tilde{\text{Li}}_p(y)$ can be computed by deforming the integration contour in (3.42) near the point z_0 at which γ' intersects $(1, \infty)$. Namely, let us cut out from $(1, \infty)$ a small interval $[z_0^-, z_0^+]$ around z_0 and replace it with a smooth curve C_ϵ from z_0^- to z_0^+ , such that, C_ϵ does not intersect the arc on γ' from z_0 to y (see Figure 1). The difference of the two values can be computed with the Cauchy residue theorem, that is,

$$\tilde{\text{Li}}_p(y) - \text{Li}_p(y) = \frac{y}{\Gamma(p)} \text{Res}_{z=y} \frac{(\text{Log } z)^{p-1}}{z - y} \frac{dz}{z} = \frac{2\pi\mathbf{i}}{\Gamma(p)} (\text{Log } y)^{p-1}.$$

Using the above formula we get

$$\text{a.c. } f_p(x) - f_p(x) = (-1)^p \left(\tilde{\text{Li}}_p(x^{-1}) - \text{Li}_p(x^{-1}) \right) = (-1)^p \frac{2\pi\mathbf{i}}{\Gamma(p)} (\text{Log } x^{-1} - 2\pi\mathbf{i})^{p-1},$$

where the shift of $\text{Log } x^{-1}$ by $-2\pi\mathbf{i}$ comes from the fact that we analytically extend $\text{Log}(y)$ along a path that crosses the negative real axis, that is, the cut in the domain of

analyticity of Log . Note that $\text{Log } x^{-1} - 2\pi\mathbf{i} = -\log x$. Therefore,

$$\text{a.c. } f_p(x) - f_p(x) = -\frac{2\pi\mathbf{i}}{\Gamma(p)} (\log x)^{p-1}.$$

On the other hand, substituting for $f_p(x)$ in the above formula the formula for $f_p(x)$ in terms of the polynomial h_p , we get the following difference equation:

$$h_p(L+1) - h_p(L) = pL^{p-1}, \quad L := \log x/2\pi\mathbf{i}, \quad p \geq 1.$$

Using $x\partial_x f_p(x) = f_{p-1}(x)$, we also get $\partial_L h_p(L) = ph_{p-1}(L)$. It is easy to check that the solution to these relations is unique and it is given by the Bernoulli polynomials, that is, $h_p(L) = B_p(L)$. \square

Let us point out the following rule for finding the branch of $\log x$ in formula (3.41). If we walk along the path connecting x and x^{-1} , then when crossing the real axis either 1 is on our left or on our right and we have

$$\log x = \begin{cases} \text{Log } x & \text{if 1 is on our left,} \\ \text{Log } x + 2\pi\mathbf{i} & \text{if 1 is on our right.} \end{cases}$$

This rule is proved by analyzing formula (3.41) for $p = 1$ in the same way as we did in the proof of Lemma 3.11.

3.3.4. Analyticity of the calibrated propagators. Let us write the calibrated propagator in the form

$$\tilde{\Omega}_{\alpha,\beta}(\lambda_1, \lambda_2) = \Omega(\tilde{\mathbf{f}}_\alpha(\lambda_1, w)_{>0}, \tilde{\mathbf{f}}_\beta(\lambda_2, w)) + P_{\alpha,\beta}(\lambda_1, \lambda_2),$$

where the term $P_{\alpha,\beta}(\lambda_1, \lambda_2)$ is a polynomial expression in $\log \lambda_i$ and $\lambda_i^{1/|\sigma|}$. By definition, $\tilde{\Omega}_{\alpha,\beta}(\lambda_1, \lambda_2)$ is a formal Laurent series of the following form:

$$\sum_{r=0}^{|\sigma|-1} \sum_{l=0}^{d-1} \lambda_1^{m+\frac{r}{|\sigma|}} (\log \lambda_1)^l \sum_{k=0}^{\infty} \tilde{\Omega}_{\alpha,\beta}^{r,l,k}(\lambda_2) \lambda_1^{-k},$$

where the value of $\log \lambda_1$ is specified by a reference path from λ° to λ_1 , the coefficients $\tilde{\Omega}_{\alpha,\beta}^{r,l,k}(\lambda_2)$ are multivalued analytic functions in λ_2 whose value is determined from the reference path between λ° and λ_1 and the straight segment $[\lambda_1, \lambda_2]$. Using Theorem 3.9, we get that the above series is convergent if the following two conditions are satisfied: (i) $|\lambda_1| > |\lambda_2|$ and (ii) the straight segment $[\lambda_1, \lambda_2]$ does not contain the origin 0. Let us construct a domain where these two conditions are satisfied. Let us fix a real number ϵ , such that, $0 < \epsilon < 1$. Put

$$\tilde{D} := \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid |\lambda_1 - \lambda_2| < \epsilon \min(|\lambda_1|, |\lambda_2|)\}$$

and let

$$\tilde{D}^+ := \{(\lambda_1, \lambda_2) \in \tilde{D} \mid |\lambda_1| > |\lambda_2|\}.$$

The calibrated propagator is a multi-valued analytic function on \tilde{D}^+ , where we choose the reference point in \tilde{D}^+ to be the point $(\lambda^\circ, \lambda^\circ (1 - \frac{\epsilon^2}{2}))$. The domains \tilde{D} and \tilde{D}^+ can be described also in the following way. Let us define

$$\Delta_\epsilon := \{x \in \mathbb{C} \mid |x - 1| < \epsilon\} \cap \{x \in \mathbb{C} \mid |x^{-1} - 1| < \epsilon\}.$$

Note that the inequality $|x^{-1} - 1| < \epsilon$ defines a disk in \mathbb{C} with center $\frac{1}{1-\epsilon^2}$ and radius $\frac{\epsilon}{1-\epsilon^2}$. In other words, Δ_ϵ is an intersection of two disks in \mathbb{C} . Moreover, the boundaries of these disks, that is, $\{|x - 1| = \epsilon\}$ and $|x^{-1} - 1| = \epsilon$, intersect at the points

$$a := 1 - \frac{\epsilon^2}{2} + \mathbf{i}\epsilon\sqrt{1 - \frac{\epsilon^2}{4}} \quad \text{and} \quad \bar{a} := 1 - \frac{\epsilon^2}{2} - \mathbf{i}\epsilon\sqrt{1 - \frac{\epsilon^2}{4}},$$

which belong to the unit circle $|x| = 1$. In particular, the unitcircle divides Δ_ϵ into two subdomains

$$\Delta_\epsilon^+ := \{x \in \Delta_\epsilon \mid |x| < 1\} \quad \text{and} \quad \Delta_\epsilon^- := \{x \in \Delta_\epsilon \mid |x| > 1\}.$$

We have an isomorphism

$$\tilde{D} \cong \mathbb{C}^* \times \Delta_\epsilon, \quad (\lambda_1, \lambda_2) \mapsto (\lambda_1, \lambda_2/\lambda_1)$$

under which the subdomain \tilde{D}^+ is mapped to $\mathbb{C}^* \times \Delta_\epsilon^+$ and the reference point of \tilde{D}^+ is mapped to $(\lambda^\circ, 1 - \frac{\epsilon^2}{2})$. From now on we require that ϵ is so small that the only solution to the equation $x^{|\sigma|} = 1$ for $x \in \Delta_\epsilon$ is $x = 1$. To be more specific, if $|\sigma| = 1$, then we choose $\epsilon := 1/2$. If $|\sigma| > 1$, then note that the length of the side of a regular $|\sigma|$ -gon inscribed in the unit circle is $2\sin(\pi/|\sigma|)$. Therefore, for ϵ we can choose any number $< 2\sin(\pi/|\sigma|)$. Let us fix $\epsilon := \sin(\pi/|\sigma|)$. Recalling Theorem 3.9 we get that the calibrated propagator can be written as follows:

$$(3.43) \quad \tilde{\Omega}_{\alpha,\beta}(\lambda_1, \lambda_2) = -(\text{Li}_\sigma(x)\alpha|\beta) + P_{\alpha,\beta}(\lambda_1, \lambda_2),$$

where $x := \lambda_2/\lambda_1$ and the RHS is viewed as an analytic function in $\tilde{D}^+ \cong \mathbb{C}^* \times \Delta_\epsilon^+$. According to Lemma 3.10, the series $\text{Li}_\sigma(x)$ can be expressed as a polynomial expression in $x^{-\rho} = e^{-\rho \text{Log } x}$ and the values of Li_ρ at the points

$$\phi_r(x) := \eta^r x^{1/|\sigma|} := \exp\left(\frac{1}{|\sigma|}(\text{Log } x + 2\pi\mathbf{i}r)\right), \quad 1 \leq r \leq |\sigma|.$$

Therefore, the calibrated propagator is an analytic function on \tilde{D}^+ which extends analytically along any path in \tilde{D} avoiding the diagonal $\{\lambda_1 = \lambda_2\} \subset \tilde{D}$.

LEMMA 3.12. *Suppose that $(\lambda_1, \lambda_2) \in \tilde{D}^+$ and that $C \subset \tilde{D} \setminus \{\lambda_1 = \lambda_2\}$ is a path from (λ_1, λ_2) to (λ_2, λ_1) , such that, C does not wind around the diagonal $\{\lambda_1 = \lambda_2\}$. Then*

$$\tilde{\Omega}_{\alpha,\beta}(\lambda_1, \lambda_2) - \tilde{\Omega}_{\beta,\alpha}(\lambda_2, \lambda_1) = -2\pi\mathbf{i}\left(\langle \alpha, \beta \rangle + k(\alpha|\beta)\right),$$

where $\tilde{\Omega}_{\beta,\alpha}(\lambda_2, \lambda_1)$ is obtained from $\tilde{\Omega}_{\beta,\alpha}(\lambda_1, \lambda_2)$ via analytic continuation along the path C and $k \in \mathbb{Z}$ depends on the choice of C .

PROOF. Using formula (3.43), we get

$$\tilde{\Omega}_{\alpha,\beta}(\lambda_1, \lambda_2) - \tilde{\Omega}_{\beta,\alpha}(\lambda_2, \lambda_1) = \left((\text{Li}_\sigma(1/x) - \text{Li}_\sigma(x)^t)\beta|\alpha\right) + P_{\alpha,\beta}(\lambda_1, \lambda_2) - P_{\beta,\alpha}(\lambda_2, \lambda_1),$$

where $x := \lambda_2/\lambda_1$ and t denotes transposition with respect to the intersection pairing $(\cdot|\cdot)$. By definition

$$P_{\alpha,\beta}(\lambda_1, \lambda_2) = \Omega(\tilde{\mathbf{f}}_\alpha(\lambda_1, w)_+, \tilde{\mathbf{f}}_\beta(\lambda_2, w)) - \Omega(\tilde{\mathbf{f}}_\alpha(\lambda_1, w)_{>0}, \tilde{\mathbf{f}}_\beta(\lambda_2, w)).$$

Using that Ω is a symplectic pairing and that $\mathcal{H}_{>0}$ is symplectic orthogonal to $\mathcal{H}_{\geq 0}$ we get

$$P_{\beta,\alpha}(\lambda_2, \lambda_1) = \Omega(\tilde{\mathbf{f}}_\beta(\lambda_2, w), \tilde{\mathbf{f}}_\alpha(\lambda_1, w)_-) - \Omega(\tilde{\mathbf{f}}_\beta(\lambda_2, w), \tilde{\mathbf{f}}_\alpha(\lambda_1, w)_{<0}).$$

Therefore,

$$P_{\alpha,\beta}(\lambda_1, \lambda_2) - P_{\beta,\alpha}(\lambda_2, \lambda_1) = \Omega(\tilde{\mathbf{f}}_\alpha(\lambda_1, w)_0, \tilde{\mathbf{f}}_\beta(\lambda_2, w)) = -\Omega(\tilde{\mathbf{f}}_\beta(\lambda_2, w)_0, \tilde{\mathbf{f}}_\alpha(\lambda_1, w)).$$

Recalling formula (3.33) we get

$$(3.44) \quad P_{\alpha,\beta}(\lambda_1, \lambda_2) - P_{\beta,\alpha}(\lambda_2, \lambda_1) = \left\langle \frac{e^{2\pi i \rho} - 1}{\rho} x^\rho \beta_0, \alpha_0 \right\rangle.$$

Recalling Lemma 3.10 we get

$$\mathrm{Li}_\sigma(x)^t = x^\rho \sum_{p=1}^{\infty} \sum_{r=1}^{|\sigma|} (-|\sigma|\rho)^{p-1} \mathrm{Li}_p(\phi_r(x)) \sigma_s^{-r},$$

where we used that $\rho^t = -\rho$ and $\sigma_s^t = \sigma_s^{-1}$. The value of $\mathrm{Li}_\sigma(1/x)$ is obtained via analytic continuation as follows. Under the isomorphism $\tilde{D} \cong \mathbb{C}^* \times \Delta_\epsilon$, the path $C = (C', C'')$, where C' is a path from λ_1 to λ_2 and C'' is a path from $x = \lambda_2/\lambda_1$ to $1/x$ that does not wind around 1. Let us recall the map

$$\phi_r : \Delta_\epsilon \rightarrow \mathbb{C}^*, \quad \phi_r(x) := \eta^r x^{1/|\sigma|} := \eta^r e^{\frac{1}{|\sigma|} \mathrm{Log} x}.$$

The analytic continuation of $\mathrm{Li}_p(\phi_r(x))$ along C'' yields $\mathrm{Li}_p(\phi_r(x^{-1})) = \mathrm{Li}_p(\phi_{-r}(x)^{-1})$. Let C_{-r} be the composition of a path inside the unit disk from $\phi_{-r}(x)$ to $\phi_r(x)$ and the path $\phi_r(C'')$. Note that $\mathrm{Li}_p(\phi_r(x^{-1}))$ is obtained from $\mathrm{Li}_p(\phi_{-r}(x))$ via the analytic continuation along C_{-r} . Using Jonquière's inversion formula (3.41) we get

$$(3.45) \quad \mathrm{Li}_p(\phi_r(1/x)) = (-1)^{p+1} \mathrm{Li}_p(\phi_{-r}(x)) + (-1)^{p+1} \frac{(2\pi i)^p}{p!} B_p\left(\frac{1}{2\pi i} \log \phi_{-r}(x)\right),$$

where $\log \phi_{-r}(x) = \mathrm{Log} \phi_{-r}(x) + 2\pi i \chi_{-r}$ with $\chi_{-r} = 0$ or 1 depending on whether 1 is on the left or on the right of the path C_{-r} . Since

$$\mathrm{Li}_\sigma(x) = x^{-\rho} \sum_{p=1}^{\infty} \sum_{r=1}^{|\sigma|} (|\sigma|\rho)^{p-1} \mathrm{Li}_p(\phi_r(x)) \sigma_s^r,$$

using formula (3.45) we get that the analytic continuation of $\mathrm{Li}_\sigma(x)$ along C'' is

$$\mathrm{Li}_\sigma(1/x) = \mathrm{Li}_\sigma(x)^t - \frac{x^\rho}{|\sigma|\rho} \sum_{r=1}^{|\sigma|} \sum_{p=1}^{\infty} \frac{(-2\pi i |\sigma|\rho)^p}{p!} B_p\left(\frac{1}{2\pi i} \log \phi_r(x)\right) \sigma_s^{-r}.$$

Recalling the definition of the Bernoulli polynomials we get that the infinite sum over p can be computed explicitly, that is,

$$(3.46) \quad \mathrm{Li}_\sigma(1/x) - \mathrm{Li}_\sigma(x)^t = -\frac{x^\rho}{|\sigma|\rho} \sum_{r=1}^{|\sigma|} \left(-2\pi i |\sigma|\rho \frac{e^{-|\sigma|\rho \log \phi_r(x)}}{e^{-2\pi i |\sigma|\rho} - 1} - 1 \right) \sigma_s^{-r}.$$

Note that if $\phi_r(x)$ is above the real axis, then 1 is on the left of the path C_r , that is, $\chi_r = 0$ and we have

$$(3.47) \quad \mathrm{Arg}(\phi_r(x)) + 2\pi \chi_r = \frac{2\pi r}{|\sigma|} + \frac{1}{|\sigma|} \mathrm{Arg}(x).$$

If $\phi_r(x)$ is below the real axis, then 1 is on the right of C_r , that is, $\chi_r = 1$ and formula (3.47) continues to hold. Therefore, $\log \phi_r(x) = \frac{1}{|\sigma|} \text{Log } x + \frac{2\pi i r}{|\sigma|}$ for $1 \leq r \leq |\sigma| - 1$ and $\log \phi_0(x) = \frac{1}{|\sigma|} \text{Log } x + 2\pi i \chi_0$, where χ_0 is 0 or 1 depending on whether 1 is on the left or on the right of the path C'' . Using these formulas for $\log \phi_r(x)$, we get that the RHS of (3.46) takes the form

$$\frac{2\pi i x^\rho}{e^{-2\pi i \rho |\sigma|} - 1} \left(\sum_{r=1}^{|\sigma|-1} e^{-\rho(\text{Log } x + 2\pi i r)} \sigma_s^{-r} + e^{-\rho(\text{Log } x + 2\pi i |\sigma| \chi_0)} \right) + \frac{x^\rho}{\rho} \left(\frac{1}{|\sigma|} \sum_{r=1}^{|\sigma|} \sigma_s^{-r} \right).$$

Recall that $\sigma = \sigma_s e^{2\pi i \rho}$ and that $\sigma_s^{|\sigma|} = 1$. Therefore, the above formula takes the form

$$\frac{2\pi i}{\sigma^{-|\sigma|} - 1} \left(\sum_{r=1}^{|\sigma|-1} \sigma^{-r} + \sigma^{-|\sigma| \chi_0} \right) + \frac{x^\rho}{\rho} \left(\frac{1}{|\sigma|} \sum_{r=1}^{|\sigma|} \sigma_s^{-r} \right).$$

Note also that $\frac{1}{|\sigma|} \left(\sum_{r=1}^{|\sigma|} \sigma_s^{-r} \right) (1 - \sigma) = \frac{1}{|\sigma|} \left(\sum_{r=1}^{|\sigma|} \sigma_s^{-r} \right) (1 - e^{2\pi i \rho})$. Using formula (3.46) and that $\chi_0 = 0$ or 1 we get

$$(\text{Li}_\sigma(1/x) - \text{Li}_\sigma(x)^t)(1 - \sigma) = 2\pi i \sigma^{1-\chi_0} - x^\rho \frac{e^{2\pi i \rho} - 1}{\rho} \left(\frac{1}{|\sigma|} \sum_{r=1}^{|\sigma|} \sigma_s^r \right).$$

Recalling formula (3.31) we get

$$((\text{Li}_\sigma(1/x) - \text{Li}_\sigma(x)^t)\beta|\alpha) = 2\pi i \langle \sigma^{1-\chi_0} \beta, \alpha \rangle - \left\langle x^\rho \frac{e^{2\pi i \rho} - 1}{\rho} \beta_0, \alpha_0 \right\rangle,$$

where we used that $\frac{1}{|\sigma|} \sum_{r=1}^{|\sigma|} \sigma_s^r$ is the projection operator $H \rightarrow H_0$. Recalling (3.44), we get

$$\tilde{\Omega}_{\alpha,\beta}(\lambda_1, \lambda_2) - \tilde{\Omega}_{\beta,\alpha}(\lambda_2, \lambda_1) = 2\pi i \langle \sigma^{1-\chi_0} \beta, \alpha \rangle.$$

In order to complete the proof of the theorem we need only to notice that if $\chi_0 = 0$, then

$$\langle \sigma \beta, \alpha \rangle = \langle \beta, \alpha \rangle + \langle (\sigma - 1)\beta, \alpha \rangle = \langle \beta, \alpha \rangle - (\beta|\alpha) = -\langle \alpha, \beta \rangle,$$

and if $\chi_0 = 1$, then $\langle \beta, \alpha \rangle = -\langle \alpha, \beta \rangle + (\alpha|\beta)$. \square

3.3.5. Analyticity of the propagators. Our goal now is to establish the analytic properties of the propagators (3.26). By definition, $\Omega_{\alpha,\beta}(t, \lambda_1, \lambda_2)$ is a formal Laurent series of the following form:

$$\sum_{r=0}^{|\sigma|-1} \sum_{l=0}^{d-1} \lambda_1^{m + \frac{r}{|\sigma|}} (\log \lambda_1)^l \sum_{k=0}^{\infty} \Omega_{\alpha,\beta}^{r,l,k}(t, \lambda_2) \lambda_1^{-k},$$

which is obtained from (3.26) by expanding each of the periods $I_\alpha^{(k)}(t, \lambda_1)$ into a Laurent series in $\lambda_1^{-1/|\sigma|}$. Note that in order for such an expansion to exist we need to choose the point (t, λ_1) to be such that $|\lambda_1| > r(t)$, where

$$r(t) := \max \{ |u| \mid u \in \text{Spec}(E_{\bullet_t}) \},$$

where $\text{Spec}(E_{\bullet_t})$ is the set of eigenvalues of the operator of Frobenius multiplication by E . Indeed, for fixed $t \in M$, the periods $I_\alpha^{(k)}(t, \lambda)$ are solutions to a Fuchsian differential equation in λ whose singularities are precisely at the points where $\lambda - E_{\bullet_t}$ is not an

invertible operator. Furthermore, in order to specify the value of $I_\beta^{(-k-1)}(t, \lambda_2)$ we require that the line segment $[(t, \lambda_1), (t, \lambda_2)]$ in $\{t\} \times \mathbb{C}$ does not intersect the discriminant. If this condition is satisfied, then the composition of the line segment $[(t, \lambda_1), (t, \lambda_2)]$ and the reference path to (t, λ_1) will be a reference path for $I_\beta^{(-k-1)}(t, \lambda_2)$.

LEMMA 3.13. *Let ϵ be a real number satisfying $0 < \epsilon < 1$. If*

$$|\lambda_1 - \lambda_2| < \epsilon \min(|\lambda_1| - r(t), |\lambda_2| - r(t)),$$

then the line segment $[(t, \lambda_1), (t, \lambda_2)]$ does not intersect the discriminant.

PROOF. Suppose that there exist a real number $x \in [0, 1]$ such that $\lambda_2 + x(\lambda_1 - \lambda_2)$ is an eigenvalue of $E_{\bullet t}$. Recalling the definition of $r(t)$, we get $|\lambda_2 + x(\lambda_1 - \lambda_2)| \leq r(t)$. Using the triangle inequality, we get

$$r(t) \geq |\lambda_2 + x(\lambda_1 - \lambda_2)| \geq |\lambda_2| - x|\lambda_1 - \lambda_2|.$$

Therefore,

$$\epsilon(|\lambda_2| - r(t)) \leq \epsilon x|\lambda_1 - \lambda_2| < |\lambda_1 - \lambda_2| < \epsilon(|\lambda_2| - r(t)).$$

This is a contradiction, so the Lemma follows. \square

Let us fix ϵ to be the same as in the definition of the domain \tilde{D} of the calibrated propagator. Motivated by the estimate in Lemma 3.13 we define

$$D := \{(t, \lambda_1, \lambda_2) \in M \times \mathbb{C}^2 \mid |\lambda_1 - \lambda_2| < \epsilon \min(|\lambda_1| - r(t), |\lambda_2| - r(t))\}.$$

Note that our definition of $\Omega_{\alpha, \beta}(t, \lambda_1, \lambda_2)$ works for all $(t, \lambda_1, \lambda_2) \in D$. The following simple lemma will allow us to determine the domain of convergence of the propagator.

LEMMA 3.14. *The following identity holds:*

$$\partial_{\lambda_1} \Omega_{\alpha, \beta}(t, \lambda_1, \lambda_2) = \frac{1}{\lambda_1 - \lambda_2} \left(I_\alpha^{(0)}(t, \lambda_1), (\lambda_2 - E_{\bullet}) I_\beta^{(0)}(t, \lambda_2) \right).$$

PROOF. Recalling the definition of the propagator we get

$$(\lambda_1 - \lambda_2) \partial_{\lambda_1} \Omega_{\alpha, \beta}(t, \lambda_1, \lambda_2) = \sum_{k=0}^{\infty} (-1)^{k+1} (\lambda_1 - \lambda_2) \partial_{\lambda_1} (I_\alpha^{(k)}(t, \lambda_1), I_\beta^{(-k-1)}(t, \lambda_2)).$$

Note that since $I_\beta^{(-k-1)}(t, \lambda_2) = \partial_{\lambda_2} I_\beta^{(-k-2)}(t, \lambda_2)$, the above formula can be rewritten as

$$\sum_{k=0}^{\infty} \left((-1)^{k+1} \left(\lambda_1 \partial_{\lambda_1} I_\alpha^{(k)}(t, \lambda_1), I_\beta^{(-k-1)}(t, \lambda_2) \right) + (-1)^k \left(I_\alpha^{(k+1)}(t, \lambda_1), \lambda_2 \partial_{\lambda_2} I_\beta^{(-k-2)}(t, \lambda_2) \right) \right).$$

Recalling the differential equations of the 2nd structure connection we get

$$\lambda_1 \partial_{\lambda_1} I_\alpha^{(k)}(t, \lambda_1) = E_{\bullet} I_\alpha^{(k+1)}(t, \lambda_1) + \left(\theta - k - \frac{1}{2} \right) I_\alpha^{(k)}(t, \lambda_1)$$

and

$$\lambda_2 \partial_{\lambda_2} I_\beta^{(-k-2)}(t, \lambda_2) = E_{\bullet} I_\alpha^{(-k-1)}(t, \lambda_2) + \left(\theta + k + \frac{3}{2} \right) I_\beta^{(-k-2)}(t, \lambda_2).$$

Substituting these formulas for the derivatives in the above sum, all terms will cancel out except for

$$- \left(\left(\theta - \frac{1}{2} \right) I_\alpha^{(0)}(t, \lambda_1), I_\beta^{(-1)}(t, \lambda_2) \right) = \left(I_\alpha^{(0)}(t, \lambda_1), \left(\theta + \frac{1}{2} \right) I_\beta^{(-1)}(t, \lambda_2) \right).$$

It remains only to use that

$$\left(\theta + \frac{1}{2}\right)I_{\beta}^{(-1)}(t, \lambda_2) = (\lambda_2 - E\bullet)I_{\beta}^{(0)}(t, \lambda_2).$$

□

Since the radius of convergence of $\Omega_{\alpha,\beta}(t, \lambda_1, \lambda_2)$ and its λ_1 -derivative are the same, the formula in Lemma 3.14 implies that the propagator series is convergent for $|\lambda_1| > \max(r(t), |\lambda_2|)$. Let us define

$$D^+ := \{(t, \lambda_1, \lambda_2) \in D \mid |\lambda_1| > |\lambda_2|\}.$$

Then the propagator series is convergent on D^+ , that is, the propagator is a multivalued analytic function in D^+ .

LEMMA 3.15. *The phase factor $W_{\alpha,\beta}(t, \lambda_1, \lambda_2)$ is convergent for all $(t, \lambda_1, \lambda_2) \in D$, that is, it is a multivalued analytic function in D .*

PROOF. Suppose that $(t, \lambda_1, \lambda_2) \in D$. Let us look at the identity

$$(3.48) \quad (\lambda_1 - \lambda_2)\partial_{\lambda_1}W_{\alpha,\beta}(t, \lambda_1, \lambda_2) = (I_{\alpha}^{(0)}(t, \lambda_1), (\lambda_2 - E\bullet)I_{\beta}^{(0)}(t, \lambda_2)) - (\lambda_1 - \lambda_2)\partial_{\lambda_1}\tilde{\Omega}_{\alpha,\beta}(\lambda_1, \lambda_2).$$

Note that (3.48) is a polynomial in $\log \lambda_i$ ($i = 1, 2$), whose coefficients are Laurent series in $\lambda_1^{-1/|\sigma|}$ and $\lambda_2^{-1/|\sigma|}$ convergent for $|\lambda_1| > r(t)$ and $|\lambda_2| > r(t)$. The phase factor is appriory a formal Laurent series of the same type in which the substitution $\lambda_1 = \lambda_2$ makes sense. It follows that (3.48) vanishes for $\lambda_1 = \lambda_2$. In order to complete the proof we need only to prove that (3.48) is divisible by $\lambda_1 - \lambda_2$. By comparing the coefficients in front of the monomials in $\log \lambda_1$ and $\log \lambda_2$ we can reduce the general case to the case when (3.48) does not involve logarithms, i.e., we may assume that (3.48) is a Laurent series in $\lambda_1^{-1/|\sigma|}$ and $\lambda_2^{-1/|\sigma|}$ convergent for $|\lambda_1| > r(t)$ and $|\lambda_2| > r(t)$. Let us consider the $|\sigma|$ -fold covering of the polydisk

$$\{(\lambda_1, \lambda_2) \mid |\lambda_1| > r(t), |\lambda_2| > r(t)\},$$

that is, $\lambda_i^{-1} = x_i^{|\sigma|}$, where (x_1, x_2) belongs to

$$\Delta := \{(x_1, x_2) \in (\mathbb{C}^*)^2 \mid |x_i| < r(t)^{1/|\sigma|}\}$$

We have to prove that if f is a holomorphic function in Δ , such that, f vanishes on the hypersurface $Z := \{x_1^{|\sigma|} = x_2^{|\sigma|}\} \subset \Delta$, then f must be divisible by $x_1^{|\sigma|} - x_2^{|\sigma|}$. Indeed, note that $h(x_1, x_2) := \frac{f(x_1, x_2)}{x_1^{|\sigma|} - x_2^{|\sigma|}}$ is a holomorphic function on $\Delta \setminus Z$. Suppose that $z \in Z$, that is, $z = (\eta a, a)$ for some $a \in \mathbb{C}^*$ and some $|\sigma|$ -root of unity η . Using Weierstrass divison theorem, we get that the germ of f in $\mathcal{O}_{\Delta, z}$ is divisible by $x_1 - \eta x_2$. Therefore, $h(x_1, x_2)$ is locally bounded near every point $z \in Z$. It remains only to recall the Riemann extension theorem. □

Using Proposition 3.5 we will prove that the phase factors and hence the propagators can be extended analytically beyond the domain D . However, let us postpone this goal for the next section because now we are in position to prove a remarkable identity which has several important consequences.

PROPOSITION 3.16. *The following identity holds*

$$(3.49) \quad (I_{\alpha}^{(0)}(t, \lambda), (\lambda - E\bullet)I_{\beta}^{(0)}(t, \lambda)) = (\alpha|\beta).$$

PROOF. Using Lemma 3.14 and since $(\lambda - \mu)\partial_\lambda W_{\alpha,\beta}(t, \lambda, \mu) \rightarrow 0$ when $\mu \rightarrow \lambda$, we get that the LHS of the formula can be written as

$$\lim_{\mu \rightarrow \lambda} (\lambda - \mu)\partial_\lambda \tilde{\Omega}_{\alpha,\beta}(\lambda, \mu).$$

Note that in the above formula we may replace the calibrated propagator $\tilde{\Omega}_{\alpha,\beta}(\lambda, \mu)$ with the energy propagator $\Omega(\tilde{\mathbf{f}}_\alpha(\lambda, w)_{>0}, \tilde{\mathbf{f}}_\beta(\mu, w))$, because the difference between the two propagators is a polynomial expression in $\log \lambda$, $\log \mu$, $\lambda^{\pm 1/|\sigma|}$, and $\mu^{\pm 1/|\sigma|}$, which clearly does not contribute to the above limit. Recalling Theorem 3.9 we get

$$-\lim_{\mu \rightarrow \lambda} (\lambda - \mu)\partial_\lambda \left(\sum_{n=1}^{\infty} \frac{(\mu/\lambda)^{n+\mathcal{N}}}{n+\mathcal{N}} \alpha|\beta \right) = \lim_{\mu \rightarrow \lambda} (\lambda - \mu) \frac{\mu/\lambda}{\lambda - \mu} ((\mu/\lambda)^\mathcal{N} \alpha|\beta) = (\alpha|\beta).$$

□

REMARK 3.17. The formula in Proposition 3.16 was discovered first by K. Saito in the settings of singularity theory (see [53]). It is a bit surprising because the LHS is an analytic expression defined via period integrals and a residue pairing, while the RHS is purely topological, that is, the intersection pairing in vanishing cohomology.

Let us recall the monodromy representation of the second structure connection – see item 2) after Proposition 3.3. Since the RHS of (3.49) is a constant invariant under the analytic continuation, we get that the intersection pairing is monodromy invariant. Moreover, using (3.49) we will prove that the monodromy group of the 2nd structure connection is a reflection group. Let us recall that with every simple loop C around a generic point on the discriminant we have associated a reflection vector φ – see the discussion after Proposition 3.4. By definition, in a neighborhood of a generic point on the discriminant we have (see formula (3.19))

$$I_\varphi^{(0)}(t, \lambda) = \frac{2\sqrt{\Delta_i}}{\sqrt{2(\lambda - u_i)}} \left(\frac{\partial}{\partial u_i} + O(\lambda - u_i) \right),$$

where $u_i = u_i(t)$ is the canonical coordinate, such that, locally near the generic point on the discriminant, the equation of the discriminant is given by $\lambda = u_i(t)$. Substituting the above expansion in

$$(I_\varphi^{(0)}(t, \lambda), (\lambda - E_\bullet)I_\varphi^{(0)}(t, \lambda)) = (\varphi|\varphi)$$

and recalling that $(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i}) = \Delta_i^{-1}$ and $(\lambda - E_\bullet)\frac{\partial}{\partial u_i} = (\lambda - u_i)\frac{\partial}{\partial u_i}$ we get that the LHS of the above equation has the form $2 + O(\lambda - u_i)$. Therefore, $(\varphi|\varphi) = 2$ for every reflection vector $\varphi \in \mathcal{R}$. Suppose now that $a \in H$ is arbitrary and let us decompose $a = a' + k\varphi$, where $a' \in H$ is invariant with respect to the monodromy r_C along the simple loop C – such a decomposition exists according to Proposition 3.4. Since the intersection pairing is monodromy invariant we have $(a'|\varphi) = (r_C(a')|r_C(\varphi)) = -(a'|\varphi) \Rightarrow (a'|\varphi) = 0$. Therefore,

$$(a|\varphi) = (a'|\varphi) + k(\varphi|\varphi) = 2k,$$

that is, $k = (a|\varphi)/2$. Finally, we get

$$r_C(a) = a' - k\varphi = a - 2k\varphi = a - (a|\varphi)\varphi.$$

In other words, the local monodromy r_C coincides with the reflection in H corresponding to the reflection vector φ . Therefore, the monodromy group of the 2nd structure connection coincides with the reflection group $W_{\mathcal{R}}$, that is, the subgroup of $\mathrm{GL}(H)$ generated by the reflections

$$(3.50) \quad r_{\varphi}(a) := a - (a|\varphi)\varphi, \quad \varphi \in \mathcal{R}.$$

REMARK 3.18. Let us recall that the Frobenius manifold M here is simply connected. If M is not simply connected, then the monodromy group $\pi_1(M, t^\circ)$ will contribute to the monodromy of the 2nd structure connection, that is, the monodromy group of the 2nd structure connection is not a reflection group in general.

3.4. Phase form

Let us compute the differential of the propagator

$$\mathcal{W}_{\alpha,\beta}(t, \lambda_1, \lambda_2) := d\Omega_{\alpha,\beta}(t, \lambda_1, \lambda_2),$$

where the differential is the de Rham differential on D^+ . Using Proposition 3.5 we get

$$(3.51) \quad \partial_{t_i}\Omega_{\alpha,\beta}(t, \lambda_1, \lambda_2) = (I_{\alpha}^{(0)}(t, \lambda_1), \phi_i \bullet I_{\beta}^{(0)}(t, \lambda_2)).$$

Recall that we fixed a flat coordinate system on M , such that, $\frac{\partial}{\partial t_1}$ is the unit vector field. We have $(\partial_{t_1} + \partial_{\lambda_1} + \partial_{\lambda_2})\Omega_{\alpha,\beta}(t, \lambda_1, \lambda_2) = 0$ due to the translation invariance of the periods $I_{\alpha}^{(k)}(t, \lambda) = I_{\alpha}^{(k)}(t - \lambda \mathbf{1}, 0)$. Therefore,

$$(\partial_{\lambda_1} + \partial_{\lambda_2})\Omega_{\alpha,\beta}(t, \lambda_1, \lambda_2) = -(I_{\alpha}^{(0)}(t, \lambda_1), I_{\beta}^{(0)}(t, \lambda_2)).$$

Finally, recalling Lemma 3.14 we get the following formula for the differential of the propagator

$$(3.52) \quad \begin{aligned} \mathcal{W}_{\alpha,\beta}(t, \lambda_1, \lambda_2) &= \sum_{i=1}^N (I_{\alpha}^{(0)}(t, \lambda_1), \phi_i \bullet I_{\beta}^{(0)}(t, \lambda_2)) dt_i + \\ &+ ((\lambda_1 - E \bullet) I_{\alpha}^{(0)}(t, \lambda_1), I_{\beta}^{(0)}(t, \lambda_2)) \frac{d(\lambda_1 - \lambda_2)}{\lambda_1 - \lambda_2} + \\ &- (I_{\alpha}^{(0)}(t, \lambda_1), I_{\beta}^{(0)}(t, \lambda_2)) d\lambda_1. \end{aligned}$$

We will refer to $\mathcal{W}_{\alpha,\beta}$ as the *phase form*. It allows us to extend analytically the propagators along paths in the domain of analyticity of $\mathcal{W}_{\alpha,\beta}$. In particular, the possible values of the propagators are governed by the periods of the phase form, i.e., integrals of $\mathcal{W}_{\alpha,\beta}$ along closed loops with a base point. The phase form is multivalued, so the notion of a period is a bit subtle, i.e., the value of the period depends on the choice of a base point. The periods of the phase form will allow us to solve an important problem, i.e., to measure the failure of the propagators to be compatible with the monodromy representation.

3.4.1. Domain of analyticity. Let us fix a domain in which the phase form is a multivalued analytic 1-form. Looking at formula (3.52) we get that we need a domain consisting of points $(t, \lambda_1, \lambda_2)$, such that, $(t, \lambda_1) \in (M \times \mathbb{C})'$ and λ_2 is sufficiently close to λ_1 so that the straight segment in $\{t\} \times \mathbb{C}$ between the points (t, λ_1) and (t, λ_2) does not intersect the discriminant. The following choice seems to be quite natural: let $(M \times \mathbb{C}^2)'$ be the set of points $(t, \lambda_1, \lambda_2) \in M \times \mathbb{C}^2$ satisfying

$$|\lambda_1 - \lambda_2| < \epsilon |\lambda_i - u| \quad \forall u \in \mathrm{Spec}(E \bullet_t) \quad \forall \text{ and } i = 1, 2,$$

where ϵ is the same number used in the definition of the domain \tilde{D} . Note that $(t, \lambda_1) \in (M \times \mathbb{C})'$ and that for fixed (t, λ_1) the set of λ_2 satisfying the above condition is the intersection of several disks (in \mathbb{C}) containing the point λ_1 .

Let us identify the propagator $\Omega_{\alpha, \beta}(t, \lambda_1, \lambda_2)$ with a multivalued analytic function on a certain subdomain of $(M \times \mathbb{C}^2)'$. Clearly, we have to remove from $(M \times \mathbb{C}^2)'$ the points for which $\lambda_1 = \lambda_2$. More precisely, we have a natural embedding

$$(M \times \mathbb{C}^2)' \rightarrow (M \times \mathbb{C}^2)', \quad (t, \lambda) \mapsto (t, \lambda, \lambda).$$

We will refer to the image of the embedding as the *diagonal* of $(M \times \mathbb{C}^2)'$. Let us denote by $(M \times \mathbb{C}^2)''$ the complement of the diagonal. Furthermore, note that the domain $D \subset (M \times \mathbb{C}^2)'$. Indeed, if $(t, \lambda_1, \lambda_2) \in D$, then using the triangle inequality we get

$$|\lambda_1 - \lambda_2| < \epsilon(|\lambda_i| - r(t)) \leq \epsilon(|\lambda_i| - |u|) \leq \epsilon|\lambda_i - u|,$$

for all $u \in \text{Spec}(E_{\bullet t})$. Since the phase form $\mathcal{W}_{\alpha, \beta}$ is a multivalued analytic 1-form on the diagonal complement $(M \times \mathbb{C}^2)''$, we get that the propagator $\Omega_{\alpha, \beta}(t, \lambda_1, \lambda_2)$ extends analytically from D^+ along any path in $(M \times \mathbb{C}^2)''$. We fix the point

$$(t^\circ, \lambda_1^\circ, \lambda_2^\circ) := \left(t^\circ, \lambda^\circ, \lambda^\circ \left(1 - \frac{\epsilon^2}{2} \right) \right) \in D^+$$

as a reference point. The value of the propagator is determined by the homotopy class of a reference path in $(M \times \mathbb{C}^2)''$ which will be called the reference path of the propagator.

The closed loops in $(M \times \mathbb{C}^2)''$ can be separated into two different types: closed loops around the diagonal and closed loops around the discriminant. More precisely, the projection map

$$(M \times \mathbb{C}^2)' \rightarrow (M \times \mathbb{C})', \quad (t, \lambda_1, \lambda_2) \mapsto (t, \lambda_1)$$

is a trivial smooth fibration. The fiber of the projection, as we already mentioned above, is an intersection of several disks in \mathbb{C} containing the point λ_1 . In particular, the diagonal embedding is a section of this fibration, which will be referred to as the *zero section*. Our main interest is in the periods of the phase form along a loop in $(M \times \mathbb{C}^2)''$ obtained as a lift of a loop in $(M \times \mathbb{C})'$. The ambiguity of choosing a lift can be described by yet another loop in the fiber winding several times around the diagonal.

3.4.2. Analytic extension around the diagonal. Suppose that $(t, \lambda_1, \lambda_2) \in D^+$ and that C is a simple loop based at $(t, \lambda_1, \lambda_2)$ that goes around the diagonal in a counter-clockwise direction, that is, the homotopy class of C can be represented by a loop with parametrization

$$(s, \mu_1, \mu_2) = (t, \lambda_1, \lambda_1 + r e^{i\theta}), \quad 0 \leq \theta \leq 2\pi,$$

where $r > 0$ is sufficiently small. Then the analytic continuation of $\Omega_{\alpha, \beta}(t, \lambda_1, \lambda_2)$ along C is given by

$$\Omega_{\alpha, \beta}(t, \lambda_1, \lambda_2) + \int_{(s, \mu_1, \mu_2) \in C} \mathcal{W}_{\alpha, \beta}(s, \mu_1, \mu_2).$$

Since the loop C is in the λ_2 -plane, only the term of $\mathcal{W}_{\alpha, \beta}$ involving $\frac{d(\lambda_1 - \lambda_2)}{\lambda_1 - \lambda_2}$ contributes to the integral. Using the Cauchy residue theorem, we get that the integral equals

$$2\pi i (I_\alpha^{(0)}(t, \lambda_1), (\lambda_1 - E_{\bullet}) I_\beta^{(0)}(t, \lambda_1)) = 2\pi i (\alpha | \beta).$$

On the other hand, we can interpret $\log(\lambda_1 - \lambda_2)$ as a multivalued analytic function on $D \setminus \{\lambda_1 = \lambda_2\}$. Indeed, given a reference path in $(M \times \mathbb{C}^2)''$ we get an induced path in

\mathbb{C}^* between $\lambda_1^\circ - \lambda_2^\circ$ and $\lambda_1 - \lambda_2$. Note that $\lambda_1^\circ - \lambda_2^\circ$ is a positive real number. Let us define the value $\log(\lambda_1^\circ - \lambda_2^\circ)$ by using the principal branch of the logarithm and the value $\log(\lambda_1 - \lambda_2)$ by analytic continuation along the induced reference path.

LEMMA 3.19. *The multivalued analytic function in D^+ given by*

$$(3.53) \quad \Omega_{\alpha,\beta}(t, \lambda_1, \lambda_2) - (\alpha|\beta) \log(\lambda_1 - \lambda_2)$$

extends to a multi-valued analytic function in D .

PROOF. If C is a simple loop around the diagonal, then the analytic continuation of $\log(\lambda_1 - \lambda_2)$ along C is $\log(\lambda_1 - \lambda_2) + 2\pi i$. Therefore, the branches of (3.53) are single valued in a neighborhood of points on the diagonal. We need to check that (3.53) does not have a singularity at $\lambda_1 = \lambda_2$. Differentiating (3.53) with respect to λ_1 , we get

$$\frac{1}{\lambda_1 - \lambda_2} (I_\alpha^{(0)}(t, \lambda_1), (\lambda_2 - E_\bullet) I_\beta^{(0)}(t, \lambda_2)) - \frac{(\alpha|\beta)}{\lambda_1 - \lambda_2}.$$

This function is regular at $\lambda_1 = \lambda_2$ due to Proposition 3.16. Therefore, the Laurent series expansion of (3.53) in λ_1^{-1} is convergent for all $|\lambda_1| > r(t)$. The Lemma follows. \square

The propagator $\Omega_{\alpha,\beta}(t, \lambda_1, \lambda_2)$ has an important symmetry, that is, there is a relation between the values at $(t, \lambda_1, \lambda_2)$ and $(t, \lambda_2, \lambda_1)$. In order to compare the values at these two points we need to fix a path C between them avoiding the diagonal. We would like to avoid using paths that wind around the discriminant. We will say that C is a *local path* if it is homotopic inside D to the straight segment in $\{t\} \times \mathbb{C}^2$ between $(t, \lambda_1, \lambda_2)$ and $(t, \lambda_2, \lambda_1)$.

LEMMA 3.20. *Suppose that $(t, \lambda_1, \lambda_2) \in D^+$ and that C is a local path in $D \setminus \{\lambda_1 = \lambda_2\}$ between $(t, \lambda_1, \lambda_2)$ and $(t, \lambda_2, \lambda_1)$. Then we have*

$$\Omega_{\alpha,\beta}(t, \lambda_1, \lambda_2) - \Omega_{\beta,\alpha}(t, \lambda_2, \lambda_1) = -2\pi i \left(\langle \alpha, \beta \rangle + k(\alpha|\beta) \right),$$

where $k \in \mathbb{Z}$ is an integer depending on the choice of C and the propagator $\Omega_{\beta,\alpha}(t, \lambda_2, \lambda_1)$ is obtained from $\Omega_{\beta,\alpha}(t, \lambda_1, \lambda_2)$ by analytic continuation along C .

PROOF. We may assume that the path C does not wind around the diagonal, because the analytic continuation of the propagator along a closed loop around the diagonal changes the value by $2\pi i(\alpha|\beta)$, that is, changes the value of k . Furthermore, according to Lemma 3.7 the propagator decomposes as a sum of the calibrated propagator $\tilde{\Omega}_{\alpha,\beta}(\lambda_1, \lambda_2)$ and a phase factor $W_{\alpha,\beta}(t, \lambda_1, \lambda_2)$. We already proved that the phase factor is a multivalued analytic function in D . Recalling part b) of Proposition 1.34 we get that $W_{\alpha,\beta}(t, \lambda_1, \lambda_2) = W_{\beta,\alpha}(t, \lambda_2, \lambda_1)$. Finally, it remain only to recall Lemma 3.12. \square

3.4.3. Paths with transverse directions. Suppose that $C \subset \mathbb{C}$ is a smooth oriented path without self-intersections. Let us fix a parametrization of C , that is, a smooth embedding $\Lambda : [0, 1] \rightarrow \mathbb{C}$ compatible with the orientation. Put

$$T^+(C) := \bigcup_{s \in C} \mathbb{R}_{\geq 0} \Lambda'(s).$$

In other words, $T^+(C)$ is the subcone of \mathbb{C} consisting of all *positively* oriented tangent vectors to C . Clearly, $T^+(C)$ is independent of the choice of the parametrization Λ . We will say that $\delta \in \mathbb{C}$ is a *transverse direction* for C if $\delta \notin T^+(C)$. Note that the set of all transverse directions for C coincides with the complement $\mathbb{C} \setminus T^+(C)$.

REMARK 3.21. The notion of transverse is slightly abused here. A transverse direction δ is allowed to be a tangent vector to C (at somepoint) as long as it has the opposite orientation. \square

Let u_1, \dots, u_N be a set of pairwise distinct complex numbers and $\Lambda_0 \in \mathbb{C}$ is a reference point, such that, $|\Lambda_0| > |u_j|$ for all $1 \leq j \leq N$. Recall that a simple loop $L \subset \mathbb{C} \setminus \{u_1, \dots, u_N\}$ around u_j based at Λ_0 is a path consisting of two pieces: a path C from Λ_0 to some point $u_j + \delta$ and a circle with radius $|\delta|$ and center u_j , where $|\delta|$ is required to be so small that the points u_i ($i \neq j$) are outside the circle. The simple loop L is defined as the path traversed by a point traveling along C from Λ_0 to $u_j + \delta$, going around u_j in counter-clockwise direction along the circle, and finally returning back to Λ_0 along the path C . We will refer to C as the *tail* of the simple loop L .

DEFINITION 3.22. Let L be a simple loop around u_i based at Λ_0 . We say that L is a simple loop *approaching u_i in a transverse direction δ* if the following conditions are satisfied:

- (i) The tail C of the simple loop L ends at a point $u_i + \delta$, such that, δ is a transverse direction for C .
- (ii) We have $|\Lambda_0| > |\lambda - \delta|$ for all $\lambda \in C$. \square

A simple loop L approaching u_i in a transverse direction can be constructed as follows. Let $[u_i, \Lambda_0]$ be the line segment between u_i and Λ_0 . Let $\delta \in \mathbb{C}$ be such that $u_i + \delta \in [u_i, \Lambda_0]$ and $|\delta| < |u_j - u_i|$ for all $j \neq i$. Note that since $|u_i| < |\Lambda_0|$, we have $|\lambda - \delta| < |\Lambda_0|$ for all $\lambda \in [u_i + \delta, \Lambda_0]$. If the line segment $[u_i + \delta, \Lambda_0]$ does not contain any of the points u_j , then we can simply take the tail of our loop to be $C = [u_i + \delta, \Lambda_0]$. Otherwise, for each $u_j \in [u_i + \delta, \Lambda_0]$, we cut a small piece from the line segment around u_j and replace it with a half-circle avoiding u_j . The resulting path C has all the required properties provided the pieces that we have removed are sufficiently small. Note that in particular, the fundamental group $\pi_1(\mathbb{C} \setminus \{u_1, \dots, u_N\}, \Lambda_0)$ is generated by simple loops with transverse directions. Therefore, it is sufficient to compute the periods of the phase form along simple loops that admit a transverse direction.

3.4.4. Connection formula. Let t be a semi-simple point, such that, the canonical coordinates $u_1(t), \dots, u_N(t)$ are pairwise distinct. Let $\Lambda_2 \in \mathbb{C}$ be such that $|\Lambda_2| > |u_j(t)|$ for all j . Suppose that C is the tail of a simple loop L around u_i based at Λ_2 approaching u_i in a transverse direction δ . Put $\lambda_2 := u_i + \delta$. Note that $\{t\} \times L$ is a simple loop around the discriminant. Let us fix a reference path (avoiding the discriminant) between $(t^\circ, \lambda^\circ(1 - \epsilon^2/2))$ and (t, Λ_2) and let β be a reflection vector corresponding to the composition of $\{t\} \times C$ and the reference path. Here $\epsilon = \sin(\pi/|\sigma|)$ is the same number as in the definition of the domain D (see Section 3.3.5). Let us consider the set $U_i(t, C)$ of all $\lambda_1 \in \mathbb{C}$ satisfying the following 3 conditions:

- (i) Put $\Lambda_1 := \lambda_1 - \lambda_2 + \Lambda_2$. Then $(t, \Lambda_1, \Lambda_2) \in D^+$.
- (ii) The following inequalities hold:

$$|\lambda_2 - u_i(t)| < |\lambda_1 - u_i(t)| < |u_j(t) - u_i(t)| \quad \forall j \neq i.$$

- (iii) Let $\Lambda(s)$ ($0 \leq s \leq 1$) be a parametrization of C with $\Lambda(0) = \Lambda_2$ and $\Lambda(1) = \lambda_2$. Put \widehat{C} for the path between $(t, \Lambda_1, \Lambda_2)$ and $(t, \lambda_1, \lambda_2)$ with parametrization

$$(3.54) \quad (t, \lambda_1 - \lambda_2 + \Lambda(s), \Lambda(s)), \quad 0 \leq s \leq 1.$$

Then $\widehat{C} \subset (M \times \mathbb{C}^2)''$.

Note that $U_i(t, C)$ is an open subset of \mathbb{C} and that λ_2 is a boundary point of $U_i(t, C)$. Condition (i) is imposed so that the propagator

$$\Omega_{\alpha, \beta}(t, \Lambda_1, \Lambda_2) = \sum_{n=0}^{\infty} (-1)^{n+1} (I_{\alpha}^{(n)}(t, \Lambda_1), I_{\beta}^{(-n-1)}(t, \Lambda_2))$$

where each summand should be expanded into a Laurent series in Λ_1^{-1} . Furthermore, condition (ii) is imposed so that the following series is convergent:

$$\Omega_{\alpha, \beta}^i(t, \lambda_1, \lambda_2) := \sum_{n=0}^{\infty} (-1)^{n+1} (I_{\alpha}^{(n)}(t, \lambda_1), I_{\beta}^{(-n-1)}(t, \lambda_2))$$

where each summand on the RHS is expanded into a Laurent series in $(\lambda_2 - u_i(t))^{1/2}$. To prove the convergence, we argue in the same way as in the proof of the convergence of the propagators. Namely, since β is a reflection vector, the period vector $I_{\beta}^{(-n-1)}(t, \lambda_2)$ has a zero at $\lambda_2 = u_i(t)$ of order at least $n + 1/2$. Therefore, the Laurent series expansions add up to at least a formal Laurent series. The derivative with respect to λ_2 is $\iota_{\partial/\partial\lambda_2} \mathcal{W}_{\alpha, \beta}$ which has a convergent Laurent series expansion at $\lambda_2 = u_i(t)$ provided that condition (ii) holds. Finally, condition (iii) guarantees that we can analytically extend the propagator $\Omega_{\alpha, \beta}(t, \Lambda_1, \Lambda_2)$ along the path \widehat{C} . The following theorem is the first main result in this chapter.

THEOREM 3.23. *Suppose that t, Λ_2, λ_2 , and C are as above. If $\lambda_1 \in U_i(t, C)$, then*

$$\int_{\widehat{C}} \mathcal{W}_{\alpha, \beta} = \Omega_{\alpha, \beta}^i(t, \lambda_1, \lambda_2) - \Omega_{\alpha, \beta}(t, \Lambda_1, \Lambda_2),$$

where $\Lambda_1 = \Lambda_2 + \lambda_1 - \lambda_2$ and the path \widehat{C} is defined by (3.54).

The goal in the rest of this section is to prove Theorem 3.23. Let us make several comments about the identity that we have to prove. First, of all the difference of the LHS and the RHS is locally independent with respect to Λ_2, λ_2 , and λ_1 . Therefore, we may choose λ_2 so close to $u_i(t)$ that the length of $\delta = \lambda_2 - u_i(t)$ is as small as we wish. Let us assume that

$$(3.55) \quad 2|\delta| (\sin(\pi/|\sigma|) + 1) < |\Lambda(s) - u_j(t)|$$

for all $j \neq i$ and $s \in [0, 1]$. Similarly, we may choose λ_1 as close to λ_2 as we wish. Let us consider the path A with parametrization $A(s) := \Lambda(1 - s) - \delta$, $0 \leq s \leq 1$, connecting the points $A(0) = u_i(t)$ and $A(1) = \Lambda_2 - \delta$. Let D_j ($j \neq i$) be a set of pairwise disjoint open disks in \mathbb{C} with centers at $u_j(t)$ and sufficiently small radii δ_j and let $D_i(A)$ be a sufficiently small open neighborhood of A in \mathbb{C} . We can slightly deform the base point Λ_2 of the tail C of the simple loop without destroying the property that δ is a transverse direction. Therefore, using the Painleve property for semi-simple Frobenius manifolds, we can arrange that the germ of the Frobenius manifold (M, t) extends to a Frobenius structure on

$$\mathbb{D}(A) = D_1 \times \cdots \times D_{i-1} \times D_i(A) \times D_{i+1} \times \cdots \times D_N.$$

Let us define the path $\tau(s)$, $0 \leq s \leq 1$ in $\mathbb{D}(A)$ by the following formulas in canonical coordinates

$$u_i(\tau(s)) = \Lambda(1 - s) - \delta, \quad u_j(\tau(s)) = u_j(t) \quad (j \neq i),$$

that is, the path τ is such that only the i th canonical coordinate varies while the remaining ones remain fixed. The main idea here is the following. Note that we have extended the Frobenius manifold along a path in such a way that the i -th canonical coordinate becomes $u_i(\tau(1)) = \Lambda_2 - \delta$ while the remaining ones remain fixed. Therefore, since $|\delta|$ can be made as small as we wish, we have a continuous deformation which makes the i th canonical coordinate the longest one. Therefore, we can reduce the proof of the theorem to the case when the canonical coordinate u_i has maximal length among all canonical coordinates. As we will see below the proof of the theorem in such a special case is quite easy. The main difficulty will be to analyze the change of the integral of the phase form.

LEMMA 3.24. *If λ_1 is sufficiently close to λ_2 , then $(\tau(s), \Lambda_1, \Lambda_2) \in D^+$ for all $0 \leq s \leq 1$.*

PROOF. We have to prove the following inequalities:

$$|\Lambda_1| > |\Lambda_2| > |u_j(\tau(s))|, \quad \forall 1 \leq j \leq N$$

and

$$|\lambda_1 - \lambda_2| < \epsilon(|\Lambda_a| - |u_j(\tau(s))|), \quad \forall a = 1, 2, \quad \forall 1 \leq j \leq N,$$

where $\epsilon = \sin(\pi/|\sigma|)$. The statement is true for $s = 0$ by definition and since $u_j(\tau(s)) = u_j(t)$ for all $j \neq i$, the only inequalities that we have to check are the ones that involve $u_i(\tau(s)) = \Lambda(1-s) - \delta$. Recalling the definition of a transverse direction for the simple loop (see Definition 3.22, condition (ii)) we get $|\Lambda_2| > |\Lambda(1-s) - \delta|$ for all $0 \leq s \leq 1$. The remaining two inequalities

$$|\lambda_1 - \lambda_2| < \epsilon(|\Lambda_a| - |u_i(\tau(s))|), \quad \forall a = 1, 2,$$

can be achieved by choosing λ_1 sufficiently close to λ_2 . \square

Put

$$\lambda_1(s) := \lambda_1 - \lambda_2 + \Lambda(1-s), \quad \lambda_2(s) := \Lambda(1-s), \quad 0 \leq s \leq 1.$$

LEMMA 3.25. *If λ_1 is sufficiently close to λ_2 , then*

$$|\lambda_2(s) - u_i(\tau(s))| < |\lambda_1(s) - u_i(\tau(s))| < |u_j(\tau(s)) - u_i(\tau(s))|, \quad \forall j \neq i.$$

PROOF. We have

$$\lambda_1(s) - u_i(\tau(s)) = \lambda_1 - \lambda_2 + \Lambda(1-s) - (\Lambda(1-s) - \delta) = \lambda_1 - \lambda_2 + \delta = \lambda_1 - u_i(t)$$

and

$$\lambda_2(s) - u_i(\tau(s)) = \Lambda(1-s) - (\Lambda(1-s) - \delta) = \lambda_2 - u_i(t).$$

We need only to prove the inequality

$$|\lambda_1 - u_i(t)| < |u_j(t) + \delta - \Lambda(1-s)|.$$

We have

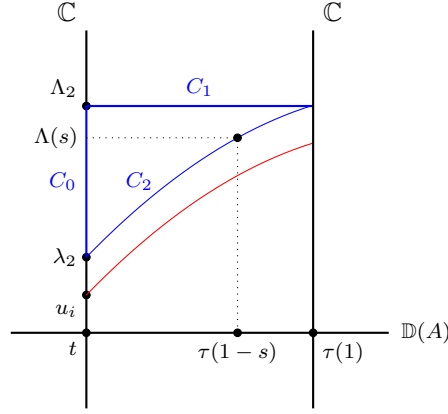
$$|\lambda_1 - u_i(t)| \leq |\lambda_2 - u_i(t)| + |\lambda_1 - \lambda_2| = |\delta| + |\lambda_1 - \lambda_2| < 2|\delta|(1 + \sin(\pi/|\sigma|)) - |\delta|$$

where the last inequality will hold provided we choose λ_1 sufficiently close to λ_2 . Finally, recalling the estimate (3.55) we get

$$2|\delta|(1 + \sin(\pi/|\sigma|)) - |\delta| < |\Lambda(1-s) - u_j(t)| - |\delta| \leq |\Lambda(1-s) - u_j(t) - \delta|. \quad \square$$

LEMMA 3.26. *If λ_1 is sufficiently close to λ_2 , then*

$$(\tau(s), \lambda_1(s), \lambda_2(s)) \in (\mathbb{D}(A) \times \mathbb{C}^2)''.$$

FIGURE 2. The paths C_0, C_1 , and C_2 .

PROOF. We have to prove the inequalities

$$|\lambda_1 - \lambda_2| < \epsilon |\lambda_a(s) - u_j(\tau(s))|, \quad a = 1, 2, \quad 1 \leq j \leq N,$$

where $\epsilon = \sin(\pi/|\sigma|)$. If $j = i$, then $|\lambda_a - u_i(\tau(s))| = |\lambda_a - u_i(t)|$ and the inequalities are true because by definition $(t, \lambda_1, \lambda_2) \in (M \times \mathbb{C}^2)''$. If $j \neq i$, then

$$|\lambda_1(s) - u_j(\tau(s))| = |\lambda_1 - \lambda_2 + \Lambda(1-s) - u_j(t)| \geq |\Lambda(1-s) - u_j(t)| - |\lambda_1 - \lambda_2|$$

and

$$|\lambda_2(s) - u_j(\tau(s))| = |\Lambda(1-s) - u_j(t)|.$$

Recalling the estimate (3.55) we get that the inequalities that we have to prove hold for λ_1 sufficiently close to λ_2 . \square

Let us consider the following paths in $(\mathbb{D}(A) \times \mathbb{C})'$:

$$\begin{aligned} C_0 &: (t, \Lambda(s)), \quad 0 \leq s \leq 1, \\ C_1 &: (\tau(1-s), \Lambda_2), \quad 0 \leq s \leq 1, \\ C_2 &: (\tau(1-s), \Lambda(s)), \quad 0 \leq s \leq 1 \end{aligned}$$

and the corresponding lifts

$$\begin{aligned} \widehat{C}_0 &: (t, \lambda_1 - \lambda_2 + \Lambda(s), \Lambda(s)), \quad 0 \leq s \leq 1, \\ \widehat{C}_1 &: (\tau(1-s), \lambda_1 - \lambda_2 + \Lambda_2, \Lambda_2), \quad 0 \leq s \leq 1, \\ \widehat{C}_2 &: (\tau(1-s), \lambda_1 - \lambda_2 + \Lambda(s), \Lambda(s)), \quad 0 \leq s \leq 1. \end{aligned}$$

Note that \widehat{C}_i ($i = 0, 1, 2$) belong to $(\mathbb{D}(A) \times \mathbb{C}^2)''$ for λ_1 sufficiently close to λ_2 : for \widehat{C}_0 this holds by definition, for \widehat{C}_1 – by Lemma 3.24, and for \widehat{C}_2 – by Lemma 3.26. Figure 2 might be helpful for visualizing the paths C_i : the picture is inside the space $\mathbb{D}(A) \times \mathbb{C}$.

LEMMA 3.27. a) The path C_2 is homotopic to $C_0 \circ C_1$ in $(\mathbb{D}(A) \times \mathbb{C})'$.

b) If λ_1 is sufficiently close to λ_2 , then the path \widehat{C}_2 is homotopic to $\widehat{C}_0 \circ \widehat{C}_1$ in $(\mathbb{D}(A) \times \mathbb{C}^2)''$

PROOF. Let us proof a). Put

$$h(s_1, s_2) = \begin{cases} (\tau(s_1 + (1 - 3s_2)(1 - s_1)), \Lambda_2) & \text{if } 0 \leq s_2 \leq 1/3, \\ \left(\tau(s_1), \Lambda \left(\frac{(3s_2 - 1)(1 - s_1)}{2 - s_1} \right) \right) & \text{if } 1/3 \leq s_2 \leq 1 - s_1/3, \\ (\tau(3 - 3s_2), \Lambda(3s_2 - 2)) & \text{if } 1 - s_1/3 \leq s_2 \leq 1. \end{cases}$$

Note that the path $h(0, s_2)$, $0 \leq s_2 \leq 1$ coincides with $C_0 \circ C_1$ while $h(1, s_2)$, $0 \leq s_2 \leq 1$ coincides with C_2 . Therefore, it remains only to prove that $h(s_1, s_2) \in (\mathbb{D}(A) \times \mathbb{C}^2)'$. By definition, $(\tau(s'), \Lambda(1 - s'))$ and $(\tau(s'), \Lambda_2)$ belong to $(\mathbb{D}(A) \times \mathbb{C})'$ for all $s' \in [0, 1]$. Only the middle case is not obvious, that is, we have to prove that

$$\Lambda(s') \neq u_i(\tau(s_1)) = \Lambda(1 - s_1) - \delta, \quad s' := \frac{(3s_2 - 1)(1 - s_1)}{2 - s_1}.$$

Note that

$$s' - (1 - s_1) = \frac{(3s_2 - 3 + s_1)(1 - s_1)}{2 - s_1} \leq 0.$$

If we assume that $\Lambda(s') = \Lambda(1 - s_1) - \delta$, then using the mean value theorem we get $\delta = \Lambda(1 - s_1) - \Lambda(s') = \Lambda'(s_0)(1 - s_1 - s')$ for some $s_0 \in [s', 1 - s_1]$. Note that $1 - s_1 \neq s'$ because $\delta \neq 0$. Therefore, $1 - s_1 - s' > 0 \Rightarrow \delta \in T^+(C)$ contradicting that δ is a transverse direction for C . This completes the proof of a).

For part b), we need only to check that if λ_1 is sufficiently close to λ_2 , then the natural lift $\widehat{h}(s_1, s_2)$ defined by

$$\begin{cases} (\tau(s_1 + (1 - 3s_2)(1 - s_1)), \lambda_1 - \lambda_2 + \Lambda_2, \Lambda_2) & \text{if } 0 \leq s_2 \leq 1/3, \\ \left(\tau(s_1), \lambda_1 - \lambda_2 + \Lambda \left(\frac{(3s_2 - 1)(1 - s_1)}{2 - s_1} \right), \Lambda \left(\frac{(3s_2 - 1)(1 - s_1)}{2 - s_1} \right) \right) & \text{if } 1/3 \leq s_2 \leq 1 - s_1/3, \\ (\tau(3 - 3s_2), \lambda_1 - \lambda_2 + \Lambda(3s_2 - 2), \Lambda(3s_2 - 2)) & \text{if } 1 - s_1/3 \leq s_2 \leq 1. \end{cases}$$

takes values in $(\mathbb{D}(A) \times \mathbb{C}^2)''$ and that $(s_1, s_2) \mapsto \widehat{h}(s_1, s_2)$ is a homotopy between $\widehat{C}_0 \circ \widehat{C}_1$ and \widehat{C}_2 . This is straightforward, so we leave the details as an exercise. \square

Now we are in position to prove Theorem 3.23. Let us first prove that

$$(3.56) \quad \int_{\widehat{C}_2} \mathcal{W}_{\alpha, \beta} + \Omega_{\alpha, \beta}^i(\tau(1), \Lambda_1, \Lambda_2) = \Omega_{\alpha, \beta}^i(t, \lambda_1, \lambda_2).$$

Let us denote by $\widehat{C}_{2,s} \subset \widehat{C}_2$ be the subpath connecting the points $(\tau(s), \lambda_1(s), \lambda_2(s))$ and $(t, \lambda_1, \lambda_2)$ and consider the integral

$$\int_{\widehat{C}_{2,s}} \mathcal{W}_{\alpha, \beta} + \Omega_{\alpha, \beta}^i(\tau(s), \lambda_1(s), \lambda_2(s)).$$

Note that here we make use of Lemma 3.25 in order to prove that the Laurent series expansion of the propagator in the powers of $\lambda_2(s) - u_i(\tau(s)) = \lambda_2 - u_i(t)$ is convergent. The derivative of the above expression with respect to s is 0 while its value at $s = 0$ coincides with the RHS of (3.56). This completes the proof of formula (3.56).

Similarly, using Lemma 3.24 we get

$$(3.57) \quad \int_{\widehat{C}_1} \mathcal{W}_{\alpha, \beta} + \Omega_{\alpha, \beta}(\tau(1), \Lambda_1, \Lambda_2) = \Omega_{\alpha, \beta}(t, \Lambda_1, \Lambda_2).$$

The key observation now is that both $\Omega_{\alpha,\beta}(\tau(1), \Lambda_1, \Lambda_2)$ and $\Omega_{\alpha,\beta}^i(\tau(1), \Lambda_1, \Lambda_2)$ coincide with the infinite series

$$\sum_{n=0}^{\infty} (-1)^{n+1} (I_{\alpha}^{(n)}(\tau(1), \Lambda_1), I_{\beta}^{(-n-1)}(\tau(1), \Lambda_2))$$

where each term in the sum is expanded into a Laurent series in Λ_1^{-1} and a Laurent series in $\Lambda_2 - u_i(\tau(1)) = \lambda_2 - u_i(t)$. The convergence of both Laurent series expansions follows from Lemma 3.24 and Lemma 3.25. Subtracting (3.57) from (3.56), we get

$$\int_{\widehat{C}_2} \mathcal{W}_{\alpha,\beta} - \int_{\widehat{C}_1} \mathcal{W}_{\alpha,\beta} = \Omega_{\alpha,\beta}^i(t, \Lambda_1, \Lambda_2) - \Omega_{\alpha,\beta}(t, \Lambda_1, \Lambda_2).$$

It remains only to recall that \widehat{C}_2 is homotopic to $\widehat{C}_0 \circ \widehat{C}_1$ (see Lemma 3.27). Therefore, the LHS of the above identity becomes

$$\int_{\widehat{C}_2} \mathcal{W}_{\alpha,\beta} - \int_{\widehat{C}_1} \mathcal{W}_{\alpha,\beta} = \int_{\widehat{C}_0} \mathcal{W}_{\alpha,\beta}. \quad \square$$

3.4.5. Analytic extension around the discriminant. Now we are in position to state and prove the second main result of this chapter. Suppose that $t \in M$ is a *generic* semi-simple point, where generic means that the canonical coordinates $\{u_i(t)\}_{i=1}^N$ are pairwise distinct: $u_i(t) \neq u_j(t)$ for $i \neq j$. Let L be a simple loop in \mathbb{C} around $u_i(t)$ that approaches $u_i(t)$ in a transverse direction. We will be interested in loops L whose base point Λ_2 is such that $|\Lambda_2| > |u_j(t)|$ for all $1 \leq j \leq N$. If Λ_1 is sufficiently close to Λ_2 , then the path

$$\widehat{L}: (t, \Lambda_1 - \Lambda_2 + \lambda, \lambda), \quad \lambda \in L$$

belongs to $(M \times \mathbb{C}^2)''$ and hence the integral of the phase form is well defined. Let us fix a reference path in D so that the values of the propagators $\Omega_{\alpha,\beta}(t, \Lambda_1, \Lambda_2)$ are uniquely fixed. Let φ be the reflection vector corresponding to the reference path and the simple loop $\{t\} \times L \subset (M \times \mathbb{C})'$.

THEOREM 3.28. *Under the above notation the following formula holds:*

$$(3.58) \quad \Omega_{\alpha,\beta}(t, \Lambda_1, \Lambda_2) - \Omega_{w(\alpha),w(\beta)}(t, \Lambda_1, \Lambda_2) + \int_{\widehat{L}} \mathcal{W}_{\alpha,\beta} = -2\pi i (\alpha|\varphi) \langle \varphi, \beta \rangle,$$

where $w = r_{\varphi}$ is the reflection (3.50) corresponding to φ .

Let us make several remarks about the identity (3.58). Using Lemma 3.19 and that $(\alpha|\beta) = (w(\alpha)|w(\beta))$ we get that the LHS is a multivalued analytic function in $(t, \Lambda_1, \Lambda_2) \in D$. Note that

$$d \int_{\widehat{L}} \mathcal{W}_{\alpha,\beta} = \mathcal{W}_{w(\alpha),w(\beta)}(t, \lambda_1, \lambda_2) - \mathcal{W}_{\alpha,\beta}(t, \lambda_1, \lambda_2).$$

Therefore, the LHS in (3.58) is locally independent of t , Λ_1 , and Λ_2 . Let us first prove the independence of the statement of Theorem 3.28 on the choice of a reference point and a reference path (inside D !) for the propagators. Let us emphasize that it is very important that the reference path of the propagators on the LHS of (3.58) is inside D , that is, it is allowed to wind around the entire discriminant, but it is not allowed to wind around a local branch of the discriminant.

LEMMA 3.29. *Suppose that (3.58) holds for specific choices of a reference point and a reference path in D , then it holds for any other choice of a reference point and a reference path in D .*

PROOF. Suppose that $(t^\#, \lambda^\#)$ is a different reference point in $(M \times \mathbb{C})'$ and $C^\#$ is an arbitrary reference path from $(t^\#, \lambda^\#)$ to (t, Λ_1) . Let A be an admissible path connecting the two reference points (t°, λ°) and $(t^\#, \lambda^\#)$ – see comment 5) after Proposition 3.3. Let C° be the reference path from (t°, λ°) to (t, Λ_1) for which formula (3.58) is already established. Note that $C' := A^{-1} \circ (C^\#)^{-1} \circ C^\circ$ is a loop winding around the discriminant. Therefore, the corresponding monodromy transformation $r_{C'} = \sigma^k$, where σ is the classical monodromy and $k \in \mathbb{Z}$. Therefore, if we change the reference point and the reference path, then the LHS of (3.58) becomes

$$\Omega_{\sigma^k(\alpha), \sigma^k(\beta)}(t, \Lambda_1, \Lambda_2) - \Omega_{\sigma^k \circ w(\alpha), \sigma^k \circ w(\beta)}(t, \Lambda_1, \Lambda_2) + \int_{\widehat{C}} \mathcal{W}_{\sigma^k(\alpha), \sigma^k(\beta)}.$$

We have $\sigma^k \circ w(\alpha) = \sigma^k \circ w \circ \sigma^{-k}(\sigma^k \alpha)$. Note that $\sigma^k \circ w \circ \sigma^{-k}$ is the reflection with respect to $\sigma^k \varphi$ and that $\sigma^k \varphi$ is the reflection vector corresponding to the simple loop $\{t\} \times C$ with a reference point to its base point (t, Λ_1) given by $C^\#$. Therefore, in order to complete the proof we need to check that

$$(\sigma^k(\alpha) | \sigma^k(\varphi)) \langle \sigma^k(\varphi), \sigma^k(\beta) \rangle = (\alpha | \varphi) \langle \varphi, \beta \rangle.$$

This follows from the fact that both the intersection and the Euler pairings are σ -invariant. \square

Let us establish next the special case of Theorem 3.28 when $\alpha = \beta = \varphi$.

LEMMA 3.30. *We have*

$$\int_{\widehat{L}} \mathcal{W}_{\varphi, \varphi} = -4\pi i.$$

PROOF. The integral that we have to evaluate coincides with the LHS of (3.58) in the case $\alpha = \beta = \varphi$. The integral along $\widehat{L} \cong L$ splits into two integrals along the tail of L and an integral along a small loop around $u_i(t)$. Since the phase form $\mathcal{W}_{\varphi, \varphi}$ is invariant with respect to the local monodromy around $u_i(t)$, the two integrals along the tail of L cancel out. Therefore, we may assume that L is a small circle around $u_i(t)$. The integral is also independent of Λ_1 and Λ_2 , so we may also assume that $\Lambda_1 = \Lambda_2$. The restriction of the phase form to $\widehat{L} \cong L$ is

$$\mathcal{W}_{\varphi, \varphi}(t, \lambda, \lambda) = -(I_\varphi^{(0)}(t, \lambda), I_\varphi^{(0)}(t, \lambda)) d\lambda.$$

On the other hand, since λ is sufficiently close to $u_i(t)$, the period has the following Laurent series expansion:

$$I_\varphi^{(0)}(t, \lambda) = \frac{2}{\sqrt{2(\lambda - u_i(t))}} \left(e_i + O(\lambda - u_i(t)) \right),$$

where $e_i = \sqrt{\Delta_i} \frac{\partial}{\partial u_i}$ is the normalized idempotent. Substituting this expansion in the above formula for the phase form we get that only the leading order term of $I_\varphi^{(0)}$ contributes. Since $(e_i, e_i) = 1$ we get

$$\int_{\widehat{L}} \mathcal{W}_{\varphi, \varphi} = - \int_L (I_\varphi^{(0)}(t, \lambda), I_\varphi^{(0)}(t, \lambda)) d\lambda = -2 \int_L \frac{d\lambda}{\lambda - u_i(t)} = -4\pi i.$$

□

Proof of Theorem 3.28 The LHS of (3.58) is locally independent of Λ_1 , so let us assume that $|\Lambda_1| > |\Lambda_2|$, that is, $(t, \Lambda_1, \Lambda_2) \in D^+$. According to Lemma 3.29 we have the freedom to choose a reference path. Let us choose the reference path in such a way that $\Omega_{\alpha, \beta}(t, \Lambda_1, \Lambda_2)$ coincides with the Laurent series expansion in Λ_1^{-1} of the propagator series

$$\sum_{n=0}^{\infty} (-1)^n (I_{\alpha}^{(n)}(t, \Lambda_1), I_{\beta}^{(-n-1)}(t, \Lambda_2)).$$

Let us introduce the following notation. The tail of the simple loop L will be denoted by C and we fix a parameterization $\Lambda(s)$ ($0 \leq s \leq 1$) of C . The end point of the tail C is $\lambda_2 := \Lambda(1)$. The loop L is required to approach u_i in a transverse direction. Let us choose Λ_1 sufficiently close to Λ_2 , such that, $\lambda_1 := \Lambda_1 - \Lambda_2 + \lambda_2$ belongs to the open subset $U_i(t, C)$, that is, the tail C satisfies the conditions of Theorem 3.23. The simple loop $L = C^{-1} \circ \gamma \circ C$, where γ is a circle with center $u_i(t)$ and radius $|\lambda_2 - u_i(t)|$. The integral of the phase form takes the form

$$(3.59) \quad \int_{\widehat{L}} \mathcal{W}_{\alpha, \beta} = \int_{\widehat{C}} \mathcal{W}_{\alpha, \beta} + \int_{\widehat{\gamma}} \mathcal{W}_{\alpha, \beta} - \int_{\widehat{C}} \mathcal{W}_{w(\alpha), w(\beta)}.$$

Let us decompose

$$\alpha = \alpha' + (\alpha|\varphi)\varphi/2, \quad \beta = \beta' + (\beta|\varphi)\varphi/2.$$

Note that α' and β' are invariant with respect to the monodromy transformation along $\{t\} \times \gamma$. Therefore, the periods $I_{\alpha'}^{(m)}(t, \lambda_1 - \lambda_2 + \lambda)$ and $I_{\beta'}^{(m)}(t, \lambda)$ are analytic for all λ inside the disk bounded by the circle γ . Therefore, using the Cauchy theorem we get that $\int_{\widehat{\gamma}} \mathcal{W}_{\alpha', \beta'} = 0$. Since $w(\alpha') = \alpha'$ and $w(\beta') = \beta'$, the integrals of $\mathcal{W}_{\alpha', \beta'}$ along the two tails in (3.59) cancel out. We get

$$(3.60) \quad \int_{\widehat{L}} \mathcal{W}_{\alpha, \beta} = \frac{(\beta|\varphi)}{2} \int_{\widehat{L}} \mathcal{W}_{\alpha', \varphi} + \frac{(\alpha|\varphi)}{2} \int_{\widehat{L}} \mathcal{W}_{\varphi, \beta'} + \frac{(\alpha|\varphi)(\beta|\varphi)}{4} \int_{\widehat{L}} \mathcal{W}_{\varphi, \varphi}.$$

We claim that

$$(3.61) \quad F_{\alpha', \varphi}(t, \mu_1, \mu_2) := (\mu_2 - u_i(t))^{-1/2} \Omega_{\alpha', \varphi}^i(t, \mu_1, \mu_2)$$

is an analytic function in (μ_1, μ_2) in a neighborhood of $(u_i(t), u_i(t))$, where $\Omega_{\alpha', \varphi}^i(t, \mu_1, \mu_2)$ denotes the Laurent series expansion in $\mu_2 - u_i(t)$ of the series

$$(3.62) \quad \sum_{n=0}^{\infty} (-1)^n (I_{\alpha'}^{(n)}(t, \mu_1), I_{\varphi}^{(-n-1)}(t, \mu_2)).$$

Indeed, we already known that $(\mu_2 - u_i(t))^{-1/2} I_{\varphi}^{(-k-1)}(t, \mu_2)$ are analytic functions in μ_2 in a neighborhood of $u_i(t)$ and that the Taylor series expansion in $\mu_2 - u_i(t)$ produces at least a formal power series. In order to determine the radius of convergence let us differentiate (3.62) with respect to μ_2 (see formula (3.52)):

$$-\frac{1}{\mu_1 - \mu_2} ((\mu_1 - E\bullet)I_{\alpha'}^{(0)}(t, \mu_1), I_{\varphi}^{(0)}(t, \mu_2)).$$

Recalling Saito's formula (3.49), we get that the above expression is analytic at $\mu_2 = \mu_1$, because $(\alpha'|\varphi) = 0$. Therefore, the radius of convergence of the Laurent series expansion

$\Omega_{\alpha',\varphi}^i(t, \mu_1, \mu_2)$ is $\min_{j:j \neq i} |u_j(t) - u_i(t)|$. In addition, since the periods $I_{\alpha'}^{(n)}(t, \mu_1)$ are analytic at $\mu_1 = u_i$, the Laurent series expansion $\Omega_{\alpha',\varphi}^i(t, \mu_1, \mu_2)$ depends analytically on μ_1 for all μ_1 sufficiently close to $u_i(t)$, that is, $|\mu_1 - u_i(t)| < |u_j(t) - u_i(t)|$ for all $j \neq i$. Our claim that (3.61) is analytic follows.

The analytic continuation of the propagator

$$\Omega_{\alpha',\varphi}^i(t, \lambda_1 - \lambda_2 + \lambda, \lambda) = (\lambda - u_i(t))^{1/2} F_{\alpha',\varphi}(t, \lambda_1 - \lambda_2 + \lambda, \lambda)$$

when λ varies along γ starting at $\lambda = \lambda_2$ is $-\Omega_{\alpha',\varphi}^i(t, \lambda_1, \lambda_2)$. Therefore,

$$\int_{\widehat{\gamma}} \mathcal{W}_{\alpha',\varphi} = -2\Omega_{\alpha',\varphi}^i(t, \lambda_1, \lambda_2).$$

On the other hand, according to Theorem 3.23

$$\int_{\widehat{C}} \mathcal{W}_{\alpha',\varphi} = \Omega_{\alpha',\varphi}^i(t, \lambda_1, \lambda_2) - \Omega_{\alpha',\varphi}(t, \Lambda_1, \Lambda_2).$$

Therefore, we get

$$\int_{\widehat{L}} \mathcal{W}_{\alpha',\varphi} = 2\Omega_{\alpha',\varphi}(t, \Lambda_1, \Lambda_2),$$

where we used formula (3.59) and the fact that $w(\alpha') = \alpha'$ and $w(\varphi) = -\varphi$. Note that the above formula and Lemma 3.30 complete the proof of (3.28) in the case when $\beta = \varphi$.

Similarly, using that $\mathcal{W}_{\varphi,\beta'} = \mathcal{W}_{\beta',\varphi}$, we get that

$$(3.63) \quad \int_{\widehat{L}} \mathcal{W}_{\varphi,\beta'} = -2\Omega_{\beta',\varphi}(t, \Lambda_2, \Lambda_1).$$

There is a slight complication here, because in order to apply Theorem 3.23 we need the condition $|\Lambda_2| > |\Lambda_1|$, while we already assumed the opposite inequality. This complication can be offset as follows. Note that $\lambda_1 - \lambda_2 + C$ is the tail of a simple loop around u_i based at Λ_1 and that if Λ_1 is sufficiently close to Λ_2 , then this loop is approaching u_i in a transverse direction. Let us fix Λ_1 and pick Λ_2 sufficiently close to Λ_1 , such that, the conclusion of Theorem 3.23 holds (in particular $|\Lambda_2| > |\Lambda_1|$). According to the argument from the previous case formula (3.63) holds for all Λ_2 belonging to some open subset of \mathbb{C} containing Λ_1 as a boundary point. However, just like we argued above, the condition $(\beta'|\varphi) = 0$ implies that the Laurent series expansion in Λ_2^{-1} of the series

$$\sum_{n=0}^{\infty} (-1)^n (I_{\beta'}^{(n)}(t, \Lambda_2), I_{\varphi}^{(-n-1)}(t, \Lambda_1))$$

is analytic at $\Lambda_2 = \Lambda_1$. The LHS of (3.63) is also analytic in a neighborhood of $\Lambda_2 = \Lambda_1$: recall formula (3.52) and note that the singularity at $\Lambda_2 = \Lambda_1$ cancels out because $(\beta'|\varphi) = 0$. Therefore, the LHS and the RHS of (3.63) are analytic functions in Λ_2 in a neighborhood of Λ_1 and they coincide on an open subset. Therefore, they must coincide in an open neighborhood of Λ_1 , that is, the equality holds for all Λ_2 sufficiently close to Λ_1 .

Recalling Lemma 3.30 we get that the integral (3.60) takes the form

$$(3.64) \quad \int_{\widehat{L}} \mathcal{W}_{\alpha,\beta} = -(\beta|\varphi) \Omega_{\alpha',\varphi}(t, \Lambda_1, \Lambda_2) - (\alpha|\varphi) \Omega_{\beta',\varphi}(t, \Lambda_2, \Lambda_1) - \pi \mathbf{i} (\alpha|\varphi) (\beta|\varphi).$$

On the other hand, using that $w(\alpha) = \alpha - (\alpha|\varphi)\varphi$ and $w(\beta) = \beta - (\beta|\varphi)\varphi$ we get

$$\Omega_{\alpha,\beta} - \Omega_{w(\alpha),w(\beta)} = (\alpha|\varphi)\Omega_{\varphi,\beta} + (\beta|\varphi)\Omega_{\alpha,\varphi} - (\alpha|\varphi)(\beta|\varphi)\Omega_{\varphi,\varphi},$$

where we suppressed the argument $(t, \Lambda_1, \Lambda_2)$ of the propagators. Let us combine each of the first two terms on the RHS of the above expression with $\frac{1}{2}$ of the last term. Recalling that $\beta - (\beta|\varphi)\varphi/2 = \beta'$ and $\alpha - (\alpha|\varphi)\varphi/2 = \alpha'$ we get

$$(3.65) \quad \Omega_{\alpha,\beta} - \Omega_{w(\alpha),w(\beta)} = (\alpha|\varphi)\Omega_{\varphi,\beta'} + (\beta|\varphi)\Omega_{\alpha',\varphi}.$$

Combining formula (3.65) and (3.64) we get that the LHS of (3.58) equals

$$(\alpha|\varphi) \left(\Omega_{\varphi,\beta'}(t, \lambda_1, \lambda_2) - \Omega_{\beta',\varphi}(t, \lambda_2, \lambda_1) \right) - \pi\mathbf{i}(\alpha|\varphi)(\beta|\varphi).$$

Recalling Lemma 3.20 we get

$$-2\pi\mathbf{i}(\alpha|\varphi)\langle\varphi,\beta'\rangle - \pi\mathbf{i}(\alpha|\varphi)(\beta|\varphi) = -2\pi\mathbf{i}(\alpha|\varphi)\langle\varphi,\beta\rangle + \pi\mathbf{i}(\alpha|\varphi)(\beta|\varphi)\langle\varphi,\varphi\rangle - \pi\mathbf{i}(\alpha|\varphi)(\beta|\varphi).$$

Since $\langle\varphi,\varphi\rangle = 1$, the last two terms in the above expression cancel out and we get precisely the RHS of (3.58). \square

Analytic theory of primitive forms

This chapter is an introduction to the analytic theory of primitive forms. We follow mostly the lecture notes of K. Saito [51]. Nevertheless, motivated by the applications to mirror symmetry for toric orbifolds (see [34, 35]), we have extended slightly the original framework.

4.1. Relative de Rham theory

The goal of this section is to recall some basic constructions from complex geometry and to fix the notation which will be used through out this chapter.

Suppose that Z is a complex manifold. If we forget the complex structure, then Z is a real smooth manifold. The corresponding tangent bundle is called the *real tangent bundle* of Z and it will be denoted by $T_Z^{\mathbb{R}}$. Let $T_z^{\mathbb{R}}Z$ denote the real tangent space of Z at z . The *complexified tangent bundle* $T_Z^{\mathbb{C}}$ of Z is the vector bundle whose fiber at a point $z \in Z$ is $T_z^{\mathbb{C}}Z := T_z^{\mathbb{R}}Z \otimes_{\mathbb{R}} \mathbb{C}$. Complex conjugation defines an \mathbb{R} -linear involution $\kappa : T_Z^{\mathbb{C}} \rightarrow T_Z^{\mathbb{C}}$, $\kappa(v) := \bar{v}$. Let $J_Z : T_Z^{\mathbb{R}} \rightarrow T_Z^{\mathbb{R}}$, $J_Z^2 = -\text{id}$ be the complex structure of Z . The endomorphism J_Z extends uniquely to a complex linear endomorphism of $T_Z^{\mathbb{C}}$. Since $J_Z^2 = -1$, the endomorphism J_Z is diagonalizable with eigenvalues $\pm \mathbf{i}$, where $\mathbf{i} := \sqrt{-1}$. Let $T_Z^{1,0} := \text{Ker}(J_Z - \mathbf{i})$ and $T_Z^{0,1} := \text{Ker}(J_Z + \mathbf{i})$ be the corresponding eigen-subbundles. Clearly $T_Z^{0,1} = \kappa(T_Z^{1,0})$ and we have a direct sum decomposition $T_Z^{\mathbb{C}} = T_Z^{1,0} \oplus T_Z^{0,1}$. It is easy to check that $T_Z := T_Z^{1,0}$ is a holomorphic vector bundle, which is also known as the *holomorphic tangent bundle* of Z . The fiber of T_Z at $z \in Z$ is denoted by T_zZ and it will be called the *holomorphic tangent space*. To avoid cumbersome notation, we put $\bar{T}_Z := T_Z^{0,1}$ – this is an anti-holomorphic vector bundle.

Suppose now that S is another complex manifold and that $p : Z \rightarrow S$ is a regular holomorphic map, i.e., the tangent map $d_z p : T_z^{\mathbb{R}}Z \rightarrow T_{p(z)}^{\mathbb{R}}S$ is surjective for all $z \in Z$. Since the map p is regular, it is straightforward to check that the relative version of the above discussion applies. Namely, by forgetting the complex structures on Z and S and viewing p as a smooth map between real smooth manifolds, we can define the *real relative tangent bundle* $T_{Z/S}^{\mathbb{R}} := \text{Ker}(dp : T_Z^{\mathbb{R}} \rightarrow p^*T_S^{\mathbb{R}})$. Since p is a holomorphic map, $dp \circ J_Z = J_S \circ dp$, so the real relative tangent bundle $T_{Z/S}^{\mathbb{R}}$ is J_Z -invariant. We refer to the restriction $J_{Z/S} := J_Z|_{T_{Z/S}^{\mathbb{R}}}$ as the relative complex structure. The endomorphism $J_{Z/S}$ extends uniquely to a complex-linear endomorphism of the *complexified relative tangent bundle* $T_{Z/S}^{\mathbb{C}} := T_{Z/S}^{\mathbb{R}} \otimes \mathbb{C}$. Again, $J_{Z/S}$ is diagonalizable with eigenvalues $\pm \mathbf{i}$ and we have a direct sum decomposition $T_{Z/S}^{\mathbb{C}} = T_{Z/S}^{1,0} \oplus T_{Z/S}^{0,1}$, where $T_{Z/S}^{1,0} := \text{Ker}(J_{Z/S} - \mathbf{i})$ and $T_{Z/S}^{0,1} := \text{Ker}(J_{Z/S} + \mathbf{i})$ are the corresponding eigen-subbundles. It is easy to check that $T_{Z/S} := T_{Z/S}^{1,0}$ is a holomorphic vector bundle, also known as the *holomorphic relative*

tangent bundle. To avoid cumbersome notation, we put $\bar{T}_{Z/S} = T_{Z/S}^{0,1}$ – this is an anti-holomorphic vector bundle on Z . Let $\mathcal{T}_Z^{\mathbb{R}}$ and $\mathcal{T}_{Z/S}^{\mathbb{R}}$ be the sheaves of smooth sections of respectively the real vector bundles $T_Z^{\mathbb{R}}$ and $T_{Z/S}^{\mathbb{R}}$. Let \mathcal{T}_Z and $\mathcal{T}_{Z/S}$ be the sheaves of holomorphic sections of respectively the holomorphic vector bundles T_Z and $T_{Z/S}$.

If E is a complex vector bundle on Z , then the dual complex vector bundle E^* is by definition the vector bundle whose fiber at a point z is the vector space of complex linear functions on E_z . Let us denote by $\mathcal{A}_{Z/S}^{p,q}$ the sheaf of smooth sections of $\wedge^p(T_{Z/S}^*) \otimes \wedge^q(\bar{T}_{Z/S}^*)$ and by $\Omega_{Z/S}^p$ the sheaf of holomorphic sections of $\wedge^p(T_{Z/S}^*)$. Let us fix local holomorphic coordinates $(x, t) := (x_0, \dots, x_n, t_1, \dots, t_m)$ on Z and $t = (t_1, \dots, t_m)$ on S , such that, the map $p : Z \rightarrow S$ has the form of a projection $(x, t) \mapsto t$. Then the sections of $\mathcal{A}_{Z/S}^{p,q}$ take the form

$$\omega = \sum_{I,J} \omega_{I,J}(x, t) dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge d\bar{x}_{j_1} \wedge \cdots \wedge d\bar{x}_{j_q},$$

where the sum is over all multi-indexes $I = (i_1, \dots, i_p)$ ($0 \leq i_1 < \cdots < i_p \leq n$) and $J = (j_1, \dots, j_q)$ ($0 \leq j_1 < \cdots < j_q \leq n$) and the coefficients $\omega_{I,J}$ are smooth complex-valued functions. Using the local coordinates, it is easy to see that the de Rham differential d_Z on Z induces two differentials $d_{Z/S}^{1,0} : \mathcal{A}_{Z/S}^{p,q} \rightarrow \mathcal{A}_{Z/S}^{p+1,q}$ and $d_{Z/S}^{0,1} : \mathcal{A}_{Z/S}^{p,q} \rightarrow \mathcal{A}_{Z/S}^{p,q+1}$. Therefore, $\mathcal{A}_{Z/S}^{p,q}$ is a double complex with anticommuting horizontal and vertical differentials. The corresponding total complex

$$\mathcal{A}_{Z/S}^m := \bigoplus_{p+q=m} \mathcal{A}_{Z/S}^{p,q}, \quad d_{Z/S} := d_{Z/S}^{1,0} + d_{Z/S}^{0,1}$$

is by definition the *smooth relative* de Rham complex. The relative version of the de Rham lemma holds and the smooth relative de Rham complex provides a soft resolution of the sheaf $p^{-1}\mathcal{O}_S$.

Similarly, using local coordinates, it is not hard to see that the de Rham differential d_Z on Z induces a differential $d_{Z/S} : \Omega_{Z/S}^p \rightarrow \Omega_{Z/S}^{p+1}$. The resulting complex is by definition the *holomorphic relative* de Rham complex.

4.2. Family of functions

4.2.1. Definition and first properties. Suppose that we have a commutative diagram

$$\begin{array}{ccccc} X \times S & \longleftarrow & Z & \xrightarrow{F} & \mathbb{C} \\ & \searrow \text{pr}_S & \downarrow p & & \\ & & S & & \end{array}$$

where S is a connected complex manifold, X is a Stein manifold, Z is an open subset of $X \times S$, F is a holomorphic function, and p is the map induced by the natural projection $\text{pr}_S : X \times S \rightarrow S$. Let $\mathcal{T}_{Z/S}$ be the sheaf of relative holomorphic vector fields on Z , that is, the sheaf of sections of the holomorphic relative tangent bundle $T_{Z/S} := \text{Ker}\{dp : T_Z \rightarrow p^*T_S\}$. The image of $\mathcal{T}_{Z/S}$ under the natural morphism $\mathcal{T}_{Z/S} \rightarrow \mathcal{O}_Z$, $\xi \mapsto \langle dF, \xi \rangle$ defines a coherent sheaf of ideals $\mathcal{J}_F \subset \mathcal{O}_Z$. The zero locus of \mathcal{J}_F defines a closed complex subspace C_F of Z with structure sheaf $\mathcal{O}_{C_F} := \mathcal{O}_Z/\mathcal{J}_F$. The complex subspace C_F is called the *relative critical set* of F .

DEFINITION 4.1. The data $(F, p : Z \rightarrow S)$ is said to be a *family* of functions over the connected complex manifold S if the following two conditions are satisfied:

- (i) The map p is a Stein map.
- (ii) The restriction $p|_{C_F} : C_F \rightarrow S$ is a proper finite map.

Let us denote by $Z_t := p^{-1}(t)$ the fiber of p at a point $t \in S$ and by $f_t := F|_{Z_t} : Z_t \rightarrow \mathbb{C}$ the restriction of F to Z_t . Note that the critical points of f_t coincide with $C_F \cap Z_t$. Since $p|_{C_F}$ is a finite map, we get that f_t is a holomorphic function with finitely many isolated critical points.

PROPOSITION 4.2. *If $(F, p : Z \rightarrow S)$ is a family, then the relative critical set C_F is a locally complete intersection and pure dimensional of dimension $m := \dim_{\mathbb{C}}(S)$.*

PROOF. Let us first prove that C_F is pure dimensional of dimension m . Suppose that $z = (x, t) \in C_F$. Since $p|_{C_F}$ is a finite map, the image $p(C_F)$ is an analytic subvariety of S and for the dimension at $t = p(z)$ we have

$$m = \dim_t S \geq \dim_t p(C_F) = \max_{w \in C_F \cap Z_t} \dim_w C_F \geq \dim_z C_F.$$

Let us prove the opposite inequality, that is, $\dim_z C_F \leq m$. Let us choose a product open neighbourhood $U \times V$ of z in Z , such that, U and V are open neighborhoods respectively in X and S equipped with coordinates respectively (x_0, x_1, \dots, x_n) and (t_1, \dots, t_m) . The relative critical set $C_F \cap (U \times V)$ is given by the zero locus of the partial derivatives $\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n}$. If $\dim_z C_F = k$, then there are holomorphic functions $f_1, \dots, f_k \in \mathcal{O}_Z(U \times V)$, such that, after shrinking U and V if necessary, the zero locus $\{f_1 = \dots = f_k = \frac{\partial F}{\partial x_0} = \dots = \frac{\partial F}{\partial x_n} = 0\}$ coincides with $\{z\}$. In particular, $k + n + 1 \geq \dim_z Z = m + n + 1$, that is, $\dim_z C_F = k \geq m$. Since the number of equations defining C_F in $U \times V$ is $n + 1$, we get that C_F is a locally complete intersection in Z . \square

COROLLARY 4.3. *If $z^\circ = (x^\circ, t^\circ) \in C_F$ is an arbitrary point and (x_0, x_1, \dots, x_n) and (t_1, \dots, t_m) are local coordinates near $x^\circ \in X$ and $t^\circ \in S$, then*

- a) $\left(\frac{\partial f_t}{\partial x_0}, \dots, \frac{\partial f_t}{\partial x_n}\right)$ is a regular sequence in $\mathcal{O}_{Z_t, x^\circ}$.
- b) $\left(\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n}\right)$ is a regular sequence in \mathcal{O}_{Z, z° .

Recall that if (A, \mathfrak{m}) is a Noetherian local ring, then a sequence (a_1, \dots, a_r) , $a_i \in \mathfrak{m}$ is said to be *regular* if a_1 is not a 0-divisor and a_i is not a 0-divisor in $A/(a_1, \dots, a_{i-1})A$ for $2 \leq i \leq r$. The *depth* of A is defined to be the maximal length of a regular sequence. The ring A is said to be Cohen–Maculay if the dimension of A equals the depth of A . The following simple lemma will be used both in the proof of Corollary 4.3 and to conclude that the local rings $\mathcal{O}_{C_F, z}$ are Cohen–Maculay.

LEMMA 4.4. *If (A, \mathfrak{m}) is a Cohen–Maculay ring and $a \in A$ is not a 0-divisor, then A/aA is a Cohen–Maculay ring of dimension $\dim(A/aA) = \dim(A) - 1$.*

PROOF. Recall that if A is a Noetherian local ring and $x \in A$ is not a 0-divisor, then $\dim(A/xA) = \dim(A) - 1$ (see [7], Corollary 11.8). Therefore, we have

$$\text{depth}(A) - 1 \leq \text{depth}(A/xA) \leq \dim(A/xA) = \dim(A) - 1.$$

Since $\text{depth}(A) = \dim(A)$, all inequalities above must be equalities. \square

Corollary 4.3 follows immediately from the following more general result:

PROPOSITION 4.5. *Suppose that A is a Noëtherian local ring of dimension m and that A is Cohen–Macualay. If $f_1, \dots, f_m \in A$ is a sequence, such that $\dim A/(f_1, \dots, f_m)A = 0$. Then (f_1, \dots, f_m) is a regular sequence.*

PROOF. We are going to assume that the reader is familiar with the basic facts about associated prime ideals, primary ideals, and Cohen–Macualay rings. For some background, we refer to [45], Sections 6 and 17.

If f_1 is a 0-divisor, then $f_1 \in \text{ann}(x)$ for some $x \in A$. Let us choose x such that $\text{ann}(x)$ is maximal, then $P := \text{ann}(x)$ is a prime ideal of A (see [45], Theorem 6.1). By definition, P is an associated prime ideal of A . Since A is Cohen–Macualay, $\dim(A/P) = \dim(A)$ (see [45], Theorem 17.3). Let g_i ($2 \leq i \leq m$) be the image of f_i in $B := A/P$ under the natural quotient map. Then $\dim(B/(g_2, \dots, g_m)B) = 0$, so the radical of the ideal (g_2, \dots, g_m) must coincide with the maximal ideal \mathfrak{m}_B of B . Therefore, the ideal (g_2, \dots, g_m) is \mathfrak{m}_B -primary (see [7], Proposition 4.2). On the other hand, according to the dimension theorem (see [7], Theorem 11.14), the number of generators in a \mathfrak{m}_B -primary ideal is $\geq \dim(B)$, that is, $m - 1 \geq \dim(B) = \dim(A) = m$ – contradiction. This proves that f_1 is not a 0-divisor. According to Lemma 4.4, the quotient A/f_1A is a Cohen–Macualay ring of dimension $m - 1$. Clearly the ring A/f_1A and the images of f_2, \dots, f_m in A/f_1A satisfy the assumptions of the proposition, so we can complete the proof by induction on m . \square

Part b) of Corollary 4.3 and Lemma 4.4 imply that $\mathcal{O}_{C_F, z}$ is a Cohen–Macualay ring, that is, C_F is a Cohen–Macualay complex space.

4.2.2. Flatness. Recall that a holomorphic map $f : X \rightarrow Y$ between two complex spaces is said to be *flat* if for every $x \in X$, $y = f(x)$, the local ring $\mathcal{O}_{X, x}$ is a flat $\mathcal{O}_{Y, y}$ -module. If X and Y are complex manifolds, then the map f is flat if and only if it is open (see [50], Section 2.3). In general, if X and Y have singularities, flat maps are open, but the converse might fail. There are various criterias for flatness available. In our settings, the following result will be very useful (see [4], Corollary 5.7):

PROPOSITION 4.6. *Suppose that (A, \mathfrak{m}_A) and (B, \mathfrak{m}_B) are Noëtherian local rings and that $\varphi : (B, \mathfrak{m}_B) \rightarrow (A, \mathfrak{m}_A)$ is a homomorphism of local rings, that is, $\varphi(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$. If*

- (i) *A is a flat B -module.*
- (ii) *The elements $F_i \in A$ ($1 \leq i \leq n$) and their images $f_i = \pi(F_i)$ ($1 \leq i \leq n$) under the quotient map $\pi : A \rightarrow A/\mathfrak{m}_B A$ satisfy the condition: every linear relation*

$$a_1 f_1 + \dots + a_n f_n = 0, \quad a_i \in A/\mathfrak{m}_B A$$

lifts to a relation $b_1 F_1 + \dots + b_n F_n = 0$ for some $b_i \in A$, such that $\pi(b_i) = a_i$.

Then $A/(F_1, \dots, F_n)$ is a flat B -module.

Let us prove that $p|_{C_F} : C_F \rightarrow S$ is flat. Let us fix local coordinates (x_0, \dots, x_n) and (t_1, \dots, t_m) as in Corollary 4.3. We claim that $A = \mathcal{O}_{Z, z^\circ}$, $B = \mathcal{O}_{S, t^\circ}$, and $F_i = \frac{\partial F}{\partial x_i}$ ($0 \leq i \leq n$) satisfy conditions (i) and (ii) of Proposition 4.6. The map $p : Z \rightarrow S$ is open, so condition (i) is satisfied. Let us check condition (ii). Note that the quotient map $A \rightarrow A/\mathfrak{m}_B A$ is just the restriction map $\mathcal{O}_{Z, z^\circ} \rightarrow \mathcal{O}_{Z_{t^\circ}, x^\circ}$, $\phi \mapsto \phi|_{t=t^\circ}$. The image of F_i is $\frac{\partial f_{t^\circ}}{\partial x_i}$. Suppose that we have a linear relation

$$\sum_{i=0}^n a_i(x) \frac{\partial f_{t^\circ}}{\partial x_i} = 0, \quad a_i \in \mathcal{O}_{Z_{t^\circ}, x^\circ}.$$

According to Corollary 4.3, the partial derivatives $\frac{\partial f_{t^\circ}}{\partial x_i}$ ($0 \leq i \leq n$) form a regular sequence. In particular, the corresponding Koszul complex is exact

$$\cdots \longrightarrow \wedge^2(\mathcal{O}_{Z_{t^\circ}, x^\circ}^{n+1}) \xrightarrow{d_2} \mathcal{O}_{Z_{t^\circ}, x^\circ}^{n+1} \xrightarrow{d_1} \mathcal{O}_{Z_{t^\circ}, x^\circ} \longrightarrow \mathcal{O}_{Z_{t^\circ}, x^\circ} / \left(\frac{\partial f_{t^\circ}}{\partial x_0}, \dots, \frac{\partial f_{t^\circ}}{\partial x_n} \right) \longrightarrow 0.$$

The linear relation implies that (a_0, \dots, a_n) is in the kernel of the differential d_1 . Therefore, there exists $a_{i,j} \in \mathcal{O}_{Z_{t^\circ}, x^\circ}$ ($0 \leq i, j \leq n$), such that $a_{i,j} = -a_{j,i}$ and $a_i = \sum_{j=0}^n a_{i,j} \frac{\partial f_{t^\circ}}{\partial x_j}$. Let us pick an arbitrary lift $b_{i,j} \in \mathcal{O}_{Z, z^\circ}$ of $a_{i,j}$ for $i < j$ and define $b_{i,j} = -b_{j,i}$ if $i > j$. Clearly, the functions $b_i := \sum_{j=0}^n b_{i,j} \frac{\partial F}{\partial x_j}$ ($0 \leq i \leq n$) will satisfy $\sum_{i=0}^n b_i \frac{\partial F}{\partial x_i} = 0$ and $b_i|_{t=t^\circ} = a_i$. This proves that condition (ii) is also satisfied. According to Proposition 4.6, the local ring $\mathcal{O}_{C_F, z^\circ}$ is a flat \mathcal{O}_{S, t° -module.

Let us prove that $p_*\mathcal{O}_{C_F}$ is a locally free \mathcal{O}_S -module. Recall that if a finitely generated module over a Noëtherian local ring is flat, then it must be free (see [7], Chapter 7, Exercise 15). In our case, $\mathcal{O}_{C_F, z^\circ}$ is a finitely generated \mathcal{O}_{S, t° -module, because $p_*\mathcal{O}_{C_F}$ is coherent. Therefore, $\mathcal{O}_{C_F, z^\circ}$ is a free \mathcal{O}_{S, t° -module. Let us denote by $\mu_{t^\circ}(z^\circ)$ the rank of $\mathcal{O}_{C_F, z^\circ}$. Put

$$\mu_{f_{t^\circ}} = \sum_{z^\circ \in C_F \cap Z_{t^\circ}} \mu_{t^\circ}(z^\circ).$$

Since $p|_{C_F}$ is a finite map, the stalk

$$(p_*\mathcal{O}_{C_F})_{t^\circ} = \bigoplus_{z^\circ \in C_F \cap Z_{t^\circ}} \mathcal{O}_{C_F, z^\circ}.$$

is a free \mathcal{O}_{S, t° -module of rank $\mu_{f_{t^\circ}}$. Since $p_*\mathcal{O}_{C_F}$ is coherent, we get that it is locally free and that the function $t \mapsto \mu_{f_t}$ is locally constant and upper semi-continuous. However, S is connected by assumption, so μ_{f_t} is an integer independent of $t \in S$, which from now on will be denoted by μ_F .

4.2.3. Complete families. Let us recall the exact sequence of vector bundles on Z

$$(4.1) \quad 0 \longrightarrow T_{Z/S} \longrightarrow T_Z \xrightarrow{dp} p^*T_S \longrightarrow 0.$$

Since Z is an open subset of $X \times S$, the above exact sequence splits. In particular, every vector field $\delta \in \Gamma(V, \mathcal{T}_S)$ can be lifted to a vector field $\widehat{\delta} \in \Gamma(p^{-1}(V), \mathcal{T}_Z)$. Let us define

$$(4.2) \quad \mathcal{T}_S \rightarrow p_*\mathcal{O}_{C_F}, \quad \delta \mapsto \widehat{\delta}(F)|_{C_F}.$$

The definition is independent of the choice of a lift, because if $\widehat{\delta}'$ and $\widehat{\delta}''$ are two different lifts, then their difference $\xi := \widehat{\delta}' - \widehat{\delta}'' \in \mathcal{T}_{Z/S}$. Recalling the definition of the relative critical set, we get that

$$\widehat{\delta}'(F) - \widehat{\delta}''(F) = \langle dF, \xi \rangle \in \mathcal{J}_F$$

vanishes on C_F . The map (4.2) is an \mathcal{O}_S -module morphism known as the *Kodaira–Spencer map*.

DEFINITION 4.7. A family $(F, p : Z \rightarrow S)$ is said to be *complete* if the Kodaira–Spencer map (4.2) is an isomorphism.

PROPOSITION 4.8. *If the family $(F, p : Z \rightarrow S)$ is complete, then the relative critical set C_F is a smooth complex manifold.*

PROOF. Suppose that $z^\circ = (x^\circ, t^\circ) \in C_F$ is an arbitrary point. Let us choose a direct product open neighborhood $U \times V$ of z° in Z and local coordinates $(x_0, \dots, x_n, t_1, \dots, t_m)$ on $U \times V$ with center at z° , such that,

$$p(x_0, \dots, x_n, t_1, \dots, t_m) = (t_1, \dots, t_m)$$

and

$$F = F(z^\circ) + \frac{1}{2}(x_0^2 + \dots + x_k^2) + g(x_{k+1}, \dots, x_n) + G(x_{k+1}, \dots, x_n, t_1, \dots, t_m),$$

where $G \in \mathfrak{m}_{S, t^\circ} \mathcal{O}_{Z, z^\circ}$ is independent of x_0, \dots, x_k , and $g \in \mathcal{O}_{U, x^\circ}$ has at least cubic terms in its Taylor's series expansion at $x = x^\circ = 0$. By definition the relative critical set C_F is defined by $\{\frac{\partial F}{\partial x_0} = \dots = \frac{\partial F}{\partial x_n} = 0\}$. Recalling the implicit function theorem, we get that it is sufficient to prove that the following matrix has maximal rank (i.e. rank $n+1$):

$$\begin{bmatrix} 1 & & & & \frac{\partial^2 F}{\partial x_0 \partial t_1}(z^\circ) & \cdots & \frac{\partial^2 F}{\partial x_0 \partial t_m}(z^\circ) \\ & \ddots & & & \vdots & & \vdots \\ & & 1 & 0 & \cdots & 0 & \frac{\partial^2 F}{\partial x_k \partial t_1}(z^\circ) & \cdots & \frac{\partial^2 F}{\partial x_k \partial t_m}(z^\circ) \\ & & & 0 & \cdots & 0 & \frac{\partial^2 F}{\partial x_{k+1} \partial t_1}(z^\circ) & \cdots & \frac{\partial^2 F}{\partial x_{k+1} \partial t_m}(z^\circ) \\ & & & & \ddots & \vdots & \vdots & \cdots & \vdots \\ & & & & & 0 & \frac{\partial^2 F}{\partial x_n \partial t_1}(z^\circ) & \cdots & \frac{\partial^2 F}{\partial x_n \partial t_m}(z^\circ). \end{bmatrix}$$

We claim that the minor formed by the partial derivatives $\frac{\partial^2 F}{\partial x_i \partial t_a}(z^\circ)$ ($k+1 \leq i \leq n, 1 \leq a \leq m$) has rank $n-k$. Since the Kodaira–Spencer map is an isomorphism, there are vector fields $\sum_{a=1}^m v_{a,i}(t) \frac{\partial}{\partial t_a}$ ($k+1 \leq i \leq n$), such that

$$(4.3) \quad \sum_{a=1}^m v_{a,i}(t) \frac{\partial F}{\partial t_a} = x_i \pmod{\mathcal{J}_F}.$$

Note that all partial derivatives $\frac{\partial^2 F}{\partial x_i \partial x_j}(z^\circ) = 0$ for all $k+1 \leq i \leq n$ and $0 \leq j \leq n$, because only the g -term in F contributes and g has at least cubic terms in z . Therefore, $\frac{\partial \phi}{\partial x_i}(z^\circ) = 0$ for all $\phi \in \mathcal{J}_F$ and $k+1 \leq i \leq n$. Differentiating (4.3) with respect to x_j for $k+1 \leq j \leq n$ and substituting $z = z^\circ$, we get $\sum_{a=1}^m v_{a,i}(z^\circ) \frac{\partial^2 F}{\partial t_a \partial x_j}(z^\circ) = \delta_{i,j}$. This relation implies that the rows of our minor are linearly independent, so its rank must be equal to the number of rows, that is $n-k$. Clearly the rank of the above matrix is $n+1$. \square

PROPOSITION 4.9. *If $(F, p : Z \rightarrow S)$ is a complete family, then the map*

$$\theta : C_F \rightarrow T^*S, \quad \langle \theta(z), \delta \rangle := \widehat{\delta}(F)(z)$$

is a closed embedding and the image $L_F := \theta(C_F)$ is a Lagrangian submanifold.

PROOF. In order to prove that θ is a closed embedding, it is sufficient to prove that $d\theta$ is injective and that θ is one-to-one. Let us prove first the injectivity. Suppose that $z^\circ \in C_F$ is an arbitrary point. Let us fix a product open neighborhood $U \times V$ of z° in Z and coordinates $x = (x_0, \dots, x_n)$ on U and $t = (t_1, \dots, t_m)$ on V . Let us also fix coordinates $(t_1, \dots, t_m, p_1, \dots, p_m)$ on $T^*V = T^*S|_V$, such that, the cotangent vectors in T_t^*S take the form $p_1 dt_1 + \dots + p_m dt_m$. The map θ takes the form

$$\theta(x, t) = \left(t_1, \dots, t_m, \frac{\partial F}{\partial t_1}, \dots, \frac{\partial F}{\partial t_m} \right).$$

The tangent space $T_{z^\circ}C_F$ consists of all vectors $v = \sum_{i=0}^n v^i \partial/\partial x_i + \sum_{a=1}^m v^{a+n} \partial/\partial t_a \in T_{z^\circ}Z$, such that

$$\sum_{i=0}^n v^i \frac{\partial^2 F}{\partial x_j \partial x_i}(z^\circ) + \sum_{a=1}^m v^{n+a} \frac{\partial^2 F}{\partial x_j \partial t_a}(z^\circ) = 0, \quad 0 \leq j \leq n.$$

Therefore,

$$d\theta(v) = \sum_{a=1}^m \left(\sum_{i=0}^n v^i \frac{\partial^2 F}{\partial t_a \partial x_i}(z^\circ) + \sum_{b=1}^m v^{n+b} \frac{\partial^2 F}{\partial t_a \partial t_b}(z^\circ) \right) \frac{\partial}{\partial p_a} + \sum_{b=1}^m v^{n+b} \frac{\partial}{\partial t_b}.$$

If we assume that $d\theta(v) = 0$, then we get $v^{n+b} = 0$ for $1 \leq b \leq m$ and

$$\sum_{i=0}^n v^i \frac{\partial^2 F}{\partial t_a \partial x_i}(z^\circ) = \sum_{i=0}^n v^i \frac{\partial^2 F}{\partial x_j \partial x_i}(z^\circ) = 0, \quad 1 \leq a \leq m, \quad 0 \leq j \leq n.$$

Note that the above equalities are a system of $m+n+1$ linear equations for v^i ($0 \leq i \leq n$). The coefficient matrix is precisely the Jacobian of the map (at $z = z^\circ$)

$$U \times V \rightarrow \mathbb{C}^{n+1}, \quad z \mapsto \left(\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n} \right).$$

The fiber of the above map over 0 is by definition $C_F \cap U \times V$ and since C_F is a smooth complex manifold, we get that the the Jacobian matrix has a maximal rank at all points $z^\circ \in C_F \cap U \times V$, that is, the rank of the Jacobian matrix is $n+1$, which implies that $v^i = 0$ for all $0 \leq i \leq n$. This completes the proof of the injectivity of $d\theta$.

Let us prove that θ is one-to-one. Suppose that $\theta(z') = \theta(z'')$. Note that $t = p(z') = p(z'')$, so z' and z'' are critical points of $f_t : Z_t \rightarrow \mathbb{C}$. Let $\{\xi_1, \dots, \xi_r\} = C_F \cap Z_t$ be the set of all critical points of f_t and suppose that $\xi_i = z'$, $\xi_j = z''$ for some $i \neq j$. The Kodaira–Spencer isomorphism induces an isomorphism between the fibers of the sheaves \mathcal{T}_S and $p_*\mathcal{O}_{C_F}$ at t , that is

$$T_t S = \mathcal{T}_{S,t}/\mathfrak{m}_{S,t} \mathcal{T}_{S,t} \xrightarrow{\cong} (p_*\mathcal{O}_{C_F})_t/\mathfrak{m}_{S,t}(p_*\mathcal{O}_{C_F})_t = \bigoplus_{i=1}^r \mathcal{O}_{C_F, \xi_i}/\mathfrak{m}_{S,t} \mathcal{O}_{C_F, \xi_i},$$

where to compute the stalks of $p_*\mathcal{O}_{C_F}$ at t we used that $p|_{C_F}$ is a finite map. Since the map $\mathcal{O}_{C_F, \xi_i}/\mathfrak{m}_{S,t} \mathcal{O}_{C_F, \xi_i} \rightarrow \mathcal{O}_{C_F, \xi_i}/\mathfrak{m}_{C_F, \xi_i} \mathcal{O}_{C_F, \xi_i} \cong \mathbb{C}$ is surjective, we get that the map

$$T_t S \rightarrow \mathbb{C}^r, \quad \frac{\partial}{\partial t_a} \mapsto \left(\frac{\partial F}{\partial t_a}(\xi_1), \dots, \frac{\partial F}{\partial t_a}(\xi_r) \right)$$

is surjective. Let us pick a tangent vector $v \in T_t S$ such that $v(F)(\xi_i) = 1$ and $v(F)(\xi_j) = 0$. Since $\frac{\partial F}{\partial t_a}(\xi_i) \neq \frac{\partial F}{\partial t_a}(\xi_j)$ for some a , we get that $\theta(\xi_i) = \sum_a \frac{\partial F}{\partial t_a}(\xi_i) dt_a \neq \sum_a \frac{\partial F}{\partial t_a}(\xi_j) dt_a = \theta(\xi_j)$ – contradiction. This proves that θ is one-to-one.

Finally, in order to see that $\theta(C_F)$ is Lagrangian, we have to prove that the pullback $\theta^*(\omega) = 0$, where ω is the standard symplectic form of T^*S . The problem is local, so let us work in local coordinates. Then $\omega = d\alpha$, where $\alpha = \sum_{a=1}^m p_a dt_a$ is the so called *action 1-form*. However, recalling the definition of θ , we get that $\theta^*(\alpha) = d(F|_{C_F})$ is an exact form. Therefore, the pullback of the symplectic form is 0. \square

4.2.4. Morse families. Let us first prove the following fact: If $(F, p : Z \rightarrow S)$ is a family of functions in the sense of Definition 4.1, such that C_F is a *reduced* complex space, then the map $p|_{C_F} : C_F \rightarrow S$ is an analytic covering, i.e., there exists a thin subset \mathcal{K} of S , called *critical locus* of the covering, such that $(p|_{C_F})^{-1}(\mathcal{K})$ is a thin subset of C_F and the restriction of $p|_{C_F}$ is a local biholomorphism $C_F \setminus p^{-1}(\mathcal{K}) \cap C_F \rightarrow S \setminus \mathcal{K}$. For the proof, note first that $p|_{C_F}$ is an open map, because we already proved that $p|_{C_F}$ is flat and every flat map is open. Alternatively, since C_F is pure dimensional and S is irreducible, we can recall the open mapping theorem (see [25], Chapter 5, Section 4.3) to conclude that $p|_{C_F}$ is open. The image $p(C_F)$ is both an open and a closed subset of S , so $p(C_F) = S$. Therefore, $p|_{C_F}$ is an open finite surjection. Since S is a complex manifold, we can recall the Covering Lemma from [25], Chapter 5, Section 2.2 to conclude that $p|_{C_F}$ is an analytic covering.

Since S is a connected complex manifold, the number of points in the fiber $(p|_{C_F})^{-1}(t)$ for $t \in S \setminus \mathcal{K}$ is a constant μ independent of t . Moreover, using the local description of a proper finite map (see [25], Chapter 2, Section 3.2), we get that the restriction of $p|_{C_F}$ gives a regular covering $C_F \setminus p^{-1}(\mathcal{K}) \cap C_F \rightarrow S \setminus \mathcal{K}$ of degree μ . Let us denote by $\Omega_{C_F/S}^1$ the sheaf of relative differentials, defined by the following exact sequence:

$$\mathcal{J}_F/\mathcal{J}_F^2 \xrightarrow{d_{Z/S}} \Omega_{Z/S}^1|_{C_F} \longrightarrow \Omega_{C_F/S}^1 \longrightarrow 0,$$

where \mathcal{J}_F is the ideal sheaf of C_F and the map $d_{Z/S}$ is induced from the composition of the relative de Rham differential and the restriction to C_F . Note that all morphisms in the above exact sequence are morphisms of \mathcal{O}_{C_F} -modules. It is well known (and easy to prove using the exactness of the Koszul complex) that if the stalks $\mathcal{J}_{F,\xi}$ of the ideal sheaf are generated by a regular sequence (F_0, \dots, F_n) , then $\mathcal{J}_{F,\xi}/\mathcal{J}_{F,\xi}^2$ is a free $\mathcal{O}_{C_F,\xi}$ -module of rank $n+1$. Therefore, in our case $\mathcal{J}_F/\mathcal{J}_F^2$ is a locally free sheaf of rank $n+1$.

LEMMA 4.10. *The map $p|_{C_F}$ is a local biholomorphism at some point $\xi \in C_F$ if and only if one of the following two equivalent conditions hold:*

- (i) ξ is a Morse critical point for f_t , where $t = p(\xi)$.
- (ii) The stalk $\Omega_{C_F/S,\xi}^1 = 0$.

PROOF. Let us fix local coordinates $(x, t) := (x_0, \dots, x_n, t_1, \dots, t_m)$ on Z near the point $\xi \in C_F$, such that, the map p takes the form of a projection $(x, t) \mapsto t$. Put $F_i = \frac{\partial F}{\partial x_i}$ for brevity.

Let us first check that conditions (i) and (ii) are equivalent. Both stalks $\mathcal{J}_{F,\xi}/\mathcal{J}_{F,\xi}^2$ and $(\Omega_{Z/S}^1|_{C_F})_\xi$ are free $\mathcal{O}_{C_F,\xi}$ -modules of rank $n+1$. The $\mathcal{O}_{C_F,\xi}$ -bases are given respectively by (F_0, \dots, F_n) and dx_0, \dots, dx_n . Therefore, the map $\mathcal{J}_{F,\xi}/\mathcal{J}_{F,\xi}^2 \rightarrow (\Omega_{Z/S}^1|_{C_F})_\xi$ induced by the relative de Rham differential is given by the Hessian matrix $\left(\frac{\partial^2 F}{\partial x_i \partial x_j} \right)_{0 \leq i, j \leq n}$. Therefore, the stalk $\Omega_{C_F/S,\xi}^1$ vanishes if and only if the determinant of the Hessian matrix is not 0 at ξ , that is, if and only if ξ is a Morse critical point.

If $p|_{C_F}$ is a local bi-holomorphism at ξ , then C_F must be non-singular at ξ . The tangent space $T_\xi C_F$ consists of all vectors $v = \sum_{i=0}^n v^i \partial/\partial x_i + \sum_{a=1}^m v^{n+a} \partial/\partial t_a$ satisfying

$$(4.4) \quad \sum_{i=0}^n v^i \frac{\partial^2 F}{\partial x_j \partial x_i}(\xi) + \sum_{a=1}^m v^{n+a} \frac{\partial^2 F}{\partial x_j \partial t_a}(\xi) = 0, \quad 0 \leq j \leq n.$$

If the Hessian matrix was degenerate, then we can find a non-zero tangent vector $v \in T_\xi C_F$, such that, all components $v^{n+a} = 0$. However, such a vector v will be in the kernel of the tangent map $d_\xi p$ – contradicting the assumption that $p|_{C_F}$ is a local biholomorphism. Therefore, the Hessian matrix is non-degenerate, that is, ξ is a Morse critical point. Conversely, if ξ is a Morse critical point, then the map defined by (F_0, \dots, F_n) has Jacobi matrix with maximal possible rank, because the Hessian matrix is a minor of size $(n+1) \times (n+1)$. The implicit function theorem implies that C_F is smooth in a neighborhood of ξ . Given an arbitrary tangent vector $\sum_{a=1}^m v^{n+a} \partial/\partial t_a \in T_t S$, where $t = p(\xi)$, we can uniquely solve the equation (4.4) for v^i ($0 \leq i \leq n$), because the Hessian matrix is non-degenerate. Therefore, the tangent map $d_\xi p : T_\xi C_F \rightarrow T_t S$ is an isomorphism. Recalling again the implicit function theorem, we get that $p|_{C_F}$ is a local biholomorphism at ξ . \square

Let us also point out an important corollary of the above proof. The support of the sheaf $\Omega_{C_F/S}^1$ is either empty or an analytic hypersurface in C_F . Indeed, if $\xi \in \text{supp}(\Omega_{C_F/S}^1)$, then locally in a neighborhood of ξ in Z , the determinant of the Hessian matrix $H(z) = \det(\frac{\partial^2 F}{\partial x_i \partial x_j})$ defines a holomorphic function, which represents a non-zero divisor in the local ring $\mathcal{O}_{C_F, \xi}$. Otherwise, if H represents a zero divisor, then H must vanish identically on some irreducible component A of C_F at ξ . This would imply that $A \subset p^{-1}(\mathcal{K}) \cap C_F$, because the points of A are not of Morse type (the Hessian vanishes), so $p|_{C_F}$ is not a local biholomorphism. However, this would imply that $p(A) \subset \mathcal{K}$ is not an irreducible component of S , contradicting the fact that $p|_{C_F}$ is an analytic covering (see [25], Chapter 9, Section 3.3). The support of $\Omega_{C_F/S}^1$ is locally given by $\{H = 0\}$ and since H is not a zero divisor, the support must be either empty or an analytic hypersurface.

Note that we can assume that $\mathcal{K} = p(\text{supp}(\Omega_{C_F/S}^1))$, because the image of the support is always contained in \mathcal{K} , so it is a thin subset and if t is not in the image of $\text{supp}(\Omega_{C_F/S}^1)$, then $\Omega_{C_F/S, \xi}^1 = 0$ for all $\xi \in C_F$ such that $p(\xi) = t$, therefore $p|_{C_F}$ is a local biholomorphism at ξ . Therefore, the conditions in the definition of an analytic covering are satisfied if we take $p(\text{supp}(\Omega_{C_F/S}^1))$ to be a critical locus. Moreover, if $\text{supp}(\Omega_{C_F/S}^1)$ is not empty, then the dimension of \mathcal{K} must be $m - 1$ at all points. Indeed, \mathcal{K} is an analytic subvariety of S , so its dimension is $< m$ and $p|_{\text{supp}(\Omega_{C_F/S}^1)} : \text{supp}(\Omega_{C_F/S}^1) \rightarrow \mathcal{K}$ is a proper finite surjective map, so $\dim(\mathcal{K}) \geq m - 1$.

DEFINITION 4.11. A family of functions $(F, p : Z \rightarrow S)$ is said to be Morse, if the relative critical set C_F is a reduced complex space, that is, C_F is an analytic variety.

If we have a Morse family then the subset $\mathcal{K} \subset S$ consisting of points $t \in S$, such that, f_t is not a Morse function is called *bifurcation set* or *caustic*. Our discussion from above implies, that \mathcal{K} is an analytic hypersurface in S and that $p|_{C_F} : C_F \rightarrow S$ is a branched analytic covering with critical locus \mathcal{K} . Finally, the support $\text{supp}(\Omega_{C_F/S}^1)$ is the *ramification divisor*, i.e., the points in C_F at which $p|_{C_F}$ is not a local biholomorphism.

4.2.5. Discriminant. Let $(F, p : Z \rightarrow S)$ be a family. Let us introduce the following map:

$$\Phi : Z \rightarrow \mathbb{C} \times S, \quad z \mapsto (F(z), p(z)).$$

We claim that Φ is open. Indeed, since Φ is a holomorphic map between complex manifolds, we need to check that the fibers

$$\Phi^{-1}(\lambda, t) = \{z \in Z_t \mid f_t(z) = \lambda\}$$

are either empty or equidimensional of dimension $\dim(Z) - \dim(\mathbb{C} \times S) = n$, which is clear. It is also easy to see that $\Phi|_{C_F}$ is a proper finite map. Indeed, this follows from the fact that the composition $\text{pr}_S \circ (\Phi|_{C_F}) = p|_{C_F}$ is a proper finite map, where $\text{pr}_S : \mathbb{C} \times S \rightarrow S$ is the projection.

Since $\Phi|_{C_F}$ is a proper map, its image $D_F := \Phi(C_F)$ is an analytic subvariety of $\mathbb{C} \times S$. Since C_F is pure dimensional of dimension m and $\Phi|_{C_F}$ is a proper finite map, we get that D_F is also pure dimensional of dimension m , that is, D_F is an analytic hypersurface in $\mathbb{C} \times S$. There is an elegant way to describe an equation that defines D_F . Let p_*F be the \mathcal{O}_S -linear map $p_*\mathcal{O}_{C_F} \rightarrow p_*\mathcal{O}_{C_F}$ defined by multiplication by F . Since $p_*\mathcal{O}_{C_F}$ is a locally free sheaf of rank μ_F , we can define

$$\Delta(\lambda, t) := \det(\lambda \text{id} - (p_*F)_t),$$

where $(p_*F)_t : (p_*\mathcal{O}_{C_F})_t \rightarrow (p_*\mathcal{O}_{C_F})_t$ is the stalk of the map p_*F . Clearly, $\Delta \in \mathcal{O}(S)[\lambda]$, that is, Δ is a monic polynomial in λ of degree μ_F , whose coefficients are holomorphic functions on S .

PROPOSITION 4.12. *The hypersurface $\{\Delta(\lambda, t) = 0\} \subset \mathbb{C} \times S$ coincides with D_F .*

PROOF. Suppose that $t^\circ \in S$ is an arbitrary point. We have to prove that $(\lambda, t^\circ) \in D_F$ if and only if $\Delta(\lambda, t^\circ) = 0$. By definition $\Delta(\lambda, t^\circ) \in \mathbb{C}[\lambda]$ is the determinant of the following linear operator:

$$\bigoplus_{z \in \text{Crit}(f_{t^\circ})} \mathcal{O}_{Z_{t^\circ}, z} / \left(\frac{\partial f_{t^\circ}}{\partial x_0}, \dots, \frac{\partial f_{t^\circ}}{\partial x_n} \right) \xrightarrow{\lambda - f_{t^\circ}} \bigoplus_{z \in \text{Crit}(f_{t^\circ})} \mathcal{O}_{Z_{t^\circ}, z} / \left(\frac{\partial f_{t^\circ}}{\partial x_0}, \dots, \frac{\partial f_{t^\circ}}{\partial x_n} \right),$$

where $\text{Crit}(f_{t^\circ}) = Z_{t^\circ} \cap C_F$ are the critical points of f_{t° , for each $z \in \text{Crit}(f_{t^\circ})$ we choose local coordinates $x = (x_0, \dots, x_n)$ on a neighborhood of z in Z_{t° in order to define $M_z := \mathcal{O}_{Z_{t^\circ}, z} / \left(\frac{\partial f_{t^\circ}}{\partial x_0}, \dots, \frac{\partial f_{t^\circ}}{\partial x_n} \right)$, and $\lambda - f_{t^\circ}$ is the map induced by multiplication. Each M_z is a finite dimensional vector space over \mathbb{C} of dimension $\mu_{f_{t^\circ}}(z)$.

Suppose now that $(\lambda, t^\circ) \notin D_F$. Then $\lambda \neq f_{t^\circ}(z)$ for all $z \in \text{Crit}(f_{t^\circ})$. We claim that multiplication by $\lambda - f_{t^\circ}$ induces an isomorphism $M_z \xrightarrow{\cong} M_z$. Indeed, since $\lambda \neq f_{t^\circ}(z)$ we get that $\lambda - f_{t^\circ}$ is invertible in $\mathcal{O}_{Z_{t^\circ}, z}$, so it is invertible as a linear map on the quotient M_z . Therefore, $\det(\lambda - f_{t^\circ}) \neq 0$. This proves that $\{\Delta = 0\} \subseteq D_F$. Conversely, if $(\lambda, t^\circ) \in D_F$, then there exists $z \in \text{Crit}(f_{t^\circ})$, such that, $\lambda = f_{t^\circ}(z)$, that is, $\lambda - f_{t^\circ} \in \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{m}_{Z_{t^\circ}, z}$ is the maximal ideal of the local ring $\mathcal{O}_{Z_{t^\circ}, z}$. If $\det(\lambda - f_{t^\circ}) \neq 0$, then $M_z = (\lambda - f_{t^\circ})M_z \subseteq \mathfrak{m}M_z \subseteq M_z$. Therefore, $M_z = \mathfrak{m}M_z$, so by Nakayama's lemma $M_z = 0$, which contradicts the fact that z is a critical point of f_{t° . \square

The analytic hypersurface D_F equipped with the structure of a complex space with structure sheaf $\mathcal{O}_{D_F} := \mathcal{O}_{\mathbb{C} \times S} / \Delta \mathcal{O}_{\mathbb{C} \times S}$ is called the *discriminant* of Φ .

4.3. Tame Families

Let $(F, p : Z \rightarrow S)$ be a family. Recall the map (see Section 4.2.5)

$$\Phi : Z \rightarrow S \times \mathbb{C}, \quad z \mapsto (p(z), F(z)).$$

Motivated by Iritani's work on mirror symmetry for toric orbifolds [34, 35], we would like to introduce a certain class of families for which the map

$$(4.5) \quad \Phi|_{Z \setminus \Phi^{-1}(D_F)} : Z \setminus \Phi^{-1}(D_F) \rightarrow S \times \mathbb{C} \setminus D_F$$

is a locally trivial fibration.

4.3.1. Relative Kähler structure. Suppose that $g^{Z/S}$ is a positive definite, symmetric, bilinear pairing on $T_{Z/S}^{\mathbb{R}}$, that is,

$$g^{Z/S} : \mathcal{T}_{Z/S}^{\mathbb{R}} \otimes \mathcal{T}_{Z/S}^{\mathbb{R}} \rightarrow \mathcal{C}_Z^{\infty},$$

where \mathcal{C}_Z^{∞} is the sheaf of smooth real-valued functions on Z and the following two conditions hold:

- (i) $g^{Z/S}(v_1, v_2) = g^{Z/S}(v_2, v_1)$ for all $v_1, v_2 \in \mathcal{T}_{Z/S}^{\mathbb{R}}$.
- (ii) $g^{Z/S}(v, v) > 0$ for all non-zero $v \in \mathcal{T}_{Z/S}^{\mathbb{R}}$.

Let $J \in \text{End}(T_{Z/S}^{\mathbb{R}})$ be the relative complex structure. We say that $g^{Z/S}$ is a *relative Kähler metric* if in addition the following two conditions hold:

- (iii) *J-invariance*: $g^{Z/S}(Jv_1, Jv_2) = g^{Z/S}(v_1, v_2)$ for all $v_1, v_2 \in \mathcal{T}_{Z/S}^{\mathbb{R}}$.
- (iv) *Symplectic Structure*: The two-form $\omega^{Z/S} \in \Gamma(Z, \wedge^2(T_{Z/S}^{\mathbb{R}})^*)$ defined by

$$\omega^{Z/S}(v_1, v_2) := g^{Z/S}(Jv_1, v_2), \quad v_1, v_2 \in \mathcal{T}_{Z/S}^{\mathbb{R}}$$

is closed, that is, $d^{Z/S}(\omega^{Z/S}) = 0$.

DEFINITION 4.13. The data $(F, p : Z \rightarrow S, g^{Z/S})$ is said to be a *Kähler family* if $(F, p : Z \rightarrow S)$ is a family and $g^{Z/S}$ is a relative Kähler metric. \square

Note that if $(F, p : Z \rightarrow S, g^{Z/S})$ is a Kähler family, then each fiber

$$Z_t := p^{-1}(t), \quad g_t := g^{Z/S}|_{Z_t}, \quad t \in S,$$

is a Kähler manifold.

From now on, until the end of this section, we fix a Kähler family $(F, p : Z \rightarrow S, g^{Z/S})$. The *relative gradient* $\nabla f \in \Gamma(Z, \mathcal{T}_{Z/S}^{\mathbb{R}})$ of a smooth function $f : Z \rightarrow \mathbb{R}$ is defined by the following formula:

$$g^{Z/S}(\nabla f, v) := \langle df, v \rangle = v(f), \quad \forall v \in \mathcal{T}_{Z/S}^{\mathbb{R}}.$$

The length of the gradient is by definition $\|\nabla f\| := g^{Z/S}(\nabla f, \nabla f)^{1/2}$. The following simple lemma will play a key role in our discussion.

LEMMA 4.14. *Suppose that $f : Z \rightarrow \mathbb{C}$ is a holomorphic function and put $u := \text{Re}(f)$ and $v := \text{Im}(f)$. Then $\|\nabla u\| = \|\nabla v\|$ and $g^{Z/S}(\nabla u, \nabla v) = 0$.*

PROOF. Let $J \in \text{End}(T_{Z/S}^{\mathbb{R}})$ be the relative complex structure. The Cauchy–Riemann equations for f yield the following identity:

$$\langle du, \xi \rangle = \langle dv, J\xi \rangle, \quad \xi \in \mathcal{T}_{Z/S}^{\mathbb{R}}.$$

Recalling the definition of the relative gradient, we get

$$g^{Z/S}(\nabla u, \xi) = g^{Z/S}(\nabla v, J\xi) = g^{Z/S}(-J\nabla v, \xi),$$

where we used the J -invariance of $g^{Z/S}$ and that $J^2 = -1$. Since $g^{Z/S}$ is non-degenerate, we get $\nabla u = -J\nabla v$. Therefore, using the J -invariance again, we get

$$\|\nabla u\|^2 = g^{Z/S}(\nabla u, \nabla u) = g^{Z/S}(J\nabla v, J\nabla v) = g^{Z/S}(\nabla v, \nabla v) = \|\nabla v\|^2.$$

Similarly, using the symplectic condition, we get

$$g^{Z/S}(\nabla u, \nabla v) = -g^{Z/S}(J\nabla v, \nabla v) = -\omega^{Z/S}(\nabla v, \nabla v) = 0. \quad \square$$

4.3.2. Kähler-complete families. Note that the real tangent bundle

$$(4.6) \quad T_Z^{\mathbb{R}} \cong T_{Z/S}^{\mathbb{R}} \oplus p^*T_S^{\mathbb{R}}.$$

To see this, let us recall that Z is an open subset of $X \times S$. Let us denote by $\text{pr}_X : X \times S \rightarrow X$ and $\text{pr}_S : X \times S \rightarrow S$ the projection maps. Then

$$(4.7) \quad T_{X \times S}^{\mathbb{R}} \cong \text{pr}_X^* T_X^{\mathbb{R}} \oplus \text{pr}_S^* T_S^{\mathbb{R}}.$$

Clearly, $T_Z^{\mathbb{R}} = T_{X \times S}^{\mathbb{R}}|_Z$, $p^*T_S^{\mathbb{R}} = \text{pr}_S^* T_S^{\mathbb{R}}|_Z$, and $T_{Z/S}^{\mathbb{R}} \cong \text{pr}_X^* T_X^{\mathbb{R}}|_Z$. Therefore, the splitting (4.6) follows from (4.7) by restriction to Z .

Using the splitting (4.6), we construct a Riemannian metric g^Z on Z as follows. Let g^S be a Riemannian metric on S . Then the pullback p^*g^S defines a metric on $p^*T_S^{\mathbb{R}}$ and we define $g^Z = g^{Z/S} \oplus p^*g^S$, that is,

$$g^Z(v_1, v_2) = g^{Z/S}(\pi(v_1), \pi(v_2)) + g^S(dp(v_1), dp(v_2)),$$

where $v_i \in T_z^{\mathbb{R}}Z$, $dp(v_i) \in T_{p(z)}^{\mathbb{R}}S$, and $\pi(v_i)$ is the projection of v_i to the relative tangent space corresponding to the splitting (4.6). The Riemannian metric g^Z allows us to turn Z into a metric space. If $K \subset S$ is a compact subset, then put $Z_K := p^{-1}(K)$. Let us equip Z_K with the metric induced from Z . Note that, since K is compact, the restrictions of any two Riemannian metrics g_1^S and g_2^S to K are equivalent, that is, there are constants C and c , such that

$$cg_1^S(v, w) \leq g_2^S(v, w) \leq Cg_1^S(v, w),$$

for all $v, w \in T_t^{\mathbb{R}}S$ and for all $t \in K$. Therefore, the condition that Z_K is a complete metric space is independent of the choice of g^S .

DEFINITION 4.15. A Kähler family $(F, p : Z \rightarrow S, g^{Z/S})$ is said to be *Kähler-complete* if for every compact subset $K \subset S$ the metric space $Z_K = p^{-1}(K)$ is complete, i.e., every Cauchy sequence has a convergent subsequence.

REMARK 4.16. If we have a family $(F, p : Z \rightarrow S)$ for which $Z = X \times S$, then we can always choose a relative Kähler metric $g^{Z/S}$, such that, the family $(F, p : Z \rightarrow S, g^{Z/S})$ is Kähler-complete. Indeed, since X is a Stein manifold, it can be embedded as a closed submanifold of \mathbb{C}^N for some $N > 0$. The standard Kähler metric on \mathbb{C}^N induces a complete Kähler metric on X . If $K \subset S$ is compact, then $Z_K = X \times K$ is a product of complete metric spaces, so it must be complete. \square

4.3.3. The relative gradient flow. Suppose that $(F, p : Z \rightarrow S, g^{Z/S})$ is a relative Kähler family. For given an open subset $V \subset S$, smooth vector field $\xi \in \Gamma(V, \mathcal{T}_S^{\mathbb{R}})$, and a complex number $\eta \in \mathbb{C}$, let us define

$$C_{\eta, \xi}(V) = \{z \in p^{-1}V \mid \|\nabla F(z)\| \leq |\eta| + |\widehat{\xi}(F)(z)|\},$$

where $\|\nabla F\| := \sqrt{\|\nabla \text{Re}(F)\|^2 + \|\nabla \text{Im}(F)\|^2}$ and $\widehat{\xi} \in \Gamma(p^{-1}V, \mathcal{T}_Z^{\mathbb{R}})$ is the lift of the vector field ξ defined via the splitting (4.6).

REMARK 4.17. If $\eta = 0$ and $\xi = 0$, then $C_{0,0}(V) = C_F \cap p^{-1}V$ is the relative critical set. We think of $C_{\eta,\xi}(V)$ as a thickening of the relative critical set. \square

DEFINITION 4.18. A Kähler family $(F, p : Z \rightarrow S, g^{Z/S})$ is said to be *tame* if for every open subset $V \subset S$, smooth vector field $\xi \in \Gamma(V, \mathcal{T}_S^{\mathbb{R}})$, and a complex number $\eta \in \mathbb{C}$, the map

$$p|_{C_{\eta,\xi}(V)} : C_{\eta,\xi}(V) \rightarrow V$$

is proper. \square

The main goal in this Section is to prove the following theorem.

THEOREM 4.19. *If $(F, p : Z \rightarrow S, g^{Z/S})$ is a tame Kähler-complete family, then the map (4.5) is a locally trivial fibration.*

The local trivializations will be constructed via the flows of certain systems of ordinary differential equations, which we would like to define now. Let $(t^\circ, \lambda^\circ) \in S \times \mathbb{C} \setminus D_F$ be an arbitrary point. Let us fix local coordinates $t = (t_1, \dots, t_m)$ in a neighborhood of $t^\circ \in S$. Let $0 < \epsilon_1 < \epsilon_2$ be small positive numbers, such that, the polydisks

$$V_i := \{t \in \mathbb{C}^m \mid |t_j - t_j^\circ| < \epsilon_i\}, \quad i = 1, 2,$$

are contained in the coordinate chart about t° and the polydisks

$$D_i := V_i \times \{\lambda \in \mathbb{C} \mid |\lambda - \lambda^\circ| < \epsilon_i\}$$

are contained in $S \times \mathbb{C} \setminus D_F$. We will prove that $\Phi^{-1}(D_1)$ is diffeomorphic to $D_1 \times Z_{t^\circ, \lambda^\circ}$, where $Z_{t, \lambda} := \Phi^{-1}(t, \lambda)$.

Given $\eta \in \mathbb{C}$ and a smooth vector field $\xi \in \Gamma(U_2, \mathcal{T}_S^{\mathbb{R}})$ we define the following real smooth vector field on $Z_{D_2} := \Phi^{-1}(D_2)$:

$$(4.8) \quad G := \widehat{\xi} + \frac{1}{\|\nabla u\|^2} \left(\operatorname{Re}(\eta - \widehat{\xi}(F)) \nabla u + \operatorname{Im}(\eta - \widehat{\xi}(F)) \nabla v \right),$$

where $u := \operatorname{Re}(F)$ and $v := \operatorname{Im}(F)$. Let us view $\Phi : Z_{D_2} \rightarrow S \times \mathbb{C}$ as a smooth map between real smooth manifolds. In particular, we identify $\mathbb{C} \cong \mathbb{R}^2$ in the standard way, that is, $\lambda = \lambda_1 + \mathbf{i}\lambda_2 \mapsto (\lambda_1, \lambda_2)$. We claim that the vector field $G \in \Gamma(\Phi^{-1}(D_2), \mathcal{T}_Z^{\mathbb{R}})$ is a Φ -lift of $\xi + \operatorname{Re}(\eta) \frac{\partial}{\partial \lambda_1} + \operatorname{Im}(\eta) \frac{\partial}{\partial \lambda_2} \in \Gamma(D_2, \mathcal{T}_{S \times \mathbb{C}}^{\mathbb{R}})$. Indeed, we have

$$d\Phi(\widehat{\xi}) = dp(\widehat{\xi}) + \widehat{\xi}(u) \frac{\partial}{\partial \lambda_1} + \widehat{\xi}(v) \frac{\partial}{\partial \lambda_2} = \xi + \widehat{\xi}(u) \frac{\partial}{\partial \lambda_1} + \widehat{\xi}(v) \frac{\partial}{\partial \lambda_2},$$

where we used that by definition $\widehat{\xi}$ is a p -lift of ξ . Since ∇u is a relative tangent field, $dp(\nabla u) = 0$ and we get

$$d\Phi(\nabla u) = du(\nabla u) \frac{\partial}{\partial \lambda_1} = \|\nabla u\|^2 \frac{\partial}{\partial \lambda_1},$$

where we used that $dv(\nabla u) = g^{Z/S}(\nabla v, \nabla u) = 0$ according to Lemma 4.14. Similarly,

$$d\Phi(\nabla v) = dv(\nabla v) \frac{\partial}{\partial \lambda_2} = \|\nabla v\|^2 \frac{\partial}{\partial \lambda_2} = \|\nabla u\|^2 \frac{\partial}{\partial \lambda_2}.$$

The above 3 identities clearly imply that $d\Phi(G) = \xi + \operatorname{Re}(\eta) \frac{\partial}{\partial \lambda_1} + \operatorname{Im}(\eta) \frac{\partial}{\partial \lambda_2}$.

Let $\psi(z, \eta, \xi; s)$ be the flow of the vector field G on the manifold Z_{D_1} through the point $z \in Z_{D_1}$, that is, $\psi(z, \eta, \xi; s)$ is a solution to the following system of ODEs

$$\begin{aligned} \frac{\partial \psi}{\partial s}(z, \eta, \xi; s) &= G(\psi(z, \eta, \xi; s)), \\ \psi(z, \eta, \xi; 0) &= z. \end{aligned}$$

Let $(R^-(z, \eta, \xi), R^+(z, \eta, \xi))$ be the maximal interval such that the solution $\psi(z, \eta, \xi; s)$ exists for all $s \in (R^-(z, \eta, \xi), R^+(z, \eta, \xi))$. Let us stress that the definition of R^\pm depends on the polydisk D_1 . We are interested in the flow lines of G inside the domain Z_{D_1} . We will actually prove that R^+ (resp. R^-) is the time that it takes for the positive (resp. negative) time flow to reach the boundary of Z_{D_1} .

Let $\Xi(t, s)$ be the flow line of the vector field ξ on V_2 through a point $t \in V_1$, that is, $\Xi(t, s)$ is a solution to the following system of ODEs

$$\begin{aligned} \frac{\partial \Xi}{\partial s}(t, s) &= \xi(\Xi(t, s)), \\ \Xi(t, 0) &= t. \end{aligned}$$

We will be interested only in vector fields ξ , such that, the flow line $\Xi(t, s)$ reaches the boundary of \bar{V}_1 in finite time for both the positive ($s > 0$) and the negative ($s < 0$) flow of ξ . Note that the flow line in D_2 of the vector field $\xi + \operatorname{Re}(\eta) \frac{\partial}{\partial \lambda_1} + \operatorname{Im}(\eta) \frac{\partial}{\partial \lambda_2}$ through a point $(t, \lambda) \in D_1$ has the form $\gamma(t, \lambda, s) := (\Xi(t, s), \lambda + \eta s)$. Let us denote by $\rho^+(t, \lambda, \eta, \xi)$ the minimal value of $s > 0$ for which $\gamma(t, \lambda, s)$ belong to the boundary of \bar{D}_1 , that is, ρ^+ is the time needed to reach the boundary of the domain \bar{D}_1 . Similarly, let $\rho^-(t, \lambda, \eta, \xi)$ be the maximal value of $s < 0$ for which $\gamma(t, \lambda, s)$ belong to the boundary of \bar{D}_1 .

Since $d\Phi(G) = \xi + \operatorname{Re}(\eta) \frac{\partial}{\partial \lambda_1} + \operatorname{Im}(\eta) \frac{\partial}{\partial \lambda_2}$, we get that

$$\Phi(\psi(z, \eta, \xi; s)) = (\Xi(p(z), s), F(z) + \eta s).$$

The above relation, since the LHS belongs to D_1 for all $s \in (R^-, R^+)$, implies that

$$\rho^-(\Phi(z), \eta, \xi) \leq R^-(z, \eta, \xi) < R^+(z, \eta, \xi) \leq \rho^+(\Phi(z), \eta, \xi).$$

The most difficult part in the proof of Theorem 4.19 is to prove that $R^\pm = \rho^\pm$.

LEMMA 4.20. *Under the above notation we have $R^\pm(z, \eta, \xi) = \rho^\pm(\Phi(z), \eta, \xi)$.*

Assuming that Lemma 4.20 is established the proof of Theorem 4.19 can be completed as follows. Assuming the notation from above and letting $D := D_1$ for brevity. Let us define

$$(4.9) \quad \Psi : Z_{t^\circ, \lambda^\circ} \times D \rightarrow \Phi^{-1}(D),$$

by the following formula

$$(4.10) \quad \Psi(z, t', \lambda') := \psi(z, \lambda' - \lambda^\circ, t' - t^\circ; 1),$$

where $t' - t^\circ$ is identified with the constant vector field $\xi = \sum_{j=1}^m (t'_j - t_j^\circ) \frac{\partial}{\partial t_j}$. The flow line of ξ through $\Phi(z) = (t^\circ, \lambda^\circ)$ is the straight line $\Xi(t^\circ, s) = t^\circ + s(t' - t^\circ)$. Therefore, the flow line

$$\gamma(t^\circ, \lambda^\circ, s) = (\Xi(t, s), \lambda + \eta s) = (t^\circ + s(t' - t^\circ), \lambda^\circ + s(\lambda' - \lambda^\circ)).$$

Since the point $(t', \lambda') \in D$ is reached at time $s = 1$, we get that $1 \in (\rho^-, \rho^+)$. Therefore, the RHS of (4.10) is well defined. The vector field G depends smoothly on the parameters (z, t', λ') , so the solution, i.e., the RHS of (4.10) depends also smoothly on the parameters (z, t', λ') . This proves that Ψ is a differentiable map.

In order to construct the inverse, first note that if $z \in Z_D$ and $(t', \lambda') = \Phi(z)$, then

$$\psi(\psi(z, \lambda^\circ - \lambda', t^\circ - t', 1), \lambda' - \lambda^\circ, t' - t^\circ, 1 - s) = \psi(z, \lambda^\circ - \lambda', t^\circ - t', s).$$

Indeed, both sides coincide for $s = 1$ and they satisfy the same differential equation with respect to s . Setting $s = 0$ in the above formula, we get that the inverse of Ψ is given by the following formula:

$$\tilde{\Psi}(z) := \left(\psi(z, \lambda^\circ - F(z), t^\circ - p(z); 1), \Phi(z) \right).$$

4.3.4. Proof of Lemma 4.20. We have to prove the existence of the flow of a certain perturbation of the relative gradient vector field. It turns out that the tameness and the Kähler-completeness conditions are sufficient for the usual statements of Morse theory to hold. More precisely, we will check that Palais' argument in [49], proving the existence of the flow of the gradient of a Morse function, applies in our settings.

Before we go into the details of the proof we need to recall two well known facts. The first one is from the theory of ODEs. Suppose that M is a real smooth manifold, G is a smooth vector field on M , and $z \in M$ is an arbitrary point. Let $\psi(z, s)$ be the flow line of G through the point z , that is,

$$\begin{aligned} \frac{\partial \psi}{\partial s}(z, s) &= G(\psi(z, s)), \\ \psi(z, 0) &= z. \end{aligned}$$

In general, the flow line exists only for s sufficiently close to 0. Let $(R^-(z), R^+(z))$ be the maximal interval for which the flow line $\psi(z, s)$ is defined for all $s \in (R^-(z), R^+(z))$. If $R^+(z) < \infty$, then given any sequence $\{s_n\}_{n=1}^\infty$ in $(R^-(z), R^+(z))$, such that, $\{s_n\}_{n=1}^\infty$ is convergent in \mathbb{R} with limit $R^+(z)$, then the sequence $\{\psi(z, s_n)\}_{n=1}^\infty$ does not have a convergent subsequence. Similarly, if $R^-(z) > -\infty$ and $s_n \rightarrow R^-(z)$, then the sequence $\{\psi(z, s_n)\}_{n=1}^\infty$ does not have a convergent subsequence.

The second fact is from the theory of metric spaces. Suppose that M is a metric space. If $z \in M$ and $r > 0$ is a real number, then we denote by $B(a, r)$ the open ball in M with center at a and radius r . A subset $A \subset M$ is said to be *totally bounded* if for every $\delta > 0$ there are finitely many points a_1, \dots, a_k of A , such that, the balls $B(a_i, \delta)$ ($1 \leq i \leq k$) cover A . Furthermore, a subset $A \subset M$ is said to be *relatively compact* if the closure of A in M is compact. If the metric space M is complete, then a subset A is relatively compact if and only if A is totally bounded (see [65], Section 0.2). In particular, if A is a totally bounded subset of a complete metric space M , then every infinite sequence in A has a subsequence convergent in M .

Let us return to our settings. For brevity, let us drop the arguments and write $R^\pm := R^\pm(z, \eta, \xi)$ and $\rho^\pm := \rho^\pm(\Phi(z), \eta, \xi)$. Suppose that $R^+ < \rho^+$. The image $\gamma(\Phi(z), s) = \Phi(\psi(z, \eta, \xi; s))$ for $s \in [0, R^+)$ has a compact closure in D_1 . Indeed, $\gamma(\Phi(z), s)$ is the flow line of a vector field on D_2 . By definition, $\gamma(\Phi(z), s)$ is defined and it belongs to D_1 for all $s \in (\rho^-, \rho^+)$. Since $R^+ \in (\rho^-, \rho^+)$, we get that the closure that we mentioned above is the smooth compact curve $\Gamma \subset D_1$ parametrized by $\gamma(\Phi(z), s)$ $0 \leq s \leq R^+$. Let us choose a compact neighborhood K of Γ , such that, $\Gamma \subset K \subset D_1$. In particular, the smooth curve $\hat{\Gamma} := \{\psi(z, \eta, \xi; s) \mid 0 \leq s < R^+\}$ is contained in $Z_K := \Phi^{-1}(K)$.

Let us consider the following integral:

$$\int_0^{R^+} \|\partial_s \psi(z, \eta, \xi; s)\|_Z ds \leq \left(\int_0^{R^+} \|\partial_s \psi(z, \eta, \xi; s)\|_Z^2 ds \right)^{1/2},$$

where we fixed a metric on S and constructed a Riemannian metric g^Z on Z as it was explained before. The norm $\|\cdot\|_Z$ is computed with respect to g^Z . The integral computes the length of the smooth curve $\widehat{\Gamma}$ and in principle could be ∞ . However, we will prove that $\widehat{\Gamma}$ is a totally bounded subset of Z_K . We have $g^Z(\widehat{\xi}, \widehat{\xi}) = g^S(\xi, \xi) = \|\xi\|_S^2$, $g^Z(\nabla u, \nabla u) = g^{Z/S}(\nabla u, \nabla u) = \|\nabla u\|^2$, and similarly $g^Z(\nabla v, \nabla v) = g^{Z/S}(\nabla v, \nabla v) = \|\nabla v\|^2$. By definition, the vector field $\widehat{\xi}$ is orthogonal to ∇u and ∇v . Recalling Lemma 4.14, we get

$$\|\partial_s \psi(z, \eta, \xi; s)\|_Z^2 = \|\xi\|_S^2 + \left(\frac{|\eta - \widehat{\xi}(F)|}{\|\nabla u\|} \right)^2 \leq \|\xi\|_S^2 + \left(\frac{|\eta| + |\widehat{\xi}(F)|}{\|\nabla u\|} \right)^2.$$

Given a real number $\delta > 0$, suppose that there exists a sequence $s_n \rightarrow R^+$, $s_n < R^+$, such that,

$$\frac{|\eta| + |\widehat{\xi}(F)(\psi(z, \eta, \xi; s_n))|}{\|\nabla u(\psi(z, \eta, \xi; s_n))\|} \geq \delta.$$

Then

$$\|\nabla u(\psi(z, \eta, \xi; s_n))\| \leq \frac{1}{\delta} |\eta| + \frac{1}{\delta} |\widehat{\xi}(F)(\psi(z, \eta, \xi; s_n))|.$$

Therefore, $\psi(z, \eta, \xi; s_n) \in C_{\frac{\eta}{|\delta|}, \frac{1}{\delta}\xi}(V_1) \cap Z_K$. The tameness condition implies that $C_{\frac{\eta}{|\delta|}, \frac{1}{\delta}\xi}(V_1) \cap Z_K$ is a compact subset of Z_{D_1} . Therefore, the sequence $\psi(z, \eta, \xi; s_n)$ has a convergent subsequence – contradiction with the fact that the flow line $\psi(z, \eta, \xi; s)$ does not extend beyond $s = R^+$. This contradiction proves that for every $\delta > 0$, there exists $r_1 \in (0, R^+)$ sufficiently close to R^+ , such that,

$$\int_{r_1}^{R^+} \|\partial_s \psi(z, \eta, \xi; s)\|_Z ds < \delta.$$

Since, $\|\partial_s \psi(z, \eta, \xi; s)\|_Z$ is continuous, and hence uniformly continuous, for $s \in [0, r_0]$, we can find finitely many numbers $0 = r_k < r_{k-1} < \dots < r_1 < r_0 := R^+$, such that,

$$\int_{r_i}^{r_{i-1}} \|\partial_s \psi(z, \eta, \xi; s)\|_Z ds < \delta$$

for all $i = 1, 2, \dots, k$. This implies that the set $\widehat{\Gamma}$ is totally bounded. According to the Kähler-complete condition, the metric space Z_K is complete. Therefore, $\widehat{\Gamma}$ is relatively compact. This however implies that if we choose an arbitrary sequence $s_n \rightarrow R^+$, $s_n < R^+$, then we will be able to choose a subsequence convergent in Z_K , and hence in Z_{D_1} as well. This is again a contradiction with the fact that the flow line $\psi(z, \eta, \xi; s)$ does not extend beyond $s = R^+$. This contradiction proves that $R^+ = \rho^+$. The argument that $R^- = \rho^-$ is completely analogous. This completes the proof of Lemma 4.20.

4.4. Modules of formal oscillatory integrals

Let $(F, p : Z \rightarrow S)$ be a family of functions in the sense of Definition 4.1. The main goal of this section is to introduce and prove the basic properties of the twisted de Rham complex. Let us denote by $(\Omega_{Z/S}^\bullet, d_{Z/S})$ the *relative* holomorphic de Rham complex, where

$$\Omega_{Z/S}^0 := \mathcal{O}_Z, \quad \Omega_{Z/S}^i := \Omega_Z^i / p^* \Omega_S^1 \wedge \Omega_Z^{i-1} \quad (i \geq 1)$$

and $d_{Z/S}$ is the differential induced from the de Rham differential d_Z on Z (see Section 4.1 for some overview of relative de Rham theory).

4.4.1. The de Rham lemma for $dF \wedge$. In order to avoid cumbersome notation we would like to denote $dF := d_{Z/S}F$. The wedge product operation $dF \wedge$ defines yet another complex $(\Omega_{Z/S}^\bullet, dF \wedge)$. We will prove that the cohomology of this complex is non-trivial only in degree $n+1$. This fact follows from the generalized de Rham lemma in [52]. The cohomology in degree $n+1$ is $\Omega_F := \Omega_{Z/S}^{n+1}/dF \wedge \Omega_{Z/S}^n$ – this is a coherent sheaf on Z whose support coincides with the relative critical set C_F .

LEMMA 4.21. *The following sequence:*

$$(4.11) \quad 0 \longrightarrow \mathcal{O}_Z \xrightarrow{dF \wedge} \Omega_{Z/S}^1 \xrightarrow{dF \wedge} \cdots \xrightarrow{dF \wedge} \Omega_{Z/S}^{n+1} \longrightarrow \Omega_F \longrightarrow 0$$

is exact.

PROOF. Suppose that $z \in Z$ is an arbitrary point. Let us choose a direct product open neighborhood $U \times V$ of z in Z and local coordinates $x = (x_0, \dots, x_n)$ and $t = (t_1, \dots, t_m)$ on respectively U and V . Since $\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n}$ is a regular sequence in $\mathcal{O}_{Z,z}$ (see Corollary 4.3), the corresponding Koszul complex is exact, that is,

$$\cdots \longrightarrow \bigwedge^i(\mathcal{O}_{Z,z}^{n+1}) \xrightarrow{\delta_i} \bigwedge^{i-1}(\mathcal{O}_{Z,z}^{n+1}) \xrightarrow{\delta_{i-1}} \cdots \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_Z / \left(\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n} \right) \longrightarrow 0$$

is an exact sequence, where the differential

$$\delta_i(e_{k_1} \wedge \cdots \wedge e_{k_i}) = \sum_{s=1}^i (-1)^{s-1} \frac{\partial F}{\partial x_s} e_{k_1} \wedge \cdots \widehat{e}_{k_s} \cdots \wedge e_{k_i}.$$

We have an isomorphism of $\mathcal{O}_{Z,z}$ -modules $\phi : \bigwedge^i(\mathcal{O}_{Z,z}^{n+1}) \xrightarrow{\cong} \Omega_{Z/S,z}^{n+1-i}$ defined by

$$\phi(e_{k_1} \wedge \cdots \wedge e_{k_i}) = \iota_{\partial/\partial x_{k_1}} \circ \cdots \circ \iota_{\partial/\partial x_{k_i}} \omega$$

where $\omega = dx_0 \wedge \cdots \wedge dx_n$ and $\iota_{\partial/\partial x_i}$ is the operation of contraction by the vector field $\frac{\partial}{\partial x_i}$. Note that

$$\phi(\delta_i(e_{k_1} \wedge \cdots \wedge e_{k_i})) = dF \wedge \phi(e_{k_1} \wedge \cdots \wedge e_{k_i}).$$

Therefore, the isomorphism ϕ transforms the Koszul complex into the exact sequence (4.11). \square

4.4.2. Twisted de Rham cohomology. For any sheaf \mathcal{F} we denote by $\mathcal{F}[[w]]w^{-k}$ the sheaf whose sections over some open subset V are given by $\Gamma(V, \mathcal{F})[[w]]w^{-k}$. We leave it to the reader to check that this construction produces a sheaf, i.e., the sheafification is not necessary. The sheaves $\mathcal{F}[[w]]w^{-k}$ ($k \in \mathbb{Z}$) form a directed system. Let $\mathcal{F}((w)) := \varinjlim \mathcal{F}[[w]]w^{-k}$ be the directed limit. Let us define the twisted de Rham complexes

$$(4.12) \quad (\Omega_{Z/S}^\bullet[[w]]w^{-k}, d_F) \quad (k \in \mathbb{Z}) \quad \text{and} \quad (\Omega_{Z/S}^\bullet((w)), d_F),$$

where the differential $d_F := wd_{Z/S} + dF \wedge$. The main goal in this section is to prove that the cohomologies of these complexes are concentrated in degree $n+1$.

PROPOSITION 4.22. *a) The cohomology groups*

$$H^i(\Omega_{Z/S}^\bullet[[w]]w^{-k}, d_F) \quad (k \in \mathbb{Z}), \quad H^i(\Omega_{Z/S}^\bullet((w)), d_F)$$

vansih for $i \neq n+1$ and for $i = n+1$ they define sheaves on Z whose support is contained in C_F .

b) We have $H^i(p_\Omega_{Z/S}^\bullet[[w]]w^{-k}, d_F) = p_*(H^i(\Omega_{Z/S}^\bullet[[w]]w^{-k}, d_F))$ for all $k, i \in \mathbb{Z}$.*

c) The higher direct images

$$R^j p_*(\Omega_{Z/S}^i[[w]]w^{-k}) = 0$$

for all $j > 0$ and all $i, k \in \mathbb{Z}$.

PROOF. Put $F^k \widehat{\Omega}^\bullet := \Omega_{Z/S}^\bullet[[w]]w^{-k}$. The idea of the proof is to compare the two spectral sequences converging to the hyperderived pushforward $\mathbb{R}^i p_*(F^k \widehat{\Omega}^\bullet)$. Recall that the later is by definition the cohomology $H^i(\text{Tot}(p_* I^{\bullet, \bullet}))$, where $I^{p, q}$ ($p, q \geq 0$) is a Cartan–Eilenberg resolution of $(F^k \widehat{\Omega}^\bullet, d_F)$ (see [62], Section 5.7.9). There are two spectral sequences converging to $\mathbb{R}^i p_*(F^k \widehat{\Omega}^\bullet, d_F)$. Their E_2 -pages are given by

$$'E_2^{i, j} := R^j p_*(H^i(F^k \widehat{\Omega}^\bullet, d_F))$$

$$''E_2^{i, j} := H^i(R^j p_* F^k \widehat{\Omega}^\bullet, d_F).$$

We will prove that both spectral sequences degenerate.

Let us prove that $'E_2^{i, j} = 0$ for $i \neq n+1$. Suppose that $z \in Z$ is an arbitrary point, $\omega \in (F^k \widehat{\Omega}^i)_z$, and $d_F(\omega) = 0$. Put $\omega = \sum_{m \geq -k} \omega_m w^m$, $\omega_m \in \Omega_{Z/S, z}^i$. Let us compare the coefficients in front of the powers of w in $d_F(\omega) = 0$. We get the following system of equations:

$$dF \wedge \omega_{-k} = 0, \quad (dF \wedge \omega_m + d_{Z/S} \omega_{m-1} = 0 \quad (m > -k)).$$

If $i < n+1$, then the exactness of the sequence (4.11) implies that $\omega_{-k} = dF \wedge \eta_{-k}$ for some $\eta_{-k} \in \Omega_{Z/S, z}^{i-1}$. Substituting this in the equation with $m = -k+1$, we get $dF \wedge (\omega_{-k+1} - d\eta_{-k}) = 0$. For the same reason, $\omega_{-k+1} - d\eta_{-k} = dF \wedge \eta_{-k+1}$. Continuing in the same way, we get that $\omega_m = d\eta_{m-1} + dF \wedge \eta_m$ for all $m > -k$, where $\eta_m \in \Omega_{Z/S, z}^{i-1}$. Therefore, $\omega = dF \left(\sum_{m \geq -k} \eta_m w^m \right)$. This proves that $H^i(\Omega_{Z/S, z}^\bullet[[w]]w^{-k}, d_F) = 0$ for all $i < n+1$ and hence for all $i \neq n+1$, because the complex is concentrated only in degrees from 0 to $n+1$.

Note that the above argument also proves that $H^{n+1}(\Omega_{Z/S, z}^\bullet[[w]]w^{-k}, d_F) = 0$ if $z \notin \text{supp}(\Omega_F) = C_F$. Therefore, the support of the sheaf

$$H^{n+1}(F^k \widehat{\Omega}^\bullet, d_F) = \Omega_{Z/S}^{n+1}[[w]]w^{-k} / d_F \left(\Omega_{Z/S}^n[[w]]w^{-k} \right)$$

is contained in C_F . Using this observation, we get

$$'E^{n+1, j} = R^j(p|_{C_F})_*(H^{n+1}(F^k \widehat{\Omega}^\bullet, d_F)).$$

Since $p|_{C_F}$ is a proper map, the stalk of the sheaf on the RHS of the above equality at a point $t \in S$ is given by the cohomology group

$$H^j(C_F \cap Z_t, H^{n+1}(F^k \widehat{\Omega}^\bullet, d_F)).$$

This cohomology group vanishes for $j > 0$, because $C_F \cap Z_t$ is a finite set of points. This proves that $'E_2^{i,j} = 0$ for all (i, j) except for

$$'E_2^{n+1,0} = p_* \left(\Omega_{Z/S}^{n+1}[[w]]w^{-k} / d_F \left(\Omega_{Z/S}^n[[w]]w^{-k} \right) \right).$$

Let us point out that the argument so far applies to the sheaf $\Omega_{Z/S}^{n+1}((w))$ too, i.e., at this point part a) is proved.

Let us compute the second spectral sequence. Note that if $0 \rightarrow \mathcal{F} \rightarrow I^\bullet$ is an injective resolution of \mathcal{F} , then $0 \rightarrow \mathcal{F}[[w]]w^{-k} \rightarrow I^\bullet[[w]]w^{-k}$ is an injective resolution of $\mathcal{F}[[w]]w^{-k}$. Therefore,

$$H^j(p^{-1}V, F^k \widehat{\Omega}^i) = H^j(p^{-1}V, \Omega_{Z/S}^i[[w]]w^{-k}).$$

Since $\Omega_{Z/S}^i$ is a coherent sheaf, we get that the above cohomology is 0 if V is an open Stein neighborhood and $j > 0$. The higher direct image $R^j p_*(F^k \widehat{\Omega}^i)$ is the sheaf associated to the presheaf $V \mapsto H^j(p^{-1}V, F^k \widehat{\Omega}^i)$. Therefore, $R^j p_*(F^k \widehat{\Omega}^i) = 0$ for all $j > 0$ and for all i . The terms in the second spectral sequence that could be non-vanishing are given by

$$''E_2^{i,0} = H^i(p_* \Omega_{Z/S}^\bullet[[w]]w^{-k}, d_F), \quad 0 \leq i \leq n+1.$$

Comparing with the first spectral sequence, we get that

$$\mathbb{R}^i p_*(F^k \widehat{\Omega}^\bullet, d_F) = H^i(p_* \Omega_{Z/S}^\bullet[[w]]w^{-k}, d_F) = 0, \quad i \neq n+1$$

and

$$\mathbb{R}^{n+1} p_*(F^k \widehat{\Omega}^\bullet, d_F) = p_* H^{n+1}(\Omega_{Z/S}^\bullet[[w]]w^{-k}, d_F) = H^{n+1}(p_* \Omega_{Z/S}^\bullet[[w]]w^{-k}, d_F). \quad \square$$

Let us introduce the following twisted de Rham cohomology groups:

$$(4.13) \quad \widehat{\mathcal{H}}_F^{(k)} := p_* \Omega_{Z/S}^{n+1}[[w]]w^{-k} / d_F \left(p_* \Omega_{Z/S}^n[[w]]w^{-k} \right) \quad (k \in \mathbb{Z})$$

and

$$(4.14) \quad \widehat{\mathcal{H}}_F := p_* \Omega_{Z/S}^{n+1}((w)) / d_F \left(p_* \Omega_{Z/S}^n((w)) \right),$$

where slightly abusing the notation we define $p_* \Omega_{Z/S}^{n+1}((w)) := (p_* \Omega_{Z/S}^{n+1})((w))$.

COROLLARY 4.23. *The hyperderived pushforward of the twisted de Rham complexes are given by the following formulas:*

$$\mathbb{R}^i p_*(\Omega_{Z/S}^\bullet[[w]]w^{-k}, d_F) = \begin{cases} 0 & \text{if } i \neq n+1, \\ \widehat{\mathcal{H}}_F^{(k)} & \text{if } i = n+1, \end{cases}$$

and

$$\mathbb{R}^i p_*(\Omega_{Z/S}^\bullet((w)), d_F) = \begin{cases} 0 & \text{if } i \neq n+1, \\ \widehat{\mathcal{H}}_F & \text{if } i = n+1. \end{cases}$$

PROOF. The first formula follows immediately from our argument in the proof of Proposition 4.22. For the second formula, let us look again at the first spectral sequence. We have

$$'E_2^{n+1,0} = p_* \left(H^{n+1}(\Omega_{Z/S}^\bullet((w)), d_F) \right) = (p|_{C_F})_* \left(\varinjlim H^{n+1}(\Omega_{Z/S}^\bullet[[w]]w^{-k}, d_F) \right).$$

Since $p|_{C_F}$ is proper and direct limits commute with proper pushforward, the above term coincides with

$$\varinjlim p_* \left(H^{n+1}(\Omega_{Z/S}^\bullet \llbracket w \rrbracket w^{-k}, d_F) \right).$$

According to part b) of Proposition 4.22, the pushforward p_* commutes with H^{n+1} . The direct limit also commutes with H^{n+1} , so we get

$$H^{n+1} \left(\varinjlim p_* \left(\Omega_{Z/S}^\bullet \llbracket w \rrbracket w^{-k} \right), d_F \right).$$

It remains only to use that $p_*(\mathcal{F} \llbracket w \rrbracket w^{-k}) = (p_*\mathcal{F}) \llbracket w \rrbracket w^{-k}$, that is, we get that the only non-vanishing term on the second page of the spectral sequence is

$$'E_2^{n+1,0} = H^{n+1} \left((p_*\Omega_{Z/S}^\bullet)(\llbracket w \rrbracket), d_F \right). \quad \square$$

PROPOSITION 4.24. *Suppose that $V \subset S$ is a contractible open Stein subset. Then*

a) $\Gamma(p^{-1}V, \Omega_F)$ is a free $\Gamma(V, \mathcal{O}_S)$ -module of rank μ_F .

b) Let $\omega_i \in \Gamma(p^{-1}V, \Omega_{Z/S}^{n+1})$ ($1 \leq i \leq \mu_F$) be a set of relative holomorphic forms that project to a $\Gamma(V, \mathcal{O}_S)$ -basis of $\Gamma(p^{-1}V, \Omega_F)$. Then the map

$$\begin{aligned} \left(\Gamma(V, \mathcal{O}_S) \llbracket w \rrbracket w^{-k} \right)^{\mu_F} &\rightarrow H^{n+1} \left(\Gamma(p^{-1}V, \Omega_{Z/S}^\bullet \llbracket w \rrbracket w^{-k}, d_F) \right) \\ (\phi_1, \dots, \phi_{\mu_F}) &\mapsto \sum_{i=1}^{\mu_F} \phi_i \omega_i \end{aligned}$$

is an isomorphism.

PROOF. a) Since $\text{supp}(\Omega_F) = C_F$ and $p|_{C_F}$ is finite, the \mathcal{O}_S -module $p_*\Omega_F = (p|_{C_F})_*\Omega_F$ is coherent and the stalks

$$(p_*\Omega_F)_t = \bigoplus_{z \in \text{Crit}(f_t)} \Omega_{F,z}.$$

Using local coordinates, it is easy to prove that the stalk $\Omega_{F,z} \cong \mathcal{O}_{C_F,z}$ for all $z \in C_F$. Therefore, $(p_*\Omega_F)_t \cong (p_*\mathcal{O}_{C_F})_t \cong \mathcal{O}_{S,t}^{\mu_F}$. Since $p_*\Omega_F$ is a coherent sheaf and the stalks are free modules of rank μ_F , we get that $p_*\Omega_F$ is locally free, that is a holomorphic vector bundle of rank μ_F . On the other hand, since V is Stein and contractible, the restriction $p_*\Omega_F|_V$ must be a trivial bundle. The statement in part a) follows.

b) Let us prove that the map is injective. Suppose that $\sum_{i=1}^{\mu_F} \phi_i \omega_i = d_F(\eta)$ for some $\eta \in \Gamma(p^{-1}V, \Omega_{Z/S}^n \llbracket w \rrbracket w^{-k})$. We will prove that $\phi_i = 0$ for all i . Put $\phi_i = \sum_{l \geq -k} \phi_{i,l} w^l$ and $\eta = \sum_{l \geq -k} \eta_{i,l} w^l$. Comparing the coefficients in front of the powers of w we get

$$(4.15) \quad \sum_{i=1}^{\mu_F} \phi_{i,l} \omega_i = dF \wedge \eta_l + d\eta_{l-1}.$$

Since $\eta_{-k-1} = 0$, the above identity for $l = -k$ becomes $\sum_{i=1}^{\mu_F} \phi_{i,-k} \omega_i = dF \wedge \eta_{-k} = 0$ in $\Gamma(p^{-1}V, \Omega_F)$. However, ω_i represent a basis in $\Gamma(p^{-1}V, \Omega_F)$, so $\phi_{i,-k} = 0$ for all i . Furthermore, $p^{-1}V$ is an open Stein subset of Z , because p is a Stein map by assumption. Therefore, taking the sections over $p^{-1}V$ in an exact sequence of coherent sheaves yields again an exact sequence. In particular, applying the functor $\Gamma(p^{-1}V, \cdot)$ to the exact sequence (4.11) gives an exact sequence. Therefore, from $dF \wedge \eta_{-k} = 0$, we get $\eta_{-k} =$

$dF \wedge \xi_{-k}$ for some $\xi_{-k} \in \Gamma(p^{-1}V, \Omega_{Z/S}^{n-1})$. Substituting in the equation with $l = -k + 1$ in (4.15), we get

$$\sum_{i=1}^{\mu_F} \phi_{i, -k+1} \omega_i = dF \wedge (\eta_{-k+1} - d\xi_{-k}).$$

Repeating the above argument, we will get $\phi_{i, -k+1} = 0$ and $\eta_{-k+1} = d\xi_{-k} + dF \wedge \eta_{-k+1}$. Clearly, the process can be continued and we will get $\phi_{i, l} = 0$ and $\eta_l = d\xi_{l-1} + dF \wedge \eta_l$ for all $l > -k$. This complete the proof of the injectivity. The proof that the map is surjective is similar and it will be left as an exercise. \square

Let us prove that the map in part b) of Proposition 4.24 induces the following isomorphisms:

$$(\mathcal{O}_S[[w]]w^{-k})^{\mu_F}|_V \cong \widehat{\mathcal{H}}_F^{(k)}|_V, \quad (\mathcal{O}_S((w)))^{\mu_F}|_V \cong \widehat{\mathcal{H}}_F|_V,$$

where V is an open contractible Stein subset of S . By definition, the sheaf $\widehat{\mathcal{H}}_F^{(k)}$ is the sheafification of the presheaf $\mathcal{F}^{(k)}$ defined by

$$\mathcal{F}^{(k)}(U) = H^{n+1}(\Gamma(p^{-1}U, \Omega_{Z/S}^\bullet)[[w]]w^{-k}, d_F),$$

for all open subsets $U \subset S$. If U is in addition Stein, contractible, and $U \subset V$, then we have the following sequence:

$$\Gamma(U, (\mathcal{O}_S[[w]]w^{-k})^{\mu_F}) \rightarrow \mathcal{F}^{(k)}(U) \rightarrow \Gamma(U, \widehat{\mathcal{H}}_F^{(k)}),$$

where the first map is the map from part b) of Proposition 4.24. Since the stalks of $\mathcal{F}^{(k)}$ and $\widehat{\mathcal{H}}_F^{(k)}$ are the same, we get that the above map induces an isomorphism between the stalks of $(\mathcal{O}_S[[w]]w^{-k})^{\mu_F}$ and $\widehat{\mathcal{H}}_F^{(k)}$ at all points $t \in V$, that is,

$$\begin{aligned} (\mathcal{O}_S[[w]]w^{-k})^{\mu_F}|_V &\rightarrow \widehat{\mathcal{H}}_F^{(k)}|_V \\ (\phi_1, \dots, \phi_{\mu_F}) &\mapsto \sum_{i=1}^{\mu_F} [\phi_i \omega_i] \end{aligned}$$

is an isomorphism of sheaves, where the square bracket stands for the cohomology class of the corresponding holomorphic form. Note that our argument also implies that

$$\Gamma(V, \widehat{\mathcal{H}}_F^{(k)}) = H^{n+1}(\Gamma(p^{-1}V, \Omega_{Z/S}^\bullet)[[w]]w^{-k}, d_F)$$

for all open contractible Stein subsets V .

By definition, $\widehat{\mathcal{H}}_F$ is a direct limit of $\widehat{\mathcal{H}}_F^{(k)}$. Since direct limits preserve isomorphisms, we get that the map

$$\begin{aligned} (\mathcal{O}_S((w)))^{\mu_F}|_V &\rightarrow \widehat{\mathcal{H}}_F|_V \\ (\phi_1, \dots, \phi_{\mu_F}) &\mapsto \sum_{i=1}^{\mu_F} [\phi_i \omega_i] \end{aligned}$$

is also an isomorphism. Let us summarize the above discussion into the following commutative diagram

$$\begin{array}{ccc}
(\mathcal{O}_S[[w]]w^{-k})^{\mu_F}|_V & \xrightarrow{\cong} & \widehat{\mathcal{H}}_F^{(k)}|_V \\
\text{mult. by } w \downarrow & & \downarrow \text{mult. by } w \\
(\mathcal{O}_S[[w]]w^{-k+1})^{\mu_F}|_V & \xrightarrow{\cong} & \widehat{\mathcal{H}}_F^{(k-1)}|_V \\
\downarrow & & \downarrow \\
(\mathcal{O}_S((w)))^{\mu_F}|_V & \xrightarrow{\cong} & \widehat{\mathcal{H}}_F|_V
\end{array}$$

Using the above diagram, we get that the following properties are satisfied:

- (i) The inclusion $\Omega_{Z/S}^\bullet[[w]]w^{-k} \subset \Omega_{Z/S}^\bullet((w))$ induces an inclusion $\widehat{\mathcal{H}}_F^{(k)} \subset \widehat{\mathcal{H}}_F$.
- (ii) Multiplication by w induces an isomorphism $\widehat{\mathcal{H}}_F^{(k)} \cong \widehat{\mathcal{H}}_F^{(k-1)}$.
- (iii) If $V \subset S$ is an open contractible Stein subset, then

$$\Gamma(V, \widehat{\mathcal{H}}_F) = \bigcup_{k \in \mathbb{Z}} \Gamma(V, \widehat{\mathcal{H}}_F^{(k)}), \quad \bigcap_{k \in \mathbb{Z}} \Gamma(V, \widehat{\mathcal{H}}_F^{(k)}) = \{0\}.$$

- (iv) The following sequence:

$$0 \longrightarrow \widehat{\mathcal{H}}_F^{(k-1)} \xrightarrow{w} \widehat{\mathcal{H}}_F^{(k)} \longrightarrow p_*\Omega_F \longrightarrow 0$$

is an exact sequence of \mathcal{O}_S -modules.

4.4.3. Gauss–Manin connection. In this section, we would like to construct a set of connection operators on $\widehat{\mathcal{H}}^{(k)}$ of the following form:

$$\nabla : \widehat{\mathcal{H}}^{(k)} \rightarrow \widehat{\mathcal{H}}^{(k+1)} \otimes \Omega_S^1[[w]] \oplus \widehat{\mathcal{H}}^{(k+2)} \otimes \mathcal{O}_S[[w]]dw.$$

We will refer to ∇ as the *Gauss–Manin connection*. The idea is to think of the sections of $\widehat{\mathcal{H}}^{(k)}$ as formal oscillatory integrals and define the connection by formally differentiating the integral.

Let us first explain how to think of the sections of $\widehat{\mathcal{H}}^{(k)}$ as formal oscillatory integrals. If $V \subset S$ is an open Stein subset, then the sections of $\widehat{\mathcal{H}}_F^{(k)}$ over V can be represented by a Laurent series $\omega = \sum_{m \geq -k} \omega_m w^m$, with coefficients $\omega_m \in \Gamma(p^{-1}V, \Omega_{Z/S}^{n+1})$. Note that the form ω is d_F -exact if and only if $e^{F/w}\omega$ is $d_{Z/S}$ -exact. The cohomology class in $\Gamma(V, \widehat{\mathcal{H}}_F^{(k)})$ represented by ω will be denoted by $[\omega]$ or $\int e^{F/w}\omega$.

We already discussed in Section 4.2.3 that there is a natural way to lift vector fields. By definition, we have an exact sequence

$$(4.16) \quad 0 \longrightarrow p^*\Omega_S^1 \otimes \Omega_Z^n \xrightarrow{\wedge} \Omega_Z^{n+1} \xrightarrow{\text{rel}} \Omega_{Z/S}^{n+1} \longrightarrow 0,$$

where rel is the natural quotient map. On the other hand, the $(n+1)$ -st exterior power of the dual of the exact sequence (4.1) gives the exact sequence

$$0 \longrightarrow p^*T_S^* \otimes \wedge^n(T_Z^*) \xrightarrow{\wedge} \wedge^{n+1}(T_Z^*) \longrightarrow \wedge^{n+1}(T_{Z/S}^*) \longrightarrow 0.$$

Comparing the corresponding exact sequence of sheaves of sections with (4.16), we get that there is a natural identification

$$\Omega_{Z/S}^{n+1} \xrightarrow{\cong} \left(\Lambda^{n+1}(\mathcal{T}_{Z/S}) \right)^*,$$

where the map is induced from the embedding $T_{Z/S} \subset T_Z$. Since the exact sequence (4.1) splits, the exact sequence (4.16) also splits. In particular, every relative form $\omega \in \Gamma(U, \Omega_{Z/S}^{n+1})$ can be lifted to an absolute form $\widehat{\omega} \in \Gamma(U, \Omega_Z^{n+1})$, such that, $\text{rel}(\widehat{\omega}) = \omega$.

If $v \in \Gamma(V, \mathcal{T}_S)$ and $\omega \in \Gamma(p^{-1}V, \Omega_{Z/S}^{n+1})[[w]]w^{-k}$, then let us define

$$(4.17) \quad w\nabla_v(\omega) = \left[\text{rel} \circ \iota_{\widehat{v}}((wd_Z + d_Z F \wedge) \widehat{\omega}) \right].$$

$$(4.18) \quad w^2 \nabla_{\frac{\partial}{\partial w}}(\omega) = \left[-F\omega + w^2 \partial_w(\omega) \right],$$

where $\iota_{\widehat{v}}$ is the operation of contraction by the vector field \widehat{v} .

LEMMA 4.25. *The definitions (4.17) and (4.18) are independent of the choice of lifts \widehat{v} and $\widehat{\omega}$ satisfying respectively $dp(\widehat{v}) = v$ and $\text{rel}(\widehat{\omega}) = \omega$.*

PROOF. In order to prove the independence of the choice of the lift \widehat{v} , it is sufficient to prove that if $\xi \in \Gamma(p^{-1}V, \mathcal{T}_{Z/S})$, then $\text{rel} \circ \iota_{\xi}(wd_Z + d_Z F \wedge) \widehat{\omega} = 0$. To begin with, note that $(wd_Z + d_Z F \wedge) \widehat{\omega} \in \Gamma(p^{-1}V, \Omega_Z^{n+2})[[w]]w^{-k}$ and $\Omega_Z^{n+2} = p^* \Omega_S^1 \wedge \Omega_Z^{n+1}$. If $\phi = p^* \psi$ is a pullback of some form $\psi \in \Gamma(V, \Omega_S^1)$, then $\iota_{\xi}(\phi) = p^*(\iota_{dp(\xi)}(\psi)) = 0$. Therefore, the contraction $\iota_{\xi}(wd_Z + d_Z F \wedge) \widehat{\omega}$ is a section of $p^* \Omega_S^1 \wedge \Omega_Z^n$, so $\text{rel} \circ \iota_{\xi}(wd_Z + d_Z F \wedge) \widehat{\omega} = 0$.

Let us prove the independence of the lift of ω . Suppose that $\widehat{\omega}'$ is another lift such that $\text{rel}(\widehat{\omega}') = \omega$. Then $\widehat{\omega}' - \widehat{\omega} \in \text{Ker}(\text{rel})$, that is, the difference is a sum of terms of the form $\phi \wedge \eta$, where $\phi = p^* \psi$ is a pullback of a 1-form $\psi \in \Gamma(V, \Omega_S^1)$ and $\eta \in \Gamma(p^{-1}V, \Omega_Z^n)$. We have

$$\text{rel} \circ \iota_{\widehat{v}}(d_Z \phi) = \text{rel}(p^* d_S(\iota_v(\psi))) = 0,$$

$$\text{rel} \circ \iota_{\widehat{v}}(-\phi \wedge d_Z \eta) = -\text{rel}(p^*(\iota_v(\psi)) d_Z \eta) = -d_{Z/S} \text{rel}(p^*(\iota_v(\psi)) \eta),$$

and

$$\text{rel} \circ \iota_{\widehat{v}}(d_Z F \wedge \phi \wedge \eta) = -d_{Z/S} F \wedge \text{rel}(p^*(\iota_v(\psi)) \eta).$$

The above three formulas imply that

$$\text{rel} \circ \iota_{\widehat{v}}(wd_Z + d_Z F \wedge) \phi \wedge \eta = -d_F(\text{rel}(p^*(\iota_v(\psi)) \eta))$$

is a d_F -exact form. In particular, the definitions (4.17) corresponding to the lifts $\widehat{\omega}$ and $\widehat{\omega}'$ coincide. \square

LEMMA 4.26. *Suppose that $\omega \in \Gamma(p^{-1}V, \Omega_{Z/S}^{n+1})[[w]]w^{-k}$ is d_F -exact, then $w\nabla_v(\omega) = w^2 \nabla_{\frac{\partial}{\partial w}}(\omega) = 0$.*

PROOF. Suppose that $\omega = d_F(\eta)$ for some $\eta \in \Gamma(p^{-1}V, \Omega_{Z/S}^n) [[w]] w^{-k}$. It is easy to check that

$$w^2 \nabla_{\frac{\partial}{\partial w}}(\omega) = \left[d_F(w^2 \partial_w(\eta)) + (w - F)\eta \right] = 0.$$

Let us check that the expression in the square brackets on the RHS in formula (4.17) is d_F -exact. Let us choose a lift $\widehat{\eta} \in \Gamma(p^{-1}V, \Omega_Z^n) [[w]] w^{-k}$, such that, $\text{rel}(\widehat{\eta}) = \eta$. Then

$$\widehat{\omega} - (wd_Z + d_Z F \wedge)(\widehat{\eta}) \in \text{Ker}(\text{rel}) = \Gamma(p^{-1}V, p^* \Omega_S^1 \wedge \Omega_Z^n),$$

that is,

$$\widehat{\omega} - (wd_Z + d_Z F \wedge)(\widehat{\eta}) = \sum_{i=1}^k \phi_i \wedge \eta_i,$$

for some $\phi_i \in \Gamma(p^{-1}V, p^*\Omega_S^1)$ and $\eta_i \in \Gamma(p^{-1}V, \Omega_Z^n)$ ($1 \leq i \leq k$). Substituting the above formula for $\widehat{\omega}$ in the definition of $w\nabla_v(\omega)$, we get

$$w\nabla_v(\omega) = \sum_{i=1}^k \left[\text{rel} \circ \iota_{\widehat{v}}(wd_Z(\phi_i \eta_i) + d_Z F \wedge \phi_i \wedge \eta_i) \right].$$

We may assume that $\phi_i = p^*\psi_i$ is a pullback of a 1-form $\psi_i \in \Gamma(V, \Omega_S^1)$. The same computation as in the proof of Lemma 4.25 yields

$$w\nabla_v(\omega) = - \sum_{i=1}^k \left[d_F(p^*(\iota_v(\psi_i))\eta_i) \right]. \quad \square$$

The Gauss–Manin connection can be defined as follows. Suppose that $V \subset S$ is an arbitrary open subset and $\omega \in \Gamma(V, \widehat{\mathcal{H}}_F^k)$ is any section. Let us choose an open covering $\{V_i\}$ of V , such that, V_i is an open Stein subset. Then $\omega|_{V_i}$ can be represented by a holomorphic form $\omega_i \in \Gamma(p^{-1}V_i, \Omega_{Z/S}^{n+1})[[w]]w^{-k}$. Moreover, since the intersections $V_i \cap V_j$ are Stein and p is a Stein map, we have that $\omega_i|_{V_i \cap V_j} - \omega_j|_{V_i \cap V_j}$ is d_F -exact, that is, the difference is $d_F(\eta_{ij})$ for some $\eta_{ij} \in \Gamma(p^{-1}(V_i \cap V_j), \Omega_{Z/S}^n)[[w]]w^{-k}$. Using Lemma 4.26, we get that the cohomology classes $w\nabla_v(\omega_i) \in \Gamma(V_i, \widehat{\mathcal{H}}_F^{(k)})$ can be glued to a section in $\Gamma(V, \widehat{\mathcal{H}}_F^{(k)})$. This section is by definition $w\nabla_v(\omega)$. Lemma 4.25 implies that $w\nabla_v(\omega)$ does not depend on the choice of local representatives ω_i . Similarly, we define $w^2\nabla_{\frac{\partial}{\partial w}}(\omega)$ by gluing $w^2\nabla_{\frac{\partial}{\partial w}}(\omega_i)$.

PROPOSITION 4.27. *The Gauss–Manin connection is flat.*

PROOF. The problem is local, so we may assume that $\omega \in \Gamma(V, \widehat{\mathcal{H}}_F^{(k)})$, where $V \subset S$ is an open Stein subset equipped with local coordinates (t_1, \dots, t_m) . We need to prove that $[w\nabla_{\partial/\partial t_a}(\omega), w\nabla_{\partial/\partial t_b}(\omega)] = 0$.

The cohomology class $w\nabla_{\partial/\partial t_a}(\omega) \in \Gamma(p^{-1}V, \Omega_{Z/S}^{n+1})[[w]]w^{-k}/\text{Im}(d_F)$. Suppose that $U \subset p^{-1}V$ is any open Stein subset. Since rel and the contraction $\iota_{\widehat{v}}$ are compatible with the restriction maps, we get that

$$w\nabla_v(\omega)|_U = \left[\text{rel} \circ \iota_{\widehat{v}_U}(wd_Z + d_Z F \wedge)(\widehat{\omega}_U) \right] \in \Gamma(U, \Omega_{Z/S}^{n+1})[[w]]w^{-k}/\text{Im}(d_F),$$

where \widehat{v}_U and $\widehat{\omega}_U$ are respectively the restrictions of \widehat{v} and $\widehat{\omega}$ to U . Using the same argument as in the proof of Lemma 4.25, we get that the above formula is independent of the choice of $\widehat{v}_U \in \Gamma(U, \mathcal{T}_Z)$ and $\widehat{\omega}_U \in \Gamma(U, \Omega_{Z/S}^{n+1})[[w]]w^{-k}$ satisfying $dp(\widehat{v}_U) = v|_{p(U)}$ and $[\text{rel}(\widehat{\omega}_U)] = \omega|_U$. In other words, in order to compute the restriction $w\nabla_v(\omega)|_U$ we can use any lifts \widehat{v}_U and $\widehat{\omega}_U$, not only the lifts obtained by restriction. Let us choose U to be a coordinate neighborhood, such that the local coordinates have the form $(x_0, x_1, \dots, x_n, t_1, \dots, t_m)$. Then we can choose $\widehat{\omega}_U$ to be of the form $g(x, t, w)dx_0 \wedge \dots \wedge dx_n$. The formula for the Gauss–Manin connection takes the form

$$w\nabla_{\partial/\partial t_a}(\omega)|_U = \left[(w\partial_{t_a}(g) + \partial_{t_a}(F)g)dx_0 \wedge \dots \wedge dx_n \right]$$

In order to check the flatness, we just need to prove that the differential operators $w \frac{\partial}{\partial t_a} + \partial_{t_a}(F)$ and $w \frac{\partial}{\partial t_b} + \partial_{t_b}(F)$ commute, which is obvious. \square

4.5. Higher-residue pairing

Let $(F, p : Z \rightarrow S)$ be a family of functions in the sense of Definition 4.1 and let $\widehat{\mathcal{H}}_F$ and $\widehat{\mathcal{H}}_F^{(k)}$ be the associated modules of formal oscillatory integrals.

4.5.1. Sheaves as topological spaces. Let us recall an alternative way to define a sheaf. Suppose that \mathcal{S} is a sheaf of Abelian groups on a topological space X . Then we associate to \mathcal{S} the topological space $|\mathcal{S}| = \bigsqcup_{x \in X} \mathcal{S}_x$, where the topology is defined as follows: The points of $|\mathcal{S}|$ will be written as pairs (x, s) , where $x \in X$ and $s \in \mathcal{S}_x$. The basis of the topology of $|\mathcal{S}|$ at a point (x, s) is given by subsets of the form $\{(y, \sigma_y) \mid y \in U\}$, where $U \subset X$ is an open subset containing x and $\sigma \in \Gamma(U, \mathcal{S})$, such that, $\sigma_x = s$. We have a natural projection $\pi : |\mathcal{S}| \rightarrow X$. The above topology turns π into a local homeomorphism. The fibers $\pi^{-1}(x) = \mathcal{S}_x$ are Abelian groups, s.t., the group operations addition and inverse define continuous maps $\mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$ and $\mathcal{S} \rightarrow \mathcal{S}$. Conversely, every local homeomorphism $\pi : S \rightarrow X$ whose fibers are Abelian groups, s.t., the group operations addition and taking inverse are continuous will define a sheaf of Abelian groups with sections defined by

$$\Gamma(U, \mathcal{S}) := \{\sigma : U \rightarrow \mathcal{S} \mid \sigma \text{ is continuous, } \pi \circ \sigma = \text{id}_U\}.$$

The construction has an appropriate modification for sheaf of rings and sheaf of \mathcal{O}_X -modules for a fixed sheaf of rings \mathcal{O}_X on X . A sheaf morphism $\phi : \mathcal{S}' \rightarrow \mathcal{S}''$ induces a continuous map $|\phi| : |\mathcal{S}'| \rightarrow |\mathcal{S}''|$ compatible with the projections, that is, $\pi'' \circ |\phi| = \pi'$. Conversely, every continuous map $\varphi : |\mathcal{S}'| \rightarrow |\mathcal{S}''|$ compatible with the projections has the form $\varphi = |\phi|$ for some morphism of sheaves $\phi : \mathcal{S}' \rightarrow \mathcal{S}''$. The following lemma is very useful for constructing morphisms of sheaves:

LEMMA 4.28. *Suppose that the map $\varphi : |\mathcal{S}'| \rightarrow |\mathcal{S}''|$ satisfies the following condition: for every $p' \in |\mathcal{S}'|$, there exists an open set $U \subset X$ and a section $s \in \Gamma(U, \mathcal{S}')$, such that $p' = (x, s_x)$ for some $x \in U$ and $\varphi \circ s \in \Gamma(U, \mathcal{S}'')$. Then φ is a sheaf morphism, i.e., it is continuous and compatible with the projections.*

The proof is straightforward, so it will be left as an exercise.

4.5.2. Classical residue pairing J_F . Using multi-dimensional residues, we will define a symmetric non-degenerate \mathcal{O}_S -bilinear pairing

$$J_F : p_* \Omega_F \times p_* \mathcal{O}_F \rightarrow \mathcal{O}_S.$$

We refer to [28], Chapter 5, Section 1 (see also [30], Section 9) for some background on multi-dimensional residues. Our strategy is to define first J_F set theoretically as a map between the corresponding topological spaces of germs. Then we will check that our definition satisfies the condition of Lemma 4.28.

Suppose that $t^\circ \in S$ is any point. For each $\xi^\circ = (x^\circ, t^\circ) \in C_F \cap Z_{t^\circ}$, let us denote by Z_{ξ° the germ of the complex manifold Z at the point ξ° . Since $Z \subset X \times S$, we can represent Z_{ξ° by a product open neighborhood $U_{\xi^\circ} \times V_{\xi^\circ}$ of ξ° in Z , such that, U_{ξ° and V_{ξ° are coordinate charts with compact closures respectively on X and S with centers

respectively x° and t° . We may assume that the charts V_{ξ° are independent of ξ° , that is, they all coincide with some open neighborhood V° of t° in S . Then

$$\mathcal{O}_{Z, \xi^\circ} \left/ \left(\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n} \right) \right. \cong \Omega_{F, \xi^\circ}, \quad \phi(x, t) \mapsto \phi(x, t) dx_0 \wedge \dots \wedge dx_n,$$

where $x = (x_0, \dots, x_n)$ are coordinates on U_{ξ° . Given two germs $\omega_i \in \Omega_{F, \xi^\circ}$ ($i = 1, 2$), let us decrease the open neighborhood $U_{\xi^\circ} \times V^\circ$ if necessary, so that we can find holomorphic representatives $\phi_i(x, t) dx_0 \wedge \dots \wedge dx_n$. Let us define (c.f. [30]) the *residue symbol*

$$\text{Res}_{Z_{\xi^\circ}/C_F} \left[\begin{array}{c} \phi_1 \phi_2 dx_0 \wedge \dots \wedge dx_n \\ \frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n} \end{array} \right] = \frac{1}{(2\pi\sqrt{-1})^{n+1}} \int_{\gamma_\epsilon^\circ} \frac{\phi_1(x, t) \phi_2(x, t) dx_0 \wedge \dots \wedge dx_n}{\frac{\partial f_t}{\partial x_0} \dots \frac{\partial f_t}{\partial x_n}},$$

where $\gamma_\epsilon^\circ \in H_{n+1}(U_{\xi^\circ} \setminus \{\xi^\circ\}, \mathbb{Z})$ is the cycle defined by $|\partial_{x_0} f_{t^\circ}| = \dots = |\partial_{x_n} f_{t^\circ}| = \epsilon$. Here $\epsilon > 0$ and the open subset V° are fixed to be so small that the following two conditions hold: for all $x \in \partial \bar{U}_{\xi^\circ}$ we have $|\partial_{x_i} f_{t^\circ}(x)| \geq 2\epsilon$ for some i and $|\partial_{x_i} f_t(x)| \geq \epsilon/2$ ($0 \leq i \leq n$) for all $(x, t) \in \gamma_\epsilon^\circ \times V^\circ$. The first condition guarantees that the cycle $\gamma_\epsilon^\circ \subset U_{\xi^\circ}$. The second condition guarantees that the critical points of f_t are away from the cycle γ_ϵ° . In particular, the residue symbol is holomorphic as a function in $t \in V^\circ$. Recalling, the transformation law for residues, we get that the residue symbol is independent of the choice of local coordinates (x_0, \dots, x_n) .

Suppose now that $\omega_i \in (p_* \Omega_F)_{t^\circ}$ ($i = 1, 2$). Since the support of the sheaf Ω_F is C_F and $p|_{C_F}$ is a finite map, we have $(p_* \Omega_F)_{t^\circ} = \bigoplus_{\xi \in C_F \cap Z_{t^\circ}} \Omega_{F, \xi}$. The sum of the residue symbols

$$(4.19) \quad \sum_{\xi^\circ \in C_F \cap Z_{t^\circ}} \text{Res}_{Z_{\xi^\circ}/C_F} \left[\begin{array}{c} \phi_1 \phi_2 dx_0 \wedge \dots \wedge dx_n \\ \frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n} \end{array} \right].$$

defines a holomorphic function on V° , which is independent of the choice of holomorphic representatives of ω_i . We define $J_{F, t^\circ}(\omega_1, \omega_2) \in \mathcal{O}_{S, t^\circ}$ to be the germ of (4.19) at $t = t^\circ$.

It remains to check that the pairing J_F satisfies the condition of Lemma 4.28. Let us denote by $\rho(t)$ ($t \in V^\circ$) the sum (4.19). By definition, ρ is a holomorphic function representing the germ $J_{F, t^\circ}(\omega_1, \omega_2)$. We claim that for all $'t^\circ \in V^\circ$ the germ of the function ρ at $'t^\circ$ coincides with $J_{F, 't^\circ}(\omega_1, \omega_2)$. The critical points of f_{t° split into several groups, such that, each group belongs to one of the open neighborhoods $U_\xi \times V^\circ$, where ξ is a critical point of f_{t° . Intuitively, if we vary the parameter t from t° to $'t^\circ$, then each critical points ξ of f_{t° will split into several critical points. If $\epsilon > 0$ is sufficiently small, then the equations $|\partial_{x_0} f_{t^\circ}(x)| = \dots = |\partial_{x_n} f_{t^\circ}(x)| = \epsilon$ define a cycle $'\gamma_\epsilon^\circ$ in U_ξ , consisting of a disjoint union of toroidal cycles corresponding to the critical points of f_{t° in U_ξ . Therefore, the sum of the residue symbols corresponding to the critical points of f_{t° in U_ξ is

$$\frac{1}{(2\pi\sqrt{-1})^{n+1}} \int_{'\gamma_\epsilon^\circ} \frac{\phi_1(x, t) \phi_2(x, t) dx_0 \wedge \dots \wedge dx_n}{\frac{\partial f_t}{\partial x_0} \dots \frac{\partial f_t}{\partial x_n}}.$$

On the other hand, the integration cycle $'\gamma_\epsilon^\circ$ is homotopic to γ_ϵ° , so the above integral coincides with the residue symbol Res_{Z_ξ/C_F} . Summing over all critical points ξ of f_{t° , we get that $J_{F, 't^\circ}(\omega_1, \omega_2)$ coincides with the germ of $\rho(t)$ at $t = 't^\circ$.

4.5.3. Formulation of higher residue pairing. Let us introduce the following involution $*$ on $P \in \mathcal{O}_S((w))$:

$$P = \sum_{k \in \mathbb{Z}} p_k w^k \mapsto P^* := \sum_{k \in \mathbb{Z}} p_k (-w)^k$$

A \mathcal{O}_S -bilinear pairing

$$K_F : \widehat{\mathcal{H}}_F \times \widehat{\mathcal{H}}_F \rightarrow \mathcal{O}_S((w))$$

is said to be a *higher residue pairing* if the following properties are satisfied:

1. For all $\omega_1, \omega_2 \in \widehat{\mathcal{H}}_F$,

$$K_F(\omega_1, \omega_2) = (-1)^{n+1} K_F(\omega_2, \omega_1)^*.$$

2. For all $P \in \mathcal{O}_S((w))$ and for all $\omega_1, \omega_2 \in \widehat{\mathcal{H}}_F$,

$$PK_F(\omega_1, \omega_2) = K_F(P\omega_1, \omega_2) = K_F(\omega_1, P^*\omega_2).$$

3. If $\omega_1, \omega_2 \in \widehat{\mathcal{H}}_F^{(0)}$, then

$$K_F(\omega_1, \omega_2) \in \mathcal{O}_S[[w]]w^{n+1}$$

and the following diagram is commutative

$$\begin{array}{ccc} \widehat{\mathcal{H}}_F^{(0)} \times \widehat{\mathcal{H}}_F^{(0)} & \xrightarrow{K_F} & \mathcal{O}_S[[w]]w^{n+1} \\ r^{(0)} \times r^{(0)} \downarrow & & \downarrow \text{mod } \mathcal{O}_S[[w]]w^{n+2} \\ p_*\Omega_F \times p_*\Omega_F & \xrightarrow{J_F} & \mathcal{O}_S \end{array}$$

where $r^{(0)} : \widehat{\mathcal{H}}_F^{(0)} \rightarrow \widehat{\mathcal{H}}_F^{(0)}/w\widehat{\mathcal{H}}_F^{(0)} = p_*\Omega_F$ is the natural quotient map.

4. The following version of the Leibnitz rule holds:

$$\begin{aligned} \xi(K_F(\omega_1, \omega_2)) &= K_F(\nabla_\xi(\omega_1), \omega_2) + K_F(\omega_1, \nabla_\xi(\omega_2)), \\ \partial_w K_F(\omega_1, \omega_2) &= K_F(\nabla_{\partial/\partial w}(\omega_1), \omega_2) - K_F(\omega_1, \nabla_{\partial/\partial w}(\omega_2)), \end{aligned}$$

for all $\omega_1, \omega_2 \in \widehat{\mathcal{H}}_F$ and all vector fields $\xi \in \mathcal{T}_S$.

THEOREM 4.29. *Suppose that $(F, p : Z \rightarrow S)$ is a family of functions in the sense of Definition 4.1. Then*

- a) *There exists a higher residue pairing.*
- b) *If the family is also complete, then there exists a unique higher residue pairing.*

4.5.4. Twisted double relative de Rham complex. Let $(F, p : Z \rightarrow S)$ be a family of functions and $V \subset S$ be an arbitrary open subset. Let us define the double complex

$$A_V^{p,q} := \Gamma(p^{-1}V \setminus C_F, \mathcal{A}_{Z/S}^{p,q}[[w]]).$$

with vertical differential $d_F^{0,1} := wd_{Z/S}^{0,1}$ and horizontal differential $d_F^{1,0} := wd_{Z/S}^{1,0} + d_{Z/S}F \wedge$.

LEMMA 4.30. *The horizontal rows of the complex $A_V^{p,q}$ are exact.*

PROOF. We claim that the sequence (specializing the q th row at $w = 0$)

$$0 \longrightarrow \Gamma(p^{-1}V \setminus C_F, \mathcal{A}_{Z/S}^{0,q}) \xrightarrow{dF \wedge} \Gamma(p^{-1}V \setminus C_F, \mathcal{A}_{Z/S,z}^{1,q}) \xrightarrow{dF \wedge} \dots$$

is exact. Since the sheaves $\mathcal{A}_{Z/S}^{p,q}$ are soft, it is sufficient to prove that the corresponding sequences of sheaves on $p^{-1}V \setminus C_F$ are exact, or equivalently the sequence of germs

$$0 \longrightarrow \mathcal{A}_{Z/S,z}^{0,q} \xrightarrow{dF \wedge} \mathcal{A}_{Z/S,z}^{1,q} \xrightarrow{dF \wedge} \dots$$

is exact for all $z \in Z \setminus C_F$. Suppose that $\eta \in \mathcal{A}_{Z/S,z}^{p,q}$ and $dF \wedge \eta = 0$. Since z is not a relative critical point, there exists an i , such that $\frac{\partial F}{\partial x_i}(z) \neq 0$. Let us write the form $\eta = dx_i \wedge \eta' + \eta''$, where η'' is a (p, q) -form that does not involve dx_i . Comparing the wedge monomials that involve dx_i in $dF \wedge \eta = 0$, we get that

$$\frac{\partial F}{\partial x_i} dx_i \wedge \eta'' + \sum_{j \neq i} \frac{\partial F}{\partial x_j} dx_j \wedge dx_i \wedge \eta' = 0.$$

Therefore, since the partial derivative $\frac{\partial F}{\partial x_i}$ does not vanish in a neighborhood of z , we can solve for η'' . Expressing η'' in terms of η' we get

$$\eta = \frac{1}{\frac{\partial F}{\partial x_i}} dF \wedge \eta'.$$

This completes the proof of the exactness claim.

Suppose now that $\omega \in A_V^{p,q}$ and that $d_F^{1,0}(\omega) = 0$. Let us write $\omega = \sum_{k \geq 0} \omega_k w^k$. Then $d_{Z/S}^{1,0}(\omega_{k-1}) + dF \wedge \omega_k = 0$ for all $k \geq 0$ (we assume $\omega_{-1} := 0$). We have to prove that $\omega = d_F^{1,0}(\eta)$. Writing $\eta = \sum_{k \geq 0} \eta_k w^k$, we get that our claim is equivalent to finding η_k , such that

$$\omega_k = d_{Z/S}^{1,0}(\eta_{k-1}) + dF \wedge \eta_k, \quad \forall k \geq 0.$$

Suppose that we have determined η_i for all $i < k$. Note that

$$dF \wedge (\omega_k - d_{Z/S}^{1,0}(\eta_{k-1})) = -d_{Z/S}^{1,0}(\omega_{k-1}) - dF \wedge d_{Z/S}^{1,0}(\eta_{k-1}) = 0,$$

because $\omega_{k-1} = d_{Z/S}^{1,0}(\eta_{k-2}) + dF \wedge \eta_{k-1}$ according to our inductive assumption. Using the exactness of the wedge operation $dF \wedge$ we get that η_k exists. \square

Let $A_V^m := \bigoplus_{p+q=m} A_V^{p,q}$ ($m \geq 0$) be the total complex with differential $d_F = d_F^{1,0} + d_F^{0,1}$. The cohomology of (A_V^\bullet, d_F) vanishes in all degrees. Indeed, for every first quadrant double complex we can construct two spectral sequences converging to the cohomology of the total complex (see [62]). One of the spectral sequences has $E_1^{p,q} = H^p(A_V^{\bullet,q}, d_F^{1,0})$, which according to the above Lemma vanishes in all degrees $p, q \geq 0$. Therefore, the cohomology of the total complex is also 0 in all degrees.

The key to the higher residue pairing is the \mathcal{O}_S -bilinear map

$$(4.20) \quad \widehat{\mathcal{H}}_F^{(0)} \times \widehat{\mathcal{H}}_F^{(0)} \rightarrow (p|_{Z \setminus C_F})_* \mathcal{A}_{Z/S}^{2n+1}[[w]] / d_{Z/S}(p|_{Z \setminus C_F})_* \mathcal{A}_{Z/S}^{2n}[[w]]$$

defined by the formula

$$([\omega_1], [\omega_2]) \mapsto [d_F^{-1}(\omega_1) \wedge \omega_2^*].$$

Let us explain how to make sense of the above formula. Suppose that $V \subset S$ is an open Stein subset, so that $[\omega_i]$ ($i = 1, 2$) can be represented by holomorphic forms $\omega_i \in$

$\Gamma(p^{-1}V, \Omega_{Z/S}^{n+1})[[w]]$. The restriction of ω_i to $p^{-1}V \setminus C_F$ determines an element in A_V^{n+1} satisfying $d_F(\omega_i) = 0$. In particular, $\omega_1 = d_F(\eta_1)$ for some $\eta_1 \in A_V^n$. We define $d_F^{-1}(\omega_1) := \eta_1$. Let us check that the map does not depend on the choices involved. Suppose that $\omega_2 = d_F(\eta_2)$ for some $\eta_2 \in \Gamma(p^{-1}V, \Omega_{Z/S}^n)[[w]]$. We have to prove that $\eta_1 \wedge (d_F(\eta_2))^*$ is a $d_{Z/S}$ -exact form. On the other hand,

$$wd_{Z/S}(\eta_1 \wedge \eta_2^*) = wd_{Z/S}(\eta_1) \wedge \eta_2^* + (-1)^n \eta_1 \wedge (wd_{Z/S}(\eta_2)^*).$$

By definition $wd_{Z/S}\eta_1 = \omega_1 - dF \wedge \eta_1$. Note that since $\omega_1 \in A_V^{n+1,0}$ and $\eta_2 \in A_V^{n,0}$, we have $\omega_1 \wedge \eta_2^* = 0$. Therefore,

$$\eta_1 \wedge (wd_{Z/S}(\eta_2)^*) = (-1)^n dF \wedge \eta_1 \wedge \eta_2^* + (-1)^n wd_{Z/S}(\eta_1 \wedge \eta_2^*).$$

Using the above formula we get that $\eta_1 \wedge \omega_2^*$ is equal to

$$\eta_1 \wedge (-wd_{Z/S} + dF \wedge)(\eta_2^*) = (-1)^{n+1} dF \wedge \eta_1 \wedge \eta_2^* + (-1)^{n+1} wd_{Z/S}(\eta_1 \wedge \eta_2^*) + \eta_1 \wedge dF \wedge \eta_2^*.$$

The terms involvin dF cancel out, because $\eta_1 \wedge dF = (-1)^n dF \wedge \eta_1$, so the above expression is exact. The independence of our definition on the choice of a holomorphic form ω_1 representing $[\omega_1]$ and on the choice $\eta_1 \in A_V^n$ satisfying $d_F \eta_1 = \omega_1$ is done in a similar way.

4.5.5. Proof of Theorem 4.29 a. It suffices to define the higher residue pairing on $\widehat{\mathcal{H}}_F^{(0)}$ and check that it satisfies the corresponding properties. If this is done, then it is easy to check that the extension of the pairing to $\widehat{\mathcal{H}}_F$ can be constructed uniquely and that all properties will continue to hold.

We follow the same strategy as in the definition of the classical residue pairing. Suppose that $t^\circ \in S$ is an arbitrary point and that $[\omega_i] \in \widehat{\mathcal{H}}_{F,t^\circ}$ ($i = 1, 2$) are two germs. Let us choose a small open Stein neighborhood V of t° in S and represent the cohomology classes $[\omega_i]$ by holomorphic forms $\omega_i \in \Gamma(p^{-1}V, \Omega_{Z/S}^{n+1})[[w]]$. For each critical point $\xi^\circ = (x^\circ, t^\circ) \in C_F \cap Z_{t^\circ}$ there exists a sufficiently small ball U_{ξ° in X with center at x° , such that, after decreasing V if necessary we can arrange that $U_{\xi^\circ} \times V \subset p^{-1}V$. Put

$$(4.21) \quad K_{F,t^\circ}(\omega_1, \omega_2) = C_n \sum_{\xi \in \text{Crit}(f_{t^\circ})} \int_{\partial \bar{U}_\xi} w d_F^{-1}(\omega_1) \wedge \omega_2^*.$$

where $C_n = (-1)^{n(n-1)/2} (2\pi\sqrt{-1})^{-n-1}$ is a normalization constant. If $t = (t_1, \dots, t_m)$ is a holomorphic coordinate system on V , then d_F commutes with $\frac{\partial}{\partial t_i}$, so the integral in (4.21) depends holomorphically on $t \in V$. Let us denote by ρ the RHS of (4.21). The same argument that we used in the construction of the classical residue pairing allows us to prove that the germ of ρ at any other point $'t^\circ \in V$ coincides with $K_{F,'t^\circ}(\omega_1, \omega_2)$. This implies that the condition of Lemma 4.28 is satisfied, so we have a map of sheaves

$$K_F : \widehat{\mathcal{H}}_F^{(0)} \times \widehat{\mathcal{H}}_F^{(0)} \rightarrow \mathcal{O}_S[[w]].$$

Let us prove that K_F satisfies properties 1–4 in the definition of a higher residue pairing (see Section 4.5.3). Suppose that $V \subset S$ is a small open Stein subset and that $\omega_i \in \Gamma(p^{-1}V, \Omega_{Z/S}^{n+1})[[w]]$ ($i = 1, 2$) are two holomorphic forms representing twisted cohomology classes in $\Gamma(p^{-1}V, \widehat{\mathcal{H}}_F^{(0)})$. There are smooth relative forms $\eta_i \in \Gamma(p^{-1}V \setminus C_F, \mathcal{A}_{Z/S}^n)[[w]]$

($i = 1, 2$), such that, $\omega_i = d_F(\eta_i)$. A straightforward computation yields

$$\eta_1 \wedge \omega_2^* + (-1)^{n+1}(\eta_2 \wedge \omega_1^*)^* = (-1)^{n+1}wd_{Z/S}(\eta_1 \wedge \eta_2^*).$$

Multiplying both sides of the above formula by wC_n , integrating over $\partial\bar{U}_\xi$, and summing up over all critical points ξ (see formula (4.21)), we get

$$K_F(\omega_1, \omega_2) - (-1)^{n+1}K_F(\omega_1, \omega_2)^* = 0.$$

This completes the proof of the first property.

The second property is obvious. Let us check the third one. Let us decompose $\eta_1 = \sum_{p=0}^n \eta_1^{n-p,p}$, where $\eta_1^{n-p,p} \in A_V^{n-p,p}$. Since $\omega_2^* \in A_V^{n+1,0}$, only the component $\eta_1^{0,n}$ contributes to the higher residue pairing (4.21). On the other hand, comparing the $(n+1-p, p)$ -components in $d_F(\eta_1) = \omega_1$, we get

$$(wd_{Z/S}^{1,0} + dF\wedge)(\eta_1^{n,0}) = \omega_1, \quad (wd_{Z/S}^{1,0} + dF\wedge)(\eta_1^{n-p,p}) = -wd_{Z/S}^{0,1}\eta_1^{n+1-p,p-1} \quad (1 \leq p \leq n).$$

Using induction on p , the above recursion implies immediately that $\eta_1^{n-p,p}$ has order $O(w^p)$ at $w = 0$. Therefore, the order of $wd_{Z/S}^{-1}(\omega_1) \wedge \omega_2^* = w\eta_1^{0,n} \wedge \omega_2^*$ is $O(w^{n+1})$. This proves the first part of property 3.

For the second part, we need to compute the leading order term of $\eta_1^{0,n} \wedge \omega_2^*$ in the expansion at $w = 0$. Since the definition of the higher residue pairing (4.21) involves only the restriction of $\eta_1^{0,n} \wedge \omega_2^*$ to each coordinate chart $Z_{\xi^\circ} := U_{\xi^\circ} \times V$, it is sufficient to compute the leading order terms only locally on each Z_{ξ° . For brevity put $dx := dx_0 \wedge \cdots \wedge dx_n$. The holomorphic forms $\omega_i = (\phi_i(x, t) + O(w))dx$ ($i = 1, 2$), where $\phi_i \in \mathcal{O}(U_{\xi^\circ} \times V)$. Put $\eta_1^{n-p,p} = \psi_p w^p + O(w^{p+1})$, where $\psi_p \in \Gamma(p^{-1}V \setminus C_F, \mathcal{A}_{Z/S}^{n-p,p})$ is the leading order term. Extracting the leading order terms in the recursion equations for $\eta_1^{n-p,p}$ we get

$$(4.22) \quad dF \wedge \psi_0 = \phi_1(x, t)dx, \quad dF \wedge \psi_p = -d_{Z/S}^{0,1}(\psi_{p-1}) \quad (1 \leq p \leq n).$$

Let us introduce the following notation: If $\{\varphi_i\}_{i=1}^m$ is a set of 1-forms on some manifold and $1 \leq j_1 < \cdots < j_k \leq m$ is an increasing sequence, then

$$\frac{\varphi_1 \wedge \cdots \wedge \varphi_n}{\varphi_{j_1} \wedge \cdots \wedge \varphi_{j_k}} := \text{sign} \begin{bmatrix} j_1 & \cdots & j_k & i_1 & \cdots & i_{m-k} \\ 1 & \cdots & k & k+1 & \cdots & m \end{bmatrix} \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_{m-k}},$$

where $i_1 < \cdots < i_{m-k}$ is the complement $\{i_1, \dots, i_{m-k}\} := \{1, 2, \dots, m\} \setminus \{j_1, \dots, j_k\}$. For brevity, let us denote the partial derivatives of F by $F_i := \frac{\partial F}{\partial x_i}$ ($0 \leq i \leq n$). The open Stein subsets $Z_{\xi^\circ, i} := \{F_i \neq 0\}$ provide a covering of $Z_{\xi^\circ} \setminus C_F$. This covering admits a set of functions that resemble a partition of unity (see [28], Chapter 5, Section 1)

$$\rho_i := \frac{|F_i|^2}{|F_0|^2 + \cdots + |F_n|^2}, \quad 0 \leq i \leq n,$$

satisfying the following properties:

- (i) $\rho_i \in C^\infty(Z_{\xi^\circ} \setminus C_F)$ ($0 \leq i \leq n$),
- (ii) $\text{supp}(\rho_i) \subset Z_{\xi^\circ, i}$ ($0 \leq i \leq n$),
- (iii) $\rho_0 + \cdots + \rho_n = 1$.

We claim that the following forms

$$\psi_p = \sum_{0 \leq i_0 < \cdots < i_p \leq n} \frac{(-1)^p p!}{F_{i_0} \cdots F_{i_p}} \left(\sum_{s=0}^p \rho_{i_s} \frac{\bar{\partial} \rho_{i_0} \wedge \cdots \wedge \bar{\partial} \rho_{i_p}}{\bar{\partial} \rho_{i_s}} \right) \wedge \frac{\omega_1}{dx_{i_0} \wedge \cdots \wedge dx_{i_p}},$$

where $\bar{\partial} := d_{Z/S}^{0,1}$, satisfy the recursion relation (4.22). The verification is straightforward, so we leave it as an exercise. The formula for ψ_n can be simplified a little bit. We have

$$\sum_{s=0}^n \rho_s \frac{\bar{\partial}\rho_0 \wedge \cdots \wedge \bar{\partial}\rho_n}{\bar{\partial}\rho_s} = \rho_0 \bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_n + \sum_{s=1}^n (-1)^s \rho_s \bar{\partial}\rho_0 \wedge \cdots \wedge \widehat{\bar{\partial}\rho_s} \wedge \cdots \wedge \bar{\partial}\rho_n.$$

Since $\bar{\partial}\rho_0 = -\sum_{i=1}^n \bar{\partial}\rho_i$, the summand on the RHS of the above formula is equal to $\rho_s \bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_n$. Using $\sum_{i=0}^n \rho_i = 1$, we get that the RHS coincides with $\bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_n$. We get that the leading order term of $\eta_1^{0,n} \wedge \omega_2^*$ is given by

$$\eta_{\phi_1 \phi_2 dx} := (-1)^n n! \bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_n \wedge \frac{\phi_1 \phi_2 dx}{F_0 \cdots F_n}.$$

The form $\frac{\phi_1 \phi_2 dx}{F_0 \cdots F_n}$ determines a Čech cohomology class in $H^n(\{Z_{\xi^\circ, i}\}_{i=0}^n, \Omega_{Z/S}^{n+1})$. The form $\eta_{\phi_1 \phi_2 dx}$ determines a Dolbeault cohomology class in the relative Dolbeault cohomology group $H_{\bar{\partial}}^{n+1, n}(Z_{\xi^\circ} \setminus C_F) := H^n(\Gamma(Z_{\xi^\circ} \setminus C_F, \mathcal{A}_{Z/S}^{n+1, \bullet}), \bar{\partial})$. Under the Dolbeault isomorphism the cohomology class corresponding to $\frac{\phi_1 \phi_2 dx}{F_0 \cdots F_n}$ coincides with the cohomology class corresponding to the form $C_n \eta_{\phi_1 \phi_2 dx}$, where C_n is a numerical constant depending only on n . Moreover, we have the following formula for the residue symbol (see [28], Chapter 5, Section 1):

$$(4.23) \quad \text{Res}_{Z_{\xi^\circ}/C_F} \left[\begin{array}{c} \phi_1 \phi_2 dx_0 \wedge dx_n \\ F_0 \cdots F_n \end{array} \right] = C_n \int_{\partial \bar{U}_{\xi^\circ}} \eta_{\phi_1 \phi_2 dx}.$$

LEMMA 4.31. *The constant $C_n = (-1)^{n(n-1)/2} (2\pi\mathbf{i})^{-n-1}$.*

PROOF. Since the constant is independent of F , it is sufficient to compare both sides of (4.23) in the case when $F(x) = x_0^2 + \cdots + x_n^2$ and $\phi_1 = \phi_2 = 1$. The LHS becomes

$$\frac{1}{(2\pi\mathbf{i})^{n+1}} \int_{|x_0|=\cdots=|x_n|=\epsilon} \frac{dx_0 \wedge \cdots \wedge dx_n}{(2x_0) \cdots (2x_n)} = 2^{-n-1}.$$

Note that $\rho_i(x) = |x_i|^2 / \|x\|^2$, where $\|x\|^2 := |x_0|^2 + \cdots + |x_n|^2$ and that

$$\bar{\partial}\rho_i \wedge dx = \frac{x_i}{\|x\|^2} \left(1 - \frac{d\|x\|^2}{\|x\|^2} \wedge \iota_{\bar{E}} \right) d\bar{x}_i \wedge dx = \frac{x_i}{\|x\|^2} \left(\iota_{\bar{E}} \frac{d\|x\|^2}{\|x\|^2} \wedge \right) d\bar{x}_i \wedge dx,$$

where $\bar{E} := \bar{x}_0 \partial_{x_0} + \cdots + \bar{x}_n \partial_{x_n}$ and $\iota_{\bar{E}}$ denotes the operation contraction by the vector field \bar{E} . The operation in the big brackets commutes with the wedging operations $\bar{\partial}\rho_j \wedge$ for all j . Therefore,

$$\bar{\partial}\rho_1 \wedge \cdots \wedge \bar{\partial}\rho_n \wedge dx = \frac{x_0 x_1 \cdots x_n}{\|x\|^{2n+2}} \iota_{\bar{E}}(d\bar{x} \wedge dx),$$

where $d\bar{x} := d\bar{x}_0 \wedge \cdots \wedge d\bar{x}_n$, and we get

$$\eta_{dx} = (-1)^n 2^{-n-1} n! \|x\|^{-2n-2} \iota_{\bar{E}}(d\bar{x} \wedge dx).$$

Note that the form $d\iota_{\bar{E}}(d\bar{x} \wedge dx) = (n+1)d\bar{x} \wedge dx$ coincides with

$$(n+1)(2\mathbf{i})^{n+1} (-1)^{n(n+1)/2} d\text{Re}(x_0) \wedge d\text{Im}(x_0) \wedge \cdots \wedge d\text{Re}(x_n) \wedge d\text{Im}(x_n)$$

and that the form $\wedge_{i=0}^n d\text{Re}(x_i) \wedge d\text{Im}(x_i)$ defines the orientation of the unit ball in \mathbb{C}^{n+1} induced from the complex manifold orientation of \mathbb{C}^{n+1} . Using Stoke's theorem and

recalling that the volume of the unit ball in \mathbb{C}^{n+1} is $\pi^{n+1}/(n+1)!$ we get

$$\int_{\mathbb{S}^{2n+1}} \eta_{dx} = (-1)^n 2^{-n-1} (n+1)! (2\mathbf{i})^{n+1} (-1)^{n(n+1)/2} \frac{\pi^{n+1}}{(n+1)!} = (-1)^{n(n-1)/2} (2\pi\mathbf{i})^{n+1} 2^{-n-1}.$$

Comparing the LHS and the RHS of (4.23) we get $C_n = (-1)^{n(n-1)/2} (2\pi\mathbf{i})^{-n-1}$. \square

Summing up formula (4.23) over all critical points ξ° of f_{t° and comparing with the definitions of J_F and K_F , we get $J_{F,t^\circ}(\phi_1 dx, \phi_2 dx) = K_{F,t^\circ}^{(0)}(\omega_1, \omega_2)$. This completes the proof of property 3.

It remains to prove property 4, that is, the Leibnitz rule with respect to the Gauss–Manin connection. The problem is local, so we may assume that $t^\circ \in S$ is a given point and that $V \subset S$ is an open coordinate neighborhood of t° with local coordinates $t = (t_1, \dots, t_m)$. For brevity put $\partial_a := \frac{\partial}{\partial t_a}$. Let us check the Leibnitz rule for vector fields of the form $w\partial_a$, that is,

$$(4.24) \quad w\partial_a K_{F,t^\circ}(\omega_1, \omega_2) = K_{F,t^\circ}(w\nabla_{\partial_a}(\omega_1), \omega_2) - K_{F,t^\circ}(\omega_1, w\nabla_{\partial_a}(\omega_2)).$$

Since K_{F,t° is defined by integrating cycles defined locally in a neighborhood of the critical points of f_{t° , it is sufficient to compute $w\nabla_{\partial_a}(\omega_i)|_{Z_\xi}$, where $Z_\xi = U_\xi \times V$ is a coordinate neighborhood of a relative critical point $\xi \in \text{Crit}(f_{t^\circ})$. Note that here, slightly abusing the notation, we think of $\Gamma(V, \widehat{\mathcal{H}}_F^{(0)})$ as the sections of the sheaf $\Omega_{Z/S}^{n+1}[[w]]/d_F\Omega_{Z/S}^n[[w]]$ on the open subset $p^{-1}V \subset Z$. The restriction of the cohomology class ω_i ($i = 1, 2$) to Z_ξ can be represented by a relative holomorphic form $\phi_i(x, t, w)dx$ ($i = 1, 2$). By definition, the restriction of the covariant derivative $w\nabla_{\partial_a}(\omega_i)$ to Z_ξ is represented by

$$(\partial_a(F)\phi_i + w\partial_a(\phi_i))dx.$$

Suppose that $\phi_1(x, t, w)dx = d_F(\eta_1)$ for some $\eta_1 \in \Gamma(Z_\xi \setminus C_F, \mathcal{A}_{Z/S}^{0,n})$. Differentiating the identity $\phi_1 dx = d_F(\eta_1)$ with respect to t_a , after some short computation, we get that

$$(\partial_a(F)\phi_1 + w\partial_a(\phi_1))dx = d_F(\partial_a(F)\eta_1 + w\partial_a(\eta_1)),$$

where $\partial_a(\eta_1) = \iota_{\partial_a} \circ d_Z(\eta_1)$ is the Lie derivative of η_1 . The residue pairing

$$K_{F,t^\circ}(w\nabla_{\partial_a}(\omega_1), \omega_2) = wC_n \sum_{\xi \in \text{Crit}(f_{t^\circ})} \int_{\partial \overline{U}_\xi} (\partial_a(F)\eta_1 + w\partial_a(\eta_1)) \wedge \omega_2^*$$

while the residue pairing

$$K_{F,t^\circ}(\omega_1, w\nabla_{\partial_a}(\omega_2)) = wC_n \sum_{\xi \in \text{Crit}(f_{t^\circ})} \int_{\partial \overline{U}_\xi} \eta_1 \wedge (\partial_a(F)\phi_i^* - w\partial_a(\phi_i^*))dx.$$

The difference of the two residue pairings, i.e., the RHS of (4.24) is

$$wC_n \sum_{\xi \in \text{Crit}(f_{t^\circ})} \int_{\partial \overline{U}_\xi} (w\partial_a(\eta_1) \wedge \omega_2^* + \eta_1 \wedge w\partial_a(\omega_2^*)).$$

The above integral is an integral on U_{ξ° (i.e. in the x -variables) depending on the parameters t . The usual Leibnitz rule applies, so we can write the above integral as

$$w^2\partial_a C_n \sum_{\xi \in \text{Crit}(f_{t^\circ})} \int_{\partial \overline{U}_\xi} \eta_1 \wedge \omega_2^* = w\partial_a K_{F,t^\circ}(\omega_1, \omega_2).$$

The Leibnitz rule for the vector field $w^2\partial/\partial w$ is checked similarly.

4.5.6. Proof of Theorem 4.29 b. Let us assume now that the family of functions $(F, p : Z \rightarrow S)$ is complete. Then the relative critical set C_F is a complex manifold. In particular, the family $(F, p : Z \rightarrow S)$ is Morse. Let us fix a point $t^\circ \in S \setminus \mathcal{K}$ and an open neighborhood V of t° in S , such that $p^{-1}(V) \cap C_F = V_1 \sqcup \cdots \sqcup V_\mu$ and $p|_{V_i} : V_i \rightarrow V$ is a biholomorphism. We claim that $\mu = \mu_F$ and that $u_i = F \circ (p|_{V_i})^{-1} : V \rightarrow \mathbb{C}$ ($1 \leq i \leq \mu$) are holomorphic coordinates on V . Let $\{\xi_i^\circ\}_{i=1}^\mu$ be the critical points of f_{t° and let us fix a direct product coordinate system $(x_0^i, \dots, x_n^i, t_1, \dots, t_m)$ in a neighborhood of ξ_i° in Z for each i ($1 \leq i \leq \mu$). The Kodaira–Spencer map takes the form

$$(4.25) \quad \begin{aligned} \mathcal{T}_{S, t^\circ} \rightarrow (p_* \mathcal{O}_{C_F})_t &= \bigoplus_{i=1}^{\mu} \mathcal{O}_{C_F, \xi_i} \cong \bigoplus_{i=1}^{\mu} \mathcal{O}_{V, t^\circ} \\ \frac{\partial}{\partial t_a} &\mapsto \left(\frac{\partial u_1}{\partial t_a}, \dots, \frac{\partial u_\mu}{\partial t_a} \right). \end{aligned}$$

Since the Kodaira–Spencer map is an isomorphism and the rank of \mathcal{T}_{S, t° is μ_F we get that $\mu = \mu_F$ and that $\det \left(\frac{\partial u_i}{\partial t_a} \right) \neq 0$.

The local coordinates $u = (u_1, \dots, u_\mu)$ constructed above are called *canonical coordinates*. Let us equip the tangent sheaf \mathcal{T}_S with multiplication \bullet , which under the Kodaira–Spencer isomorphism coincides with the natural product of functions in the sheaf $p_* \mathcal{O}_{C_F}$. Note that the direct sum in the Kodaira–Spencer isomorphism (4.25) is a direct sum of algebras and that the vector field $\frac{\partial}{\partial u_i}$ corresponds to the idempotent $(0, \dots, 1, \dots, 0)$, where only the i th entry is 1 and the rest are 0. Therefore, if $t^\circ \in S \setminus \mathcal{K}$, then the stalk \mathcal{T}_{S, t° is a semi-simple algebra, that is,

$$\mathcal{T}_{S, t^\circ} = \mathcal{O}_{S, t^\circ} \frac{\partial}{\partial u_1} \oplus \cdots \oplus \mathcal{O}_{S, t^\circ} \frac{\partial}{\partial u_\mu}, \quad \frac{\partial}{\partial u_i} \bullet \frac{\partial}{\partial u_j} = \delta_{i,j} \frac{\partial}{\partial u_j}.$$

After this preliminary remarks, we are in position to prove part b of Theorem 4.29. It is sufficient to prove the uniqueness of the germ of the higher residue pairing K_{F, t° at a point $t^\circ \in S \setminus \mathcal{K}$. Let us fix a sufficiently small contractible Stein neighborhood V of t° , such that, we have canonical coordinates $u = (u_1, \dots, u_\mu)$. According to Proposition 4.24, there exists a set of holomorphic forms $\omega_i \in \Gamma(p^{-1}V, \Omega_{Z/S}^{n+1})$ ($1 \leq i \leq \mu$) representing an \mathcal{O}_V -basis of the sheaf $p_* \Omega_F$. Let ξ_i° ($1 \leq i \leq \mu$) be the critical points of f_{t° . Using local coordinates, it is easy to see that Ω_{F, ξ_i° is a free $\mathcal{O}_{C_F, \xi_i^\circ}$ module of rank 1. Let $\theta_i \in \Omega_{F, \xi_i^\circ}$ be a generator. We have the following decomposition:

$$(p_* \Omega_F)_{t^\circ} = \bigoplus_{i=1}^{\mu} \Omega_{F, \xi_i^\circ} = \bigoplus_{i=1}^{\mu} \mathcal{O}_{C_F, \xi_i^\circ} \theta_i.$$

On the other hand, since $p|_{C_F}$ is a local biholomorphism at ξ_i° , we have $\mathcal{O}_{C_F, \xi_i^\circ} \cong \mathcal{O}_{S, t^\circ}$. Let $[\omega_j, \xi_i^\circ] \in \Omega_{F, \xi_i^\circ}$ be the equivalence class of the germ of the form ω_j at ξ_i° . Then $([\omega_j, \xi_1^\circ], \dots, [\omega_j, \xi_\mu^\circ]) \in (p_* \Omega_F)_{t^\circ}$ ($1 \leq i \leq \mu$) and θ_j ($1 \leq j \leq \mu$) are two \mathcal{O}_{S, t° -bases of $(p_* \Omega_F)_{t^\circ}$. After an appropriate \mathcal{O}_{S, t° -linear change we may arrange that the two bases coincide. Under the Kodaira–Spencer isomorphism $p_* \Omega_F$ becomes a \mathcal{T}_S -module and the forms ω_i satisfy the following relations:

$$(4.26) \quad \partial_{u_i} \bullet [\omega_j] = \delta_{i,j} [\omega_j], \quad 1 \leq i, j \leq \mu,$$

where $\partial_{u_i} = \frac{\partial}{\partial u_i}$ and $[\omega_i]$ is the equivalence class of ω_i in $p_* \Omega_F$.

According to Proposition 4.24, the forms ω_i ($1 \leq i \leq \mu$) represent an $\mathcal{O}_V[[w]]$ -basis of $\widehat{\mathcal{H}}_F^{(0)}|_V$. The Gauss–Manin connection takes the form

$$w\nabla_{\partial_i}[\omega_j] = \sum_{k=1}^{\mu} C_{ij}^k(u, w)[\omega_k]$$

and

$$w^2\nabla_{\partial_w}[\omega_j] = \sum_{k=1}^{\mu} C_j^k(u, z)[\omega_k],$$

where $\partial_w = \frac{\partial}{\partial w}$. Let us put the information about the Gauss–Manin connection and the higher residue pairing into the set of $\mu + 2$ size $\mu \times \mu$ matrices $K(u, w)$ and $C_a(u, w)$ ($0 \leq a \leq \mu$), whose (i, j) -entries are defined as follows:

$$\begin{aligned} K_{ij}(u, w) &:= K_F([\omega_i], [\omega_j]), \\ (C_a(u, w))_{ij} &:= C_{aj}^i(u, w), \quad 1 \leq a \leq \mu, \\ (C_0(u, w))_{ij} &:= C_j^i(u, w), \quad 1 \leq a \leq \mu. \end{aligned}$$

Let $K(u, w) = \sum_{p=0}^{\infty} K_p(u)w^{p+n+1}$ and $C_a(u, w) = \sum_{p=0}^{\infty} C_{a;p}(u)w^p$ be the corresponding series expansions. Let E_{ij} be the matrix with only one non-zero entry which is in position (i, j) and it is equal to 1. Relations (4.26) imply that $C_{i;0} = E_{ii}$ ($1 \leq i \leq \mu$), while the definition of the canonical coordinates implies that $C_{0;0} = -\sum_{i=1}^{\mu} u_i E_{ii}$. The Leibnitz rule for the Gauss–Manin-connection implies that

$$\begin{aligned} w\partial_{u_i}K(u, w) &= C_i(u, w)^T K(u, w) - K(u, w)C_i(u, -w), \\ w^2\partial_w K(u, w) &= C_0(u, w)^T K(u, w) - K(u, w)C_0(u, -w). \end{aligned}$$

Let us assume that we have two pairings K' and K'' satisfying the axioms of a higher residue pairing. Then the matrix $K(u, w) = K'(u, w)K''(u, w)^{-1}$ satisfies the following differential equations:

$$\begin{aligned} w\partial_{u_i}K(u, w) &= [C_i(u, w)^T, K(u, w)], \\ w^2\partial_w K(u, w) &= [C_0(u, w)^T, K(u, w)]. \end{aligned}$$

Let us expand $K(u, w) = \sum_{p=0}^{\infty} K_p(u)w^p$. Since the leading order term in the expansion in the powers of w of a higher residue pairing is fixed to be the classical residue pairing, we get that $K_0 = 1$. We will prove by induction on p that $K_p = 0$ for all $p > 0$. Comparing the coefficients in front of the powers of w , we get the following recursion relations:

$$[K_0, E_{ii}] = [K_0, C_{0;0}] = 0,$$

$$\partial_{u_i}(K_p) = [E_{ii}, K_{p+1}] + \sum_{q=1}^{p+1} [C_{i;q}^T, K_{p+1-q}], \quad p \geq 0, \quad 1 \leq i \leq \mu,$$

and

$$(p+1)K_{p+1} = [C_{0;0}, K_{p+2}] + \sum_{q=1}^{p+2} [C_{0;q}^T, K_{p+2-q}], \quad p \geq 1.$$

The first set of equations is trivially satisfied, because K_0 is the identity matrix. Suppose that $K_1 = K_2 = \dots = K_p = 0$. Using the second set of equations we get that $[E_{ii}, K_{p+1}] = 0$ for all $1 \leq i \leq \mu$, which implies that K_{p+1} is a diagonal matrix. Note that the matrices $C_{0;0}$ and K_{p+2-q} ($1 \leq q \leq p+2$) in the last set of equations are diagonal. Comparing the diagonal entries, we get $K_{p+1} = 0$. \square

4.6. Primitive forms and Frobenius structures

Suppose that $(F, p : Z \rightarrow S)$ is a complete family. Using the Kodaira–Spencer isomorphism $\mathcal{T}_S \cong p_* \mathcal{O}_{C_F}$ we define

- (i) *Frobenius multiplication* \bullet in \mathcal{T}_S corresponding to the natural multiplication in the structure sheaf \mathcal{O}_{C_F} .
- (ii) *Unit vector field* $e \in \Gamma(S, \mathcal{T}_S)$ corresponding to $1 \in p_* \mathcal{O}_{C_F}$.
- (iii) *Euler vector field* $E \in \Gamma(S, \mathcal{T}_S)$ corresponding to $F|_{C_F} \in p_* \mathcal{O}_{C_F}$.

Let us recall the sheaves on S of formal oscillatory integrals $\widehat{\mathcal{H}}_F$ and $\widehat{\mathcal{H}}_F^{(0)}$, the Gauss–Manin connection ∇ , and the higher residue pairing K_F .

4.6.1. The idea of a primitive form. Let us recall the sheaf $\Omega_F := p_* \Omega_{Z/S}^{n+1} / dF \wedge p_* \Omega_{Z/S}^n$. According to the direct image theorem for finite maps we have the following formula for the stalks of Ω_F :

$$\Omega_{F,t} = \bigoplus_{\xi \in C_F \cap p^{-1}(t)} \mathcal{O}_{C_F, \xi} dx_0 \wedge \dots \wedge dx_n,$$

where $x = (x_0, \dots, x_n)$ is a local coordinate system on $Z_t = p^{-1}(t)$ with center at ξ . Every section $\varphi \in \Gamma(S, \Omega_F)$ can be represented locally near a point $t \in S$ with a set of holomorphic forms $\varphi_\xi(x, t) = g_\xi(x, t) dx_0 \wedge \dots \wedge dx_n$ corresponding to the critical points ξ of $f_t := F|_{Z_t}$. We say that $\varphi \in \Gamma(S, \Omega_F)$ is a *holomorphic volume form* if $g_\xi(0, t) \neq 0$ for all t and ξ , that is, $g_\xi(x, t)$ is an invertible element of $\mathcal{O}_{C_F, \xi}$. More generally, we will say that $\omega \in \Gamma(S, \widehat{\mathcal{H}}_F^{(0)})$ is a holomorphic volume form if the image of ω in $\Omega_F = \widehat{\mathcal{H}}_F^{(0)} / w \widehat{\mathcal{H}}_F^{(0)}$ under the natural quotient map is a holomorphic volume form in Ω_F .

Suppose now that $\omega \in \Gamma(S, \widehat{\mathcal{H}}_F^{(0)})$ is a holomorphic volume form. Using Proposition 4.24 and the fact that the family is complete, we get that the map

$$(4.27) \quad \Pi : \mathcal{T}_S[[w]] \xrightarrow{\cong} \widehat{\mathcal{H}}_F^{(0)} \quad v \mapsto w \nabla_v \omega$$

is an isomorphism of \mathcal{O}_S -modules. Let us fix a local coordinate system $t = (t_1, \dots, t_m)$ on S defined on an open neighborhood of a point $t^\circ \in S$. Let us denote by $\partial_i := \partial/\partial t_i$ ($1 \leq i \leq m$) the coordinate vector fields and by $\nabla_i := \nabla_{\partial_i}$ the corresponding covariant derivatives with respect to the Gauss–Manin connection. The period isomorphism implies that the covariant derivatives $w \nabla_{\partial_i} \omega$ ($1 \leq i \leq m$) form a $\mathcal{O}_S[[w]]$ -basis of $\widehat{\mathcal{H}}_F^{(0)}$. In particular, the Gauss–Manin connection can be written as

$$(4.28) \quad w \nabla_i w \nabla_j \omega = \sum_{k=1}^m \Gamma_{ij}^k(t, w) w \nabla_k \omega,$$

$$(4.29) \quad w^2 \nabla_w w \nabla_i \omega = \sum_{k=1}^m U_i^k(t, w) w \nabla_k \omega,$$

where $\nabla_w := \nabla_{\partial/\partial w}$, $\Gamma_{ij}^k(t, w) = \sum_{p=0}^{\infty} \Gamma_{ij;p}^k(t)w^p$, and $U_i^k(t, w) = \sum_{p=0}^{\infty} U_{i;p}^k(t)w^p$. We will check later on that $\Gamma_{ij;0}^k$ are the structure constants of the Frobenius multiplication on \mathcal{T}_S and that $\Gamma_{ij;1}^k$ are the Christophel symbols corresponding to the Levi-Civite connection of the classical residue pairing on \mathcal{T}_S . Moreover, $U_{i;0}^k$ is the matrix of the linear operator of Frobenius multiplication by $-E$ and $U_{i;1}^k$ is the grading operator corresponding to the Euler vector field. The remaining terms of the Gauss-Manin connection $\Gamma_{ij;p}^k(t)$ and $U_{i;p}^k(t)$ ($p \geq 2$) do not have a meaningful interpretation. The question that lead to the discovery of the notion of a primitive form is whether we can choose the holomorphic volume form ω in such a way that the remaining higher order terms of the Gauss-Manin connection vanish.

DEFINITION 4.32. A holomorphic volume form $\omega \in \Gamma(S, \widehat{\mathcal{H}}_F^{(0)})$ is said to be a *primitive form* if the following conditions are satisfied:

- (i) $K_F^{(p)}(w\nabla_i\omega, w\nabla_j\omega) = 0$ for all $1 \leq i, j \leq m$ and $p \geq 1$.
- (ii) $K_F^{(p)}(w\nabla_i w\nabla_j\omega, w\nabla_k\omega) = 0$ for all $1 \leq i, j, k \leq m$ and $p \geq 2$.
- (iii) $K_F^{(p)}(w^2\nabla_w w\nabla_i\omega, w\nabla_j\omega) = 0$ for all $1 \leq i, j \leq m$ and $p \geq 2$.
- (iv) There exists a constant $r \in \mathbb{C}$, such that

$$(w\nabla_w + \nabla_E)\omega = r\omega.$$

- (v) $w\nabla_e\omega = \omega$.

The number r in condition (iv) is said to be the *homogeneous degree* of the primitive form.

THEOREM 4.33. If ω is a primitive form of homogeneous degree r , then

$$g(v', v'') := K_F^{(0)}(w\nabla_{v'}\omega, w\nabla_{v''}\omega), \quad v', v'' \in \mathcal{T}_S$$

is a non-degenerate bi-linear pairing on \mathcal{T}_S and the data $(S, \bullet, (,), E, e)$ is a Frobenius structure of conformal dimension $D := n + 1 - 2r$.

PROOF. It is sufficient to check that the axioms of a Frobenius manifold (see Definition 1.1) are satisfied locally in a neighborhood of each point $t^\circ \in S$. Let $t = (t_1, \dots, t_m)$ be a local coordinate system defined in a neighborhood of t° . Let $g_{ij}(t) = K_F^{(0)}(w\nabla_i\omega, w\nabla_j\omega)$ be the matrix of the residue pairing. The residue pairing is non-degenerate, so $(g_{ij}(t))_{1 \leq i, j \leq m}$ is an invertible matrix. Using axiom (i) in Definition 4.32 we get

$$K_F(w\nabla_i\omega, w\nabla_j\omega) = g_{ij}(t)w^{n+1}, \quad 1 \leq i, j \leq m,$$

$$K_F(w\nabla_i w\nabla_j\omega, w\nabla_k\omega) = \sum_{l=1}^m \sum_{p=0}^{\infty} \Gamma_{ij;p}^l(t)g_{lk}(t)w^{n+1+p},$$

and

$$K_F(w^2\nabla_w w\nabla_i\omega, w\nabla_j\omega) = \sum_{k=1}^m \sum_{p=0}^{\infty} U_{i;p}^k(t)g_{kj}(t)w^{n+1+p}.$$

Recalling axioms (ii) and (iii) from the definition of a primitive form we get that $\Gamma_{ij;p}^l(t) = 0$ and $U_{i;p}^k(t) = 0$ for all $p \geq 2$.

Let us express the coefficients $\Gamma_{ij;p}^k(t)$ and $U_{ij;p}^k(t)$ for $p = 0, 1$ in terms of the residue pairing $g_{ij}(t)$ and the Frobenius multiplication \bullet . To begin with, let $C_{ij}^k(t)$ be the structure constants of \bullet , that is, $\partial_i \bullet \partial_j =: \sum_{k=1}^m C_{ij}^k(t) \partial_k$. We claim that $\Gamma_{ij;0}^k(t) = C_{ij}^k(t)$. To prove this, let us represent ω by a holomorphic form in $\Gamma(p^{-1}V, \Omega_{Z/S}^{n+1}[[w]])$. Let us choose an open covering of $p^{-1}V$ consisting of open subsets of the form $\mathcal{U}_\alpha := U_\alpha \times V \cap p^{-1}V$ ($\alpha \in \mathcal{A}$), where U_α ($\alpha \in \mathcal{A}$) is a holomorphic atlas of X (recall that Z is an open subset of $X \times S$). Then $\omega|_{\mathcal{U}_\alpha} = g_\alpha(x, t, w) dx$, where we put for brevity $dx := dx_0 \wedge \cdots \wedge dx_n$, where $x = (x_0, \dots, x_n)$ are the coordinates on U_α . By definition

$$(4.30) \quad (w \nabla_i w \nabla_j \omega)|_{\mathcal{U}_\alpha} = \left(\partial_i F \partial_j F + O(w) \right) g_\alpha(x, t, w) dx \pmod{wd + dF \wedge}.$$

Recalling the definition of \bullet , we get

$$\partial_i F \partial_j F = \sum_{k=1}^m C_{ij}^k(t) \partial_k F + \sum_{a=0}^n \phi_a(x, t) \partial_{x_a} F.$$

Note that

$$\phi_a(x, t) (\partial_{x_a} F) g_\alpha(x, t, w) dx = -w \frac{\partial \phi_a(x, t) g_\alpha(x, t, w)}{\partial x_a} \pmod{wd + dF \wedge}.$$

and that

$$C_{ij}^k(t) \partial_k F g_\alpha(x, t, w) dx = C_{ij}^k(t) w \nabla_k \omega|_{\mathcal{U}_\alpha} \pmod{w \widehat{\mathcal{H}}_F^{(0)}}.$$

Recalling (4.30) and the definition (4.28), we get $\Gamma_{ij;0}^k(t) = C_{ij}^k(t)$.

Let us determine the coefficient $\Gamma_{ij;1}^k(t)$. In order to do this, let us define a connection on the tangent bundle \mathcal{T}_V by

$$\widetilde{\nabla}_{\partial_i} \partial_j = \sum_{k=1}^m \Gamma_{ij;1}^k(t) \partial_k.$$

We claim that $\widetilde{\nabla}$ is the Levi-Civita connection of the residue pairing g_{ij} . \therefore Recalling the definition (4.28) we get that $\Gamma_{ij;1}^k = \Gamma_{ji;1}^k$, that is, $\widetilde{\nabla}$ is Torsion free. We need only to check the compatibility with the residue pairing. We have

$$(4.31) \quad \partial_i g_{kl} = \left(K_F(w \nabla_i w \nabla_k \omega, w \nabla_l \omega) - K_F(w \nabla_k \omega, w \nabla_i w \nabla_l \omega) \right) w^{-n-2},$$

where we used the Leibnitz rule for the higher residue pairing. On the other hand, we have

$$w \nabla_i w \nabla_k \omega = \sum_{j=1}^m \left(C_{ik}^j(t) + w \Gamma_{ik;1}^j(t) \right) w \nabla_j \omega = \sum_{j=1}^m C_{ik}^j(t) w \nabla_j \omega + w^2 \nabla_{\widetilde{\nabla}_i \partial_k} \omega.$$

and similarly

$$w \nabla_i w \nabla_l \omega = \sum_{j=1}^m C_{il}^j(t) w \nabla_j \omega + w^2 \nabla_{\widetilde{\nabla}_i \partial_l} \omega.$$

Let us substitute these formula in (4.31). The terms involving the structure constants C_{ik}^j and C_{il}^j cancel out. The remaining terms contribute

$$\left(K_F(w^2 \nabla_{\widetilde{\nabla}_i \partial_k} \omega, w \nabla_l \omega) - K_F(w \nabla_k \omega, w^2 \nabla_{\widetilde{\nabla}_i \partial_l} \omega) \right) w^{-n-2} = g(\widetilde{\nabla}_i \partial_k, \partial_l) + g(\partial_k, \widetilde{\nabla}_i \partial_l).$$

This completes the proof that $\tilde{\nabla}$ is the Levi–Civita connection, which we denote by $\nabla^{\text{L.C.}}$, of the residue pairing.

Note that the formulas for $\Gamma_{ij;p}^k(t)$ yield the following identity:

$$(4.32) \quad w \nabla_v w \nabla_j \omega = w \nabla_{v \bullet \partial_j + w \nabla_v^{\text{L.C.}}(\partial_j)} \omega,$$

for all $v \in \mathcal{T}_S$.

The coefficients $U_{i;0}^k(t)$ are the entries of the matrix of the linear operator of Frobenius multiplication by $-E$, that is,

$$(4.33) \quad E \bullet \partial_i = - \sum_{k=1}^m U_{i;0}^k(t) \partial_k.$$

The proof of this fact is very similar to the proof that $\Gamma_{ij;0}^k(t) = C_{ij}^k(t)$ and we leave it as an exercise. Let us prove that the coefficients $U_{i;1}^j(t)$ satisfy

$$(4.34) \quad \nabla_{\partial_i}^{\text{L.C.}} E = (1+r)\partial_i - \sum_{j=1}^m U_{i;1}^j(t) \partial_j.$$

\therefore Using the definition (4.29) we get

$$(4.35) \quad K_F(w^2 \nabla_w w \nabla_i \omega, w \nabla_k \omega) = \sum_{j=1}^m \left(U_{i;0}^j(t) g_{jk}(t) w^{n+1} + U_{i;1}^j(t) g_{jk}(t) w^{n+2} \right).$$

On the other hand, using the flatness of the Gauss–Manin connection, we have

$$w^2 \nabla_w w \nabla_i = w \nabla_i w^2 \nabla_w + w^2 \nabla_i,$$

while the homogeneity of the primitive form yields $w^2 \nabla_w \omega = -w \nabla_E \omega + r w \omega$. We get

$$(4.36) \quad K_F(w^2 \nabla_w w \nabla_i \omega, w \nabla_k \omega) = w^{n+2} (1+r) g_{ik}(t) - K_F(w \nabla_i w \nabla_E \omega, w \nabla_k \omega).$$

Using again the flatness of the Gauss–Manin connection we get

$$\begin{aligned} w \nabla_i w \nabla_E \omega &= w \nabla_E w \nabla_i \omega + w^2 \nabla_{[\partial_i, E]} \omega \\ &= w \nabla_{E \bullet \partial_i + w (\nabla_E^{\text{L.C.}}(\partial_i) + [\partial_i, E])} \omega \\ &= w \nabla_{E \bullet \partial_i + w \nabla_{\partial_i}^{\text{L.C.}}(E)} \omega, \end{aligned}$$

where for the 2nd equality we used formula (4.32) and for the third one we used the torsion freeness of the Levi–Civita connection. Substituting the above formula in (4.36) we get

$$K_F(w^2 \nabla_w w \nabla_i \omega, w \nabla_k \omega) = -g(E \bullet \partial_i, \partial_k) w^{n+1} + \left((1+r) g_{ik}(t) - \nabla_{\partial_i}^{\text{L.C.}}(E) \right) w^{n+2}$$

Comparing the above formula with (4.35) and recalling also (4.33), we get (4.34). This completes our task to express the coefficients of the Gauss–Manin connection in terms of the Frobenius multiplication and the residue pairing.

Next, let us show that E is an Euler vector field, that is,

$$E g_{kl} = g([E, \partial_k], \partial_l) + g(\partial_k, [E, \partial_l]) + (2-D) g(\partial_k, \partial_l),$$

where $D := n + 1 - 2r$. \therefore By definition $g_{kl} = K_F(w\nabla_k\omega, w\nabla_l\omega)w^{-n-1}$. Recalling the Leibnitz rule for the higher residue pairing and that

$$\begin{aligned} w\nabla_E w\nabla_k\omega &= w\nabla_k w\nabla_E\omega + w^2\nabla_{[E, \partial_k]}\omega \\ &= w\nabla_k (rw - w^2\nabla_w)\omega + w^2\nabla_{[E, \partial_k]}\omega \\ &= -w^2\nabla_w w\nabla_k\omega + w^2\nabla_{(r+1)\partial_k + [E, \partial_k]}\omega \end{aligned}$$

we get

$$\begin{aligned} Eg_{kl} &= (K_F(-w^2\nabla_w w\nabla_k\omega + w^2\nabla_{(r+1)\partial_k + [E, \partial_k]}\omega, w\nabla_l\omega) - \\ &\quad K_F(w\nabla_k\omega, -w^2\nabla_w w\nabla_l\omega + w^2\nabla_{(r+1)\partial_l + [E, \partial_l]}\omega)) w^{-n-2}. \end{aligned}$$

The terms involving the covariant derivative with respect to w contribute

$$w^{-n-2}(-w^2\partial_w)K_F(w\nabla_k\omega, w\nabla_l\omega) = -w^{-n}\partial_w(g_{kl}w^{n+1}) = -(n+1)g_{kl},$$

while the remaining terms contribute

$$g((r+1)\partial_k + [E, \partial_k], \partial_l) + g(\partial_k, (r+1)\partial_l + [E, \partial_l]) = (2r+2)g_{kl} + g([E, \partial_k], \partial_l) + g(\partial_k, [E, \partial_l]).$$

This completes the proof that E is an Euler vector field.

Now the proof can be completed as follows. Let

$$\Pi : \mathcal{T}_S[[w]] \xrightarrow{\cong} \widehat{\mathcal{H}}_F^{(0)}, \quad v \mapsto w\nabla_v\omega,$$

be the period isomorphism. The pullback $\Pi^*\nabla$ of the Gauss–Manin connection is defined by

$$\begin{aligned} \Pi(w\Pi^*\nabla_{\partial_i}\partial_j) &:= w\nabla_{\partial_i}\Pi(\partial_j), \\ \Pi(w^2\Pi^*\nabla_{\partial_w}\partial_i) &:= w^2\nabla_{\partial_w}\Pi(\partial_i). \end{aligned}$$

Since

$$w\nabla_{\partial_i}\Pi(\partial_j) = w\nabla_{\partial_i}w\nabla_{\partial_j}\omega = \sum_{k=1}^m \left(\sum_{p=0}^{\infty} \Gamma_{ij;p}^k(t)w^p \right) w\nabla_k\omega.$$

We get

$$w\Pi^*\nabla_{\partial_i}\partial_j = \sum_{k=1}^m \left(\Gamma_{ij;0}^k(t) + w\Gamma_{ij;1}^k(t) \right) \partial_k.$$

Similarly,

$$w^2\Pi^*\nabla_{\partial_w}\partial_i = \sum_{k=1}^m \left(U_{i;0}^k(t) + wU_{i;1}^k(t) \right) \partial_k.$$

Recalling our formulas for $\Gamma_{ij;p}^k$ and $U_{i;0}^k$ we get

$$\begin{aligned}\sum_{k=1}^m \Gamma_{ij;0}^k(t) \partial_k &= \partial_i \bullet \partial_j, \\ \sum_{k=1}^m \Gamma_{ij;1}^k(t) \partial_k &= \nabla_{\partial_i}^{\text{L.C.}} \partial_j, \\ \sum_{k=1}^m U_{i;0}^k(t) \partial_k &= -E \bullet \partial_i.\end{aligned}$$

Let us recall that the grading operator $\theta : \mathcal{T}_S \rightarrow \mathcal{T}_S$ is by definition

$$\theta(\partial_i) = \nabla_{\partial_i}^{\text{L.C.}} E - (1 - D/2) \partial_i = \nabla_{\partial_i}^{\text{L.C.}} E - (1 + r - (n + 1)/2) \partial_i.$$

Therefore,

$$\sum_{k=1}^m U_{i;1}^k(t) \partial_k = (1 + r) \partial_i - \nabla_{\partial_i}^{\text{L.C.}} E = \left(-\theta + \frac{1}{2}(n + 1) \right) \partial_i.$$

Therefore, the pullback of the Gauss–Manin connection takes the following form:

$$(4.37) \quad \Pi^* \nabla_{\partial_i} = \nabla_{\partial_i}^{\text{L.C.}} + w^{-1} \partial_i \bullet$$

$$(4.38) \quad \Pi^* \nabla_{\partial_w} = \partial_w - w^{-1} \left(\theta - \frac{1}{2}(n + 1) \right) - w^{-2} E \bullet.$$

Up to changing $w \rightarrow -w$ and shifting θ by $\frac{n+1}{2}$, we get that $\Pi^* \nabla$ coincides with the Dubrovin’s connection (see Definition 1.1). The flatness of the Gauss–Manin connection implies the flatness of the Dubrovin’s connection. In order complete the proof of the theorem, it remains only to check that the unit vector field e is flat. Indeed, using the flatness of the Gauss–Manin connection and the torsion freeness of the Levi–Civita connection, we get

$$w \nabla_v w \nabla_j = w \nabla_j w \nabla_v + w^2 \nabla_{[v, \partial_j]}$$

and

$$w \nabla_v^{\text{L.C.}} \partial_j = w \nabla_{\partial_j}^{\text{L.C.}} v + w[v, \partial_j]$$

for every vector field $v \in \mathcal{T}_S$. Therefore, formula (4.32) yields

$$w \nabla_j w \nabla_v \omega = w \nabla_{v \bullet \partial_j + w \nabla_{\partial_j}^{\text{L.C.}} v} \omega.$$

Let us substitute $v = e$ and recall Axiom (v) of the primitive form. We get $w \nabla_{\nabla_{\partial_j}^{\text{L.C.}} e} \omega = 0$, that is, since the period map is an isomorphism, $\nabla_{\partial_j}^{\text{L.C.}} e = 0$. \square

Let us point out that under some further assumptions for the family of functions $(F, p : Z \rightarrow S)$ the above proof gives also a method for constructing a solution to the Dubrovin’s connection in terms of oscillatory integrals. For example, in most applications we are interested in the homology groups

$$\mathbb{L}_{t,w} := \varprojlim_M H_{n+1}(Z_t, \{x \in Z_t \mid \text{Re}(F(x)/w) < -M\}; \mathbb{Z}),$$

where the inverse limit is taken over all positive real numbers. The goal of the extra assumptions that one has to make is to arrange that the union $\mathbb{L} := \cup_{(t,w) \in S \times \mathbb{C}^*} \mathbb{L}_{t,w}$

has a structure of a local system on $S \times \mathbb{C}^*$ and that the primitive form and its covariant derivatives have a moderate growth at infinity. Then the corresponding oscillatory integrals are convergent and we can define the following vector field $J_\Gamma(t, w)$ on S :

$$(J_\Gamma(t, w), \partial_i) := w \partial_i \left((-2\pi w)^{-\frac{n+1}{2}} \int_\Gamma e^{F/w} \omega \right),$$

where $(\ , \)$ is the residue pairing. Formulas (4.37)–(4.38) yield

$$\begin{aligned} w \nabla_{\partial_i}^{\text{L.C.}} J_\Gamma(t, w) &= \partial_i \bullet J_\Gamma(t, w) \\ (w \partial_w + \nabla_E^{\text{L.C.}}) J_\Gamma(t, w) &= \theta J_\Gamma(t, w). \end{aligned}$$

In other words, $J_\Gamma(t, w)$ is a solution to the Dubrovin's connection.

4.6.2. Opposite subspaces. The problem of proving the existence of a primitive form for a given family of functions can be divided into two parts. First, let us fix an arbitrary point $t^\circ \in S$. Note that the definition of a primitive form makes sense also if we replace S with the formal completion of the germ (S, t°) of the complex manifold S at t° . In this case we say that the primitive form is *formal*. It turns out that formal primitive forms can be constructed quite easily from the so called *opposite subspaces*. The latter were introduced by Li-Li-Saito in [42], although the notion of an opposite subspace is closely related to the notion of a *good section* in [53] and [54]. The 2nd part of the problem is to determine whether the formal primitive form extends to an analytic one. This is a very difficult problem and in principle one has to make some extra assumptions for the family of functions.

The goal in this section is to explain the construction of a formal primitive form at a given point $t^\circ \in S$ following the ideas of [42]. To begin with, let us fix the following notation. Let us fix local coordinates $t = (t_1, \dots, t_m)$ near t° , such that, t° is identified with the origin. Let $Z_0 := Z_{t^\circ}$ and $f := F|_{Z_0}$. The fibers of the sheaves $\widehat{\mathcal{H}}_F$ and $\widehat{\mathcal{H}}_F^{(0)}$ at t° will be denoted by respectively $\widehat{\mathcal{H}}_f$ and $\widehat{\mathcal{H}}_f^{(0)}$. The restriction of the higher residue pairing to $t = 0$ will be denoted by K_f .

The vector space $\widehat{\mathcal{H}}_f$ has a natural symplectic structure, that is,

$$\Omega(\phi_1, \phi_2) := \text{Res}_{z=0} K_f(\phi_1, \phi_2) w^{-n-1} dw,$$

where the residue is taken formally as the coefficient in front of w^{-1} . The skew-symmetry $\Omega(\phi_1, \phi_2) = -\Omega(\phi_2, \phi_1)$ follows from property 1 of the higher residue pairing (see Section 4.5.3), while the non-degeneracy of Ω follows from the fact that the classical residue pairing is non-degenerate. Note that property 3 of the higher residue pairing (see Section 4.5.3) implies that $\widehat{\mathcal{H}}_f^{(0)}$ is a Lagrangian subspace of $\widehat{\mathcal{H}}_f^{(0)}$.

DEFINITION 4.34. A Lagrangian subspace $P \subset \widehat{\mathcal{H}}_f$ is said to be an *opposite subspace* if (i) $\widehat{\mathcal{H}}_f = \widehat{\mathcal{H}}_f^{(0)} \oplus P$ and (ii) $w^{-1}P \subset P$. \square

Note that we have a natural isomorphism

$$\widehat{\mathcal{H}}_f^{(0)} / w \widehat{\mathcal{H}}_f^{(0)} \cong \Omega_f := \Gamma(Z_0, \Omega_{Z_0}^{n+1}) / df \wedge \Gamma(Z_0, \Omega_{Z_0}^n).$$

The main properties of an opposite subspace can be stated as follows.

PROPOSITION 4.35. *If $P \subset \widehat{\mathcal{H}}_f$ is an opposite subspace, then*

a) The quotient map

$$\widehat{\mathcal{H}}_f^{(0)} \rightarrow \widehat{\mathcal{H}}_f^{(0)}/w\widehat{\mathcal{H}}_f^{(0)} \cong \Omega_f$$

induces an isomorphism $\widehat{\mathcal{H}}_f^{(0)} \cap wP \cong \Omega_f$. Let

$$\sigma : \Omega_f \rightarrow \widehat{\mathcal{H}}_f^{(0)} \cap wP \subset \widehat{\mathcal{H}}_f^{(0)}$$

be the corresponding inverse.

b) We have $P = \sigma(\Omega_f)[w^{-1}]w^{-1}$ and $\widehat{\mathcal{H}}_f^{(0)} = \sigma(\Omega_f)[[w]]$.

c) $K_f(\phi_1, \phi_2) \in \mathbb{C}w^{n+1}$ for all $\phi_1, \phi_2 \in \widehat{\mathcal{H}}_f^{(0)} \cap wP$.

PROOF. a) First, let us prove that the map is injective. Suppose that $\phi \in \widehat{\mathcal{H}}_f^{(0)} \cap wP$ is mapped to 0 in Ω_f , that is, $\phi \in w\widehat{\mathcal{H}}_f^{(0)}$. It follows that $\phi \in w(\widehat{\mathcal{H}}_f^{(0)} \cap P) = 0$. For the surjectivity, we need to prove that for a given $\phi \in \widehat{\mathcal{H}}_f^{(0)}$ there exists $\psi \in \widehat{\mathcal{H}}_f^{(0)}$, such that, $\phi + w\psi \in \widehat{\mathcal{H}}_f^{(0)} \cap wP$. Using condition (i) from the definition of an opposite subspace (see Definition 4.34), we get that $w^{-1}\phi = \psi_1 + \psi_2$, for some $\psi_1 \in \widehat{\mathcal{H}}_f^{(0)}$ and $\psi_2 \in P$. Note that $\psi = -\psi_1$ has the required property.

b) The inclusion $\sigma(\Omega_f)[w^{-1}]w^{-1} \subset P$ is obvious. Let us prove the opposite inclusion. Note that if $\phi \in P$ then there exists $k > 0$, such that, $w^k\phi \in \widehat{\mathcal{H}}_f^{(0)}$. We argue by induction on k that $\phi \in \sigma(\Omega_f)[w]w^{-1}$. If $k = 1$, then $w\phi \in \widehat{\mathcal{H}}_f^{(0)} \cap wP = \sigma(\Omega_f)$, that is, $\phi \in w^{-1}\sigma(\Omega_f)$. If $k > 1$, then let us recall again condition (i) from Definition 4.34. We have $w\phi = \psi_1 + \psi_2$ for some $\psi_1 \in \widehat{\mathcal{H}}_f^{(0)}$ and $\psi_2 \in P$. Note that $w^{k-1}\psi_2 = w^k\phi - w^{k-1}\psi_1 \in \widehat{\mathcal{H}}_f^{(0)}$. Therefore, using the inductive assumption, we get

$$\phi \in w^{-1}\psi_1 + \sigma(\Omega_f)[w]w^{-1}.$$

On the other hand, $\psi_1 = w(\phi - w^{-1}\psi_2)$. Recalling condition (ii) from Definition 4.34, we get $w^{-1}\psi_2 \in P$. Hence $\psi_1 \in \widehat{\mathcal{H}}_f^{(0)} \cap wP = \sigma(\Omega_f)$.

c) Suppose that $\phi_1, \phi_2 \in \widehat{\mathcal{H}}_f^{(0)} \cap wP$. Let us expand

$$K_f(\phi_1, \phi_2) = \sum_{l=0}^{\infty} K_f^{(l)}(\phi_1, \phi_2)w^{n+1+l}.$$

We need to prove that $K_f^{(l)}(\phi_1, \phi_2) = 0$ for $l > 0$. Note that

$$K_f^{(l)}(\phi_1, \phi_2) = -\text{Res}_{w=0} K_f^{(l)}(w^{-l}\phi_1, w^{-1}\phi_2)w^{-n-1}dw = -\Omega(w^{-l}\phi_1, w^{-1}\phi_2).$$

Since $\phi_1, \phi_2 \in wP$ and $w^{-1}P \subset P$, we get $w^{-l}\phi_1 \in P$ and $w^{-1}\phi_2 \in P$. The vanishing claim follows from the fact that P is a Lagrangian subspace. \square

DEFINITION 4.36. A subspace $P \subset \widehat{\mathcal{H}}_f$ is said to be *homogeneous* if $w\nabla_{\partial_w}(P) \subset P$. \square

If P is a homogeneous opposite subspace, then there are two important linear operators θ_P and $\rho_P \in \text{End}(\Omega_f)$ defined as follows. According to Proposition 4.35 the isomorphism $\sigma : \Omega_f \rightarrow \widehat{\mathcal{H}}_f^{(0)} \cap wP$ extends $\mathbb{C}((w))$ -linearly to an isomorphism

$$\widehat{\sigma}_P : \Omega_f((w)) \rightarrow \widehat{\mathcal{H}}_f.$$

Note that part c) of Proposition 4.35 can be stated equivalently as

$$K_f(\widehat{\sigma}_P(\varphi_1), \widehat{\sigma}_P(\varphi_2)) = J_f(\varphi_1, \varphi_2^*)w^{n+1}, \quad \varphi_1, \varphi_2 \in \Omega_f((w)),$$

where $*$ is the involution of $\Omega_f((w))$ induced by $w \mapsto -w$ and J_f is the classical residue pairing on Ω_f .

LEMMA 4.37. *If P is a homogeneous opposite subspace, then under the isomorphism $\widehat{\sigma}_P$ the Gauss–Manin connection takes the form*

$$\nabla_{\partial_w} = \frac{\partial}{\partial w} - w^{-2}\rho - w^{-1}\theta_P,$$

for some linear endomorphisms θ_P and $\rho \in \text{End}(\Omega_f)$. Moreover, ρ is induced by multiplication by f , so in particular, it is independent of P .

PROOF. We have to prove that if $\omega \in \widehat{\mathcal{H}}_f^{(0)} \cap wP$, then

$$w\nabla_{\partial_w}\omega = w^{-1}\omega_1 + \omega_2,$$

for some uniquely determined $\omega_1, \omega_2 \in \widehat{H}_f^{(0)} \cap wP$. If this is proved, then we can simply set $\rho(\omega) := -\omega_1$ and $\theta_P(\omega) = -\omega_2$. Let us prove our claim. Since P is $w\nabla_{\partial_w}$ -invariant. We get that $w\nabla_{\partial_w}\omega \in wP$. Using the decomposition $\widehat{\mathcal{H}}_f = \widehat{\mathcal{H}}_f^{(0)} \oplus P$, we can decompose uniquely $w\nabla_{\partial_w}\omega = \widetilde{\omega}_1 + \widetilde{\omega}_2$, where $\widetilde{\omega}_1 \in \widehat{\mathcal{H}}_f^{(0)}$ and $\widetilde{\omega}_2 \in P$. Note that $w^{-1}P \subset P$ implies that $P \subset wP$, so $\widetilde{\omega}_1 = w\nabla_{\partial_w}\omega - \widetilde{\omega}_2 \in wP$, that is, $\widetilde{\omega}_1 \in \widehat{\mathcal{H}}_f^{(0)} \cap wP$. Note that $w\widetilde{\omega}_2 = w^2\nabla_{\partial_w}\omega - w\widetilde{\omega}_1 \in \widehat{\mathcal{H}}_f^{(0)} \cap wP$. Therefore, we can set $\omega_1 := w\widetilde{\omega}_2$ and $\omega_2 := \widetilde{\omega}_1$.

The statement about the endomorphism ρ can be proved as follows. Suppose that $\omega = \sigma(\varphi)$, then $\omega = \varphi \pmod{w\widehat{\mathcal{H}}_f^{(0)}}$. Recalling the definition of the Gauss–Manin connection, we get

$$-w^{-1}\rho(\varphi) - \theta_P(\varphi) = w\nabla_{\partial_w}\omega = -w^{-1}f\varphi \pmod{\widehat{\mathcal{H}}_f^{(0)}}.$$

The above identity implies that $\rho(\varphi) = f\varphi \pmod{w\widehat{\mathcal{H}}_f^{(0)}}$, which is exactly what we have to prove. \square

Let us introduce the notion of a *formal primitive form*. Let us define

$$\widehat{F} \in \Gamma(Z_0, \mathcal{O}_{Z_0})[[t_1, \dots, t_m]]$$

by taking the Taylor series expansion of F in the variables t_1, \dots, t_m . More precisely, let $\{U_\alpha\}$ be an open covering of Z_0 , such that, for every α there exists a sufficiently small open neighborhood V_α of t° in S , such that, $U_\alpha \times V_\alpha \subset Z$ and t_1, \dots, t_m are still holomorphic coordinates on V_α . Then $F|_{U_\alpha \times V_\alpha}$ is a holomorphic functions, so by taking the Taylor series expansion in t_1, \dots, t_m for each fixed $x \in U_\alpha$, we obtain an element of $\Gamma(U_\alpha, \mathcal{O}_{Z_0})[[t_1, \dots, t_m]]$. Clearly, these local expansions glue and define an element \widehat{F} . Put

$$\widehat{\mathcal{H}}_{\widehat{F}} := H^{n+1} \left(\Gamma(Z_0, \Omega_{Z_0}^\bullet)((w))[[t_1, \dots, t_m]], wd + d\widehat{F} \wedge \right)$$

and

$$\widehat{\mathcal{H}}_{\widehat{F}}^{(0)} := H^{n+1} \left(\Gamma(Z_0, \Omega_{Z_0}^\bullet)[[w, t_1, \dots, t_m]], wd + d\widehat{F} \wedge \right).$$

Note that the above cohomology groups are obtained by applying our construction of sheaves of formal oscillatory integrals to the case of a formal family of functions $(\widehat{F}, \widehat{p} : \widehat{Z} \rightarrow \widehat{S})$, where the total space is

$$\widehat{Z} := Z_0, \quad \mathcal{O}_{\widehat{Z}} = \mathcal{O}_{Z_0}[[t_1, \dots, t_m]],$$

the base is

$$\widehat{S} := \{t^\circ\}, \quad \mathcal{O}_{\widehat{S}} := \mathbb{C}[[t_1, \dots, t_m]],$$

and the projection \widehat{p} is just the contraction map $Z_0 \rightarrow \{t^\circ\}$. Furthermore, the definitions of the Gauss–Manin connection and the higher-residue pairing apply also in the formal settings, i.e., we have connection operators

$$\nabla_\xi : \widehat{\mathcal{H}}_{\widehat{F}} \rightarrow \widehat{\mathcal{H}}_{\widehat{F}},$$

where

$$\xi \in \mathcal{O}_{\widehat{S}}((w)) \frac{\partial}{\partial w} + \mathcal{O}_{\widehat{S}}((w)) \frac{\partial}{\partial t_1} + \dots + \mathcal{O}_{\widehat{S}}((w)) \frac{\partial}{\partial t_m},$$

and higher-residue pairing

$$K_{\widehat{F}} : \widehat{\mathcal{H}}_{\widehat{F}}^{(0)} \times \widehat{\mathcal{H}}_{\widehat{F}}^{(0)} \rightarrow \mathcal{O}_{\widehat{S}}[[w]]w^{n+1}.$$

The notion of a primitive form $\omega \in \widehat{\mathcal{H}}_{\widehat{F}}^{(0)}$ is defined as before. Theorem 4.33 implies that if $\omega \in \widehat{\mathcal{H}}_{\widehat{F}}^{(0)}$ is a primitive form of homogeneous degree r , then \widehat{S} is a formal Frobenius manifold of conformal dimension $D = n + 1 - 2r$.

Suppose now that (P, φ) is a pair of a homogeneous opposite subspace $P \subset \widehat{\mathcal{H}}_f$ and $\varphi \in \Omega_f$ is a holomorphic volume form, such that, $\theta_P(\varphi) = -r\varphi$. We would like to construct a primitive form $\omega \in \widehat{\mathcal{H}}_{\widehat{F}}^{(0)}$. The first step is to extend the opposite subspace P to a formal family of opposite subspaces.

LEMMA 4.38. a) *The map $\omega \mapsto e^{(F-f)/w} \omega$ induces an isomorphism*

$$\widehat{\mathcal{H}}_{\widehat{F}} \cong \widehat{\mathcal{H}}_f[[t_1, \dots, t_m]].$$

b) *The isomorphism in a) is compatible with the higher-residue pairing, that is,*

$$K_{\widehat{F}}(\omega_1, \omega_2) = K_f(e^{(F-f)/w} \omega_1, e^{(F-f)/w} \omega_2).$$

c) *The isomorphism in a) is compatible with the Gauss–Manin connection, that is,*

$$\begin{aligned} w \nabla_{\partial/\partial w} e^{(F-f)/w} \omega &= e^{(F-f)/w} w \nabla_{\partial/\partial w} \omega \\ w \frac{\partial}{\partial t_a} e^{(F-f)/w} \omega &= e^{(F-f)/w} w \nabla_{\partial/\partial t_a} \omega. \end{aligned}$$

PROOF. a) Multiplication by $e^{(F-f)/w}$ defines an isomorphism

$$\Gamma(Z_0, \Omega_{Z_0}^\bullet)((w))[[t_1, \dots, t_m]] \rightarrow \Gamma(Z_0, \Omega_{Z_0}^\bullet)((w))[[t_1, \dots, t_m]]$$

that intertwines the following two differentials:

$$(wd + df \wedge) e^{(F-f)/w} = e^{(F-f)/w} (wd + d\widehat{F} \wedge).$$

Passing to cohomology we get the isomorphism stated in part a).

b) By definition

$$(4.39) \quad K_{\widehat{F}}(\omega_1, \omega_2) = C_n \sum_{\xi \in \text{Crit}(f)} \int_{\partial \bar{U}_\xi} d_{\widehat{F}}^{-1}(\omega_1) \wedge \omega_2^*,$$

where $\overline{U}_\xi \subset Z_0$ is a sufficiently small ball in Z_0 with center at ξ and $C_n = (-1)^n (2\pi\mathbf{i})^{-n-1}$. On the other hand, if $\eta_1 = d_{\widehat{F}}^{-1}(\omega_1)$, that is, $d_{\widehat{F}}(\eta_1) = \omega_1$, then

$$e^{(F-f)/w}\omega_1 = d_f \left(e^{(F-f)/w}\eta_1 \right) \text{ and } \left(e^{(F-f)/w}\omega_2 \right)^* = e^{-(F-f)/w}\omega_2^*.$$

Therefore,

$$d_{\widehat{F}}^{-1}(\omega_1) \wedge \omega_2^* = d_f^{-1} \left(e^{(F-f)/w}\omega_1 \right) \wedge \left(e^{(F-f)/w}\omega_2 \right)^*.$$

Substituting in formula (4.39) we get $K_{\widehat{F}}(\omega_1, \omega_2) = K_f(e^{(F-f)/w}\omega_1, e^{(F-f)/w}\omega_2)$.

c) The proof of part c) is straightforward and it will be omitted. \square

Let $\omega^\circ \in \Gamma(Z_0, \Omega_{Z_0}^{n+1})[[w]]$ be a holomorphic form representing the cohomology class $\sigma(\varphi) \in \widehat{\mathcal{H}}_f^{(0)} \cap wP$. Note that the class of ω° in $\widehat{\mathcal{H}}_{\widehat{F}}^{(0)}$ is a holomorphic volume form and hence the corresponding period map $\mathcal{T}_{\widehat{S}}[[w]] \rightarrow \widehat{\mathcal{H}}_{\widehat{F}}^{(0)}$ is an isomorphism. Using that $\widehat{\mathcal{H}}_f^{(0)} = \widehat{\mathcal{H}}_{\widehat{F}}^{(0)} / (t_1, \dots, t_m) \widehat{\mathcal{H}}_{\widehat{F}}^{(0)}$ we get that there are uniquely determined $\omega_i^\circ \in \widehat{\mathcal{H}}_f^{(0)} \cap wP$ ($1 \leq i \leq m$), such that,

$$(4.40) \quad \frac{\partial F}{\partial t_i} \omega^\circ = \omega_i^\circ \pmod{w \widehat{\mathcal{H}}_f^{(0)}, t_1, \dots, t_m},$$

where the LHS should be viewed as an element in $\widehat{\mathcal{H}}_{\widehat{F}}^{(0)}$. Moreover, since $\frac{\partial}{\partial t_i}$ form a $\mathbb{C}[[w]]$ -basis of $\mathcal{T}_{\widehat{S}}[[w]]$ and the period map is an isomorphism, we get that ω_i° ($1 \leq i \leq m$) form a $\mathbb{C}[[w]]$ -basis of $\widehat{\mathcal{H}}_f^{(0)}$ and $\mathbb{C}((w))$ -basis of $\widehat{\mathcal{H}}_f$. Therefore

$$e^{(F-f)/w} \frac{\partial F}{\partial t_j} \omega^\circ = \sum_{i=1}^m \omega_i^\circ E_{ij}(t, w),$$

where the above identity should be viewed in $\widehat{\mathcal{H}}_f[[t_1, \dots, t_m]]$ (see Lemma 4.38, a)) and $E_{ij}(t, w) \in \mathbb{C}((w))[[t_1, \dots, t_m]]$. Let $E(t, w)$ be $m \times m$ matrix whose entries are $E_{ij}(t, w)$. Note that by construction $E(0, w) = 1 + O(w)$, therefore, the matrix $E(t, w)$ has a Birkhof factorization $E(t, w) = T(t, w)A(t, w)^{-1}$, where $T(t, w) = 1 + O(w^{-1})$ and $A(t, w) = A_0(t) + A_1(t)w + \dots$, where $A_0(t)$ is an invertible matrix with entries in $\mathbb{C}[[t_1, \dots, t_m]]$. Let us define $\omega_j \in \widehat{\mathcal{H}}_{\widehat{F}}^{(0)}$ to be the cohomology class of the form

$$(4.41) \quad \sum_{i=1}^m \frac{\partial F}{\partial t_i} \omega^\circ A_{ij}(t, w),$$

where $A_{ij}(t, w)$ denotes the (i, j) -entry of the matrix $A(t, w)$.

REMARK 4.39. The notion of an opposite subspace $P \subset \widehat{\mathcal{H}}_f$ extends naturally to the formal completion $\widehat{\mathcal{H}}_{\widehat{F}}$. The forms ω_j that we have just constructed span an opposite subspace, that is,

$$\bigoplus_{j=1}^m \omega_j \mathcal{O}_{\widehat{S}}[[w^{-1}]]w^{-1} \subset \widehat{\mathcal{H}}_{\widehat{F}}$$

is an opposite subspace.

Let us define the vector fields $\delta_j := \sum_{i=1}^m A_{ij}(t, 0) \frac{\partial}{\partial t_i}$ and let $C_a(t) = (C_{aij}(t))_{1 \leq i, j \leq m}$ be the matrix of Frobenius multiplication by $\frac{\partial}{\partial t_a}$ in the basis δ_j , that is,

$$\frac{\partial}{\partial t_a} \bullet \delta_j = \sum_{i=1}^m C_{aij}(t) \delta_i.$$

LEMMA 4.40. *Let $\omega_j \in \widehat{\mathcal{H}}_{\widehat{F}}^{(0)}$ be the cohomology classes constructed above from the pair (P, φ) . Then*

$$\begin{aligned} w \nabla_{\partial/\partial t_a} \omega_j &= \sum_{k=1}^m \omega_k C_{akj}(t), \\ (w \nabla_{\partial/\partial w} + \nabla_E) \omega_j &= - \sum_{k=1}^m \omega_k \theta_{kj}, \end{aligned}$$

where θ_{kj} are the entries of the matrix of the linear operator θ_P , that is,

$$\sigma \circ \theta_P \circ \sigma^{-1}(\omega_j^\circ) = \sum_{k=1}^m \theta_{kj} \omega_k^\circ.$$

PROOF. Using Lemma 4.38, part c), we get

$$w \frac{\partial}{\partial t_a} e^{(F-f)/w} \omega_j = e^{(F-f)/w} w \nabla_{\partial/\partial t_a} \omega_j = e^{(F-f)/w} \left(\sum_{i=1}^m \omega_i C_{aij}(t) + \dots \right),$$

where the dots stand for terms in $w \widehat{\mathcal{H}}_{\widehat{F}}^{(0)}$. For the leading term in the expression after the 2nd equality we used that modulo terms in $w \widehat{\mathcal{H}}_{\widehat{F}}^{(0)}$, the form $\delta_j(F) \omega^\circ$ represents the cohomology class ω_j .

On the other hand, according to our construction $e^{(F-f)/w} \omega_j = \sum_{i=1}^m \omega_i^\circ T_{ij}(t, w)$. Therefore,

$$w \frac{\partial}{\partial t_a} e^{(F-f)/w} \omega_j = e^{(F-f)/w} \sum_{i=1}^m \omega_i (T^{-1} w \partial_{t_a} T)_{ij}.$$

Therefore, $T(t, w)^{-1} w \partial_{t_a} T(t, w) = C_a(t) + \dots$, where the dots involve only positive powers of w and hence they must vanish, because the LHS is a power series in w^{-1} , that is, $w \partial_{t_a} T(t, w) = T(t, w) C_a(t)$. The first differential equation that we have to prove follows.

Let us prove the 2nd differential equation. Recalling again Lemma 4.38, part c), we get

$$(w \nabla_{\partial/\partial w} + E) e^{(F-f)/w} \omega_j = e^{(F-f)/w} (w \nabla_{\partial/\partial w} + \nabla_E) \omega_j = e^{(F-f)/w} \sum_{i=1}^m \omega_i U_{ij}(t, w),$$

where $U_{ij}(t, w) \in \mathbb{C}[[w, t_1, \dots, t_m]]$. On the other hand, using again that $e^{(F-f)/w} \omega_j = \sum_{i=1}^m \omega_i^\circ T_{ij}(t, w)$, we get

$$(4.42) \quad (w \nabla_{\partial/\partial w} + E) e^{(F-f)/w} \omega_j = \sum_{i=1}^m \left((w \nabla_{\partial/\partial w} \omega_i^\circ) T_{ij} + \omega_i^\circ (w \partial_w + E) T_{ij} \right).$$

Recalling Lemma 4.37, we get

$$w\nabla_{\partial/\partial w}\omega_i^\circ = -\sum_{k=1}^m \omega_k^\circ(\rho_{ki}w^{-1} + \theta_{ki}),$$

where ρ_{ki} are the entries of the matrix of the linear operator ρ with respect to the basis of Ω_f induced by ω_i° ($1 \leq i \leq m$). Substituting the above formula in (4.42) and using that $\omega_k^\circ = e^{(F-f)/w} \sum_{j=1}^m \omega_j(T^{-1})_{jk}$, we get

$$U(t, w) = -T(t, w)^{-1}(\rho w^{-1} + \theta)T(t, w) + T(t, w)^{-1}(w\partial_w + E)T(t, w)$$

Let us expand $U(t, w) = \sum_{k=0}^{\infty} U_k(t)w^k$ and compare the coefficients in front of the powers of w . Since $T(t, w) = 1 + O(w^{-1})$ we get that $U_0(t) = -\theta$ and $U_k(t) = 0$ for $k > 0$. \square

Let us summarize our construction. Starting with a pair (P, φ) of a homogeneous opposite subspace $P \subset \widehat{\mathcal{H}}_f$ and a holomorphic volume form $\varphi \in \Omega_f$, we have introduced the following objects:

- (i) A holomorphic form $\omega^\circ \in \Gamma(Z_0, \Omega_{Z_0}^{n+1})[[w]]$, such that, the cohomology class of ω° in $\widehat{\mathcal{H}}_f^{(0)}$ coincides with $\sigma(\varphi)$.
- (ii) Cohomology classes $\omega_i^\circ \in \widehat{\mathcal{H}}_f^{(0)} \cap wP$ ($1 \leq i \leq m$) uniquely determined by the condition (4.40).
- (iii) Cohomology classes $\omega_j \in \widehat{\mathcal{H}}_{\widehat{F}}^{(0)}$ ($1 \leq j \leq m$) and a matrix series $T(t, w) = 1 + \sum_{k=1}^{\infty} T_k(t)w^{-k}$, such that, $T(0, w) = 1$ and

$$e^{(F-f)/w}\omega_j = \sum_{i=1}^m \omega_i^\circ T_{ij}(t, w).$$

Since ω_i° form a basis of $\widehat{\mathcal{H}}_f^{(0)} \cap wP$, we have $\sigma(\varphi) =: \sum_{i=1}^m c_i \omega_i^\circ$ for some uniquely determined constants $c_i \in \mathbb{C}$.

THEOREM 4.41. *Let (P, φ) be as above and $\theta_P(\varphi) = -r\varphi$. Then the cohomology class $\omega := \sum_{i=1}^m c_i \omega_i \in \widehat{\mathcal{H}}_{\widehat{F}}^{(0)}$ is a primitive form of homogeneous degree r .*

PROOF. Since ω coincides with $\sigma(\varphi)$ modulo terms in $(t_1, \dots, t_m)\widehat{\mathcal{H}}_{\widehat{F}}^{(0)}$ and φ is a holomorphic volume form, we get that ω is a holomorphic volume form. Therefore, the period map

$$\mathcal{T}_{\widehat{S}}[[w]] \rightarrow \widehat{\mathcal{H}}_{\widehat{F}}^{(0)}, \quad \partial/\partial t_i \mapsto w\nabla_{\partial/\partial t_i}\omega$$

is an isomorphism. Both $w\nabla_{\partial/\partial t_i}\omega$ ($1 \leq i \leq m$) and ω_i ($1 \leq i \leq m$) are $\mathbb{C}[[w, t_1, \dots, t_m]]$ -bases of $\widehat{\mathcal{H}}_{\widehat{F}}^{(0)}$ and modulo $(t_1, \dots, t_m, w)\widehat{\mathcal{H}}_{\widehat{F}}^{(0)}$ the two bases coincide. Using Lemma 4.40, we get

$$w\nabla_{\partial/\partial t_i}\omega = \sum_{j=1}^m c_j w\nabla_{\partial/\partial t_i}\omega_j = \sum_{j,k=1}^m \omega_k C_{ikj}(t) c_j =: \sum_{k=1}^m \omega_k R_{ki}(t),$$

where $R_{ki}(t) := \sum_{j=1}^m C_{ikj}(t)c_j$. Let $R(t)$ be the matrix with entries $R_{ki}(t)$. Note that $R(0) = 1$, so $R(t)$ is an invertible matrix. Therefore, in the case of ω , the first three axioms of a primitive form in Definition 4.32 are equivalent to the following two conditions:

$$(4.43) \quad K_{\widehat{F}}^{(p)}(\omega_i, \omega_j) = 0 \quad \forall p > 0$$

and

$$(4.44) \quad K_{\widehat{F}}^{(p)}((w^2 \nabla_{\partial/\partial w} + w \nabla_E) \omega_i, \omega_j) = 0 \quad \forall p > 1.$$

Condition (4.44) follow from condition (4.43) thanks to Lemma 4.40. In order to prove (4.43), we first recall Lemma 4.38, part b) and we get

$$K_{\widehat{F}}(\omega_i, \omega_j) = \sum_{a,b}^m K_f(\omega_a^\circ, \omega_b^\circ) T_{ai}(t, w) T_{bj}(t, -w).$$

However, recalling Proposition 4.35, part c) and using that $T(t, w) = 1 + O(w^{-1})$, we get that the RHS in the above equality has the form $w^{n+1}(J_f(\omega_i, \omega_j) + O(w^{-1}))$. On the other hand, the LHS does not contain terms of type $w^{n+1}O(w^{-1})$, so these terms must vanish, that is,

$$(4.45) \quad K_{\widehat{F}}(\omega_i, \omega_j) = J_f(\omega_i^\circ, \omega_j^\circ) w^{n+1}.$$

This proves (4.43). Let us prove that ω satisfies axiom (iv) in Definition 4.32. Recalling Lemma 4.40 we get

$$(4.46) \quad (w \nabla_{\partial/\partial w} + \nabla_E) \omega = - \sum_{i,k=1}^m \omega_k \theta_{ki} c_i.$$

Let us consider the image of the above identity under the quotient map

$$\widehat{\mathcal{H}}_{\widehat{F}}^{(0)} \rightarrow \widehat{\mathcal{H}}_{\widehat{F}}^{(0)} / (w, t_1, \dots, t_m) \widehat{\mathcal{H}}_{\widehat{F}}^{(0)}.$$

The image of $w \nabla_{\partial/\partial w} \omega$ is $-\theta_P(\varphi) - \rho(\varphi) w^{-1}$, while the image of $\nabla_E \omega$ is $\rho(\varphi)$. Therefore, the image of the LHS is $-\theta_P(\varphi) = r\varphi$. Comparing with the image of the RHS we get

$$r\varphi = - \sum_{k=1}^m \omega_k^\circ (\theta_{k1} c_1 + \dots + \theta_{km} c_m) \text{ mod } w \widehat{\mathcal{H}}_f^{(0)}.$$

Applying the section $\sigma : \Omega_f \rightarrow \widehat{\mathcal{H}}_f^{(0)} \cap wP$ and recalling that $\sigma(\varphi) = \sum_{k=1}^m \omega_k^\circ c_k$, we get $\sum_{i=1}^m \theta_{ki} c_i = -rc_k$, which together with (4.46) implies that ω satisfies axiom (iv). The last axiom follows immediately from Lemma 4.40. Indeed, since e is the unit of the Frobenius multiplication we have $w \nabla_e \omega_j = \omega_j \Rightarrow w \nabla_e \omega = \omega$. \square

Weighted homogeneous singularities

This chapter is an application of the theory developed in Chapter 4 to weighted homogeneous singularities. Namely, we will classify the Frobenius structures on the space of miniversal deformations of a weighted homogeneous singularity that can be constructed via the theory of primitive forms. Furthermore, we will give a geometric interpretation of the solutions of the 2nd structure connection in terms of period integrals.

5.1. Families of weighted homogeneous singularities

Let $X = \mathbb{C}^{n+1}$ be the standard complex vector space with coordinates x_0, x_1, \dots, x_n . Let $c = (c_0, c_1, \dots, c_n)$ be a tuple of rational numbers, such that, $0 < c_i \leq \frac{1}{2}$. By assigning weight c_i to each variable x_i we turn the polynomial ring $\mathbb{C}[x_0, x_1, \dots, x_n]$ into a \mathbb{Q} -graded algebra. More precisely, if $\kappa = (k_0, k_1, \dots, k_n)$ is a $(n+1)$ -tuple of non-negative integers, then we define the monomial $x^\kappa := x_0^{k_0} x_1^{k_1} \dots x_n^{k_n}$. The number $c \cdot \kappa := c_0 k_0 + c_1 k_1 + \dots + c_n k_n$ is called the *weight* of x^κ . Then we have

$$\mathbb{C}[x_0, x_1, \dots, x_n] = \bigoplus_{d \in \mathbb{Q}_{\geq 0}} \mathbb{C}[x_0, x_1, \dots, x_n]_d,$$

where $\mathbb{C}[x_0, x_1, \dots, x_n]_d$ is the subspace of polynomials spanned by monomials of weight d . Equivalently, a polynomial $f \in \mathbb{C}[x_0, x_1, \dots, x_n]_d$ if and only if

$$f(\lambda^{c_0} x_0, \lambda^{c_1} x_1, \dots, \lambda^{c_n} x_n) = \lambda^d f(x_0, x_1, \dots, x_n)$$

for every real $\lambda > 0$. The elements of $\mathbb{C}[x_0, x_1, \dots, x_n]_d$ are called *weighted homogeneous polynomials* of weight d .

Suppose now that f is a weighted homogeneous polynomial of weight 1, such that, 0 is an isolated critical point of f . Slightly abusing the terminology, we will refer to such a polynomial f as a *weighted homogeneous singularity*. The algebra

$$(5.1) \quad H_f := \mathbb{C}[x_0, x_1, \dots, x_n] / (f_{x_0}, f_{x_1}, \dots, f_{x_n}), \quad f_{x_i} := \frac{\partial f}{\partial x_i},$$

is called the *local algebra* of f . The condition that 0 is an isolated critical point is equivalent to the condition that the local algebra H_f is finite dimensional as a \mathbb{C} -vector space. The dimension μ of H_f is called the *multiplicity* of the critical point. Note that since $(f_{x_0}, f_{x_1}, \dots, f_{x_n})$ is a homogeneous ideal, the \mathbb{Q} -grading of $\mathbb{C}[x_0, x_1, \dots, x_n]$ induces a \mathbb{Q} -grading of H_f . Let us fix a set of monomials x^κ ($\kappa \in \mathcal{B}$) that represent a basis of H_f . Following physicists terminology, we split the basis \mathcal{B} into 3 groups \mathcal{B}_{rel} , \mathcal{B}_{mar} , and \mathcal{B}_{irr} depending on whether the weight of the monomial x^κ is respectively < 1 , $= 1$, or > 1 . The monomials x^κ with $\kappa \in \mathcal{B}_{\text{rel}}$ are called *relevant* and the function

$$F(x, t) = f(x) + \sum_{\kappa \in \mathcal{B}_{\text{rel}}} t_\kappa x^\kappa$$

is called a *relevant deformation* of f . Here the deformation parameters t_κ are allowed to be arbitrary complex numbers, that is, $t = (t_\kappa)$ is a point in $S_{\text{rel}} := \mathbb{C}^{\mathcal{B}_{\text{rel}}}$. Our next goal is to prove the following proposition.

PROPOSITION 5.1. *Let F be a relevant deformation of a weighted homogeneous singularity f . Put $Z_{\text{rel}} = X \times S_{\text{rel}}$. Then $(F, p : Z_{\text{rel}} \rightarrow S_{\text{rel}})$ is a family of functions in the sense of Definition 4.1.*

PROOF. First, let us prove that $p|_{C_F} : C_F \rightarrow S_{\text{rel}}$ is a proper map. It is sufficient to prove that if $K \subset S_{\text{rel}}$ is a bounded subset, then $p^{-1}(K) \cap C_F$ is bounded. Let $\mathbb{D} = \{x \in \mathbb{C}^{n+1} \mid |x_i| \leq 1 \forall i\}$ be the compact polydisk. Given a critical point $(\xi, t) \in C_F$, let us denote by r the maximal number in the set $\{|\xi_0|^{1/c_0}, |\xi_1|^{1/c_1}, \dots, |\xi_n|^{1/c_n}\}$. Let $d_\kappa := c \cdot \kappa$ be the weight of the monomial x^κ . Then, since f_{x_i} is weighted homogeneous of weight $1 - c_i$, we have

$$r^{c_i-1} F_{x_i}(\xi, t) = f_{x_i}(y_0, y_1, \dots, y_n) + \sum_{\kappa \in \mathcal{B}_{\text{rel}}} k_i t_\kappa r^{d_\kappa-1} (y_0)^{k_0} \dots y_{i-1}^{k_{i-1}} y_i^{k_i-1} y_{i+1}^{k_{i+1}} \dots y_n^{k_n},$$

where $y_i := r^{-c_i} \xi_i$ ($0 \leq i \leq n$). The LHS must be 0, because (ξ, t) is a critical point. Since $t \in K$ and K is bounded, we get that

$$(5.2) \quad |f_{x_i}(y_0, y_1, \dots, y_n)| \leq \text{const}_K \sum_{\kappa \in \mathcal{B}_{\text{rel}}} r^{d_\kappa-1},$$

where the constant const_K depends only on K and used that $|y_i| \leq 1$ for all i . Note that according to our definition of r , we have $|y_i| = 1$ for some i , that is, the point $y = (y_0, y_1, \dots, y_n)$ belongs to the boundary $\partial\mathbb{D}$. If the set $p^{-1}(K) \cap C_F$ is not bounded, then we will be able to find a sequence $(\xi^{(l)}, t^{(l)})$ of points in it, such that, $r^{(l)} := \max(|\xi_i^{(l)}|^{1/c_i}) \rightarrow \infty$. Let $y_i^{(l)} := (r^{(l)})^{-c_i} \xi_i^{(l)}$. Then $y^{(l)} := (y_0^{(l)}, \dots, y_n^{(l)})$ is a sequence of points in the compact set $\partial\mathbb{D}$. Passing to a subsequence if necessary, we may assume that $y^{(l)} \rightarrow y^\circ$. Using the estimates (5.2) and the fact that $d_\kappa < 1$ ($\because x^\kappa$ is a relevant monomial), we get that y° must be a critical point of f . But $y^\circ \neq 0$, because y° is on the boundary of the polydisk \mathbb{D} . Using the homogeneity of f again we get that all points on the complex line $\mathbb{C}y^\circ$ are critical points of f . This is a contradiction with the requirement that 0 is an isolated critical point of f .

It remains to prove that if $t^\circ \in S_{\text{rel}}$ is a fixed point, then the function $f_{t^\circ} = F|_{X \times \{t^\circ\}}$ has finitely many critical points. To begin with, let us prove that

$$(5.3) \quad A := \mathbb{C}[x_0, x_1, \dots, x_n] / (\partial_{x_0} f_{t^\circ}, \dots, \partial_{x_n} f_{t^\circ})$$

is a finite dimensional vector space. Suppose that x^α is a monomial of weight $c \cdot \alpha \geq m$, where m is the maximal possible weight of a monomial x^κ with $\kappa \in \mathcal{B}$. Then we have

$$x^\alpha = \sum_{\kappa \in \mathcal{B}} a_\kappa x^\kappa + \sum_{i=0}^n g_i(x) f_{x_i},$$

where a_κ are complex numbers and g_i are weighted homogeneous polynomials of weight $c \cdot \alpha - 1 + c_i$. Substituting $f_{x_i} = F_{x_i} - \sum_{\kappa \in \mathcal{B}_{\text{rel}}} t_\kappa \partial_{x_i}(x^\kappa)$, we get that x^α , modulo the ideal $(F_{x_0}, \dots, F_{x_n})$ coincides with

$$\sum_{\kappa \in \mathcal{B}} a_\kappa x^\kappa - \sum_{i=0}^n \sum_{\kappa \in \mathcal{B}_{\text{rel}}} t_\kappa g_i(x) \partial_{x_i}(x^\kappa).$$

Note that $g_i(x)\partial_{x_i}(x^\kappa)$ is homogeneous of weight $c\cdot\alpha - 1 + c\cdot\kappa$ and since $\kappa \in \mathcal{B}_{\text{rel}}$, $c\cdot\kappa < 1$, so the weight of $g_i(x)x^\kappa$ is smaller than the weight of x^α . Repeating this procedure for the monomials of $g_i(x)x^\kappa$ whose weight is $> m$ and so on we get that x^α can be expressed in the form

$$\sum_{\beta} a_{\beta}(t)x^{\beta} + \sum_{i=0}^n G_i(x,t)F_{x_i}(x,t),$$

where the 1st sum is over all β whose weight $c\cdot\beta \leq m$. Specializing $t = t^\circ$ we get that the set of monomials x^β of weight $\leq m$ represents a set of elements in the local algebra A (see (5.3)) that span A as a \mathbb{C} -vector space. This proves that A is a finite dimensional vector space and hence it is an Artin algebra. On the other hand, the critical points of f_{t° correspond to the maximal ideals of A , so it remains only to recall that an Artin algebra has finitely many maximal ideals. \square

Our next goal is to construct tame deformations of f (see Definition 4.18). Put

$$\mathcal{B}_{\text{tame}} := \{\kappa \in \mathcal{B} \mid c\cdot\kappa < 1 - c_i \forall i\}.$$

Note that $\mathcal{B}_{\text{tame}} \subseteq \mathcal{B}_{\text{rel}}$. We will refer to $S_{\text{tame}} = \mathbb{C}^{\mathcal{B}_{\text{tame}}} \subset S_{\text{rel}}$ as the space of *tame* deformations. Let us equip \mathbb{C}^{n+1} with the standard Kähler metric g , that is, the complexification of g is given by

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial \bar{x}_j}\right) = \delta_{i,j}/2, \quad g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = g\left(\frac{\partial}{\partial \bar{x}_i}, \frac{\partial}{\partial \bar{x}_j}\right) = 0.$$

Clearly g induces a relative Kähler metric on $Z_{\text{tame}} = X \times S_{\text{tame}}$ relative to S_{tame} . Slightly abusing the notation we will denote the relative Kähler metric by g too. After a short computation we get the following formulas for the relative gradient of a holomorphic function $F : Z_{\text{tame}} \rightarrow \mathbb{C}$ and its norm

$$\nabla(F) = 2 \sum_{i=0}^n F_{x_i} \frac{\partial}{\partial \bar{x}_i}, \quad \|\nabla(F)\|^2 = 2 \sum_{i=0}^n |F_{x_i}|^2.$$

PROPOSITION 5.2. *The family $(F, p : Z_{\text{tame}} \rightarrow S_{\text{tame}}, g)$ is Kähler-complete and tame (see Definitions 4.15 and 4.18).*

PROOF. The fact that the family is Kähler complete is clear (see Remark 4.16). The proof that it is tame is very similar to the proof of the properness of the projection $p|_{C_F}$ in Proposition 5.1. Let us recall the definition of tame from Definition 4.18. It is sufficient to check that the condition in the Definition is satisfied for $\xi = \frac{\partial}{\partial t_\alpha}$, where $\alpha \in \mathcal{B}_{\text{tame}}$. Furthermore, the main difficulty is to prove that the set $C_{\eta,\xi}(V)$ is bounded for every bounded subset $V \subset S_{\text{tame}}$. Let us concentrate on proving this and leave the rest of the details as an exercise. Suppose that $(x, t) \in C_{\eta,\xi}(V)$. Since $\sqrt{2}|F_{x_i}| \leq \|\nabla(F)\|$, we get

$$\sqrt{2} \left| f_{x_i}(x) + \sum_{\kappa \in \mathcal{B}_{\text{tame}}} t_\kappa k_i x^{\kappa - e_i} \right| \leq \eta + |x|^\alpha,$$

where k_i is the i th component of κ and e_i is a multi-index whose components are all 0, except for the i th one, which is 1. Using the triangle inequality, we get

$$|f_{x_i}(x)| \leq \eta_1 + \eta_2 |x|^\alpha + \eta_3 \sum_{\kappa \in \mathcal{B}_{\text{tame}}} |x|^{\kappa - e_i},$$

where $\eta_1 = \eta/\sqrt{2}$, $\eta_2 = 1/\sqrt{2}$, and η_3 are constants whose value is not important – η_3 is chosen such that $|k_i t_\kappa| \leq \eta_3$ for all $t \in V$. Let us write $x_i = r^{c_i} y_i$, where $y = (y_0, \dots, y_n)$ is a point on the boundary of the polydisk $\mathbb{D} = \{y \in \mathbb{C}^{n+1} \mid |y_i| \leq 1\}$. Using that f_{x_i} is homogeneous of weight $1 - c_i$, we get

$$|f_{x_i}(y)| \leq r^{c_i-1} \eta_1 + r^{c_i \alpha + c_i - 1} \eta_2 |y|^\alpha + \eta_3 \sum_{\kappa \in \mathcal{B}_{\text{tame}}} r^{c_i \kappa - 1} |y|^{\kappa - e_i}.$$

Since α and κ are tame all the powers of r in the above formula are negative. In particular, the RHS converges to 0 when $r \rightarrow \infty$. Just like in the proof of Proposition 5.1, if we assume that $C_{\eta, \xi}(V)$ is not bounded, then we will be able to find a convergent sequence of points $y^{(l)}$ on the boundary of the polydisk \mathbb{D} , such that, the above inequality holds for $y = y^{(l)}$ and $r = r^{(l)}$, where $r^{(l)} \rightarrow \infty$ as $l \rightarrow \infty$. We get that the limit $y^\circ = \lim y^{(l)}$ is a critical point of f on the boundary of \mathbb{D} . Using that f is homogeneous, we get a contradiction with the requirement that 0 is an isolated critical point of f . \square

Our next goal is to determine the highest possible weight of a homogeneous element in the local algebra H_f . Let us denote by $\text{Hess}_f(x)$ the determinant of the *Hessian matrix* of f , that is, the matrix of size $(n+1) \times (n+1)$ whose (i, j) entry is $\frac{\partial^2 f}{\partial x_i \partial x_j}$ ($0 \leq i, j \leq n$). The determinant $\text{Hess}_f(x)$ is also known as the *Hessian* of f . Note that $\text{Hess}_f(x)$ is a weighted homogeneous polynomial of weight $D := \sum_{i=0}^n (1 - 2c_i)$. We would like to prove the following proposition.

PROPOSITION 5.3. *a) The maximal possible weight of a homogeneous element in H_f is D .*

b) The subspace of H_f consisting of elements of weight D is one dimensional and the class of $\text{Hess}_f(x)$ gives a basis.

The proof of the above proposition relies on constructing a Morse family deformation of f and using the non-degeneracy of the classical residue pairing. Let us first construct a Morse family. To begin with, let us assume that the weights c_i are ordered in an increasing order, that is, $c_0 \leq c_1 \leq \dots \leq c_n$. Let k be the unique integer number, such that, $c_{k-1} < \frac{1}{2}$ and $c_k = c_{k+1} = \dots = c_n = \frac{1}{2}$. Note that the weighted homogeneous polynomial f has the form

$$(5.4) \quad f(x) = \frac{1}{2} \sum_{i,j=k}^n a_{ij} x_i x_j + \sum_{i=k}^n f_i^{(1)}(x_0, \dots, x_{k-1}) x_i + f^{(2)}(x_0, \dots, x_{k-1}),$$

where $f_i^{(1)}$ and $f^{(2)}$ are weighted homogeneous polynomials of weights respectively $\frac{1}{2}$ and 1. The coefficients a_{ij} are constants independent of x satisfying $a_{ij} = a_{ji}$. Moreover, the matrix whose (i, j) entry is a_{ij} must be non-degenerate, otherwise the function f has non-isolated critical points of type $(0, \dots, 0, \xi_k, \dots, \xi_n)$.

REMARK 5.4. Note that after changing the coordinates of \mathbb{C}^{n+1} we can arrange that all $f_i^{(1)} = 0$ and that (a_{ij}) is the identity matrix. In other words, every weighted homogeneous singularity is R -equivalent to the direct sum of a Morse function and a weighted homogeneous singularity for which all weights are strictly less than $\frac{1}{2}$.

REMARK 5.5. The rank of the Hessian matrix of f at $x = 0$ is precisely $n + 1 - k$. The number k is also known as the *corank* of f .

LEMMA 5.6. *The classes of the variables x_0, \dots, x_{k-1} in H_f are linearly independent.*

PROOF. Suppose that $\phi(x) := \sum_{i=0}^{k-1} b_i x_i$ is a linear function whose class in H_f is 0, that is, there exist weighted homogeneous polynomials $g_i(x)$ ($0 \leq i \leq n$), such that, $\phi(x) = \sum_{i=0}^n g_i(x) f_{x_i}$. However, such an identity is impossible, because the weight of f_{x_i} is $1 - c_i \geq \frac{1}{2}$, while $\phi(x)$ is a sum of linear functions of homogeneous weight $< \frac{1}{2}$. \square

We claim that $F(x, t) = f(x) + \sum_{i=0}^{k-1} t_i x_i$ is a Morse family, where $t = (t_0, \dots, t_{k-1}) \in S_{\text{lin}} := \mathbb{C}^k$. Indeed, the critical set C_F is by definition the complex subspace of $Z_{\text{lin}} := \mathbb{C}^{n+1} \times S_{\text{lin}}$ defined by

$$C_F = \{(x, t) \in Z_{\text{lin}} \mid F_{x_0} = \dots = F_{x_n} = 0\}.$$

The first k equations, that is, $F_{x_i} = 0$ ($0 \leq i \leq k-1$) yield

$$t_i = f_{x_i} = \sum_{j=k}^n \partial_{x_i} f_j^{(1)}(x_0, \dots, x_{k-1}) x_j + \partial_{x_i} f^{(2)}(x_0, \dots, x_{k-1}),$$

while the remaining $n+1-k$ ones have the form

$$\sum_{j=k}^n a_{sj} x_j + f_s^{(1)}(x_0, \dots, x_{k-1}) = 0, \quad k \leq s \leq n.$$

The above equations can be solved uniquely for x_j ($k \leq j \leq n$) and t in terms of x_0, \dots, x_{k-1} . Therefore, the projection $(x, t) \mapsto (x_0, \dots, x_{k-1})$ induces an isomorphism of complex manifolds $C_F \cong \mathbb{C}^k$. In particular, C_F is a reduced complex space and hence the corresponding family F is Morse (see Section 4.2.4). In other words, with respect to x , $F(x, t)$ is a Morse function for generic $t \in S_{\text{lin}}$.

Let us prove Proposition 5.3. Recall that every family of functions is equipped with a non-degenerate residue pairing. Let us consider the classical residue pairing J_F for the family $(F, p : Z_{\text{lin}} \rightarrow S_{\text{lin}})$ (see Section 4.5.2). The residue pairing

$$(5.5) \quad J_F(dx, \text{Hess}_F dx) \in \mathcal{O}_{S_{\text{lin}}},$$

where $dx := dx_0 \wedge \dots \wedge dx_n$, is easy to compute. Indeed, for generic $t \in S_{\text{lin}}$, the function $F(x, t)$ is Morse as a function in x and the residue symbol

$$\text{Res}_{Z_\xi/C_F} \left[\text{Hess}_{f_t}(x) dx_0 \wedge \dots \wedge dx_n \right],$$

$$\frac{\partial f_t}{\partial x_0}, \dots, \frac{\partial f_t}{\partial x_n}$$

where $f_t(x) := F(x, t)$ and ξ is a critical point of f_t , can be computed by switching to Morse coordinates for f_t . Let $x = \phi(y)$ be the change to Morse coordinates, that is, $f \circ \phi(y) = y_0^2 + \dots + y_n^2$. The transformation law of the residue pairing implies that the above residue symbol takes the form

$$(5.6) \quad \frac{1}{(2\pi\mathbf{i})^{n+1}} \int_{|y_0|=\dots=|y_n|=\epsilon} \text{Hess}_{f_t}(\phi(y)) \det^2(\partial\phi/\partial y) \frac{dy_0 \wedge \dots \wedge dy_n}{2y_0 \dots 2y_n}.$$

Let us differentiate the identity $f \circ \phi(y) = y_0^2 + \dots + y_n^2$ with respect to y_k and y_l and substitute $y = 0$. Since $\xi = \phi(0)$ is a critical point of f_t , we get

$$\sum_{i,j=0}^n \frac{\partial^2 f_t}{\partial x_i \partial x_j}(\phi(0)) \frac{\partial \phi_i}{\partial y_k}(0) \frac{\partial \phi_j}{\partial y_l}(0) = 2\delta_{k,l},$$

where $\phi(y) =: (\phi_0(y), \dots, \phi_n(y))$. Therefore, recalling the Cauchy residue theorem, we get that the residue integral (5.6) is 1. The conclusion is that the value of the residue pairing (5.5) at a generic point t is a constant equal to the number of critical points, i.e.,

the Milnor number μ of f . Since the residue pairing is a holomorphic function in t , we get that the pairing (5.5) must be μ for all values of t .

Now let us specialize our discussion to the value of the residue pairing at $t = 0$. Let us identify H_f with the fiber of $p_*\Omega_F$ at $t = 0$ by $x^\kappa \mapsto x^\kappa dx$. Then the residue pairing induces a non-degenerate bilinear pairing J_f on H_f

$$J_f(x^\alpha, x^\beta) = \frac{1}{(2\pi\mathbf{i})^{n+1}} \int_{|f_{x_0}|=\dots=|f_{x_n}|=\epsilon} \frac{x^{\alpha+\beta} dx_0 \wedge \dots \wedge dx_n}{f_{x_0} \cdots f_{x_n}}.$$

Let us change the coordinate inside the integral on the RHS by rescaling $x_i = \lambda^{c_i} y_i$, where $\lambda > 0$ is a real number. Then the RHS is given by the same integral except that the integrand is rescaled by $\lambda^{c \cdot (\alpha+\beta) - D}$. Therefore, the residue pairing is non-zero only if the weights of x^α and x^β add up to D . Part a) of Proposition 5.3 follows immediately, for if there was a monomial x^α representing an element of H_f of weight $> D$, then this monomial would be orthogonal to H_f and since the residue pairing is non-degenerate we get that x^α must be 0 in the local algebra H_f . For part b), we already know that $J_f(1, \text{Hess}_f(x)) = \mu$, so $\text{Hess}_f(x)$ can not be 0 in H_f . If φ is a 2nd element in H_f of weight D , then $\varphi - \mu^{-1} J_f(\varphi, 1) \text{Hess}_f(x)$ is orthogonal to 1 and to all monomials of weight > 0 (\cdot it has weight D). Hence this linear combination is orthogonal to H_f and by the non-degeneracy of J_f , we get that φ is proportional to $\text{Hess}_f(x)$.

Let us conclude this section with the following useful corollary from the above discussion.

COROLLARY 5.7. *The residue pairing $J_f(x^\alpha, x^\beta)$ is non-zero only if $c \cdot \alpha + c \cdot \beta = D$. In the latter case, we have the following identity in the local algebra: $\mu x^\alpha x^\beta = J_f(x^\alpha, x^\beta) \text{Hess}_f(x)$.*

5.2. Frobenius structures for weighted homogeneous singularities

Let $f(x_0, \dots, x_n)$ be a weighted homogeneous singularity of corank k , that is, the weights of the variables x_0, \dots, x_{k-1} are $< \frac{1}{2}$ and the weights of x_k, \dots, x_n are $\frac{1}{2}$. We have introduced 3 families of functions with bases respectively $S_{\text{lin}} \subseteq S_{\text{tame}} \subseteq S_{\text{rel}}$. Furthermore, note that for given an opposite subspace $P \subset \mathcal{H}_f$ and a holomorphic volume form $\varphi \in \Omega_f$, by following the construction in Section 4.6.2, we can associate a formal Frobenius manifold to the following formal family of functions $(\widehat{F}, \widehat{p}: \widehat{Z} \rightarrow \widehat{S})$, where

$$\widehat{Z} = \mathbb{C}^{n+1}, \quad \mathcal{O}_{\widehat{Z}} := \mathcal{O}_{\mathbb{C}^{n+1}}[[t_\kappa(\kappa \in \mathcal{B})]],$$

$$\widehat{S} = \{0\}, \quad \mathcal{O}_{\widehat{S}} := \mathbb{C}[[t_\kappa(\kappa \in \mathcal{B})]],$$

and

$$\widehat{F}(x, t) = f(x) + \sum_{\kappa \in \mathcal{B}} t_\kappa x^\kappa.$$

We are going to prove that the formal Frobenius manifold is in fact analytic, i.e., there exists an open subset $U \subseteq \mathbb{C}^{\mathcal{B}}$, such that, $S_{\text{rel}} = \mathbb{C}^{\mathcal{B}_{\text{rel}}} \subset U$ and a uniquely determined Frobenius structure on U , such that, the formal germ of the Frobenius manifold U at 0 is isomorphic to the formal Frobenius manifold associated with the pair (P, φ) . The idea is to use the construction theorem of Hertling and Manin, which will be recalled next in Section 5.2.1.

5.2.1. The construction theorem of Hertling and Manin. Suppose that we are given a data $(M, \mathbb{K}, \nabla^r, C, \theta, \mathcal{U}, g)$, where

- (i) M is a complex manifold.
- (ii) \mathbb{K} is a holomorphic vector bundle on M and ∇^r is a flat connection on \mathbb{K} .
- (iii) $C : T_M \rightarrow \text{End}(\mathbb{K})$ is a morphism of holomorphic vector bundles, that is, for every holomorphic vector field $X \in \Gamma(V, T_M)$ defined on some open subset $V \subset M$, we have an associated morphism $C_X : \mathbb{K}|_V \rightarrow \mathbb{K}|_V$ of holomorphic vector bundles.
- (iv) θ and $\mathcal{U} : \mathbb{K} \rightarrow \mathbb{K}$ are morphisms of holomorphic vector bundles, that is, θ and \mathcal{U} are global sections of $\text{End}(\mathbb{K})$.
- (v) g is a non-degenerate bi-linear pairing on \mathbb{K} , such that,

$$\begin{aligned} g(C_X a, b) &= g(a, C_X b), \\ g(\mathcal{U} a, b) &= g(a, \mathcal{U} b), \\ g(\theta a, b) &= -g(a, \theta b), \\ Xg(a, b) &= g(\nabla_X^r a, b) + g(a, \nabla_X^r b). \end{aligned}$$

for all $X \in \mathcal{T}_M$ and $a, b \in \mathbb{K}$.

DEFINITION 5.8. The data $(M, \mathbb{K}, \nabla^r, C, \theta, \mathcal{U}, g)$ is said to be a *Frobenius type structure* on M if the connection ∇ on the vector bundle $\mathbb{C}^* \times \mathbb{K} \rightarrow \mathbb{C}^* \times M$ defined by

$$(5.7) \quad \nabla_X := \nabla_X^r + w^{-1}C_X, \quad X \in \mathcal{T}_M,$$

$$(5.8) \quad \nabla_{\partial/\partial w} := \frac{\partial}{\partial w} - w^{-1}\theta - w^{-2}\mathcal{U}$$

is flat, where w is the standard coordinate function on \mathbb{C}^* . □

The map C is usually called *Higgs field*, while θ is called *grading operator*.

REMARK 5.9. If $(M, \mathbb{K}, \nabla^r, C, \theta, \mathcal{U}, g)$ is a Frobenius type structure, then the set of endomorphisms \mathcal{U} and $C_X (X \in \mathcal{T}_M)$ pairwise commute, while the endomorphism θ is ∇^r -flat, that is,

$$\nabla_X^r(\theta(a)) = \theta(\nabla_X^r(a)), \quad \forall X \in \mathcal{T}_M, a \in \mathbb{K}. \quad \square$$

If $t \in M$, then we denote by \mathbb{K}_t the fiber of \mathbb{K} at t . Suppose that $\zeta \in \mathbb{K}$. We will say that the fiber \mathbb{K}_t is $\{C_X (X \in \mathcal{T}_M), \mathcal{U}\}$ -generated by ζ if every vector in \mathbb{K}_t can be written as a linear combination of vectors of the form $C_{X_1} \circ C_{X_2} \circ \dots \circ C_{X_s} \circ \mathcal{U}^l(\zeta)$, where $X_1, \dots, X_s \in \mathcal{T}_{M,t}$ are germs of holomorphic vector fields and $l \geq 0$ is an integer. The construction theorem of Hertling and Manin can be stated as follows.

THEOREM 5.10 (Hertling-Manin construction). *Suppose that $(M, \mathbb{K}, \nabla^r, C, \theta, \mathcal{U}, g)$ is a Frobenius type structure and that there exists a vector $\zeta \in \mathbb{K}_t$, such that,*

- (i) *The map $C\zeta : T_t M \rightarrow \mathbb{K}_t, X \mapsto C_X(\zeta)$ is injective.*
- (ii) *The fiber \mathbb{K}_t is $\{C_X (X \in \mathcal{T}_M), \mathcal{U}\}$ -generated by ζ .*
- (iii) *$\theta(\zeta) = \frac{D}{2}\zeta$ for some complex number D .*

Then the germ of the complex manifold M at t and the monomorphism $C\zeta$ can be extended uniquely to respectively a Frobenius manifold $(\widetilde{M}, (\cdot, \cdot), \bullet, e, E)$ and an isomorphism $T\widetilde{M}|_M \rightarrow \mathbb{K}$, such that, C_X coincides with the Frobenius multiplication by X , \mathcal{U} with the Frobenius multiplication by the Euler vector field E , θ with the grading operator

of the Frobenius manifold, ζ with the unit vector field e , and g with the Frobenius pairing $(\ , \)$. \square

5.2.2. Opposite subspaces and Frobenius type structures. Suppose that $(F, p : Z_{\text{rel}} \rightarrow S_{\text{rel}})$ is the family of relevant deformations of f , that is, $S_{\text{rel}} = \mathbb{C}^{\mathcal{B}_{\text{rel}}}$, $Z_{\text{rel}} := \mathbb{C}^{n+1} \times S_{\text{rel}}$, and

$$F(x, t) = f(x) + \sum_{\kappa \in \mathcal{B}_{\text{rel}}} t_{\kappa} x^{\kappa}.$$

Due to the homogeneity of f , there is a natural degree operator $\text{deg} : \Omega_f \rightarrow \Omega_f$, that is,

$$(5.9) \quad \text{deg}(x^{\kappa} dx) := (c \cdot \kappa + c_0 + \cdots + c_n) x^{\kappa} dx,$$

where $dx := dx_0 \wedge \cdots \wedge dx_n$ and the number appearing on the RHS in the brackets is precisely the weight of the form $x^{\kappa} dx$.

PROPOSITION 5.11. *Let $P \subset \widehat{\mathcal{H}}_f$ be a homogeneous opposite subspace. Then the corresponding degree operator (see Lemma 4.37) $\theta_P = -\text{deg}$.*

PROOF. Let $\omega_{\kappa}^{\circ} := \sigma(x^{\kappa} dx)$ be the lifts of the basis of Ω_f represented by the forms $x^{\kappa} dx$ ($\kappa \in \mathcal{B}$). The cohomology classes $[x^{\kappa} dx]$ in $\widehat{\mathcal{H}}_f^{(0)}$ form a $\mathbb{C}[[w]]$ -basis. Therefore,

$$(5.10) \quad \omega_{\beta}^{\circ} = \sum_{\alpha} [x^{\alpha} dx] s_{\alpha\beta}(w),$$

for some $s_{\alpha\beta}(w) \in \mathbb{C}[[w]]$. Note that the operator $\rho = 0$, because the class of f in H_f is 0. Let $\theta_{\alpha\beta}$ be the entries of the operator θ_P with respect to the basis $x^{\alpha} dx$ of Ω_f . Then

$$w \nabla_{\partial_w} (\omega_{\beta}^{\circ}) = - \sum_{\alpha} \theta_{\alpha\beta} \omega_{\alpha}^{\circ}.$$

On the other hand, note that (see also the proof of Lemma 6.3 below)

$$w \nabla_{\partial_w} [x^{\alpha} dx] = d_{\alpha} [x^{\alpha} dx],$$

where $d_{\alpha} := c \cdot \alpha + c_0 + \cdots + c_n$ is the weight of $x^{\alpha} dx$. Let us apply $w \nabla_{\partial_w}$ to the RHS of (5.10). We get

$$\sum_{\alpha} [x^{\alpha} dx] (d_{\alpha} + w \partial_w) s_{\alpha\beta}(w),$$

Let $s(w)$ be the matrix with entries $s_{\alpha\beta}(w)$ and δ – the diagonal matrix with entries d_{α} . Then the above formulas imply

$$-s(w) \theta_P = \delta s(w) + w \partial_w s(w).$$

Specializing $w = 0$ and using that $s(0)$ is the identity matrix – by definition the forms ω_{α}° and $x^{\alpha} dx$ coincide modulo w – we get that $\theta_P = -\delta$. It remains only to use that δ is the matrix of the degree operator deg in the basis $x^{\alpha} dx$. \square

Let (P, φ) be a pair of a homogeneous opposite subspace $P \subset \widehat{\mathcal{H}}_f$ and a holomorphic volume form $\varphi \in \Omega_f$, such that, $\theta_P(\varphi) = -r\varphi$ (see Section 4.6.2). According to the above proposition φ must be homogeneous of degree r , that is, φ can be represented by a weighted homogeneous holomorphic volume form $\psi(x) dx$ of weight r . On the other hand, since φ is a holomorphic volume form, we must have $\psi(0) \neq 0$. Therefore, ψ must be weighted homogeneous of weight 0. In other words, up to a constant φ coincides with the class of the standard holomorphic volume form dx . In particular, the weight of the form

φ must be $r = c_0 + \cdots + c_n$. There is a natural candidate for a Frobenius type structure in these settings, which we would like to present next.

Put $M = S_{\text{rel}}$ and let \mathbb{K} be the vector bundle whose sheaf of sections is $p_*\Omega_F$ – here we used that $p_*\Omega_F$ is a locally free sheaf (see Proposition 4.24). The Higgs field C is defined by composing the Kodaira–Spencer map $\mathcal{T}_M \rightarrow p_*\mathcal{O}_{C_F}$ and the map which to a function $\psi \in p_*\mathcal{O}_{C_F}$ associates the endomorphism of $p_*\Omega_F$ given by multiplication by ψ . In other words, C_X is the operator of multiplication by $X(F)$. Similarly, \mathcal{U} is the operator of multiplication by F . Furthermore, the grading operator θ is defined by using the weighted-homogeneity of f . Namely, note that in order to define the grading operator θ of a Frobenius type structure, it is sufficient to define it at a fixed point of M . Indeed, since θ is ∇^r -flat (see Remark 5.9) the values of θ at the remaining points of M are uniquely determined. In our case, we choose $t = 0$. The fiber of \mathbb{K} at $t = 0$ is Ω_f and we define $\theta := \frac{n+1}{2} - \text{deg}$. Furthermore, for pairing g , let us take the classical residue pairing J_F . Note that the skew-symmetry property of θ follows from Corollary 5.7, while the Frobenius properties of the Higgs fields C_X and \mathcal{U} are easy to verify from the definition of J_F . Finally, it remains to define the connection ∇^r .

To begin with, let us recall the section $\sigma : \Omega_f \rightarrow \widehat{\mathcal{H}}_f^{(0)}$ constructed from the opposite subspace P in Proposition 4.35. Let us redenote σ by σ_f and think of Ω_f and $\widehat{\mathcal{H}}_f^{(0)}$ as the fibers of the vector bundles $p_*\Omega_F$ and $\widehat{\mathcal{H}}_F^{(0)}$ over $t = 0$. Following the construction before Remark 4.39, let us extend σ_f to an embedding of vector bundles $\sigma_F : p_*\Omega_F \rightarrow \widehat{\mathcal{H}}_F^{(0)}$. Namely, let $\omega^\circ \in \Gamma(\mathbb{C}^{n+1}, \Omega_{\mathbb{C}^{n+1}}^{n+1})[[w]]$ be a holomorphic volume form representing the cohomology class $\sigma_f(\varphi)$. Put $\omega_\kappa^\circ = \sigma_f(x^\kappa \varphi)$. Since the cohomology classes ω_α° ($\alpha \in \mathcal{B}$) form a $\mathbb{C}[[w]]$ -basis of $\widehat{\mathcal{H}}_f$, we have

$$\exp\left(\sum_{\kappa \in \mathcal{B}_{\text{rel}}} t_\kappa x^\kappa / w\right) [x^\beta \omega^\circ] = \sum_{\alpha \in \mathcal{B}} \omega_\alpha^\circ E_{\alpha\beta}(t, w),$$

where $[\]$ on the LHS denotes the cohomology class in $\widehat{\mathcal{H}}_F$. The above identity should be viewed in $\widehat{\mathcal{H}}_f[[t_\kappa(\kappa \in \mathcal{B}_{\text{rel}})]]$ (see Lemma 4.38, a). The coefficients $E_{\alpha\beta}(t, w) \in \mathbb{C}((w))[[t_\kappa(\kappa \in \mathcal{B}_{\text{rel}})]]$ defined by the above identity are the key to the entire construction. It is convenient to assign weight $1 - c \cdot \kappa$ to each variable t_κ and weight 1 to the formal variable w . Then it is easy to check that $E_{\alpha\beta}(t, w)$ must be homogeneous of weight $c \cdot \beta - c \cdot \alpha$. Since the weights of the variables t_κ ($\kappa \in \mathcal{B}_{\text{rel}}$) are positive, the coefficients in front of the powers of w in $E_{\alpha\beta}(t, w)$ are polynomials in t . Let $E(t, w)$ be the matrix with entries $E_{\alpha\beta}(t, w)$. Since $E(0, w) = 1 + O(w)$, the matrix $E(t, w)$ has a Birkhoff factorization $E(t, w) = T(t, w)A(t, w)^{-1}$, where $T(t, w) = 1 + O(w^{-1})$ and $A(t, w) = A_0(t) + A_1(t)w + \cdots$, where $A_k(t)$ are matrices with entries in $\mathbb{C}[[t_\kappa(\kappa \in \mathcal{B}_{\text{rel}})]]$ with $A_0(t)$ being invertible. Moreover, the Birkhoff factorization is compatible with the grading, i.e., the entries $T_{\alpha\beta}(t, w)$ and $A_{\alpha\beta}(t, w)$ are weighted homogeneous of weight $c \cdot \beta - c \cdot \alpha$. In particular, the entries of $A(t, w)$ must be polynomials in t and w . Therefore, the forms

$$\omega_\beta := \sum_{\alpha \in \mathcal{B}} x^\alpha A_{\alpha\beta}(t, w) \omega^\circ$$

are analytic and represent cohomology classes $[\omega_\beta] \in \widehat{\mathcal{H}}_F^{(0)}$. According to Proposition 4.24, the vector bundle $p_*\Omega_F$ is trivial and the classes of the forms $x^\alpha dx$ ($\alpha \in \mathcal{B}$) provide a trivialization. Let us denote by $[[\omega_\beta]]$ the image in $\widehat{\mathcal{H}}_F^{(0)}/w\widehat{\mathcal{H}}_F^{(0)} = p_*\Omega_F$ of the cohomology

class $[\omega_\beta]$ under the natural quotient map, that is,

$$[\omega_\beta] = \sum_{\alpha \in \mathcal{B}} x^\alpha A_{\alpha\beta}(t, 0) [\omega^\circ].$$

Since the matrix $A(t, 0)$ is invertible, the classes $[\omega_\beta]$ ($\beta \in \mathcal{B}$) provide also a trivialization of $p_*\Omega_F$. Let us define the map

$$(5.11) \quad \sigma_F : p_*\Omega_F \rightarrow \widehat{\mathcal{H}}_F^{(0)}, \quad [\omega_\beta] \mapsto [\omega_\beta].$$

Note that at $t = 0$, since $A(0, w) = E(0, w)^{-1}$, we have $\omega_\beta|_{t=0} = \omega_\beta^\circ$, that is, σ_F is an extension of σ_f . The points $t \in S_{\text{rel}}$ for which σ_F induces an injective map between the fibers at t is an open subset V of S_{rel} containing 0. Moreover, the set V is invariant under the rescaling $t \mapsto \lambda \cdot t$, where $(\lambda \cdot t)_\kappa := \lambda^{1-c \cdot \kappa} t_\kappa$ and λ is a positive real number. By definition of a relevant deformation $1 - c \cdot \kappa > 0$, so the rescaling invariance of V forces $V = S_{\text{rel}}$. We get that the map σ_F is an injective morphism of holomorphic vector bundles. The connection ∇^r on $\mathbb{K} = p_*\Omega_F$ is defined by requiring that $[\omega_\beta]$ are flat sections, that is, $\nabla_X^r [\omega_\beta] = 0$ for all $X \in \mathcal{T}_M$.

PROPOSITION 5.12. *The data $(M, \mathbb{K}, \nabla^r, C, \theta, \mathcal{U}, g)$ constructed above is a Frobenius type structure.*

PROOF. The proof is essentially the same as the union of the proofs of Lemma 4.40 and Theorem 4.41. Let us just outline the main steps and leave it as an exercise to adjust to the current settings the arguments from the proofs of Lemma 4.40 and Theorem 4.41.

Let us denote by $C_\kappa(t)$ the matrix with entries $C_{\kappa\alpha\beta}(t)$ defined by

$$x^\kappa [\omega_\beta] = \sum_{\alpha \in \mathcal{B}} [\omega_\alpha] C_{\kappa\alpha\beta}(t), \quad \beta \in \mathcal{B}.$$

The Gauss-Manin connection takes the following form in the frame $[\omega_\beta]$ ($\beta \in \mathcal{B}$):

$$(5.12) \quad w \nabla_{\partial/\partial t_\kappa} [\omega_\beta] = \sum_{\alpha \in \mathcal{B}} [\omega_\alpha] C_{\kappa\alpha\beta}(t),$$

$$(5.13) \quad (w \nabla_{\partial/\partial w} + \nabla_E) [\omega_\beta] = (c \cdot \beta + c_0 + \cdots + c_n) [\omega_\beta],$$

where $E = \sum_{\kappa \in \mathcal{B}_{\text{rel}}} (1 - c \cdot \kappa) t_\kappa \partial_{t_\kappa}$ is the Euler vector field. The proof of the above differential equations is identical to the proof of Lemma 4.40. Let us also point out the following important byproducts of the proof. The matrix series $T(t, w)$ satisfies the following differential equations

$$\begin{aligned} w \partial_{t_\kappa} T(t, w) &= T(t, w) C_\kappa(t), \\ (w \partial_w + E) T(t, w) &= [\text{deg}, T(t, w)], \end{aligned}$$

where slightly abusing the notation we denoted by deg the diagonal matrix with entries $c \cdot \alpha + c_0 + \cdots + c_n$, that is, this is the matrix of the degree operator (5.9) in the basis $[x^\alpha dx]$ ($\alpha \in \mathcal{B}$). The 2nd differential equation implies that $T(t, w) w^{\text{deg}}$ is a solution to a differential equation in w that has only two singularities, i.e., a Fuchsian singularity at $w = \infty$ and an irregular singularity at $w = 0$. Therefore, for each $t \in S_{\text{rel}}$ the operator series $T(t, w)$ is analytic for all $|w| > 0$. Moreover, since the matrices $C_\kappa(t)$ are polynomial and hence analytic in t , the operator series $T(t, w)$ is holomorphic for all $t \in S_{\text{rel}}$.

Let us extend the map (5.11) $\mathcal{O}_M[[w]]$ -linearly to a map $\widehat{\sigma}_F : p_*\Omega_F[[w]] \rightarrow \widehat{\mathcal{H}}_F^{(0)}$. According to Proposition 4.24 the map $\widehat{\sigma}_F$ is an isomorphism. The connection in the

definition of a Frobenius type structure (see Definition 5.8) can be interpreted also as a connection on $\mathbb{K}[[w]]$. We claim that in the current settings this connection is a pullback of the Gauss–Manin connection via $\widehat{\sigma}_F$. Indeed, using (5.12) we get

$$(\widehat{\sigma}_F^{-1} \nabla_{\partial_{t_\kappa}} \widehat{\sigma}_F)[\omega_\beta] = \widehat{\sigma}_F^{-1} \nabla_{\partial_{t_\kappa}}[\omega_\beta] = \sum_{\alpha \in \mathcal{B}} [\omega_\alpha] C_{\kappa\alpha\beta}(t) = C_{\partial_{t_\kappa}}[\omega_\beta],$$

where we used that the Higgs field $C_{\partial_{t_\kappa}}$ is the operator of multiplication by $\partial_{t_\kappa}(F) = x^\kappa$. Since by definition $\nabla_{\partial_{t_\kappa}}^r[\omega_\beta] = 0$, the above formula implies that $\widehat{\sigma}_F^{-1} \nabla_X \widehat{\sigma}_F$ coincides with (5.7). Note that

$$(\widehat{\sigma}_F^{-1} w \nabla_E \widehat{\sigma}_F)[\omega_\beta] = E(F)[\omega_\beta] = F[\omega_\beta],$$

where we used that $F = E(F) + \sum_{i=0}^n c_i x_i F_{x_i}$ and the fact that the derivatives F_{x_i} act trivially on $p_* \Omega_F$. By definition, the operator of multiplication by F is \mathcal{U} . Therefore, the operator $\widehat{\sigma}_F^{-1} \nabla_E \widehat{\sigma}_F = \nabla_E^r + w^{-1} \mathcal{U}$. Furthermore, using (5.13) we get

$$(\widehat{\sigma}_F^{-1} w \nabla_{\partial_w} \widehat{\sigma}_F)[\omega_\beta] = \left(\deg - w^{-1} \mathcal{U} \right) [\omega_\beta].$$

Comparing with (5.8) and recalling that $\theta = \frac{n+1}{2} - \deg$, we get that the difference of the connection operators $\widehat{\sigma}_F^{-1} \nabla_{\partial_w} \widehat{\sigma}_F$ and (5.8) is the scalar matrix $\frac{n+1}{2} \text{Id}$.

It remains only to prove that the residue pairing J_F is ∇^r -flat, that is, that for all $\alpha, \beta \in \mathcal{B}$ the pairings $J_F([\omega_\alpha], [\omega_\beta])$ are constants independent of t . This follows from the identity

$$K_F([\omega_\alpha], [\omega_\beta]) = J_f(\omega_\alpha^\circ, \omega_\beta^\circ) w^{n+1},$$

which can be proved in the same way as (4.45). Indeed, the above identity implies that the higher residue pairings $K_F^{(p)}([\omega_\alpha], [\omega_\beta]) = 0$ for $p > 0$, while for $p = 0$ we get

$$J_F([\omega_\alpha], [\omega_\beta]) = K_F^{(0)}([\omega_\alpha], [\omega_\beta]) = J_f(\omega_\alpha^\circ, \omega_\beta^\circ). \quad \square$$

5.2.3. Extension of the Frobenius type structure. We would like to prove that the Frobenius type structure from Proposition 5.12 can be extended to a Frobenius structure defined in some open neighborhood of $M = \mathbb{C}^{\mathcal{B}_{\text{rel}}}$ in $\mathbb{C}^{\mathcal{B}}$. The existence of an extension is guaranteed by the construction theorem of Hertling and Manin. Namely, it is straightforward to check that the Frobenius type structure from Proposition 5.12 satisfies conditions (i)–(iii) in Theorem 5.10. However, in our settings the proof of Theorem 5.10 simplifies significantly, so we would like to give a direct proof.

Let us assume that the weighted homogeneous singularity f has the form (5.4). Note that in the local algebra H_f the classes of the variables x_i with $k \leq i \leq n$ can be expressed in terms of the variables x_0, \dots, x_{k-1} . Therefore, we may assume that the multi-index set \mathcal{B} consists only of tuples $\beta = (b_0, \dots, b_n)$, such that, $b_i = 0$ for $k \leq i \leq n$. Let us compare the formal Frobenius manifold $(\widehat{S} := \{0\}, \mathcal{O}_{\widehat{S}} = \mathbb{C}[[t_\beta (\beta \in \mathcal{B})]])$ constructed from the pair (P, φ) (see Theorem 4.41) and the Frobenius type manifold constructed in Proposition 5.12. The Frobenius type structure is encoded into the following connection on $p_* \Omega_F[[w]]$

$$\nabla = d + \sum_{\kappa \in \mathcal{B}_{\text{rel}}} w^{-1} C_\kappa(t) dt_\kappa + \left(w^{-1} \deg - w^{-2} \mathcal{U}(t) \right),$$

where we trivialized $p_*\Omega_F$ via the frame $[\omega_\beta]$ ($\beta \in \mathcal{B}$), $C_\kappa(t)$ is the matrix of the operator of multiplication by x^κ , and \mathcal{U} is the matrix of the operator of multiplication by F , that is,

$$\begin{aligned} x^\kappa[\omega_\beta] &= \sum_{\alpha} C_{\kappa\alpha\beta}(t)[\omega_\alpha], \\ F(x, t)[\omega_\beta] &= \sum_{\alpha} \mathcal{U}_{\alpha\beta}(t)[\omega_\alpha]. \end{aligned}$$

Let us consider the miniversal deformation of f

$$\widehat{F}(x, t, y) = F(x, t) + \sum_{\beta \in \mathcal{B}_{\text{mar}} \sqcup \mathcal{B}_{\text{irr}}} y_\beta x^\beta,$$

where we denoted by y the set of deformation parameters corresponding to deformations that are not relevant. The formal Frobenius structure is encoded into the following connection

$$\widehat{\nabla} = d + w^{-1} \left(\sum_{\kappa \in \mathcal{B}_{\text{rel}}} \widehat{C}_\kappa(t, y) dt_\kappa + \sum_{\beta \in \mathcal{B}_{\text{mar}} \sqcup \mathcal{B}_{\text{irr}}} \widehat{C}_\beta(t, y) dy_\beta \right) + \left(w^{-1} \text{deg} - w^{-2} \widehat{\mathcal{U}}(t, y) \right),$$

where the notation is as follows. Firstly, this is a connection on the sheaf $p_*\Omega_{\widehat{F}}[w]$ written in the trivializing frame $[\widehat{\omega}_\beta]$ ($\beta \in \mathcal{B}$), where the forms $\widehat{\omega}_\beta$ are constructed in the same way as ω_β (see also (4.41)). The connection matrices $\widehat{C}_\beta(t, y)$ ($\beta \in \mathcal{B}$) are just the matrices of multiplication by x^β in $p_*\Omega_{\widehat{F}}$ in the basis $\{[\widehat{\omega}_\beta]\}$ and $\widehat{\mathcal{U}}$ is the operator of multiplication by \widehat{F} . Note that the restriction of $\widehat{\nabla}$ to $y = 0$ is precisely ∇ .

We would like to prove that the connection $\widehat{\nabla}$ is analytic. More precisely, the entries of \widehat{C}_β and $\widehat{\mathcal{U}}$ are homogeneous functions in t and y . Since the weights of the t -variables are positive, we get that the entries of the connection matrices are formal power series in y whose coefficients are polynomial in t . Suppose that $t^\circ \in \mathbb{C}^{\mathcal{B}_{\text{rel}}}$ is an arbitrary point. We would like to prove that there exists a positive real number r° (depending on t°), such that, the connection matrices $\widehat{C}_\beta(t, y)$ and $\widehat{\mathcal{U}}(t, y)$ are convergent for every $(t, y) \in \mathbb{C}^{\mathcal{B}}$, such that, $|t_\kappa - t_\kappa^\circ| < r^\circ \forall \kappa \in \mathcal{B}_{\text{rel}}$ and $|y_\beta| < r^\circ \forall \beta \in \mathcal{B}_{\text{mar}} \sqcup \mathcal{B}_{\text{irr}}$. To begin with, note that it is sufficient to prove the convergence of $\widehat{C}_i(t, y) := \widehat{C}_{e_i}(t, y)$ ($0 \leq i \leq k-1$), where $e_i \in \mathcal{B}$ is the multi-index for which only the i th component is 1 and the remaining ones are 0. Indeed, due to the homogeneity of f , we have

$$\widehat{F} = \left(\sum_{i=0}^n c_i x_i \partial_{x_i} + \sum_{\kappa \in \mathcal{B}_{\text{rel}}} (1 - c \cdot \kappa) t_\kappa \partial_{t_\kappa} + \sum_{\beta \in \mathcal{B}_{\text{mar}} \sqcup \mathcal{B}_{\text{irr}}} (1 - c \cdot \beta) y_\beta \partial_{y_\beta} \right) \widehat{F}.$$

Therefore, the operator $\widehat{\mathcal{U}}$ can be expressed in terms of the Higgs fields

$$\widehat{\mathcal{U}} = \sum_{\kappa \in \mathcal{B}_{\text{rel}}} (1 - c \cdot \kappa) t_\kappa \widehat{C}_\kappa(t, y) + \sum_{\beta \in \mathcal{B}_{\text{mar}} \sqcup \mathcal{B}_{\text{irr}}} (1 - c \cdot \beta) y_\beta \widehat{C}_\beta(t, y).$$

Furthermore, if $\beta = (b_0, \dots, b_{k-1}, 0, \dots, 0)$, then since \widehat{C}_β is the operator of multiplication by x^β and \widehat{C}_i is the operator of multiplication by x_i , we get

$$\widehat{C}_\beta(t, y) = \widehat{C}_0(t, y)^{b_0} \dots \widehat{C}_{k-1}(t, y)^{b_{k-1}}.$$

In order to prove the convergence of $\widehat{C}_i(t, y)$ we recall the Cauchy-Kowalevski theorem (see [17], Theorem 1.41). We follow the notation of Hertling and Manin ([32], formulas (2.42)–(2.43)):

THEOREM 5.13 (Cauchy-Kowalevski Theorem). *Given a positive integer N , matrices*

$$A_i \in M(N \times N, \mathbb{C}\{t_0, \dots, t_{k-1}, y, x_1, \dots, x_N\}) \quad (0 \leq i \leq k-1)$$

and

$$B \in M(N \times 1, \mathbb{C}\{t_0, \dots, t_{k-1}, y, x_1, \dots, x_N\})$$

there exists a unique vector

$$\Phi \in M(N \times 1, \mathbb{C}\{t_0, \dots, t_{k-1}, y\})$$

with

$$\begin{aligned} \frac{\partial \Phi}{\partial y} &= \sum_{i=0}^{k-1} A_i(t, y, \Phi) \frac{\partial \Phi}{\partial t_i} + B(t, y, \Phi), \\ \Phi(t, 0) &= 0, \end{aligned}$$

where $M(N \times K, R)$ denotes the ring of matrices of size $N \times K$ whose entries belong to R .

The Cauchy-Kowalevski theorem should be applied successively for each variable y_β ($\beta \in \mathcal{B}_{\text{mar}} \sqcup \mathcal{B}_{\text{irr}}$). Let us perform just the first step in this process. The remaining steps are completely analogous. In other words, let us pick some non-relevant deformation parameter y_β and set the remaining $y_\alpha = 0$ for $\alpha \neq \beta$. For simplicity, we put $y := y_\beta$. The flatness of $\widehat{\nabla}$ implies that

$$(5.14) \quad \frac{\partial \widehat{C}_i}{\partial y} = \frac{\partial \widehat{C}_\beta}{\partial t_i} = \partial_{t_i} \left(\widehat{C}_0(t, y)^{b_0} \cdots \widehat{C}_{k-1}(t, y)^{b_{k-1}} \right).$$

Let us denote by Φ the vector whose components $\Phi_{i\alpha\gamma}$ are indexed by tripples (i, α, γ) , such that, $0 \leq i \leq k-1$ and $\alpha, \gamma \in \mathcal{B}$. Let Φ_i be the matrix of size $\mathcal{B} \times \mathcal{B}$ whose (α, γ) -entry is $\Phi_{i\alpha\gamma}$. It is easy to see that under the substitution $\Phi_i(t, y) := \widehat{C}_i(t, y) - C_i(t)$ the PDE (5.14) transforms into a PDE that has the form of the PDE in the Cauchy-Kowalevski theorem. Indeed, let us denote by $f_{\alpha\gamma}(t, \Phi)$ the (α, γ) -entry of the matrix $(\Phi_0 + C_0(t))^{b_0} \cdots (\Phi_{k-1} + C_{k-1}(t))^{b_{k-1}}$. Clearly, $f_{\alpha\gamma}(t, \Phi)$ is a polynomial in Φ and t and in particular an element in $\mathbb{C}\{t_0 - t_0^c, \dots, t_{k-1} - t_{k-1}^c, \Phi\}$. Using the chain rule it is straightforward to verify that (5.14) is equivalent to saying that $\Phi_i(t, y) = \widehat{C}_i(t, y) - C_i(t)$ is a solution to the following PDE

$$\frac{\partial \Phi}{\partial y} = \sum_{i=0}^{k-1} A_i(t, \Phi) \frac{\partial \Phi}{\partial t_i} + B(t, \Phi),$$

where $A_i(t, \Phi)$ is a matrix with entry $(i' \alpha' \gamma', i'' \alpha'' \gamma'')$ given by

$$A_{i, i' \alpha' \gamma', i'' \alpha'' \gamma''}(t, \Phi) := \delta_{i, i'} \frac{\partial f_{\alpha' \gamma'}}{\partial \Phi_{i'' \alpha'' \gamma''}}(t, \Phi)$$

and $B(t, \Phi)$ is a vector column with $i\alpha\gamma$ -entry given by

$$B_{i\alpha\gamma}(t, \Phi) := \frac{\partial f_{\alpha\gamma}}{\partial t_i}(t, \Phi).$$

Note also that if we follow the algorithm used to define the two connections $\widehat{\nabla}$ and ∇ , then we see immediately that the restriction of the connection $\widehat{\nabla}$ to $y = 0$ coincides with ∇ . In other words, $\widehat{C}_i(t, 0) = C_i(t)$. It remains only to recall the Cauchy-Kowalevski theorem.

5.3. Homotopy type of the Milnor fiber

The goal of this section is to prove that the Milnor fiber of a weighted homogeneous singularity has the homotopy type of a bouquet of spheres. In particular, we will introduce the notion of vanishing cycles and Milnor lattice.

5.3.1. Morse theory for non-compact manifolds. We will make use of Palais' generalization of Morse theory to non-compact Riemannian manifolds. Let us recall the appropriate results from [49]. Suppose that M is a complete smooth Riemannian manifold and that $\phi : M \rightarrow \mathbb{R}$ is a Morse function. Let us denote by g the Riemannian metric on M and by $\nabla\phi$ the corresponding *gradient* vector field, that is, $\langle d\phi, v \rangle = g(\nabla\phi, v)$ for every vector field v on M .

DEFINITION 5.14 (Condition (C)). The function ϕ is said to satisfy condition (C) if for any subset $S \subset M$ on which

- (i) ϕ is bounded on S , that is, there exists a constant K , such that $|f(s)| < K$ for all $s \in S$,
- (ii) $\nabla\phi$ is not bounded away from 0, that is, for every $\epsilon > 0$, there exists $s \in S$, such that, $\|\nabla\phi(s)\| < \epsilon$,

then there is a critical point of ϕ *adherent* to S , that is, there exists a sequence $\{s_n\}$ in S converging in M to a critical point of ϕ . \square

If condition (C) is satisfied, then it is easy to see that ϕ has isolated critical values and that for every critical value u of ϕ there are only finitely many critical points on the critical level $\phi^{-1}(u)$. Put $M_a := \phi^{-1}(-\infty, a]$. It turns out that if condition (C) holds, then the conclusions of Morse theory describing the change of the homotopy type of M_a as a goes through a critical value u continue to hold. Let us give a slightly more precise statement. Recall that the index (resp. co-index) of a Morse critical point ξ of a function ϕ is the number counted with multiplicity of negative (resp. positive) eigenvalues of the Hessian matrix of ϕ at the point ξ . Furthermore, let us denote by \mathbb{D}^k the unit ball in \mathbb{R}^k . The product $\mathbb{D}^k \times \mathbb{D}^l$ is called a *handle* of type (k, l) .

DEFINITION 5.15. Let X be a smooth manifold with boundary and $Y \subset X$ a closed submanifold with boundary. We say that X is obtained from Y by the disjoint attachments of handles of type $((k_1, l_1), \dots, (k_s, l_s))$ if there exist pairwise disjoint closed subsets $h_1, \dots, h_s \subset X$ and homeomorphisms $\psi_i : \mathbb{D}^{k_i} \times \mathbb{D}^{l_i} \rightarrow h_i$ ($1 \leq i \leq s$), such that,

- (i) $X = Y \cup h_1 \cup \dots \cup h_s$.
- (ii) The restriction of ψ_i induces a smooth isomorphism $\mathbb{S}^{k_i-1} \times \mathbb{D}^{l_i} \cong h_i \cap \partial Y$, where \mathbb{S}^{k_i-1} is the boundary of \mathbb{D}^{k_i} .
- (iii) The restriction of $\sqcup_i \psi_i$ induces a smooth isomorphism

$$\bigsqcup_i \text{Int}(\mathbb{D}^{k_i}) \times \mathbb{D}^{l_i} \cong X \setminus Y,$$

where $\text{Int}(S)$ denotes the interior of S .

The pairs $(\psi_i, \mathbb{D}^{k_i} \times \mathbb{D}^{l_i})$ will be called *handles* of Y and if the above conditions holds, then we will write $X = Y \cup_{i=1}^s \cup_{\psi_i} \mathbb{D}^{k_i} \times \mathbb{D}^{l_i}$. \square

Suppose that ϕ is a Morse function satisfying condition (C). Let u be a critical value of ϕ and let ξ_1, \dots, ξ_s be the critical points of ϕ on the level $\phi^{-1}(u)$. Let k_i and l_i be respectively the index and the co-index of ϕ at ξ_i . For given $\epsilon > 0$, let us

define $W = \phi^{-1}(u - 2\epsilon, +\infty)$. For each $a > u - 2\epsilon$, put $W_a := \phi^{-1}(u - 2\epsilon, a]$ and $\widetilde{W}_a := \widetilde{\phi}^{-1}(u - 2\epsilon, a]$.

THEOREM 5.16 (Palais). *If $\epsilon > 0$ is sufficiently small, then there exists a smooth function $\widetilde{\phi} : W \rightarrow \mathbb{R}$ with the following properties:*

- (i) *The support of $\widetilde{\phi} - \phi|_W$ is a disjoint union of open subsets V_i ($1 \leq i \leq s$) and each V_i is contained in a compact neighborhood of the critical point ξ_i .*
- (ii) *$\widetilde{W}_{u+\epsilon} = W_{u+\epsilon}$ and $\widetilde{W}_{u-\epsilon}$ is obtained from $W_{u-\epsilon}$ by disjointly attaching handles of type $((k_1, l_1), \dots, (k_s, l_s))$.*
- (iii) *The flow of the vector field $\frac{\nabla \phi}{\nabla \phi(\phi)}$ defines a deformation retract $\widetilde{W}_{u+\epsilon} \rightarrow \widetilde{W}_{u-\epsilon}$.*

Note that since \mathbb{D}^{l_i} are contractible, conditions (ii) and (iii) imply that $M_{u+\epsilon} = M_{-2\epsilon} \cup W_{u+\epsilon}$ is homotopic to a topological space obtained from $M_{u-\epsilon} = M_{-2\epsilon} \cup W_{u-\epsilon}$ by attaching cells \mathbb{D}^{k_i} of dimension k_i .

For the sake of completeness, let us also recall the definition of $\widetilde{\phi}$. First, let $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth monotone decreasing function, such that,

$$\lambda(x) = \begin{cases} 1 & \text{if } x \leq \frac{1}{2}, \\ 0 & \text{if } x \geq 1, \\ \in (0, 1) & \text{if } \frac{1}{2} < x < 1. \end{cases}$$

REMARK 5.17. The region in \mathbb{R}^2 defined by

$$x^2 - y^2 - \frac{3}{2}\epsilon \lambda\left(\frac{x^2}{\epsilon}\right) \leq -\epsilon,$$

where $\epsilon > 0$ is a positive number, is obtained from the region $x^2 - y^2 \leq -\epsilon$ by attaching a handle of type $(1, 1)$. In the special case when $M = \mathbb{R}^2$ and $\phi(x, y) = x^2 - y^2$, the function

$$\widetilde{\phi}(x, y) := x^2 - y^2 - \frac{3}{2}\epsilon \lambda\left(\frac{x^2}{\epsilon}\right). \quad \square$$

Let $\varphi_i : U_i \rightarrow M$ be a Morse coordinate chart on M , that is, U_i is an open neighborhood of $0 \in \mathbb{R}^n$, $\varphi_i(0) = \xi_i$, and

$$\phi \circ \varphi_i(x) = u + x_1^2 + \dots + x_l^2 - x_{l+1}^2 - \dots - x_n^2.$$

Let $\langle \cdot, \cdot \rangle$ be a positive definite symmetric bi-linear pairing on \mathbb{R}^n , where $n = \dim_{\mathbb{R}}(M)$. The linear structure of \mathbb{R}^n induces a trivialization of the tangent bundle $TU_i \cong U_i \times \mathbb{R}^n$. In order to make a distinction between points in U_i and tangent vectors to U_i , let us agree to represent the points by vector-rows and the tangent vectors by vector-columns. Using this trivialization, we identify $\langle \cdot, \cdot \rangle$ with a flat Riemannian metric on U_i . Note that in the local chart U_i , the Riemannian metric g on M has the form

$$g_{\varphi_i(x)}(d\varphi_i(u), d\varphi_i(v)) = \langle G_i(x)u, v \rangle, \quad u, v \in \mathbb{R}^n,$$

where $G_i(x)$ is a $n \times n$ matrix depending smoothly on $x \in U_i$. If $x = (x_1, \dots, x_n) \in U_i$, then let us denote by x^T the vector column with entries x_1, \dots, x_n . The map $x \mapsto x^T$ should be viewed as a vector field on U_i . Then we have

$$\phi \circ \varphi_i(x) = u + \langle P_i x^T, x^T \rangle - \langle (1 - P_i)x^T, x^T \rangle,$$

where P_i is a uniquely determined $n \times n$ matrix, self-adjoint with respect to the pairing $\langle \cdot, \cdot \rangle$. It can be proved that P_i is a projection matrix, that is, $P_i^2 = P_i$. There exists a

choice of the pairing $\langle \cdot, \cdot \rangle$, such that, the operators P_i and $G_i(0)$ commute. Let us fix such a choice $\langle \cdot, \cdot \rangle_i$ and denote by $\mathbb{D}_i^n(0, r) = \{x \in \mathbb{R}^n \mid \langle x^T, x^T \rangle_i \leq r^2\}$ the corresponding ball in \mathbb{R}^n with radius r and center 0. First, let us choose $\delta > 0$ so small that $\mathbb{D}_i^n(0, 2\delta) \subset U_i$ for all $1 \leq i \leq s$ and then let us pick $\epsilon < \delta^2$. Put

$$\tilde{\phi}(p) = \phi(p) - \frac{3\epsilon}{2} \lambda(\langle P_i \varphi_i^{-1}(p)^T, \varphi_i^{-1}(p)^T \rangle / \epsilon),$$

where $p \in \varphi(\mathbb{D}_i^n(0, 2\delta)) \cap W$. Let V_i be the set of points $p \in \varphi(\mathbb{D}_i^n(0, 2\delta)) \cap W$ for which $\tilde{\phi}(p) \neq \phi(p)$. If $p \in V_i$, then we must have $\langle Px^T, x^T \rangle < \epsilon$, where $x = \varphi^{-1}(p)$. Since $p \in W$, we also have

$$u - 2\epsilon \leq \phi(p) = u + \langle P_i x^T, x^T \rangle_i - \langle (1 - P_i)x^T, x^T \rangle_i < u + \epsilon - \langle (1 - P_i)x^T, x^T \rangle_i.$$

Therefore, $\langle (1 - P_i)x^T, x^T \rangle_i < 3\epsilon$ and

$$\langle x^T, x^T \rangle_i = \langle P_i x^T, x^T \rangle_i + \langle (1 - P_i)x^T, x^T \rangle_i < 4\epsilon < 4\delta^2.$$

This proves that $V_i \subset \mathbb{D}_i^n(0, 2\sqrt{\epsilon}) \subset \text{Int}(\mathbb{D}_i^n(0, 2\delta))$ and hence we can extend the definition of $\tilde{\phi}(p)$ for all $p \in W$ by defining $\tilde{\phi}(p) = \phi(p)$ for all $p \notin \mathbb{D}_i^n(0, 2\sqrt{\epsilon}) \cap W$.

Let us point out that after introducing the above notation the proof of Palais' theorem is not so hard. In fact, parts (i) and (ii) are straightforward to verify. In order to prove (iii), one has first to prove that by decreasing ϵ if necessary, we can arrange that the function $\nabla\phi(\tilde{\phi})$ is positive in the domain $\tilde{\phi}^{-1}(u - 5\epsilon/4, u + 5\epsilon/4)$. This is a local problem in which the requirement that P_i and $G_i(0)$ commute plays an essential role (see [49], Section 12, Proposition (1)). Then the rest of the proof amounts to proving that both the negative and the positive time flow lines of the vector field $\frac{\nabla\phi}{\nabla\phi(\tilde{\phi})}$ through a point of the domain $\tilde{\phi}^{-1}[u - \epsilon, u + \epsilon]$ exits the domain in a finite time. The proof of this fact is identical to the proof of Lemma 4.20.

5.3.2. Homotopy type of the Milnor fiber. Let us return back to the settings where $f \in \mathbb{C}[x_0, \dots, x_n]$ is a weighted homogeneous singularity. Let k be the co-rank of f and suppose that the variables x_i are enumerated in such a way that their weights $c_0 \leq \dots \leq c_{k-1} < c_k = \dots = c_n = \frac{1}{2}$. In this section we will be interested in the family $(F, p : Z_{\text{lin}} \rightarrow S_{\text{lin}})$ of linear deformations of f , that is, $Z_{\text{lin}} = \mathbb{C}^{n+1} \times \mathbb{C}^k$, $S_{\text{lin}} = \mathbb{C}^k$, and

$$F(x, t) := f(x) + t_0 x_0 + \dots + t_{k-1} x_{k-1},$$

where $t = (t_0, \dots, t_{k-1})$. According to Proposition 5.2, the above family is Kähler complete and tame. Therefore, according to Theorem 4.19, the map

$$\Phi : \mathbb{C}^{n+1} \times \mathbb{C}^k \rightarrow \mathbb{C} \times \mathbb{C}^k, \quad (x, t) \mapsto (F(x, t), t)$$

induces a locally trivial smooth fibration

$$(5.15) \quad \mathbb{C}^{n+1} \times \mathbb{C}^k \setminus \Phi^{-1}(D_F) \rightarrow \mathbb{C} \times \mathbb{C}^k \setminus D_F,$$

where $D_F = \Phi(C_F)$ is the discriminant of Φ (see Section 4.2.5). The fiber of Φ over a point (λ, t) is the complex hypersurface $Z_{\lambda, t} := \{x \in \mathbb{C}^{n+1} \mid F(x, t) = \lambda\}$. Note that this is a non-singular hypersurface if and only if $(\lambda, t) \notin D_F$. The fibration (5.15) will be called the *Milnor fibration* of F .

REMARK 5.18. We restricted ourselves to linear deformations only, but the constructions in this section work for tame deformations as well.

We would like to prove that the non-singular fibers $Z_{\lambda,t}$ have the homotopy type of a bouquet of spheres $\bigvee_{i=1}^{\mu} \mathbb{S}^n$. The idea of the proof is due to Milnor – [48], Theorem 6.5 (see also [6]). Moreover, we would like to introduce the notion of a vanishing cycle and to construct a \mathbb{Z} -basis of $H_n(Z_{\lambda,t}, \mathbb{Z})$ consisting of vanishing cycles.

Since (5.15) is a locally trivial fibration the homotopy types of all non-singular fibers $Z_{\lambda,t}$ are the same. Moreover, the disjoint unions

$$(5.16) \quad H^n := \bigcup_{(\lambda,t) \notin D_F} H^n(Z_{\lambda,t}, \mathbb{C}) \quad \text{and} \quad H_n := \bigcup_{(\lambda,t) \notin D_F} H_n(Z_{\lambda,t}, \mathbb{C})$$

have a natural structure of holomorphic vector bundles on $\mathbb{C} \times \mathbb{C}^k \setminus D_F$ equipped with flat connections. We will refer to these bundles as respectively the *vanishing cohomology* and the *vanishing homology bundles* and to the corresponding flat connection as a *Gauss–Manin* connection. In order to solve the problems stated above we may choose any non-singular fiber that we wish.

LEMMA 5.19. *There exists $t^\circ \in \mathbb{C}^k$, such that, $F(x, t^\circ)$ is a Morse function whose critical values have pairwise distinct imaginary parts.*

PROOF. We already know that the family $(F, p : Z_{\text{lin}} \rightarrow S_{\text{lin}})$ is Morse – see Section 5.1. Therefore, the projection map p induces an analytic covering $p|_{C_F} : C_F \rightarrow \mathbb{C}^k$ whose branching locus consists of $t \in \mathbb{C}^k$, such that, $F(x, t)$ is not a Morse function – see Lemma 4.10. Let $t \in \mathbb{C}^k$ be a non-branching point and let $V \subset \mathbb{C}^k$ be a sufficiently small open neighborhood, such that, $p^{-1}(V) \cap C_F = \sqcup_{i=1}^{\mu} V_i$, where each V_i is biholomorphic to V , that is, $p|_{V_i} : V_i \rightarrow V$ is a biholomorphism. Let $\xi^{(i)}$ be the inverse of $p|_{V_i}$, then $\xi^{(i)}(t) = (\xi_0^{(i)}(t), \dots, \xi_n^{(i)}(t))$ ($1 \leq i \leq \mu$, $t \in V$) are the critical points of $F(x, t)$ and $u_i(t) = F(\xi^{(i)}(t), t)$ are the corresponding critical values. Differentiating with respect to t_a and using that $\xi^{(i)}(t)$ is a critical point, we get $\frac{\partial u_i}{\partial t_a} = \xi_a^{(i)}(t)$ for all $0 \leq a \leq k-1$. Note that the first k coordinates of $\xi^{(i)}(t)$ uniquely determine the remaining $n+1-k$ ones. Since the critical points are pairwise distinct, we get that if $i \neq j$, then u_i and u_j are different holomorphic functions on V and hence $\text{Im}(u_i)$ and $\text{Im}(u_j)$ are different real analytic functions. For t° we can choose any point in the complement of the real analytic hypersurface in V defined by $\prod_{1 \leq i < j \leq \mu} (\text{Im } u_i(t) - \text{Im } u_j(t)) = 0$. \square

Let us choose t° as in Lemma 5.19, that is, $F(x, t^\circ)$ is a Morse function and its critical values u_i° ($1 \leq i \leq \mu$) have pairwise distinct imaginary parts. Let λ° be a positive real number, such that, $-\lambda^\circ < \text{Re}(u_i^\circ) < \lambda^\circ$ for all $1 \leq i \leq \mu$. Let us apply Morse theory to $M = \mathbb{C}^{n+1}$ and $\phi(x) := -\text{Re } F(x, t^\circ)$.

LEMMA 5.20. *The function ϕ satisfies condition (C).*

PROOF. Suppose that $S \subset \mathbb{C}^k$ is a subset, such that, $\|\nabla\phi(x)\| < K$ for all $x \in S$ for some constant K and that $\nabla\phi$ is not bounded away from 0, that is, there exists a sequence $s_n \in S$, such that, $\|\nabla\phi(s_n)\| \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, linear deformations are tame – see Proposition 5.2 and $\|\nabla F(x)\| = \sqrt{2}\|\nabla\phi(x)\|$. Therefore, the set $\{x \in \mathbb{C}^n \mid \|\nabla\phi(x)\| \leq K\}$ is a compact subset of \mathbb{C}^k containing S and hence the sequence $\{s_n\}$. However, in a compact subset every sequence has a convergent subsequence. Therefore, the sequence $\{s_n\}$ has a convergent subsequence, which clearly converges to a critical point of $F(x, t^\circ)$. It remains only to note that for a holomorphic function F the critical points of F and $\text{Re}(F)$ are the same. \square

It is an easy exercise to prove that all critical points of ϕ have index $n + 1$ and co-index $n + 1$. If $\lambda \in \mathbb{C}$, then let us denote by $M_\lambda := \phi^{-1}(-\infty, \operatorname{Re}(\lambda)]$. Recalling Theorem 5.16, we get that for every critical value u_i° there exists an $\epsilon > 0$ and a handle $\psi_i : \mathbb{D}^{n+1} \times \mathbb{D}^{n+1} \rightarrow M_{-u_i^\circ + \epsilon}$, such that, the boundary $\mathbb{S}^n \times \mathbb{D}^{n+1}$ is attached to $M_{-u_i^\circ - \epsilon}$. Let $\mathbb{D}^{n+1}(0, R)$ be the ball in \mathbb{R}^{n+1} with center 0 and radius R . We claim that for every $R \geq 1$, by choosing ϵ sufficiently small so that all handles are mapped in a sufficiently small neighborhood of the corresponding critical point, we can extend ψ_i to a handle

$$\tilde{\psi}_i : \mathbb{D}^{n+1}(0, R) \times \mathbb{D}^{n+1} \rightarrow M_{-u_i^\circ + \epsilon},$$

such that, the boundary $\partial\mathbb{D}^{n+1}(0, R) \times \mathbb{D}^{n+1}$ is attached to $M_{-u_i^\circ - \epsilon - R + 1}$. Indeed, let $G(x, t)$ denotes the time- t flow line of the vector field $-\frac{\nabla\phi}{\nabla\phi(\phi)}$ starting at the point x , that is, $G(x, 0) = x$. Since $\partial_t(F(G(x, t))) = 1$ we have $F(G(x, t)) = F(x) + t$. Let us define

$$\tilde{\psi}_i(x, y) = G(\psi_i(x/r, y), r - 1), \quad r := \|x\| \geq 1, \quad y \in \mathbb{D}^{n+1}.$$

We have

$$F(G(\psi_i(x/r, y), r - 1)) = F(\psi_i(x/r, y)) + r - 1.$$

Since we can arrange that $F(\psi_i(x/r, y))$ is in a sufficiently small neighborhood of u_i° and all critical values u_j° have pairwise different imaginary parts we get that the flow lines used to define the extension $\tilde{\psi}_i$ can not converge to a critical value, that is, they are defined for all $r \geq 1$. Since $(x/r, y) \in \partial\mathbb{D}^{n+1} \times \mathbb{D}^{n+1}$ the point $\psi_i(x/r, y) \in \partial M_{-u_i^\circ - \epsilon}$, that is,

$$\phi(G(\psi_i(x/r, y), r - 1)) = -\operatorname{Re} u_i^\circ - \epsilon - r + 1.$$

Let us choose the radius R for the extension $\tilde{\psi}_i$ to be $R_i = 1 + \lambda^\circ - \operatorname{Re} u_i^\circ - \epsilon$. Then the boundary $\mathbb{S}^n(0, R_i) \times \mathbb{D}^{n+1}$ of the handle $\tilde{\psi}_i$ is attached to $M_{-\lambda^\circ}$.

PROPOSITION 5.21. *The fibers of the Milnor fibration (5.15) have the homotopy type of a bouquet of μ spheres, where μ is the Milnor number of f .*

PROOF. It is sufficient to prove the proposition for $Z_{\lambda^\circ, t^\circ}$, where λ° and t° are chosen as above. We will assume that $n \geq 2$ and leave the case $n = 1$ as an exercise. Note that

$$M_{-\lambda^\circ} = \{x \in \mathbb{C}^{n+1} \mid \operatorname{Re} F(x, t^\circ) \geq \lambda^\circ\}$$

is the total space of the restriction of the Milnor fibration to $\{\operatorname{Re} \lambda \geq \lambda^\circ\} \times \{t^\circ\} \subset \mathbb{C} \times \mathbb{C}^k$. Since λ° is a deformation retract of $\{\operatorname{Re} \lambda \geq \lambda^\circ\}$ and smooth fibrations have the homotopy lifting property, the fiber $Z_{\lambda^\circ, t^\circ}$ is also a deformation retract of $M_{-\lambda^\circ}$. For a similar reason, M_{λ° is a deformation retract of \mathbb{C}^{n+1} . Finally, according to Theorem 5.16

$$M_{\lambda^\circ} = M_{-\lambda^\circ} \cup_{i=1}^\mu \cup_{\tilde{\psi}_i} \mathbb{D}^{n+1}(0, R_i) \times \mathbb{D}^{n+1}.$$

Attaching cells of dimension $n + 1$ does not change the homotopy groups π_i with $0 \leq i \leq n - 1$. Therefore, $\pi_i(M_{-\lambda^\circ}) = \pi_i(M_{\lambda^\circ}) = 0$ for all $0 \leq i \leq n - 1$. By the Hurewicz theorem (here we have to use that $n \geq 2!$), the natural map $\pi_n(M_{-\lambda^\circ}) \rightarrow H_n(M_{-\lambda^\circ}, \mathbb{Z})$ is an isomorphism. We claim that the homology classes of the spheres $\mathbb{S}^n(0, R_i)$, embedded in $M_{-\lambda^\circ}$ via $\tilde{\psi}_i$ form a \mathbb{Z} -basis of $H_n(M_{-\lambda^\circ}, \mathbb{Z})$. If this is proved, then the proof can be completed as follows. Let us construct a map

$$(5.17) \quad (\mathbb{S}^n \vee \dots \vee \mathbb{S}^n, \text{base point}) \rightarrow (M_{-\lambda^\circ}, \text{base point})$$

where the number of spheres in the bouquet is μ and the base point of $M_{-\lambda^\circ}$ is chosen arbitrary. The i th copy of \mathbb{S}^n in the bouquet is homotopic to $I \vee \mathbb{S}^n$, where $I = [0, 1]$ is

an interval. We choose an isomorphism between $\mathbb{S}^n \cong \tilde{\psi}_i(\mathbb{S}^n(0, R_i))$ and extend it to a map $I \vee \mathbb{S}^n \rightarrow M_{-\lambda^\circ}$ in such a way that the interval I is mapped to a path connecting the sphere $\tilde{\psi}_i(\mathbb{S}^n(0, R_i))$ with the base point. The map (5.17) induces an isomorphism between the homotopy groups. Recalling the Whitehead's theorem, we get that (5.17) is a homotopy equivalence.

It remains only to prove our claim about $H_n(M_{-\lambda^\circ}, \mathbb{Z})$. Let us consider the long exact sequence in homology for the pair $(M_{\lambda^\circ}, M_{-\lambda^\circ})$

$$0 = H_{n+1}(M_{\lambda^\circ}) \longrightarrow H_{n+1}(M_{\lambda^\circ}, M_{-\lambda^\circ}) \xrightarrow{\partial} H_n(M_{-\lambda^\circ}) \longrightarrow H_n(M_{\lambda^\circ}) = 0$$

where all homology groups are with coefficients in \mathbb{Z} and the homology of M_{λ° vanishes because M_{λ° is homotopic to \mathbb{C}^{n+1} . We get that the boundary morphism in the long exact sequence gives an isomorphism $H_{n+1}(M_{\lambda^\circ}, M_{-\lambda^\circ}) \cong H_n(M_{-\lambda^\circ})$. The pair $(M_{\lambda^\circ}, M_{-\lambda^\circ})$ has the homotopy extension property. Therefore

$$H_{n+1}(M_{\lambda^\circ}, M_{-\lambda^\circ}) = \tilde{H}_{n+1}(M_{\lambda^\circ}/M_{-\lambda^\circ}) = \tilde{H}_{n+1}\left(\bigvee_i \mathbb{D}^{n+1}/\partial\mathbb{D}^{n+1}\right) = \bigoplus_{i=1}^{\mu} \tilde{H}_{n+1}(\mathbb{S}^{n+1}).$$

This proves that the relative homology group $H_{n+1}(M_{\lambda^\circ}, M_{-\lambda^\circ}) \cong \mathbb{Z}^\mu$ and that a \mathbb{Z} -basis is given by the homology classes of the handles $\tilde{\psi}_i$. \square

5.3.3. Picard–Lefschetz formula. Let us recall some basic terminology from the so-called Picard–Lefschetz theory. We refer to Chapter 2 in [6] for more details. Let us consider the homology group $H_n(Z_{\lambda^\circ, t^\circ}, \mathbb{Z})$ where (λ°, t°) are as in the above discussion. The Milnor fiber $Z_{\lambda^\circ, t^\circ}$ is a smooth complex manifold of dimension $2n$. By Poincaré duality for non-compact manifolds the cap product with the fundamental class of $Z_{\lambda^\circ, t^\circ}$ gives an isomorphism

$$\text{PD} : H_{\text{cpt}}^n(Z_{\lambda^\circ, t^\circ}, \mathbb{Z}) \rightarrow H_n(Z_{\lambda^\circ, t^\circ}, \mathbb{Z}),$$

where H_{cpt}^\bullet denotes cohomology with compact support. Using this isomorphism and the natural pairing between cohomology and homology we define the *intersection pairing*

$$\alpha \circ \beta := \langle \text{PD}^{-1}(\alpha), \beta \rangle.$$

The homology group $H_n(Z_{\lambda^\circ, t^\circ}, \mathbb{Z})$ equipped with the intersection pairing is called the *Milnor lattice*. Parallel transport with respect to the Gauss–Manin connection defines a representation

$$\pi_1(\mathbb{C} \times S_{\text{lin}} \setminus D_F, (\lambda^\circ, t^\circ)) \rightarrow \text{End}(H_n(Z_{\lambda^\circ, t^\circ}, \mathbb{Z}))$$

known as the *monodromy representation*. The image W of the monodromy representation is called the *monodromy group* of f .

The Milnor fiber $Z_{\lambda^\circ, t^\circ}$ is homotopy equivalent to $M_{-\lambda^\circ}$. The spheres $\mathbb{S}^n(0, R_i) \subset M_{-\lambda^\circ}$ contained in the boundary of the handle $\tilde{\psi}_i$ form a \mathbb{Z} -basis of $H_n(M_{-\lambda^\circ}, \mathbb{Z}) \cong H_n(Z_{\lambda^\circ, t^\circ}, \mathbb{Z})$. These spheres have the property that under the gradient flow of $-\text{Re} F$ they flow to a critical point of F , that is, they vanish. The smallest W -invariant subset of $H_n(Z_{\lambda^\circ, t^\circ}, \mathbb{Z})$ containing the homology classes of the spheres $\mathbb{S}^n(0, R_i)$ is called the set of *vanishing cycles*. By definition, for each vanishing cycle α there exists a path C approaching a generic point on the discriminant D_F , such that, under the parallel transport along C the cycle α vanishes. Given a vanishing cycle α and a corresponding path C , let us construct a simple loop by approaching the discriminant along C , just before hitting the discriminant D_F we make a small counterclockwise loop around D_F ,

and then return to the reference point (λ°, t°) again along C . Slightly abusing the notation we denote the simple loop by C and refer to α as the vanishing cycle corresponding to the simple loop C .

PROPOSITION 5.22. *a) If α is a vanishing cycle, then the self-intersection number $\alpha \circ \alpha = (-1)^{n(n-1)/2}(1 + (-1)^n)$.*

b) If C is a simple loop around the discriminant, then there exists a vanishing cycle α , such that, α is a vanishing cycle corresponding to the loop C . Moreover, the monodromy transformation representing the simple loop C is given by the following formula:

$$x \mapsto x + (-1)^{(n+1)(n+2)/2}(x \circ \alpha) \alpha, \quad \forall x \in H_n(Z_{\lambda^\circ, t^\circ}, \mathbb{Z}).$$

Part a) of the above proposition is an elementary local computation (see [6], Lemma 1.4). The proof of part b) is more difficult – one has to investigate the properties of the so-called variation operators. We refer to [6], Section 1.3 for further details. Let us point out that the formula in part b) is known as the Picard–Lefschetz formula.

5.4. Hodge theory on orbifolds

Let us recall the notion of an orbifold. For more systematic study of this subject we refer to [4, 3] and the references there in. There are various points of view. We would like to outline the theory in such a way that one can easily extend results for complex manifolds to complex orbifolds. More precisely, we will need the orbifold version of primitive cohomology, Hodge theory, and Poincaré residue.

5.4.1. Complex orbifolds. Let \mathcal{G} be a small category and let \mathcal{G}_0 and \mathcal{G}_1 be respectively the set of objects and the set of morphisms of \mathcal{G} .

DEFINITION 5.23. The category \mathcal{G} is said to be a *complex orbifold groupoid* if \mathcal{G}_0 and \mathcal{G}_1 are equipped with the structure of complex manifolds, such that, the following conditions are satisfied:

- (i) All morphisms in \mathcal{G} are isomorphisms and the inverse map $\iota : \mathcal{G}_1 \rightarrow \mathcal{G}_1, g \mapsto g^{-1}$ is holomorphic.
- (ii) The source and the target maps $s, t : \mathcal{G}_1 \rightarrow \mathcal{G}_0$ are local bi-holomorphisms.
- (iii) The map $\text{id} : \mathcal{G}_0 \rightarrow \mathcal{G}_1$ which to an object x assigns the identity morphism $x \rightarrow x$ in \mathcal{G} is holomorphic.
- (iv) Note that the fiber product

$$\mathcal{G}_2 := \mathcal{G}_{1s} \times_t \mathcal{G}_1 = \{(h, g) \in \mathcal{G}_1 \times \mathcal{G}_1 \mid s(h) = t(g)\}$$

is a smooth manifold, because s is a submersion. Then composition of morphisms induces a holomorphic map

$$\mathcal{G}_{1s} \times_t \mathcal{G}_1 \rightarrow \mathcal{G}_1, \quad (h, g) \mapsto h \circ g.$$

- (v) The map $(s, t) : \mathcal{G}_1 \rightarrow \mathcal{G}_0 \times \mathcal{G}_0$ is proper. □

Let us point out that if we relax condition (ii) by requiring that the source and the target maps s and t are submersions, then the resulting groupoid is called a *complex Lie groupoid*. If \mathcal{G} is a complex Lie groupoid, then the *orbit space* $|\mathcal{G}|$ of \mathcal{G} is defined to be the quotient \mathcal{G}_0 / \sim , where the equivalence relation is defined by $x \sim y$ if there exists $h \in \mathcal{G}_1$, such that, $s(h) = x$ and $t(h) = y$. Let $\pi : \mathcal{G}_0 \rightarrow |\mathcal{G}|$ be the quotient map. We equip $|\mathcal{G}|$ with the quotient topology, i.e., a subset $V \subset |\mathcal{G}|$ is open iff $U := \pi^{-1}V$ is open in \mathcal{G}_0 .

Note that in addition $U = \pi^{-1}V$ is \mathcal{G} -invariant, that is, $s^{-1}(U) = t^{-1}(U)$. We say that a function $f : U \rightarrow \mathbb{C}$ is \mathcal{G} -invariant if $f \circ s = f \circ t$ on $s^{-1}(U) = t^{-1}(U)$. Let us denote by $\mathcal{O}_{|\mathcal{G}|}$ the sheaf on $|\mathcal{G}|$ whose sections over an open subset V are given by the holomorphic \mathcal{G} -invariant functions on $U := \pi^{-1}V$.

THEOREM 5.24. *If \mathcal{G} is a complex orbifold groupoid, then $(|\mathcal{G}|, \mathcal{O}_{|\mathcal{G}|})$ is a reduced normal complex space, i.e., the orbit space $|\mathcal{G}|$ is a normal analytic variety.*

An analytic variety X is said to be an *orbifold* if there exists a complex orbifold groupoid \mathcal{G} , such that, $X \cong |\mathcal{G}|$. The proof of Theorem 5.24 is non-trivial. The interested reader is referred to Grauert's paper [26] and to the references there in. The algebraic version of Theorem 5.24 is equivalent to the well known fact that if G is a finite group acting on a finite dimensional vector space, then the ring of G -invariant polynomials is finitely generated (e.g. see [4]). One can obtain the analytic result from the algebraic one.

The main example of a complex Lie groupoid is the following. Suppose that U is a complex manifold and that G is a complex Lie group acting holomorphically on U . Put $\mathcal{G}_0 := U$ and $\mathcal{G}_1 := G \times U$. The structure maps are defined by

$$s(g, x) := x, \quad t(g, x) := g \cdot x, \quad \text{id}(x) := (1, x), \quad \iota(g, x) = (g^{-1}, x),$$

where $g \in G$ and $x \in U$. The resulting complex Lie groupoid is called *translation groupoid* and it will be denoted by $[U/G]$. If the group G is finite then $[U/G]$ is a complex orbifold groupoid.

DEFINITION 5.25. Suppose that \mathcal{G} is a complex Lie groupoid, X is a complex manifold, and G is a finite group acting on X . We say that the translation groupoid $[X/G]$ is *equivalent* to \mathcal{G} if there are embeddings of complex manifolds $\phi_0 : X \rightarrow \mathcal{G}_0$ and $\phi_1 : G \times X \rightarrow \mathcal{G}_1$, such that,

- (i) The pair of maps (ϕ_0, ϕ_1) define a groupoid homomorphism $[X/G] \rightarrow \mathcal{G}$, that is, the two maps are compatible with the structure maps of the two groupoids.
- (ii) The following diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\phi_1} & \mathcal{G}_1 \\ (s,t) \downarrow & & \downarrow (s,t) \\ X \times X & \xrightarrow{(\phi_0, \phi_0)} & \mathcal{G}_0 \times \mathcal{G}_0 \end{array}$$

is Cartesian.

- (iii) The map

$$\mathcal{G}_1 \times_{\phi_0} X \rightarrow \mathcal{G}_0, \quad (h, x) \mapsto t(h)$$

is a principal G -bundle, where G acts on the fiber product by

$$g \cdot (h, x) := (h \circ \phi_1(g, x)^{-1}, g \cdot x). \quad \square$$

Let us point out that the fiber product in condition (iii) consists of pairs $(h, x) \in \mathcal{G}_1 \times X$, such that $s(h) = \phi_0(x)$. Since s is a submersion, the fiber product is a smooth submanifold of $\mathcal{G}_1 \times X$. Condition (ii) implies that ϕ_1 defines an isomorphism between the stabilizer subgroup $G_x := \{g \in G \mid g \cdot x = x\}$ and the isotropy group $\text{Aut}(y) := s^{-1}(y) \cap t^{-1}(y)$, where $x \in X$ and $y = \phi_0(x)$. This fact implies that the action of G on

$\mathcal{G}_{1s} \times_{\phi_0} X$ is free. Finally, condition (ii) implies also that ϕ_0 induces a homeomorphism between the orbit spaces $|\mathcal{G}| \cong X/G$.

We would like to explain a method that would allow us to construct an orbifold groupoid by gluing translation groupoids $\mathcal{X}_i := [X_i/G_i]$, where X_i is a complex manifold and G_i is a finite group. Suppose that \mathcal{G} is a complex Lie groupoid and let $X := |\mathcal{G}|$ be its orbit space. If $U \subset \mathcal{G}_0$ is an open subset, then we can define the restriction $\mathcal{G}|_U$ as the complex Lie groupoid whose objects are $(\mathcal{G}|_U)_0 := U$ and $(\mathcal{G}|_U)_1 := s^{-1}(U) \cap t^{-1}(U)$. Suppose that $\{U_i\}$ is a covering of X satisfying the following conditions:

- (i) There exists an open subset $\mathcal{G}_{0i} \subset \mathcal{G}_0$, such that, the restriction \mathcal{G}_i of \mathcal{G} to \mathcal{G}_{0i} is equivalent to an orbifold translation groupoid $[X_i/G_i]$ with G_i a finite group, that is, there exists a groupoids homomorphism $[X_i/G_i] \rightarrow \mathcal{G}_i$ which is equivalence in the sense of Definition 5.25
- (ii) $U_i = |\mathcal{G}_i| \cong X_i/G_i$.
- (iii) Put

$$X_{ij} := \{(x^i, h, y^j) \in X_i \times \mathcal{G}_1 \times X_j \mid x^i = s(h), y^j = t(h)\}.$$

Then the two projections

$$\pi_{ij,i} : X_{ij} \rightarrow X_i, \quad (x^i, h, y^j) \mapsto x^i$$

and

$$\pi_{ij,j} : X_{ij} \rightarrow X_j, \quad (x^i, h, y^j) \mapsto y^j$$

are local biholomorphisms.

We refer to X_i as orbifold charts and to any coordinate system x^i on X_i as orbifold coordinates. Note that X_{ij} coincides with the objects of the orbifold groupoid fibre product $[X_i/G_i] \times_{\mathcal{G}} [X_j/G_j]$. For that reason X_{ij} is called the orbifold intersection of X_i and X_j . The functions $\pi_{ij,i}$ and $\pi_{ij,j}$ are the orbifold version of transition functions. If conditions (i)–(iii) are satisfied, then we can construct an orbifold groupoid \mathcal{X} as follows. Put

$$\mathcal{X}_0 := \sqcup_i X_i, \quad \mathcal{X}_1 := \sqcup_{i,j} X_{ij},$$

where in the 2nd disjoint union we allow $i = j$: then $X_{ii} \cong G_i \times X_i$. The structure maps of \mathcal{X} are defined as follows:

$$s(x^i, h, y^j) := x^i, \quad t(x^i, h, y^j) := y^j, \quad \iota(x^i, h, y^j) := (y^j, h^{-1}, x^i), \quad \text{id}(x^i) := (x^i, \text{id}_{x^i}, x^i),$$

where $(x^i, h, y^j) \in X_{ij}$. The composition is defined by

$$(y^j, h, z^k) \circ (x^i, g, y^j) := (x^i, h \circ g, z^k),$$

where $(x^i, g, y^j) \in X_{ij}$ and $(y^j, h, z^k) \in X_{jk}$. It is straightforward to check that \mathcal{X} is a complex orbifold groupoid and that its orbit space $|\mathcal{X}| = X$.

5.4.2. Orbifold de Rham theory. Let us assume that we are in the settings of the above section: X is an analytic variety that can be identified with the orbit space of a complex Lie groupoid \mathcal{G} and there exists an open covering U_i of X that can be lifted to an orbifold groupoid \mathcal{X} . Let us recall that an *orbifold vector bundle* on X is a vector bundle $E \rightarrow \mathcal{X}_0$ and an isomorphism $a : s^*E \rightarrow t^*E$ satisfying the following cocycle condition. If $h \in \mathcal{X}_1$, then let a_h be the isomorphism between $(s^*E)_h = E_{s(h)}$ and $(t^*E)_h = E_{t(h)}$, then for any two composable morphisms h_1 and h_2 , that is, $s(h_1) = t(h_2)$ we have $a_{h_1} \circ a_{h_2} = a_{h_1 \circ h_2}$. In terms of the orbifold charts, an orbifold vector bundle is

determined by $E_i := E|_{X_i}$ and isomorphisms $a_{ij} : \pi_{ij,i}^* E_i \cong \pi_{ij,j}^* E_j$ satisfying the cocycle condition

$$p_{ik}^* a_{ik} = p_{jk}^* a_{jk} \circ p_{ij}^* \pi_{ij},$$

where p_{ab} are the projections to X_{ab} from the orbifold tripple intersection

$$X_{ijk} := \{(x^i, h, y^j, g, z^k) \mid s(h) = x^i, t(h) = y^j = s(g), t(g) = z^k\}$$

defined by

$$\begin{aligned} p_{ij}(x^i, h, y^j, g, z^k) &:= (x^i, h, y^j), \\ p_{jk}(x^i, h, y^j, g, z^k) &:= (y^j, g, z^k), \\ p_{ik}(x^i, h, y^j, g, z^k) &:= (x^i, g \circ h, z^k). \end{aligned}$$

Finally, a section s of E consists of sections $s_i \in \Gamma(X_i, E_i)$, such that, $a_{ij}(\pi_{ij,i}^* s_i) = \pi_{ij,j}^* s_j$. Slightly abusing the terminology we will refer to s_i as the *restriction* of s to X_i . The space of sections of E will be denoted by $\Gamma(X, E)$. Note that by specializing $i = j$, we get that E_i is a G_i -equivariant vector bundle on X_i and that each s_i is a G_i -invariant section.

The real tangent bundles $T^{\mathbb{R}} X_i$ define naturally an orbifold vector bundle $T_{\mathcal{X}}^{\mathbb{R}}$ on X . Indeed, the pullbacks $\pi_{ij,i}^* T^{\mathbb{R}} X_i$ and $\pi_{ij,j}^* T^{\mathbb{R}} X_j$ are naturally identified by $a_{ij} := d\pi_{ij,j} \circ d\pi_{ij,i}^{-1}$ and the latter satisfy the cocycle condition because $p_{ij}^* \pi_{ij,j} = p_{jk}^* \pi_{jk,j}$. The vector bundle $T_{\mathcal{X}}^{\mathbb{R}}$ is called the *real orbifold tangent bundle*. Similarly, we can define the complexified orbifold tangent bundle $T_{\mathcal{X}}^{\mathbb{C}}$, the holomorphic orbifold tangent bundle $T_{\mathcal{X}} := T_{\mathcal{X}}^{(1,0)}$ and the anti-holomorphic orbifold tangent bundle $\bar{T}_{\mathcal{X}} := T_{\mathcal{X}}^{(0,1)}$.

Now it is straightforward to extend the definition of the de Rham and the Dolbeault complexes to orbifolds. Let $A^n(X)$ be the vector space of global smooth sections of $\wedge^n(T_{\mathcal{X}}^{\mathbb{C}})^*$ and $A^{p,q}(X)$ be the space of global smooth sections of $\wedge^p(T_{\mathcal{X}}^*) \otimes \wedge^q(\bar{T}_{\mathcal{X}}^*)$. The vector space $A^n(X)$ admits a real structure defined by $T_{\mathcal{X}}^{\mathbb{R}}$, that is, a form $\omega \in A^n(X)$ is said to be real if the sequence of contractions $\iota_{v_1} \circ \dots \circ \iota_{v_n}(\omega)$ yield a real valued function for any sequence $v_1, \dots, v_n \in T_{\mathcal{X}}^{\mathbb{R}}$ of real vector fields. Let $A^n(X, \mathbb{R})$ be the space of real forms, then $A^n(X) = A^n(X, \mathbb{R}) + \mathbf{i}A^n(X, \mathbb{R})$ and we have complex conjugation: if $\omega = \omega_1 + \mathbf{i}\omega_2$ with $\omega_1, \omega_2 \in A^n(X, \mathbb{R})$, then $\bar{\omega} := \omega_1 - \mathbf{i}\omega_2$. Furthermore, note that

$$A^r(X) = \bigoplus_{p+q=r} A^{p,q}(X), \quad \overline{A^{p,q}(X)} = A^{q,p}(X).$$

and that the de Rham differential $d : A^r(X) \rightarrow A^{r+1}(X)$ decomposes as $d = \partial + \bar{\partial}$, where $\partial : A^{p,q}(X) \rightarrow A^{p+1,q}(X)$, $\bar{\partial} : A^{p,q}(X) \rightarrow A^{p,q+1}(X)$. The de Rham and the Dolbeault cohomology groups are defined by

$$H_{\text{dR}}^r(X, \mathbb{C}) := H^r(A^{\bullet}(X), d), \quad H^{p,q}(X) := H^q(A^{p,\bullet}(X), \bar{\partial}).$$

According to Satake [55], the de Rham cohomology group $H_{\text{dR}}^r(X, \mathbb{C}) \cong H^r(X, \mathbb{C})$. From now on we denote the de Rham cohomology group simply by $H^r(X, \mathbb{C})$. Let us point out that the de Rham differential preserves the real structure, that is, $d : A^r(X, \mathbb{R}) \rightarrow A^{r+1}(X, \mathbb{R})$. Therefore, the real de Rham cohomology group $H_{\text{dR}}^r(X, \mathbb{R}) := H^r(A^{\bullet}(X, \mathbb{R}), d)$ defines a real structure in $H_{\text{dR}}^r(X, \mathbb{C})$. Again, we have an isomorphism $H_{\text{dR}}^r(X, \mathbb{R}) \cong H^r(X, \mathbb{R})$, so we denote the real de Rham cohomology group simply by $H^r(X, \mathbb{R})$.

Suppose that X is compact. Then we can integrate differential forms as follows. Let ρ_i be a partition of unity subordinate to the open covering U_i of X . The quotient

map $X_i \rightarrow X_i/G_i \cong U_i$ is a finite proper map. Therefore, the pullback $\tilde{\rho}_i$ of ρ_i to X_i is a smooth function with compact support. If $\omega \in A^r(X)$, then orbifold integration is defined by

$$\int_X \omega := \sum_i \frac{1}{|G_i|} \int_{X_i} \tilde{\rho}_i \omega_i,$$

where ω_i is the restriction of ω to X_i and the above integral makes sense because X_i is a manifold and $\tilde{\rho}_i$ has compact support. Note that the integral is non-zero only if the degree r of ω coincides with the real dimension of X . Finally, orbifold integration induces a \mathbb{C} -bilinear pairing on $H^\bullet(X, \mathbb{C})$

$$(\phi_1, \phi_2) := \int_X \phi_1 \wedge \phi_2, \quad \phi_1, \phi_2 \in H^\bullet(X, \mathbb{C}),$$

which will be called the *orbifold Poincare pairing*.

Finally, let us define the orbifold version of a Poincare residue (see [27]). Suppose that Y is an orbifold hypersurface in X , that is, there exists an open covering U_i of X with corresponding orbifold charts X_i , such that, the orbifold intersection $Y_i := X_i \cap Y := X_i \times_X Y$ is a smooth analytic hypersurface in X_i . By choosing a finer covering U_i if necessary we can arrange that the ideal sheaf of Y_i in X_i is generated by a holomorphic function $f_i \in \mathcal{O}(X_i)$. We say that a smooth form $\omega \in A^r(X \setminus Y)$ has a *logarithmic pole* along Y if the restrictions $\omega_i \in A^r(X_i \setminus Y_i)$ of ω to the orbifold charts X_i satisfy $f_i \omega_i \in A^r(X_i)$ and $df_i \wedge \omega_i \in A^{r+1}(X_i)$, that is, $f_i \omega_i$ and $df_i \wedge \omega_i$ extend to smooth forms on X_i for all i . Since in a neighborhood of Y_i we can find a holomorphic coordinate system on X_i , such that, one of the coordinates is f_i , we have the following decomposition

$$\omega_i = \alpha_i \wedge \frac{df_i}{f_i} + \beta_i,$$

where α_i and β_i are smooth forms on X_i and β_i does not contain monomials divisible by df_i . Put

$$\text{Res}_Y \omega_i := \alpha_i|_{Y_i} \in A^{r-1}(Y_i).$$

The forms $\text{Res}_Y \omega_i$ are local representatives of an orbifold differential form in $A^{r-1}(Y)$, which will be denoted by $\text{Res}_Y \omega$ and will be called the *residue* of ω along Y .

5.4.3. Hodge theory on compact Kähler orbifolds. With the above definitions at hands it is a routine exercise to extend the harmonic theory of differential forms (e.g. see [63]) to orbifolds. Let us just state the results relevant for our purposes. In order for the harmonic theory to work smoothly, just like in the manifold case, let us assume that X is compact and Kähler. The latter means that there exists a real $(1, 1)$ -form $\omega \in A^{1,1}(X)$, which is closed and positive definite. If we fix orbifold coordinates $x^i = (x_1^i, \dots, x_n^i)$ on the orbifold chart X_i , then

$$\omega_i = \frac{\mathbf{i}}{2} \sum_{a,b=1}^n h_{ab}^i dx_a^i \wedge d\bar{x}_b^i,$$

where ω_i is the restriction of ω to X_i and the matrix h^i with entries h_{ab}^i is a positive definite Hermitian matrix. Let

$$L : A^{p,q}(X) \rightarrow A^{p+1,q+1}(X), \quad L(\phi) := \omega \wedge \phi,$$

be the so-called *Lefschetz operator*. Since ω is a real form, the Lefschetz operator is real, i.e., it preserves the real structure $A^\bullet(X, \mathbb{R})$. We have the following *primitive cohomology groups*:

$$P^r(X, k) := \{\xi \in H^r(X, k) \mid L^{n-r+1}\xi = 0\}, \quad k = \mathbb{R}, \mathbb{C},$$

and

$$P^{p,q}(X) := \{\xi \in H^{p,q}(X) \mid L^{n-p-q+1}\xi = 0\},$$

where we used that the operator L commutes with both d and $\bar{\partial}$ and therefore it induces a linear operator in both the de Rham and the Dolbeault cohomologies. By definition, $P^r(X, k) = 0$ for $r > n$ and $P^{p,q}(X) = 0$ for $p + q > n$. The following theorem is known as the *Lefschetz decomposition theorem*.

THEOREM 5.26. *If X is a compact Kähler orbifold, then*

$$H^r(X, k) = \bigoplus_{s=(r-n)^+}^{\lfloor r/2 \rfloor} L^s P^{r-2s}(X, k), \quad k = \mathbb{R}, \mathbb{C},$$

where if x is a real number, then we denote by $x^+ := \max(x, 0)$ and by $\lfloor x \rfloor$ the integer part of x . \square

For a proof we refer to [63], Corollary 3.12. The next theorem is known as the *Hodge decomposition theorem*.

THEOREM 5.27. *If X is a compact Kähler orbifold, then*

$$H^r(X, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(X), \quad \overline{H^{p,q}(X)} = H^{q,p}(X)$$

and

$$P^r(X, \mathbb{C}) = \bigoplus_{p+q=r} P^{p,q}(X), \quad \overline{P^{p,q}(X)} = P^{q,p}(X). \quad \square$$

For a proof we refer to [63], Theorem 4.1. Finally, let us recall also the so-called Hodge–Riemann bilinear relations. Let us define the following bilinear form on $P^r(X, \mathbb{C})$

$$(5.18) \quad S(\xi, \eta) := \int_X L^{n-r}\xi \wedge \eta = \int_X \omega^{n-r} \wedge \xi \wedge \eta, \quad \xi, \eta \in P^r(X, \mathbb{C}).$$

Note that S is a real pairing, i.e., if $\xi, \eta \in P^r(X, \mathbb{R})$, then $S(\xi, \eta) \in \mathbb{R}$.

THEOREM 5.28. *If X is a compact Kähler orbifold, then*

a) *the Hodge decomposition of $P^r(X, \mathbb{C})$ is orthogonal with respect to the Hermitian pairing*

$$\langle \xi, \eta \rangle := \mathbf{i}^r S(\xi, \bar{\eta}), \quad \xi, \eta \in P^r(X, \mathbb{C}),$$

that is, $\langle \xi, \eta \rangle = 0$ for all $\xi \in P^{p,q}$ and $\eta \in P^{p',q'}$ with $(p, q) \neq (p', q')$.

b) *The restriction of the above Hermitian pairing to $P^{p,q}(X)$ is positive definite for even q and negative definite for odd q .* \square

For a proof we refer to [63], Theorem 5.3. It is common to reformulate the Hodge decomposition theorem and the Hodge–Riemann bilinear relations as follows.

DEFINITION 5.29. Suppose that H is a complex vector space equipped with a real structure $H_{\mathbb{R}}$. A *Polarized Hodge Structure* on H of weight r is the data of a decreasing filtration F^p ($p \in \mathbb{Z}$) of H

$$F^p = 0 \quad \text{for } p \gg 0, \quad F^{p+1} \subseteq F^p, \quad F^p = H \quad \text{for } p \ll 0$$

and a real $(-1)^r$ -symmetric form S , that is, $S(x, y) = (-1)^r S(y, x)$, such that,

- (i) $H = F^p \oplus \overline{F^{r-p+1}}$ for all $p \in \mathbb{Z}$,
- (ii) $S(F^p, \overline{F^{r-p+1}}) = 0$, for all p ,
- (iii) $\mathbf{i}^{r^2+2r-2p} S(x, \bar{x}) > 0$ for all $x \in F^p \cap \overline{F^{r-p}} \setminus \{0\}$. \square

The Hodge decomposition theorem and the Hodge–Riemann bilinear relations can be reformulated by saying that $F^p := P^{p, r-p}(X, \mathbb{C}) \oplus \cdots \oplus P^{r, 0}(X, \mathbb{C})$ and the bilinear form (5.18) define a Polarized Hodge Structure of weight r on $P^r(X, \mathbb{C})$ with real structure $P^r(X, \mathbb{R})$.

5.4.4. Weighted-projective orbifolds. Let d_0, d_1, \dots, d_{n+1} be a sequence of positive integers. The *weighted-projective space* is by definition the quotient space $\mathbb{P}^{d_0, d_1, \dots, d_{n+1}} := (\mathbb{C}^{n+2} \setminus 0)/\mathbb{C}^*$, where the action of \mathbb{C}^* is defined by

$$t \cdot (Z_0, \dots, Z_{n+1}) = (t^{d_0} Z_0, \dots, t^{d_{n+1}} Z_{n+1}).$$

Note that the stabilizer subgroups of the \mathbb{C}^* -action are finite. It is a general fact that if a reductive complex Lie group acts on a complex manifold with finite stabilizers then the corresponding quotient space has a natural orbifold structure. Let us construct a complex orbifold groupoid \mathcal{X} , such that, $|\mathcal{X}| = \mathbb{P}^{d_0, d_1, \dots, d_{n+1}}$. To begin with, let us fix the open covering $U_i := \{Z_i \neq 0\} \subset \mathbb{P}^{d_0, d_1, \dots, d_{n+1}}$. Put $X_i := \mathbb{C}^{n+1}$ and let us denote by $x^i := (x_0^i, \dots, x_{i-1}^i, x_{i+1}^i, \dots, x_{n+1}^i)$ the standard coordinates on X_i . Note that the map

$$\pi_i : X_i \rightarrow U_i, \quad x^i \mapsto [x_0^i, \dots, x_{i-1}^i, 1, x_{i+1}^i, \dots, x_{n+1}^i],$$

induces an isomorphism $U_i \cong X_i/\mu_{d_i}$, where μ_{d_i} is the multiplicative group of order d_i acting on X_i by

$$(\eta \cdot x^i)_s := \eta^{-d_s} x_s^i, \quad s \in \{0, 1, \dots, n+1\} \setminus \{i\}.$$

The relation between the coordinates x^i and x^j of respectively X_i and X_j is given by

$$[x_0^i, \dots, x_{i-1}^i, 1, x_{i+1}^i, \dots, x_{n+1}^i] = [x_0^j, \dots, x_{j-1}^j, 1, x_{j+1}^j, \dots, x_{n+1}^j].$$

Therefore, there exists a non-zero complex number $t \in \mathbb{C}^*$, such that,

$$(5.19) \quad x_s^j = t^{-d_s} x_s^i \quad (s \neq i, j),$$

$$(5.20) \quad x_i^j = t^{-d_i},$$

$$(5.21) \quad x_j^i = t^{d_j}.$$

We would like to think of x^i and x^j as objects and of the solutions t of the above equations for fixed x^i and x^j as the morphisms from x^i to x^j . In other words, we define $\mathcal{X}_0 := \sqcup_{i=0}^{n+1} X_i$ and $\mathcal{X}_1 := \sqcup_{i,j=0}^{n+1} X_{ij}$, where $X_{ii} := \mu_{d_i} \times X_i$ and X_{ij} for $i \neq j$ is the subvariety of $X_i \times X_j \times \mathbb{C}^*$ defined by the equations (5.19)–(5.21). Note that if we set $y_s^{ij} := x_s^i$ for $s \neq i, j$ and $t^{ij} := t$, then the projection map

$$X_{ij} \rightarrow \mathbb{C}^n \times \mathbb{C}^*, \quad (x^i, x^j, t) \mapsto (y^{ij}, t^{ij})$$

is an isomorphisms. This shows that X_{ij} is a complex manifold. The source map $s : X_{ij} \rightarrow X_i$ and the target map $t : X_{ij} \rightarrow X_j$ are defined by

$$s(x^i, x^j, t) := x^i, \quad t(x^i, x^j, t) := x^j$$

for the case when $i \neq j$ and by

$$s(\eta, x^i) := x^i, \quad t(\eta, x^i) := \eta \cdot x^i,$$

where the action of μ_{d_i} on X_i is as defined above, that is, $(\eta \cdot x^i)_s = \eta^{-d_s} x^i_s$. Furthermore, the identity map $X_i \rightarrow X_{ii}$ is defined by $x^i \mapsto (1, x^i)$ and the inverse $X_{ij} \rightarrow X_{ji}$ by $(x^i, x^j, t) \mapsto (x^j, x^i, t^{-1})$. Finally, if $h = (x^j, x^k, t^{jk}) \in X_{jk}$ and $g = (x^i, x^j, t^{ij}) \in X_{ij}$ then the composition $h \circ g := (x^i, x^k, t^{ik}) \in X_{ik}$, where $t^{ik} = t^{ij}t^{jk}$. It is straightforward to check that all conditions in the definition of a complex orbifold groupoid are satisfied, that is, \mathcal{X} is a complex orbifold groupoid. Moreover, the orbit space $|\mathcal{X}| \cong \mathbb{P}^{d_0, \dots, d_{n+1}}$.

We say that M is a *weighted-projective orbifold* if M is an analytic subvariety of $\mathbb{P}^{d_0, d_1, \dots, d_{n+1}}$, such that, $\pi_i^{-1}(M \cap U_i)$ is a smooth analytic subvariety of X_i for all $i = 0, 1, \dots, n+1$.

5.5. Hodge structure on the Milnor fiber

Let us return to the settings of singularity theory, i.e., let $f \in \mathbb{C}[x_0, \dots, x_n]$ be a weighted homogeneous singularity. We would like to prove that the middle cohomology group $H^n(f^{-1}(1), \mathbb{C})$ carries a polarized Hodge structure. This result is due to Steenbrink and Hertling. Namely, Steenbrink proved the existence of a mixed Hodge structure on the middle cohomology group for any singularity, while Hertling proved that the higher residue pairing of K. Saito induces a polarizing form. The proof relies on a very deep result of Schmidt, known as the nilpotent orbit theorem. However, in the case of weighted homogeneous singularities we will give an elementary proof.

5.5.1. Compactification of the Milnor fiber. As usual the arguments for the cases $n = 0$ and 1 need a modification, so let us assume that $n > 1$ and leave the remaining two cases as an exercise. Let us write the weights of the variables x_i as $c_i = \frac{d_i}{d}$, where $d_i, d \in \mathbb{Z}$ and $\gcd(d_0, \dots, d_n) = 1$. Following Steenbrink [58], we compactify the Milnor fiber by embedding it in the weighted projective space $\mathbb{P}^{d_0, \dots, d_{n+1}}$, where $d_{n+1} := 1$. Note that the chart $U_{n+1} = \mathbb{C}^{n+1}$. Let us embed the Milnor fiber $V := f^{-1}(1)$ in U_{n+1} in the obvious way $x_i^{n+1} = x_i$ ($0 \leq i \leq n$). The Zariski closure of V in $\mathbb{P}^{d_0, \dots, d_{n+1}}$ is given by

$$\bar{V} = \{Z \in \mathbb{P}^{d_0, \dots, d_{n+1}} \mid f(Z_0, \dots, Z_n) - Z_{n+1}^d = 0\}.$$

Set theoretically $\bar{V} = V \sqcup V_\infty$, where

$$V_\infty = \bar{V} \cap \{Z_{n+1} = 0\} = \{Z \in \mathbb{P}^{d_0, \dots, d_n} \mid f(Z_0, \dots, Z_n) = 0\}.$$

Using that the only critical point of f is at $x = 0$, it is easy to prove that both \bar{V} and V_∞ are suborbifolds of $\mathbb{P}^{d_0, \dots, d_{n+1}}$ (see also [58], Lemma 1). Let us examine the cohomological long exact sequence of the pair $(\bar{V}, \bar{V} \setminus V_\infty)$:

$$\begin{aligned} 0 &\longrightarrow H^n(\bar{V}, \bar{V} \setminus V_\infty) \longrightarrow H^n(\bar{V}) \longrightarrow H^n(V) \longrightarrow \\ &\longrightarrow H^{n+1}(\bar{V}, \bar{V} \setminus V_\infty) \longrightarrow H^{n+1}(\bar{V}) \longrightarrow 0 \end{aligned}$$

where all cohomology groups are taken with coefficients in $k = \mathbb{R}$ or \mathbb{C} and we used that $H^i(V) = 0$ for $i = n - 1$ and $n + 1$. The Thom isomorphism theorem generalizes to orbifold vector bundles. Therefore, by excision and the Thom isomorphism theorem for the orbifold normal bundle to V_∞ in \bar{V} we get

$$\begin{aligned} H^{n-2}(V_\infty) &\cong H^n(\bar{V}, \bar{V} \setminus V_\infty), \\ H^{n-1}(V_\infty) &\cong H^{n+1}(\bar{V}, \bar{V} \setminus V_\infty). \end{aligned}$$

where both isomorphisms are constructed as follows. Let $\theta \in H^2(N, N \setminus V_\infty)$ be the Thom class of the normal orbundle N to V_∞ in \bar{V} . By excision, we may assume that N is a tubular neighborhood of V_∞ in \bar{V} . Let $p : N \rightarrow V_\infty$ be the projection. Then the above isomorphisms have the form $\alpha \mapsto p^*\alpha \wedge \theta$. Since θ has a support in N , the form $i_*(\alpha) := p^*\alpha \wedge \theta$ extends naturally to a closed differential form in \bar{V} , which is also known as the *pushforward* of α along i . Here $i : V_\infty \rightarrow \bar{V}$ is the inclusion map. The exact sequence from above yields

$$(5.22) \quad 0 \longrightarrow \text{Cok}(i_*^{n-2}) \longrightarrow H^n(V) \longrightarrow \text{Ker}(i_*^{n-1}) \longrightarrow 0,$$

where $i_*^k : H^k(V_\infty) \rightarrow H^{k+2}(\bar{V})$ denotes the pushforward in degree k . On the other hand, just like in the case of ordinary projective space, it can be proved that the Thom class of the suborbifold $\{Z_{n+1} = 0\}$ can be represented by a Kähler form on $\mathbb{P}^{d_0, \dots, d_{n+1}}$. The restriction of the Kähler form to \bar{V} and V_∞ is still a Kähler form, so we have the notion of primitive cohomology. We have the following isomorphisms:

$$(5.23) \quad P^n(\bar{V}, \mathbb{C}) \cong \text{Cok}(i_*^{n-2}), \quad \text{Ker}(i_*^{n-1}) = P^{n-1}(V_\infty, \mathbb{C}),$$

where the first isomorphism is induced from the quotient map $H^n(\bar{V}, \mathbb{C}) \rightarrow \text{Cok}(i_*^{n-2})$. Here, the first isomorphism in (5.23) is a consequence of the Lefschetz decomposition theorem (see Theorem 5.26) and the second one is by the definition of primitive cohomology.

Let us describe the maps in the exact sequence (5.22). The first map is induced from the retraction $H^n(\bar{V}) \rightarrow H^n(V)$, while the 2nd one is induced from the composition

$$(5.24) \quad H^n(V) \longrightarrow H^{n+1}(\bar{V}, \bar{V} \setminus V_\infty) \xrightarrow{\cong} H^{n-1}(V_\infty)$$

where the first map is the boundary morphism in the long exact sequence of the pair $(\bar{V}, \bar{V} \setminus V_\infty)$ and the 2nd one is the Thom isomorphism, i.e., integration along the fiber. Let N be an orbifold tubular neighborhood of V_∞ in \bar{V} , such that, N is an orbifold disk bundle over V_∞ . If ϕ is a closed differential form representing a cohomology class in $H^n(V)$ and $\gamma \subset V_\infty$ is a cycle representing a homology class in $H_{n-1}(V_\infty)$, then let us define

$$T^\vee : H^n(V) \rightarrow H^{n-1}(V_\infty), \quad T^\vee(\phi)(\gamma) := \int_{\partial N|_\gamma} \phi,$$

where the orientation of the boundary $\partial N|_\gamma$ is the orientation induced from the complex orbifold orientation of N . Recalling the definitions of the boundary morphism and the operation of integration along the fiber, we get that the composition of the morphisms (5.24) coincides with the map $\phi \mapsto T^\vee(\phi)$. The boundary ∂N is an orbifold S^1 -bundle over V_∞ and if we put $T(\gamma) := \partial N|_\gamma$, then we obtain a map $T : H_{n-1}(V_\infty) \rightarrow H_n(V)$ known as the *tube mapping* (see [27]). Therefore, we can also say that the composition

(5.24) is the dual of the tube mapping. Finally, let us point out that up to a factor of $\pm 2\pi\mathbf{i}$ the map T^\vee coincides with the so-called *residue map*

$$(5.25) \quad \text{Res} : H^n(V) \rightarrow H^{n-1}(V_\infty), \quad \text{Res}(\phi)(\gamma) := \frac{(-1)^{n-1}}{2\pi\mathbf{i}} \int_{T(\gamma)} \phi.$$

The exact sequence (5.22) takes the form

$$0 \longrightarrow P^n(\bar{V}, \mathbb{C}) \longrightarrow H^n(V, \mathbb{C}) \xrightarrow{2\pi\mathbf{i}\text{Res}} P^{n-1}(V_\infty, \mathbb{C}) \longrightarrow 0.$$

The above sequence splits in a natural way. Namely, let us consider the restriction of the vanishing cohomology bundle to $\mathbb{C}^* \times \{0\} \subset \mathbb{C} \times \mathbb{C}^k \setminus D_F$. It is a vector bundle whose total space is $\bigcup_{\lambda \in \mathbb{C}^*} H^n(f^{-1}(\lambda), \mathbb{C})$. Parallel transport with respect to the Gauss–Manin connection along the circle $|\lambda| = 1$ defines a linear operator $M : H^n(V, \mathbb{C}) \rightarrow H^n(V, \mathbb{C})$ known as the *classical monodromy* operator. Let $H^n(V, \mathbb{C})_{\neq 1}$ be the direct sum of all generalized eigensubspaces of M whose eigenvalue is $\neq 1$ and $H^n(V, \mathbb{C})_1$ be the generalized eigensubspace with eigenvalue 1.

PROPOSITION 5.30. *a) The restriction map $H^n(\bar{V}) \rightarrow H^n(V)$ induces an isomorphism $P^n(\bar{V}, k) \cong H^n(V, k)_{\neq 1}$, where $k = \mathbb{R}$ or \mathbb{C} .*

b) The dual of the tube mapping $T^\vee = 2\pi\mathbf{i}\text{Res}$ induces an isomorphism $H^n(V, k)_1 \cong P^{n-1}(V_\infty, k)$, where $k = \mathbb{R}$, or \mathbb{C} .

The proof of Proposition 5.30 is based on an explicit computation due to Steenbrink. Let us recall the relevant results. We assume the notation from Section 5.1. Put

$$\omega_\alpha = \frac{x^\alpha}{(f(x) - 1)^{\lceil l(\alpha) \rceil}} dx \quad (\alpha \in \mathfrak{B})$$

where $dx = dx_0 \wedge \cdots \wedge dx_n$, $l(\alpha) := \sum_{i=0}^n c_i(\alpha_i + 1)$ is the weight of the form $x^\alpha dx$, and $\lceil l \rceil$ denotes the ceil of l . The forms ω_α extend to rational differential forms on $\mathbb{P}^{d_0, \dots, d_{n+1}}$. Changing the orbifold coordinates $x_a = x_a^{n+1} = (x_{n+1}^i)^{-d_a} x_a^i$ ($a \neq i$) and $x_i = (x_{n+1}^i)^{-d_i}$ and using that f is weighted homogeneous, we get that

$$\omega_\alpha = (-1)^{n-i+1} \frac{(x_0^i)^{\alpha_0} \cdots (x_{i-1}^i)^{\alpha_{i-1}} (x_{i+1}^i)^{\alpha_{i+1}} \cdots (x_n^i)^{\alpha_n}}{(f(x_0^i, \dots, x_{i-1}^i, 1, x_{i+1}^i, \dots, x_n^i) - (x_{n+1}^i)^d)^{-\lceil l(\alpha) \rceil}} \frac{dx^i}{(x_{n+1}^i)^{1 - (\lceil l(\alpha) \rceil - l(\alpha))d}},$$

where $dx^i := dx_0^i \wedge \cdots \wedge dx_{i-1}^i \wedge dx_{i+1}^i \wedge \cdots \wedge dx_n^i$. The above formula proves that if $l(\alpha)$ is not an integer, then ω_α is a rational form on $\mathbb{P}^{d_0, \dots, d_{n+1}}$ with a pole along the hypersurface \bar{V} . Moreover, the restriction of ω_α to $\{Z_{n+1} = 0\}$ is 0. If $l(\alpha) \in \mathbb{Z}$, then ω_α has a pole of order 1 at $\{Z_{n+1} = 0\}$. Let us denote by $\eta_\alpha := \text{Res}_{Z_{n+1}=0}(\omega_\alpha)$ for $l(\alpha) \in \mathbb{Z}$. Then η_α is a rational form on $\mathbb{P}^{d_0, \dots, d_n}$ with a pole along the hypersurface V_∞ . By extending Griffith's theory of rational integrals (see [27]) to orbifolds, Steenbrink proved the following proposition (see [58], Lemma 4 and Lemma 5).

PROPOSITION 5.31. *a) The forms $\text{Res}_{\bar{V}}(\omega_\alpha)$ ($l(\alpha) \notin \mathbb{Z}$) give a basis of $P^n(\bar{V}, \mathbb{C})$ and the subset of forms, such that, $l(\alpha) < p + 1$ form a basis of $F^{n-p}P^n(\bar{V}, \mathbb{C})$.*

b) The forms $\text{Res}_{V_\infty}(\eta_\alpha)$ ($l(\alpha) \in \mathbb{Z}$) give a basis of $P^{n-1}(V_\infty, \mathbb{C})$ and the subset of forms, such that, $l(\alpha) \leq p$ form a basis of $F^{n-p}P^{n-1}(V_\infty, \mathbb{C})$. \square

Given a holomorphic form $\omega \in \Omega^{n+1}(\mathbb{C}^{n+1})$ let us recall the so-called *geometric sections* of the vanishing cohomology bundle

$$(5.26) \quad s(\omega, \lambda) := \int \frac{\omega}{df} \in H^n(f^{-1}(\lambda), \mathbb{C}),$$

where ω/df denotes a holomorphic n -form η defined in a tubular neighborhood of $f^{-1}(\lambda)$, such that, $\omega = df \wedge \eta$. The choice of η is not unique, but its restriction to $f^{-1}(\lambda)$ is uniquely determined. Put $\varphi_\alpha := x^\alpha dx/df|_V$. Due to the homogeneity of $x^\alpha dx$ and f , we have

$$\text{Res}_V(\omega_\alpha) = \frac{(-1)^n}{2\pi\mathbf{i}} \int_{|\lambda-1|=\epsilon} \frac{\lambda^{l(\alpha)-1} d\lambda}{(\lambda-1)^{\lceil l(\alpha) \rceil}} \varphi_\alpha = (-1)^n \binom{l(\alpha)-1}{\lceil l(\alpha) \rceil - 1} \varphi_\alpha.$$

Proof of Proposition 5.30. It is sufficient to prove the proposition only for $k = \mathbb{C}$, because both maps in part a) and b) preserve the real structure. Recalling Proposition 5.31 we get that the holomorphic forms φ_α with $l(\alpha) \notin \mathbb{Z}$ form a basis for the image of $P^n(\bar{V}, \mathbb{C})$, while the forms φ_α with $l(\alpha) \in \mathbb{Z}$ map via the residue map Res to the basis η_α of $P^{n-1}(V_\infty, \mathbb{C})$. On the other hand, the number of the forms φ_α coincides with the Milnor number of the singularity f and hence with the dimension of $H^n(V, \mathbb{C})$. Therefore, the forms φ_α represent a basis of $H^n(V, \mathbb{C})$. Note that by homogeneity

$$s(x^\alpha dx, \lambda) = \lambda^{l(\alpha)-1} \int \varphi_\alpha,$$

where the LHS is interpreted as a global holomorphic section of the vanishing cohomology bundle, while on the RHS $\int \varphi_\alpha \in H^n(V, \mathbb{C})$ is interpreted as a multi-valued flat section. The formula shows that the cohomology class represented by φ_α is an eigenvector of M with eigenvalue $e^{-2\pi i l(\alpha)}$. In particular, the classical monodromy operator is diagonalizable and the forms φ_α with $l(\alpha) \notin \mathbb{Z}$ represent a basis of $H^n(V, \mathbb{C})_{\neq 1}$, while φ_α with $l(\alpha) \in \mathbb{Z}$ represent a basis of $H^n(V, \mathbb{C})_1$. \square

The Hodge structure in primitive cohomology allows us to equip the cohomology group $H^n(V, \mathbb{C})$ with a Hodge structure. Namely, following Steenbrink, we define

$$F^p H^n(V, \mathbb{C}) := \{A \in H^n(V, \mathbb{C}) \mid A = s(\omega, 1) \text{ for some } \omega \text{ such that } \deg(\omega) \leq n+1-p\}.$$

Note that the above filtration is M -invariant and therefore Proposition 5.30 implies that the above filtration induces Polarized Hodge Structures on $H^n(V, \mathbb{C})_{\neq 1}$ and $H^n(V, \mathbb{C})_1$ of weights respectively n and $n+1$.

5.5.2. Polarizing form and the higher residue pairing. The problem that we would like to solve now is to find explicit formula for the bilinear form on $H^n(V, \mathbb{C})$ corresponding to the polarizing forms of $P^n(\bar{V}, \mathbb{C})$ and $P^{n-1}(V_\infty, \mathbb{C})$ via the isomorphisms in Proposition 5.30. As a byproduct we get a proof of Hertling's result in the case of weighted homogeneous singularity, that is, the higher residue pairing provides a polarizing form for Steenbrink's Hodge structure on vanishing cohomology.

Let us define positive even integers a_i, b_i ($0 \leq i \leq n$), a , and b , such that,

$$\frac{1}{c_i} = \frac{b_i}{b}, \quad \gcd(b_0, \dots, b_n) = 2$$

and

$$\frac{1}{1-c_i} = \frac{a_i}{a}, \quad \gcd(a_0, \dots, a_n) = 2.$$

Put

$$\|x\|_b := (|x_0|^{b_0} + \dots + |x_n|^{b_n})^{1/b},$$

where $x = (x_0, \dots, x_n) \in \mathbb{C}^{n+1}$ and

$$\mathbb{S}_b := \{x \in \mathbb{C}^{n+1} \mid \|x\|_b = 1\}.$$

It is straightforward to check that \mathbb{S}_b is a smooth manifold diffeomorphic to the standard unit sphere \mathbb{S}^{2n+1} . Suppose that $\omega_i = x^{\beta^{(i)}} dx$ ($i = 1, 2$) are two weighted homogeneous forms with weights

$$\text{wt}(\omega_i) := \sum_{s=0}^n (\beta_s^{(i)} + 1) c_s.$$

There are differential forms $\psi_k^{(i)} \in \bigoplus_{p+q=n} A^{p,q}(\mathbb{C}^{n+1} \setminus \{0\})$, such that, $\omega_i = (wd + df \wedge)(\psi_0^{(i)} + \psi_1^{(i)} w + \dots)$ – see Section 4.5.4. In other words, the forms $\psi_s^{(i)}$ satisfy the following recursion relations:

$$\omega_i = df \wedge \psi_0^{(i)}, \quad d\psi_{s-1}^{(i)} = -df \wedge \psi_s^{(i)} \quad (s > 0).$$

Let us give an explicit algorithm for constructing such forms. Put

$$\rho_i(x) = \frac{|f_i(x)|^{a_i-2} \overline{f_i(x)}}{\sum_{j=0}^n |f_j(x)|^{a_j}},$$

where $f_i := \frac{\partial f}{\partial x_i}$. Note that ρ_i are smooth functions on $\mathbb{C}^{n+1} \setminus \{0\}$ satisfying

$$\sum_{i=0}^n \rho_i(x) \frac{\partial f}{\partial x_i}(x) = 1.$$

In order to state the remaining properties of $\rho_i(x)$, let us equip \mathbb{C}^{n+1} with the \mathbb{R}_+ -action

$$t \cdot (x_0, \dots, x_n) := (t^{c_0} x_0, \dots, t^{c_n} x_n), \quad t \in \mathbb{R}_+$$

and the \mathbb{S}^1 -action

$$\lambda \cdot (x_0, \dots, x_n) := (\lambda^{d_0} x_0, \dots, \lambda^{d_n} x_n), \quad \lambda \in \mathbb{S}^1.$$

LEMMA 5.32. a) We have $\rho_i(t \cdot x) = t^{c_i-1} \rho_i(x)$ for all $t \in \mathbb{R}_+$.

b) We have $\rho_i(\lambda \cdot x) = \lambda^{d_i-d} \rho_i(x)$ for all $\lambda \in \mathbb{S}^1$.

c) We have $\overline{E}(\rho_i) = 0$, where

$$\overline{E} := \sum_{i=0}^n c_i \overline{x}_i \frac{\partial}{\partial \overline{x}_i}.$$

The proof of the above lemma is straightforward, so we leave it as an exercise.

LEMMA 5.33. Suppose that $\omega \in A^p(\mathbb{C}^{n+1} \setminus \{0\})$ is an arbitrary form satisfying $df \wedge \omega = 0$. Then the form

$$\psi := \sum_{i=0}^n \rho_i(x) \iota_{\partial/\partial x_i}(\omega)$$

satisfies the following properties.

a) $\omega = df \wedge \psi$.

b) If $\iota_{\overline{E}}(\omega) = 0$, then $\iota_{\overline{E}}(\psi) = 0$.

c) If $\mathcal{L}_{\overline{E}}\omega = 0$, then $\mathcal{L}_{\overline{E}}(\psi) = 0$, where $\mathcal{L}_X := d \circ \iota_X + \iota_X \circ d$ is the Lie derivative in the direction of the vector field X .

PROOF. Parts b) and c) follow immediately from Lemma 5.32, c). Let us prove a). We have

$$df \wedge \psi = \sum_{i=0}^n \rho_i(x) \left((df \wedge) \circ \iota_{\partial/\partial x_i} + \iota_{\partial/\partial x_i} \circ (df \wedge) \right) \omega.$$

On the other hand, since $(dx_i \wedge) \circ \iota_{\partial/\partial x_j} + \iota_{\partial/\partial x_j} \circ (dx_i \wedge) = \delta_{i,j}$, we have

$$(df \wedge) \circ \iota_{\partial/\partial x_i} + \iota_{\partial/\partial x_i} \circ (df \wedge) = \frac{\partial f}{\partial x_i}.$$

It remains only to recall that $\sum_{i=0}^n \rho_i(x) f_i(x) = 1$. \square

Suppose that $\omega = x^\beta dx$ is a weighted homogeneous form. Put

$$(5.27) \quad \psi_0 := \sum_{i=0}^n \rho_i(x) \iota_{\partial/\partial x_i}(\omega),$$

$$(5.28) \quad \psi_{s+1} := - \sum_{i=0}^n \rho_i(x) \iota_{\partial/\partial x_i}(d\psi_s).$$

Lemma 5.33 implies that the sequence ψ_s ($s \geq 0$) satisfies the following properties:

- (i) $\omega = df \wedge \psi_0$ and $d\psi_s = -df \wedge \psi_{s+1}$.
- (ii) The forms ψ_s are weighted homogeneous of weight $\text{wt}(\omega) - s - 1$, that is, under the rescaling $x_i \mapsto t^{c_i} x_i$ the form $\psi_s \mapsto t^{\text{wt}(\omega) - s - 1} \psi_s$.
- (iii) $\iota_{\overline{E}}(\psi_s) = 0$ for all $s \geq 0$.

Let us apply the above construction to $\beta = \beta^{(i)}$ and define the sequences $\psi_s^{(i)}$, $s \geq 0$.

LEMMA 5.34. *If the higher residue pairing $K_f(\omega_1, \omega_2)$ is non-zero, then*

$$\text{wt}(\omega_1) + \text{wt}(\omega_2) = k + 1, \quad k \in \mathbb{Z}$$

and the following formula holds:

$$(-1)^{n(n+1)/2} K_f(\omega_1, \omega_2) = \frac{(-1)^{k_2}}{(2\pi\mathbf{i})^{n+1}} w^{k+1} \int_{\mathbb{S}_b} \psi_{k_1}^{(1)} \wedge \psi_{k_2}^{(2)} \wedge df,$$

where (k_1, k_2) is any pair of non-negative integers, such that, $k_1 + k_2 = k$.

PROOF. By definition

$$(-1)^{n(n+1)/2} K_f(\omega_1, \omega_2) = (-1)^n (2\pi\mathbf{i})^{-n-1} w \int_{\mathbb{S}_b} (\psi_0^{(1)} + \psi_1^{(1)} w + \dots) \wedge \omega_2.$$

Let us change the integration variables via $x_i = e^{2\pi\mathbf{i}\theta c_i} y_i$, where $\theta \in \mathbb{R}$ is any real number. Note that the term $\psi_s^{(1)} \wedge \omega_2$ is rescaled by

$$\exp\left(2\pi\mathbf{i}\theta(\text{wt}(\omega_1) + \text{wt}(\omega_2) - s - 1)\right).$$

In order for the integral to be non-zero, the above scalar must be 1 for all θ . This proves that the weights of ω_1 and ω_2 add up to an integer, which we denote by $k + 1$, and the pairing takes the form

$$(-1)^{n(n+1)/2} K_f(\omega_1, \omega_2) = (-1)^n (2\pi\mathbf{i})^{-n-1} w^{k+1} \int_{\mathbb{S}_b} \psi_k^{(1)} \wedge \omega_2.$$

The rest of the lemma follows easily by integration by parts. Indeed, since $\omega_2 = df \wedge \psi_0^{(2)}$, the above formula is precisely the case $k_1 = k$ and $k_2 = 0$. Suppose that the formula is proved for some $k_1 > 0$ and $k_2 = k - k_1$. We have

$$\psi_{k_1}^{(1)} \wedge df = (-1)^n df \wedge \psi_{k_1}^{(1)} = -(-1)^n d\psi_{k_1-1}^{(1)}$$

and

$$-(-1)^n d\psi_{k_1-1}^{(1)} \wedge \psi_{k_2}^{(2)} = d\left(-(-1)^n \psi_{k_1-1}^{(1)} \wedge \psi_{k_2}^{(2)}\right) + \psi_{k_1-1}^{(1)} \wedge d\psi_{k_2}^{(2)}.$$

It remains only to recall that $d\psi_{k_2}^{(2)} = -df \wedge d\psi_{k_2+1}^{(2)}$. \square

Let us fix $\epsilon > 0$ and define

$$K_\epsilon := \{x \in \mathbb{S}_b \mid |f(x)| < \epsilon\}$$

and

$$V_\epsilon := \{y \in \mathbb{C}^{n+1} \mid f(y) = 1, \|y\|_b \leq \epsilon^{-1}\}.$$

LEMMA 5.35. *The map*

$$\varphi : \mathbb{S}^1 \times V_\epsilon \rightarrow \mathbb{S}_b \setminus K_\epsilon, \quad (\lambda, y) \mapsto \lambda \cdot \|y\|_b^{-1} \cdot y$$

is a regular covering of degree d .

PROOF. Given $x \in \mathbb{S}_b \setminus K_\epsilon$ let us find all $(\lambda, y) \in \mathbb{S}^1 \times V_\epsilon$, such that, $\varphi(\lambda, y) = x$. Since

$$f(x) = \lambda^d \|y\|_b^{-1} f(y) = \lambda^d \|y\|_b^{-1}$$

we must have $\|y\|_b = |f(x)|^{-1}$ and $\lambda^d = f(x)/|f(x)|$. We get that (λ, y) are given by

$$(5.29) \quad \lambda = \eta(f(x)/|f(x)|)^{-1/d}, \quad y_i = \lambda^{-d_i} |f(x)|^{-c_i} x_i,$$

where $\eta \in \mu_d$ is a d -th root of 1. Note that if λ and y are defined by (5.29), then

$$f(y) = \lambda^{-d} |f(x)|^{-1} f(x) = 1$$

and

$$\|y\|_b = \left(\sum_{i=0}^n |x_i|^{b_i} |f(x)|^{-c_i b_i} \right)^{1/b} = \|x\|_b |f(x)|^{-1} = |f(x)|^{-1} \leq \epsilon,$$

that is, $(\lambda, y) \in \mathbb{S}^1 \times V_\epsilon$. Finally, it remains only to notice that formulas (5.29) provide a local inverse for φ . \square

PROPOSITION 5.36. *Let us assume the notation in Lemma 5.34. The following formula holds:*

$$(-1)^{n(n+1)/2} K_f(\omega_1, \omega_2) = \frac{(-1)^{k_2}}{(2\pi\mathbf{i})^n} w^{k+1} \lim_{\epsilon \rightarrow 0} \int_{V_\epsilon} \psi_{k_1}^{(1)} \wedge \psi_{k_2}^{(2)}$$

where (k_1, k_2) is any pair of non-negative integers, such that,

$$k_1 + k_2 = k := \text{wt}(\omega_1) + \text{wt}(\omega_2) - 1.$$

PROOF. Put $\omega := \psi_{k_1}^{(1)} \wedge \psi_{k_2}^{(2)}$. We have to compute

$$\int_{\mathbb{S}_b} \omega \wedge df = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{S}_b \setminus K_\epsilon} \omega \wedge df = \frac{1}{d} \lim_{\epsilon \rightarrow 0} \int_{\mathbb{S}^1 \times V_\epsilon} \varphi^*(\omega \wedge df),$$

where for the last equality we used Lemma 5.35. Let us compute $\varphi^*(\omega \wedge df)$. Suppose that $x = (x_0, \dots, x_n)$ and $y = (y_0, \dots, y_n)$ are two copies of the standard holomorphic coordinates of \mathbb{C}^{n+1} and $\lambda \in \mathbb{S}^1$. The map $x = \varphi(\lambda, y)$ written in components takes the form $x_i = \lambda^{d_i} \|y\|_b^{-c_i} y_i$ ($0 \leq i \leq n$). Using that f is weighted homogeneous, we get

$$\varphi^* f = f(\lambda \cdot \|y\|_b^{-1} \cdot y) = \lambda^d \|y\|_b^{-1} f(y) = \lambda^d \|y\|_b^{-1},$$

where we used that $f(y) = 1$ for $y \in V_\epsilon$. Let us compute the pullback of ω . Recalling the definition of $\psi_s^{(i)}$ (see the text just before Lemma 5.34) we get that ω is a smooth $2n$ -form on $\mathbb{C}^{n+1} \setminus \{0\}$ satisfying the following properties:

- (i) ω is weighted homogeneous of weight -1 , that is, under the rescaling $x_i \mapsto t^{c_i} x_i$, the form $\omega \mapsto t^{-1} \omega$.
- (ii) We have $\iota_{\bar{E}}(\omega) = 0$.
- (iii) We have $\mathcal{L}_{\bar{E}}(\omega) = 0$.

Property (iii) implies that if we view ω as a rational form in x_i and \bar{x}_i ($0 \leq i \leq n$), then ω is weighted homogeneous of weight 0 with respect to the variables \bar{x}_i . Therefore, if $\lambda \in \mathbb{S}^1$, then under the rescaling $x_i \mapsto \lambda^{d_i} x_i$ ($0 \leq i \leq n$) the form $\omega \mapsto \lambda^{-d} \omega$.

Let us write

$$\omega =: \sum_{I,J} \omega_{I,J}(x) dx_I \wedge d\bar{x}_J,$$

where the coefficients $\omega_{I,J}(x)$ are smooth functions on $\mathbb{C}^{n+1} \setminus \{0\}$, I and J are subsequences of $0, 1, \dots, n$, and $dx_I = \wedge_{i \in I} dx_i$ and $d\bar{x}_J = \wedge_{j \in J} d\bar{x}_j$. The coefficients $\omega_{I,J}(x)$ have the following homogeneity properties:

$$\omega_{I,J}(\lambda \cdot x) = \lambda^{-d-d_I+d_J} \omega_{I,J}(x), \quad \lambda \in \mathbb{S}^1,$$

where $d_I := \sum_{i \in I} d_i$ and $d_J := \sum_{j \in J} d_j$, and

$$\omega_{I,J}(t \cdot x) = t^{-1-c_I-c_J} \omega_{I,J}(x), \quad t \in \mathbb{R}_+,$$

where $c_I := \sum_{i \in I} c_i$ and $c_J := \sum_{j \in J} c_j$. Therefore, the pullback that we would like to compute takes the form

$$\begin{aligned} \varphi^*(\omega \wedge df) &= \sum_{I,J} \omega_{I,J}(y) \frac{d(\varphi^* f)}{\varphi^* f} \wedge \\ &\quad \bigwedge_{i \in I} \lambda^{-d_i} \|y\|_b^{c_i} d(\lambda^{d_i} \|y\|_b^{-c_i} y_i) \bigwedge_{j \in J} \lambda^{d_j} \|y\|_b^{c_j} d(\lambda^{-d_j} \|y\|_b^{-c_j} \bar{y}_j). \end{aligned}$$

Note that

$$\lambda^{-d_i} \|y\|_b^{c_i} d(\lambda^{d_i} \|y\|_b^{-c_i} y_i) = dy_i + c_i y_i \frac{d(\varphi^* f)}{\varphi^* f}$$

and

$$\lambda^{d_j} \|y\|_b^{c_j} d(\lambda^{-d_j} \|y\|_b^{-c_j} \bar{y}_j) = d\bar{y}_j + c_j \bar{y}_j \frac{d(\overline{\varphi^* f})}{\overline{\varphi^* f}}.$$

Therefore,

$$\varphi^*(\omega \wedge df) = \frac{d(\varphi^*f)}{\varphi^*f} \wedge \left(\omega + \frac{d(\overline{\varphi^*f})}{\varphi^*f} \wedge \iota_{\overline{E}}(\omega) \right) = \frac{d(\varphi^*f)}{\varphi^*f} \wedge \omega,$$

where we used that $\iota_{\overline{E}}(\omega) = 0$. Furthermore,

$$\frac{d(\varphi^*f)}{\varphi^*f} \wedge \omega = d \frac{d\lambda}{\lambda} \wedge \omega - \frac{d\|y\|_b}{\|y\|_b} \wedge \omega = d \frac{d\lambda}{\lambda} \wedge \omega,$$

where we used that ω is a $2n$ -form, i.e., top degree form on V_ϵ , so $\frac{d\|y\|_b}{\|y\|_b} \wedge \omega = 0$ on V_ϵ . Finally, we get that

$$\frac{1}{d} \int_{\mathbb{S}^1 \times V_\epsilon} \varphi^*(\omega \wedge df) = 2\pi \mathbf{i} \int_{V_\epsilon} \omega.$$

Now the formula that we have to prove follows easily from Lemma 5.34. \square

5.5.3. Polarizing form on $H^n(V, \mathbb{C})_{\neq 1}$. Let us examine more carefully the isomorphism $P^n(\overline{V}, \mathbb{C}) \cong H^n(V, \mathbb{C})_{\neq 1}$. Suppose that $\omega = x^\alpha dx$ is a weighted homogeneous form whose weight is not an integer. Then $(\omega/df)|_V$ is a holomorphic n -form on V which determines a cohomology class $[\omega/df]$ in $H^n(V, \mathbb{C})_{\neq 1}$. Let ψ_s ($s \geq 0$) be the sequence of smooth n -forms defined by (5.27)–(5.28).

LEMMA 5.37. *Suppose that $k < \text{wt}(\omega) < k + 1$.*

a) *The differential form ψ_k extends to a smooth differential form on \overline{V} vanishing at V_∞ . In particular, ψ_k represents a primitive cohomology class $[\psi_k] \in P^n(\overline{V}, \mathbb{C})$.*

b) *Under the isomorphism $P^n(\overline{V}, \mathbb{C}) \cong H^n(V, \mathbb{C})_{\neq 1}$ (see part a) of Proposition 5.30) the class $[\psi_k]$ is mapped to*

$$(-1)^k (\text{wt}(\omega) - 1) \cdots (\text{wt}(\omega) - k) [\omega/df].$$

PROOF. We already know that ψ_k is a smooth form on $V = U_{n+1} \cap \overline{V}$, where $U_{n+1} = \{Z_{n+1} \neq 0\}$ is one of the orbifold charts of $\mathbb{P}^{d_0, \dots, d_{n+1}}$ (see Section 5.5.1). Let us check that ψ_k is smooth in the remaining coordinate charts U_i ($0 \leq i \leq n$). Let us consider the case when $i = 0$. The remaining cases are similar. To avoid cumbersome notation, let us denote the orbifold coordinates x_i^0 by y_i ($1 \leq i \leq n + 1$). The coordinate change takes the form

$$x_0 = y_{n+1}^{-d_0}, \quad x_i = y_{n+1}^{-d_i} y_i \quad (1 \leq i \leq n).$$

The holomorphic form ω transforms into

$$\omega = (-1)^{n+1} d_0 y_{n+1}^{-\text{wt}(\omega)d} y_1^{\alpha_1} \cdots y_n^{\alpha_n} dy_1 \wedge \cdots \wedge dy_n \wedge \frac{dy_{n+1}}{y_{n+1}}.$$

Furthermore, the Euler vector field

$$E = \sum_{i=0}^n c_i x_i \frac{\partial}{\partial x_i} = -\frac{1}{d} y_{n+1} \frac{\partial}{\partial y_{n+1}}$$

and the operator $\sum_{i=0}^n \rho_i(x) \iota_{\partial/\partial x_i}$ takes the form

$$y_{n+1}^d \left(-\frac{1}{d_0} \rho_0(1, y_1, \dots, y_n) \iota_{y_{n+1} \partial_{y_{n+1}}} + \sum_{i=1}^n \left(\rho_i(1, y_1, \dots, y_n) - \frac{d_i}{d_0} \rho_0(1, y_1, \dots, y_n) \right) \iota_{y_i \partial_{y_i}} \right).$$

Part a) of the lemma follows easily. Indeed, recalling the definition (5.27)–(5.28), we get that ψ_k has order of vanishing at $y_{n+1} = 0$ at least $(k+1)d - \text{wt}(\omega)d - 1 > 0$.

To prove part b) of the Lemma, let us compute the residue $\text{Res}_{\overline{V}} \frac{k!x^\alpha dx}{(f-1)^{k+1}}$. As we already discussed in Section 5.5.1, the restriction of the residue to V is precisely

$$(5.30) \quad (-1)^n (\text{wt}(\omega) - 1) \cdots (\text{wt}(\omega) - k) \left[\frac{\omega}{df} \right].$$

On the other hand, since $x^\alpha dx = df \wedge \psi_0$, using integration by parts we get that the residue equals

$$\text{Res}_{\overline{V}} \frac{k!df \wedge \psi_0}{(f-1)^{k+1}} = -\text{Res}_{\overline{V}} (k-1)!d \left(\frac{1}{(f-1)^k} \right) \psi_0 = \text{Res}_{\overline{V}} (k-1)! \frac{d\psi_0}{(f-1)^k}.$$

However, since $d\psi_s = -df \wedge \psi_{s+1}$ we can repeat the above computation until we get

$$(-1)^{k-1} \text{Res}_{\overline{V}} \frac{d\psi_{k-1}}{f-1} = (-1)^k \text{Res}_{\overline{V}} \frac{df \wedge \psi_k}{f-1} = (-1)^{k+n} \psi_k.$$

Therefore, the restriction of $(-1)^{k+n} \psi_k$ to V is a differential form representing the cohomology class (5.30), which is exactly what we had to prove. \square

Proposition 5.36 and Lemma 5.37 yield the following corollary.

COROLLARY 5.38. *Let $S_{\overline{V}}$ be the bi-linear form on $H^n(V, \mathbb{C})_{\neq 1}$ induced from the polarizing form on $P^n(\overline{V}, \mathbb{C})$. Then*

$$S_{\overline{V}} \left(\frac{[\omega_1/df]}{\Gamma(\{\text{wt}(\omega_1)\})}, \frac{[\omega_2/df]}{\Gamma(\{\text{wt}(\omega_2)\})} \right)$$

coincides with

$$(-1)^{k_1+n(n+1)/2} (2\pi i)^n K_f^{(k_1+k_2-n)} \left(\frac{\omega_1}{\Gamma(\text{wt}(\omega_1))}, \frac{\omega_2}{\Gamma(\text{wt}(\omega_2))} \right)$$

for all weighted homogeneous forms ω_1 and ω_2 of non-integer weight, where $\{\alpha\} = \alpha - [\alpha]$ denotes the fractional part of α and $k_i = [\text{wt}(\omega_i)]$.

5.5.4. Polarizing form on $H^n(V, \mathbb{C})_1$. Similarly, let us examine more carefully the isomorphism $P^{n-1}(V_\infty, \mathbb{C}) \cong H^n(V, \mathbb{C})_1$. Suppose that $\text{wt}(\omega) = k$ is an integer. Then $(\omega/df)|_V$ is a holomorphic n -form on V which determines a cohomology class $[\omega/df]$ in $H^n(V, \mathbb{C})_1$. In order to avoid cumbersome notation, let us put $\mathbb{P} := \mathbb{P}^{d_0, \dots, d_{n+1}}$, $\mathbb{P}_\infty := \mathbb{P}^{d_0, \dots, d_n}$, and $\mathbf{0} := [0, \dots, 0, 1] \in \mathbb{P}$. We have a natural projection map

$$\pi : \mathbb{P} \setminus \{\mathbf{0}\} \rightarrow \mathbb{P}_\infty, \quad [Z_0, \dots, Z_{n+1}] \mapsto [Z_0, \dots, Z_n].$$

REMARK 5.39. Note that $\mathbb{P} \setminus \{\mathbf{0}\}$ is the total space of the orbifold vector bundle $\mathcal{O}_{\mathbb{P}_\infty}(1) := (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C}/\mathbb{C}^*$, where $\lambda \in \mathbb{C}^*$ acts by

$$\lambda \cdot (z_0, \dots, z_n, t) := (\lambda^{d_0} z_0, \dots, \lambda^{d_n} z_n, \lambda t). \quad \square$$

Let us recall the open covering $\{U_i\}_{0 \leq i \leq n+1}$ of \mathbb{P} , the orbifold charts $\pi_i : X_i \rightarrow U_i$, and the orbifold coordinates x^i on X_i – see Section 5.4.4. The open subsets U_i ($0 \leq i \leq n$) provide a covering of $\mathbb{P} \setminus \{\mathbf{0}\}$ and the local equation of \mathbb{P}_∞ in the orbifold chart X_i is $x_{n+1}^i = 0$.

LEMMA 5.40. *Suppose that $\text{wt}(\omega) = k$ is an integer. The restriction of the forms ψ_s to the orbifold chart X_i ($0 \leq i \leq n$) have the form*

$$(x_{n+1}^i)^{(k-1-s)d} \left(\alpha_s(x_0^i, \dots, x_n^i) \frac{dx_{n+1}^i}{x_{n+1}^i} + \beta_s(x_0^i, \dots, x_n^i) \right),$$

where α_s and β_s are smooth forms independent of the variable x_{n+1}^i .

PROOF. Let us consider only the case $i = 0$. The remaining cases are similar. Let us denote the coordinates $x_i^0 =: y_i$ ($1 \leq i \leq n + 1$). Just like in the proof of Lemma 5.37 we get

$$\omega = (-1)^{n+1} d_0 y_{n+1}^{-k d} y_1^{\alpha_1} \dots y_n^{\alpha_n} dy_1 \wedge \dots \wedge dy_n \wedge \frac{dy_{n+1}}{y_{n+1}}.$$

while the contraction operation $\sum_i \rho_i(x) \iota_{\partial/\partial x_i}$ takes the form

$$y_{n+1}^d \left(-\frac{1}{d_0} \rho_0(1, y_1, \dots, y_n) \iota_{y_{n+1} \partial_{y_{n+1}}} + \sum_{i=1}^n \left(\rho_i(1, y_1, \dots, y_n) - \frac{d_i}{d_0} \rho_0(1, y_1, \dots, y_n) \right) \iota_{y_i \partial_{y_i}} \right).$$

Now the lemma follows easily from the definitions (5.27)–(5.28) by induction on s . \square

In particular, the above lemma implies that the form $\psi_{k-1}|_V$ extends to a meromorphic form on \bar{V} with a pole of order 1 along V_∞ . Therefore, $\text{Res}_{V_\infty}(\psi_{k-1}|_V)$ determines a primitive cohomology class in $P^{n-1}(V_\infty, \mathbb{C})$.

LEMMA 5.41. *Suppose that $\text{wt}(\omega) = k$ is an integer. Under the isomorphism $H^n(V, \mathbb{C})_1 \cong P^{n-1}(V_\infty, \mathbb{C})$ (see part b) of Proposition 5.30) the cohomology class $\text{Res}_{V_\infty}(\psi_{k-1}|_V)$ corresponds to*

$$\frac{(-1)^{k-1}}{2\pi i} (\text{wt}(\omega) - 1) \dots (\text{wt}(\omega) - k + 1) [\omega/df].$$

The proof of Lemma 5.41 is similar to the proof of part b) of Lemma 5.37 and we leave it as an exercise.

PROPOSITION 5.42. *Let S_{V_∞} be the bi-linear form on $H^n(V, \mathbb{C})_1$ induced from the polarizing form on $P^{n-1}(V_\infty, \mathbb{C})$. Then*

$$(-1)^n S_{V_\infty} \left([\omega_1/df], [\omega_2/df] \right)$$

coincides with

$$(-1)^{k_1+n(n+1)/2} (2\pi i)^{n+1} d K_f^{(k_1+k_2-1-n)} \left(\frac{\omega_1}{\Gamma(\text{wt}(\omega_1))}, \frac{\omega_2}{\Gamma(\text{wt}(\omega_2))} \right)$$

for all weighted homogeneous forms ω_1 and ω_2 of integer weights $k_i := \text{wt}(\omega_i)$.

PROOF. Let $\omega_i = x^{\beta^{(i)}} dx$ ($i = 1, 2$) be weighted homogeneous forms whose weights $k_i := \text{wt}(\omega_i)$ are integer numbers. We would like to express (see Proposition 5.36)

$$(5.31) \quad \lim_{\epsilon \rightarrow 0} \int_{V_\epsilon} \psi_{k_1}^{(1)} \wedge \psi_{k_2-1}^{(2)}$$

in terms of the Poincaré pairing on \mathbb{P}_∞ . Since $\psi_{k_1-1}^{(1)}$ is weighted homogeneous of weight 0 we have $(d\iota_E + \iota_E d)\psi_{k_1-1}^{(1)} = 0$. On the other hand, since $d\psi_{k_1-1}^{(1)} = -df \wedge \psi_{k_1}^{(1)}$ we get

$$\iota_E d\psi_{k_1-1}^{(1)} = -\iota_E(df \wedge \psi_{k_1}^{(1)}) = -f\psi_{k_1}^{(1)} + df \wedge \iota_E \psi_{k_1}^{(1)}.$$

Restricting to V_ϵ where $f = 1$ and $df = 0$, we get

$$d\iota_E(\psi_{k_1-1}^{(1)})\Big|_{V_\epsilon} = -\iota_E d\psi_{k_1-1}^{(1)}\Big|_{V_\epsilon} = \psi_{k_1}^{(1)}\Big|_{V_\epsilon}.$$

Using Stoke's theorem we transform (5.31) into

$$\lim_{\epsilon \rightarrow 0} \int_{\partial V_\epsilon} \iota_E(\psi_{k_1-1}^{(1)}) \wedge \psi_{k_2-1}^{(2)} = \lim_{\epsilon \rightarrow 0} \sum_{i=0}^n \frac{1}{d_i} \int_{\partial V_\epsilon \cap X_i} \iota_E(\psi_{k_1-1}^{(1)}) \wedge \psi_{k_2-1}^{(2)}.$$

According to Lemma 5.40 we have

$$\psi_{k_l-1}^{(l)} = \alpha^{(l)}(x_0^i, \dots, x_n^i) \wedge \frac{dx_{n+1}^i}{x_{n+1}^i} + \beta^{(l)}(x_0^i, \dots, x_n^i), \quad l = 1, 2,$$

where $\alpha^{(l)}$ and $\beta^{(l)}$ are smooth forms on X_i independent of the variable x_{n+1}^i , that is, the forms $\alpha^{(l)}$ and $\beta^{(l)}$ are pullbacks via the projection $\pi : X_i \rightarrow X_i \cap \mathbb{P}_\infty$. Recall that $x_i = (x_{n+1}^i)^{-d_i}$ and $x_j = (x_{n+1}^i)^{-d_j} x_j^i$ for $j \neq i$. Therefore, the Euler vector field $E = -\frac{1}{d} x_{n+1}^i \frac{\partial}{\partial x_{n+1}^i}$ and we get

$$(5.32) \quad \int_{\partial V_\epsilon \cap X_i} \iota_E(\psi_{k_1-1}^{(1)}) \wedge \psi_{k_2-1}^{(2)} = \frac{(-1)^n}{d} \int_{\partial V_\epsilon \cap X_i} \left(\alpha^{(1)} \wedge \alpha^{(2)} \wedge \frac{dx_{n+1}^i}{x_{n+1}^i} + \alpha^{(1)} \wedge \beta^{(2)} \right).$$

In the orbifold coordinate chart X_i , the submanifold $\partial V_\epsilon \cap X_i$ is defined by

$$f(x_0^i, \dots, 1, \dots, x_n^i) = (x_{n+1}^i)^d, \quad |x_{n+1}^i|^d = \epsilon \left(1 + \sum_{0 \leq j \neq i \leq n} |x_j^i|^{b_j} \right)^{1/b}.$$

Note that the restriction of the projection $\pi|_{\partial V_\epsilon}$ is a finite map of degree d and that the 1-form

$$\frac{dx_{n+1}^i}{x_{n+1}^i} = \frac{1}{d} \frac{df(x_0^i, \dots, 1, \dots, x_n^i)}{f(x_0^i, \dots, 1, \dots, x_n^i)}$$

is a pullback via the projection map $\pi|_{\partial V_\epsilon}$. The integral (5.32) takes the form

$$\lim_{\epsilon \rightarrow 0} \frac{(-1)^n}{d} \int_{\pi(\partial V_\epsilon) \cap X_i} \left(\alpha^{(1)} \wedge \alpha^{(2)} \wedge \frac{df(x_0^i, \dots, 1, \dots, x_n^i)}{f(x_0^i, \dots, 1, \dots, x_n^i)} + \alpha^{(1)} \wedge \beta^{(2)} \right).$$

In order to describe the image $\pi(\partial V_\epsilon)$, let us denote by $N_\epsilon \subset \mathbb{P}_\infty$ the tubular neighborhood of V_∞ consisting of all $z = [z_0, \dots, z_n] \in \mathbb{P}_\infty$, such that,

$$|f(z_0, \dots, z_n)| \leq \epsilon (|z_0|^{b_0} + \dots + |z_n|^{b_n})^{1/b}.$$

The projection $\pi : \partial V_\epsilon \rightarrow \partial N_\epsilon$ is a regular covering of degree d . Note however that the orientation of the boundary ∂N_ϵ is opposite to the orientation induced from ∂V_ϵ via the map π , because the image of V_ϵ under the projection π is the complement $\mathbb{P}_\infty \setminus N_\epsilon$. Therefore, the above formula becomes

$$\lim_{\epsilon \rightarrow 0} \frac{(-1)^{n+1}}{d} \int_{\partial N_\epsilon \cap X_i} \left(\alpha^{(1)} \wedge \alpha^{(2)} \wedge \frac{df(x_0^i, \dots, 1, \dots, x_n^i)}{f(x_0^i, \dots, 1, \dots, x_n^i)} + \alpha^{(1)} \wedge \beta^{(2)} \right).$$

Since in the limit $\epsilon \rightarrow 0$ the $(2n-1)$ -dimensional orbifold ∂N_ϵ becomes the $(2n-2)$ -dimensional orbifold V_∞ the contribution to the above integral of $\alpha^{(1)} \wedge \beta^{(2)}$ is 0 for dimensional reasons. For the remaining part of the integral we have

$$(-1)^{n+1} \lim_{\epsilon \rightarrow 0} \frac{1}{d} \int_{\partial N_\epsilon \cap X_i} \alpha^{(1)} \wedge \alpha^{(2)} \wedge \frac{df(x_0^i, \dots, 1, \dots, x_n^i)}{f(x_0^i, \dots, 1, \dots, x_n^i)} = (-1)^{n+1} \frac{2\pi \mathbf{i}}{d} \int_{V_\infty \cap X_i} \alpha^{(1)} \wedge \alpha^{(2)}.$$

Now the proof of the proposition can be completed as follows. According to Proposition 5.36 we have

$$(-1)^{n(n+1)/2} K_f(\omega_1, \omega_2) = \frac{(-1)^{k_2-1}}{(2\pi\mathbf{i})^n} w^{k_1+k_2} \lim_{\epsilon \rightarrow 0} \int_{V_\epsilon} \psi_{k_1}^{(1)} \wedge \psi_{k_2-1}^{(2)},$$

which according to the above computation is equal to

$$\frac{(-1)^{k_2+n}}{(2\pi\mathbf{i})^n} w^{k_1+k_2} \frac{2\pi\mathbf{i}}{d} \sum_{i=0}^n \frac{1}{d_i} \int_{V_\infty \cap X_i} \alpha^{(1)} \wedge \alpha^{(2)} = \frac{(-1)^{k_2+n}}{d(2\pi\mathbf{i})^{n-1}} w^{k_1+k_2} S_{V_\infty}([\alpha^{(1)}], [\alpha^{(2)}]),$$

where $[\alpha^{(l)}] \in H^{n-1}(V_\infty, \mathbb{C})$ denotes the cohomology class represented by the differential form $\alpha^{(l)}$. On the other hand, since

$$\alpha^{(l)}|_{V_\infty} = \text{Res}_{V_\infty}(\psi_{k_l-1}^{(l)}),$$

according to Lemma 5.41 the cohomology class $[\alpha^{(l)}]$ corresponds via the isomorphism $P^{n-1}(V_\infty, \mathbb{C}) \cong H^n(V, \mathbb{C})_1$ to $(-1)^{k_l-1} (2\pi\mathbf{i})^{-1} (k_l-1)! [\omega_l/df]$, that is,

$$S_{V_\infty}([\alpha^{(1)}], [\alpha^{(2)}]) = \frac{(-1)^{k_1+k_2}}{(2\pi\mathbf{i})^2} \Gamma(\text{wt}(\omega_1)) \Gamma(\text{wt}(\omega_2)) S_{V_\infty}([\omega_1/df], [\omega_2/df]). \quad \square$$

The formulas in Proposition 5.42 and Corollary 5.38 give an explicit description of a polarizing form in terms of the higher residue pairing. Namely, the form $S_{\overline{V}}$ is a polarization form for the Hodge structure on $H^n(f^{-1}(1), \mathbb{C})_{\neq 1}$, while $-S_{V_\infty}$ is a polarization form for $H^n(f^{-1}(1), \mathbb{C})_1$. The reason for the negative sign in the 2nd case comes from the fact that under the residue isomorphism $2\pi\mathbf{i} \text{Res} : H^n(f^{-1}(1), \mathbb{C})_1 \rightarrow P^{n-1}(V_\infty, \mathbb{C})$ the group $F^p \cap \overline{F^q}$ is identified with $P^{p-1, q-1}(V_\infty, \mathbb{C})$. This has two implications. First, $F^p \cap \overline{F^q}$ is non-empty only if $p-1+q-1 = n-1$, that is, $p+q = n+1$, which proves that the weight of the Hodge structure on $H^n(f^{-1}(1), \mathbb{C})_1$ is $n+1$ (not $n-1$). Second, we have

$$-\mathbf{i}^{(n+1)^2+2q} S_{V_\infty}(u, \overline{u}) = \mathbf{i}^{(n-1)^2+2(q-1)} S_{V_\infty}(\alpha, \overline{\alpha}) > 0,$$

where $u \in F^p \cap \overline{F^q} \setminus 0$ and $\alpha := 2\pi\mathbf{i} \text{Res}(u)$.

5.5.5. The polarizing form of Hertling. Finally, let us compare the above polarizing form with the one found by Hertling (see Theorem 10.30 in [31]). Let us recall the *Seifert form* of the singularity f . Let us choose $\epsilon > 0$ sufficiently small, so that V_ϵ has the same homotopy type as $V = f^{-1}(1)$. We have the following sequence

$$H^n(V, \mathbb{Z}) \cong H^n(V_\epsilon, \mathbb{Z}) \xrightarrow{PD} H_n(V_\epsilon, \partial V_\epsilon, \mathbb{Z}) \xrightarrow{\text{var}} H_n(V_\epsilon, \mathbb{Z}) \cong H_n(V, \mathbb{Z}),$$

where the map PD is the Poincaré duality isomorphism, i.e., cap product with the fundamental cycle of the manifold with boundary V_ϵ , and var is the so-called *variation operator* (see [6], Section 1.1). Using Alexander duality one can prove that the composition $\text{var} \circ PD$ is an isomorphism. The Seifert form is defined by

$$L(A, B) := \langle A, \text{var} \circ PD(B) \rangle, \quad A, B \in H^n(V, \mathbb{Z}),$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between cohomology and homology. It turns out that the higher residue pairing can be expressed in terms of the Seifert form as follows.

PROPOSITION 5.43. *If ω_1 and ω_2 are weighted homogeneous forms, then the pairing*

$$(-1)^{n(n+1)/2} (2\pi\mathbf{i})^{n+1} K_f\left(\frac{\omega_1}{\Gamma(\text{wt}(\omega_1))}, \frac{\omega_2}{\Gamma(\text{wt}(\omega_2))}\right)$$

coincides with

$$e^{-\pi\mathbf{i}\text{wt}(\omega_1)} L([\omega_1/df], [\omega_2/df]) w^{\text{wt}(\omega_1)+\text{wt}(\omega_2)}. \quad \square$$

Comparing the formula in the above proposition with the formulas in Corollary 5.38 and Proposition 5.42 we get

$$(5.33) \quad S_{\nabla}(A, B) = L(A, (M-1)^{-1}B), \quad A, B \in H^n(V, \mathbb{C})_{\neq 1}$$

and

$$(5.34) \quad S_{V_\infty}(A, B) = (-1)^n d L(A, B), \quad A, B \in H^n(V, \mathbb{C})_1$$

The statement of Proposition 5.43 is the same as Theorem 4.3.3 in [33]. We refer to Remark 4.3.4 in [33] for some further comments.

REMARK 5.44. We expect that formulas (5.33) and (5.34) can be proved directly. If this is done, then Corollary 5.38 and Proposition 5.42 would give yet another proof of Proposition 5.43. \square

The polarizing form of Hertling is defined by the following formulas:

$$S(A, B) := (-1)^{n(n-1)/2} L(A, (M-1)^{-1}B), \quad A, B \in H^n(V, \mathbb{C})_{\neq 1}$$

and

$$S(A, B) := -(-1)^{n(n-1)/2} L(A, B), \quad A, B \in H^n(V, \mathbb{C})_1.$$

The factor of d in (5.34) is not so important, but the signs $(-1)^{n(n-1)/2}$ and $(-1)^{n(n+1)/2}$ by which S differs from respectively S_{∇} and $-S_{V_\infty}$ deserve an explanation. The definition of the Hodge–Riemann bilinear relations in Hertling’s work do not agree with the definition in [63], that is, with Definition 5.29. Namely, if $A \in F^p H^n(V, \mathbb{C})_{\neq 1} \cap \overline{F^q H^n(V, \mathbb{C})}_{\neq 1}$ then the convention in [31], Definition 10.16 yields $\mathbf{i}^{2p-n} S(A, \overline{A}) > 0$, while the convention in Definition 5.29 yields $\mathbf{i}^{n^2+2n-2p} S_{\nabla}(A, \overline{A}) > 0$. The ratio of the two powers of \mathbf{i} is exactly $\mathbf{i}^{n^2-n} = (-1)^{n(n-1)/2}$. Similarly, if $A \in F^p H^n(V, \mathbb{C})_1 \cap \overline{F^q H^n(V, \mathbb{C})}_1$, then $\mathbf{i}^{2p-n-1} S(A, \overline{A}) > 0$, while in our case $\mathbf{i}^{(n+1)^2+2(n+1-p)} (-S_{V_\infty}(A, \overline{A})) > 0$. The ratio of the two powers of \mathbf{i} is exactly $\mathbf{i}^{(n+1)^2-n-1} = (-1)^{n(n+1)/2}$.

5.6. Classification of opposite subspaces

Using the Hodge structure on $H^n(f^{-1}(1), \mathbb{C})$ we will prove the existence of a homogeneous opposite subspace. Moreover, we will prove that a certain finite dimensional unipotent Lie group acts transitively and faithfully on the space of homogeneous opposite subspaces. Hence we will obtain a complete classification of all homogeneous opposite subspaces. Finally, for given opposite subspace we would like to give a geometric interpretation of the solutions to the second structure connection in terms of period integrals.

5.6.1. The Brieskorn lattice. Let $\Omega^p[\mathbb{C}^{n+1}]$ be the vector space of polynomial p -forms on \mathbb{C}^{n+1} . The quotient

$$H'' := \Omega^{n+1}[\mathbb{C}^{n+1}]/df \wedge d\Omega^{n-1}[\mathbb{C}^{n+1}]$$

will be called the *algebraic Brieskorn lattice* of f . Let us recall also the twisted de Rham cohomology groups $\widehat{\mathcal{H}}_f, \widehat{\mathcal{H}}_f^{(0)}$ equipped with the higher residue pairing K_f and the Gauss-Manin connection – see Section 4.6.2. We are going to use the Brieskorn lattice to construct an embedding

$$(5.35) \quad \psi : H^n(f^{-1}(1), \mathbb{C}) \rightarrow \widehat{\mathcal{H}}_f^{(0)}.$$

This is a very important map. It would allow us to transfer the Hodge structure in $\widehat{\mathcal{H}}_f^{(0)}$ which would provide a very powerful tool for constructing opposite subspaces. Let us denote by $H^n := \cup_{\lambda \in \mathbb{C}^*} H^n(f^{-1}(\lambda), \mathbb{C})$ the vanishing cohomology bundle. Recall that H^n is equipped with a flat Gauss–Manin connection ∇ . There is a natural map

$$s : H'' \rightarrow \Gamma(\mathbb{C} \setminus \{0\}, H^n), \quad \omega \mapsto s(\omega, \lambda),$$

where $s(\omega, \lambda)$ is the geometric section (5.26). The Brieskorn lattice has a natural structure of a $\mathbb{C}[\partial_\lambda^{-1}]$ -module

$$\partial_\lambda^{-1} \cdot \omega := df \wedge d^{-1}\omega, \quad \omega \in H'',$$

where $d^{-1}\omega$ denotes n -form $\eta \in \Omega^n[\mathbb{C}^n]$, such that, $d\eta = \omega$. The choice of η is not unique but the equivalence class of $df \wedge \eta$ in H'' is uniquely determined by ω . The main property of the above definition is that

$$(5.36) \quad \nabla_{\partial/\partial\lambda} s(\partial_\lambda^{-1}\omega, \lambda) = s(\omega, \lambda), \quad \omega \in H''.$$

The proof of the above formula is left as an exercise (see also Section 10.2.4 in [6]).

PROPOSITION 5.45. *The natural quotient map which to a holomorphic form $\omega \in \Omega^{n+1}[\mathbb{C}^{n+1}]$ associates its cohomology class $[\omega] \in \widehat{\mathcal{H}}_f^{(0)}$ induces an embedding $[\] : H'' \rightarrow \widehat{\mathcal{H}}_f^{(0)}$ satisfying $[\partial_\lambda^{-1}\omega] = -w[\omega]$.*

PROOF. The formula $[\partial_\lambda^{-1}\omega] = -w[\omega]$ follows easily from the definitions. The difficult part is to prove that if $\omega = (wd + df \wedge)\eta$ for $\omega \in \Omega^{n+1}[\mathbb{C}^{n+1}]$ and $\eta \in \Gamma(\mathbb{C}^{n+1}, \Omega_{\mathbb{C}^{n+1}}^{n+1})[[w]]$, then $\omega = df \wedge d\phi$ for some $\phi \in \Omega^{n-1}[\mathbb{C}^{n+1}]$. Note that if we assign weight 1 to w , then the twisted de Rham differential $wd + df$ will have weight 1. Therefore, we may assume that ω is a weighted homogeneous form and that $\eta = \eta_0 + \eta_1 w + \dots$ where η_k is a weighted homogeneous form of weight $\text{wt}(\omega) - k - 1$. Since the weights of x_i are positive the forms η_k must be polynomial of positive weight. Therefore, $\eta_k = 0$ for $k \geq \text{wt}(\omega) - 1$. We claim that there are weighted homogeneous forms $\phi_k \in \Omega^{n-1}[\mathbb{C}^{n+1}]$, such that, $\eta_k = df \wedge \phi_{k+1} + d\phi_k$. We argue by decreasing induction on k . For k sufficiently negative we can simply choose $\phi_k = 0$. Suppose that we determined η_i and ϕ_i for all $i \geq k$. Since

$$d\eta_{k-1} = -df \wedge \eta_k = -df \wedge (df \wedge \phi_{k+1} + d\phi_k) = -df \wedge d\phi_k = d(df \wedge \phi_k),$$

we get $d(\eta_{k-1} - df \wedge \phi_k) = 0$. Since \mathbb{C}^{n+1} is a Stein manifold, the holomorphic de Rham complex can be used to compute the cohomology of \mathbb{C}^{n+1} . Since $H^n(\mathbb{C}^{n+1}) = 0$ we get that $\eta_{k-1} - df \wedge \phi_k = d\phi_{k-1}$ for some holomorphic $(n-1)$ -form ϕ_{k-1} . It remains only to notice that we may assume that ϕ_{k-1} is weighted homogeneous and hence polynomial and that $\omega = df \wedge \eta_0 = df \wedge d\phi_0$. \square

Put $\mathfrak{h} := H^n(f^{-1}(1), \mathbb{C})$ for the fiber of H^n at $\lambda = 1$. Parallel transport along the unit circle $|\lambda| = 1$ in counter clockwise direction defines a linear operator $M \in \text{End}(H)$ which we called the classical monodromy operator. According to Propositions 5.30 and 5.31 the geometric sections $s(x^\alpha dx, 1)$ ($\alpha \in \mathcal{B}$) form a basis of \mathfrak{h} . Since $s(x^\alpha dx, \lambda) = \lambda^{l(\alpha)-1} s(x^\alpha dx, 1)$ is invariant under the parallel transport with respect to ∇ , we get that $s(x^\alpha dx, 1)$ is an eigenvector for the classical monodromy with eigenvalue $e^{-2\pi i l(\alpha)}$. In other words, M is a diagonalizable operator. Put $\mathcal{N} := -\frac{1}{2\pi i} \log M$, where the value of \log is chosen in such a way that all eigenvalues of \mathcal{N} are in the interval $(-1, 0]$. Let us define

$$s_{\text{reg}} : H'' \rightarrow \mathfrak{h}[\lambda], \quad s_{\text{reg}}(\omega) := \lambda^{-\mathcal{N}} s(\omega, \lambda),$$

where in the definition of s_{reg} we assume that λ is sufficiently close to 1, the fibers $H^n(f^{-1}(\lambda), \mathbb{C})$ are identified with \mathfrak{h} by parallel transport with respect to the Gauss-Manin connection, and $\lambda^{-\mathcal{N}} := e^{-\mathcal{N} \log \lambda}$ is defined via the principal branch of the logarithm. Let us check that $s_{\text{reg}}(\omega)$ has only integer powers of λ . If ω is weighted homogeneous of weight $\text{wt}(\omega) = k + \alpha$, where $k \in \mathbb{Z}$, $-1 < \alpha \leq 0$, then $\mathcal{N}s(\omega, 1) = \alpha s(\omega, 1)$ and we have

$$(5.37) \quad s_{\text{reg}}(\omega) = \lambda^{-\mathcal{N}} s(\omega, \lambda) = \lambda^{k-1} s(\omega, 1).$$

Finally, there is a natural way to turn $\mathfrak{h}[\lambda]$ into a $\mathbb{C}[\partial_\lambda^{-1}]$ -module, such that, the map s_{reg} becomes a morphism of $\mathbb{C}[\partial_\lambda^{-1}]$ -modules. Namely, suppose that $A \in \mathfrak{h}$ is an eigenvector with eigenvalue $e^{-2\pi i \alpha}$ with $-1 < \alpha \leq 0$, then we define

$$\partial_\lambda^{-1} \cdot (A\lambda^k) := \lambda^{-\mathcal{N}} \int_0^\lambda (x^\mathcal{N} A x^k) dx := \frac{A\lambda^{k+1}}{\alpha + k + 1}.$$

PROPOSITION 5.46. *The map $s_{\text{reg}} : H'' \rightarrow \mathfrak{h}[\lambda]$ is an injective morphism of $\mathbb{C}[\partial_\lambda^{-1}]$ -modules.*

PROOF. Let us first prove that s_{reg} is a $\mathbb{C}[\partial_\lambda^{-1}]$ -modules morphism. Suppose that ω is a weighted homogeneous form of weight $\text{wt}(\omega) = k + \alpha$, where $-1 < \alpha \leq 0$. Then $\partial_\lambda^{-1} \cdot \omega = df \wedge d^{-1}\omega$ is weighted homogeneous of weight $\text{wt}(\omega) + 1 = k + 1 + \alpha$. Put $s(\omega, \lambda) = \lambda^{k+\alpha-1} A$ and $s(\partial_\lambda^{-1} \cdot \omega, \lambda) = \lambda^{k+\alpha} B$. Formula (5.36) implies that $A = B(k + \alpha)$. Therefore,

$$s_{\text{reg}}(\partial_\lambda^{-1} \cdot \omega) = \lambda^k B = \frac{\lambda^k}{k + \alpha} A = \partial_\lambda^{-1} \cdot s_{\text{reg}}(\omega, \lambda),$$

where we used formula (5.37).

Suppose that $\omega \in \Omega^{n+1}[\mathbb{C}^{n+1}]$ is a weighted homogeneous form, such that, the corresponding geomeric sections $s(\omega, \lambda) = 0$. Recalling Corollary 5.38 and Proposition 5.42 we get that $K_f(\omega, \cdot) = 0$. Since the higher residue pairing on $\widehat{\mathcal{H}}_f^{(0)}$ is non-degenerate, we get that the class of ω in $\widehat{\mathcal{H}}_f^{(0)}$ is 0. It remains only to recall Proposition 5.45. \square

If $A(\lambda) \in \mathfrak{h}[\lambda]$, then slightly abusing the notation we will say that $A(\lambda) \in H''$ if $A(\lambda) \in s_{\text{reg}}(H'')$. The Steenbrink's Hodge filtration on \mathfrak{h} can be characterized also in the following way.

LEMMA 5.47. *The Steenbrink's Hodge filtration $F^p \mathfrak{h}$ satisfies the following identity:*

$$F^p \mathfrak{h} = \{A \in \mathfrak{h} \mid \partial_\lambda^{-(n-p)} \cdot A \in H''\}.$$

PROOF. Suppose first that $A \in F^p\mathfrak{h}$. Since the Hodge filtration is M -invariant we may assume that A is an eigenvector of M with eigenvalue $e^{-2\pi i\alpha}$, $-1 < \alpha \leq 0$ and that $A = s(\omega, 1)$ for some weighted homogeneous form ω , such that, $\text{wt}(\omega) = k + \alpha$ for some integer $k \leq n + 1 - p - \alpha$. By definition $s_{\text{reg}}(\omega, \lambda) = \lambda^{k-1}A$ and

$$\partial_\lambda^{-(n-p)} \cdot A = \frac{A\lambda^{n-p}}{(\alpha+1)\cdots(\alpha+n-p)} = \partial_\lambda^{n-p-k+1} \frac{\lambda^{k-1}A}{(\alpha+1)\cdots(\alpha+k-1)},$$

where we used that $n - p - k + 1 \geq 0$. Since s_{reg} is a morphism of $\mathbb{C}[\partial_\lambda^{-1}]$ -modules, the above formula implies that

$$\partial_\lambda^{-(n-p)} \cdot A = \frac{s_{\text{reg}}(\partial_\lambda^{-(n-p-k+1)} \cdot \omega)}{(\alpha+1)\cdots(\alpha+k-1)} \in s_{\text{reg}}(H'').$$

In the opposite direction, suppose that $\partial_\lambda^{-(n-p)} \cdot A = s_{\text{reg}}(\omega)$. The general case is easily reduced to the case when A is an eigenvector of M with eigenvalue $e^{-2\pi i\alpha}$, $-1 < \alpha \leq 0$. Let us write $\omega = \sum_i \omega_i$, where ω_i are weighted homogeneous forms. Since $s(\omega_i, 1)$ is an eigenvector of M with eigenvalue $e^{-2\pi i \text{wt}(\omega_i)}$, we get that each ω_i has weight of the form $\text{wt}(\omega_i) = k_i + \alpha$ for some $k_i \in \mathbb{Z}$. Therefore

$$\frac{A\lambda^{n-p}}{(\alpha+1)\cdots(\alpha+n-p)} = \partial_\lambda^{-(n-p)} \cdot A = \sum_i s_{\text{reg}}(\omega_i) = \sum_i \lambda^{k_i-1} s(\omega_i, 1).$$

By comparing the powers of λ , we get that only the terms for which $k_i - 1 = n - p$ contribute, that is, $k_i = n + 1 - p \Rightarrow \text{wt}(\omega_i) = k_i + \alpha \leq n + 1 - p \Rightarrow s(\omega_i, 1) \in F^p\mathfrak{h}$. Setting $\lambda = 1$, we get that $A \in F^p\mathfrak{h}$ too. \square

After all these preparations we define the map (5.35) by the following formula

$$(5.38) \quad \psi(A) = [(-w)^{n+1-\text{deg}} s_{\text{reg}}^{-1}(\partial_\lambda^{-k} \cdot (-w)^{\mathcal{N}} A)],$$

where $\text{deg} : H'' \rightarrow H''$ is the linear operator induced from (5.9) and k is sufficiently big such that $\partial_\lambda^{-k} \cdot A \in H''$. The definition is independent of the choice of k because

$$\text{deg} \circ \partial_\lambda^{-1} = \partial_\lambda^{-1} \circ (\text{deg} + 1)$$

and $[\partial_\lambda^{-1}\omega] = -w[\omega]$ for all $\omega \in H''$. Since $F^0\mathfrak{h} = \mathfrak{h}$, Lemma 5.47 implies that we may choose $k = n$.

Let us recall Hertling's polarizing form S , that is,

$$S(A, B) = \begin{cases} (-1)^{n(n-1)/2} S_{\overline{V}}(A, B) & \text{if } A, B \in \mathfrak{h}_{\neq 1}, \\ -\frac{(-1)^{n(n+1)/2}}{d} S_{V_\infty}(A, B) & \text{if } A, B \in \mathfrak{h}_1. \end{cases}$$

PROPOSITION 5.48. *The higher residue pairing $K_f^{(m)}(\psi(A_1), \psi(A_2))$ could be non-zero only if $A_1, A_2 \in \mathfrak{h}_{\neq 1}$ and $m = n$, or $A_1, A_2 \in \mathfrak{h}_1$ and $m = n + 1$. The following formula holds:*

$$K_f^{(m)}(\psi(A_1), \psi(A_2)) = \frac{1}{(2\pi i)^m} S(A_1, A_2),$$

where $m = n$ in the first case and $m = n + 1$ in the second case.

PROOF. Suppose that A_i ($i = 1, 2$) are eigenvectors of M with eigenvalues $e^{-2\pi i\alpha_i}$ where $-1 < \alpha_i \leq 0$. Let $\omega_i \in \Omega^{n+1}[\mathbb{C}^{n+1}]$ be a weighted homogeneous form such that

$$s_{\text{reg}}(\omega_i) = \partial_\lambda^{-n} \cdot A_i = \frac{\lambda^n A_i}{(\alpha_i + 1)\cdots(\alpha_i + n)}.$$

Since $s(\omega_i, \lambda) = \lambda^{\text{wt}(\omega_i)-1} s(\omega_i, 1)$ and $\mathcal{N}(A_i) = \alpha_i A_i$, we get $\text{wt}(\omega_i) = n + 1 + \alpha_i$ and

$$(5.39) \quad [\omega_i/df] = s(\omega_i, 1) = \frac{A_i}{(\alpha_i + 1) \cdots (\alpha_i + n)}.$$

By definition

$$\psi(A_i) = (-w)^{n+1-\text{wt}(\omega_i)+\alpha_i} [\omega_i] = [\omega_i].$$

The first claim in the proposition follows easily because the higher residue pairing $K_f^{(m)}(\omega_1, \omega_2)$ is non-zero only if $\text{wt}(\omega_1) + \text{wt}(\omega_2) = m + n + 1$ (see Lemma 5.34). There are two cases: either $\alpha_1 + \alpha_2 = -1$ and then $m = n$, or $\alpha_1 = \alpha_2 = 0$ and then $m = n + 1$. For the second part of the proposition, suppose first that $\alpha_1 + \alpha_2 = -1$. Let us apply the formula from Corollary 5.38

$$K_f^{(n)}\left(\frac{\omega_1}{\Gamma(\text{wt}(\omega_1))}, \frac{\omega_1}{\Gamma(\text{wt}(\omega_1))}\right) = \frac{(-1)^{n+n(n+1)/2}}{(2\pi\mathbf{i})^n} S_V\left(\frac{[\omega_1/df]}{\Gamma(1+\alpha_1)}, \frac{[\omega_2/df]}{\Gamma(1+\alpha_2)}\right).$$

Recalling formula (5.39) and the relation $S_V = (-1)^{n(n-1)/2} S$ we get that the RHS of the above formula coincides with

$$\frac{1}{(2\pi\mathbf{i})^n} S\left(\frac{A_1}{\Gamma(n+1+\alpha_1)}, \frac{A_2}{\Gamma(n+1+\alpha_2)}\right).$$

The first case of the formula that we would like to prove follows.

Suppose now that $\alpha_1 = \alpha_2 = 0$. Recalling the formula from Proposition 5.42 we get

$$K_f^{(n+1)}\left(\frac{\omega_1}{\Gamma(\text{wt}(\omega_1))}, \frac{\omega_1}{\Gamma(\text{wt}(\omega_1))}\right) = \frac{-(-1)^{n(n+1)/2}}{d(2\pi\mathbf{i})^{n+1}} S_{V_\infty}([\omega_1/df], [\omega_2/df]).$$

Recalling formula (5.39) and the relation $S_{V_\infty} = -d(-1)^{n(n+1)/2} S$ we get that the RHS of the above formula coincides with

$$\frac{1}{(2\pi\mathbf{i})^{n+1}} S\left(\frac{A_1}{\Gamma(n+1)}, \frac{A_2}{\Gamma(n+1)}\right).$$

This completes the proof of the proposition. \square

5.6.2. Opposite filtrations and opposite subspaces. Suppose that $U_p\mathfrak{h}$ ($p \in \mathbb{Z}$) is an increasing M -invariant filtration of \mathfrak{h} . The M -invariance imply that

$$U_p\mathfrak{h} = \bigoplus_{s \in \mathbb{S}^1} U_p\mathfrak{h}_s, \quad \mathfrak{h}_s := \text{Ker}(M - s\text{Id}),$$

where $U_p\mathfrak{h}_s = U_p\mathfrak{h} \cap \mathfrak{h}_s$.

DEFINITION 5.49. An *opposite filtration* on \mathfrak{h} is an increasing M -invariant filtration $U_p\mathfrak{h}$ ($p \in \mathbb{Z}$) satisfying the following conditions:

- (i) The filtration is finite: $U_p\mathfrak{h} = 0$ for $p \ll 0$ and $U_p\mathfrak{h} = \mathfrak{h}$ for $p \gg 0$.
- (ii) We have $\mathfrak{h} = \bigoplus_{p \in \mathbb{Z}} F^p\mathfrak{h} \cap U_p\mathfrak{h}$.
- (iii) If s is an eigenvalue of M , then

$$S(U_p\mathfrak{h}_s, U_{m-1-p}\mathfrak{h}_s) = 0, \quad \forall p \in \mathbb{Z},$$

where $m = n$ for $s \neq 1$ and $m = n + 1$ for $s = 1$. \square

The idea to use opposite filtrations to construct opposite subspaces is due to M. Saito [54] (see also [31], Theorem 7.16).

PROPOSITION 5.50. *If $U_p\mathfrak{h}$ ($p \in \mathbb{Z}$) is an opposite filtration, then the subspace*

$$P := \text{Span}_{\mathbb{C}} \{ \psi(A)w^{-p-k-1} \mid p \in \mathbb{Z}, A \in F^p\mathfrak{h} \cap U_p\mathfrak{h}, k \geq 0 \}$$

is a homogeneous opposite subspace.

The rest of this section will be dedicated to the proof of the above proposition. Let us point out that the inverse is also correct, i.e., any homogeneous opposite subspace P is obtained in the above way and the above formula for P provides a one-to-one correspondence between opposite filtrations and homogeneous opposite subspaces. The interested reader is invited to reverse the argument that follows (see also [33], Proposition 4.3.6).

Suppose that $\{A_i\}_{i=1}^{\mu}$ is a basis of \mathfrak{h} , such that, $A_i \in F^{p_i}\mathfrak{h} \cap U_{p_i}\mathfrak{h}$. Since both filtrations F^p and U_p are M -invariant we may also arrange that A_i is an eigenvector of M with eigenvalue $e^{-2\pi i\alpha_i}$ where $-1 < \alpha_i \leq 0$. By definition $\partial_{\lambda}^{-(n-p_i)} \cdot A_i \in H''$. Therefore, there exists a weighted homogeneous form $\omega_i \in \Omega^{n+1}[\mathbb{C}^{n+1}]$, such that,

$$s_{\text{reg}}(\omega_i) = \partial_{\lambda}^{-(n-p_i)} \cdot A_i = \frac{\lambda^{n-p_i} A_i}{(\alpha_i + 1) \cdots (\alpha_i + n - p_i)}.$$

Since $\mathcal{N}(A_i) = \alpha_i A_i$, we get that $\text{wt}(\omega_i) = n + 1 - p_i + \alpha_i$ and

$$\psi(A_i) = (-w)^{n+1-\text{wt}(\omega_i)+\alpha_i} [\omega_i] = (-w)^{p_i} [\omega_i].$$

We claim that the forms $[\omega_i]$ form a $\mathbb{C}[[w]]$ -basis of $\widehat{\mathcal{H}}_f^{(0)}$. In order to prove this it is sufficient to prove that the classes $[[\omega_i]] \in \Omega_f = \widehat{\mathcal{H}}_f^{(0)}/w\widehat{\mathcal{H}}_f^{(0)}$ are linearly independent which is equivalent to proving that the matrix with entries $K_f^{(0)}(\omega_i, \omega_j)$ is non-degenerate. The pairing $K_f^{(0)}(\omega_i, \omega_j)$ is non-zero only if $\text{wt}(\omega_i) + \text{wt}(\omega_j) = n + 1$. There are two cases: $\alpha_i + \alpha_j = -1$ and $\alpha_i = \alpha_j = 0$. In the first case $p_i + p_j = n$ and according to Proposition 5.48 we have

$$K_f^{(0)}(\omega_i, \omega_j) = (-1)^{p_i} K_f^{(n)}(\psi(A_i), \psi(A_j)) = \frac{(-1)^{p_i}}{(2\pi i)^n} S(A_i, A_j).$$

In the second case $p_i + p_j = n + 1$ and again according to Proposition 5.48 we have

$$K_f^{(0)}(\omega_i, \omega_j) = (-1)^{p_i} K_f^{(n+1)}(\psi(A_i), \psi(A_j)) = \frac{(-1)^{p_i}}{(2\pi i)^{n+1}} S(A_i, A_j).$$

Suppose now that we have a pair (i, j) , such that, $S(A_i, A_j) \neq 0$. Since S is M invariant we either have $\alpha_i + \alpha_j = -1$ or $\alpha_i = \alpha_j = 0$. In the first case, since S is a polarizing form for the Hodge filtration $F^p\mathfrak{h}_{\neq 1}$ which has weight n and $A_i \in F^{p_i}\mathfrak{h}$ we get that $p_i + p_j \leq n$. On the other hand, condition (iii) in Definition 5.49 implies that $p_i + p_j \geq n$, that is, $p_i + p_j = n$. Similarly, in the second case we will get that $p_i + p_j = n + 1$. This proves that the matrix of the residue pairing $K_f^{(0)}$ in the basis $[[\omega_i]]$ is a product of the diagonal matrix whose i th entry is $i^{2p_i-m}(2\pi)^{-m}$ ($m = n$ or $n + 1$ depending on whether $\alpha_i \neq 0$ or $\alpha_i = 0$) and the matrix of the polarizing form S in the basis A_i . Since the polarizing form is non-degenerate, we get that the matrix of the residue pairing is also non-degenerate which is what we had to prove.

The rest of the proof is straightforward. Since $w^{-p_i}\psi(A_i)$ is a $\mathbb{C}[[w]]$ -basis it is easy to prove that $w^{-p_i-1}\psi(A_i)$ is a $\mathbb{C}[[w^{-1}]]$ -basis of P and we have a direct sum decomposition $\widehat{\mathcal{H}}_f = \widehat{\mathcal{H}}_f^{(0)} \oplus P$. Since the forms ω_i representing $w^{-p_i}\psi(A_i)$ are weighted homogeneous,

$w\nabla_{\partial/\partial w}[\omega_i] = \text{wt}(\omega_i)[\omega_i]$. Therefore, P is a homogeneous subspace. Finally, let us check that P is a Lagrangian subspace, that is,

$$(5.40) \quad \text{Res}_{w=0} K_f(\psi(A_i)w^{-p_i-k_i-1}, \psi(A_j)w^{-p_j-k_j-1})w^{-n-1}dw = 0, \quad \forall k_i, k_j \geq 0.$$

There are two cases: $\alpha_i + \alpha_j = -1$ or $\alpha_i = \alpha_j = 0$. In the first case, assuming the above notation, we have

$$K_f(\psi(A_i)w^{-p_i-k_i-1}, \psi(A_j)w^{-p_j-k_j-1}) = \frac{(-1)^{p_j+k_j+1}}{(2\pi\mathbf{i})^n} w^{-p_i-p_j-k_i-k_j-1+2n} S(A_i, A_j),$$

where we used Proposition 5.48. Since $S(A_i, A_j)$ could be non-zero only if $p_i + p_j = n$, we may assume that the power of w in the above formula is $-k_i - k_j - 1 + n \Rightarrow$ the residue (5.40) is 0. The argument in the second case is similar and it is left as an exercise. This completes the proof of Proposition 5.50

Let us point out that using complex conjugation with respect to the real structure $H^n(f^{-1}(1), \mathbb{R})$ we can define an opposite filtration in the following way:

$$U_p \mathfrak{h}_{\neq 1} := \overline{F^{n-p} \mathfrak{h}_{\neq 1}}, \quad p \in \mathbb{Z},$$

and

$$U_p \mathfrak{h}_1 := \overline{F^{n+1-p} \mathfrak{h}_1}, \quad p \in \mathbb{Z}.$$

Therefore, according to Proposition 5.50 there exists a homogeneous opposite subspace.

5.6.3. Classification of homogeneous opposite subspaces. Let us recall the vector space $\Omega_f := \widehat{\mathcal{H}}_f^{(0)}/w\widehat{\mathcal{H}}_f^{(0)}$ and the grading operator $\theta = \frac{n+1}{2} - \text{deg}$ where the degree operators is defined by (5.9). The operator θ is diagonalizable and we denote by $\Omega_f^\alpha := \text{Ker}(\theta + \alpha)$ the eigensubspace of θ with eigenvalue $-\alpha$. Let us order the set of numbers α , such that, $\Omega_f^\alpha \neq 0$ in an increasing sequence and let us repeat each α as many times as $\dim_{\mathbb{C}} \Omega_f^\alpha$. We get a sequence $\alpha_1 \leq \dots \leq \alpha_\mu$ known as the *Steenbrink's spectrum* of f . Furthermore, the residue pairing $K_f^{(0)}$ on $\widehat{\mathcal{H}}_f^{(0)}$ induces a non-degenerate pairing $(\ , \)$ on Ω_f which will be called the *residue pairing*. Let us introduce also the *Euler pairing* (see (3.29))

$$\langle a, b \rangle := \frac{1}{2\pi} (a, e^{\pi i \theta} b), \quad a, b \in \Omega_f.$$

Since the degree of the Hessian of f is $D = \sum_{i=0}^n (1 - 2c_i)$, using Corollary 5.7, we get that if $a \in \Omega_f^\alpha$, $b \in \Omega_f^\beta$, and $(a, b) \neq 0$, then $\alpha + \beta = 0$. In particular, the Steenbrink's spectrum has the following symmetry $\alpha_i + \alpha_{\mu+1-i} = 0$. Let us introduce also the following vector subspaces of Ω_f :

$$\Omega_f^{<\alpha} := \bigoplus_{\beta: \beta < \alpha} \Omega_f^\beta, \quad \Omega_f^{\leq \alpha} := \bigoplus_{\beta: \beta \leq \alpha} \Omega_f^\beta, \quad \alpha \in \mathbb{Q}.$$

The subspaces $\Omega_f^{>\alpha}$ and $\Omega_f^{\geq \alpha}$ are defined in a similar fashion. Let $\text{Aut}(\Omega_f^{\leq \bullet}, \langle \ , \ \rangle)$ be the group of all endomorphisms $g \in \text{End}(\Omega_f)$, such that,

- (i) If $v \in \Omega_f^\alpha$, then $g(v) \in v + \Omega_f^{<\alpha}$.
- (ii) $\langle g(a), g(b) \rangle = \langle a, b \rangle$ for all $a, b \in \Omega_f$.

The definition of the group $\text{Aut}(\Omega_f^{\leq \bullet}, \langle \cdot, \cdot \rangle)$ can be stated equivalently using Givental's symplectic space formalism (see Section 1.4) associated with the vector space Ω_f and the residue pairing (\cdot, \cdot) . Namely, an element $g \in \text{End}(\Omega_f)$ belongs to $\text{Aut}(\Omega_f^{\leq \bullet}, \langle \cdot, \cdot \rangle)$ if and only if

(i) The operator series $R_g(w) := w^\theta g w^{-\theta}$ has the form

$$R_g(w) = 1 + R_{g,1}w + R_{g,2}w^2 + \cdots, \quad R_{g,k} \in \text{End}(\Omega_f).$$

(ii) $R_g(w)$ is a symplectic transformation: $R_g(w)R_g(-w)^T = 1$.

Let us recall that if $P \subset \widehat{\mathcal{H}}_f$ is an opposite subspace then we have constructed a section $\sigma_P : \Omega_f \rightarrow \widehat{\mathcal{H}}_f^{(0)}$ (see Proposition 4.35).

PROPOSITION 5.51. *a) If $P \subset \widehat{\mathcal{H}}_f$ is a homogeneous opposite subspace, then $\widehat{\sigma}_P$ induces an isomorphism $\Omega_f[w] \cong H''$.*

b) The group $\text{Aut}(\Omega_f^{\leq \bullet}, \langle \cdot, \cdot \rangle)$ acts transitively and faithfully on the set of all homogeneous opposite subspaces. The action is given by the following formula:

$$g \cdot P = \widehat{\sigma}_P \circ R_g(w)(\Omega_f[w^{-1}]w^{-1}),$$

where $\widehat{\sigma}_P : \Omega_f((w)) \rightarrow \widehat{\mathcal{H}}_f$ is the $\mathbb{C}((w))$ -linear extension of σ_P .

PROOF. Let $\omega_i \in \Omega^{n+1}[\mathbb{C}^{n+1}]$ ($1 \leq i \leq \mu$) be weighted homogeneous forms whose classes $[\omega_i] \in \Omega_f^{\alpha_i}$ form a basis of Ω_f where α_i ($1 \leq i \leq \mu$) is the Steenbrink spectrum of f . Note that the cohomology classes $[\omega_i]$ ($1 \leq i \leq \mu$) form a $\mathbb{C}[[w]]$ -basis of $\widehat{\mathcal{H}}_f^{(0)}$.

Let $P \subset \widehat{\mathcal{H}}_f$ be a fixed homogeneous opposite subspace and let σ_P and $\widehat{\sigma}_P$ be the corresponding section $\Omega_f \rightarrow \widehat{\mathcal{H}}_f^{(0)}$ and its $\mathbb{C}((w))$ -linear extension $\Omega_f((w)) \rightarrow \widehat{\mathcal{H}}_f^{(0)}$. Part a) is an easy consequence of Proposition 5.11. Namely, we have

$$(5.41) \quad w \nabla_{\partial/\partial w} \circ \widehat{\sigma} = \widehat{\sigma} \circ (w \partial_w + \text{deg}),$$

for $\sigma = \sigma_P$. Therefore, the classes $\sigma_P([\omega_i])$ are homogeneous of degree $\text{wt}(\omega_i) = \frac{n+1}{2} + \alpha_i$. Since the variable w has degree 1, the cohomology class $\sigma_P([\omega_i])$ can be written as

$$\sum_{j=1}^{\mu} \sum_{k=0}^{\infty} s_{jik} [\omega_j] w^k, \quad s_{jik} \in \mathbb{C},$$

where only finitely many s_{jik} are non-zero because the following identity must hold: $\text{wt}(\omega_i) = k + \text{wt}(\omega_j)$. Since $w^k[\omega_j] = [(-\partial_\lambda)^{-k} \cdot \omega_j]$ is in (the image of) the Brieskorn lattice we get that $\sigma_P([\omega_i])$ is in the Brieskorn lattice. In particular, $\widehat{\sigma}_P$ maps $\Omega_f[w]$ into the Brieskorn lattice. It remains only to prove that the map $\widehat{\sigma}_P$ induces a surjective map $\Omega_f[w] \rightarrow H''$ where we identify H'' with its image in $\widehat{\mathcal{H}}_f^{(0)}$. This follows easily from the fact that $\sigma_P([\omega_i]) = [\omega_i] \pmod{w \widehat{\mathcal{H}}_f^{(0)}}$ which implies that $\sigma_P([\omega_i])$ ($1 \leq i \leq \mu$) is a $\mathbb{C}[[w]]$ -basis of $\widehat{\mathcal{H}}_f^{(0)}$. This completes the proof of part a).

Let us prove part b). Suppose that Q is another homogeneous opposite subspace. Put $R := \widehat{\sigma}_P^{-1} \circ \widehat{\sigma}_Q$. This is a $\mathbb{C}((w))$ -linear operator of $\Omega_f((w))$ leaving the subspace $\Omega_f[w]$ invariant. In particular, we have

$$R[\omega_j] := \sum_{i=1}^{\mu} R_{ij}(w) [\omega_i], \quad R_{ij}(w) \in \mathbb{C}[w]$$

or equivalently

$$\widehat{\sigma}_Q(\llbracket \omega_j \rrbracket) = \sum_{i=1}^{\mu} R_{ij}(w) \widehat{\sigma}_P(\llbracket \omega_i \rrbracket).$$

On the other hand, formula (5.41) holds for both $\sigma = \sigma_P$ and σ_Q . Differentiating the above identity with $w \nabla_{\partial/\partial w}$ we get

$$w \partial_w (R_{ij}(w)) = (\text{wt}(\omega_j) - \text{wt}(\omega_i)) R_{ij}(w).$$

This proves that $R_{ij}(w) = w^{\alpha_j - \alpha_i} g_{ij}$ for some $g_{ij} \in \mathbb{C}$. Let $g \in \text{End}(\Omega_f)$ be the linear operator whose matrix in the basis $\llbracket \omega_i \rrbracket$ ($1 \leq i \leq \mu$) is the matrix with entries g_{ij} . We claim that $g \in \text{Aut}(\Omega_f^{\leq \bullet}, \langle \cdot, \cdot \rangle)$ and that $Q = g \cdot P$. Clearly, we have $R = w^\theta g w^{-\theta}$. In order to prove that $g \in \text{Aut}(\Omega_f^{\leq \bullet}, \langle \cdot, \cdot \rangle)$, we need only to check that R is a symplectic transformation. However, note that both $\widehat{\sigma}_P$ and $\widehat{\sigma}_Q$ are isomorphisms of symplectic vector spaces because

$$K_f(\widehat{\sigma}(\phi_1(w)), \widehat{\sigma}(\phi_2(w))) w^{-n-1} = (\phi_1(w), \phi_2(-w))$$

for $\sigma = \sigma_P, \sigma_Q$. Indeed, if $\phi_1, \phi_2 \in \Omega_f$, then the above formula becomes precisely Proposition 4.35, c). The general case follows easily from the general properties of the higher residue pairing – see property 2 in Section 4.5.3. Therefore R is a symplectic transformation of $\Omega_f(w)$. Finally,

$$Q = \widehat{\sigma}_Q(\Omega_f[w^{-1}]w^{-1}) = \widehat{\sigma}_P \circ R(\Omega_f[w^{-1}]w^{-1}) = g \cdot P. \quad \square$$

5.7. Period integrals

Let f be a weighted homogeneous singularity and let $U_p \mathfrak{h}$ be an opposite filtration. Let $(F, p : Z_{\text{lin}} \rightarrow S_{\text{lin}})$ be the space of linear deformations of f – see Section 5.3.2. The opposite filtration allows us to construct an opposite subspace in $\widehat{\mathcal{H}}_f^{(0)}$ and hence, according to Proposition 5.12, S_{lin} is equipped with a Frobenius type structure. Moreover, according to Section 5.2.3, the Frobenius type structure can be extended to a Frobenius structure on an open subset B of \mathbb{C}^μ containing $S_{\text{lin}} = \mathbb{C}^k$. Let $I_a^{(m)}(t, \lambda)$ be the solutions of the second structure connection defined by (3.11) where the choice of the calibration will be specified later on. We are going to prove that if we restrict to $t \in S_{\text{lin}}$, then the functions $I_a^{(m)}(t, \lambda)$ can be identified with period integrals of the Milnor fiber $Z_{\lambda, t}$. Moreover, this identification would allow us to prove that the reflection vectors in H_f correspond to the vanishing cycles and that the monodromy of the Frobenius manifold (3.12) coincides with the monodromy of the singularity f .

5.7.1. Period map. Let $U_p \mathfrak{h}$ be an opposite filtration to the Hodge filtration $F^p \mathfrak{h}$ on $\mathfrak{h} = H^n(f^{-1}(1), \mathbb{C})$. Let $P \subset \widehat{\mathcal{H}}_f$ be the corresponding opposite subspace – see Proposition 5.50. Put

$$\psi_U := \bigoplus_{p \in \mathbb{Z}} (-w)^p \psi|_{F^p \mathfrak{h} \cap U_p \mathfrak{h}}.$$

We have the following diagram of isomorphisms

$$H^n(f^{-1}(1), \mathbb{C}) \xrightarrow{\psi_U} \widehat{\mathcal{H}}_f^{(0)} \cap wP \xrightarrow{\pi} \Omega_f \cong H_f,$$

where the second isomorphism is induced from the quotient map $\pi : \widehat{\mathcal{H}}_f^{(0)} \rightarrow \widehat{\mathcal{H}}_f^{(0)}/w\widehat{\mathcal{H}}_f^{(0)}$ and the isomorphism $\Omega_f \cong H_f$ depends on the choice of a holomorphic volume form $\varphi = \nu dx$ where $\nu \in \mathbb{C}$ is a non-zero complex number: $g \in H_f \mapsto g\varphi \in \Omega_f$. The general theory allows us to associate a Frobenius structure to the pair (P, φ) – see Section 4.6.2. Furthermore, let us recall the isomorphism

$$\text{var} \circ \text{PD} : H^n(f^{-1}(1), \mathbb{Z}) \rightarrow H_n(f^{-1}(1), \mathbb{Z})$$

defined by Poincare duality and the variation isomorphism – see Section 5.5.5. Finally, let us introduce the following notation. If a is a real number, then $\lceil a \rceil$ denotes the ceiling of a , that is, the smallest integer greater or equal than a and $\langle a \rangle := a - \lceil a \rceil$. Note that $-1 < \langle a \rangle \leq 0$. If A is a diagonalizable linear operator acting on a finite dimensional vector space, then we denote by $\langle A \rangle$ and $\lceil A \rceil$ the linear operators which are defined on the eigenvectors v of A by $\langle A \rangle v := \langle \lambda \rangle v$ and $\lceil A \rceil v := \lceil \lambda \rceil v$ where $Av = \lambda v$. Let us define the linear map

$$\Pi : H_n(f^{-1}(1), \mathbb{C}) \rightarrow \Omega_f,$$

such that,

$$(5.42) \quad \llbracket \psi_U \circ (\text{var} \circ \text{PD})^{-1}(\gamma) \rrbracket = (-1)^{n(n+1)/2} \mathbf{i}^{n+1} (2\pi)^{-1-l} \Gamma(1 + \langle \text{deg} \rangle) e^{\pi i \theta} \Pi(\gamma),$$

where $l := n/2$ and $\theta := \frac{n+1}{2} - \text{deg}$. Let us recall that if ω is a holomorphic form then we denote by $[\omega]$ the cohomology class of ω in $\widehat{\mathcal{H}}_f^{(0)}$ and by $\llbracket \omega \rrbracket$ the class of ω in $\Omega_f = \widehat{\mathcal{H}}_f^{(0)}/w\widehat{\mathcal{H}}_f^{(0)}$. The main motivation to introduce the map Π comes from the following proposition.

PROPOSITION 5.52. *If $\omega \in H''$ is a holomorphic form whose cohomology class $[\omega] \in \widehat{\mathcal{H}}_f^{(0)} \cap wP$, then the following formula holds*

$$\left(\llbracket \omega \rrbracket, \frac{\lambda^{\theta+l-1/2}}{\Gamma(\theta+l+1/2)} \Pi(\gamma) \right) = (2\pi)^{-l} \int_{\gamma_\lambda} \frac{\omega}{df},$$

for all $\gamma \in H_n(f^{-1}(1), \mathbb{C})$, where $\gamma_\lambda \in H_n(f^{-1}(\lambda), \mathbb{C})$ denotes the parallel transport of γ along the reference path used to specify the value of the LHS.

PROOF. Since $H_n(f^{-1}(1), \mathbb{C})$ has a basis consisting of vectors of the form $\gamma = \text{var} \circ \text{PD}(B)$ with $B \in F^q \mathfrak{h} \cap U_q \mathfrak{h}$, $M(B) = e^{-2\pi i \beta} B$, $-1 < \beta \leq 0$, we may assume that γ has such a form. Put $\omega_B := \psi_U(B) = (-w)^q \psi(B)$ and note that

$$s(\omega_B, 1) = [\omega_B/df] = \frac{B}{(\beta+1) \cdots (\beta+n-q)}$$

and $\text{wt}(\omega_B) = n+1-q+\beta$. Similarly, since Ω_f has a basis consisting of classes $\llbracket \omega_A \rrbracket$ where $[\omega_A] := \psi_U(A) = (-w)^p \psi(A)$ for $A \in F^p \mathfrak{h} \cap U_p \mathfrak{h}$, $M(A) = e^{-2\pi i \alpha} A$, $-1 < \alpha \leq 0$, we may assume that $[\omega] = [\omega_A]$ for some A . We have

$$s(\omega_A, 1) = [\omega_A/df] = \frac{A}{(\alpha+1) \cdots (\alpha+n-q)}$$

and $\text{wt}(\omega_A) = n+1-p+\alpha$.

The RHS is straightforward to compute

$$\frac{1}{(2\pi)^l} \lambda^{\text{wt}(\omega)-1} \langle s(\omega, 1), \gamma \rangle = \frac{1}{(2\pi)^l} \lambda^{\text{wt}(\omega)-1} L([\omega/df], B).$$

Let us compute the LHS. We have

$$\Pi(\gamma) = c_n^{-1} \frac{e^{-\pi i \deg}}{\Gamma(1 + \langle \deg \rangle)} \llbracket \omega_B \rrbracket = c_n^{-1} \frac{e^{-\pi i(n+1-q+\beta)}}{\Gamma(1 + \beta)} \llbracket \omega_B \rrbracket$$

where $c_n = (-1)^{n(n+1)/2} \mathbf{i}^{n+1} (2\pi)^{-1-l}$. Note that

$$\left(\theta + l - \frac{1}{2}\right) \llbracket \omega_B \rrbracket = (q - \beta - 1) \llbracket \omega_B \rrbracket.$$

The LHS of the identity that we have to prove takes the form

$$c_n^{-1} \frac{e^{-\pi i(n+1-q+\beta)}}{\Gamma(1 + \beta)} \frac{\lambda^{q-\beta-1}}{\Gamma(q - \beta)} K^{(0)}([\omega_A], [\omega_B]).$$

The residue pairing could be non-zero only if the weights of the forms ω_A and ω_B add up to $n + 1$, that is, $p + q = n + 1 + \alpha + \beta$. Recalling Proposition 5.48 we get that $K^{(0)}([\omega_A], [\omega_B])$ is equal to

$$K^{(0)}((-w)^{-p}\psi(A), (-w)^{-q}\psi(B)) = (-1)^p K^{(p+q)}(\psi(A), \psi(B)) = \frac{(-1)^p}{(2\pi \mathbf{i})^m} S(A, B),$$

where $m := p + q = n + 1 + \alpha + \beta$. Note that $q - \beta = n + 1 - p + \alpha = \text{wt}(\omega)$ and

$$\frac{A}{\Gamma(q - \beta)} = \frac{1}{\Gamma(1 + \alpha)} \frac{A}{(\alpha + 1) \cdots (\alpha + n - p)} = \frac{[\omega/df]}{\Gamma(1 + \alpha)}$$

Therefore, the LHS becomes

$$\frac{c_n^{-1}}{(2\pi \mathbf{i})^m} \frac{(-1)^{n+1-p-q} e^{-\pi i \beta}}{\Gamma(1 + \alpha) \Gamma(1 + \beta)} \lambda^{\text{wt}(\omega)-1} S([\omega/df], B).$$

Now the computation splits into two cases. First, if $\alpha + \beta = -1$, then $m = n$ and

$$S([\omega/df], B) = \frac{(-1)^{n(n-1)/2}}{e^{-2\pi i \beta} - 1} L([\omega/df], B).$$

A simple computation shows that

$$\Gamma(1 + \alpha) \Gamma(1 + \beta) = 2\pi \mathbf{i} \frac{e^{-\pi i \beta}}{e^{-2\pi i \beta} - 1}.$$

Therefore, the LHS takes the form

$$-(-1)^{n(n-1)/2} \frac{c_n^{-1}}{(2\pi \mathbf{i})^{n+1}} \lambda^{\text{wt}(\omega)-1} L([\omega/df], B).$$

The constant becomes

$$-(-1)^{n(n-1)/2} \frac{c_n^{-1}}{(2\pi \mathbf{i})^{n+1}} = \frac{1}{(2\pi)^l},$$

that is, we get that the LHS is equal to the RHS. The second case is when $\alpha = \beta = 0$. Then $m = n + 1$ and the LHS takes the form

$$\frac{c_n^{-1}}{(2\pi \mathbf{i})^{n+1}} \lambda^{\text{wt}(\omega)-1} S([\omega/df], B) = -(-1)^{n(n-1)/2} \frac{c_n^{-1}}{(2\pi \mathbf{i})^{n+1}} \lambda^{\text{wt}(\omega)-1} L([\omega/df], B).$$

Clearly, the constant factor is as in the previous case, so the above expression also coincides with the RHS. \square

According to Proposition 5.52, the map Π defined by formula (5.42) determines completely the periods of the Milnor fiber $f^{-1}(1)$. For that reason we will refer to Π as the *period map* of f corresponding to the opposite filtration $U_\bullet \mathfrak{h}$.

5.7.2. Euler pairing and its symmetrization. Let us recall that Ω_f is equipped with the Euler pairing (see (3.29))

$$\langle \llbracket \omega_1 \rrbracket, \llbracket \omega_2 \rrbracket \rangle = \frac{1}{2\pi} K_f^{(0)}(\llbracket \omega_1 \rrbracket, e^{\pi i \theta} \llbracket \omega_2 \rrbracket).$$

Let us introduce also the $(-1)^n$ -symmetrization of the Euler pairing

$$(a|b) := \langle a, b \rangle + (-1)^n \langle b, a \rangle$$

which will be called the intersection pairing. We would like to prove that under the period map Π these two pairings are identified with respectively the Seifert form and the intersection pairing of the singularity f .

PROPOSITION 5.53. *The following formulas hold:*

$$\langle \Pi(\gamma_1), \Pi(\gamma_2) \rangle = -(-1)^{n(n-1)/2} L(\gamma_1, \gamma_2)$$

and

$$(\Pi(\gamma_1)|\Pi(\gamma_2)) = (-1)^{n(n-1)/2} \gamma_1 \circ \gamma_2$$

for all $\gamma_1, \gamma_2 \in H_n(f^{-1}(1), \mathbb{C})$.

PROOF. We may assume that $\gamma_i = \text{var} \circ \text{PD}(A_i)$ where $A_i \in F^{p_i} \mathfrak{h} \cap U_{p_i} \mathfrak{h}$, $M(A_i) = e^{-2\pi i \alpha_i} A_i$, and $-1 < \alpha_i \leq 0$. Let us denote $[\omega_i] := \psi_U(A_i) = (-w)^{p_i} \psi(A_i)$ where ω_i is a weighted homogeneous form of weight $n+1-p_i+\alpha_i$.

Let us prove the first identity. By definition the RHS is $-(-1)^{n(n-1)/2} \langle A_1, \gamma_2 \rangle$ – here the angle brackets denote the natural pairing between cohomology and homology. Recall that

$$\frac{A_1}{(\alpha_1 + 1) \cdots (\alpha_1 + n - p_1)} = [\omega_1/df].$$

Recalling Proposition 5.52 we get that the RHS of our identity is equal to

$$-(-1)^{n(n-1)/2} (2\pi)^l (\alpha_1 + 1) \cdots (\alpha_1 + n - p_1) K_f^{(0)}(\llbracket \omega_1 \rrbracket, \frac{1}{\Gamma(p_2 - \alpha_2)} \Pi(\gamma_2)),$$

where we used that

$$(\theta + l + 1/2) \Pi(\gamma_2) = ((n+1)/2 - \text{wt}(\omega_2) + l + 1/2) \Pi(\gamma_2) = (p_2 - \alpha_2) \Pi(\gamma_2).$$

On the other hand, $p_2 - \alpha_2 = n+1-p_1+\alpha_1$ because the residue pairing is non-zero only if the weights of the forms add up to $n+1$. Therefore, the above formula takes the form

$$-(-1)^{n(n-1)/2} (2\pi)^l K_f^{(0)}(\llbracket \omega_1 \rrbracket, \frac{1}{\Gamma(1 + \alpha_1)} \Pi(\gamma_2)).$$

By definition

$$\llbracket \omega_1 \rrbracket = \psi_U(A_1) = (-1)^{n(n+1)/2} \mathbf{i}^{n+1} (2\pi)^{-1-l} \Gamma(1 + \alpha_1) e^{\pi i \deg} \Pi(\gamma_1).$$

Note that

$$-(-1)^{n(n-1)/2} (-1)^{n(n+1)/2} \mathbf{i}^{n+1} e^{\pi i \deg} = e^{\pi i(-(n+1)/2 + \deg)} = e^{-\pi i \theta}.$$

Since $\theta^T = -\theta$ the formula that we have to prove follows easily from the above formulas.

The second formula is a consequence of the well known formula for the intersection form in terms of the Seifert form (see [6]), that is,

$$a \circ b = -L(a, b) - (-1)^n L(b, a), \quad \forall a, b \in H_n(f^{-1}(1), \mathbb{C}). \quad \square$$

5.7.3. Period vectors. Let us continue to work with a fixed opposite filtration $U_p \mathfrak{h}$, the corresponding opposite subspace P , and a fixed choice $\varphi = \nu dx \in \Omega_f$ of a holomorphic volume form. The pair (P, φ) determines a Frobenius type structure on $S_{\text{lin}} = \mathbb{C}^k$ which can be extended to a Frobenius structure on an open subset B of $\mathbb{C}^{\mathcal{B}} = \mathbb{C}^\mu$ containing S_{lin} – see Sections 5.2.2 and 5.2.3 for more details. Let us recall the construction of weighted homogeneous holomorphic forms ω_I ($I \in \mathcal{B}$), such that, their cohomology classes $[\omega_I] \in \widehat{\mathcal{H}}_F^{(0)}$ determine a $\mathbb{C}[[w]]$ -trivialization of $\widehat{\mathcal{H}}_F^{(0)}$. Namely, as we already discussed in Section 5.2.3 we may assume that the indices $I \in \mathcal{B}$ have non-zero entries only in the first k variables. The classes of the forms $x^I dx$ in Ω_f form a basis. Let us choose weighted homogeneous forms $\omega_I^\circ \in \Omega^{n+1}[\mathbb{C}^{n+1}]$, such that, $[\omega_I^\circ] = \sigma_f(\llbracket x^I dx \rrbracket)$ where $\sigma_f : \Omega_f \rightarrow \widehat{\mathcal{H}}_f^{(0)} \cap wP$ is the section corresponding to the opposite subspace P . Let ω° be a weighted homogeneous form, such that, $[\omega^\circ] = \sigma_f(\varphi)$. Then the forms ω_I are defined as follows. First, we construct a matrix $E(t, w)$ of size $\mathcal{B} \times \mathcal{B}$ whose entries $E_{IJ}(t, w)$ are defined by

$$\exp\left(\sum_{a=0}^{k-1} t_a x_a/w\right) x^J \omega^\circ = \sum_{I \in \mathcal{B}} \omega_I^\circ E_{IJ}(t, w),$$

where the identity should be viewed in $\widehat{\mathcal{H}}_f^{(0)} \llbracket t_0, \dots, t_{k-1} \rrbracket$. The matrix $E(t, w)$ admits a Birkhoff factorization $E(t, w) = T(t, w)A(t, w)^{-1}$ with $T(t, w) = 1 + O(w^{-1})$ and $A(t, w) = A_0(t) + A_1(t)w + \dots$ and we define

$$\omega_I := \sum_{J \in \mathcal{B}} x^J A_{IJ}(t, w) \omega^\circ.$$

Due to homogeneity, as it was explained in Section 5.2.2, the forms ω_I are polynomial in x, t , and w and if we assign weight 1 to w , weight c_i to x_i ($0 \leq i \leq n$), and weight $1 - c_a$ to t_a ($0 \leq a \leq k - 1$), then ω_I becomes a weighted homogeneous form of weight $\text{wt}(\omega_I^\circ) = c \cdot I + c_0 + \dots + c_n$. In particular, ω_I are holomorphic forms on \mathbb{C}^{n+1} depending holomorphically on t . Note that if ω is a holomorphic form on \mathbb{C}^{n+1} , then the forms $w\omega$ and $-dF \wedge d^{-1}\omega$ represent the same class in $\widehat{\mathcal{H}}_F^{(0)}$. Therefore, we may assume that the forms ω_I are independent of w .

On the other hand, let us recall the vanishing cohomology bundle H^n (see (5.16)) equipped with a flat Gauss-Manin connection. Let us consider a deformation of the Brieskorn lattice which would allow us to construct holomorphic sections of H^n . Namely, put $\Omega_{Z_{\text{lin}}/S_{\text{lin}}}^p[\mathbb{C}^k \times \mathbb{C}^{n+1}]$ for the vector space of polynomial p -forms on \mathbb{C}^{n+1} depending polynomially on the parameters t_0, \dots, t_{k-1} . Let us define

$$H_F'' := \Omega_{Z_{\text{lin}}/S_{\text{lin}}}^{n+1}[\mathbb{C}^k \times \mathbb{C}^{n+1}]/dF \wedge d\Omega_{Z_{\text{lin}}/S_{\text{lin}}}^{n-1}[\mathbb{C}^k \times \mathbb{C}^{n+1}].$$

We will refer to H_F'' as the *deformed Brieskorn lattice*. The deformed Brieskorn lattice H_F'' is naturally a $\mathbb{C}[t_0, \dots, t_n, \partial_\lambda^{-1}]$ -module, where the action of t_a ($0 \leq a \leq k - 1$) is by multiplication and the action of ∂_λ^{-1} is defined by

$$\partial_\lambda^{-1} \cdot \omega = dF \wedge d^{-1}\omega.$$

Just like in the non-deformed case we have the following important fact.

PROPOSITION 5.54. *The map*

$$H_F'' \rightarrow \Gamma(S_{\text{lin}}, \widehat{\mathcal{H}}_F^{(0)}), \quad \omega \mapsto [\omega]$$

is injective and we have

$$p(t)[\omega] = [p(t)\omega] \quad \forall p(t) \in \mathbb{C}[t_0, \dots, t_{k-1}], \quad [\partial_\lambda^{-1} \cdot \omega] = -w[\omega].$$

PROOF. The proof is the same as the proof of Proposition 5.45. The only difference is that we have extra variables t_0, \dots, t_{k-1} . However, if we assign weights $1 - c_a$ to t_a and extend the notion of weighted homogeneous to include the variables t_a too, then the argument remains the same. The key fact which makes the argument work is that the weights of t_a are positive. \square

Furthermore, if $\omega \in \Omega_{Z_{\text{lin}}/S_{\text{lin}}}^{n+1}[\mathbb{C}^k \times \mathbb{C}^{n+1}]$, then we have the notion of a *geometric section*

$$s(\omega, \lambda, t) := \int \frac{\omega}{dF} \in H^n(Z_{\lambda, t}, \mathbb{C}), \quad (\lambda, t) \in \mathbb{C} \times S_{\text{lin}} \setminus D_F,$$

where ω/dF denotes any form η defined in a tubular neighborhood of $Z_{\lambda, t}$ in \mathbb{C}^{n+1} , such that, $\omega = dF \wedge \eta$. Again the choice of η is not unique but its restriction $\eta|_{Z_{\lambda, t}}$ is a uniquely determined holomorphic n -form on the hypersurface $Z_{\lambda, t}$. The map $(\lambda, t) \mapsto s(\omega, \lambda, t)$ is a holomorphic section of H^n which we denote by $[\omega/dF]$ and we will refer to as the geometric section corresponding to the form ω . We have the following formulas for the derivatives of the geometric sections with respect to the Gauss–Manin connection:

$$\begin{aligned} \nabla_{\partial/\partial\lambda}[\partial_\lambda^{-1} \cdot \omega/dF] &= [\omega/dF] \quad (0 \leq a \leq k-1), \\ \nabla_{\partial/\partial t_a}[\partial_\lambda^{-1} \cdot \omega/dF] &= [(-\partial_{t_a} F \omega + \text{Lie}_{\partial/\partial t_a} \omega)/dF], \end{aligned}$$

where $\text{Lie}_v \omega$ denotes the Lie derivative of the form ω with respect to the vector field v . The proof of the above formulas is elementary and we leave it as an exercise (see also [6]).

Finally, let us recall (see Section 5.2.2) that the classes $[\omega_I]$ ($I \in \mathcal{B}$) of the forms ω_I in $p_*\Omega_F \cong \widehat{\mathcal{H}}_F^{(0)}/w\widehat{\mathcal{H}}_F^{(0)}$ determine a trivialization of the vector bundle $p_*\Omega_F$. Let $C_a(t)$ be the matrix of the operator of multiplication by $\frac{\partial F}{\partial t_a} = x_a$, that is,

$$x_a[\omega_J] = \sum_{I \in \mathcal{B}} [\omega_I] C_{aIJ}(t).$$

After all these preparations we can state the following proposition.

PROPOSITION 5.55. *The following formulas hold:*

$$(5.43) \quad \nabla_{\partial/\partial t_a}[\omega_J/dF] = - \sum_I \nabla_{\partial/\partial\lambda}[\omega_J/dF] C_{aIJ}(t) \quad (0 \leq a \leq k-1)$$

$$(5.44) \quad (\lambda \nabla_{\partial/\partial\lambda} + \nabla_E)[\omega_J/dF] = (\text{wt}(\omega_J) - 1)[\omega_J/dF],$$

where $E = \sum_{a=0}^{k-1} (1 - c_a) t_a \frac{\partial}{\partial t_a}$ is the Euler vector field.

PROOF. This proposition is an easy corollary from formulas (5.12)–(5.13) and Proposition 5.54. Indeed, according to formula (5.12) we have

$$w \nabla_{\partial/\partial t_a}[\omega_J] = \sum_I [\omega_I] C_{aIJ}(t).$$

Equivalently,

$$\left[\frac{\partial F}{\partial t_a} \omega_J - \partial_\lambda^{-1} \cdot \text{Lie}_{\partial/\partial t_a} \omega_J - \sum_I \omega_I C_{aIJ}(t) \right] = 0.$$

According to Proposition 5.54 the form in the square brackets above is 0 in H_F'' . Therefore, the corresponding geometric section is also 0, that is,

$$\left[\left(\frac{\partial F}{\partial t_a} \omega_J - \partial_\lambda^{-1} \cdot \text{Lie}_{\partial/\partial t_a} \omega_J \right) / dF \right] = \sum_I [\omega_I / dF] C_{aIJ}(t).$$

Recalling the formulas for the derivatives of the geometric sections we get that the LHS of the above identity is

$$-\nabla_{\partial/\partial t_a} [\partial_\lambda^{-1} \cdot \omega_I / dF].$$

Differentiating by $\nabla_{\partial/\partial \lambda}$ we get formula (5.43). The second formula (5.44) is proved in a similar way. \square

Using Proposition 5.55 we would like to construct a solution to the 2nd structure connection in terms of the geometric sections. Let us recall the Frobenius type structure on S_{lin} consisting of the holomorphic vector bundle $\mathbb{K} := p_* \Omega_F$, the flat connection ∇^r on \mathbb{K} defined by the requirement that $[\omega_I]$ ($I \in \mathcal{B}$) are flat sections, the Higgs field

$$C : \mathcal{T}_{S_{\text{lin}}} \rightarrow \text{End}(\mathbb{K}), \quad \partial/\partial t_a \mapsto C_a,$$

where C_a is the operator of multiplication by $\frac{\partial F}{\partial t_a}$, the operator $\mathcal{U} := C_E =$ the endomorphism of \mathbb{K} of multiplication by F . Using the flat connection ∇^r we trivialize $p_* \Omega_F \cong S_{\text{lin}} \times \Omega_f$. By definition, in this trivialization, the sections $[\omega_I]$ correspond to the constant sections $[\omega_I^\circ] = [x^I dx]$. Let us denote by $[\omega] \in \Gamma(S_{\text{lin}}, p_* \Omega_F)$ the section which in the flat trivialization corresponds to the constant section $\varphi = \nu dx$, that is, $[\omega] := \nu [\omega_{\mathbf{0}}]$, where $\mathbf{0}$ is the multi-index in \mathcal{B} whose entries are all 0. The remaining two ingredients of the Frobenius type structure are the grading operator $\theta \in \text{End}(\mathbb{K})$ and the non-degenerate bilinear pairing g on \mathbb{K} . Both θ and g are ∇^r -flat and in the flat trivialization are given respectively by $\theta = \frac{n+1}{2} - \text{deg}$ and g is just the residue pairing on Ω_f . According to Proposition 5.12 the data $(S_{\text{lin}}, \mathbb{K}, \nabla^r, C, \theta, \mathcal{U}, g)$ that we just defined is a Frobenius type structure.

According to the construction theorem of Hertling and Manin (see also Section 5.2.3), there exists a unique Frobenius manifold B containing S_{lin} as a closed complex submanifold and an isomorphism

$$\Phi : TB|_{S_{\text{lin}}} \rightarrow \mathbb{K},$$

such that,

- (i) $\Phi(e) = [\omega]$, where e is the unit vector field.
- (ii) If $v' \in T_t S_{\text{lin}} \subset T_t B$ where $t \in S_{\text{lin}}$, then $\Phi(v' \bullet_t v'') = C_{v'}(\Phi(v''))$ for all $v'' \in T_t B$ where \bullet_t is the Frobenius multiplication in $T_t B$.
- (iii) Under the isomorphism $\Phi_0 : T_0 B \rightarrow \Omega_f$ the grading operator θ and the pairing g coincide with respectively the grading operator and the Frobenius pairing of B .

Suppose that $I^{(m)}(t, \lambda)$ is a solution to the 2nd structure connection of the Frobenius manifold B . Using the flat trivialization of $p_* \Omega_F$ and the isomorphism Φ we obtain a trivialization of $TB|_{S_{\text{lin}}} \cong S_{\text{lin}} \times \Omega_f$. Therefore, we can and we will interpret the restriction

of $I^{(m)}(t, \lambda)$ to $(\lambda, t) \in \mathbb{C} \times S_{\text{lin}} \setminus D_F$ as a multivalued analytic function with values in Ω_f . Note that the restriction satisfies the following system of differential equations

$$\begin{aligned} (\lambda - \mathcal{U})\partial_{t_a} I^{(m)}(t, \lambda) &= -C_a \left(\theta - m - \frac{1}{2} \right) I^{(m)}(t, \lambda) \quad (0 \leq a \leq k-1), \\ (\lambda - \mathcal{U})\partial_\lambda I^{(m)}(t, \lambda) &= \left(\theta - m - \frac{1}{2} \right) I^{(m)}(t, \lambda) \end{aligned}$$

where \mathcal{U} and C_a are the linear operators in Ω_f which under the flat trivialization $p_*\Omega_F \cong S_{\text{lin}} \times \Omega_f$ correspond to \mathcal{U} and $C_{\partial/\partial t_a}$. Note that since $\mathcal{U} = C_E$, the above system can be written equivalently as

$$(5.45) \quad \partial_{t_a} I^{(m)}(t, \lambda) = -C_a \partial_\lambda I^{(m)}(t, \lambda) \quad (0 \leq a \leq k-1),$$

$$(5.46) \quad (\lambda \partial_\lambda + E) I^{(m)}(t, \lambda) = \left(\theta - m - \frac{1}{2} \right) I^{(m)}(t, \lambda).$$

Using geometric sections we can construct solutions to the above system in the following way. If $\alpha \in H_n(f^{-1}(1), \mathbb{C})$ is an arbitrary cycle and $m \in -l + \mathbb{Z}$ with $l := n/2$, then we define the multivalued Ω_f -valued function $I_\alpha^{(m)}(t, \lambda)$ by the following formula:

$$([\omega_I^\circ], I_\alpha^{(-l)}(t, \lambda)) := (2\pi)^{-l} \partial_\lambda^{m+l} \int_{\alpha_{\lambda,t}} \frac{\omega_I}{dF}, \quad \forall I \in \mathcal{B},$$

where $(\ , \)$ is the residue pairing on Ω_f , if $m+l < 0$ then we think of ω_I as an element of H_F'' and define the RHS by

$$(2\pi)^{-l} \int_{\alpha_{\lambda,t}} \frac{\partial_\lambda^{m+l} \cdot \omega_I}{dF},$$

and $\alpha_{\lambda,t} \in H_n(Z_{\lambda,t}, \mathbb{C})$ is the parallel transport of α along a reference path. We will refer to the functions $I_\alpha^{(m)}(t, \lambda)$ ($m \in -l + \mathbb{Z}$) as the *period vectors* of α . The value of the period vector depends on the choice of a reference path in $\mathbb{C} \times S_{\text{lin}} \setminus D_F$ connecting the points $(1, 0)$ and (λ, t) . Using Proposition 5.55 it is straightforward to prove that the period vectors $I_\alpha^{(m)}(t, \lambda)$ satisfy the differential equations (5.45)–(5.46). Moreover, according to Proposition 5.52

$$I_\alpha^{(m)}(0, \lambda) = \frac{\lambda^{\theta-m-1/2}}{\Gamma(\theta-m+1/2)} \Pi(\alpha).$$

The RHS coincides with our definition of a calibrated period – see (3.9). In particular, if we choose the calibration $S(t, w)$ of the Frobenius manifold B to be such that $S(0, w) = \text{Id}$, then the period vector $I_\alpha^{(m)}(t, \lambda)$ coincides with $I_{\Pi(\alpha)}^{(m)}(t, \lambda)$ where the latter is defined by formula (3.11).

Note that the Frobenius structure on B depends only on the co-rank k of the singularity f . By adding to f squares of new variables we can arrange that the number n takes any value $\geq k$ that we wish. By choosing n to be odd or even, the above construction gives a geometric interpretation for the periods (3.11) for respectively $m \in \frac{1}{2} + \mathbb{Z}$ and $m \in \mathbb{Z}$. We will be interested only in the period vectors with $m \in \mathbb{Z}$, that is, we will assume that $n = 2l$ is an even number.

COROLLARY 5.56. *If $n = 2l$ is an even number, then under the isomorphism $\Phi_0^{-1} \circ \Pi : H_n(f^{-1}(1), \mathbb{C}) \cong T_0 B$ we have the following identifications*

- (a) The Euler pairing $\langle \cdot, \cdot \rangle$ of the Frobenius structure (see (3.29)) coincides with $-(-1)^l L$, where L is the Seifert form of the singularity f .
- (b) The intersection pairing $(\cdot | \cdot)$ of the Frobenius structure (see (3.30)) coincides with $(-1)^l \times$ intersection pairing.
- (c) The reflection vectors and the monodromy of the Frobenius manifold coincide with respectively the set of vanishing cycles and the monodromy group of the singularity f . \square

PROOF. Statements (a) and (b) of the above corollary follow immediately from Proposition 5.53. Part (c) follows from the above discussion, that is, the period vector $I_\alpha^{(m)}(t, \lambda)$ coincides with the solution of the 2nd structure connection $I_{\Pi(\alpha)}^{(m)}(t, \lambda)$ defined by formula (3.11). Therefore, the isomorphism Π intertwines the monodromy representations of the 2nd structure connection and the Gauss–Manin connection of the vanishing homology bundle. If $\Pi(\alpha)$ is a reflection vector, then there exists a simple loop C around a generic point on the discriminant D_F , such that, the monodromy transformation along C transforms $\Pi(\alpha)$ into $-\Pi(\alpha) \Rightarrow$ the cycle α is transformed into $-\alpha$. Recalling the Picard–Lefschetz formula (see Proposition 5.22), we get that $\alpha = \frac{(\alpha|\gamma)}{2} \gamma$ must be proportional to the vanishing cycle γ corresponding to the simple loop C . Finally, since $\alpha \circ \alpha = (-1)^l (\alpha|\alpha) = (-1)^l 2$ coincides with the self-intersection number of a vanishing cycle, we get that $\alpha = \pm\gamma$ must be a vanishing cycle. \square

REMARK 5.57. Using the so-called twisted Picard–Lefschetz theory (see [23]) we can give a geometric interpretation of the solutions $I_a^{(m)}(t, \lambda)$ of the 2nd structure connection defined by (3.11) for all $m \in \mathbb{C}$. However, the significance of the period vectors for which $m \notin \mathbb{Z}$ from the point of view of the theory of integrable systems and representation theory of vertex operator algebras is not known to us.

Simple singularities

6.1. Frobenius structures

There are many characterizations of simple singularities. The one that seems to be the most relevant for our purposes is the following.

DEFINITION 6.1. A weighted homogeneous singularity is said to be *simple* if all deformations are relevant, that is, $\mathcal{B} = \mathcal{B}_{\text{rel}}$.

In the rest of this chapter, we will be working with a simple singularity f . Since $\mathcal{B} = \mathcal{B}_{\text{rel}}$, we will drop the index rel from the notation, i.e., according to Proposition 5.1, we have a family of functions $(F, p : Z \rightarrow B)$, where $Z := \mathbb{C}^{n+1} \times B$, $B := \mathbb{C}^{\mathcal{B}}$, and $F(x, t) = f(x) + \sum_{\kappa \in \mathcal{B}} t_{\kappa} x^{\kappa}$.

6.1.1. Kodaira–Spencer isomorphism.

PROPOSITION 6.2. *If f is a simple singularity, then $(F, p : Z \rightarrow B)$ is a complete family (see Definition 4.7).*

PROOF. We have to prove that the Kodaira–Spencer map

$$\mathcal{T}_B \rightarrow p_* \mathcal{O}_{C_F}, \quad \partial_{t_{\kappa}} \mapsto \frac{\partial F}{\partial t_{\kappa}}$$

is an isomorphism of \mathcal{O}_B -modules. We already know that $p_* \mathcal{O}_{C_F}$ is a locally free sheaf of rank $\mu_F = \dim(B)$ – see Section 4.2.2. In other words, the Kodaira–Spencer map is a map between two holomorphic vector bundles on B . Let K be the set of all $t \in B$, such that, the induced map between the fibers of the two vector bundles at t fails to be an isomorphism. Note that K is a closed subset of B . The fiber of \mathcal{T}_B at t is just the tangent space $T_t B \cong \mathbb{C}^{\mathcal{B}}$, while the fiber of $p_* \mathcal{O}_B$ at t is the algebra

$$H_t := \mathbb{C}[x_0, \dots, x_n] / (\partial_{x_0} f_t, \dots, \partial_{x_n} f_t),$$

where $f_t(x) := F(x, t)$. The Kodaira–Spencer map between the fibers at t takes the form

$$\phi_t : \mathbb{C}^{\mathcal{B}} \rightarrow H_t, \quad a = (a_{\kappa}) \mapsto \sum_{\kappa} a_{\kappa} x^{\kappa} \pmod{\partial_{x_0} f_t, \dots, \partial_{x_n} f_t}.$$

By definition, ϕ_0 is an isomorphism so $0 \notin K$. Using that f is weighted homogeneous it is easy to prove that if $t \in K$, then $\lambda \cdot t \in K$ for all positive real numbers λ , where $(\lambda \cdot t)_{\kappa} := \lambda^{1-c \cdot \kappa} t_{\kappa}$. If K is not empty, since $c \cdot \kappa < 1$, by letting $\lambda \rightarrow 0$ we would get that there is a sequence of points in K converging to 0, that is, we would have that $0 \in K$ – contradiction. \square

Using the Kodaira–Spencer isomorphism, let us equip the tangent bundle \mathcal{T}_B with the multiplication induced from the multiplication of functions in $p_* \mathcal{O}_B$. The unit $1 \in p_* \mathcal{O}_B$ induces a vector field $e = \frac{\partial}{\partial t_0}$, where $\mathbf{0} \in \mathcal{B}$ is the multi-index with all entries equal to 0.

Furthermore, the function $F \in p_*\mathcal{O}_B$ induces a vector field $E \in \Gamma(B, \mathcal{T}_B)$, which will be called the *Euler* vector field. Since $f(x)$ is weighted homogeneous, we have

$$(6.1) \quad \left(\sum_{i=0}^n c_i x_i \frac{\partial}{\partial x_i} + \sum_{\kappa \in \mathcal{B}} (1 - c \cdot \kappa) t_\kappa \frac{\partial}{\partial t_\kappa} \right) F(x, t) = F(x, t).$$

Recalling the Kodaira–Spencer isomorphism, the above identity yields that

$$E = \sum_{\kappa \in \mathcal{B}} (1 - c \cdot \kappa) t_\kappa \frac{\partial}{\partial t_\kappa}.$$

6.1.2. Primitive forms. Our next goal is to prove that up to a constant factor the only primitive form is the standard holomorphic volume form $\omega := dx_0 \wedge \cdots \wedge dx_n$. This follows almost immediately for homogeneity reasons. To begin with, let us look closer at the homogeneity properties of the sheaf $\mathcal{H}_F^{(0)}$ and the higher-residue pairing K_F .

LEMMA 6.3. *Suppose that $\phi_k(x, t)$, $k \geq 0$, are holomorphic functions on Z , such that,*

$$(6.2) \quad \left(\sum_{i=0}^n c_i x_i \frac{\partial}{\partial x_i} + \sum_{\kappa \in \mathcal{B}} (1 - c \cdot \kappa) t_\kappa \frac{\partial}{\partial t_\kappa} \right) \phi_k(x, t) = (r - k - c_0 - \cdots - c_n) \phi_k(x, t).$$

Then the series $\varphi := \sum_{k=0}^{\infty} \phi_k(x, t) w^k \omega$ represents a homogeneous element of $\mathcal{H}_F^{(0)}$ of homogeneous degree r (see Definition 4.32).

PROOF. By definition the cohomology class $[\varphi]$ is said to be homogeneous of degree r if $(w\nabla_w + \nabla_E)[\varphi] = r[\varphi]$, where ∇ is the Gauss–Manin connection, $\nabla_w := \nabla_{\partial/\partial w}$, and $[\varphi]$ denotes the cohomology class represented by the form φ . We have

$$(w\nabla_w + \nabla_E)[\varphi] = \sum_{k=0}^{\infty} w^{-1+k} \left[\left((-F + E(F) + k) \phi_k(x, t) + E(\phi_k(x, t)) \right) \omega \right].$$

Using (6.1) we can write $E(F) - F = -\sum c_i x_i F_{x_i}$. Note that

$$[-x_i F_{x_i} \phi_k(x, t) \omega] = w[(\phi_k(x, t) + x_i \partial_{x_i} \phi_k(x, t)) \omega].$$

The formula for $(w\nabla_w + \nabla_E)[\varphi]$ takes the form

$$\sum_{k=0}^{\infty} \left[\left(c_0 + \cdots + c_n + k + E + \sum_{i=0}^n c_i x_i \partial_{x_i} \right) \phi_k(x, t) w^k \omega \right].$$

Recalling (6.2), we get that the above class coincides with $r[\varphi]$, which is what we had to prove. \square

It is convenient to assign weight $1 - c \cdot \kappa$ to t_κ for all $\kappa \in \mathcal{B}$ and to introduce the notion of weighted homogeneous functions on Z . Then condition (6.2) has a simple interpretation, that is, the function $\phi_k(x, t)$ is weighted homogeneous of weight (or degree) $r - k - c_0 - \cdots - c_n$. Using Proposition 4.24 we can prove easily that the inverse of Lemma 6.3 holds, i.e., every homogeneous cohomology class in $\mathcal{H}_F^{(0)}$ can be represented by a holomorphic form $\varphi = \sum_{k=0}^{\infty} \phi_k(x, t) w^k \omega$, such that, ϕ_k are holomorphic functions on Z satisfying (6.2). Indeed, according to Proposition 4.24, the classes $[x^\kappa \omega]$, $\kappa \in \mathcal{B}$ form a $\mathcal{O}_B[w]$ -basis of $\mathcal{H}_F^{(0)}$. Suppose that $[\varphi] = \sum_{\kappa} a_\kappa(t, w) [x^\kappa \omega] \in \mathcal{H}_F^{(0)}$ is homogeneous of

degree r . According to Lemma 6.3 $[x^\kappa \omega]$ is homogeneous of weight $c \cdot \kappa + c_0 + \cdots + c_n$. We get that each coefficient $a_\kappa(t, w)$ satisfies

$$(w\partial_w + E)a_\kappa(t, w) = (r - c \cdot \kappa - c_0 - \cdots - c_n)a_\kappa(t, w).$$

Expanding in the powers of w , that is, $a_\kappa(t, w) =: \sum_{l=0}^{\infty} a_{\kappa,l}(t)w^l$, we get that each coefficient $a_{\kappa,l}(t)$ is weighted homogeneous of weight $(r - l - c \cdot \kappa - c_0 - \cdots - c_n)$. Put $\phi_l(x, t) := \sum_{\kappa} a_{\kappa,l}(t)x^\kappa$, then ϕ_l satisfies (6.2) and $\sum_{l=0}^{\infty} \phi_l(x, t)w^l \omega$ represents φ .

Suppose now that $\varphi = \sum_{k=0}^{\infty} \phi_k(x, t)w^k \omega$ is a primitive form. Since φ is a holomorphic volume form, we get that $\phi_0(x, t)$ must be an invertible element in $\mathcal{O}_{C_F,0}$, that is, $\phi_0(0,0) \neq 0$. On the other hand, since a primitive form is homogeneous of some degree r , we may assume that $\phi_k(x, t)$ are weighted homogeneous functions on Z of weight $r - k - c_0 - \cdots - c_n$. Note that the weights of all variable x_i and t_κ are positive! Therefore, the weighted homogeneous function $\phi_0(x, t)$ must have weight 0, that is, $\phi_0(x, t)$ is a constant independent of x and t . We also get that $r = c_0 + \cdots + c_n$. The latter implies that the functions $\phi_k(x, t)$ have weight $-k$, which is negative for $k > 0$. Since we can not have non-zero weighted homogeneous functions on Z of negative weight, we must have $\phi_k(x, t) = 0$ for all $k > 0$. We get that if a primitive form exists, then it must be a constant multiple of ω .

PROPOSITION 6.4. *The standard volume form ω is primitive.*

PROOF. We have to check that conditions (i)–(v) in Definition 4.32 are satisfied. We already know that (iv) is satisfied, while (v) is trivially satisfied. For $\alpha \in \mathcal{B}$, let us denote $\nabla_\alpha = \nabla_{\partial/\partial t_\alpha}$ the covariant derivative with respect to the Gauss–Manin connection. Note that the cohomology class $w\nabla_\alpha \omega$ is homogeneous of degree $r + c \cdot \alpha$, because t_α has weight $1 - c \cdot \alpha$. Using the Leibnitz rule we get

$$(w\partial_w + E)K(w\nabla_\alpha \omega, w\nabla_\beta \omega) = (2r + c \cdot (\alpha + \beta))K(w\nabla_\alpha \omega, w\nabla_\beta \omega).$$

Therefore, the pairing $K^{(p)}(w\nabla_\alpha \omega, w\nabla_\beta \omega)$ is a weighted homogeneous function on B of weight

$$2r + c \cdot (\alpha + \beta) - p - n - 1.$$

Since $n + 1 - 2r = \sum_{i=0}^n (1 - 2c_i)$ coincides with the weight of the Hessian of f which according to Proposition 5.3 is $\geq c \cdot \alpha$ and $c \cdot \beta < 1$, we get that the above weight is negative for $p \geq 1$. Since the weights of t_α are positive, we get that the above higher residue pairings must vanish, that is, condition (i) in Definition 4.32 holds. The remaining two conditions (ii) and (iii) are verified in a similar way, so we leave the details as an exercise. \square

Recalling Theorem 4.33 we get that B has a Frobenius structure of conformal dimension $D = \sum_{i=0}^n (1 - 2c_i)$. Since the higher residue pairings $K^{(p)}(x^I dx, x^J dx) = 0$ for $p > 0$ and $I, J \in \mathcal{B}$ we get that

$$(6.3) \quad P = \text{Span}_{\mathbb{C}} \{w^{-l-1}[x^I dx] \mid I \in \mathcal{B}, \quad l \geq 0\}$$

is an opposite subspace. In fact, the group $\text{Aut}(\Omega_f^{\leq \bullet}, \langle \cdot, \cdot \rangle) = \{1\}$. Indeed, suppose that $g \in \text{Aut}(\Omega_f^{\leq \bullet}, \langle \cdot, \cdot \rangle)$ and let g_{IJ} be the entries of the matrix of g with respect to the basis $[x^I dx]$ ($I \in \mathcal{B}$) of Ω_f . Note that the (I, J) -entry of $R_g(w) = w^\theta g w^{-\theta}$ is $g_{IJ} w^{c \cdot J - c \cdot I}$. By definition, $R_g(w)$ is a power series in $w \Rightarrow c \cdot J - c \cdot I$ must be an integer. However,

for simple singularities $0 < c \cdot I, c \cdot J < 1$, so the only possibility for g_{IJ} to be non-zero is that $I = J$, that is, $g = 1$. Therefore, according to Proposition 5.51, there exists a unique homogeneous opposite subspace, that is, (6.3) is the unique homogeneous opposite subspace. Note that the Frobenius structure on B coincides with the Frobenius structure associated to the pair $(P, \llbracket dx \rrbracket)$.

6.1.3. Vanishing cycles and root systems. Suppose that $n = 2l$ is an even number. We would like to prove that the set of vanishing cycles of f is a root system of type ADE . Recall that a root system \mathcal{R} of rank N is a subset of the standard real vector space \mathbb{R}^N equipped with a *positive definite* symmetric bilinear form (\mid) satisfying the following conditions:

- (i) The set \mathcal{R} spans \mathbb{R}^N .
- (ii) If $\alpha \in \mathcal{R}$ and $k\alpha \in \mathcal{R}$, then $k = \pm 1$.
- (iii) If $\alpha, \beta \in \mathcal{R}$, then $r_\alpha(\beta) := \beta - 2\frac{(\alpha|\beta)}{(\alpha|\alpha)} \in \mathcal{R}$.
- (iv) If $\alpha, \beta \in \mathcal{R}$, then $2\frac{(\alpha|\beta)}{(\alpha|\alpha)} \in \mathbb{Z}$.

Let us consider the real vector space $H^n(f^{-1}(1), \mathbb{R}) \cong \mathbb{R}^N$ where $N = \mu$ is the Milnor number. Put $(\alpha|\beta) := (-1)^l \alpha \circ \beta$, and let \mathcal{R} be the set of vanishing cycles. All axioms (i)–(iv) are clearly satisfied (see Proposition 5.22). The only non-trivial condition to check is that (\mid) is positive definite. This follows from the Hodge–Riemann bilinear relations for the Steenbrink’s Hodge structure. We claim that

$$l < \text{wt}(x^I dx) < l + 1 \quad \forall I \in \mathcal{B}.$$

Indeed, let us write $D = 1 - \frac{2}{h}$ where h is a rational number. Since D is the weight of the Hessian we must have that $h \geq 2$. As we will see later on h is in fact an integer and it coincides with the Coxeter number. We have $\sum_{i=0}^n c_i = l + \frac{1}{h}$ and $\text{wt}(x^I) \leq D \Rightarrow$

$$l < \sum_{i=0}^n c_i < \text{wt}(x^I) + \sum_{i=0}^n c_i \leq l + \frac{1}{h} + D = l + 1 - \frac{1}{h}.$$

Our claim follows. We get that $\mathfrak{h}_1 = 0$ and that the Hodge filtration has the form $0 = F^{l-1}\mathfrak{h} \subset F^l\mathfrak{h} = \mathfrak{h}$. The Hodge–Riemann bilinear relations take the form

$$\mathbf{i}^{n^2+2l} S_{\overline{\mathfrak{V}}}(A, \overline{A}) > 0,$$

where \overline{A} is the complex conjugate of A with respect to the real structure $H^n(f^{-1}(1), \mathbb{R})$. The factor $\mathbf{i}^{n^2+2l} = (-1)^l$, while

$$S_{\overline{\mathfrak{V}}}(A, B) = L(A, (M - 1)^{-1}B) = A \circ B,$$

where the second equality is a well known formula for the intersection pairing in terms of the Seifert form (see [6]). Therefore, if A is real then the Hodge–Riemann bilinear relations yield $(A|A) > 0$, that is, the cohomological intersection pairing is positive definite. Since the intersection pairings in cohomology and homology are intertwined via the duality isomorphism $\text{var} \circ \text{PD} : H^n(f^{-1}(1), \mathbb{R}) \rightarrow H^n(f^{-1}(1), \mathbb{R})$, we get that the homological intersection pairing is also positive definite. This proves that the set of vanishing cycles is a root system.

We can say a little bit more. Namely, since $(\alpha|\alpha) = 2$ for all vanishing cycles, we get that all roots have the same length. Recalling the classification of root systems, we get the following two facts

- (1) The set of vanishing cycles \mathcal{R} consists of all cycle $\alpha \in H_n(f^{-1}(1), \mathbb{Z})$, such that, $(\alpha|\alpha) = 2$.
- (2) The set of vanishing cycles \mathcal{R} must be a root system of type A , D , or E .

We will say that the simple singularity is of type A , D , or E depending on whether the set of vanishing cycles is a root system of type respectively A , D , or E . Up to an isomorphism the simple singularities are classified by the following polynomials:

$$f(x) = g(x_0, x_1) + x_2^2 + \dots + x_{n+1}^2,$$

where the polynomial $g(x_0, x_1)$ is in Table 1.

Type	$g(x)$	Exponents	h
A_N	$x_0^{N+1} + x_1^2$	$1, 2, \dots, N$	$N + 1$
D_N	$x_0^{N-1} + x_0 x_1^2$	$1, 3, \dots, 2N - 3, N - 1$	$2N - 2$
E_6	$x_0^4 + x_1^3$	$1, 4, 5, 7, 8, 11$	12
E_7	$x_0^3 x_1 + x_1^3$	$1, 5, 7, 9, 11, 13, 17$	18
E_8	$x_0^5 + x_1^3$	$1, 7, 11, 13, 17, 19, 23, 29$	30

TABLE 1. Simple singularities.

There is a general method for computing a basis of vanishing cycles and the intersection matrix for functions in two variables due to A'Campo [2] and Gusein-Zade [29] (see also [6]). By applying this method to the polynomials in Table 1 we can prove that the function g has a Morsification \tilde{g} whose critical values $\tilde{u}_1, \dots, \tilde{u}_N$ are pairwise distinct, such that, if we fix a disk $U \subset \mathbb{C}$ containing all critical values and we choose a point λ_0 on the boundary of U , then there exists a set of smooth paths C_1, \dots, C_N satisfying the following conditions:

- (i) C_i has no self-intersections and it connects λ_0 with the critical value \tilde{u}_i .
- (ii) If $i \neq j$, then $C_i \cap C_j = \{\lambda_0\}$.
- (iii) Let α_i be the vanishing cycle of $\tilde{g}(x_0, x_1) + x_2^2$ corresponding to the path C_i . The set $\{\alpha_1, \dots, \alpha_N\}$ is a set of simple roots.

By changing the enumeration of the critical values \tilde{u}_i if necessary, we can arrange that the paths C_i are enumerated according to the order in which they exit the point λ_0 , that is, if we draw a small clock-wise oriented circle with center at λ_0 and we look at the arc of that circle inside U , then the arc will intersect first C_1 , then C_2 , etc. Let γ_i be the simple loop corresponding to C_i that goes anti-clockwise around \tilde{u}_i . Note that the composition $\gamma_1 \circ \dots \circ \gamma_N$ is homotopic to the circle ∂U . Since the classical monodromy M is the monodromy transformation along ∂U , we get that $M = r_{\alpha_1} \dots r_{\alpha_N}$, that is, M is a *Coxeter transformation*.

REMARK 6.5. The vanishing cycles $\alpha_1, \dots, \alpha_N$ corresponding to a set of paths C_1, \dots, C_N satisfying conditions (i), (ii), and such that the enumeration of the paths agrees with the enumeration of the critical values $\tilde{u}_1, \dots, \tilde{u}_N$ (in the above sense) is called a *distinguished basis*. Our claim from above is that a simple singularity admits a distinguished basis consisting of simple roots. \square

REMARK 6.6. The method of Gusein-Zade [29, 6] gives a distinguished basis which however might fail to consist of simple roots. In particular, the Dynkin diagram might

have a non-standard form. In order to transform the distinguished basis into one consisting of simple roots one has to apply the so-called operations α_m and β_{m+1} ($1 \leq m < N$) – see [6], Sections 2.6 and 4.1. \square

6.2. HQEs for the total descendent potential

We would like to state the main result of this chapter (see also [24]). Namely, the total descendent potential of a simple singularity f satisfies a system of Hirota quadratic equations (HQEs). These equations were identified in [16] with the Hirota bilinear equations of the principal Kac–Wakimoto hierarchy corresponding to the root system of vanishing cycles.

6.2.1. A_1 -singularity. Let us first consider the special case of an A_1 -singularity. The equations in this case are equivalent to the HQEs of the KdV hierarchy (see Section 3.1). Therefore, the proof in this case follows trivially from the Kontsevich’s theorem for \mathcal{D}_{pt} .

Suppose that $f(x) = x^2$ and that the primitive form is $\omega = \sqrt{2}\Delta^{-1/2}dx$ where Δ is a non-zero complex number. Then the Frobenius manifold is $B = \mathbb{C}$. Let t be the standard coordinate on B and $F(x, t) = x^2 + t$ be the miniversal unfolding. The Frobenius structure is given by

$$\partial_t \bullet \partial_t = \partial_t, \quad (\partial_t, \partial_t) = \frac{1}{\Delta}.$$

Note that the canonical coordinate $u = t$. The total descendent potential takes the form

$$\mathcal{D}_{A_1}(\hbar, \mathbf{q}) = \mathcal{D}_{\text{pt}}(\hbar \Delta, \mathbf{q}(z) + z) \in \mathbb{C}_{\hbar}[[q_0, q_1 + 1, q_2, \dots]],$$

where \mathcal{D}_{pt} is the Witten–Kontsevich tau-function (1.65) and $\mathbf{q}(z) = q_0 + q_1 z + q_2 z^2 + \dots$. Put $H = T_0 B$ and let us recall that we have an identification

$$H \cong \Omega_f, \quad \frac{\partial}{\partial t} \mapsto \frac{\partial F}{\partial t} \omega = \sqrt{2}\Delta^{-1/2}[[dx]].$$

The period vector (with $l = 0$) takes the form

$$([[dx]], I_{\alpha}^{(0)}(\lambda)) = \int_{\alpha_{\lambda}} \frac{dx}{df} = \lambda^{-1/2},$$

where $\alpha_{\lambda} = [x_+(\lambda)] - [x_-(\lambda)] \in \tilde{H}_0(f^{-1}(\lambda), \mathbb{C})$, $x_{\pm}(\lambda) := \pm\sqrt{\lambda}$. Since $([[dx]], [[dx]]) = 1/2$ we get $I_{\alpha}^{(0)}(\lambda) = 2\lambda^{-1/2}[[dx]]$. The remaining periods are obtained by differentiating and anti-differentiating, that is,

$$I_{\alpha}^{(k)}(\lambda) = \frac{(-1)^k}{2^{k-1}} (2k-1)!! \lambda^{-k-1/2} [[dx]], \quad k > 0,$$

and

$$I_{\alpha}^{(-k-1)}(\lambda) = 2^{k+1} \frac{\lambda^{k+1/2}}{(2k+1)!!} [[dx]], \quad k \geq 0.$$

Note that the period map

$$\Pi : \tilde{H}_0(f^{-1}(\lambda), \mathbb{C}) \rightarrow \Omega_f \cong H$$

takes the form

$$\Pi(\alpha) := 2\Gamma(1/2)[[dx]] = \sqrt{2\pi\Delta} \frac{\partial}{\partial t}.$$

Let $\mathcal{H} = H((z^{-1}))$ be the Givental's symplectic vector space. Let us recall the vertex operators associated to the period vectors (see Section 3.2.3). Put

$$\mathbf{f}^\alpha(\lambda, z) = \sum_{n \in \mathbb{Z}} I_\alpha^{(n)}(\lambda)(-z)^n$$

and $\Gamma^\alpha(\lambda) = e^{\mathbf{f}^\alpha(\lambda, z)} e^{\mathbf{f}_+^\alpha(\lambda, z)}$, where the quantization rules are determined by

$$\left(\frac{\partial}{\partial t} z^k \right)^\wedge = -\sqrt{\hbar} \frac{\partial}{\partial q_k}, \quad (dt(-z)^{-k-1})^\wedge = q_k / \sqrt{\hbar},$$

where $dt = \Delta \partial / \partial t$, that is, we identify $T_0^* B \cong T_0 B$ via the residue pairing. After a short computation we get

$$\Gamma^\alpha(\lambda) = \exp \left(\sum_{k=0}^{\infty} \frac{q_k}{\sqrt{\hbar} \Delta (2k+1)!!} (2\lambda)^{k+1/2} \right) \exp \left(-2 \sum_{k=0}^{\infty} (2k-1)!! \sqrt{\hbar} \Delta \partial_{q_k} (2\lambda)^{-k-1/2} \right).$$

The equations take the form

$$(6.4) \quad \text{Res} \frac{d\lambda}{\lambda} \left(\sum_{\pm} \Gamma^{\pm\alpha}(\lambda) \otimes \Gamma^{\mp\alpha}(\lambda) \right) \Phi \otimes \Phi = 16 \left(\ell + \frac{1}{8} \right) \Phi \otimes \Phi,$$

where

$$(6.5) \quad \ell := \sum_{k=0}^{\infty} \left(k + \frac{1}{2} \right) (q_k \otimes 1 - 1 \otimes q_k) (\partial_{q_k} \otimes 1 - 1 \otimes \partial_{q_k}),$$

Res denotes the coefficient in front of λ^0 . Under the substitutions

$$\lambda = \frac{\zeta^2}{2}, \quad q_k = \sqrt{\hbar} \Delta (2k+1)!! T_{2k+1} \quad (k \geq 0)$$

the equations (6.4) coincide with the HQEs of the principal Kac–Wakimoto hierarchy of type A_1 (see [40]) which is known to be the same as the KdV hierarchy. In other words, under the substitution $q_k = \sqrt{\hbar} \Delta (2k+1)!! T_{2k+1}$ ($k \geq 0$), the solutions Φ of (6.4) coincide with the tau-dunctions of the KdV hierarchy. Recalling Kontsevich's theorem for \mathcal{D}_{pt} we get that \mathcal{D}_{A_1} is a solution to the equations (6.4).

Let us recall that the tau-functions of KdV hierarchy can be characterized by HQEs that are reduction of the HQEs of the KP hierarchy (see Section 3.1). Therefore, the total descendent potential \mathcal{D}_{A_1} is a solution to yet another system of equations, that is,

$$(6.6) \quad \text{Res} \lambda^r \frac{d\lambda}{\sqrt{\lambda}} \left(\Gamma^{\alpha/2}(\lambda) \otimes \Gamma^{-\alpha/2}(\lambda) - \Gamma^{-\alpha/2}(\lambda) \otimes \Gamma^{\alpha/2}(\lambda) \right) \Phi \otimes \Phi = 0$$

for all integers $r \geq 0$. We will use that \mathcal{D}_{A_1} is a solution to both systems (6.4) and (6.6).

6.2.2. HQEs for a simple singularity. Suppose now that $f(x) = g(x_0, x_1) + x_2^2 + \dots + x_n^2$ is a simple singularity with $n = 2l$ even. Let $\phi_i(x)$ ($1 \leq i \leq N$) be a set of weighted homogeneous polynomials representing a basis of the local algebra H_f of f . Equivalently, the classes $[\phi_i(x) dx]$ ($1 \leq i \leq N$) form a basis of Ω_f where $dx := dx_0 \wedge \dots \wedge dx_n$. Reordering ϕ_i if necessary we may assume that the weight of ϕ_i is increasing as the index i increases. In particular, the weight of $\phi_N(x)$ coincides with the conformal dimension D . Let us write $D =: 1 - \frac{2}{h}$ and $\text{wt}(\phi_i(x)) =: \frac{m_i - 1}{h}$ where $h \geq 2$ and $m_i \geq 1$ are rational

numbers. By using a linear change of the basis ϕ_i we may also arrange that the residue pairing in Ω_f takes the form

$$(\phi_i(x)dx, \phi_j(x)dx) = \delta_{i+j, N+1}.$$

Since the residue pairing of two forms is non-zero only if their weights add up to $n + 1$, we get that the numbers m_i satisfy $m_i + m_{N+1-i} = h$.

Let us recall that the set of vanishing cycles \mathcal{R} is a root system of type A , D , or E . We claim that h is the *Coxeter number* of \mathcal{R} and that $m_1 \leq m_2 \leq \dots \leq m_N$ are the *Coxeter exponents*. In particular, h and m_i ($1 \leq i \leq N$) are integer numbers. Indeed, note that the geometric sections $[\phi_i(x)/df]$ ($1 \leq i \leq N$) provide an eigenbasis for the classical monodromy operator M with eigenvalues respectively $e^{-2\pi i m_i/h}$ ($1 \leq i \leq N$), where we used that $\text{wt}(\phi_i(x)dx) = l + m_i/h$. On the other hand, the existence of a distinguished basis of simple roots implies that M is a Coxeter transformation. By definition, the Coxeter number is the order $|M|$ of M and the Coxeter exponents $1 \leq m'_1 \leq \dots \leq m'_N \leq |M|$ are defined by the requirement that the sequence $e^{2\pi i m'_i/|M|}$ ($1 \leq i \leq N$) coincide with the eigenvalues of M counted with multiplicities. Therefore, $m'_i/|M| = m_i/h$ for all i . However, since $m'_1 = 1 = m_1$, we get $h = |M|$ and hence $m_i = m'_i$.

Let $B = \mathbb{C}^N$ be the space of miniversal deformations

$$F(x, t) = f(x) + \sum_{i=1}^N t_i \phi_i(x), \quad t = (t_1, \dots, t_N) \in \mathbb{C}^N.$$

Let us choose the primitive form $\omega = dx$ and equip B with a Frobenius structure (see Section 6.1). The vector space $H = T_0 B$ has several interpretations/identifications which we will use whenever relevant. First of all using the flat structure on B , we identify H with the space of flat vector fields on B . Furthermore, using the Kodaira–Spencer isomorphism we identify H with the local algebra H_f , that is, $\partial/\partial t_i \mapsto \left. \frac{\partial F}{\partial t_i} \right|_{t=0}$. Finally, using the primitive form we identify $H_f \cong \Omega_f$, $\phi \mapsto \phi\omega$. The *period vectors* taking values in $\Omega_f \cong H$ are defined by

$$(\phi_i(x)dx, I_\alpha^{(-l)}(\lambda)) := (2\pi)^{-l} \int_{\alpha_\lambda} \phi_i(x) \frac{dx}{df}$$

where $\alpha \in H_n(f^{-1}(1), \mathbb{C})$ and $\alpha_\lambda \subset f^{-1}(\lambda)$ is the parallel transport of α along a reference path. The remaining periods $I_\alpha^{(k)}(\lambda) = \partial_\lambda^{k+l} I_\alpha^{(-l)}(\lambda)$ for $k \in \mathbb{Z}$ are uniquely defined so that

$$(6.7) \quad I_\alpha^{(k)}(\lambda) = \frac{\lambda^{\theta-k-1/2}}{\Gamma(\theta-k+1/2)} \Pi(\alpha),$$

where $\theta = \frac{n+1}{2} - \text{deg}$ is the grading operator and

$$\Pi : H_n(f^{-1}(1), \mathbb{C}) \rightarrow \Omega_f$$

is a linear isomorphism. In other words, using the map Π we obtain an identification between the period vectors and the calibrated periods (3.9). Note that the map Π can be described by the following formula:

$$(6.8) \quad \left(\frac{[\phi_i(x)dx]}{\Gamma(\text{wt}(\phi_i(x)dx))}, \Pi(\alpha) \right) = (2\pi)^{-l} \int_\alpha \phi_i(x) \frac{dx}{df}, \quad \kappa \in \mathcal{B}.$$

Therefore, Π coincides with the period map (5.42).

Let $\mathcal{H} = H((z^{-1}))$ be Givental's symplectic loop space. Let $\Gamma^\alpha(\lambda)$ be the vertex operators associated to the period vectors, i.e., put

$$\mathbf{f}^\alpha(\lambda, z) = \sum_{n \in \mathbb{Z}} I_\alpha^{(n)}(\lambda)(-z)^n$$

and $\Gamma^\alpha(\lambda) = e^{\mathbf{f}_-^\alpha(\lambda, z)} \widehat{e^{\mathbf{f}_+^\alpha(\lambda, z)}}$, where the quantization rules are determined by

$$\left(\frac{\partial}{\partial t_i} z^k \right)^\wedge = -\sqrt{\hbar} \frac{\partial}{\partial q_{k,i}}, \quad (dt_i(-z)^{-k-1})^\wedge = q_{k,i}/\sqrt{\hbar}, \quad 1 \leq i \leq N, \quad k \geq 0,$$

where we identify $T_0^*B \cong T_0B = H$ via the residue pairing which allows us to view $\{dt_i\}$ as a basis of H dual to $\partial/\partial t_i$ with respect to the residue pairing.

The HQEs of our interest will have the following form

$$(6.9) \quad \text{Res} \frac{d\lambda}{\lambda} \left(\sum_{\alpha \in \mathcal{R}} a_\alpha \Gamma^\alpha(\lambda) \otimes \Gamma^{-\alpha}(\lambda) \right) \Phi \otimes \Phi = \frac{N(h+1)}{12h} (\Phi \otimes \Phi) +$$

$$(6.10) \quad \sum_{k=0}^{\infty} \sum_{i=1}^N \left(k + \frac{m_i}{h} \right) (q_{k,i} \otimes 1 - 1 \otimes q_{k,i}) (\partial_{q_{k,i}} \otimes 1 - 1 \otimes \partial_{q_{k,i}}) (\Phi \otimes \Phi),$$

where the coefficients a_α are defined in terms of the phase factors as follows. Let us consider the product of vertex operators

$$\Gamma^\alpha(\lambda_1) \Gamma^\alpha(\lambda_2) = B_{\alpha,\alpha}(\lambda_1, \lambda_2) : \Gamma^\alpha(\lambda_1) \Gamma^\alpha(\lambda_2) :$$

where $B_{\alpha,\alpha}(\lambda_1, \lambda_2) = e^{\Omega(\mathbf{f}_+^\alpha(\lambda_1, z), \mathbf{f}_+^\alpha(\lambda_2, z))}$ is the exponential of the propagator (see Section 3.3.1). Then

$$(6.11) \quad a_\alpha = \lim_{\lambda' \rightarrow \lambda} (1 - \lambda/\lambda')^{-2} B_{\alpha,\alpha}(\lambda', \lambda).$$

The coefficient a_α can be computed explicitly as follows. To begin with, note that by analytically continuing the identity in Proposition 5.52 we get that the period map Π intertwines the classical monodromy operator M and the operator σ defined by (3.27), that is, $\Pi \circ M = e^{2\pi i(\theta + \frac{1}{2})} \circ \Pi$. In the case of a simple singularities the energy propagator coincides with the calibrated propagator. Therefore, using Theorem 3.9 and Lemma 3.10, we get

$$B_{\alpha,\alpha}(\lambda', \lambda) = \prod_{r=1}^h \left(1 - \eta^r (\lambda/\lambda')^{1/h} \right)^{(M^r \alpha | \alpha)}$$

where $\eta = e^{2\pi i/h}$. Substituting in formula (6.11) we get

$$a_\alpha = \frac{1}{h^2} \prod_{r=1}^{h-1} (1 - \eta^r)^{(M^r \alpha | \alpha)}.$$

The main goal in this chapter is to prove the following theorem.

THEOREM 6.7. *The total descendent potential $\mathcal{D}(\hbar, \mathbf{q})$ is a solution to the system of HQEs (6.9)–(6.10).*

6.3. Proof of the HQEs

Now we come to one of the main points in this book. Namely, we would like to demonstrate how to apply the techniques developed in Chapter 3 to prove that the total descendent potential satisfies HQEs. As we will see the argument is quite general. There are no explicit formulas involved. The problem of constructing HQEs for other singularities is in understanding the relation between the set of vanishing cycles and the representation theory of lattice vertex algebras.

6.3.1. The coset Virasoro construction. The term (6.10) has an interpretation in terms of the so-called *coset representation* of the Virasoro algebra¹ which will be used in our proof. Let us point out that the proof of Theorem 6.7 in [24] uses yet another interpretation of (6.10) in terms of quantization of quadratic Hamiltonians.

Let us define $\gamma_i \in H_n(f^{-1}(1), \mathbb{C})$, such that,

$$\Pi(\gamma_i) = \Gamma(1 - m_i/h) \llbracket \phi_i(x) dx \rrbracket, \quad 1 \leq i \leq N.$$

Note that γ_i form an eigenbasis for the classical monodromy operator. After a direct computation we get that the Euler pairing

$$\langle \gamma_i, \gamma_j \rangle = \frac{1}{2\pi} (\Pi(\gamma_i), e^{\pi i \theta} \Pi(\gamma_j)) = \frac{1}{1 - \eta^{m_j}} \delta_{i+j, N+1},$$

where $\eta = e^{2\pi i/h}$. In particular, the intersection pairing takes the form

$$(\gamma_i | \gamma_j) = \langle \gamma_i, \gamma_j \rangle + \langle \gamma_j, \gamma_i \rangle = 1,$$

where we used that $m_i + m_{N+1-i} = h$. Therefore, the vectors $\gamma^i := \gamma_{N+1-i}$ form a basis of $H_n(f^{-1}(1), \mathbb{C})$ dual to the basis $\{\gamma_i\}$ with respect to the intersection pairing $(|)$.

The period vectors $I_{\gamma_i}^{(m)}(\lambda)$ can be computed explicitly. After a direct computation we get

$$I_{\gamma_i}^{(k+1)}(\lambda) = (-1)^{k+1} \frac{m_i}{h} \left(\frac{m_i}{h} + 1 \right) \cdots \left(\frac{m_i}{h} + k \right) \lambda^{-k-1-m_i/h} \llbracket \phi_i(x) dx \rrbracket, \quad \forall k \geq 0,$$

and

$$I_{\gamma_i}^{(-k)}(\lambda) = \frac{\lambda^{k-m_i/h}}{\left(k - \frac{m_i}{h} \right) \cdots \left(1 - \frac{m_i}{h} \right)} \llbracket \phi_i(x) dx \rrbracket, \quad \forall k \geq 0.$$

We will be interested in the differential operators

$$\begin{aligned} (\partial_\lambda \mathbf{f}^{\gamma_i}(\lambda, z))^\wedge &= \sum_{k=0}^{\infty} \frac{m_i}{h} \left(\frac{m_i}{h} + 1 \right) \cdots \left(\frac{m_i}{h} + k \right) \lambda^{-k-1-m_i/h} \sqrt{\hbar} \frac{\partial}{\partial q_{k,i}} + \\ &\quad \sum_{k=0}^{\infty} \frac{\lambda^{k-m_i/h}}{\left(k - \frac{m_i}{h} \right) \cdots \left(1 - \frac{m_i}{h} \right)} q_{k, N+1-i} / \sqrt{\hbar}. \end{aligned}$$

It can be checked that the coefficients in front of the powers of λ generate a Heisenberg Lie algebra. Following the physics terminology, the operator series above is a free boson. There is a standard way to construct free bosons on the tensor square of the Fock space. Namely, put

$$\phi_i(\lambda) := (\partial_\lambda \mathbf{f}^{\gamma_i}(\lambda, z))^\wedge \otimes 1 - 1 \otimes (\partial_\lambda \mathbf{f}^{\gamma_i}(\lambda, z))^\wedge, \quad 1 \leq i \leq N,$$

¹We are thankful to Bojko Bakalov for pointing out to us this fact

and

$$\phi^i(\lambda) := (\partial_\lambda \mathbf{f}^{\gamma^i}(\lambda, z))^{\wedge} \otimes 1 - 1 \otimes (\partial_\lambda \mathbf{f}^{\gamma^i}(\lambda, z))^{\wedge}, \quad 1 \leq i \leq N.$$

The coefficients in front of the powers of λ generate a Heisenberg Lie algebra and the operator series $\phi_i(\lambda)$ and $\phi^i(\lambda)$ are free bosons. Let us define the operator series

$$(6.12) \quad \frac{1}{2} \sum_{i=1}^N : \phi_i(\lambda) \phi^i(\lambda) := \frac{1}{2} \sum_{i=1}^N \left(\phi_i^-(\lambda) \phi^i(\lambda) + \phi^i(\lambda) \phi_i^+(\lambda) \right),$$

where

$$\phi_i^\pm(\lambda) = (\partial_\lambda \mathbf{f}_\pm^{\gamma^i}(\lambda, z))^{\wedge} \otimes 1 - 1 \otimes (\partial_\lambda \mathbf{f}_\pm^{\gamma^i}(\lambda, z))^{\wedge}, \quad 1 \leq i \leq N.$$

Note that $\phi_i^+(\lambda)$ is a vector field while $\phi_i^-(\lambda)$ is a linear function, that is, the notation $:$ in formula (6.12) is the standard normal ordering in which the differentiation operation should be applied first. The coefficients in front of λ^{-n-2} in (6.12) close a Lie algebra isomorphic to the Virasoro algebra. We will be interested only in the coefficient in front of λ^{-2} . Namely, using the explicit formulas for $(\partial_\lambda \mathbf{f}^{\gamma^i}(\lambda, z))^{\wedge}$ we get that the operator (6.10) coincides with

$$(6.13) \quad \text{Res } \lambda d\lambda \left(\frac{1}{2} \sum_{i=1}^N : \phi_i(\lambda) \phi^i(\lambda) : \right).$$

The above construction of a representation of the Virasoro algebra is an example of the so-called *coset construction* in the theory of vertex operator algebras. We refer to [38] for further details.

6.3.2. From descendents to ancestors. Let us trivialize the tangent bundle $TB \cong B \times H$, where $H = T_0B$, by using the flat structure. Let $S(t, z) := 1 + S_1(t)z^{-1} + \dots$, $S_k(t) \in \text{End}(H)$ be the calibration of the Frobenius structure, such that, $S_k(0) = 0$ for $k \geq 1$. The calibration allows us to construct flat coordinates on B as follows. We have a map

$$\tau : B \rightarrow H, \quad t \mapsto \tau(t) := S_1(t)\mathbf{1},$$

where $\mathbf{1}$ is the unit of $H \cong H_f$. Using the identification $H \cong H_f$, we have a decomposition $\tau(t) = \sum_i \tau_i(t) \phi_i(x)$. The functions $\tau_i(t)$ ($1 \leq i \leq N$) form a flat coordinate system on B , such that $\tau_i(0) = 0$ for all i .

If $t \in B$ is a semi-simple point, then we have a canonical coordinate system (u_1, \dots, u_N) , such that,

$$\frac{\partial}{\partial u_i} \bullet \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_j}, \quad \left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right) = \delta_{ij} / \Delta_j.$$

Let us recall the linear operator $\Psi_t : \mathbb{C}^N \rightarrow H$ which maps the i th standard vector to the normalized idempotent

$$\sqrt{\Delta_i} \partial / \partial u_i = \sum_{a=1}^N \sqrt{\Delta_i} \frac{\partial \tau_a}{\partial u_i} \frac{\partial}{\partial \tau_a} = \sum_{a=1}^N \sqrt{\Delta_i} \frac{\partial \tau_a}{\partial u_i} \phi_a(x).$$

Let us recall the asymptotic solution $\Psi_t(1 + R_1(t)z + \dots)e^{U/z}$ to the Dubrovin's connection and the operator $R_t := 1 + \sum_{k=1}^{\infty} \Psi_t R_k(t) \Psi_t^{-1} z^k$. Recall that R_t is a symplectic

transformation of $\mathcal{H} := H((z^{-1}))$. The formula for the total descendent potential (see Section 1.6.4) takes the form

$$\mathcal{D}(\hbar, \mathbf{q}) = e^{F^1(t)} \widehat{S}_t^{-1} \widehat{R}_t \prod_{j=1}^N \mathcal{D}_{\text{pt}}(\hbar \Delta_j, \mathbf{q}(u_j) + z),$$

where $F^1(t)$ is the genus-1 potential, $S_t := S(t, z)$, and $\mathbf{q} = \sum_{k \geq 0} \sum_I q_{k,I} z^k \partial / \partial \tau_I$. Here we think of \mathbf{q} as a flat vector field and denote by $\mathbf{q}(u_j)$ the derivative of u_j with respect to \mathbf{q} . The total descendent potential is an element of the Fock space

$$(6.14) \quad \widehat{\mathcal{O}}_{\tau(t)-z} = \mathbb{C}(\hbar) \llbracket q_0 - \tau(t), q_1 + \mathbf{1}, q_2, \dots \rrbracket$$

where $\mathbf{1} = x^0$ is the unit in $H_f \cong H$. Applying the vertex operator $\Gamma^\alpha(\lambda) \mathcal{D}$ we get an element in

$$(6.15) \quad e^{(I_\alpha^{(-1)}(\lambda), \tau(t)) - (I_\alpha^{(-2)}(\lambda), \mathbf{1})} \mathbb{C}((\lambda^{-1/h}))(\hbar) \llbracket q_0 - \tau(t), q_1 + \mathbf{1}, q_2, \dots \rrbracket,$$

where the exponential term is due to the shifts in q_0 and q_1 . Note however that the exponential terms in the action of $\Gamma^\alpha(\lambda) \otimes \Gamma^{-\alpha}(\lambda)$ cancel out, that is, $(\Gamma^\alpha(\lambda) \otimes \Gamma^{-\alpha}(\lambda)) \mathcal{D} \otimes \mathcal{D}$ belongs to the Fock space

$$\mathbb{C}((\lambda^{-1/h}))(\hbar) \llbracket q'_0 - \tau(t), q''_0 - \tau, q'_1 + \mathbf{1}, q''_1 + \mathbf{1}, q'_2, q''_2, \dots \rrbracket.$$

The first step in the proof is to conjugate the vertex operators $\Gamma^\alpha(\lambda) \otimes \Gamma^{-\alpha}(\lambda)$ by $\widehat{S}_t^{-1} \otimes \widehat{S}_t^{-1}$. Note that \widehat{S}_t^{-1} is an isomorphism between the Fock space (6.14) and

$$(6.16) \quad \widehat{\mathcal{O}}_{-z} = \mathbb{C}(\hbar) \llbracket q_0, q_1 + \mathbf{1}, q_2, \dots \rrbracket.$$

Therefore, the expression $(\widehat{S}_t \otimes \widehat{S}_t) (\Gamma^\alpha(\lambda) \otimes \Gamma^{-\alpha}(\lambda)) \mathcal{D} \otimes \mathcal{D}$ takes values in

$$(6.17) \quad \mathbb{C}((\lambda^{-1/h}))(\hbar) \llbracket q'_0, q''_0, q'_1 + \mathbf{1}, q''_1 + \mathbf{1}, q'_2, q''_2, \dots \rrbracket.$$

Let recall the operator series $\mathbf{f}^\alpha(t, \lambda, z)$ and the corresponding vertex operators $\Gamma^\alpha(t, \lambda)$ – see Section 3.2.3. Note that since at $t = 0$ the calibration S_t is the identity, we have $\mathbf{f}^\alpha(0, \lambda, z) = \mathbf{f}^\alpha(\lambda, z)$ and $\Gamma^\alpha(0, \lambda) = \Gamma^\alpha(\lambda)$. Recalling formulas (3.22) and (3.24) we get

$$\widehat{S}_t \Gamma^\alpha(\lambda) \widehat{S}_t^{-1} = e^{\frac{1}{2} W_{\alpha, \alpha}(t, \lambda, \lambda)} \Gamma^\alpha(t, \lambda),$$

where slightly abusing the notation we identify the vanishing cycle α with its image $\Pi(\alpha) \in H$. Therefore, we have

$$\Gamma^\alpha(\lambda) \otimes \Gamma^\alpha(\lambda) \widehat{S}_t^{-1} \otimes \widehat{S}_t^{-1} = e^{W_{\alpha, \alpha}(t, \lambda, \lambda)} \widehat{S}_t^{-1} \otimes \widehat{S}_t^{-1} \Gamma^\alpha(t, \lambda) \otimes \Gamma^\alpha(t, \lambda).$$

Let us also recall that $W_{\alpha, \alpha}(t, \lambda, \lambda)$ can be expressed in terms of the phase form (3.52). Namely, recalling Proposition 3.5, a) and using that $W_{\alpha, \alpha}(t, \lambda_1, \lambda_2)$ vanishes at $t = 0$, we get

$$(6.18) \quad W_{\alpha, \alpha}(t, \lambda_1, \lambda_2) = \int_0^t \sum_{i=1}^N (I_\alpha^{(0)}(s, \lambda_1), \phi_i \bullet_s I_\alpha^{(0)}(s, \lambda_2)) ds_i.$$

Using the translation invariance of the period vectors we get

$$(6.19) \quad W_{\alpha, \alpha}(t, \lambda, \lambda) = \int_{-\lambda \mathbf{1}}^{t - \lambda \mathbf{1}} \sum_{i=1}^N (I_\alpha^{(0)}(s, 0), \phi_i \bullet_s I_\alpha^{(0)}(s, 0)) ds_i.$$

Let us conjugate the Virasoro term (6.10) with $\widehat{S}_t^{-1} \otimes \widehat{S}_t^{-1}$. Put

$$\phi_i(t, \lambda) := (\partial_\lambda \mathbf{f}^{\gamma_i}(t, \lambda, z))^\wedge \otimes 1 - 1 \otimes (\partial_\lambda \mathbf{f}^{\gamma_i}(t, \lambda, z))^\wedge, \quad 1 \leq i \leq N,$$

and

$$\phi^i(t, \lambda) := (\partial_\lambda \mathbf{f}^{\gamma^i}(t, \lambda, z))^\wedge \otimes 1 - 1 \otimes (\partial_\lambda \mathbf{f}^{\gamma^i}(t, \lambda, z))^\wedge, \quad 1 \leq i \leq N.$$

By comparing the linear terms in f in formula (1.62) we get the following conjugation rule:

$$\widehat{\mathbf{f}}\widehat{S}^{-1} = \widehat{S}^{-1}(\mathbf{S}\mathbf{f})^\wedge.$$

Therefore,

$$\phi_i(0, \lambda) \widehat{S}_t^{-1} \otimes \widehat{S}_t^{-1} = \widehat{S}_t^{-1} \otimes \widehat{S}_t^{-1} \phi_i(t, \lambda)$$

and the same relation still holds if we replace ϕ_i with ϕ^i . In order to conjugate the normally ordered product (6.12) let us make use of the following relation

$$\phi_i(t, \lambda_1)\phi^i(t, \lambda_2) =: \phi_i(t, \lambda_1)\phi^i(t, \lambda_2) : + [\phi_i^+(t, \lambda_1), \phi^i(t, \lambda_2)].$$

Since $[\widehat{f}, \widehat{g}] = \Omega(f, g)$ (see Proposition 1.33, b)), the above commutator takes the form

$$[\phi_i^+(t, \lambda_1), \phi^i(t, \lambda_2)] = 2\partial_{\lambda_1}\partial_{\lambda_2}\Omega_{\gamma_i, \gamma^i}(t, \lambda_1, \lambda_2).$$

Therefore,

$$\begin{aligned} : \phi_i(0, \lambda_1)\phi^i(0, \lambda_2) : &= \widehat{S}_t^{-1} \otimes \widehat{S}_t^{-1} \phi_i(t, \lambda_1)\phi^i(t, \lambda_2) : + \\ & 2\partial_{\lambda_1}\partial_{\lambda_2}\left(\Omega_{\gamma_i, \gamma^i}(t, \lambda_1, \lambda_2) - \Omega_{\gamma_i, \gamma^i}(0, \lambda_1, \lambda_2)\right). \end{aligned}$$

Note that according to Lemma 3.7, the above formula can be specialized to $\lambda_1 = \lambda_2 = \lambda$. Put

$$C(t, \lambda) := \lambda^2 \sum_{i=1}^N \partial_{\lambda_1}\partial_{\lambda_2}\left(\Omega_{\gamma_i, \gamma^i}(t, \lambda_1, \lambda_2) - \Omega_{\gamma_i, \gamma^i}(0, \lambda_1, \lambda_2)\right)\Big|_{\lambda_1=\lambda_2=\lambda},$$

and $b_\alpha(t, \lambda) := a_\alpha e^{W_{\alpha, \alpha}(t, \lambda, \lambda)}$. Then Theorem 6.7 is equivalent to the following identity

$$(6.20) \quad \text{Res} \frac{d\lambda}{\lambda} \Omega_{\mathcal{R}}(t, \lambda) \mathcal{A}_t \otimes \mathcal{A}_t = \frac{N(h+1)}{12h} \mathcal{A}_t \otimes \mathcal{A}_t,$$

where

$$\Omega_{\mathcal{R}}(t, \lambda) := \sum_{\alpha \in \mathcal{R}} b_\alpha(t, \lambda) \Gamma^\alpha(t, \lambda) \otimes \Gamma^{-\alpha}(t, \lambda) - \frac{\lambda^2}{2} \sum_{i=1}^N : \phi_i(t, \lambda)\phi^i(t, \lambda) : - C(t, \lambda)$$

and

$$(6.21) \quad \mathcal{A}_t(\hbar, \mathbf{q}) = \widehat{R}_t \prod_{i=1}^N \mathcal{D}_{\text{pt}}(\hbar\Delta_i, \mathbf{q}(u_i) + z)$$

is the total ancestor potential.

6.3.3. The total ancestor potential for A_1 -singularity. Let us assume the notation from Section 6.2.1. The calibration operator has the form $S(t, z) = e^{t/z}$ and the total ancestor potential is by definition

$$\mathcal{A}_t(\hbar, \mathbf{q}) = e^{(t/z)\hat{\cdot}} \mathcal{D}_{\text{pt}}(\hbar\Delta, \mathbf{q}(z) + z).$$

It is straightforward to compute that

$$(1/z)\hat{\cdot} = \frac{q_0}{2\hbar\Delta} + \sum_{m=0}^{\infty} q_{m+1} \partial_{q_m}$$

which is exactly the Virasoro operator L_{-1} with \hbar re-scaled by Δ (see Section 1.6.1). The string equation implies that the total ancestor potential coincides with the total descendent one. Therefore, the total descendent potential $\mathcal{D}_{\text{pt}}(\hbar\Delta, \mathbf{q}(z) + z)$ is a solution to (6.20).

Let us compute the operator $\Omega_{\mathcal{R}}(t, \lambda)$ in the case of A_1 -singularity. Formula (6.18) takes the form

$$W_{\alpha, \alpha}(t, \lambda_1, \lambda_2) = \int_0^t \frac{2ds}{\sqrt{(\lambda_1 - s)(\lambda_2 - s)}},$$

where α is the vanishing cycle. We get $W_{\alpha, \alpha}(t, \lambda, \lambda) = 2 \log(\lambda/(\lambda - t))$. Since $a_\alpha = \frac{1}{16}$, we get

$$b_\alpha(t, \lambda) = \frac{1}{16} \frac{\lambda^2}{(\lambda - t)^2}.$$

The constant

$$C(t, \lambda) = \lambda^2 \partial_{\lambda_1} \partial_{\lambda_2} W_{\alpha, \alpha/2}(t, \lambda_1, \lambda_2) \Big|_{\lambda_1 = \lambda_2 = \lambda} = \frac{\lambda^2}{4(\lambda - t)^2} - \frac{1}{4}.$$

The Laurent series expansion of $C(t, \lambda)$ at $\lambda = \infty$ has only negative powers of λ^{-1} . Therefore,

$$\text{Res} \frac{d\lambda}{\lambda} C(t, \lambda) = 0.$$

The HQEs for the total ancestor potential of A_1 -singularity takes the form

$$(6.22) \quad \text{Res} \frac{\lambda d\lambda}{(\lambda - t)^2} \Omega_{A_1}(t, \lambda) \mathcal{A}_t \otimes \mathcal{A}_t = \frac{1}{8} \mathcal{A}_t \otimes \mathcal{A}_t,$$

where

$$\begin{aligned} \Omega_{A_1}(t, \lambda) = & \frac{1}{16} \left(\Gamma^\alpha(\lambda - t) \otimes \Gamma^{-\alpha}(\lambda - t) + \Gamma^{-\alpha}(\lambda - t) \otimes \Gamma^\alpha(\lambda - t) \right) \\ & - \frac{(\lambda - t)^2}{4} : \phi_\alpha(\lambda - t)^2 : \end{aligned}$$

and

$$\phi_\alpha(\lambda - t) = \partial_\lambda \left(\mathbf{f}^\alpha(\lambda - t, z)\hat{\cdot} \otimes 1 - 1 \otimes \mathbf{f}^\alpha(\lambda - t, z)\hat{\cdot} \right).$$

Similarly, conjugating the HQEs (6.6) with $e^{(t/z)\hat{\cdot}} \otimes e^{(t/z)\hat{\cdot}}$ yields the following HQEs:

$$(6.23) \quad \text{Res} \lambda^r \frac{d\lambda}{\sqrt{\lambda - t}} \left(\Gamma^{\alpha/2}(\lambda - t) \otimes \Gamma^{-\alpha/2}(\lambda - t) - \Gamma^{-\alpha/2}(\lambda - t) \otimes \Gamma^{\alpha/2}(\lambda - t) \right) \mathcal{A}_t \otimes \mathcal{A}_t = 0$$

for all $r \geq 0$.

6.3.4. Analytic properties of the HQEs. The residue operation $\text{Res} \frac{d\lambda}{\lambda}$ in (6.20) is defined formally as the coefficient in front of λ^0 . More precisely, the expression $\Omega_{\mathcal{R}} \mathcal{A}_t \otimes \mathcal{A}_t$ takes value in (6.17) and the HQEs are equivalent to saying that the coefficient in front of λ^0 coincides with the RHS of (6.20). On the other hand, using that the total ancestor potential is tame (see Section 1.6.3) one can easily prove that the expression

$$\begin{aligned} \Gamma^\alpha(t, \lambda) \mathcal{A}_t \otimes \Gamma^{-\alpha}(t, \lambda) \mathcal{A}_t &= \exp \left(\sum_{k=0}^{\infty} (I_\alpha^{(-k-1)}, q'_k - q''_k) / \sqrt{\hbar} \right) \times \\ &\times \mathcal{A}_t(\hbar, q'_0 - \sqrt{\hbar} I_\alpha^{(0)}, q'_1 + \sqrt{\hbar} I_\alpha^{(1)}, \dots) \mathcal{A}_t(\hbar, q''_0 + \sqrt{\hbar} I_\alpha^{(0)}, q''_1 - \sqrt{\hbar} I_\alpha^{(1)}, \dots) \end{aligned}$$

is a formal power series in $q'_0, q''_0, q'_1 + \mathbf{1}, q''_1 + \mathbf{1}, \dots$ with coefficients formal Laurent series in $\hbar^{1/2}$ whose coefficients are polynomial expressions in the period vectors $I_\alpha^{(m)} := I_\alpha^{(m)}(t, \lambda)$. The phase factors $W_{\alpha, \beta}(t, \lambda, \lambda)$ and the second order partial derivatives

$$\partial_{\lambda_1} \partial_{\lambda_2} W_{\alpha, \beta}(t, \lambda_1, \lambda_2) |_{\lambda_1 = \lambda_2 = \lambda}$$

are multivalued analytic functions in $(t, \lambda) \in B \times \mathbb{C} \setminus \text{discr}$ (see (6.18)). Therefore, the coefficients $b_\alpha(t, \lambda)$ and $C(t, \lambda)$ are also multivalued analytic. Note that the operator $\Omega_{\mathcal{R}}(t, \lambda)$ is invariant under the analytic continuation around $\lambda = \infty$ because the analytic continuation permutes the summands according to the action $\alpha \mapsto M\alpha$ of the Coxeter transformation M on the root system \mathcal{R} , while $C(t, \lambda)$ remains invariant. We get that the coefficient in front of each monomial in $\hbar^{\pm 1/2}$ and $q'_0, q''_0, q'_1 + \mathbf{1}, q''_1 + \mathbf{1}, \dots$ in (6.20) is a convergent Laurent series in λ^{-1} . In particular, the formal residue $\text{Res} \frac{d\lambda}{\lambda}$ coincides with the analytic one: $-\text{Res}_{\lambda=\infty} \frac{d\lambda}{\lambda}$.

LEMMA 6.8. *The operator $\Omega_{\mathcal{R}}(t, \lambda)$ is single valued, i.e., it is analytic in $(t, \lambda) \in B \times \mathbb{C} \setminus \text{discr}$.*

PROOF. Suppose that t is a generic semi-simple point, such that, the critical values $u_1(t), \dots, u_N(t)$ of $F(x, t)$ are pairwise distinct. Let L be a simple loop in $\{t\} \times \mathbb{C} \setminus \{u_1(t), \dots, u_N(t)\}$ based at (t, λ) and going around one of the critical values, say $u_i(t)$. Let us require also that L approaches $u_i(t)$ in a transverse direction (see Definition 3.22). Since the fundamental group of $\mathbb{C} \setminus \{u_1(t), \dots, u_N(t)\}$ is generated by such loops, it is sufficient to prove that $\Omega_{\mathcal{R}}$ is invariant under the analytic continuation along L . Let $\varphi \in H_n(f^{-1}(1), \mathbb{Z})$ be the vanishing cycle corresponding to the simple loop conjugate to L via the reference path from $(0, 1)$ to (t, λ) which determines the value of $\Omega_{\mathcal{R}}(t, \lambda)$. Let $r_\varphi : H_n(Z_{\lambda, t}, \mathbb{Z}) \rightarrow H_n(Z_{\lambda, t}, \mathbb{Z})$ be the monodromy transformation representing the loop L . By definition the analytic continuation of $\Gamma^\alpha(t, \lambda)$ is $\Gamma^{r_\varphi(\alpha)}(t, \lambda)$. The function $C(t, \lambda)$ can be identified with the pairing of a geometric section with the cycle $\sum_{i=1}^N \gamma_i \otimes \gamma^i$. Therefore, the analytic continuation of $C(t, \lambda)$ along L amounts to replacing the cycle $\sum_{i=1}^N \gamma_i \otimes \gamma^i$ by

$$\sum_{i=1}^N r_\varphi(\gamma_i) \otimes r_\varphi(\gamma^i) = \sum_{i=1}^N \gamma_i \otimes \gamma^i$$

where the equality follows from the fact that r_φ preserves the intersection pairing $(|)$ and that $\{\gamma_i\}$ and $\{\gamma^i\}$ are dual bases with respect to $(|)$. It remains only to check that

the analytic continuation of $b_\alpha(t, \lambda)$ is $b_{r_\varphi(\alpha)}(t, \lambda)$. Using formula (6.19) we get that the analytic continuation of $b_\alpha(t, \lambda)$ is equal to $b_{r_\varphi(\alpha)}(t, \lambda)$ iff

$$W_{\alpha, \alpha}(t, \lambda, \lambda) - W_{r_\varphi(\alpha), r_\varphi(\alpha)}(t, \lambda, \lambda) + \int_{L_b} \sum_{i=1}^N (I_\alpha^{(0)}(s, 0), \phi_i \bullet_s I_\alpha^{(0)}(s, 0)) ds_i = \log(a_{r_\varphi(\alpha)}/a_\alpha),$$

where L_b is the path in B parameterized by $t - x\mathbf{1}$, $(t, x) \in L$ and the above identity should be considered modulo $2\pi i\mathbb{Z}$. Using Lemma 3.7 and recalling the definition (3.52) of the phase form we get that the LHS of the above identity can be written as

$$\lim_{\lambda' \rightarrow \lambda} \left(\Omega_{\alpha, \alpha}(t, \lambda', \lambda) - \Omega_{r_\varphi(\alpha), r_\varphi(\alpha)}(t, \lambda', \lambda) + \int_{\widehat{L}} \mathcal{W}_{\alpha, \alpha} \right. \\ \left. \Omega_{r_\varphi(\alpha), r_\varphi(\alpha)}(0, \lambda', \lambda) - \Omega_{\alpha, \alpha}(0, \lambda', \lambda) \right),$$

where \widehat{L} is the loop in $B \times \mathbb{C}^2$ parameterized by $(t, x + \lambda' - \lambda, x)$, $(t, x) \in L$. Thanks to Theorem 3.28, the first line in the above formula is an integer multiple of $2\pi i$. Recalling the definition (6.11), we get that the limit of the 2nd line in the above formula is precisely $\log(a_{r_\varphi(\alpha)}/a_\alpha)$. This is exactly what we had to prove. \square

Suppose that $t \in B$ is a generic semi-simple point, such that, the critical values $u_1(t), \dots, u_N(t)$ are pairwise distinct. Lemma 6.8 implies that the coefficient in front of each monomial in $\hbar^{\pm 1/2}$ and $q'_0, q''_0, q'_1 + \mathbf{1}, q''_1 + \mathbf{1}, \dots$ in (6.20) is an analytic function in $\mathbb{C} \setminus \{u_1(t), \dots, u_N(t)\}$. Recalling the residue theorem we transform (6.20) into

$$(6.24) \quad \sum_{i=1}^N \operatorname{Res}_{\lambda=u_i(t)} \frac{d\lambda}{\lambda} \Omega_{\mathcal{R}}(t, \lambda) \mathcal{A}_t \otimes \mathcal{A}_t = \left(\frac{N(h+1)}{12h} + \operatorname{Res}_{\lambda=0} \frac{d\lambda}{\lambda} C(t, \lambda) \right) \mathcal{A}_t \otimes \mathcal{A}_t.$$

The proof of (6.24) is a local computation near each critical value $u_i(t)$. Let us fix a critical value $u_i(t)$ and assume that λ is sufficiently close to $u_i(t)$. Let us fix a reference path C to (t, λ) so that the values of all vertex operators $\Gamma^\alpha(t, \lambda)$ and all coefficients $b_\alpha(t, \lambda)$ are fixed. Moreover, according to Theorem 3.23 we may choose the reference path

$$C : (\tau(s), \Lambda(s)) \quad (0 \leq s \leq 1)$$

between $(\tau(0), \Lambda(0)) = (0, 1)$ and $(\tau(1), \Lambda(1)) = (t, \lambda)$ in such a way that

$$\Omega_{\alpha, \varphi}^i(t, \lambda', \lambda) = \Omega_{\alpha, \varphi}(t, \lambda', \lambda)$$

for all λ' sufficiently close to λ satisfying

$$|\lambda - u_i(t)| < |\lambda' - u_i(t)| < |u_j(t) - u_i(t)| \quad \forall j \neq i,$$

where $\alpha \in H_n(f^{-1}(1), \mathbb{C})$ is an arbitrary cycle and φ is the vanishing cycle corresponding to the reference path C . More precisely, for C we can choose a path consisting of two pieces: a path around the discriminant (i.e. in the domain D) from $(0, 1)$ to a point $(t, \lambda_0) \in \{t\} \times \mathbb{C}$ and a path in $\{t\} \times \mathbb{C}$ from (t, λ_0) to (t, λ) which can be identified with the tail of a simple loop around $u_i(t)$ approaching $u_i(t)$ in a transverse direction. Let us recall that $\Omega_{\alpha, \varphi}^i(t, \lambda', \lambda)$ is the Laurent series expansion in the powers of $(\lambda - u_i(t))^{1/2}$ of the infinite series

$$\sum_{n=0}^{\infty} (-1)^n (I_\alpha^{(n)}(t, \lambda'), I_\varphi^{(-n-1)}(t, \lambda)),$$

while

$$\Omega_{\alpha,\varphi}(t, \lambda', \lambda) = \Omega_{\alpha,\varphi}(0, \lambda' - \lambda + 1, 1) + \int_{\widehat{C}} \mathcal{W}_{\alpha,\varphi},$$

where

$$\widehat{C} : (\tau(s), \lambda' - \lambda + \Lambda(s), \Lambda(s)) \quad (0 \leq s \leq 1)$$

is the lift of the reference path C .

If $(\alpha|\varphi) = 0$, then the vertex operator $\Gamma^\alpha(t, \lambda)$ is analytic at $\lambda = u_i(t)$ (see Proposition 3.4) and therefore it does not contribute to the residue. If $(\alpha|\varphi) \neq 0$, then since \mathcal{R} is an ADE root system, either $(\alpha|\varphi) = \pm 1$ or $\alpha = \pm\varphi$.

6.3.5. From ancestors to KdV. The subset of all $\alpha \in \mathcal{R}$ satisfying $(\alpha|\varphi) = \pm 1$ splits into pairs $(\alpha, r_\varphi(\alpha))$, such that, $(\alpha|\varphi) = 1$. The goal in this subsection is to prove that the residue of the 1-form

$$(6.25) \quad \frac{d\lambda}{\lambda} \left(b_\alpha(t, \lambda) \Gamma^\alpha(t, \lambda) \otimes \Gamma^{-\alpha}(t, \lambda) + b_{r_\varphi(\alpha)}(t, \lambda) \Gamma^{r_\varphi(\alpha)}(t, \lambda) \otimes \Gamma^{-r_\varphi(\alpha)}(t, \lambda) \right) \mathcal{A}_t \otimes \mathcal{A}_t$$

at $\lambda = u_i(t)$ vanishes.

Let us introduce the following functions:

$$v_\alpha^i(t, \lambda) := \lim_{\epsilon \rightarrow 0} \exp \left(\int_{t+(\lambda-u_i-\epsilon)\mathbf{1}}^t \left(I_\alpha^{(0)}(s, \lambda) \bullet I_\alpha^{(0)}(s, \lambda) + (\alpha|\varphi)^2 \frac{du_i(s)}{2(u_i(s) - \lambda)} \right) \right),$$

where ϵ is a complex number such that $\text{Arg}(\epsilon) = \text{Arg}(\lambda - u_i)$ and the integration path is the straight line segment $s = t + x\mathbf{1}$, $x \in [\lambda - u_i(t) - \epsilon, 0]$. Here the integrand $I_\alpha^{(0)}(s, \lambda) \bullet I_\alpha^{(0)}(s, \lambda)$ takes values in $H \cong T_s B \cong T_s^* B$, where the first isomorphism is defined by the flat structure trivialization of TB and the second one is defined by the residue (Frobenius) pairing. In other words, the integrand is identified with a 1-form on B depending on the parameter λ . The functions $v_\alpha^i(t, \lambda)$ are defined for all λ sufficiently close to $u_i(t)$. We are not going to use this fact, but note that $v_\alpha^i(t, \lambda)$ can be extended analytically along any path in $\mathbb{C} \times B \setminus \text{discr}$.

PROPOSITION 6.9. *Suppose that $\alpha \in H_n(f^{-1}(1), \mathbb{C})$ is any cycle and let $\alpha' := \alpha - (\alpha|\varphi)\varphi/2$. The following identity holds:*

$$\Gamma^\alpha(t, \lambda) \widehat{R}_t = \left(\frac{v_{\alpha'}^i(t, \lambda)}{v_\alpha^i(t, \lambda)} \right)^{1/2} \Gamma^{\alpha'}(t, \lambda) \widehat{R}_t \Gamma_{\text{pt}}^{(\alpha|\varphi)/2}(u_i(t), \lambda).$$

PROOF. Note that the vertex operator Γ^α factorizes as follows (see Proposition 1.33):

$$\Gamma^\alpha(t, \lambda) = e^{-\Omega(\mathbf{f}_+^{\alpha'}, \mathbf{f}^\varphi) (\alpha|\varphi)/2} \Gamma^{\alpha'}(t, \lambda) \Gamma^{(\alpha|\varphi)\varphi/2}(t, \lambda),$$

where the propagator $\Omega(\mathbf{f}_+^{\alpha'}(t, \lambda), \mathbf{f}^\varphi(t, \lambda))$ is interpreted via the Laurent series expansion of the period vectors at $\lambda = u_i(t)$. We will prove below that the propagator is a convergent Laurent series in $(\lambda - u_i(t))^{1/2}$. Therefore, the above identity is an identity between linear operators from the space of tame series $\widehat{\mathcal{O}}_{\mathcal{H}_+^{-z}}^{\text{tame}}$ into the space of formal power series in $q_0, q_1 + \mathbf{1}, q_2, \dots$ with coefficients formal Laurent series in $\hbar^{1/2}$ whose coefficients are convergent Laurent series in $(\lambda - u_i(t))^{1/2}$. Recalling formula (3.23) we get

$$\Gamma^{(\alpha|\varphi)\varphi/2}(t, \lambda) \widehat{R}_t = e^{(\alpha|\varphi)^2 V(\mathbf{f}_+^\varphi, \mathbf{f}_-^\varphi)/8} \widehat{R}_t \Gamma_{\text{pt}}^{(\alpha|\varphi)/2}(u_i(t), \lambda).$$

Therefore, in order to prove the proposition we need only to verify that

$$(6.26) \quad -(\alpha|\varphi)\Omega(\mathbf{f}_+^{\alpha'}, \mathbf{f}^\varphi) + \frac{1}{4}(\alpha|\varphi)^2 V(\mathbf{f}_-^\varphi, \mathbf{f}_-^\varphi) = \log(v_{\alpha'}^i(t, \lambda)/v_\alpha^i(t, \lambda)).$$

The quadratic form (see Proposition 3.5)

$$V(\mathbf{f}_-^\varphi, \mathbf{f}_-^\varphi) = \lim_{\epsilon \rightarrow 0} \int_t^{t+(\lambda-u_i-\epsilon)\mathbf{1}} \left(I_\varphi^{(0)}(s, \lambda) \bullet I_\varphi^{(0)}(s, \lambda) + \frac{2du_i(s)}{u_i(s) - \lambda} \right).$$

It remains to compute the symplectic pairing $\Omega(\mathbf{f}_+^{\alpha'}, \mathbf{f}^\varphi)$. We have

$$d\Omega(\mathbf{f}_+^{\alpha'}(t, \lambda), \mathbf{f}^\varphi(t, \lambda)) = I_{\alpha'}^{(0)}(t, \lambda) \bullet I_\varphi^{(0)}(t, \lambda),$$

where d is the differential with respect to t (see (3.51)). The above formula proves that the propagator is a convergent power series in $(\lambda - u_i(t))^{1/2}$. In fact, since by definition $(\alpha'|\varphi) = 0$, the series $\mathbf{f}_+^{\alpha'}(t, \lambda)$ is regular at $\lambda = u_i(t)$, while $\mathbf{f}_-^\varphi(t, \lambda)$ is a power series in $(\lambda - u_i(t))^{1/2}$. Therefore, the propagator is a convergent power series in $(\lambda - u_i(t))^{1/2}$ and the following formula holds:

$$\Omega(\mathbf{f}_+^{\alpha'}(t, \lambda), \mathbf{f}^\varphi(t, \lambda)) = \lim_{\epsilon \rightarrow 0} \int_{t+(\lambda-u_i(t)-\epsilon)\mathbf{1}}^t I_{\alpha'}^{(0)}(s, \lambda) \bullet I_\varphi^{(0)}(s, \lambda).$$

The LHS of formula (6.26) takes the form

$$-\lim_{\epsilon \rightarrow 0} \int_{t+(\lambda-u_i(t)-\epsilon)\mathbf{1}}^t \left((\alpha|\varphi)I_{\alpha'}^{(0)} \bullet I_\varphi^{(0)} + \frac{1}{4}(\alpha|\varphi)^2 I_\varphi^{(0)} \bullet I_\varphi^{(0)} + (\alpha|\varphi)^2 \frac{du_i(s)}{2(u_i(s) - \lambda)} \right),$$

where for brevity we suppressed the dependence of the period vectors $I^{(0)}$ on (s, λ) . By definition $(\alpha|\varphi)\varphi = 2(\alpha - \alpha')$, so the above formula takes the form

$$-\lim_{\epsilon \rightarrow 0} \int_{t+(\lambda-u_i(t)-\epsilon)\mathbf{1}}^t \left(2I_{\alpha'}^{(0)} \bullet I_{\alpha-\alpha'}^{(0)} + I_{\alpha-\alpha'}^{(0)} \bullet I_{\alpha-\alpha'}^{(0)} + (\alpha|\varphi)^2 \frac{du_i(s)}{2(u_i(s) - \lambda)} \right).$$

Recalling the definition of $v_{\alpha'}^i$ and v_α^i it is straightforward to check that the above formula coincides with the RHS of (6.26). \square

Let us substitute (6.21) into (6.25). Note that if we decompose $\alpha = \alpha' + (\alpha|\varphi)\varphi/2$, just like in Proposition 6.9, then $r_\varphi(\alpha) = \alpha' - (\alpha|\varphi)\varphi/2$. Applying the formula from Proposition 6.9 we get that (6.25) is equal to

$$(6.27) \quad v_{\alpha'}^i(t, \lambda)(\Gamma^{\alpha'}(t, \lambda) \otimes \Gamma^{\alpha'}(t, \lambda)) (\widehat{R}_t \otimes \widehat{R}_t)$$

$$(6.28) \quad \frac{d\lambda}{\lambda} \left(\frac{b_\alpha(t, \lambda)}{v_\alpha^i(t, \lambda)} \Gamma_{\text{pt}}^{\varphi/2} \otimes \Gamma_{\text{pt}}^{-\varphi/2} + \frac{b_{r_\varphi(\alpha)}(t, \lambda)}{v_{r_\varphi(\alpha)}^i(t, \lambda)} \Gamma_{\text{pt}}^{-\varphi/2} \otimes \Gamma_{\text{pt}}^{\varphi/2} \right)$$

$$(6.29) \quad \prod_{j=1}^N \mathcal{D}_{\text{pt}}(\hbar\Delta_j, \mathbf{q}'(u_j) + z) \mathcal{D}_{\text{pt}}(\hbar\Delta_j, \mathbf{q}''(u_j) + z).$$

The cycle α' is invariant with respect to the local monodromy. Therefore, all periods $I_{\alpha'}^{(m)}(t, \lambda)$ are analytic at $\lambda = u_i(t)$. In particular, the first line (6.27) is analytic at $\lambda = u_i(t)$. The operator in the big brackets on the second line (6.28) acts on the i th term

in the product (6.29). Recalling the HQEs for KdV, i.e., formula (6.6), we get that if we prove that

$$\frac{b_\alpha(t, \lambda)}{v_\alpha^i(t, \lambda)} = -\frac{b_{r_\varphi(\alpha)}(t, \lambda)}{v_{r_\varphi(\alpha)}^i(t, \lambda)} = B(t)(\lambda - u_i(t))^{-1/2},$$

where $B(t)$ is some function independent of λ , then it will follow that the expression (6.27)–(6.29) is analytic at $\lambda = u_i(t)$ for all t , such that, $u_i(t) \neq 0$. There is a slight complication here when $u_i(t) = 0$ because then the form $d\lambda/\lambda$ has a pole. This complication can be resolved as follows. Let us consider the residue of (6.27)–(6.29) at $\lambda = u_i(t)$ as a function in t . It is a holomorphic function in a neighborhood where the canonical coordinates u_1, \dots, u_N exist which must be 0 for all points t , such that, $u_i(t) \neq 0$ and $u_j(t) \neq u_k(t)$ for $j \neq k$. By continuity, the residue must be identically 0 for all t in a neighborhood where the canonical coordinates exist.

Let us first prove that

$$\frac{b_\alpha(t, \lambda)}{v_\alpha^i(t, \lambda)} = B(t)(\lambda - u_i(t))^{-1/2}.$$

Equivalently, we have to prove that

$$(6.30) \quad \partial_\lambda \log b_\alpha(t, \lambda) - \partial_\lambda \log v_\alpha^i(t, \lambda) = -\frac{1}{2(\lambda - u_i(t))}.$$

We have $\log b_\alpha(t, \lambda) = \log a_\alpha + W_{\alpha, \alpha}(t, \lambda, \lambda)$. The coefficient a_α is a constant independent of λ . The phase factor $W_{\alpha, \alpha}$ is a quadratic expression in the period vectors. Using the translation invariance of the period vectors we get

$$\partial_\lambda \log b_\alpha(t, \lambda) = -\partial_{t_1} W_{\alpha, \alpha}(t, \lambda, \lambda) = -(I_\alpha^{(0)}(t, \lambda), I_\alpha^{(0)}(t, \lambda)),$$

where for the 2nd equality we used Proposition 3.5. Using the translation invariance of the period vectors we can write

$$\log v_\alpha^i(t, \lambda) = \int_{t-(u_i+\epsilon)\mathbf{1}}^{t-\lambda\mathbf{1}} \left(I_\alpha^{(0)}(s, 0) \bullet I_\alpha^{(0)}(s, 0) + (\alpha|\varphi)^2 \frac{du_i(s)}{2u_i(s)} \right).$$

Differentiating the above formula with respect to λ we get

$$\partial_\lambda \log v_\alpha^i(t, \lambda) = -(I_\alpha^{(0)}(t, \lambda), I_\alpha^{(0)}(t, \lambda)) + (\alpha|\varphi)^2 \frac{1}{2(\lambda - u_i(t))}.$$

The identity (6.30) follows because $(\alpha|\varphi) = 1$.

Finally, it remains to prove that

$$(6.31) \quad \frac{b_\alpha(t, \lambda)}{b_{r_\varphi(\alpha)}(t, \lambda)} \frac{v_{r_\varphi(\alpha)}^i(t, \lambda)}{v_\alpha^i(t, \lambda)} = -1.$$

Let $L \subset \{t\} \times \mathbb{C} \subset B \times \mathbb{C}$ be a closed loop based at (t, λ) of the form $L = L_\epsilon^{-1} \circ C_\epsilon \circ L_\epsilon$ where $L_\epsilon := \{t\} \times (\text{the line segment } [\lambda, u_i(t) + \epsilon])$ and $C_\epsilon := \{t\} \times (\text{the circle with center } u_i(t) \text{ and radius } \epsilon)$. Let us fix λ' sufficiently close to λ and denote by \widehat{L}_ϵ and \widehat{C}_ϵ the paths in $B \times \mathbb{C}^2$ with parameterization $(t, x + \lambda' - \lambda, x)$, where the parameter x is such that (t, x) vary along respectively L_ϵ and C_ϵ . Recalling the definition of v_α^i , $v_{r_\varphi(\alpha)}^i$, and the phase form $\mathcal{W}_{\alpha, \alpha}$ we get

$$\log v_{r_\varphi(\alpha)}^i(t, \lambda) - \log v_\alpha^i(t, \lambda) = \lim_{\epsilon \rightarrow 0} \lim_{\lambda' \rightarrow \lambda} \left(\int_{\widehat{L}_\epsilon} \mathcal{W}_{\alpha, \alpha} - \int_{\widehat{L}_\epsilon} \mathcal{W}_{r_\varphi(\alpha), r_\varphi(\alpha)} \right).$$

On the other hand, the coefficient $b_{r_\varphi(\alpha)}(t, \lambda)$ coincides with the analytic continuation of $b_\alpha(t, \lambda)$ along L (see the proof of Lemma 6.8), that is,

$$b_{r_\varphi(\alpha)}(t, \lambda) = b_\alpha(t, \lambda) \exp\left(\lim_{\lambda' \rightarrow \lambda} \int_{\widehat{L}} \mathcal{W}_{\alpha, \alpha}\right).$$

Therefore the LHS of (6.31) coincides with

$$\exp\left(\lim_{\epsilon \rightarrow 0} \lim_{\lambda' \rightarrow \lambda} \left(-\int_{\widehat{L}} \mathcal{W}_{\alpha, \alpha} + \int_{\widehat{L}_\epsilon} \mathcal{W}_{\alpha, \alpha} - \int_{\widehat{L}_\epsilon} \mathcal{W}_{r_\varphi(\alpha), r_\varphi(\alpha)}\right)\right) = \exp\left(-\lim_{\epsilon \rightarrow 0} \lim_{\lambda' \rightarrow \lambda} \int_{\widehat{C}_\epsilon} \mathcal{W}_{\alpha, \alpha}\right).$$

Let us decompose the cycle $\alpha = \alpha' + (\alpha|\varphi)\varphi/2$. Note that since the period vectors $I_{\alpha'}^{(m)}(t, \lambda)$ are analytic in a neighborhood of $\lambda = u_i(t)$ the only contribution to the above limit will come from

$$\frac{1}{4}(\alpha|\varphi)^2 \lim_{\epsilon \rightarrow 0} \lim_{\lambda' \rightarrow \lambda} \int_{\widehat{C}_\epsilon} \mathcal{W}_{\varphi, \varphi} = -\pi \mathbf{i} (\alpha|\varphi)^2 = -\pi \mathbf{i},$$

where in the first identity we used Lemma 3.30. This completes the proof of the identity (6.31) and hence our claim that the residue of (6.25) at $\lambda = u_i(t)$ vanishes is proved.

6.3.6. From ancestors to the Kac–Wakimoto form of KdV. Let us compute the residue at $\lambda = u_i(t)$ of the 1-form

$$(6.32) \quad \frac{d\lambda}{\lambda} b_\varphi(t, \lambda) \left(\Gamma^\varphi(t, \lambda) \otimes \Gamma^{-\varphi}(t, \lambda) + \Gamma^{-\varphi}(t, \lambda) \otimes \Gamma^\varphi(t, \lambda) \right) \mathcal{A}_t \otimes \mathcal{A}_t.$$

Conjugating with the operator $\widehat{R}_t \otimes \widehat{R}_t$ (see Proposition 6.9) we get that the above 1-form is equal to

$$(6.33) \quad (\widehat{R}_t \otimes \widehat{R}_t) \frac{d\lambda}{\lambda} \frac{b_\varphi(t, \lambda)}{v_\varphi^i(t, \lambda)} \left(\Gamma_{\text{pt}}^\varphi \otimes \Gamma_{\text{pt}}^{-\varphi} + \Gamma_{\text{pt}}^{-\varphi} \otimes \Gamma_{\text{pt}}^\varphi \right)$$

$$(6.34) \quad \prod_{j=1}^N \mathcal{D}_{\text{pt}}(\hbar \Delta_j, \mathbf{q}'(u_j) + z) \mathcal{D}_{\text{pt}}(\hbar \Delta_j, \mathbf{q}''(u_j) + z).$$

Let us compute the ratio $b_\varphi(t, \lambda)/v_\varphi^i(t, \lambda)$. Recall that

$$b_\varphi(t, \lambda) = \lim_{\lambda' \rightarrow \lambda} (1 - \lambda/\lambda')^2 \exp\left(\Omega_{\varphi, \varphi}(t, \lambda', \lambda)\right).$$

Let us express the coefficient $v_\varphi^i(t, \lambda)$ as an integral of the phase form along a path. Let us consider the path $\widehat{\Delta}_\epsilon$ from $(t, \lambda' - \lambda + u_i(t) + \epsilon, u_i(t) + \epsilon)$ to (t, λ', λ) defined by the straight line in $\{t\} \times \mathbb{C}^2$, that is, $\widehat{\Delta}_\epsilon$ admits the parameterization

$$(t, \lambda' - x, \lambda - x), \quad x \in [\lambda - u_i(t) - \epsilon, 0].$$

Here, just like in the previous section, ϵ is a complex number, such that, $\text{Arg}(\epsilon) = \text{Arg}(\lambda - u_i(t))$. We have

$$(6.35) \quad \int_{\widehat{\Delta}_\epsilon} \mathcal{W}_{\varphi, \varphi} = \int_{\lambda - u_i(t) - \epsilon}^0 (I_\varphi^{(0)}(t, \lambda' - x), I_\varphi^{(0)}(t, \lambda - x)) dx$$

and

$$\int_{t+(\lambda-u_i-\epsilon)\mathbf{1}}^t \frac{2du_i(s)}{u_i(s) - \lambda} = \int_\epsilon^{\lambda - u_i(t)} \frac{2d\xi}{\xi},$$

where we used the parameterization $s = t + (\lambda - u_i - \xi)\mathbf{1}$. Comparing the above formulas with the definition of $v_\varphi^i(t, \lambda)$ we get

$$v_\varphi^i(t, \lambda) = \lim_{\epsilon \rightarrow 0} \lim_{\lambda' \rightarrow \lambda} \exp \left(\int_{\widehat{\Delta}} \mathcal{W}_{\varphi, \varphi} + \int_{\lambda' - \lambda + \epsilon}^{\lambda - u_i(t)} \frac{2d\xi}{\xi} \right).$$

Note that the order of the limits in the above formula is important. It turns out that it is possible to compute explicitly the error of exchanging the order of the two limits. This is actually one of the key observations in [16].

LEMMA 6.10. *Up to an integer multiple of $2\pi\mathbf{i}$ the following formula holds:*

$$(6.36) \quad \left(\lim_{\epsilon \rightarrow 0} \lim_{\lambda' \rightarrow \lambda} - \lim_{\lambda' \rightarrow \lambda} \lim_{\epsilon \rightarrow 0} \right) \left(\int_{\widehat{\Delta}_\epsilon} \mathcal{W}_{\varphi, \varphi} + \int_{\lambda' - \lambda + \epsilon}^{\lambda - u_i} \frac{2d\xi}{\xi} \right) = \log 16.$$

where $u_i = u_i(t)$, that is, we suppressed the dependence of u_i on t .

PROOF. Let us recall formula (6.35) and the Laurent series expansion of the period vectors

$$I_\varphi^{(0)}(t, \lambda - x) = \frac{2}{\sqrt{2(\lambda - x - u_i)}} \left(\frac{du_i}{\sqrt{\Delta_i}} + O(\lambda - x - u_i) \right).$$

Substituting this expansion in (6.35) it is not hard to see that only the leading order terms contribute to the LHS of (6.36). In other words, the LHS of (6.36) coincides with

$$\left(\lim_{\epsilon \rightarrow 0} \lim_{\lambda' \rightarrow \lambda} - \lim_{\lambda' \rightarrow \lambda} \lim_{\epsilon \rightarrow 0} \right) \left(\int_{\lambda - u_i - \epsilon}^0 \frac{2dx}{\sqrt{\lambda' - x - u_i} \sqrt{\lambda - x - u_i}} + \int_{\lambda' - \lambda + \epsilon}^{\lambda - u_i} \frac{2d\xi}{\xi} \right).$$

The first integral is equal to

$$-4 \log \left(\sqrt{\lambda' - x - u_i} + \sqrt{\lambda - u_i - x} \right) \Big|_{x=\lambda - u_i - \epsilon}^{x=0} = -4 \log \frac{\sqrt{\lambda' - u_i} + \sqrt{\lambda - u_i}}{\sqrt{\lambda' - \lambda + \epsilon} + \sqrt{\epsilon}}$$

and the sum of the two integrals become

$$4 \log \left(\frac{\sqrt{\lambda' - \lambda + \epsilon} + \sqrt{\epsilon}}{\sqrt{\lambda' - \lambda + \epsilon}} \frac{\sqrt{\lambda - u_i}}{\sqrt{\lambda' - u_i} + \sqrt{\lambda - u_i}} \right).$$

The commutator of the two limits is straightforward to compute, i.e., the limit $\lambda' \rightarrow \lambda$ is 0, while the limit $\lim_{\lambda' \rightarrow 0} \lim_{\epsilon \rightarrow 0}$ is $-4 \log 2$. \square

Note that

$$\lim_{\epsilon \rightarrow 0} \int_{\widehat{\Delta}_\epsilon} \mathcal{W}_{\varphi, \varphi} = \Omega_{\varphi, \varphi}^i(t, \lambda', \lambda)$$

because both sides have the same derivative with respect to λ and they both vanish at $\lambda = u_i(t)$. Therefore, using Lemma 6.10, we get the following formula

$$v_\varphi^i(t, \lambda) = 16(\lambda - u_i(t))^2 \lim_{\lambda' \rightarrow \lambda} (\lambda' - \lambda)^{-2} e^{\Omega_{\varphi, \varphi}^i(t, \lambda', \lambda)}.$$

Since $\Omega_{\varphi, \varphi}(t, \lambda', \lambda) = \Omega_{\varphi, \varphi}^i(t, \lambda', \lambda)$, we get

$$\frac{b_\varphi(t, \lambda)}{v_\varphi^i(t, \lambda)} = \frac{\lambda^2}{16(\lambda - u_i(t))^2}.$$

Let us compute explicitly the vertex operators $\Gamma_{\text{pt}}^{\pm\varphi}(u_i, \lambda)$, where for brevity we omit the argument t and write $u_i = u_i(t)$. We have

$$\mathbf{f}^\varphi(t, \lambda, z) = R_t(z) \mathbf{f}_{\text{pt}}^\varphi(u_i, \lambda, z),$$

where

$$\mathbf{f}_{\text{pt}}^\varphi(u_i, \lambda, z) = \pm \sum_{m \in \mathbb{Z}} (-z \partial_\lambda)^m \sqrt{2\Delta_i(\lambda - u_i)} \partial_{u_i}.$$

Note that if we identify the tangent and the cotangent spaces $T_t B \cong T_t^* B$ via the Frobenius pairing, then $\sqrt{\Delta_i} \partial_{u_i} = du_i / \sqrt{\Delta_i}$. Recalling the quantization rules we get

$$\left((-z)^{-k-1} \sqrt{\Delta_i} \partial_{u_i} \right)^\wedge = \frac{1}{\sqrt{\Delta_i}} \left((-z)^{-k-1} du_i \right)^\wedge = \sum_{a=1}^N \frac{\partial u_i}{\partial \tau_a} q_{k,a} / \sqrt{\hbar \Delta_i} = q_k(u_i) / \sqrt{\hbar \Delta_i},$$

where (τ_1, \dots, τ_N) are the flat coordinates on B (see Section 6.3.2) and $k \geq 0$. Using the commutation relation $[\widehat{f}, \widehat{g}] = \Omega(f, g)$, we get that

$$\left((-z)^k \sqrt{\Delta_i} \partial_{u_i} \right)^\wedge = (-1)^{k+1} \sqrt{\hbar \Delta_i} \frac{\partial}{\partial q_k(u_i)}.$$

Here we can think of $q_k(u_i)$ ($k \geq 0, 1 \leq i \leq N$) as a new set of formal variables related to $q_{k,a}$ via the substitution

$$q_k(u_i) = q_{k,1} \frac{\partial u_i}{\partial \tau_1} + \dots + q_{k,N} \frac{\partial u_i}{\partial \tau_N}.$$

This is an invertible substitution because the Jacobian matrix of the transition from flat to canonical coordinates is invertible. We get that the vertex operator $\Gamma_{\text{pt}}^\varphi =: e^{\widehat{\mathbf{f}^\varphi(u_i, \lambda, z)}}$ coincides with the vertex operator of an A_1 -singularity, that is, in the notation of Section 6.2.1 $\Gamma_{\text{pt}}^\varphi = \Gamma^\alpha(\lambda - u_i)$ with q_k replaced by $q_k(u_i)$ and Δ by Δ_i . In particular, the vertex operators $\Gamma_{\text{pt}}^{\pm\varphi}$ act only on the i th term in the product (6.34). Recalling the HQEs (6.22) we get that in order to compute the residue at $\lambda = u_i$ of the 1-form (6.33)–(6.34) we may replace (6.33) by

$$(\widehat{R}_t \otimes \widehat{R}_t) d\lambda \left(\frac{\lambda}{4} : \phi_{\text{pt}}(u_i, \lambda)^2 : + \frac{1}{8(\lambda - u_i)} \right),$$

where

$$\phi_{\text{pt}}(u_i, \lambda) = \partial_\lambda \left(\mathbf{f}_{\text{pt}}^\varphi(u_i, \lambda, z)^\wedge \otimes 1 - 1 \otimes \mathbf{f}_{\text{pt}}^\varphi(u_i, \lambda, z)^\wedge \right).$$

By extracting the linear terms in f in the conjugation formula (1.63) we get the following rule: $\widehat{R} \widehat{f} = \widehat{R} f \widehat{R}$. Therefore,

$$\widehat{R}_t \otimes \widehat{R}_t \phi_{\text{pt}}(u_i, \lambda_1) \phi_{\text{pt}}(u_i, \lambda_2) = \phi_\varphi(t, \lambda_1) \phi_\varphi(t, \lambda_2) \widehat{R}_t \otimes \widehat{R}_t,$$

where

$$\phi_\varphi(t, \lambda) = \partial_\lambda \left(\mathbf{f}^\varphi(t, \lambda, z)^\wedge \otimes 1 - 1 \otimes \mathbf{f}^\varphi(t, \lambda, z)^\wedge \right).$$

On the other hand,

$$\phi_\varphi(t, \lambda_1) \phi_\varphi(t, \lambda_2) =: \phi_\varphi(t, \lambda_1) \phi_\varphi(t, \lambda_2) : + 2\partial_{\lambda_1} \partial_{\lambda_2} \Omega(\mathbf{f}_+^\varphi(t, \lambda_1, z), \mathbf{f}^\varphi(t, \lambda_2, z))$$

and

$$\phi_{\text{pt}}(u_i, \lambda_1) \phi_{\text{pt}}(u_i, \lambda_2) =: \phi_{\text{pt}}(u_i, \lambda_1) \phi_{\text{pt}}(u_i, \lambda_2) : + 2\partial_{\lambda_1} \partial_{\lambda_2} \Omega(\mathbf{f}_{\text{pt},+}^\varphi(u_i, \lambda_1, z), \mathbf{f}_{\text{pt}}^\varphi(u_i, \lambda_2, z)),$$

where $::$ is the normally ordered product. Therefore,

$$\widehat{R}_t \otimes \widehat{R}_t : \phi_{\text{pt}}(u_i, \lambda_1) \phi_{\text{pt}}(u_i, \lambda_2) := \phi_\varphi(t, \lambda_1) \phi_\varphi(t, \lambda_2) : \widehat{R}_t \otimes \widehat{R}_t + 2\partial_{\lambda_1} \partial_{\lambda_2} \left(\Omega_{\varphi, \varphi}(t, \lambda_1, \lambda_2) - \Omega_{\varphi, \varphi}^{\text{pt}}(u_i, \lambda_1, \lambda_2) \right) \widehat{R}_t \otimes \widehat{R}_t$$

where $\Omega_{\varphi, \varphi}^{\text{pt}}$ is the propagator for A_1 -singularity with (in the notation of Section 6.2.1) $\Delta = \Delta_i$ and $t = u_i$. The residue of the 1-form (6.32) takes the form

$$(6.37) \quad \left(\text{Res}_{\lambda=u_i} d\lambda \left(\frac{\lambda}{4} : \phi_\varphi(t, \lambda) \phi_\varphi(t, \lambda) : + \frac{\lambda}{2} \partial_{\lambda_1} \partial_{\lambda_2} \left(\Omega_{\varphi, \varphi}(t, \lambda_1, \lambda_2) - \Omega_{\varphi, \varphi}^{\text{pt}}(u_i, \lambda_1, \lambda_2) \right) \Big|_{\lambda_1=\lambda_2=\lambda} \right) + \frac{1}{8} \right) \mathcal{A}_t \otimes \mathcal{A}_t.$$

6.3.7. Residue of the Virasoro term. Let us single out the terms in

$$(6.38) \quad -\frac{\lambda d\lambda}{2} \sum_{i=1}^N : \phi_{\gamma_i}(t, \lambda) \phi_{\gamma^i}(t, \lambda)$$

which contribute to the residue at $\lambda = u_i$. Let us decompose $\gamma_i = \alpha_i + (\gamma_i|\varphi)\varphi/2$ and $\gamma^i = \alpha^i + (\gamma^i|\varphi)\varphi/2$, where α_i and α^i are invariant with respect to the local monodromy around $\lambda = u_i$. Note that

$$\varphi = \sum_{i=1}^N \gamma_i(\gamma^i|\varphi) = \sum_{i=1}^N (\alpha_i + (\gamma_i|\varphi)\varphi/2)(\gamma^i|\varphi)$$

and since

$$\sum_{i=1}^N (\gamma_i|\varphi)(\gamma^i|\varphi) = (\varphi|\varphi) = 2$$

the above equality implies that

$$\sum_{i=1}^N \alpha_i(\gamma^i|\varphi) = 0.$$

similarly $\sum_{i=1}^N \alpha^i(\gamma_i|\varphi) = 0$. Therefore, (6.38) can be written in the following form

$$-\frac{\lambda d\lambda}{2} \left(\sum_{i=1}^N : \phi_{\alpha_i}(t, \lambda) \phi_{\alpha^i}(t, \lambda) : + \frac{1}{2} : \phi_\varphi(t, \lambda) \phi_\varphi(t, \lambda) : \right)$$

The terms involving $\phi_{\alpha_i}(t, \lambda)$ and $\phi_{\alpha^i}(t, \lambda)$ are analytic at $\lambda = u_i$. Note that the remaining term cancels a corresponding term in the residue (6.37).

6.3.8. Proof of Theorem 6.7. We have already reduced the proof of Theorem 6.7 to proving that the total ancestor potential satisfies the HQEs (6.24). Using the same argument as in Section 6.3.5 we may reduce the proof to the case when $u_i(t) \neq 0$ for all i . Recalling the computations from Sections 6.3.5, 6.3.6, and 6.3.7, we get that it remains

only to check the following identity:

$$\begin{aligned} & \sum_{i=1}^N \operatorname{Res}_{\lambda=u_i} \lambda d\lambda \left(-\lambda^{-2} C(t, \lambda) + \frac{1}{2} \partial_{\lambda_1} \partial_{\lambda_2} \left(\Omega_{\varphi, \varphi}(t, \lambda_1, \lambda_2) - \Omega_{\varphi, \varphi}^{\text{pt}}(u_i, \lambda_1, \lambda_2) \right) \Big|_{\lambda_1=\lambda_2=\lambda} \right) = \\ & -\frac{N}{8} + \frac{N(h+1)}{12h} + \operatorname{Res}_{\lambda=0} \frac{d\lambda}{\lambda} C(t, \lambda). \end{aligned}$$

Let us simplify the LHS. Recalling the definition of $C(t, \lambda)$ and decomposing $\gamma_j = \alpha_j + (\gamma_j|\varphi)\varphi/2$ and $\gamma^j = \alpha^j + (\gamma^j|\varphi)\varphi/2$ we get

$$\begin{aligned} \lambda^{-2} C(t, \lambda) = & \partial_{\lambda_1} \partial_{\lambda_2} \left(\sum_{j=1}^N \left(\Omega_{\alpha_j, \alpha^j}(t, \lambda_1, \lambda_2) - \Omega_{\alpha_j, \alpha^j}(0, \lambda_1, \lambda_2) \right) + \right. \\ & \left. + \frac{1}{2} \left(\Omega_{\varphi, \varphi}(t, \lambda_1, \lambda_2) - \Omega_{\varphi, \varphi}(0, \lambda_1, \lambda_2) \right) \right) \Big|_{\lambda_1=\lambda_2=\lambda}. \end{aligned}$$

Note that

$$\Omega_{\alpha_j, \alpha^j}(t, \lambda_1, \lambda_2) - \Omega_{\alpha_j, \alpha^j}(0, \lambda_1, \lambda_2) \Big|_{\lambda_1=\lambda_2=\lambda}$$

is analytic at $\lambda = u_i$ because the periods $I_{\alpha_j}^{(m)}(t, \lambda)$ and $I_{\alpha^j}^{(m)}(t, \lambda)$ are analytic at $\lambda = u_i$. Therefore, the LHS takes the form

$$(6.39) \quad \sum_{i=1}^N \operatorname{Res}_{\lambda=u_i} \frac{\lambda d\lambda}{2} \partial_{\lambda_1} \partial_{\lambda_2} \left(\sum_{j=1}^N \Omega_{\varphi, \varphi}(0, \lambda_1, \lambda_2) - \Omega_{\varphi, \varphi}^{\text{pt}}(u_i, \lambda_1, \lambda_2) \right) \Big|_{\lambda_1=\lambda_2=\lambda}$$

Let us recall that the derivatives of the propagator can be expressed in terms of the periods (see Lemma 3.14)

$$\partial_{\lambda_1} \partial_{\lambda_2} \Omega_{\alpha, \beta}(t, \lambda_1, \lambda_2) = \partial_{\lambda_2} \left(\frac{1}{\lambda_1 - \lambda_2} (I_{\alpha}^{(0)}(t, \lambda_1), (\lambda_2 - E \bullet) I_{\beta}^{(0)}(t, \lambda_2)) \right).$$

We have (see Proposition 3.16)

$$\frac{1}{\lambda_1 - \lambda_2} (I_{\varphi}^{(0)}(0, \lambda_1), (\lambda_2 - E \bullet) I_{\varphi}^{(0)}(0, \lambda_2)) = \frac{2}{\lambda_1 - \lambda_2} + \dots$$

where the dots stand for terms regular at $\lambda_1 = \lambda_2$ and that do not contribute to the residue. Recall that

$$I_{\text{pt}}^{(0)}(u_i, \lambda) = \frac{2}{\sqrt{2(\lambda - u_i)}} \frac{du_i}{\sqrt{\Delta_i}}.$$

We get

$$\partial_{\lambda_1} \Omega_{\varphi, \varphi}^{\text{pt}}(u_i, \lambda_1, \lambda_2) = \frac{2}{\lambda_1 - \lambda_2} \frac{\sqrt{\lambda_2 - u_i}}{\sqrt{\lambda_1 - u_i}} = \frac{2}{\lambda_1 - \lambda_2} - \frac{1}{\lambda_1 - u_i} - \frac{\lambda_1 - \lambda_2}{4(\lambda_1 - u_i)^2} + \dots$$

The residue (6.39) takes the form

$$-\sum_{i=1}^N \operatorname{Res}_{\lambda=u_i} \frac{\lambda d\lambda}{8(\lambda - u_i)^2} = -\frac{N}{8}.$$

Comparing with the RHS of the identity that we would like to prove we get that we need to verify that

$$\operatorname{Res}_{\lambda=0} \frac{d\lambda}{\lambda} C(t, \lambda) = -\frac{N(h+1)}{12h}.$$

Note that the differential with respect to t

$$dC(t, \lambda) = \lambda^2 \sum_{i=1}^N I_{\gamma_i}^{(1)}(t, \lambda) \bullet I_{\gamma_i}^{(1)}(t, \lambda)$$

has a zero at $\lambda = 0$ of order at least 2. Therefore, the residue that we would like to compute is independent of t . Let us pick $t = \mathbf{1}$. We get

$$C(\mathbf{1}, \lambda) = \lambda^2 \partial_{\lambda_1} \partial_{\lambda_2} \sum_{i=1}^N \left(\Omega_{\gamma_i, \gamma_i}(0, \lambda_1 - 1, \lambda_2 - 1) - \Omega_{\gamma_i, \gamma_i}(0, \lambda_1, \lambda_2) \right) \Big|_{\lambda_1 = \lambda_2 = \lambda},$$

where we used the translation invariance $I^{(m)}(\mathbf{1}, \lambda) = I^{(m)}(0, \lambda - 1)$. The above expression can be computed explicitly. We have

$$I_{\gamma_i}^{(0)}(0, \lambda) = \lambda^{-m_i/h} \llbracket \phi_i(x) dx \rrbracket$$

and $\gamma^i = \gamma_{N+1-i}$, so

$$\partial_{\lambda_1} \Omega_{\gamma_i, \gamma_i}(0, \lambda_1, \lambda_2) = \frac{(\lambda_2/\lambda_1)^{m_i/h}}{\lambda_1 - \lambda_2}.$$

A straightforward computation yields

$$C(\mathbf{1}, \lambda) = \frac{1}{2} \sum_{i=1}^N \frac{m_i}{h} \left(\frac{m_i}{h} - 1 \right) \left(1 - \frac{\lambda^2}{(\lambda - 1)^2} \right).$$

The residue

$$\operatorname{Res}_{\lambda=0} \frac{d\lambda}{\lambda} C(t, \lambda) = \frac{1}{2} \sum_{i=1}^N \frac{m_i}{h} \left(\frac{m_i}{h} - 1 \right).$$

The Coxeter exponents m_i ($1 \leq i \leq N$) of type A , D , or E are well known (see Table 1). One can easily check that the above sum coincides with $-\frac{N(h+1)}{12h}$. \square

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