LECTURE NOTES ON GROMOV–WITTEN THEORY AND REPRESENTATIONS OF AFFINE LIE ALGEBRAS

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1. Introduction

The Gromov–Witten (shortly GW) invariants of a compact Kähler manifold X are defined as the number of holomorphic maps from Riemann surfaces of fixed genus to X satisfying, various constraints. The rigorous definition of the invariants is a difficult problem. One has to develop integration theory on the moduli spaces $\overline{\mathcal{M}}_{g,n}(X,d)$ of equivalence classes of stable maps $f : (\Sigma, p_1, \dots, p_n) \to X$, where Σ is a genus-g, nodal Riemann surface, $p_i, 1 \le i \le n$ are marked points, and $f_*([\Sigma_i]) = d \in H_2(X;\mathbb{Z})$. Such spaces are in general not manifolds or orbifolds. In fact they might have several irreducible components of different dimensions. Nevertheless, they have the properties of compact complex orbifolds. To the best of my knowledge, there are three different approaches which were invented in order to deal with the fundamentals of GW theory. The first one is entirely algebraic, it is based on the intersection theory on Delign-Mumford stacks (see [13]), and it amounts to constructing a virtual fundamental cycle (see [11], [2]). The other two approaches are both analytic, but quite different in nature. One can identify the moduli space as the zero locus of a section (constructed via Cauchy-Riemann equations) of a Banach bundle. The problem is that the section is not transversal. A natural idea is to deform the section so that we achieve transversality (see [12]). Finally, the third approach is still under construction, and it consists of generalizing the notion of smoothness, so that we can actually call the moduli spaces manifolds (polyfolds to be precise) (see [7]).

In my lectures I'm not going to address any of the foundational issues. All of the above mentioned approaches are quite technical and they will take us in a different direction. My goal is to show what are the moduli spaces good for and to point towards some new directions for researching. The Gromov–Witten invariants are organized in a generating function $\mathcal{D}_X$ called the total descendent potential of X. There are two problems which are my main motivation. The first one is the so called Virasoro conjecture. It says that a certain sequence of linear differential operators, commuting as the holomorphic vector fields on the circle, annihilates $\mathcal{D}_X$. This problem is still open and it is proved only for certain types of manifolds. The second problem is motivated by the Witten's conjecture (see [14]), proved by Kontsevich (see [10]), which says that the Gromov–Witten invariants of a point are governed (i.e. uniquely determined) by the KdV hierarchy. It is known that the Gromov–Witten invariants of $\mathbb{C}P^1$ are governed by the Extended Toda Hierarchy and for some other classes of manifolds (e.g. toric manifolds, Grassmanians, flag manifolds) it is expected that a similar hierarchy should exist.

My first goal is to prove Givental's formula for the total descendent potential of $\mathbb{C}P^n$. The proof (due to A. Givental) is motivated by the localization argument of M. Kontsevich [9]. However, it relies also on the so called quantum Riemann-Roch theorem as well as on the so called materialization phenomena in equivariant quantum cohomology. I am going to follow closely the articles [3, 5, 6] and the notes that I took from Givental's class when I was a graduate student at Berkeley. Then I want to turn to the applications of this formula. First, this is almost an immediate corollary from the formula, proof of the Virasoro constraints for $\mathbb{C}P^n$. Then I want to explain a very interesting relation between the representation theory of affine Lie algebras and Picard–Lefschetz periods of simple singularities. In particular, I will describe a tool which looks very promising for addressing the problem of finding an integrable hierarchy which governs the GW invariants of $\mathbb{C}P^n$. Most of the theorems I am planning to prove can be generalized to toric manifolds, but just to simplify the exposition I'm going to work only with $\mathbb{C}P^n$.

2. Gromov–Witten theory of the point

In this lecture I would like to point out several results which I would like to generalize for $\mathbb{C}P^n$.

2.1. **Witten's conjecture.** Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of equivalence classes of nodal Riemann surfaces of genus g, equipped with n marked points. The equivalence relation is given by a diffeomorphism (i.e. re-parameterization of the surface) $\phi : (\Sigma', j') \to (\Sigma, j)$ such that $\phi^* j = j'$ and $\phi(p_i') = p_i$. By definition, $\overline{\mathcal{M}}_{g,n}$ is empty if $2g-2+n \leq 0$, i.e., $(g,n) = (0,0), (0,1), (0,2), (1,0)$. For all other pairs (g,n), the points in the moduli space have only finite order automorphisms and in fact $\overline{\mathcal{M}}_{g,n}$ can be equipped with the structure of a compact complex orbifold of complex dimension $3g-3+n$ (equipping $\overline{\mathcal{M}}_{g,n}$ with an orbifold structure can be done with the theory of Strebel differentials see [10] as well as the preprint [15]).

Put

$$\langle \psi^{k_1}, \ldots, \psi^{k_n} \rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \ldots \psi_n^{k_n},$$

where ψ_i is the 1-st Chern class of the Line bundle $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}$ of cotangent lines at the i-th marked point. By definition the integral is 0 if the moduli space is empty or the degree of the ψ-classes does not match the dimension of $\overline{\mathcal{M}}_{g,n}$. Let

$$\mathcal{D}_{\rm pt}(\mathbf{t}) = \exp\Big(\sum \frac{1}{n!}\langle \psi^{k_1},\dots,\psi^{k_n}\rangle_{g,n} t_{k_1}\dots t_{k_n}\epsilon^{2g-2}\Big),$$

where the sum is over all non-negative integers $g, n, k_1, \dots, k_n$, and $\mathbf{t} = (t_0, t_1, \dots)$ is a sequence of formal variables. As it will be seen later, it is convenient to work with another sequence of formal variables $\mathbf{q} = (q_0, q_1, \dots)$ such that $q_k = t_k$, for $k \neq 1$ and $q_1 = t_1 - 1$.

Let

$$\Gamma^{\pm} = \exp\Big(\pm\sum_{k\geq 0}\frac{(2\lambda)^{k+1/2}}{(2k+1)!!}\,\frac{q_k}{\epsilon}\Big)\exp\Big(\mp\sum_{k\geq 0}\frac{(2k-1)!!}{(2\lambda)^{-k-1/2}}\,\epsilon\partial_{q_k}\Big).$$

Then we say that $\tau(q_0, q_1, \dots; \epsilon)$ is a tau-function of the KdV hierarchy if the following equations are satisfied:

$$\mathrm{Res}_{\lambda=\infty}\lambda^n\frac{d\lambda}{\sqrt{\lambda}}\Big(\Gamma^+\otimes\Gamma^- - \Gamma^-\otimes\Gamma^+\Big)\tau\otimes\tau) = 0, \quad n\geq 0,$$

where the equations should be interpreted as follows: $\tau\otimes\tau = \tau(\mathbf{q}')\tau(\mathbf{q}'')$, the vertex operator preceding (resp. following) the tensor product acts on $\tau(\mathbf{q}')$ (resp. $\tau(\mathbf{q}'')$). Furthermore, let $\mathbf{q}' = \mathbf{q}+\mathbf{y}$ and $\mathbf{q}'' = \mathbf{q}-\mathbf{y}$. Notice that

$$\Gamma^{\pm}\otimes\Gamma^{\mp} = \exp\Big(\pm 2\sum_{k\geq 0}\frac{(2\lambda)^{k+1/2}}{(2k+1)!!}\,\frac{y_k}{\epsilon}\Big)\exp\Big(\mp\sum_{k\geq 0}\frac{(2k-1)!!}{(2\lambda)^{-k-1/2}}\,\epsilon\partial_{y_k}\Big),$$

and $\tau(\mathbf{q}')\tau(\mathbf{q}'') = \tau(\mathbf{q}+\mathbf{y})\tau(\mathbf{q}-\mathbf{y})$. Using the Taylor's formula we expand in the powers of $\mathbf{y}$. The result is a power series in $\mathbf{y}$ with coefficients Laurent series in λ^{-1}, whose coefficients are quadratic polynomials in the partial derivatives of τ. The vanishing of the residues for all $n \geq 0$ means that all quadratic polynomials in front of the negative powers of λ must vanish. For example, if $n = 0$ then the coefficient in front of y_1 gives us the celebrated KdV equation:

$$\partial_{q_1}v = vv_x + \frac{\epsilon^2}{12}v_{xxx}, \quad v = \epsilon^2(\log\tau)_{xx},$$

where $x := q_0$ and the subscript x means partial derivative with respect to x. Similarly, by comparing the coefficients in front of other powers of $\mathbf{y}$ we get that v is a solution to a system of PDE's of the following type:

$$\partial_{q_n}v = P_n(v, v_x, v_{xx}, \dots), \quad n\geq 1,$$

where P_n are differential polynomials in v. It is not obvious, but it can be shown that the above system of PDE's coincide with the so called KdV hierarchy, which justifies why we called τ a tau-function of KdV.

Theorem 2.1 (Witten's conjecture). *The generating function $\mathcal{D}_{\rm pt}$ is a tau-function of the KdV hierarchy.*

The theorem was first proved by Kontsevich. His proof had some technical subtleties which later on were overcome. One proof that I like a lot is based on Mirzahani's formula for the volume of the moduli space $\overline{\mathcal{M}}_{g,n}(L_1,\dots,L_n)$ of Riemann surfaces with boundaries of fixed lengths. It was notice by N. Do that Mirzahani's formula can be used to avoid the subtleties in the Kontsevich's argument. Namely, if we let $L_i\to\infty$ then the leading terms of Mirzahani's formula can be identified with a sum over ribbon graphs. This observation allows us to write $\mathcal{D}_{\rm pt}$ as a sum over ribbon graphs. The later coincide with the asymptotic of a certain matrix integral and it is easy to see that we have a tau-function of KdV (see [10]).

Let us show how the KdV equation can be used to compute the intersection number $\langle\psi\rangle_{1,1}=\int_{\overline{\mathcal{M}}_{1,1}}\psi_1$. Start by differentiating the KdV equation by q_3, then substitute $\mathbf{t}=0$ and compare the coefficients in front of ϵ^2. We get the following identity:

$$\langle 1,1,\psi,\psi^3\rangle_{1,4}=\langle 1,1,\psi^3\rangle_{1,3}\langle 1,1,1\rangle_{0,3}+\frac{1}{12}\langle 1,1,1,1,1,\psi^3\rangle_{1,1}. \tag{2.1}$$

Later on (see Section 4) we prove the following identities:

$$\langle\psi^{k_1},\dots,\psi^{k_n},1\rangle_{g,n+1}=\sum_{i=1}^n\langle\psi^{k_1},\dots,\psi^{k_i-1},\dots,\psi^{k_n}\rangle_{,g,n}$$

and

$$\langle\psi^{k_1},\dots,\psi^{k_n},\psi\rangle_{g,n+1}=(2g-2+n)\langle\psi^{k_1},\dots,\psi^{k_n}\rangle_{g,n},$$

known respectively as *the string* and *the dilaton equations*. With the help of these two identities it is easy to see that (2.1) implies that: $\langle\psi\rangle_{1,1}=1/24$.

The string equations and the dilaton equation imply the following constraints on the generating function $\mathcal{D}_{\rm pt}$. Put

$$L_{-1}:=\frac{t_0^2}{2}-\frac{\partial}{\partial t_0}+\sum_{m\geq 0}t_{m+1}\frac{\partial}{\partial t_m}=\frac{q_0^2}{2}+\sum_{m\geq 0}q_{m+1}\frac{\partial}{\partial q_m},$$

and

$$L_0=-\frac{3}{2}\frac{\partial}{\partial t_1}-\frac{1}{16}+\sum_{m\geq 0}(m+\frac{1}{2})t_m\frac{\partial}{\partial t_m}=-\frac{1}{16}+\sum_{m\geq 0}(m+\frac{1}{2})q_m\frac{\partial}{\partial q_m}.$$

Then $L_{-1}\mathcal{D}_{\rm pt}=L_0\mathcal{D}_{\rm pt}=0$. In fact there is a whole sequence of differential operators L_n, $n\geq -1$, commuting like holomorphic vector fields on the circle (i.e. $z^{n+1}\partial/\partial z$), such that

Theorem 2.2 (Virasoro constraints)**.** *$L_n\mathcal{D}_{\rm pt}=0$, $n\geq -1$.*

This theorem follows from a result of Kac and Schwarz, which says that a tau function of the KdV hierarchy satisfying the string equation satisfies Virasoro constraints.

2.2. **Another point of view of KdV.** The Hirota quadratic equations (shortly HQE) which were given above can be interpreted as the Plücker relations of the embedding of some infinite dimensional Grassmanian into some projective space. Yet another interpretation of HQE can be obtained via the representation theory of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. By definition, the affine Lie algebra $\widehat{\mathfrak{g}}$ corresponding to a simple lie algebra $\mathfrak{g}$ is defined by

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] + \mathbb{C}\, K + \mathbb{C}\, d,$$

where K is a central element, $d = t\partial_t$ and the commutator is defined by

$$[X \otimes t^n, Y \otimes t^m] = [X, Y] \otimes t^{n+m} + n\delta_{n,-m}(X, Y)K,$$

where $(\ ,\ )$ is an invariant bi-linear form in $\mathfrak{g}$. We will be interested only in affine Lie algebras of type A, D, or E, in which case the invariant form can be characterized uniquely by demanding that all roots have length $\sqrt{2}$.

Put $\widehat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}\, K + \mathbb{C}\, d$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{h}$. Pick a set of simple roots $\alpha_1, \ldots, \alpha_l$ and let $\alpha_0^\vee = K - \theta^\vee$ where $l = \dim_{\mathbb{C}} \mathfrak{h}$, θ is the longest root of $\mathfrak{g}$ and for a root $\alpha \in \mathfrak{h}^*$ we denote by $\alpha^\vee \in \mathfrak{h}$ the corresponding coroot, which in the A, D, or E case coincides with the dual to α with respect to the invariant form. It turns out (this is one of the things that I'm planning to prove) that the irreducible, integrable, highest weight representations of $\widehat{\mathfrak{g}}$ are classified by weights $\Lambda \in \widehat{\mathfrak{h}}^*$, s.t., $\langle \Lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}, 0 \leq i \leq l$. The representation corresponding to the weight Λ_0 defined by: $\langle \Lambda_0, \mathfrak{h} \rangle = \langle \Lambda_0, d \rangle = 0$, and $\langle \Lambda_0, K \rangle = 1$, is called the basic representation. The basic representation can be realized in many different ways by differential operators acting on certain Fock spaces. All these realizations are parametrized by conjugacy classes of the Weyl group of $\mathfrak{g}$. This is something I'm planning to explain later as well. For example, in the case of $\widehat{\mathfrak{sl}}_2$ we have the following realization. Let

$$H_{2k+1} = \begin{bmatrix} 0 & t^k \\ t^{k+1} & 0 \end{bmatrix}, \quad A_{2k} = \begin{bmatrix} -t^k & 0 \\ 0 & t^k \end{bmatrix}, \quad A_{2k+1} = \begin{bmatrix} 0 & t^k \\ -t^{k+1} & 0 \end{bmatrix}, \quad k \in \mathbb{Z}.$$

The basic representation of $\widehat{\mathfrak{sl}}_2$ is realized on the vector space $L(\Lambda_0) = \mathbb{C}[[x_1, x_3, \ldots]]$ of formal power series by the following formulas:

$$A(z) = \textstyle\sum_{n\in\mathbb{Z}} z^{-n} A_n \mapsto \frac{1}{2}\Big(\Gamma^+(z) - 1\Big),$$

$$H_j \mapsto \tfrac{\partial}{\partial x_j}, \quad H_{-j} \mapsto j x_j,$$

$$K \mapsto 1, \quad d \mapsto \tfrac{1}{2} A_0 - \textstyle\sum_j j x_j \frac{\partial}{\partial x_j},$$

where j (including the summations below) is a positive odd number and

$$\Gamma^{\pm}(z) = \exp\Big(\pm 2 \sum_j z^j x_j\Big) \exp\Big(\mp 2 \sum_j \frac{z^{-j}}{j} \frac{\partial}{\partial x_j}\Big).$$

Theorem 2.3. *A function $\tau \in L(\Lambda_0)$ belongs to the orbit $\widehat{\mathrm{SL}_2}\cdot 1$ iff it satisfies the following HQE:*

$$\mathrm{res}_{z=\infty}\frac{dz}{z}\Big(\Gamma^+(z)\otimes\Gamma^-(z)\Big)(\tau\otimes\tau) = (8l+1)(\tau\otimes\tau),$$

where $l = \frac{1}{2}\sum_j j(x'_j - x''_j)(\partial_{x'_j} - \partial_{x''_j})$.

Finally, one can show that the solutions of the above HQE correspond to tau-functions of the KdV hierarchy via the substitution: $q_k = \epsilon(2k+1)!!x_{2k+1}$.

3. The virtual tangent bundle

Let $\overline{\mathcal{M}}_{g,n}(X;d)$ be the moduli space of equivalence classes of stable maps

$$f : (\Sigma, p_1, \ldots, p_n) \to X,$$

where Σ is a genus g nodal Riemann surface, p_i are marked points, and f is a continuous map, holomorphic on each irreducible component, and $f_*[\Sigma] = d \in H_2(X;\mathbb{Z})$. By definition, stable means that the group of automorphisms of the configuration $(\Sigma, p_1, \ldots, p_n; f)$ is finite. Two stable maps $f : \Sigma \to X$ and $f' : \Sigma' \to X$ are equivalent if there is a diffeomorphism $\phi : \Sigma' \to \Sigma$ such that $j' = \phi^* j$, $\phi(p'_i) = p_i$ and $f\circ\phi = f'$, where j' and j are the complex structures respectively on Σ and Σ'.

The moduli spaces are compact, however in most cases they are not manifolds or orbifolds. The reason for this is that the infinitesimal deformations of a stable map (Σ, p, f) might have obstructions, so we can't always extend them to actual deformations. Nevertheless, one can define a homology cycle, called virtual fundamental cycle, such that the integration theory on the moduli space is the same as if $\overline{\mathcal{M}}_{g,n}(X,d)$ were compact complex orbifolds. In this section we compute the virtual tangent sheaf $\mathcal{T} \in K^0(\overline{\mathcal{M}}_{g,n}(X,d))$ whose stalk at a stable map is the formal difference of the infinitesimal deformations and the obstructions. The virtual fundamental cycle is in some sense the Euler class of the obstruction bundle. As we will see below the deformations and the obstructions of stable maps are classified by certain vector spaces whose dimension might vary, i.e., the obstructions in general give rise to a sheaf not to a bundle, so the construction of the virtual fundamental cycle is more complicated then just taking the Euler class of a bundle. In case the obstructions give rise to a bundle then the virtual fundamental cycle is the Euler class.

We consider a simplified version of the deformation theory of a stable map. Namely, let (Σ, p, f), $p = (p_1, \ldots, p_n)$, be a fixed stable map. We classify the infinitesimal deformations of the map f and their obstructions, keeping the Riemann surface and the marked points fixed. Choose an open covering $\{V_i\}$ of Σ by holomorphic disks and let U_i be coordinate charts of X such that $f(V_i) \subset U_i$. In each chart U_i we pick coordinates and so on each V_i the map

f is represented by a collection of holomorphic functions $u_i = (u_i^1, \dots, u_i^D)$, $D = \dim_{\mathbb{C}} X$. Finally, let g_{ji} be the transition functions between the charts U_i and U_j, i.e., $u_j = g_{ji}(u_i)$.

Case 1: 1-st order deformations. Let $\overline{u}_i = u_i + \epsilon v_i$ be first order deformations. Compare the coefficient in front of ϵ in the gluing identity $\bar{u}_j^a = g_{ji}^a(\bar{u}_i^1, \dots, \bar{u}_i^D)$. We get:

$$v_j^a = \sum_{b=1}^{D} \frac{\partial g_{ji}^a}{\partial u_i^b} v_i^b \quad , \tag{3.1}$$

which implies that the vector fields $\sum_a v_j^a \frac{\partial}{\partial u_j^a} \in \Gamma(V_j, f^*TX)$ glue to give a global section of f^*TX, i.e., the infinitesimal deformations are classified by $H^0(\Sigma, f^*TX)$.

Case 2: 2-nd order deformations. Let $\bar{u}_i = u_i + \epsilon v_i + \epsilon^2 w_i$ be a second order deformation. Comparing the coefficients in front of ϵ^2 in the gluing identity $\bar{u}_j^a = g_{ji}^a(\bar{u}_i)$ we get:

$$w_j^a = \sum_b \frac{\partial g_{ji}^a}{\partial u_i^b} w_i^b + \frac{1}{2} \frac{\partial^2 g_{ji}^a}{\partial u_i^b \partial u_i^c} v_i^b v_i^c,$$

i.e.,

$$\sum_a w_j^a \frac{\partial}{\partial u_j^a} = \sum_b w_i^b \frac{\partial}{\partial u_i^b} + \frac{1}{2} \sum_{a,b,c} \frac{\partial^2 g_{ji}^a}{\partial u_i^b \partial u_i^c} v_i^b v_i^c \frac{\partial}{\partial u_j^a}, \tag{3.2}$$

The LHS and the first sum on the RHS are elements respectively of $H^0(V_i, f^*TX)$ and $H^0(V_j, f^*TX)$. We denote the second term on the RHS by w_{ji}. A direct computation (using also formula (3.1)) shows that $w_{ki} = w_{kj} + w_{ji}$, i.e., $w = (w_{ji})$ give rise to a Cech cocycle. Let $[w] \in H^1(\Sigma, f^*TX)$ be the corresponding cohomology class, then formula (3.2) means that $[w] = 0$, so the obstructions belong to the cohomology group $H^1(\Sigma, f^*TX)$.

Let $\mathcal{T}_\Sigma$ be the sheaf of holomorphic vector fields on Σ which vanish at the marked points and at the nodes. A similar argument shows that $H^1(\Sigma, \mathcal{T}_\Sigma)$ classifies the deformations of the complex structure on Σ, and $H^0(\Sigma, \mathcal{T}_\Sigma)$ are the automorphisms of (Σ, p). Finally, for $s \in \mathrm{Sing}(\Sigma)$ let T_s' and T_s'' be the tangent spaces at s to the two branches of Σ that meet at s. Then $T_s' \otimes T_s''$ can be identified with a space of infinitesimal deformations of (Σ, p, f) which come from resolving s. Namely, let x and y be coordinates on the two branches and let $\epsilon \partial_x \otimes \partial_y \in T_s' \otimes T_s''$. In a neighborhood of s the Riemann surface is given by the equation $xy = 0$ and we resolve the singularity by deforming the equation into $xy = \epsilon$.

The above discussion motivates the following definition:

$$\mathcal{T}_{(\Sigma,p,f)} := H^1(\Sigma,\mathcal{T}_\Sigma) - H^0(\Sigma,\mathcal{T}_\Sigma) + \bigoplus_{s\in \mathrm{Sing}\Sigma} T_s'\otimes T_s'' + H^0(\Sigma, f^*TX) - H^1(\Sigma, f^*TX),$$

so when we vary (Σ, p, f) we get an element $\mathcal{T}\in K^0(\overline{\mathcal{M}}_{g,n}(X,d))$. Using the Riemann-Roch formula:

$$\dim_{\mathbb{C}} H^0(\Sigma, E) - \dim_{\mathbb{C}} H^1(\Sigma,E) = \mathrm{rk}(E)(1-g) + \int_\Sigma c_1(E),$$

we can easily compute that the virtual rank of $\mathcal{T}$, i.e., the virtual dimension of the moduli space $\overline{\mathcal{M}}_{g,n}(X,d)$ is: $3g-3+n+D(1-g)+\int_d c_1(TX)$.

Example 3.1. *If the degree is 0, i.e., the maps contracts the curve to a point. We have $\overline{\mathcal{M}}_{g,n}(X,0)=\overline{\mathcal{M}}_{g,n}\times X$. On the other hand $H^0(\Sigma, f^*TX)=T_{f(\Sigma)}X$, $H^1(\Sigma, f^*TX) = H^0(\Sigma,\mathcal{O}_\Sigma)\otimes T_{f(\Sigma)}X$, so the tangent bundle is given by*

$$\mathcal{T} = \mathcal{T}_{\overline{\mathcal{M}}_{g,n}} + TX - \mathbb{E}\otimes TX,$$

where $\mathbb{E}$ is the rank g bundle on $\overline{\mathcal{M}}_{g,n}$ whose fiber at (Σ,p) is given by $H^1(\Sigma,\mathcal{O}_\Sigma)$ (the dual to this bundle is known as the Hodge bundle*). Since the obstructions form a bundle we have that the virtual fundamental cycle is the Poincare dual to the Euler class, i.e.,*

$$\int_{\overline{\mathcal{M}}_{g,n}(X,0)} \alpha = \int_{\overline{\mathcal{M}}_{g,n}\times X} \alpha \smile \mathrm{Euler}(\mathbb{E}\otimes TX).$$

Example 3.2. *If X is a manifold whose tangent spaces are spanned by global vector fields $H^0(X,TX)$ (e.g. Grassmanians, flag manifolds) then $H^1(\Sigma, f^*TX)=0$ for all genus-0 curves Σ. This implies that the obstructions vanish so the moduli space $\overline{\mathcal{M}}_{0,n}(X,d)$ is a compact complex orbifold.*

4. Universal identities in Gromov–Witten theory

Let X be a projective manifold whose cohomology algebra has only even degree non-zero classes. We denote by $\{\phi_a\}, 1\leq a\leq N$ a basis of $H^*(X)$. By definition the descendent GW invariants of X are the following correlators:

$$\langle \phi_{a_1}\psi^{k_1},\ldots,\phi_{a_n}\psi^{k_n}\rangle_{g,n,d} := \int_{[\overline{\mathcal{M}}_{g,n}(X,d)]} \psi_1^{k_1}\ldots\psi_n^{k_n}\mathrm{ev}^*(\phi_{a_1}\otimes\ldots\otimes\phi_{a_n}), \tag{4.1}$$

where ψ_i is the first Chern class of the line bundle $\mathbb{L}_i\to\overline{\mathcal{M}}_{g,n}(X,d)$ whose fiber at (Σ,p,f), $p=(p_1,\ldots,p_n)$ is the cotangent line $T^*_{p_i}\Sigma$, $\mathrm{ev}(\Sigma,p,f):=(f(p_1),\ldots,f(p_n))\in X^n$ is the evaluation map and the integration is over the virtual fundamental cycle. In this section we prove some identities between the correlators (4.1).

4.1. **The universal curve.** Let (Σ, p, f) be a stable map. A point on some irreducible component is called special if it is either a marked point or a nodal point of Σ. Notice that the condition that the map is stable means that for each irreducible component Σ' of Σ contracted by f the configuration (Σ', special points) is stable, i.e., $2g'-2+n'>0$, where g' is the genus of Σ' and n' is the number of special points.

Let $\pi:\overline{\mathcal{M}}_{g,n+1}(X,d)\longrightarrow\overline{\mathcal{M}}_{g,n}(X,d)$ be the map forgetting the last marked point and contracting all unstable components. In fact π is the universal curve , i.e., if $\sigma=(\Sigma,p,f)\in\overline{\mathcal{M}}_{g,n}(X,d)$ is a stable map then there is a canonical identification between the fiber $\pi^{-1}(\sigma)$ and Σ. Indeed, if $\pi(\Sigma',p',f')=(\Sigma,p,f)$ then we either have $\Sigma'=\Sigma$, $f'=f$ or $\Sigma'=\Sigma\cup\Sigma''$, where the irreducible component Σ'' is a copy of $\mathbb{C}P^1$, the map f contracts Σ'', and the only marked points on Σ'' are p_{n+1} and p_i for some $i, 1\leq i\leq n$. In the first case we map (Σ',p',f') to p_{n+1} and in the second to p_i.

Let us denote by $s_i:\overline{\mathcal{M}}_{g,n}(X,d)\longrightarrow\overline{\mathcal{M}}_{g,n+1}(X,d)$ the section which to each $\sigma=(\Sigma,p,f)$ assigns the point in $\pi^{-1}(\sigma)\cong\Sigma$ which corresponds to the marked point p_i. Finally, let $D_i:=[s_i(\overline{\mathcal{M}}_{g,n}(X,d))]$ be the divisor corresponding to the section s_i.

4.2. **The string equation.** The following identity is known as the string equation:

$$\langle\phi_{a_1}\psi^{k_1},\dots,\phi_{a_n}\psi^{k_n},1\rangle_{g,n+1,d}=\sum_{i=1}^{n}\langle\phi_{a_1}\psi^{k_1},\dots,\phi_{a_i}\psi^{k_i-1},\dots\phi_{a_n}\psi^{k_n},1\rangle_{g,n,d}.$$

The proof is based on the following relations in the cohomology of $\overline{\mathcal{M}}_{g,n+1}(X,d)$:

Fact 1: if $1\leq i\neq j\leq n$ then $[D_i][D_j]=0$, where for a divisor D we denote by $[D]$ its Poincaré dual. This is obvious because from the definitions it follows that D_i and D_j are disjoint.

Fact 2: For $1\leq i\leq n$ we have $\psi_i[D_i]=0$. The reason for this is that the restriction of the bundle $\mathbb{L}_i$ to D_i is a trivial bundle. Indeed, notice that D_i can be viewed as the image of a gluing map

$$\mathrm{gl}_i:\overline{\mathcal{M}}_{g,n}(X,d)\times\overline{\mathcal{M}}_{0,3}\longrightarrow\overline{\mathcal{M}}_{g,n+1}(X,d).$$

On the other hand $\mathrm{gl}_i^*\mathbb{L}_i$ is a cotangent line bundle on $\overline{\mathcal{M}}_{0,3}=\mathrm{pt}$, and hence it is trivial.

Fact 3: Let $\overline{\mathbb{L}}_i\to\overline{\mathcal{M}}_{g,n+1}(X,d)$ be the pullback via π of the bundle $\mathbb{L}_i\to\overline{\mathcal{M}}_{g,n}(X,d)$. Then $\psi_i=\overline{\psi}_i+[D_i]$, where $\overline{\psi}_i=c_1(\overline{\mathbb{L}}_i)$.

Note that the line bundles $\mathbb{L}_i$ and $\overline{\mathbb{L}}_i$ are the same outside of the divisor locus D_i. Let (Σ',p',f') be a point on D_i. Write $\Sigma'=\Sigma\cup\Sigma''$, where Σ'' is a copy of $\mathbb{C}P^1$ which carries the marked points p_i and p_{n+1}. We pick local coordinates x and y respectively on Σ and Σ'' near the nodal point, let y_i be a local coordinate on Σ'' near p_i. We may assume that $y_i=1/y$. Let ϵ be the coordinate on the moduli space corresponding to resolving the node via $xy = \epsilon$. We have the following relations:

$$dx = -\epsilon y^{-2} dy = \epsilon dy_i, \text{ i.e., } dy_i = \epsilon^{-1} dx.$$

On the other hand dx and dy_i are local sections of $\overline{\mathbb{L}}_i$ and $\overline{\mathbb{L}}_i$ which generate the corresponding sheaves of holomorphic sections and $\epsilon = 0$ is the local equation of the divisor D_i. Using the correspondence between divisors and line bundles on complex manifolds we get that: $\mathbb{L}_i = \overline{\mathbb{L}}_i \otimes \mathcal{L}(D_i)$, i.e., $\psi_i = \overline{\psi}_i + [D_i]$.

To finish the proof of the string equation it remains only to notice that

$$\psi_i^k = \psi_i^{k-1}(\overline{\psi}_i + [D_i]) = \psi_i^{k-1}\overline{\psi}_i = \ldots = \overline{\psi}_i^{k-1}(\overline{\psi}_i + [D_i]).$$

4.3. **Pushforward in cohomology.** Let $p : X \to Y$ be a proper map between complex manifolds. Then there is a well defined pushforward map $p_* : H^*(X) \to H^{*-2r}(Y)$ where $r = \dim_{\mathbb{C}}(X) - \dim_{\mathbb{C}}(Y)$. Intuitively, p_* is integration along the fiber. The precise definition can be done in three steps.

Step 1: If $\pi : X \times Y \to Y$ is the projection on the second factor and $\theta \in H^*(X \times Y)$ is a cohomology class whose restriction to each fiber $X \times \{y\}$ is compactly supported, then we define $\pi_*(\theta)$ by integrating over the fiber. More precisely for each $y \in Y$ we take a coordinate chart V near y such that, the restriction of θ to $X \times V$ can be represented by $\sum_I \theta_I(y) \wedge dy_{i_1} \wedge \ldots \wedge dy_{i_s}$, where the summation is over all multiindeces $I = (i_1, \ldots, i_s)$ and $\theta_I(y)$ is a differential form on X. Then the restriction of $\pi_*(\theta)$ to V is given by $\sum_I \Big(\int_X \theta_I(y) \Big) dy_{i_1} \wedge \ldots \wedge dy_{i_s}$. It is easy to see that all these local differential forms glue together and the resulting differential form represents the cohomology class $\pi_*(\theta)$.

Step 2: If $i : X \to Y$ is a closed embedding. Then by the Thom isomorphism we have $H^*(X) \cong H^{*+r}(N, N - X)$, where N is the normal bundle to X and r is the codimension of X in Y. By excision we have $H^*(N, N - X) = H^*(U, U - X) = H^*(Y, Y - X)$, where U is a tubular neighborhood of X in Y. If $\theta \in H^*(X)$ then $i_*(\theta) \in H^*(Y)$ is defined via the above sequence of isomorphisms and the natural map $H^*(Y, Y - X) \to H^*(Y)$.

Step 3: If $p : X \to Y$ is an arbitrary proper map. Then we factor p as $X \xrightarrow{g} X \times Y \xrightarrow{\pi} Y$, where $g(x) = (x, p(x))$ is the graph-map of p. We define $p_* := \pi_* \circ g_*$.

Let us remark that p_* is a $H^*(Y)$-module homomorphism, i.e., $p_*(a \smile p^*b) = p_*(a) \smile b$. Also in case X and Y are compact then p_* can be equivalently defined via Poincaré duality: $p_*(a) = P.D.[p_*(P.D.[a])]$.

4.4. **Dilaton equation.** Now we prove the dilaton equation:

$$\langle \phi_{a_1}\psi^{k_1}, \ldots, \phi_{a_n}\psi^{k_n}, \psi \rangle_{g,n+1,d} = (2g - 2 + n) \langle \phi_{a_1}\psi^{k_1}, \ldots, \phi_{a_n}\psi^{k_n} \rangle_{g,n,d}$$

First, let us prove that $\pi_*(\psi_{n+1}) = 2g-2+n$. We know that $\pi_*(\psi_{n+1})$ is a number, so it is enough to integrate ψ_{n+1} along any of the fibers $\pi^{-1}(\sigma)$, $\sigma = (\Sigma, p, f) \in \overline{\mathcal{M}}_{g,n}(X,d)$. We may assume that Σ is a smooth curve.

Let $\phi : \pi^{-1}(\sigma) \xrightarrow{\cong} \Sigma$ be the canonical identification. For each point $\sigma' = (\Sigma', p', f') \in \overline{\mathcal{M}}_{g,n+1}(X,d)$ we denote by l_{n+1} the cotangent line at the $(n+1)$-st marked point. If $\sigma' \in D_i$ then $\Sigma' = \Sigma \cup \Sigma''$ (see the discussion above about the universal curve) we denote by $\bar{l}_i$ the cotangent line to Σ at the nodal point common for Σ and Σ''. Let $\overline{\mathbb{L}}_{n+1}$ be the line bundle whose fiber at σ' is l_{n+1} if $\sigma' \notin D_i$ for all $1 \le i \le n$, or $\bar{l}_i$ if $\sigma' \in D_i$. It is not hard to see that the restriction of $\overline{\mathbb{L}}_{n+1}$ to the fiber $\pi^{-1}(\sigma)$ is canonically isomorphic to $\phi^*(T^*\Sigma)$. On the other hand the same comparison argument as in the proof of the string equation shows that $\mathbb{L}_{n+1} = \overline{\mathbb{L}}_{n+1} \otimes \mathcal{L}(D_1 + \ldots + D_n)$. Taking the Chern classes, and then integrating along the fiber $\pi^{-1}(\sigma) \cong \Sigma$ we get $\pi_*(\psi_{n+1}) = \int_\Sigma c_1(T^*X) + n = 2g-2+n$.

Notice that the restriction of $\mathbb{L}_{n+1}$ to D_i is a trivial bundle, so $\psi_{n+1}[D_i] = 0$. Therefore $\psi_i^k \psi_{n+1} = \overline{\psi}_i^k \psi_{n+1}$, because $\psi_i - \overline{\psi}_i = [D_i]$. Now the dilaton equation is easy to prove. We have:

$$\int_{\overline{\mathcal{M}}_{g,n+1}(X,d)} \phi_{a_1}\psi_1^{k_1} \ldots \phi_{a_n}\psi_n^{k_n}\psi_{n+1} = \int_{\overline{\mathcal{M}}_{g,n+1}(X,d)} \pi^*\left(\phi_{a_1}\psi_1^{k_1} \ldots \phi_{a_n}\psi_n^{k_n}\right)\psi_{n+1}$$

and using that $\int_{\overline{\mathcal{M}}_{g,n+1}(X,d)} = \int_{\overline{\mathcal{M}}_{g,n}(X,d)} \circ \pi_*$ we get

$$\int_{\overline{\mathcal{M}}_{g,n}(X,d)} \phi_{a_1}\psi_1^{k_1} \ldots \phi_{a_n}\psi_n^{k_n}\pi_*\psi_{n+1} = (2g-2+n)\int_{\overline{\mathcal{M}}_{g,n}(X,d)} \phi_{a_1}\psi_1^{k_1} \ldots \phi_{a_n}\psi_n^{k_n}.$$

4.5. **The divisor equation.** Using the same ideas as in the proof of the string and the dilaton equations, one can easily prove that if $p \in H^2(X)$ then

$$\begin{aligned}\langle \phi_{a_1}\psi_1^{k_1}, \ldots, \phi_{a_n}\psi_n^{k_n}, p\rangle_{g,n+1,d} &= \Big(\int_d p\Big)\langle \phi_{a_1}\psi_1^{k_1}, \ldots, \phi_{a_n}\psi_n^{k_n}\rangle_{g,n,d} + \\ &\quad + \sum_{i=1}^n \langle \phi_{a_1}\psi_1^{k_1}, \ldots, (\phi_{a_i}\cdot p)\psi_i^{k_i-1}, \ldots, \phi_{a_n}\psi_n^{k_n}\rangle_{g,n,d}\end{aligned}$$

The above identity is called *the divisor equation.* In fact, the same formula holds for all cohomology classes p of degree less or equal than 2.

5. Genus-0 GW theory

Let $\phi_a, 0 \le a \le N-1$ be a basis of $H^*(X;\mathbb{C})$. We assume that $\phi_0 = 1$ and that $\phi_1, \ldots, \phi_r$, $r = \dim_{\mathbb{C}} H^2(X;\mathbb{C})$ is a basis of the second-degree cohomology group. The genus-0 total descendent potential of X is defined by:

$$\mathcal{F}^{(0)}(\mathbf{t}) = \sum_{n,d} \frac{Q^d}{n!} \langle \mathbf{t}(\psi), \dots, \mathbf{t}(\psi) \rangle_{0,n,d},$$

where $\mathbf{t}(\psi) = \sum_{k,a} t_k^a \phi_a \psi^k$ and Q^d is an element of the group algebra of $H_2(X;\mathbb{Z})$. We may assume that $\phi_i \in H^2(X;\mathbb{Z})$, $1 \leq i \leq r$ are such that their Poincaré duals $\alpha_i \in H_2(X;\mathbb{Z})$ form a $\mathbb{Z}$-basis. We put $Q_i = Q^{\alpha_i}$ and then $Q^d = Q_1^{d_1} \dots Q_r^{d_r}$, where $d = d_1\alpha_1 + \dots + d_r\alpha_r$, i.e., $\mathcal{F}^{(0)}(\mathbf{t}) \in \mathbb{C}[[t_k^a]] \otimes \mathbb{C}[[Q_1^{\pm 1}, \dots, Q_r^{\pm 1}]]$. In fact if X is a Grassmanian then one can choose α_i to be represented by effective curve classes, and since $c_1(TX)$ is a multiple of the Kähler form we have $\int_{\alpha_i} c_1(TX) > 0$ which implies that $\mathcal{F}^{(0)}(\mathbf{t}) \in \mathbb{C}[[t_k^a]] \otimes \mathbb{C}[Q_1, \dots, Q_r]$, i.e., we may assume that Q_i are complex parameters.

5.1. **Genus-0 topological recursion relations.** Our first goal is to prove the so called Topological recursion relations (TRR for brevity):

$$(5.1)\quad \frac{1}{n!}\langle \phi_a\psi^{i+1}, \phi_b\psi^j, \phi_c\psi^k, \mathbf{t}(\psi)\dots, \mathbf{t}(\psi)\rangle_{0,n+3,d} = \sum_{\substack{n_1+n_2=n \\ d_1+d_2=d}} \sum_{\mu,\nu=0}^{N-1} \frac{g^{\mu\nu}}{n_1!n_2!} \times$$
$$\langle \phi_a\psi^i, \phi_\mu, \mathbf{t}(\psi), \dots, \mathbf{t}(\psi)\rangle_{0,n_1+2,d_1} \langle \phi_\nu, \phi_b\psi^j, \phi_c\psi^k, \mathbf{t}(\psi)\dots, \mathbf{t}(\psi)\rangle_{0,n_2+3,d_2},$$

where $g_{\mu\nu} = \int_X \phi_\mu\phi_\nu$ are the entries of the matrix of the Poincaré pairing and $g^{\mu\nu}$ are the entries of its inverse.

Let $\mathrm{ct} : \overline{\mathcal{M}}_{0,n+3}(X,d) \to \overline{\mathcal{M}}_{0,3}$ be the map forgetting the map, the last n marked points, and contracting all unstable components. Let $(\Sigma, p, f) \in \overline{\mathcal{M}}_{0,n+3}(X,d)$. Note that if we forget f and the last n marked points then only one of the irreducible components of Σ is stable (and hence is not contracting by ct). We call this distinguished component the central component of Σ. Let D be the set of all stable maps such that the first marked point is not on the central component. The same comparison argument as in the proof of the string equations shows that: $\psi_1 - \overline{\psi}_1 = [D]$, where $\overline{\psi}_1$ is the first Chern class of the pullback of the cotangent lines bundle $\mathbb{L}_1 \to \overline{\mathcal{M}}_{0,3}$. Notice that $\overline{\psi}_1 = 0$. It follows that the RHS of (5.1) can be written in the following form:

$$(5.2)\qquad \frac{1}{n!}\int_{[D]} \phi_a\psi_1^i\phi_b\psi_2^j\phi_c\psi_3^k\mathbf{t}(\psi_4)\dots\mathbf{t}(\psi_{n+3}).$$

On the other hand, given a point $(\Sigma, p, f) \in D$ we can split the curve into two parts Σ' and Σ'' such that Σ' is a tree of $\mathbb{C}P^1$s which carries the first marked point and such that under the contraction map it is contracted to a point on the central component. Σ'' is the complement of Σ'. Thus there is a natural map gl which to each stable map $(\Sigma, p, f) \in D$ assigns an element of the preimage of the diagonal of the following map:

$$\overline{\mathcal{M}}_{0,n_1+1+\circ}(X,d_1)\times\overline{\mathcal{M}}_{0,\bullet+2+n_2}(X,d_2)\xrightarrow{\mathrm{ev}_\circ\times\mathrm{ev}_\bullet} X\times X.$$

The map gl is a $\binom{n}{n_1}$-covering because if we split the last n marked points of Σ into two groups then there are exactly that many ways to re-number them so that the order of the marked points in each group does not change. Since the Poincaré dual to the diagonal in $X\times X$ has the form $\sum_{\mu,\nu} g^{\mu\nu}\phi_\mu\otimes\phi_\nu$ we see that (5.2) is transformed into:

$$\sum_{\substack{n_1+n_2=n\\ d_1+d_2=d}}\frac{1}{n_1!n_2!}\int_{\overline{\mathcal{M}}_{0,n_1+1+\circ}(X,d_1)\times\overline{\mathcal{M}}_{0,\bullet+2+n_2}(X,d_2)}\sum_{\mu,\nu} g^{\mu\nu}\mathrm{ev}_\circ^*\phi_\mu\mathrm{ev}_\bullet^*\phi_\nu(\ldots),$$

where the dots stand for the integrand in (5.2). Formula (5.1) follows.

5.2. **From two to one point descendents.** Let $\tau=\sum_a\tau^a\phi_a\in H^*(X)$ be a cohomology class. We view τ^a as coordinate functions on $H^*(X)$. The correlators

$$\langle\phi_{a_1}\psi^{k_1}\ldots\phi_{a_s}\psi^{k_s}\rangle_{0,s}(\tau)=\sum_{n,d}\frac{Q^d}{n!}\langle\phi_{a_1}\psi^{k_1}\ldots\phi_{a_s}\psi^{k_s},\tau,\ldots,\tau\rangle_{0,s+n,d},$$

are called s-point descendent GW invariants.

The 1- and 2-point descendents are organized into generating series of the type $S_\tau(z)=1+S_1(\tau)z^{-1}+S_2(\tau)z^{-2}+\ldots$ and $W_\tau(z,w)=\sum_{k,l}W_{kl}(\tau)z^{-k}w^{-l}$, where $S_k(\tau)$ and $W_{kl}(\tau)$ are linear transformations in $H^*(X)$, defined by:

$$(\phi_a,S_\tau\phi_b)=(\phi_a,\phi_b)+\sum_{k\geq 0}\langle\phi_a,\phi_b\psi^k\rangle_{0,2}(\tau)z^{-k-1}$$

and

$$(\phi_a,W_\tau(z,w)\phi_b)=\sum_{k,l\geq 0}\langle\phi_a\psi^k,\phi_b\psi^l\rangle_{0,2}(\tau)z^{-k}w^{-l}.$$

Lemma 5.1. *The following formula holds:*

$$W_\tau(z,w)=\frac{{}^{\mathrm{t}}S_\tau(z)S_\tau(w)-1}{z^{-1}+w^{-1}},$$

where the transpose of S is with respect to the Poincaré pairing.

Proof. We need to verify that

$$(\phi_a,W_\tau(z,w)\phi_b)\left(z^{-1}+w^{-1}\right)+(\phi_a,\phi_b)=(S_\tau(z)\phi_a,S_\tau(w)\phi_b).$$

Using the String equation (SE for brevity) it is easy to verify that the LHS of the above identity coincides with

$$\sum_{k,l\geq 0}\langle\phi_a\psi^k,\phi_b\psi^l,1\rangle_{0,3}(\tau)z^{-k}w^{-l}. \tag{5.3}$$

We split the summation range in the above sum into four groups. First if $k = l = 0$ then the corresponding summand is just (ϕ_a, ϕ_b). The summands corresponding to $k, l \geq 1$ can be simplified first with TRR and then they add up to the following sum:

$$\sum_{\mu,\nu} \sum_{k,l\geq 1} \langle \phi_a \psi^{k-1}, \phi_\mu \rangle_{0,2}(\tau) g^{\mu\nu} \langle \phi_\nu, \phi_b \psi^l, 1\rangle_{0,3}(\tau) z^{-k} w^{-l}. \tag{5.4}$$

By definition we have

$$\sum_{k\geq 1} \langle \phi_a \psi^{k-1}, \phi_\mu \rangle_{0,2}(\tau) z^{-k} = (\phi_\mu, (S_\tau(z) - 1)\phi_a)$$

and

$$\sum_{l\geq 1} \langle \phi_\nu, \phi_b \psi^l, 1\rangle_{0,3}(\tau) w^{-l} = \sum_{l\geq 1} \langle \phi_\nu, \phi_b \psi^{l-1} \rangle_{0,2}(\tau) w^{-l} = (\phi_\nu, (S_\tau(w) - 1)\phi_b),$$

where for the first equality we used SE. Therefore the sum (5.4) equals

$$\sum_{\mu,\nu} (\phi_\mu, (S_\tau(z) - 1)\phi_a) g^{\mu\nu} (\phi_\nu, (S_\tau(w) - 1)\phi_b) = ((S_\tau(z) - 1)\phi_a, (S_\tau(w) - 1)\phi_b).$$

Similarly, the summands in (5.3) corresponding to $k \geq 1$, $l = 0$ add up to $((S_\tau(z) - 1)\phi_a, \phi_b)$, and the ones corresponding to $k = 0$ and $l \geq 1$ to $(\phi_a, (S_\tau(w) - 1)\phi_b)$. The lemma follows. □

The importance of the following corollary will become clear after we introduce the Givental's quantization formalism.

Corollary 5.2. *The 1-point series satisfies the following condition:* ${}^{\mathrm{t}}S_\tau(-z) S_\tau(z) = 1$.

5.3. **Quantum differential equations.** We view $H := H^*(X)$ as a manifold and we trivialize the tangent bundle TH by identifying ϕ_a with the coordinate vector fields $\partial/\partial\tau^a$. In each tangent space we define a multiplication $\bullet_\tau$, called the *quantum cup product*, by the formula:

$$(\phi_a \bullet \phi_b, \phi_c) = \langle \phi_a, \phi_b, \phi_c \rangle_{0,3}(\tau).$$

Notice that if we let (this is the genus-0 GW potential of X)

$$F^{(0)}(\tau) = \sum_{n,d} \frac{Q^d}{n!} \langle \tau, \ldots, \tau \rangle_{0,n,d}$$

then the RHS in the definition of $\bullet_\tau$ coincides with the third partial derivatives $\frac{\partial^3 F^{(0)}}{\partial\tau^a \partial\tau^b \partial\tau^c}$. It is clear that $\bullet_\tau$ is commutative and it satisfies the Frobenius property: $(a \bullet_\tau b, c) = (a, b \bullet_\tau c)$, i.e., each tangent space is a Frobenius algebra. Also, from the string equation it follows that $1 = \partial/\partial\tau^0$ is a unity. The only thing that is not so easy to see is that $\bullet_\tau$ is associative.

Lemma 5.3. *The 1-point descendent series $S_\tau(z)$ satisfies the following system of ODEs:*

$$z\partial_{\tau^a} S_\tau(z) = (\phi_a \bullet_\tau) S_\tau(z), \quad 0 \le a \le N-1, \tag{5.5}$$

where $\phi_a \bullet_\tau$ is the linear operator of quantum multiplication by $\phi_a = \partial/\partial\tau^a$.

Proof. The equality (5.5) is equivalent to:

$$\sum_{k=0}^{\infty} \langle \phi_a, \phi_b, \phi_c \psi^k \rangle_{0,3}(\tau) z^{-k} = (S_\tau(z)\phi_c, \phi_a \bullet_\tau \phi_b) z^{-k}.$$

On the other hand, thanks to the TRR, the LHS in the above equality is equivalent to:

$$\langle \phi_a, \phi_b, \phi_c \rangle_{0,3}(\tau) + \sum_{k=1}^{\infty} \sum_{\mu,\nu} \langle \phi_a, \phi_b, \phi_\mu \rangle_{0,3}(\tau) g^{\mu\nu} \langle \phi_\nu, \phi_c \psi^{k-1} \rangle_{0,3}(\tau)$$

Using the definitions of the quantum cup product and the 1-point series $S_\tau(z)$, we get that the above expression equals to:

$$(\phi_a \bullet_\tau \phi_b, \phi_c) + \sum_{\mu,\nu} (\phi_a \bullet_\tau \phi_b, \phi_\mu) g^{\mu\nu} \left((S_\tau(z) - 1)\phi_c, \phi_\nu \right).$$

The lemma follows. $\square$

Let M be a small ball centered at 0 in $\mathbb{C}^N$. Assume that g is a non-degenerate bi-linear pairing on TM, A is a holomorphic section of $T^*M^{\otimes 2} \otimes TM$, i.e., the tangent spaces T_tM are equipped with a multiplication $\bullet_t$ which depends holomorphically on $t \in M$, e is a vector field on M such that its restriction to T_tM is a unity with respect to $\bullet_t$, and finally E is a vector field on M.

Definition 5.4. The data (M, g, A, e, E) form a Frobenius structure on M of *conformal dimension* $D \in \mathbb{C}$, if the following conditions are satisfied.

(1) g and $\bullet$ satisfy the Frobenius property: $g(X \bullet Y_1, Y_2) = g(Y_1, X \bullet Y_2)$,
(2) The one-parameter group corresponding to E acts on M by conformal transformations of g, i.e., $\mathcal{L}_E g = (2-D)g$,
(3) e is a flat vector field: $\nabla^{\text{L.C.}} e = 0$, where $\nabla^{\text{L.C.}}$ is the Levi-Civitá connection of g,
(4) The connection operator

$$\nabla = \nabla^{\text{L.C.}} - z^{-1} \sum_{i=1}^{N} \left(\frac{\partial}{\partial t_i} \bullet_t \right) dt_i + \left(z^{-2}(E \bullet_t) - z^{-1}\mu \right) dz, \tag{5.6}$$

where $\mu := \nabla^{\text{L.C.}}(E) - (1 - \frac{D}{2})\text{Id} : TM \to TM$ is *the Hodge grading operator,* is flat, i.e., $\nabla^2 = 0$.

The flatness of the family of connection operators implies that $\bullet_t$ is commutative and associative and that there exists a function $F(\tau)$, called potential of the Frobenius structure, such that the structure constants of the multiplication $\bullet_t$ are given by the third partial derivatives of F, i.e., $g(\partial/\partial\tau^a \bullet_t \partial/\partial\tau^b, \partial/\partial\tau^c) = \partial^3 F/(\partial\tau^a\partial\tau^b\partial\tau^c)$, where $\tau = (\tau^1, \dots, \tau^N)$ is a flat coordinate system on M.

Theorem 5.5. *Let $H = H^*(X)$. Then the Poincaré pairing, the quantum multiplication, the cohomology class* 1*, and the vector field:*

$$E = \sum_{a=0}^{N-1} (1 - \deg_{\mathbb{C}}\phi_a)\tau^a \frac{\partial}{\partial\tau^a} + c_1(TX),$$

form a Frobenius structure on H of conformal dimension $D := \dim_{\mathbb{C}} X$.

Proof. The only thing that we need to check is that the corresponding connection is flat, i.e., $[\nabla_{\partial/\partial\tau^a}, \nabla_{\partial/\partial\tau^b}] = 0$ and $[\nabla_{\partial/\partial\tau^a}, \nabla_{\partial/\partial z}] = 0$. The first commuator vanishes thanks to Lemma 5.3. The vanishing of the second one is equivalent to:

$$\partial_{\tau^a}(E\bullet_\tau) = [\mu, (\phi_a \bullet_\tau)] + (\phi_a\bullet_\tau).$$

The above equality is equivalent to:

$$\partial_{\tau^a}\langle\phi_b, \phi_c, E\rangle_{0,3}(\tau) = (1 + d_b + d_c - D)\langle\phi_a, \phi_b, \phi_c\rangle_{0,3}(\tau),$$

where $d_a, 0 \le a \le N-1$ is the complex degrees of ϕ_a and we used that the Hodge grading operator satisfies ${}^{\mathrm{t}}\mu = -\mu$, and $\mu(\phi_a) = (\frac{D}{2} - d_a)\phi_a$. On the other hand the LHS is equal to: $(1-d_a)\langle\phi_a, \phi_b, \phi_c\rangle_{0,3}(\tau) + E\langle\phi_a, \phi_b, \phi_c\rangle_{0,3}(\tau)$, so we need to verify that

$$E\langle\phi_a, \phi_b, \phi_c\rangle_{0,3}(\tau) = (d_a + d_b + d_c - D)\langle\phi_a, \phi_b, \phi_c\rangle_{0,3}(\tau),$$

which follows esily from the dimension formula

$$\dim_{\mathbb{C}}\overline{\mathcal{M}}_{0,n}(X; d) = D - 3 + n + \int_d c_1(TX),$$

and the divisor equation. □

5.4. **Examples.** In the computations, it is convenient to use that the associativity of the quantum cup product is equivalent to the following system of PDEs for the potential $F = F^{(0)}$: for all a, b, c, and d we have

$$\sum_{\mu,\nu} g^{\mu\nu}(F_{ab\mu}F_{cd\nu} - F_{ad\mu}F_{bc\nu}) = 0,$$

where the indices mean partial derivatives. These are the so called WDVV-equations, and as we will see below they can be used to solve some interesting enumerative problems. To prove them note that the above system is equivalent to the identities:

$$((\phi_a \bullet_\tau \phi_b) \bullet_\tau \phi_c, \phi_d) = (\phi_a \bullet_\tau (\phi_b \bullet_\tau \phi_c), \phi_d).$$

On the other hand, the above identities follow from the associativity and the Frobenius property of $\bullet_\tau$.

Example 1: If $X = \mathrm{pt}$. Then $H = \mathbb{C}$, $\tau = \tau_0\phi_0$ and for dimensional reasons we have $F^{(0)}(\tau) = \frac{\tau_0^3}{3!}$. The quantum multiplication coincides with the standard multiplication in $\mathbb{C}$. The system of QDE reads: $z\partial_{\tau_0} S_\tau(z) = S_\tau(z)$, and we get $S_\tau(z) = e^{\tau_0/z}$.

Example 2: If $X = \mathbb{C}P^1$. Now $H = \mathbb{C}[p]/p^2$, where p is the hyperplane class, $\tau = \tau_0 + \tau_1 p$. Using string and divisor equations we get the following formula: $F^{(0)}(\tau) = \frac{1}{2}\tau_0^2\tau_1 + Qe^{\tau_1}$. From here we can compute the structure constants of $\bullet_\tau$ and we get that $(H, \bullet_\tau) \cong \mathbb{C}[p]/\langle p^2 - Qe^{\tau_1}\rangle$.

Example 3: If $X = \mathbb{C}P^2$. Similarly to the above case we have: $\tau = \tau_0 + \tau_1 p + \tau_2 p^2$ and the string and divisor equation reduce the computation of $F^{(0)}$ to computing correlators involving only p^2, i.e.,

$$F^{(0)}(\tau) = \frac{1}{2}(\tau_0^2\tau_2 + \tau_0\tau_1^2) + \sum_{d=1}^{\infty}(Qe^{\tau_1})^d \frac{\tau_2^{3d-1}}{(3d-1)!}N_d,$$

where $N_d = \langle p^2, \ldots, p^2\rangle_{0,3d-1,d}$ can be interpreted as the number of degree d rational curves in $\mathbb{C}P^2$ passing through $3d-1$ points in general position.

Among all $WDVV$-equations only one is not trivial: $a = b = 2$, and $c = d = 1$, which gives us $F_{222} = F_{211}^2 - F_{221}F_{111}$. Comparing the coefficients in front of Q we get the following recursion relation:

$$\frac{N_d}{(3d-4)!} = \sum_{d_1+d_2=d}\Big(\frac{N_{d_1}d_1^2}{(3d_1-2)!}\frac{N_{d_2}d_2^2}{(3d_2-2)!} - \frac{N_{d_1}N_{d_2}d_1d_2^3}{(3d_1-3)!(3d_2-1)!}\Big).$$

Starting with $N_1 = 1$, the above relation determines N_d, for all $d > 1$.

6. Givental's quantization formalism

Let H be any complex vector space, equipped with a non-degenerate bilinear form $(\ ,\ )$. Let $\mathcal{H} = H((z^{-1}))$ be the space of Laurent series in z^{-1} with coefficients in H. We turn the vector space $\mathcal{H}$ into a symplectic vector space via the symplectic form:

$$\Omega(f, g) = \mathrm{res}_{z=0}(f(-z), g(z))dz, \qquad \text{for all } f, g \in \mathcal{H}.$$

This form induces naturally a symplectic form ω on the manifold $\mathcal{H}$ by demanding that $(\mathcal{H}, \Omega) \cong (T_0\mathcal{H}, \omega|_{T_0\mathcal{H}})$. A Darboux coordinate system on $\mathcal{H}$ can be constructed as follows. Let $\{\phi_a\}_{a=0}^{N-1}$ be a bases of H and denote by $\{\phi^a\}$

the corresponding Poincaré dual bases, i.e., $(\phi^a, \phi_b) = \delta_{ab}$. Let $p_{k,a}$ and q_k^a be the functions on $\mathcal{H}$ defined by:

$$f = \sum_{k=0}^{\infty} \sum_{a=0}^{N-1} \Big(p_{k,a}(f)\phi^a(-z)^{-k-1} + q_k^a(f)\phi_a z^k \Big).$$

In such coordinates we have: $\phi^a(-z)^{-k-1} = \partial/\partial p_{k,a}$, $\phi_a z^k = \partial/\partial q_k^a$ and the form $\omega = \sum_{k,a} dp_{k,a} \wedge dq_k^a$.

Let $\mathfrak{sp}(\mathcal{H}, \Omega) = \{A : \mathcal{H} \to \mathcal{H} \mid \Omega(Af, g) + \Omega(f, Ag) = 0$ be the Lie algebra of infinitesimal symplectic transformations. Given an element $A \in \mathfrak{sp}(\mathcal{H}, \Omega)$ we define a linear vector field $X_A(f) = Af \in \mathcal{H} \cong T_f\mathcal{H}$. It is not hard to check that the symplectic condition is equivalent to the fact that X_A is Hamiltonian, i.e., there exists a function h_A on $\mathcal{H}$ such that $dh_A = \iota_{X_A}\omega$. On the other hand notice that

$$\Omega(Af, f) = \iota_f \iota_{Af}\omega = \iota_f dh_A = \Big(\sum_{k,a} p_{k,a}\frac{\partial}{\partial p_{k,a}} + q_k^a \frac{\partial}{\partial q_k^a} \Big) h_A = 2h_A,$$

where $f \in \mathcal{H}$ is identified with the vector field $\sum_{k,a}(p_{k,a}(f)\partial/\partial p_{k,a} + q_k^a(f)\partial/\partial q_k^a)$, and the last equality follows from the fact that h_A is quadratic in p and q, because X_A is linear. So we have the following formula: $h_A = \frac{1}{2}\Omega(Af, f)$.

Given an infinitesimal symplectic transformation A we define a differential operator $\widehat{A}$ acting on the space of formal power series in $q_k^a, k \geq 0, 0 \leq a \leq N-1$ whose coefficents are formal Laurent series in ϵ. We refer to the above space as the Fock space and we denote it by $\mathrm{Fun}(\mathcal{H}_+)$. We use the Weyl quantization rules: $\widehat{q}_k^a = q_k^a/\epsilon$ and $\widehat{p}_{k,a} = \epsilon\partial/\partial q_k^a$. Monomial expressions in p and q are quantized by representing each p (resp. q) by the corresponding differentiation (resp. multiplication) operator and moving all differentiation operators before the multiplication ones. We define $\widehat{A} := \widehat{h_A}$. Notice that the quantization of quadratic Hamiltonians is a projective representation of Lie algebras, i.e.,

$$[\widehat{F}, \widehat{G}] = \{F, G\}\widehat{\ } + C(F, G),$$

where the cocycle is defined by:

$$C(p_a p_b, q_a q_b) = -C(q_a q_b, p_a p_b) = \begin{cases} 1 & \text{if } a \neq b \\ 2 & \text{otherwise,} \end{cases}$$

and C vanishes for all other pairs of quadratic Darboux monomials.

Our next goal is to describe the quantized action of a particular class of symplectic transformations on $\mathrm{Fun}(\mathcal{H}_+)$. By definition, the twisted loop group $\mathcal{L}^{(2)}\mathrm{GL}(H)$ consists of all symplectic trnasformations of $\mathcal{H}$ of the form $M(z) = \sum_k M_k z^k$. Notice that the symplectic condition is equivalent to ${}^{\mathrm{t}}M(-z)M(z) = 1$. The elements of $\mathcal{L}^{(2)}\mathrm{GL}(H)$ of the form $S(z) = 1 + S_1 z^{-1} + S_2 z^{-2} + \ldots$

(resp. $R(z) = 1 + R_1 z + R_2 z^2 + \ldots$) are called lower-triangular (resp. upper-triangular). To be precise, the elements of the twisted loop group are Laurent polynomials in z, and S and R are elements of different formal completions of $\mathcal{L}^{(2)}\mathrm{GL}(H)$ (in particular, the product SR does not make sense).

Theorem 6.1. *Let $S = e^{A(z)}$ be a lower-traingular symplectic transformation and $\mathcal{F} \in \mathrm{Fun}(\mathcal{H}_+)$ be an element of the Fock space. Then*

$$\widehat{S}^{-1}\mathcal{F} = e^{\frac{1}{2\epsilon^2}W(\mathbf{q},\mathbf{q})}\mathcal{F}([S\mathbf{q}]_+),$$

where $\widehat{S} = e^{\widehat{A}}$, f_+ means the series obtained from f by truncating the terms with negative powers of z, and $W(\mathbf{q},\mathbf{q}) = \sum_{k,l}(W_{kl}q_l, q_k)$, where $q_n = \sum_a q_n^a \phi_a$, is defined by:

$$\sum_{k,l\geq 0} W_{kl} z^{-k} w^{-l} = \frac{{}^{\mathrm{t}}S(z)S(w) - 1}{z^{-1} + w^{-1}}.$$

Proof. Write $A(z) = \sum_{k\geq 1} A_k z^{-k}$. Then it is not hard to see that the corresponding quadratic Hamiltonian is given by: $-\frac{1}{2}(A\mathbf{q}, \mathbf{q}(-z)) - (A\mathbf{p}, \mathbf{q}(-z))$, i.e.,

$$h_A = \frac{1}{2}\sum_{m,l}(-1)^{m+1}(A_{m+l+1}q_l, q_m) + \sum_{k,l\geq 0}(-1)^k (A_k p_l, q_{k+l}),$$

where

$$\mathbf{q}(z) = \sum_k q_k z^k = \sum_{k,a} q_k^a \phi_a z^k,$$

and

$$\mathbf{p}(z) = \sum_k p_k (-z)^{-k-1} = \sum_{k,a} p_{k,a}\phi^a (-z)^{-k-1}.$$

Put $\mathcal{G}(t,\mathbf{q}) = e^{-t\widehat{A}}\mathcal{F}$. We compute $\mathcal{G}$ for all t. In particular, the Theorem would follow from the case $t = 1$.

Notice that $\mathcal{G}$ is a solution to the differential equation $\partial_t \mathcal{G} = -\widehat{A}\mathcal{G}$, which after the substitution $g = \log \mathcal{G}$, turns into:

$$\frac{\partial g}{\partial t} = \frac{1}{2\epsilon^2}(A\mathbf{q}, \mathbf{q}(-z)) + \sum_{k,a}(A\phi^a(-z)^{-k-1}, \mathbf{q}(-z))\frac{\partial g}{\partial q_k^a}. \tag{6.1}$$

This is a 1-st order PDE which we solve by the method of the characteristics.

Step 1: first, we solve the homogeneus equation, i.e.,

$$\frac{\partial g}{\partial t} = \sum_{k,a}(A\phi^a(-z)^{-k-1}, \mathbf{q}(-z))\frac{\partial g}{\partial q_k^a}.$$

The auxiliarly system of ODE's is

$$\frac{\partial q_k^a}{\partial t} = -(A\phi^a(-z)^{-k-1}, \mathbf{q}(-z)) \quad \Leftrightarrow \quad \frac{\partial \mathbf{q}}{\partial t} = -[A\mathbf{q}]_+.$$

Notice that $[A[\dots[A\mathbf{q}]_+]]_+ = [A^n\mathbf{q}]_+$, where on the LHS A is repeated n times. Therefore, the system of ODE's has the following solution: $\mathbf{q}(t) = [e^{-tA}\mathbf{c}]_+$, where $\mathbf{c} = \mathbf{q}(0) \in \mathcal{H}_+ = H[z]$ is an initial condition. The method of the characteristics is based on the fact that the solutions $g(t, \mathbf{q})$ of the PDE are constant along the curves $(t, \mathbf{q}(t)) \in \mathbb{C} \times \mathcal{H}_+$. From here we find that if $(t, \mathbf{q}) \in \mathbb{C} \times \mathcal{H}_+$ is any point then the curve $(s, \mathbf{q}(s))$ with initial condition $(0, [e^{tA}\mathbf{q}]_+)$ will pass through the point $(t, \mathbf{q})$. Therefore, the general solution of the PDE is given by: $g(t, \mathbf{q}) = f([e^{tA}\mathbf{q}]_+)$, where f is an arbitrary function on $\mathcal{H}_+$.

Step 2: a direct computation shows that the function

$$W_t(\mathbf{q}, \mathbf{q}) = \frac{1}{2\epsilon^2} \sum_{k,l} (W_{kl}(t) q_l, q_k),$$

defined by the formula:

$$\sum_{k,l \geq 0} W_{kl}(t) z^{-k} w^{-l} = \frac{e^{{}^t A(z) t} e^{A(w)t} - 1}{z^{-1} + w^{-1}}$$

is a solution to (6.1).

So the general solution to (6.1) is given by $g(t, \mathbf{q}) = W_t(\mathbf{q}, \mathbf{q}) + f([e^{tA}\mathbf{q}]_+)$. Notice that for $t = 0$ we have $\mathcal{G} = \mathcal{F}$, and $W_0(\mathbf{q}, \mathbf{q}) = 0$, so $f = \log \mathcal{F}$. The theorem follows. □

7. From descendent to ancestor GW invariants

In this section we assume that X is a projective manifold, $H = H^*(X; \mathbb{C})$, and $(\ ,\ )$ is the Poincaré peiring. Let $\alpha_i(\psi, \overline{\psi}) = \sum_{k,m} \alpha_i^{k,m} \psi^k \overline{\psi}^m$ be a polynomial in ψ and $\overline{\psi}$ whose coefficients $\alpha_i^{k,m}$ are cohomology classes in $H^*(X)$, and $\tau \in H$ is a fixed cohomology class. Then the correlator:

$$\langle \alpha_1(\psi, \overline{\psi}), \dots, \alpha_n(\psi, \overline{\psi}) \rangle_{g,n}(\tau) \tag{7.1}$$

represents the following sum of integrals over moduli spaces:

$$\sum_{d,l} \sum_{k_.,m_.} \frac{Q^d}{l!} \int_{\overline{\mathcal{M}}_{g,n+l}(X;d)} \psi_1^{k_1} \overline{\psi}_1^{m_1} \dots \psi_n^{k_n} \overline{\psi}_n^{m_n} \mathrm{ev}^*(\alpha_1^{k_1,m_1} \otimes \alpha_n^{k_n,m_n} \otimes \tau^{\otimes l}),$$

where the sum is over all $d \in H_2(X, \mathbb{Z})$, $l \geq 0$ and multi-indices $k_. = (k_1, \dots, k_n)$ and $m_. = (m_1, \dots, m_n)$. The cohomology classes in the integrand are:

- ψ_i the 1-st Chern class of the line bundle $\mathbb{L}_i$ of cotangent lines at the i-th marked points,

– $\overline{\psi}_i$ is the pullback of the ψ_i-class on $\overline{\mathcal{M}}_{g,n}$ via the (forgetfull) map $\pi : \overline{\mathcal{M}}_{g,n+l}(X,d) \to \overline{\mathcal{M}}_{g,n}$ which forgets the map, the last l marked points, and contracts all unstable components
– $\mathrm{ev} : \overline{\mathcal{M}}_{g,n+l}(X,d) \to X^{n+l}$ is the evaluation map.

By definition, the corelator (7.1) is 0 if the moduli space $\overline{\mathcal{M}}_{g,n}$ is empty, i.e., for $(g,n) \in \{(0,0),(0,1),(0,2),(1,0)\}$.

The goal in this section is to express correlators of the type (7.1) in terms of correlators that involve only $\overline{\psi}$-classes. This can be achieved thanks to the following lemma.

Lemma 7.1. *Assume that $\alpha \in H^*(X)$ and (g,n) is a stable pair (i.e. $\overline{\mathcal{M}}_{g,n}$ is non-empty). Then the following formula holds:*

$$\begin{aligned}&\langle \alpha\psi^{k+1}\overline{\psi}^m, \alpha_2(\psi,\overline{\psi}),\ldots,\alpha_n(\psi,\overline{\psi})\rangle_{g,n}(\tau) = \\ &= \langle \alpha\psi^k\overline{\psi}^{m+1} + S_{k+1}\alpha\overline{\psi}^m, \alpha_2(\psi,\overline{\psi}),\ldots,\alpha_n(\psi,\overline{\psi})\rangle_{g,n}(\tau),\end{aligned}$$

where $S_\tau(z) = 1 + S_1(\tau)z^{-1} + \ldots$ is the 1-point descendent series defined in Subsection 5.2.

Proof. Let D_1 be the divisor in $\overline{\mathcal{M}}_{g,n+l}(X,d)$ of all points $(\Sigma, p_., f)$ such that the first marked point p_1 is not on the same irreducible component as any of the points p_i, $2 \le i \le n$. Notice that $\psi_1 = \overline{\psi}_1 + [D_1]$ and that the divisor D_1 can be identified with the image of the gluing map:

$$\mathrm{gl} : \bigsqcup_{\substack{l'+l''=l\\ d'+d''=d}} \overline{\mathcal{M}}_{g,n-1+l'+\circ}(X,d') \times_X \overline{\mathcal{M}}_{0,1+l''+\bullet}(X,d'') \to \overline{\mathcal{M}}_{g,n+l}(X,d),$$

where in the fiber product the maps from the moduli spaces to X are given by the evaluations at the marked points $\circ$ and $\bullet$. Writing $\psi_1^{k+1}\overline{\psi}_1^m = \psi_1^k\overline{\psi}_1^{m+1} + [D_1]\psi_1^k\overline{\psi}^m$ we get that the integral

$$\int_{\overline{\mathcal{M}}_{g,n+l}(X,d)} \mathrm{ev}_1^*(\alpha)\psi_1^{k+1}\overline{\psi}_1^m\alpha_2\ldots\alpha_n\tau^{\otimes l}$$

equals to

$$\begin{aligned}&\int_{\overline{\mathcal{M}}_{g,n+l}(X,d)} \mathrm{ev}_1^*(\alpha)\psi_1^k\overline{\psi}_1^{m+1}\alpha_2\ldots\alpha_n\tau^{\otimes l} + \sum_{\substack{l'+l''=l\\ d'+d''=d}} \frac{l!}{l'!l''!}\sum_{\mu,\nu} g^{\mu\nu} \times \\ &\times \int_{\overline{\mathcal{M}}_{g,n-1+l'+\circ}(X,d')} \alpha_2\ldots\alpha_n\tau^{\otimes l'}\mathrm{ev}_\circ^*(\phi_\mu)\overline{\psi}_\circ^m \int_{\overline{\mathcal{M}}_{0,1+l''+\bullet}(X,d'')} \mathrm{ev}_1^*(\alpha)\psi_1^k\tau^{\otimes l''}\mathrm{ev}_\bullet^*(\phi_\nu),\end{aligned}$$

where the combinatorial factor $\binom{l}{l'}$ comes from the fact that in the gluing map gl the union of the l' marked points on the 1-st moduli space and the l'' marked points on the second one have to be renumbered with the numbers from $n+1$

to $n+l$. Notice that the expression $\sum_{\mu\nu} g^{\mu\nu}\phi_\mu \otimes \phi_\nu$ is the Poincaré dual to the diagonal in $X \times X$. The lemma follows. $\square$

Let $\mathbf{t}(z) = \sum_{k,a} t_k^a \phi_a z^k$. Then the lemma from above implies the following identity:

$$\langle \mathbf{t}(\psi), \dots, \mathbf{t}(\psi)\rangle_{g,n}(\tau) = \langle [S_\tau \mathbf{t}]_+(\psi), \dots, [S_\tau \mathbf{t}]_+(\psi)\rangle_{g,n}(\tau).$$

Notice that the LHS in the above equality can be written also in the form

$$\sum_{d,l} \frac{Q^d}{l!} \langle \mathbf{t}(\psi)+\tau, \dots, \mathbf{t}(\psi)+\tau\rangle_{g,n,d}.$$

This follows from the Taylor's formula. Now we are ready to describe the connection between descendent and ancestor GW invariants. Let

$$\widetilde{\mathcal{D}}(\mathbf{t}) = \exp\Big(\sum_{g,n} \frac{\epsilon^{2g-2}}{n!} \langle \mathbf{t}(\psi), \dots, \mathbf{t}(\psi)\rangle_{g,n}(0)\Big),$$

where the summation is over all (g,n), including the unstable ones. We are slightly abusing the notations because in (7.1) we defined the correlator to be 0 if (g,n) is an unstable pair. The ancestor invariants are organized in the following generating series:

$$\widetilde{\mathcal{A}}_\tau(\mathbf{t}) = \exp\Big(\sum_{g,n} \frac{\epsilon^{2g-2}}{n!} \langle \mathbf{t}(\overline{\psi}), \dots, \mathbf{t}(\overline{\psi})\rangle_{g,n}(\tau)\Big),$$

where the summation is over all stable pairs (g,n). From the discussion above it is clear that we have: $\widetilde{\mathcal{D}}(\mathbf{t}+\tau) = C_\tau(\mathbf{t})\widetilde{\mathcal{A}}_\tau([S_\tau \mathbf{t}]_+)$, where

$$\log C_\tau(\mathbf{t}) = \Big(\langle\ \rangle_{0,0}(\tau) + \langle\ \mathbf{t}(\psi)\rangle_{0,1}(\tau) + \frac{1}{2}\langle\ \mathbf{t}(\psi), \mathbf{t}(\psi)\rangle_{0,2}(\tau)\Big)\epsilon^{-2} + \langle\ \rangle_{1,0}(\tau).$$

Both generating functions $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{A}}$ are formal series in $\tau, t_0, t_1, \dots$ whose coefficients are formal Laurent series in ϵ. It turns out that if we want to express the relation between descendents and ancestors in terms of the quantization formalism from the previous lecture, we need to identify $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{A}}$ with elements of the Fock space $\mathrm{Fun}(\mathcal{H}_+)$ of all formal power series in $\tau, q_0, q_1+1, q_2, \dots$, whose coefficients are formal Laurent series in ϵ. The identifications are given by the so called *dilaton shift*: $\mathbf{q}(z) = \mathbf{t}(z) - z$, i.e.,

$$\mathcal{D}(\mathbf{q}) = \widetilde{\mathcal{D}}(z+\mathbf{q}(z)), \quad \mathcal{A}_\tau(\mathbf{q}) = \widetilde{\mathcal{A}}_\tau(z+\mathbf{q}(z)).$$

We refer to $\mathcal{D}$ (resp. $\mathcal{A}_\tau$) as the total descendent (resp. ancestor) potential. In the new notation our formula takes the form:

$$\mathcal{D}(\mathbf{q}) = C_\tau(\mathbf{q}(z)+z-\tau)\mathcal{A}_\tau(-z+[S_\tau(\mathbf{q}(z)+z-\tau)]_+).$$

First, let us simplify the argument in the ancestor potential:

$$-z + [S_\tau\, \mathbf{q}(z)]_+ + z + S_1 1 - \tau = [S_\tau\, \mathbf{q}(z)]_+.$$

Where we used that

$$(S_1 1, \phi_a) = \langle 1, \phi_a \rangle_{0,2}(\tau) = \langle 1, \phi_a, \tau \rangle_{0,3,0} = \int_X \phi_a \tau,$$

i.e., $S_1(\tau)1 = \tau$.

On the other hand, using the dilaton equation, it is not hard to verify that

$$\begin{aligned} \langle \psi - \tau, \mathbf{q}(\psi) \rangle_{0,2}(\tau) &= -\langle \mathbf{q}(\psi) \rangle_{0,1}(\tau) \\ \langle \psi - \tau, \psi - \tau \rangle_{0,2}(\tau) &= -\langle \psi - \tau \rangle_{0,1}(\tau) \\ \langle \psi - \tau \rangle_{0,1}(\tau) &= -2\langle \rangle_{0,0}(\tau). \end{aligned}$$

From this formulas we get

$$\log C_\tau(\mathbf{q}(z) + z - \tau) = \langle \rangle_{1,0}(\tau) + \frac{1}{2\epsilon^2} \langle \mathbf{q}(\psi), \mathbf{q}(\psi) \rangle_{0,2}(\tau).$$

Recalling Theorem 6.1 and the formula relating 1- to 2-point descendents we get the following formula:

$$\mathcal{D} = e^{F^{(1)}(\tau)} \widehat{S}_\tau^{-1} \mathcal{A}_\tau = \exp\Big(F^{(1)}(\tau) + \frac{1}{2\epsilon^2} \langle \mathbf{q}(\psi), \mathbf{q}(\psi) \rangle_{0,2}(\tau) \Big) \mathcal{A}_\tau([S_\tau \mathbf{q}]_+), \tag{7.2}$$

where $F^{(1)}(\tau) = \langle \ \rangle_{1,0}(\tau)$ is the genus-1 GW potential.

7.1. **Reconstruction of genus-0 descendents.** Notice that the integral

$$\int_{\overline{\mathcal{M}}_{g,n+l}(X,d)} \overline{\psi}_1^{k_1} \dots \overline{\psi}_n^{k_n} \mathrm{ev}^*(\phi_{a_1} \dots \phi_{a_n} \tau^{\otimes l})$$

is 0 if $k_i \geq 1$, $1 \leq i \leq n$ for dimensional reasons: the product of the $\overline{\psi}$-classes is a cohomology class of degree $\geq n$ pulled back from $\overline{\mathcal{M}}_{0,n}$, which has deimension $n-3$. Now formula (7.2) implies that if we choose $\tau = \tau(\mathbf{q})$ such that $[S_\tau \mathbf{q}]_0 = 0$ then the total genus-0 desecendent potential is given by $\mathcal{F}^{(0)}(\mathbf{q}) = \frac{1}{2} \langle \mathbf{q}(\psi), \mathbf{q}(\psi) \rangle_{0,2}(\tau)$. In other words, we need to find a formal power series $\tau(\mathbf{q})$ which is a solution to the equation:

$$q_0 + S_1(\tau)(q_1 + 1) + S_2(\tau) q_2 + \ldots = \tau,$$

where we used that $S_1(\tau)1 = \tau$. One way to do this is to denote the LHS from above by $F(\tau, \mathbf{q})$ and think of F as a map $\tau \mapsto F(\tau, \mathbf{q})$ in the space of formal series. Then we have to find a fixed point of F. Starting with an initial approximation: $\tau^{(1)} = q_0$, the succesive iterations, $\tau^{(n+1)} = F(\tau^{(n)}, \mathbf{q})$ will generate a sequence which is convergent in the appropriate formal sense to a fixed point $\tau(\mathbf{q})$ of F.

For example, if X is a point. Then $S_\tau(z) = e^{\tau/z}$ and using the formula for 2-point descendents in terms of 1-point descendents we get

$$\mathcal{F}^{(0)}_{\mathrm{pt}}(\mathbf{q}) = \frac{1}{2} \langle \mathbf{q}(\psi), \mathbf{q}(\psi) \rangle_{0,2}(\tau) = \frac{1}{2} \sum_{k,l \geq 0} \frac{\tau^{k+l+1}}{k+l+1} \frac{q_k}{k!} \frac{q_l}{l!},$$

where τ is a solution to

$$q_0 + (q_1+1)\tau + q_2\frac{\tau^2}{2!} + \ldots = \tau.$$

The first three iterations have the form:

$$\tau^{(1)} = q_0, \quad \tau^{(2)} = q_0 + (q_1+1)q_0 + q_2\frac{q_0^2}{2!} + \ldots,$$

$$\tau^{(3)} = q_0 + (q_1+1)q_0 + (q_1+1)^2 q_0 + q_2\frac{q_0^2}{2!} + \ldots$$

8. The quantum Riemann–Roch theorem

Let $E \to X$ be a vector bundle. Put $\mathbb{E}_{g,n,d} = \pi_* \mathrm{ev}_{n+1}^*(E)$ – it is an element of the K-group $K(\overline{\mathcal{M}}_{g,n}(X,d)$. In this lecture, following Mumford (see also [3]) we compute the Chern character $\mathrm{ch}_k(\mathbb{E}_{g,n,d})$. We will use the Grothendieck–Riemann–Roch theorem:

Theorem 8.1. *Let $p : X \to Y$ be a proper map between complex manifolds and $E \in K(X)$ then we have*

$$\mathrm{ch}(\pi_*(E)) = \pi_*(\mathrm{ch}(E) \cup \mathrm{td}^\vee(\Omega_{X/Y})),$$

where $\Omega_{X/Y}$ is the sheaf on X of relative differentials.

The duall Tod class $\mathrm{td}^\vee(L)$ of a line bundle L is by definition:

$$\frac{x}{e^x - 1} = \sum_{r\geq 0} \frac{B_r}{r!} x^r,$$

where $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{12}, \ldots$ are the so called Bernoulli numbers and $x = c_1(L)$. For an arbitrary vector bundle the Tod class is defined by demanding that $\mathrm{td}^\vee$ obeys the multiplicative property:

$$\text{if} \quad 0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0 \quad \text{then} \quad \mathrm{td}^\vee(E) = \mathrm{td}^\vee(E')\mathrm{td}^\vee(E'')$$

For proof and further details we refer to Fulton's book. We apply this theorem to the universal curve. We are not going to justify why Theorem 8.1 can be used in our settings.

Let us give the answer. Consider the diagram:

$$\begin{tikzcd} Z \arrow[r, "j"] & \mathcal{C} := \overline{\mathcal{M}}_{g,n+1}(X,d) \arrow[r, "\mathrm{ev}_{n+1}"] \arrow[d, "\pi"] & X \\ & \overline{\mathcal{M}} := \overline{\mathcal{M}}_{g,n}(X,d) & \end{tikzcd}$$

where Z is the locus of all singular points of the fibers of π. Note that if $t = (\Sigma, p_., f) \in \overline{\mathcal{M}}_{g,n}(X,d)$ then the fiber $\pi^{-1}(t) \cong \Sigma$, so the singular points correspond to the nodes of Σ and we can identify Z with the total range of the gluing maps:

$$\bigsqcup_{\substack{g'+g''=g,\\ n'+n''=n,\\ d'+d''=d}} \overline{\mathcal{M}}_{g',n'+\circ}(X,d') \times_X \overline{\mathcal{M}}_{0,\bullet+1+\bullet}(X,0) \times_X \overline{\mathcal{M}}_{g'',\circ+n''}(X,d'') \xrightarrow{j} \overline{\mathcal{M}}_{g,n+1}(X,d)$$

$$\bigsqcup \overline{\mathcal{M}}_{g-1,n+\circ+\circ}(X,d) \times_{X\times X} \overline{\mathcal{M}}_{0,\bullet+\bullet+1}(X,0)$$

where the maps from the moduli spaces to X necessary to define the fiber products are given by the evaluation maps at the appropriate gluing points.

Theorem 8.2. *Let $\psi_+ = \psi''_\circ$ and $\psi_- = \psi'_\circ$ be the cotangent line classes on Z. Then we have the following formula for* $\mathrm{ch}(\mathbb{E}_{g,nd})$:

$$\sum_{r\geq 0} \frac{B_r}{r!} \pi_* \Big(\mathrm{ev}^*_{n+1} \mathrm{ch}(E) \Big(\psi^r_{n+1} - \sum_{i=1}^{n} (\sigma_i)_* \psi_i^{r-1} + \frac{1}{2} j_* \sum_{a+b=r-2} (-1)^a \psi_+^a \psi_-^b \Big) \Big),$$

where $\sigma_i, 1 \leq i \leq n$ are the sections of the universal curve corresponding to the marked points.

The proof of the theorem amounts to computing the Tod class of the sheaf of relative differentials $\Omega_{\mathcal{C}/\overline{\mathcal{M}}}$ of the universal curve.

8.1. **Two exact sequences.** According to Fin Knudsen, the sheaf $\Omega_{\mathcal{C}/\overline{\mathcal{M}}}$ has a two term vector bundles resolution: $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \Omega_{\mathcal{C}/\overline{\mathcal{M}}} \to 0$. The corresponding determinant sheaf $\omega_{\mathcal{C}/\overline{\mathcal{M}}} = \det(\Omega_{\mathcal{C}/\overline{\mathcal{M}}}) := \wedge^{\mathrm{top}}\mathcal{E}_0 \otimes \wedge^{\mathrm{top}}\mathcal{E}_1^\vee$ is the relative dualizing sheaf. In particular, if we restrict $\omega_{\mathcal{C}/\overline{\mathcal{M}}}$ to a fiber $\pi^{-1}(t) \cong \Sigma$ of the universal curve π then for any vector bundle $\mathcal{E}$ on Σ we have $H^i(\Sigma, \mathcal{E}) = H^{1-i}(\Sigma, \omega_\Sigma \otimes \mathcal{E}^\vee)^\vee$.

Let us compare the sheaves Ω and ω locally. Outside of the singular locus Z, the sheaf Ω is locally free which implies that the two sheaves are canonically identified. Let $C \to S$ be a family of stable maps, and so each fiber C_t represents a point $t \in \overline{\mathcal{M}}$ and it is also identified with the fiber $\pi^{-1}(t)$. Locally, the singular locus of C has the form $xy = \epsilon$, where x and y are local coordinates on C (near the node) and ϵ is a local coordinate on S. Since $\Omega_{C/S} = (\mathcal{O}_C\, dx + \mathcal{O}_C\, dy)/d(xy)\mathcal{O}_C$ we have:

$$\omega_{C/S} = (\mathcal{O}_C dx \wedge dy) \otimes (d(xy)\mathcal{O}_C)^\vee \cong \mathrm{Hom}_{\mathcal{O}_C}(d(xy)\mathcal{O}_C, \mathcal{O}_C dx \wedge dy).$$

Let $\zeta = dx/x$ for $x \neq 0$, or $-dy/y$ for $y \neq 0$. Then we see that $\omega_{C/S} = \mathcal{O}_C\, \zeta$. Comparing the local expressions of $\Omega_{C/S}$ and $\omega_{C/S}$ we get that $\Omega_{C/S} =$

$I_{\rm Sing}\omega_{C/S}$, where $I_{\rm Sing}$ is the ideal sheaf of the singular locus of the fibers of $C \to S$. Therefore we have $\Omega_{\mathcal{C}/\overline{\mathcal{M}}} = I_Z \omega_{\mathcal{C}/\overline{\mathcal{M}}}$. From the exact sequence:

$$0 \longrightarrow I_Z \longrightarrow \mathcal{O}_{\mathcal{C}} \longrightarrow j_*\mathcal{O}_Z \longrightarrow 0,$$

tensoring by $\omega_{\mathcal{C}/\overline{\mathcal{M}}}$ we get

$$0 \longrightarrow \Omega_{\mathcal{C}/\overline{\mathcal{M}}} \longrightarrow \omega_{\mathcal{C}/\overline{\mathcal{M}}} \longrightarrow j_*(\mathcal{O}_Z \otimes j^*\omega_{\mathcal{C}/\overline{\mathcal{M}}}) \longrightarrow 0.$$

On the other hand $\omega_{\mathcal{C}/\overline{\mathcal{M}}} = \mathbb{L}_{n+1}(-D_1 - \ldots - D_n)$, where D_i are the divisors in $\mathcal{C}$ corresponding to the sections σ_i (for proof see Knudsen). Notice that the divisors D_i do not intersect Z and that $j^*\mathbb{L}_{n+1}$ is a trivial bundle. So the above exact sequence turns into:

$$0 \longrightarrow \Omega_{\mathcal{C}/\overline{\mathcal{M}}} \longrightarrow \omega_{\mathcal{C}/\overline{\mathcal{M}}} \longrightarrow j_*\mathcal{O}_Z \longrightarrow 0. \tag{8.1}$$

Finally, put $D = D_1 + \ldots + D_n$. Then we have an exact sequence: $0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0$, tensoring with $\mathbb{L}_{n+1}$ (I am slightly abusing the notations by not distinguishing between vector bundles and locally free sheaves) we get:

$$0 \longrightarrow \omega_{\mathcal{C}/\overline{\mathcal{M}}} \longrightarrow \mathbb{L}_{n+1} \longrightarrow \bigoplus_{i=1}^{n}(\sigma_i)_*\mathcal{O}_{D_i} \longrightarrow 0. \tag{8.2}$$

8.2. **The Tod class.** Using the multiplicative property of the Tod class and the exact sequences (8.1) and (8.2) we get:

$$\mathrm{td}^\vee(\Omega_{\mathcal{C}/\overline{\mathcal{M}}}) = \Big(\mathrm{td}^\vee(\mathbb{L}_{n+1}) - 1 + 1\Big)\prod_{i=1}^{n}\Big(\frac{1}{\mathrm{td}^\vee(\mathcal{O}_{D_i})} - 1 + 1\Big)\Big(\frac{1}{\mathrm{td}^\vee(j_*\mathcal{O}_Z)} - 1 + 1\Big).$$

As it will become clear from the computations below, the above product actually equals to:

$$\mathrm{td}^\vee(\mathbb{L}_{n+1}) + \sum_{i=1}^{n}\Big(\frac{1}{\mathrm{td}^\vee(\mathcal{O}_{D_i})} - 1\Big) + \frac{1}{\mathrm{td}^\vee(j_*\mathcal{O}_Z)} - 1. \tag{8.3}$$

The first term is by definition

$$\frac{\psi_{n+1}}{e^{\psi_{n+1}} - 1} = \sum_{r\geq 0}\frac{B_r}{r!}\psi_{n+1}^r. \tag{8.4}$$

For the second term we used the exact sequence:

$$0 \longrightarrow \mathcal{O}(-D_i) \longrightarrow \mathcal{O} \longrightarrow (\sigma_i)_*\mathcal{O}_{D_i} \longrightarrow 0.$$

The multiplicative property of the Tod class gives us that the i-th summand in the sum in (8.3) is:

$$\mathrm{td}^\vee(\mathcal{O}(-D_i)) - 1 = (-D_i)\Big(\sum_{r\geq 1}\frac{B_r}{r!}(-D_i)^{r-1}\Big),$$

where slightly abusing the notations we identify the divisor D_i with its Poincaré dual cohomology class.

On the other hand the divisor D_i can be identified with the image of the gluing map $\sigma_i : \overline{\mathcal{M}}_{g,n}(X,d) \cong \overline{\mathcal{M}}_{g,n}(X,d) \times_X \overline{\mathcal{M}}_{0,\bullet+2}(X,0) \to \overline{\mathcal{M}}_{g,n+1}(X,d)$, where the domains of the corresponding stable maps are glued respectively at the i-th marked point and the $\bullet$ marked point. Therefore the pullback via σ_i of the normal bundle to D_i is $\mathbb{L}_i^\vee \otimes \mathbb{L}_\bullet^\vee$. In particular $\sigma_i^* D_i = -\psi_i - \psi_\bullet = -\psi_i$, where we used that $\mathbb{L}_\bullet$ is a trivial bundle. Since the Poincaré dual to D_i is the same as $(\sigma_i)_* 1$ we find (using also the projection formula: $\alpha f_* \beta = f_*(f^* \alpha\, \beta)$) that the above formula transforsm into

$$-(\sigma_i)_* \Big(\sum_{r \geq 1} \frac{B_r}{r!} \psi_i^{r-1} \Big).$$

It remains only to compute the last term in (8.3). We will use that Z is the intersection of two divisors D_+ and D_- which can be identified respectively with the images of the gluing maps:

$$j_+ : \bigsqcup \overline{\mathcal{M}}_{g',n'+1+\bullet}(X,d') \times_X \overline{\mathcal{M}}_{g'',\circ+n''}(X,d'') \longrightarrow \overline{\mathcal{M}}_{g,n+1}(X,d)$$

and

$$j_- : \bigsqcup \overline{\mathcal{M}}_{g',n'+\circ}(X,d') \times_X \overline{\mathcal{M}}_{g'',\bullet+1+n''}(X,d'') \longrightarrow \overline{\mathcal{M}}_{g,n+1}(X,d)$$

Notice that the map j, which consists of gluing two pairs of points, factors through both j_+ and j_-, namely: $j = j_+ \sigma_+ = j_- \sigma_-$, where the maps σ_+ and σ_- are the gluing maps respectively at the first and the second pairs of points. So we can identify $Z = D_+ \cap D_-$ as the zero locus of a transversal section of the rank 2 bundle $\mathcal{O}(D_+) \oplus \mathcal{O}(D_-)$. Using the Koszul resoltuion we find:

$$0 \longrightarrow \mathcal{O}(-D_+ - D_-) \longrightarrow \mathcal{O}(-D_+) \oplus \mathcal{O}(-D_-) \longrightarrow \mathcal{O} \longrightarrow j_* \mathcal{O}_Z \longrightarrow 0$$

Again using the multiplicity of the Tod class we get:

$$\frac{1}{\mathrm{td}^\vee(j_* \mathcal{O}_Z)} - 1 = \frac{D_+ D_-}{D_+ + D_-} \Big(\frac{e^{D_+ + D_-} - 1}{(e^{D_+} - 1)(e^{D_-} - 1)} - \frac{1}{D_+} - \frac{1}{D_-} \Big).$$

On the other hand the epression in the brackets could be written as

$$\frac{1}{e^{D_+} - 1} + \frac{1}{e^{D_-} - 1} + 1 - \frac{1}{D_+} - \frac{1}{D_-} = \sum_{r \geq 2} \frac{B_r}{r!} (D_+^{r-1} + D_-^{r-1})$$

Notice that $j_* 1 = 2 D_+ D_-$. The factor of two here comes from the fact that each irreducible component of Z is covered by two gluing maps of the type $\overline{\mathcal{M}}' \times_X \overline{\mathcal{M}}''$ and $\overline{\mathcal{M}}'' \times_X \overline{\mathcal{M}}'$, or in case that $\overline{\mathcal{M}}' = \overline{\mathcal{M}}''$ then the fiber product $\overline{\mathcal{M}}' \times \overline{\mathcal{M}}'$ has an additional $\mathbb{Z}_2$ symmetry, so the corresponding irreducible component of Z instead of being isomorphic to $\overline{\mathcal{M}}' \times_X \overline{\mathcal{M}}'$ is in fact isomorphic to $(\overline{\mathcal{M}}' \times_X \overline{\mathcal{M}}')/\mathbb{Z}_2$.

Just like above we can see that $j_+^*D_+ = -\psi'_\bullet - \psi''_\circ$ and since $(\sigma_+)^*\psi'_\bullet = 0$ we get that $j^*D_+ = -\psi_+$. Similarly, $j^*D_- = -\psi_-$. From here Theorem 8.2 follows easily. $\square$

8.3. **The virtual bundles $\mathbb{E}_{g,n,d}$.** Sinc$X$ is a projective manifold we can find a positive line bundle L, such that $E\otimes L^{\otimes N}$ is generated by global sections (for some N sufficeintly large). In particular, we have an exact sequence of the following type:

$$0 \longrightarrow \mathrm{Ker} \longrightarrow H^0(X, E\otimes L^{\otimes N}) \longrightarrow E\otimes L^{\otimes N} \longrightarrow 0,$$

tensoring with $(L^\vee)^{\otimes N}$ and then pulling back to Σ we get:

$$0 \longrightarrow f^*A \longrightarrow f^*B \longrightarrow f^*E \longrightarrow 0,$$

where f^*B is a direct sum of negative line bundles. In particular $H^0(\Sigma, f^*B) = 0$, so by taking a long exact cohomology sequence we get:

$$0 \longrightarrow H^0(\Sigma, f^*E) \longrightarrow H^1(\Sigma, f^*A) \longrightarrow H^1(\Sigma, f^*B) \longrightarrow H^1(\Sigma, f^*E) \longrightarrow 0$$

From here we get that as an element in K-theory $\mathbb{E}_{g,n,d} = B_{g,n,d} - A_{g,n,d}$ is a difference of two vector bundles.

Lemma 8.3. *We have the following relations:*

$$j^*\mathbb{E}_{g,n+1,d} = \mathbb{E}_{g-1,n+\circ+\circ,d} - \mathrm{ev}^*E + \sum \Big(\mathbb{E}_{g',n'+\circ,d'} + \mathbb{E}_{g'',\bullet+n'',d''} - \mathrm{ev}^*E\Big)$$

and $\pi^*\mathbb{E}_{g,n,d} = \mathbb{E}_{g,n+1,d}$.

To prove this lemma, one may assume that $H^0(\Sigma, f^*E) = 0$ for every holomorphic map $f:\Sigma\to X$ and that $\mathbb{E}_{g,n,d}$ is a bundle with fiber $H^1(\Sigma, f^*E)$. The later due to Serre duality is $H^0(\Sigma, \omega_\Sigma\otimes f^*E^\vee)^\vee$. Recall that the dualizing sheaf ω_Σ for a nodal curve coincides with the sheaf of differentials holomorphic away from the nodes and such that near a node the sections could have simple (logarithmic) poles at such that the residues at the two branches add up to 0. Now the lemma follows easily by comparing the fibers of the corresponding bundles.

8.4. **The quantum Riemann-Roch theorem.** Let X be a projective manifold of complex dimension D. We fix a basis $\phi_a, 0\le a\le N-1$ in $H^*(X)$. We recall the quntization formalism from before: symplectic vector space $\mathcal{H}$, the Lie algebra correspondence $A\mapsto h_A$ between infintesimal symplectic transformations and quadratic functions on $\mathcal{H}$. Also, we define Darboux coordinates on $\mathcal{H}$ by $f = \sum_{k,a} p_{k,a}(f)\phi^a(-z)^{-k-1} + q_k^a(f)\phi_a z^k$.

Let $s_k, k\ge 0$ be a sequence of formal variables. We define a multiplicative characteristic class: $c(E) = e^{\sum s_k \mathrm{ch}_k(E)}$. By definition the twisted GW invariants are the following correlators:

$$\langle c_{g,n,d}; \mathbf{t}, \ldots, \mathbf{t} \rangle_{g,n,d} = \int_{[\overline{\mathcal{M}}_{g,n}(X,d)]} c(\mathbb{E}_{g,n,d}) \wedge \mathbf{t}(\psi_1) \ldots \mathbf{t}(\psi_n),$$

where $\mathbf{t}(\psi_i) = \sum_{k,a} t_k^a \mathrm{ev}_i^*(\phi_a)\psi_i^k$. The twisted total descendent potential is defined by:

$$\mathcal{D}_s = \exp \sum_{g,n,d} \frac{\epsilon^{2g-2}}{n!} Q^d \langle c_{g,n,d}; \mathbf{t}, \ldots, \mathbf{t} \rangle_{g,n,d}.$$

Recall that the Fock space was defined as the space of formal series in $q_0, q_1 + 1, q_2, \ldots$. We identify $\mathcal{D}_s$ with an element of the Fock space by the twisted dilaton shift: $\mathbf{q}(z) = \sqrt{c(E)}(\mathbf{t}(z) - 1)$, i.e.,

$$q_k = \sqrt{c(E)}\, t_k, \quad k \neq 1, \quad q_1 = \sqrt{c(E)}\, (t_1 - 1).$$

Theorem 8.4. *The following formula holds:*

$$\begin{aligned} \mathcal{D}_s &= C(s) \exp\Big(\sum_{\substack{m>0\\ l\geq 0}} \frac{B_{2m}}{(2m)!} s_{l+2m-1} \left(\mathrm{ch}_l(E) z^{2m-1}\right)\widehat{\ } \Big) \times \\ &\quad \times \exp\Big(\sum_{l\geq 0} s_l \left(\mathrm{ch}_{l+1}(E) z^{-1}\right)\widehat{\ } \Big) \mathcal{D}_0, \end{aligned}$$

where

$$\log C(s) = \frac{1}{48} \sum_{k\geq 0} s_k \Big(\int_X e(X)\mathrm{ch}_k(E) + 2 \int_X \mathrm{ch}_{k+1} c_{D-1}(TX) \Big),$$

and the Bernouli numbers B_{2m} are defined by:

$$\frac{x}{e^x - 1} = -\frac{x}{2} + \sum_{m\geq 0} \frac{B_{2m}}{(2m)!} x^{2m}.$$

8.5. **Differential equation for the RHS.** According to our quantization formalism, if $A = A_1 z + A_2 z^2 + \ldots$ and $B = B_1 z^{-1}$ are infinitesimal symplectic transformations, then we have:

$$\widehat{A} = \frac{\epsilon^2}{2} \sum (-1)^i (A_{i+j+1}\phi^a, \phi^b) \frac{\partial^2}{\partial q_i^a \partial q_j^b} + (-1)^i (A_i \phi^a, \phi_b) q_j^b \frac{\partial}{\partial q_{i+j}^a},$$

where the summation is over all $i, j \geq 0$ and $0 \leq a, b \leq N-1$, and

$$\widehat{B} = -\frac{1}{2\epsilon^2}(B_1 q_0, q_0) - \sum_{k\geq 0} (B_1\phi^a, \phi_b) q_{k+1}^b \frac{\partial}{\partial q_k^a}.$$

On the other hand if F and G are quadratic Hamiltonians, then we have

$$[\widehat{F}, \widehat{G}] = (\{F, G\})\widehat{\ } + C(F, G),$$

where the cocycle is non-zero only for the following pairs of Darboux quadratic monomials: $C(p_a^2, q_a^2) = 2$ and $C(p_ap_b, q_aq_b) = 1$, for $a \neq b$. In particular for the infinitesimal symplectic transformations A and B we have:

$$[\widehat{A}, \widehat{B}] = [A, B]\widehat{\ } - \frac{1}{2}\mathrm{tr}(A_1B_1).$$

This implies that $e^{\widehat{A}}\widehat{B}e^{-\widehat{A}} = e^{\mathrm{ad}(A)}\widehat{B} = \widehat{B} - \frac{1}{2}\mathrm{tr}(A_1B_1)$. Keeping this remark in mind it is easy to see that the RHS of the formula we want to prove satisfies the following differential equations:

$$\frac{\partial}{\partial s_k}\mathrm{RHS} = \Big(\sum_{\substack{r=0\\ r\neq 1}}^{k+1} \frac{B_r}{r!}(\mathrm{ch}_{k+1-r}(E)z^{r-1})\widehat{\ }\Big)\mathrm{RHS} + \frac{1}{48}L(s)\mathrm{RHS}, \tag{8.5}$$

where the linear (with respect to s) term $L(s)$ is given by the following formula:

$$\int_X \left(e(X)\mathrm{ch}_k(E) + 2c_{D-1}(TX)\mathrm{ch}_{k+1}(E) - 2e(X)\Big(\sum_{l\geq 0} s_{l+1}\mathrm{ch}_l(E)\Big)\mathrm{ch}_{k+1}(E)\right).$$

Let us remark that in order to derive the above differential equation one should also use that $\mathrm{tr}(\alpha\cup) = \int_X \alpha e(X)$, where $e(X)$ is the Euler class of the tangent bundle.

So we need to check that $\mathcal{D}_s$ satisfies the above differential equation. Let us differentiate $\log \mathcal{D}_s$:

$$\frac{\partial}{\partial s_k}\log\mathcal{D}_s = \sum_{g,n,d}\frac{\epsilon^{2g-2}Q^d}{n!}$$
$$\Big(\langle \mathrm{ch}_k(\mathbb{E}_{g,n,d}) \wedge c_{g,n,d}; \mathbf{t}, \ldots, \mathbf{t}\rangle_{g,n,d} + \langle c_{g,n+1,d}; \frac{\partial \mathbf{t}}{\partial s_k}, \mathbf{t}, \ldots, \mathbf{t}\rangle_{g,n+1,d}\Big).$$

According to Theorem 8.2 we have that $\mathrm{ch}_k(\mathbb{E}_{g,n,d})$ equals to:

$$\sum_{r=0}^{k+1}\frac{B_r}{r!}\pi_*\Big(\mathrm{ev}_{n+1}^*\mathrm{ch}_{k+1-r}(E)\,\Big(\psi_{n+1}^r - \sum_{i=1}^n(\sigma_i)_*\psi_i^{r-1} + \frac{1}{2}j_*\sum_{a+b=r-2}(-1)^a\psi_+^a\psi_-^b\Big)\Big).$$

Now, one has to consider three cases depending whether r is $0, 1$, or ≥ 2. Since the ideas and the techniques in each of these three cases are the same, we consider only the case when $r \geq 2$. More precisely we check that

$$\Big(\sum_{g,n,d}\frac{\epsilon^{2g-2}Q^d}{n!}\langle\Big(\psi_{n+1}^r - \sum_{i=1}^n(\sigma_i)_*\psi_i^{r-1} + \frac{1}{2}j_*\sum_{a+b=r-2}(-1)^a\psi_+^a\psi_-^b\Big)\wedge$$
$$\pi^*c_{g,n,d}; \bar{\mathbf{t}}, \ldots, \bar{\mathbf{t}}, \mathrm{ch}_{k+1-r}(E)\rangle_{g,n+1,d}\Big)\mathcal{D}_s = \big(\mathrm{ch}_{k+1-r}(E)z^{r-1}\big)\widehat{\ }\mathcal{D}_s, \tag{8.6}$$

where $\overline{\mathbf{t}} = \pi^*\mathbf{t}$. Notice that

$$\overline{\psi}_i^k = \overline{\psi}_i^{k-1}(\psi_i - (\sigma_i)_*1) = \psi_i\overline{\psi}_i^{k-1} - (\sigma_i)_*\psi_i^{k-1} = \ldots = \psi_i^k - (\sigma_i)_*\psi_i^{k-1}.$$

Let us simplify

$$\sum_{g,n,d} \frac{\epsilon^{2g-2}Q^d}{n!} \sum_{a+b=r-2} \frac{(-1)^a}{2} \langle j_*\psi_+^a\psi_-^b \wedge c_{g,n+1,d}; \overline{\mathbf{t}}, \ldots, \overline{\mathbf{t}}, \mathrm{ch}_{k+1-r}(E)\rangle_{g,n+1,d}\mathcal{D}_s,$$

We may assume that $\overline{\psi}_i^k = \psi_i^k$, becaus the cohomology class $(\sigma_i)_*\psi_i^{k-1}$ is supported on the divisor $D_i = \mathrm{Im}(\sigma_i)$ and the singular locus is disjoint from D_i. Notice that the above correlator could be written as a sum of correlators of two types:

$$\Big\langle c_{g',n'+\circ,d'}; \mathbf{t}, \ldots, \mathbf{t}, \mathrm{ch}_{k+1-r}(E)\frac{\phi^\mu\psi_-^b}{\sqrt{c(E)}}\Big\rangle_{g',n'+\circ,d'} \Big\langle c_{g'',\bullet+n'',d''}; \mathbf{t}, \ldots, \mathbf{t}, \frac{\phi_\mu\psi_+^a}{\sqrt{c(E)}}\Big\rangle_{g'',\bullet+n'',d''},$$

where $g' + g'' = g, n' + n'' = n, d' + d'' = d$, and

$$\Big\langle c_{g-1,n+\circ+\bullet,d}; \mathbf{t}, \ldots, \mathbf{t}, \mathrm{ch}_{k+1-r}(E)\frac{\phi^\mu}{\sqrt{c(E)}}, \frac{\phi_\mu}{\sqrt{c(E)}}\Big\rangle_{g-1,n+\circ+\bullet,d}.$$

On the other hand we have $\partial_{q_k^\nu}\mathbf{t}(z) = \frac{\phi_\nu z^k}{\sqrt{c(E)}}$. So if we put $\mathcal{F} = \log \mathcal{D}_s$ then the sum from above turns into:

$$\frac{\epsilon^2}{2} \sum_{a+b=r-2} (-1)^a \left(\mathrm{ch}_{k+1-r}(E)\phi^\mu, \phi^\nu\right) \Big(\frac{\partial\mathcal{F}}{\partial q_\nu^b}\frac{\partial\mathcal{F}}{\partial q_\mu^a} + \frac{\partial^2\mathcal{F}}{\partial q_\nu^b\partial q_\mu^a}\Big)e^{\mathcal{F}}.$$

Notice that

$$\Big(\frac{\partial\mathcal{F}}{\partial q_\nu^b}\frac{\partial\mathcal{F}}{\partial q_\mu^a} + \frac{\partial^2\mathcal{F}}{\partial q_\nu^b\partial q_\mu^a}\Big)e^{\mathcal{F}} = \frac{\partial^2}{\partial q_\nu^b\partial q_\mu^a}\, e^{\mathcal{F}}$$

and that

$$\frac{\epsilon^2}{2} \sum_{a+b=r-2} (-1)^a \left(\mathrm{ch}_{k+1-r}(E)\phi^\mu, \phi^\nu\right) \frac{\partial^2}{\partial q_\nu^b\partial q_\mu^a}$$

coincides with the quntization of the p^2-squares terms of the operator $\mathrm{ch}_{k+1-r}(E)z^{r-1}$. The rest of the details are left to the reader. □

9. The quantization operator

Let M be a small ball centered at 0 in $\mathbb{C}^N$, equipped with a Frobenius structure, i.e., a family of commutative associative multiplications $\bullet_\tau : T_\tau M \otimes T_t M \to T_t M$, a flat, non-degenerate, complex bilinear pairing $(\ ,\ )_t : T_t M \otimes T_t M \to \mathbb{C}$, satisfying the integrability conditions listed in Definition 5.4, except possibly for the absense of an Euler vector field.

Definition 9.1. *We say that a Frobenius structure is semi-simple if there are local vector fields $e_i, 1 \leq i \leq N$ such that the multiplication and the pairing assume a diagonal form:*

$$e_i \bullet_t e_j = \delta_{ij} e_j, \quad (e_i, e_j)_t = \frac{\delta_{ij}}{\Delta_i}.$$

Theorem 9.2. *If M is semi-simple then there are local coordinates $u^i, 1 \leq i \leq N$, called* canonical, *such that $e_i = \partial/\partial u^i$.*

Proof. We need just to check that the vector fields e_i pairwise commute. Let $\nabla^{\mathrm{LC}}_{e_i} e_j = \Gamma^q_{ij} e_q$ (summation over repeating upper and lower indexes is assumed). By definition the flatness of the connection $\nabla = \nabla^{\mathrm{LC}} - z^{-1} \sum_a (\partial_{t^a} \bullet)\, dt^a$ means: $\left(\nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i}\right) e_k = \nabla_{[e_i, e_j]} e_k$. Comparing the coefficients in front of z^{-1} we get:

$$\nabla^{\mathrm{LC}}_{e_i} (e_j \bullet e_k) + e_i \bullet \nabla^{\mathrm{LC}}_{e_j} e_k - [i \leftrightarrow j] = [e_i, e_j] \bullet e_k,$$

where the term in the square brackets is obtained from the preceeding expression by swapping i and j. The above identity transforms into:

$$\delta_{jk} \Gamma^q_{ij} e_q + \Gamma^q_{jk} \delta_{iq} e_q - [i \leftrightarrow j] = [e_i, e_j] \bullet e_k.$$

Notice that the RHS in the above equality is exactly the coefficient in front of e_k in $[e_i, e_j]$. On the other hand on the LHS the coefficient in front of e_k is

$$\delta_{jk} \Gamma^k_{ij} + \Gamma^k_{jk} \delta_{ik} - \left(\delta_{ik} \Gamma^k_{ij} + \Gamma^k_{ik} \delta_{jk}\right) = 0.$$

$\square$

Lemma 9.3. *If the Frobenius structure on M is confromal with Euler vector field E, then*

$$[E, X \bullet Y] - [E, X] \bullet Y - X \bullet [E, Y] = X \bullet Y,$$

for every vector fields X and Y.

Proof. By definition the connection operators (acting on sections of the bundle $TM \to M \times \mathbb{C}^*$) $\nabla_X = \nabla^{\mathrm{L.C.}}_X - z^{-1} X \bullet$ and $\nabla_{\partial/\partial z} = \partial/\partial z - (z^{-2} E \bullet - z^{-1} \mu)$ satisfy a flatness condition. In particular, we have

$$\nabla_E \nabla_X Y - \nabla_X \nabla_E Y = \nabla_{[E,X]} Y \tag{9.1}$$

and

$$\nabla_{\partial/\partial z} \nabla_X Y - \nabla_X \nabla_{\partial/\partial z} Y = \nabla_{[\partial/\partial z, X]} Y = 0. \tag{9.2}$$

To prove the lemma, one has to extract two identities, obtained from (9.1) and (9.2) by comparing the coefficients in front of z^{-1} and z^{-2} respectively:

$$\nabla^{\mathrm{LC}}_E (X \bullet Y) + E \bullet (\nabla^{\mathrm{LC}}_X Y) - \left(\nabla^{\mathrm{LC}}_X (E \bullet Y) + X \bullet (\nabla^{\mathrm{LC}}_E Y)\right) = [E, X] \bullet Y,$$

$$E \bullet (\nabla^{\mathrm{LC}}_X Y) + X \bullet Y + \nabla^{\mathrm{LC}}_{X \bullet Y} E = X \bullet \nabla_Y E + \nabla^{\mathrm{LC}}_X (E \bullet Y),$$

where we used that by definition $\mu(Z) = \nabla^{\mathrm{LC}}_Z E$. Subtracting these two identities and then using that the Levi-Cevita connection is torsion free, i.e. $\nabla^{\mathrm{LC}}_X Y - \nabla^{\mathrm{LC}}_Y X = [X,Y]$ we get the identity we wanted to prove. $\square$

Corollary 9.4. *For a conformal, semisimple Frobenius structure, one can choose canonical coordinates u^i such that the Euler vector field assumes the form: $E = \sum_i u^i \frac{\partial}{\partial u^i}$.*

Proof. Let $u = (u^1, \ldots, u^N)$ be any canonical coordinate system. Let $E = \sum E_i(u)\frac{\partial}{\partial u^i}$. Recalling Lemma 9.3 with $X = \partial/\partial u^i$ and $Y = \partial/\partial u^j$ we get

$$-\delta_{ij}\frac{\partial E_q}{\partial u^j}(\partial/\partial u^q) + \frac{\partial E_i}{\partial u^j}(\partial/\partial u^i) + \frac{\partial E_j}{\partial u^i}(\partial/\partial u^j) = \delta_{ij}(\partial/\partial u^j).$$

From here it follows that $\partial E_i/\partial u^j = \delta_{ij}$, i.e., up to some constants $E_i = u^i, 1 \le i \le N$. Replacing u^i by appropriate translations of u^i we obtained the desired canonical coordinate system $\square$

Let us assume that M has a semi-simple Frobenius structure. We denote by $u = (u^1, \ldots, u^N)$ a canonical coordinate system, by $\tau = (\tau^0, \ldots, \tau^{N-1})$ a flat coordinate system and by $\partial_a = \partial/\partial\tau^a$ the corresponding flat vector fields. In case the Frobenius structure is conformal we require that the canonical coordinates are chosen in such a way that $E = \sum_i u^i \frac{\partial}{\partial u^i}$.

Using the canonical coordinates we trivialize the tangent bundle TM, by the following family of linear operators, parametrized by $\tau \in M$:

$$\Psi_\tau : \mathbb{C}^N \to T_\tau M, \quad \Psi e_i = \sqrt{\Delta_i}\frac{\partial}{\partial u^i}. \tag{9.3}$$

Put $U = \mathrm{diag}(u^1, \ldots, u^N)$. Our next goal is to describe the formal asymptotical solutions, near $z = 0$ to the system

$$\begin{aligned}
&z\partial_a J = (\partial_a \bullet)J, \quad 0 \le a \le N-1,\\
&(z\partial_z + L_E)J = \mu J,\\
&J = \Psi_\tau(1 + R_1(\tau)z + R_2(\tau)z^2 + \ldots)e^{U/z},
\end{aligned}$$

where the second equation should be considered only if the Frobenius structure is conformal, and $R_k(\tau)$ are matrices which together with U are idenitfied with linear operators on $\mathbb{C}^N$ via the standard basis $e_i, 1 \le i \le N$. Also, we identify Ψ with a $N \times N$ matrix via the standard basis in $\mathbb{C}^N$ and the flat basis ∂_a in $T_\tau M$.

Lemma 9.5. *a) Let g be the matrix of the flat metric, i.e., $g_{ab} = (\partial_a, \partial_b)$. Then the following formula holds: ${}^{\mathrm{t}}\Psi_\tau g \Psi_\tau = 1$. In particular, $\Psi^{-1}d\Psi$ is an anti-symmetric matrix.*

b) Let $A = \sum_a (\partial_a \bullet)d\tau^a$. Then $A = \Psi dU \Psi^{-1}$.

Proof. a) It follows from

$$\delta_{ij} = (\Psi e_i, \Psi e_j) = \sum_{a,b} \Psi_{ia}\Psi_{jb}(\partial_a, \partial_b).$$

b) The identity follows easily becaus if we think of the matrix A as a linear operator then since $A = \sum_i (\partial/\partial u^i \bullet) du^i$ we see that in the basis $e_i = \sqrt{\Delta_i}\partial/\partial u^i$ the matrix of A becomes dU, so the formula is just the standard change of basis in linear algebra. $\square$

Theorem 9.6. Existence: *The formal asymptotical solution always exists.* Uniqueness: *a) If we require that* ${}^{\mathrm{t}}R(-z)R(z) = 1$*, then* R *is uniquely determined up to multiplication from the right by* $\exp\Big(\sum_{m\geq 1} A_{2m-1}z^{2m-1}\Big)$*, where* A_{2m-1} *are constant diagonal matrices.*

b) If the Frobenius structure is semisimple then R *is uniquely determined and it automatically satisfies the symplectic condition:* ${}^{\mathrm{t}}R(-z)R(z) = 1$.

Proof. Using Lemma 9.5, part b) we get that the system of differential equations for J is equivalent to:

$$(d + \Omega)R = z^{-1}[dU, R], \quad \Omega = \Psi^{-1}d\Psi.$$

Comapring the coefficients in front of z^k we get:

$$(d + \Psi^{-1}d\Psi)R_k = [dU, R_{k+1}], \quad k \geq 0. \tag{9.4}$$

The equation corresponding to $k = 0$ reads: $(du^i - du^j)R_1^{ij} = (\Omega)^{ij}$. From here we find $R_1^{ij} = (\Psi^{-1}\partial_{u^i}\Psi)^{ij}$ for $i \neq j$. Now one has to check that if R_1 is defined by the above formula then $[dU, R_1] = \Omega$. This follows from the fact that $dU \wedge \Omega + \Omega \wedge dU = d(dU) = 0$, which implies that $\Omega^{ij} \wedge (du^i - du^j) = 0$. From the case $k = 1$, by comparing the diagonal entries we find that $dR_1^{ii} = -\sum_j (\Omega)^{ij}R_1^{ji}$. Notice that in the above sum the RHS depends only on the offdiagonal entries of R_1, so the diagonal entries are determined up to a constant.

Continuing in the same fashion we see that the the entries of R_k are uniquely determined except for the diagonal ones which are determined up to a constant, i.e., that R can be recovered uniquely up to a factor $\exp\Big(\sum_k A_k z^k\Big)$.

Put $P = {}^{\mathrm{t}}R(-z)R(z) = 1 + P_1 z + \ldots$. Then one checks that $dP = z^{-1}[dU, P]$, i.e., $dP_k = [dU, P_{k+1}]$. From $k = 0$ case we get P_1 is diagonal and then $dP_1 = 0$ because $[dU, P_2]$ is off-diagonal. Continuing in the same way, we get that all P_k's are constant diagonal matrices. Also, from

$$P_k = (-1)^{k\,\mathrm{t}}R_k + (-1)^{k-1\,\mathrm{t}}R_{k-1}R_1 + \ldots + R_k$$

we get ${}^{\mathrm{t}}P_k = (-1)^k P_k$, so $P_{2m+1} = 0$. Now the first existence statement follows easily.

For the second one, we use that $L_E R_k = -kR_k$. By the Cartan's magic formula:

$$-kR_k^{ii} = \iota_E dR_k = \iota_E\Big(-\sum_j \Omega^{ij}R_k^{ji}\Big) = \sum_j (u^j - u^i)R_1^{ij}R_k^{ji}.$$

□

We trivialize the tangent bundle $TM \cong M \times H$, $H := T_0M$ via the flat metric. Put $\phi_a = \partial_a$, basis of H. Using the flat metric instead of intersection pairing, we addopt the same quantization formalism on $\mathcal{H} = H((z^{-1}))$ as before.

Lemma 9.7. *Let $R = 1 + R_1 z + R_2 z^2 + \dots$ be a symplectic transformation, i.e. ${}^{\mathrm{t}}R(-z)R(z) = 1$. Then*

$$(\widehat{R}\mathcal{F})(\mathbf{q}) = \left(e^{\frac{\epsilon^2}{2}V(\partial_q,\partial_q)}\mathcal{F}\right)(R^{-1}\mathbf{q}),$$

where the Laplacian $V(\partial_q,\partial_q) = \sum(\phi^a, V_{kl}\phi^b)\partial_{q_k^a}\partial_{q_l^b}$ is defined by

$$\sum_{k,l} V_{kl}(-z)^k(-w)^l = \frac{{}^{\mathrm{t}}R(z)R(w) - 1}{z+w}.$$

In the above lemma $\mathcal{F}$ is an element of the Fock space on the variables q_k^a, $k \geq 0$, $0 \leq a \leq N-1$. However, this action does not always makes sense. For example if $\mathcal{F}(\mathbf{q}) = q_0^a + q_1^a + \dots$ then $\mathcal{F}(R^{-1}\mathbf{q})$ is a linear function in $\mathbf{q}$ whose coefficient in front of (say) q_0^b is $\sum_k {}^{\mathrm{t}}(R_k\phi_a, \phi^b)(-1)^k$, and the later sum might be diveregent.

We prove that the action of R is well defined on the class of the so called tame asymptotical functions. By definition, an expression of the type

$$\mathcal{F} = \exp\sum_{g=0}^{\infty}\epsilon^{2g-2}\mathcal{F}^{(g)}(\mathbf{q}),$$

is called *an asymptotical function.* Furthermore, such a function is called *tame* if each $\mathcal{F}^{(g)}$ is a sum of monomials of the type $q_{k_1}^{a_1}\dots q_{k_r}^{a_r}$ satisfying $k_1+\dots+k_r \leq 3g-3+r$. For the purposes of GW theory one has to incorporate everywhere the dilation shift, however, we leave this to the reader.

We are going to use the following combinatorial fact. Let $V = \{V_{ij}\}, i,j \geq 0$ be an infinite symmetric matrix. Given a graph Γ whose flags (flag is a pair of incident vertex and edge of Γ) are labeled by the integers $i \geq 0$. For each edge $e \in E(\Gamma)$ put $V_e = \epsilon V_{ij}$, where i,j are the labels of the two flags incident with e. Also, for each vertex $v \in V(\Gamma)$ we put $\partial_v = \partial_{i_1}\dots\partial_{i_r}$ where $i_1,\dots,i_r$ are the labels of the flags incident with v.

Lemma 9.8. *a) The following formula holds:*

$$e^{\frac{\epsilon^2}{2}\sum_{i,j\geq 0} V_{ij}\frac{\partial^2}{\partial q_i \partial q_j}} e^{\mathcal{F}} = \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{e\in E(\Gamma)} V_e \prod_{v\in V(\Gamma)} \partial_v \mathcal{F},$$

where the sum is taken over all, possibly disconnected, graphs.

b) The logarithm of the LHS is given by the same formula except that the summation is over all connected graphs.

Lemma 9.9. *The symplectic transformation R preserves the class of tame asymptotical functions.*

Proof. Assume that $\mathcal{F}$ is a tame asymptotical function. Let $\ln \mathcal{F} = \sum \epsilon^{2g-2} q_{k_1}^{a_1}\dots q_{k_r}^{a_r}$. The tameness of $\mathcal{F}$ is equivalent to $k_1+\ldots+k_r \leq 3g-3+r$.

We have

$$\ln\left(e^{\frac{\epsilon^2}{2}\sum_{ij}V_{ij}\partial_i\partial_j}\mathcal{F}\right) = \sum \epsilon^{2g-2} q_{k_1}^{a_1}\dots q_{k_r}^{a_r}.$$

On the RHS we fix a monomial $\epsilon^{2g-2}q_{k_1}^{a_1}\dots q_{k_r}^{a_r}$. Using Lemma 9.8, b) we write the LHS as a sum over connected graphs Γ, for each vertex v of Γ we choose a monomial $\epsilon^{2g_v-2}q_{k_1^v}^{a_1^v}q_{k_2^v}^{a_2^v}\dots$ from the expansion of $\ln\mathcal{F}$. Let us see what Γ and corresponding vertex contributions contribute to the monomial on the RHS that we fixed. We use the following notations: e_v – number of edges incident with $v\in V(\Gamma)$, the set of labels $\{(k_1^v,a_1^v),(k_2^v,a_2^v),\dots\}$ splits into two parts labels that correspond to the flags incident with v and labels (k_i,a_i) associated with the the monomial $\epsilon^{2g-2}q_{k_1}^{a_1}\dots q_{k_r}^{a_r}$. Let r_v be the number of the second type of labels (notice $\sum_v r_v = r$) and denote by k_v the sum of all k_is of these labels. Finally, l_v is the sum of ks from all labels (k,a) of the flags incident with v.

From the tameness of $\mathcal{F}$ we have:

$$3g_v - 3 + e_v + r_v \geq k_v + l_v.$$

On the other hand, using the genus relation $2g-2 = \sum_v(2g_v-2)+2|E(\Gamma)|$, we get

$$g-1 = \sum_v (g_v-1) + |E(\Gamma)| \geq |E(\Gamma)| - |V(\Gamma)| \geq -1,$$

where we used that Γ is connected. In particular, $g\geq 0$. Also, $3g-3+r = \sum_v(3g_v-3)+3|E(\Gamma)|+\sum_v r_v \geq \sum_v(k_v+l_v-e_v)+3|E(\Gamma)| = \sum_v(k_v+l_v) + |E(\Gamma)| \geq \sum_v k_v = \sum_i k_i$.

The fact that the transformation $\mathbf{q}\mapsto R^{-1}\mathbf{q}$ preserves the class of tame functions is obvious. □

Assume that $\tau\in M$ is a semi-simple point. Then we have two Fock spaces: one associated to $H = T_\tau M$ and the flat metric, and another one associated to $\mathbb{C}^N$ and the standard complex bi-linear form. The two spaces are naturally identified via Ψ_τ. Explicitly, we have: $\mathbf{q}(z) = \sum_{k,a} q_k^a \partial_a z^k \in H[z]$ and $Q(z) = \sum_{k,i} Q_k^i e_i z^k \in \mathbb{C}^N[z]$. Then the identification Ψ, which we also denote by $\widehat{\Psi}$, is given by the following substitution: $q_k^a = \sum_i \Psi_{ai} Q_k^i$.

Lemma 9.10. *Let $S_\tau = 1 + S_1(\tau)z^{-1} + \ldots$ be a solution to the system of differential equations $z\partial_a S_\tau = (\partial_a \bullet_\tau)S, 0 \leq a \leq N-1$. There exists a constant $C(\tau)$ such that the composition of differential operators, whenever their composition makes sense,*

$$e^{C(\tau)} \widehat{S}_\tau^{-1} \widehat{\Psi}_\tau \widehat{R} e^{(U/z)^\wedge}$$

is independent of τ. In fact the constant is given by: $C(\tau) = \int^\tau \sum_{i=1}^N R_1^{ii} du^i$.

10. Equivariant cohomology and fixed-point localization

Let $T = (S^1)^{n+1}$ be the $(n+1)$-dimensional torus acting on a compact complex manifold X. We assume further that the fixed points set $X^T = \{x \in X \mid T \cdot x = x\}$ is a submanifold (possibly disconnected) of X.

10.1. **Basic definitions.** Let $ET \to BT$ be the universal T-bundle. The equivariant cohomology $H_T^*(X)$, of a topological space X equipped with a T-action, is by definition the cohomology (we work only with coefficients in $\mathbb{C}$) of $(ET \times X)/T$, where T acts diagonally, i.e., $t \cdot (v, x) = (t \cdot v, t \cdot x)$. Since the above construction is functorial, every T-equivariant map $X \to Y$ induces a ring homeomorphism $H_T^*(Y) \to H_T^*(X)$. In particular, the contraction map $X \to \mathrm{pt}$ turns every $H_T^*(X)$ into a $H_T^*(\mathrm{pt})$-module.

Lemma 10.1. *a) The algebra $H_T^* := H_T^*(\mathrm{pt})$ is naturally isomorphic to the symmetric algebra $S(\mathfrak{t}^*)$, where $\mathfrak{t}$ is the Lie algebra of T.*

b) If $T' \subset T$ then there is a natural map $\phi : H_T^* \to H_{T'}^*$. Furthermore, there is $f \in H_T^*$ such that $f \neq 0$ and $\phi(f) = 0$.

Proof. First, consider the case $T = S^1$. Recall that $ES^1 = S^\infty$ is the set of unitary vectors in $\mathbb{C}^\infty$ and $BS^1 = \mathbb{C}P^\infty$ is the set of all complex lines in $\mathbb{C}^\infty$. Notice that if we take the standard representation of S^1 on $\mathbb{C}$, then the corresponding line bundle $(ES^1 \times \mathbb{C})/S^1$ is the tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{C}P^\infty$, so the standard generator of $H^*(\mathbb{C}P^\infty)$ is $c_1(O(1)) = c_1((ES^1 \times \mathbb{C}^*)/S^1)$. For the genral case, let ν_i, $0 \leq i \leq n$ be a basis in $\mathfrak{t}^*$, dual to the standard basis of $\mathfrak{t} \cong \mathbb{R}^{n+1}$, i.e., $\nu_i(e_j) = \delta_{ij}$. The set of characters $\mathrm{Hom}(T, \mathbb{C}^*)$ is naturally identified with the lattice $\mathfrak{t}_\mathbb{Z}^* = \mathbb{Z}\nu_0 + \ldots + \mathbb{Z}\nu_n$: if $\chi \in \mathfrak{t}_\mathbb{Z}$ then $\chi(e^X) := e^{\chi(X)}$. Using the well known description of the cohomology of $\mathbb{C}P^\infty$, it is easy to see that the map:

$$\chi \in \mathfrak{t}_\mathbb{Z}^* \mapsto c_1(L_\chi) \in H^2(BT; \mathbb{Z}), \quad L_\chi = (ET \times_\chi \mathbb{C}^*)/T,$$

where the torus representation is the one dual to the representation corresponding to χ, induces the isomorphism $S(\mathfrak{t}^*) \cong H^*(BT) = H_T^*$. This proves part a).

In other words we may think of H_T^* as the algebra of functions on $\mathfrak{t}$. Moreover, if $T' \subset T$, then the map $BT' = ET/T' \to BT$ induces a ring homeomorphism $S(\mathfrak{t}^*) \to S((\mathfrak{t}')^*)$ which is in fact a restriction to the subvector space $\mathfrak{t}'$ of $\mathfrak{t}$. Part b) of the lemma follows. $\square$

Theorem 10.2 (Borel's localization)**.** *The inclusion map $X^T \to X$ induces a ring isomorphism:*

$$H_T^*(X) \otimes_{H_T^*} \mathbb{C}(\nu_0, \ldots, \nu_n) \cong H_T^*(X^T) \otimes_{H_T^*} \mathbb{C}(\nu_0, \ldots, \nu_n),$$

where we used that $S(\mathfrak{t}^*) = H_T^* = \mathbb{C}[\nu_0, \ldots, \nu_n]$.

Proof. We follow the argument from Hsiang's book. Since X is compact we can find a covering of T-invariant open subsets U_i, $1 \leq i \leq N$, V such that V is a tubular neighborhood of X^T, and U_i are tubular neighborhoods of T-orbits Tx_i, $x_i \in X$. Put $U = U_1 \cup \ldots \cup U_N$.

Step 1: there exists $f \in H_T^*$ such that $f \cdot H_T^*(U) = 0$. Since U_i is a deformation retract of Tx_i, we have

$$H_T^*(U_i) = H_T^*(Tx_i) = H^*(ET/T_{x_i}) = H^*(BT_{x_i}),$$

where $T_{x_i} = \{t \in T \mid tx_i = x_i\}$. So we can find $f_i \in H_T^*$ such that $f_i|_{\mathfrak{t}_{x_i}} = 0$, in particular $f_i \cdot H_T^*(U_i) = 0$.

Put $f = f_1 \ldots f_N$. Then $f \cdot H_T^*(U_i) = 0$ for all i. Using the functoriality of the equivariant construction we get that the maps $U_i \cap U_j \to U_i \to X \to \mathrm{pt}$, gives us $f \cdot H_T^*(U_i \cap U_j) = 0$. The equivaraint version of the Mayer-Vietories sequence holds, so we have:

$$\ldots \longrightarrow H_T^*(U_i \cap U_j) \xrightarrow{\delta} H_T^*(U_i \cup U_j) \xrightarrow{\beta} H_T^*(U_i) \oplus H_T^*(U_j) \longrightarrow \ldots$$

where all the maps are H_T^*-module homeomorphisms. If $m \in H_T^*(U_i \cup U_j)$ is arbitrary, then we have $\beta(fm) = (fm', fm'') = 0$, so $fm = \delta(m'')$ and we get $f^2 m = f\delta(m'') = \delta(fm'') = 0$. In other words, $f^2 \cdot H_T^*(U_i \cup U_j) = 0$, for all $1 \leq i, j \leq N$. So inductively, we get $f^N \cdot H_T^*(U) = 0$.

Step 2. From the long exact sequence of the pair (X, X^T)

$$\ldots \longrightarrow H_T^*(X, X^T) \longrightarrow H_T^*(X) \longrightarrow H_T^*(X^T) \longrightarrow H_T^*(X, X^T) \longrightarrow \ldots$$

we get that it is enough to prove that $H_T^*(X, X^T) \otimes_{H_T^*} \mathbb{C}(\nu_0, \ldots, \nu_n) = 0$, i.e., that for every $m \in H_T^*(X, X^T)$ there exists $f \in H_T^*$ such that $fm = 0$.

In fact we prove that f, chosen as in Step 1 works. We have the following commutative diagram:

$$\begin{tikzcd}
 & H_T^* \otimes H_T^*(X,V) \arrow[d] & \\
H_T^*(X,U) \otimes H_T^*(X,V) \arrow[r, "\alpha"] \arrow[d, "\cup"] & H_T^*(X) \otimes H_T^*(X,V) \arrow[r, "\beta"] \arrow[d, "\cup"] & H_T^*(U) \otimes H_T^*(X,V) \arrow[d, "\cup"] \\
H_T^*(X, U \cup V) \arrow[r] & H_T^*(X,V) \arrow[r] & H_T^*(U, U \cap V)
\end{tikzcd}$$

Since $\beta(f \cdot 1) = 0$, we have $f = \alpha(m'')$, so $fm = \alpha(m'') \cup m = \alpha(m'' \cup m) = 0$, because $m'' \cup m \in H_T^*(X, U \cup V) = 0$. $\square$

10.2. **Equivariant integration.** If $\alpha \in H_T^*(X)$ then we define $\int_X \alpha \in H_T^*$ to be the pushforward of α via the projection of the fibration $X_T \to BT$.

Theorem 10.3. *Assume that X is a compact complex manifold equipped with a T-action such that the fixed points set is a submanifold of X. Then the following formula holds:*

$$\int_X \alpha = \int_{X^T} \frac{i^*\alpha}{E_T(i)}, \quad \alpha \in H_T^*(X) \otimes \mathbb{Q}(\nu_0, \dots, \nu_n)$$

where $i : X^T \hookrightarrow X$ is the embedding of the fixed point locus and $E_T(i)$ is the equivariant Euler class of the corresponding normal bundle.

Proof. We may assume that X^T is connected. First notice that $E_T(i)$ is invertible in $H_T^*(X) \otimes \mathbb{Q}(\nu_0, \dots, \nu_n)$. Indeed, let $N \to X^T$ be the normal bundle. Then each fiber N_x, $x \in X^T$ is equipped with a T-action and so it splits into a sum of T-invariant complex lines $(L_i)_x \cong \mathbb{C}$, $i = 1, 2, \dots, r$, $r = \operatorname{codim}(X^T)$. Moreover, if we denote by $\chi_{i,x} \in \mathfrak{t}_{\mathbb{Z}}^*$ the characters corresponding to the T-action on $(L_i)_x$, then since $\chi_{i,x}$ depends continuously on x and $\mathfrak{t}_{\mathbb{Z}}^*$ is a discrete set we get that $\chi_{i,x}$ is independent of x, i.e., the bundle N splits into a direct sum of T-equivariant line bundles L_i. Now it is easy to see that $(ET \times N)/T = \bigoplus_{i=1}^r (L_i \boxtimes L_{-\chi_i})$, so $E_T(i) = \prod_{i=1}^r (c_1(L_i) - \chi_i)$.

Put $\beta = i^*\alpha$. Then we have $i^* i_* \beta = E_T(i)\beta$, i.e.,

$$i^* i_* \frac{\beta}{E_T(i)} = \beta = i^*\alpha.$$

On the other hand, according to the Borel localization theorem, i^* is an isomorphism. Thus $\alpha = i_*(\beta/E_T(i))$. Let $\pi : X_T \to BT$, then $\int_X \alpha = \pi_*(\alpha) = (\pi \circ i)_*(\beta/E_T(i)) = \int_{X_T} \beta/E_T(i)$. □

10.3. **The equivariant cohomology of the projective space.** Assume that $X = \mathbb{C}P^n$. We equip X with the following T-action:

$$t \cdot [z_0, \ldots, z_n] = [t_0 z_0, \ldots, t_n z_n], \quad \text{where} \quad t = (t_0, \ldots, t_n) \in T.$$

It is not hard to see that X_T is the projectivization of the bundle $\bigoplus_{i=0}^n L_{-\nu_i}$. In particular X_T has a tautological bundle $\mathcal{O}(-1)$, whose Chern class will be denoted by $-p$, and the equivariant cohomology of X is given by:

$$\mathbb{C}[p, \nu_0, \ldots, \nu_n]/(p - \nu_0) \ldots (p - \nu_n).$$

We can also compute the equivariant intersection pairing. Notice that X^T consists of $n+1$ points: $p_i = [0, \ldots, 1, \ldots, 0]$, where 1 is on the i-th place, $0 \le i \le n$. It follows from the definitions that $(ET \times \{p_i\})/T$ is the image of the section of $\mathbb{P}(\bigoplus_{i=0}^n L_{-\nu_i}) \to BT$, determined by the i-th line. In particular, the restriction of $\mathcal{O}(-1)$ to $(ET \times \{p_i\})/T \cong BT$ is $L_{-\nu_i}$ and so the restriction of $p \in H_T^*(X)$ to the fixed point p_i is ν_i. Put

$$\phi_i = \prod_{j : j \neq i} \frac{p - \nu_i}{\nu_j - \nu_i}, \quad 0 \le i \le n.$$

In order to compute the equivariant pairing we need also to know the equivariant Euler class of the tangent spaces $T_{p_i}X$. Since $x_j = z_j/z_i$, $j \neq i$ are local coordinates near p_i, we have that the torus T acts on $T_{p_i}X$ with characters $\nu_j - \nu_i$. This implies that the equivariant Euler class of $T_{p_i}X$ is $e_i := \prod_{j:j\neq i}(\nu_i - \nu_j)$.

Using the equivariant integration formula we get

$$(\phi_i, \phi_j) := \int_X \phi_i \phi_j = \frac{\delta_{ij}}{e_i}.$$

11. Fixed-point localization in Gromov–Witten theory

From now on we assume that $X = \mathbb{C}P^n$. The goal in this lecture is to prove that the equivariant quantum cohomology of $\mathbb{C}P^n$ is semi-simple and that the corresponding system of quantum differential equations admits an asymptotical solution, whose ingredients, U and R coincide with certain contributions to the fixed point localization formula for the genus-0 GW theory of X.

11.1. **The combinatorial model for the fixed-points locus.** The moduli space $\overline{\mathcal{M}}_{g,n}(X, d)$ admits a T-action: $(t \cdot f)(x) = t(f(x))$, where $t \in T$ and f is a stable map. Moreover, the cotangent lines bundles $\mathbb{L}_i$ admit a T-action, so we have the equivariant version of GW theory.

Let us describe the fixed-point locus. Assume that $f : (\Sigma, x_1, \ldots, x_m) \to X$ is a stable map. Let $\{\Sigma_r\}$ be the set of irreducible components of Σ. Recall that a special point on an irreducible component of Σ is called special if it is a marked point or a nodal point on Σ. The following statements are easy to verify:

(1) Each special point is mapped to a fixed point.
(2) If $(\Sigma_r, \text{special points})$ is stable then $f(\Sigma_r)$ coincides with some of the fixed points.
(3) If $(\Sigma_r, \text{special points})$ is not stable, then Σ_r is a copy of $\mathbb{C}P^1$ with one or two special points. The map $f|_{\Sigma_r}$ has the following form: let w be a local coordinate near a special point on Σ_r and assume that this special point is mapped to p_i. Then the image of Σ_r is the complex line passing through p_i and some other fixed point p_j, and the map is given by $z = w^d$, where $z = z_j/z_i$ and $d \geq 1$ is some integer.

It follows that to a generic point (Σ, f) in the fixed point locus we can naturally associate the following graph Γ: the edges correspond to non-contracted components, and the vertices correspond to parts of Σ which are pre-images of the fixed points. Also, each vertex is labeled, by a genus and an integer $i \in \{0, 1, \ldots, n\}$ which encodes to which fixed point p_i the vertex is mapped. Each flag, i.e., an incident edge-vertex pair is labeled by the character $\chi := (\nu_i - \nu_j)/d$, where the vertex is mapped to the fixed point p_i and j and d are the same as above (see the 3-rd property of the fixed point locus). We will refer to Γ as *the combinatorial model* of the stable map.

We denote by $\overline{\mathcal{M}}_\Gamma$ the fixed points in $\overline{\mathcal{M}}_{g,n}(X, d)$ whose combinatorial model is Γ. The stable maps with different combinatorial models belong to different connected components of the fixed point locus. Notice that for each flag (v, e) the tangent lines to the edge at the corresponding special point (common for the vertex and the edge) form a trivial bundle on $\overline{\mathcal{M}}_\Gamma$ whose equivariant Euler class is χ.

11.2. **Materialization.**

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