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修士論文題目

Gamma integral structure for the blowup of $\mathbb{P}^n$ at a point

一点における$\mathbb{P}^n$の爆発のガンマ整構造

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論文内容の要旨

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Gamma Integral Structure for the Blowup of $\mathbb{P}^n$ at a Point

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A.Bayerの結果によれば、滑らかな射影的多様体Xが半単純の量子コホモロジーを有するのであれば、何点かにおけるXの爆発についても同じことが言えます。したがって、新しい半単純量子コホモロジー代数は爆発操作を適用して構築されます。量子コホモロジー、より一般的には半単純フロベニウス多様体の理論における反射ベクトルの概念は、ミラー対称性から生まれた物である。つまり、ミラー対称現象からみると、反射ベクトルは消滅サイクルに対応します。それらは、不変の双線型写像が正定値である必要がない場合以外、ルート系のすべての性質を持っています。最も一般的な目標は、半単純フロベニウス多様体に関連する反射ベクトルのシステムを分類することです。

この論文の出発点は、反射ベクトルの集合に爆発操作を適用した場合の影響を調査することでした。半単純フロベニウス多様体の反射ベクトルの定義は、第二構造接続から与えられます。第二構造接続はミラー対称性がGauss–Manin接続に対応します。一般的なケースを理解するには、種数零Gromov–Witten不変量が第二構造接続？への寄与を理解する必要があります。種数零Gromov–Witten不変量の 度は例外因子でサポートされていますこのような不変量は、例外因子の管状近傍にのみ依存するため、特定の例を検討することで理解できます。この論文では、可能な限り単純な目標$\mathbb{P}^n$を採用しました。$Q_1 = e_1^{\tau_1}$と$Q_2 = e^{\tau_2}$がノビコフ変数であると仮定します。$\Psi_\tau(E)$を$e^{-\tau_1 p_1 - \tau_2 p_2}\Psi(E)$に定義します。$\Psi : K^\circ(\mathrm{Bl}(\mathbb{P}^n)) \to H^*(\mathrm{Bl}(\mathbb{P}^n))$ はチャーン指標の入谷Γ類の写像である。

本研究の主な結果は、$\Psi_\tau(\mathcal{O})$は反射ベクトルであることです。$\mathcal{O}$はの一点における$\mathbb{P}^n$の爆発の構造層。次に、ミラーファミリーの臨界値$\{u_1(Q), \cdots, u_{2n}(Q)\}$を研究し

ます。除数$Q_1Q_2=0$を回る$\mathbb{C}^2$の閉ループに沿った解析接続は、特定置換群によって作用します。この群作用には2つの軌道があることが解明し、モノドロミーを使用すると、その内の1つで臨界値に対応する反射ベクトルが簡単に作成できます。もう一つの軌道の臨界値に対応する反射ベクトルの構築ははるかに困難であり、まだ研究中です。

Abstract

According to A. Bayer, if a smooth projective variety X has semi-simple quantum cohomology, then the blow-up of X at any number of points also has semi-simple quantum cohomology. Therefore, new semi-simple quantum cohomology algebras can be constructed by applying blow-up operation. The notion of a reflection vector in quantum cohomology and more generally in the theory of semi-simple Frobenius manifolds is motivated by mirror symmetry. Namely, under the mirror symmetry phenomenon reflection vectors correspond to vanishing cycles. They have all the properties of a root system, except that the invariant bilinear pairing does not have to be positive definite. The most general goal is to classify the system of reflection vectors associated with semi-simple Frobenius manifolds. The starting point of my thesis was to investigate the effect of applying the blow-up operation on the set of reflection vectors. The definition of a reflection vector for a semi-simple Frobenius manifold is given via the so called 2nd structure connection. The latter, under mirror symmetry corresponds to the Gauss–Manin connection. It turns out that in order to understand the general case, one has to understand the contribution to the second structure connection coming from genus-0 Gromov–Witten invariants whose degree is supported in the exceptional divisor. Such invariants depend only on a tubular neighborhood of the exceptional divisor, so we can understand them by considering a specific example. In my thesis, I took the simplest possible target, that is, the projective space $\mathbb{P}^n$. Suppose that $Q_1 = e_1^{\tau_1}$ and $Q_2 = e^{\tau_2}$ are the Novikov variables. Put $\Psi_\tau(E) :== e^{-\tau_1 p_1 - \tau_2 p_2}\Psi(E)$, where $\Psi : K^0(\mathrm{Bl}(\mathbb{P}^n)) \to H^*(\mathrm{Bl}(\mathbb{P}^n))$ is Iritani's Γ-class modification of the Chern character map. Our main result is that $\Psi_\tau(\mathcal{O})$ is a reflection vector, where $\mathcal{O}$ is the structure sheaf of $\mathrm{Bl}(\mathbb{P}^n)$.

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Contents

CHAPTER 1

Introduction

One of the main reasons why Frobenius manifold are important comes from the examples of Quantum Cohomology in Gromov-Witten theory. The latter gives us a Frobenius manifold structure. Under the semi-simplicity condition the Frobenius structure determines all higher-genus Gromov-Witten invariants. This was conjectured by Givental, who proved several special cases using fixed point localization. Teleman proved Givental's conjecture in general. The resulting theory is known as Givental-Teleman higher-genus reconstruction. The higher-genus reconstruction can be defined for any semi-simple Frobenius manifold. The generating function for Gromov-Witten invariants is called total descendant potential. According to the Givental-Teleman reconstruction, the generating function is expressed in terms of a differential operator, constructed from Dubrovin connection, and a product of tau-functions of the KdV hierarchy. These tau-functions come from the generating function of Gromov-Witten invariants for the point. One of the problems in general that we would like to solve is to construct a system of Hirota bilinear equations for the total descendent potential. In fact, there is a general construction of vertex operators suggested by Givental in his paper [**12**]. Givental's construction is straightforward to generalize for any semi-simple Frobenius manifold. The vertex operators corresponding to reflection vectors are the ones that conjugate to the vertex operators for the KdV hierarchy. Therefore we expect that the vertex operators corresponding to the reflection vectors would play a key role in constructing Hirota quadratic equations. Hence, it comes our interest in classifying reflection vectors corresponding to semi-simple Frobenius manifolds. We expect that vertex operators corresponding to reflection vector should be used to construct integrable hierarchies in the form of Hirota bilinear equations for the total descendent potential of Givental.

Besides the above motivation, the problem of classifying reflection vectors has some other applications too. The reflection vectors would allow us to compute the monodromy group of the Frobenius manifold. The invariant theory of the monodromy group is a key to understanding the analytic properties of the semi-simple Frobenius manifold. The main examples of Frobenius manifolds come from Quantum cohomology and singularity theory. Especially, the reflection vectors in singularity theory satisfy almost all axioms of a root system. More precisely, the only axiom that fails is that the invariant bilinear form is positive definite.

We know form [**15**] that if X has semi-simple quantum cohomology then $H^{p,q}(H;\mathbb{C}) = 0$ for $p \neq q$. Combined with the result of Bayer proved in [**2**] we have that

Theorem 1.1. *[**2**] Whenever X has semi-simple quantum cohomology, the same is true for the blow-up of X at any number of points.*

Furthermore, Bayer conjectured that

Conjecture 1.2. *Whenever X has semi-simple quantum cohomology, the same is true for the blow-up of X at any subvariety that itself has semi-simple quantum cohomology.*

Therefore Blow-up operation gives a tool to construct new semi-simple Quantum cohomology algebras. We would like to investigate whether this construction can be performed abstractly in the category of semi-simple Frobenius manifolds. Moreover, we would like to understand how the set of reflection vectors changes under the blow-up operation. Suppose that

$X :=$ manifold with a semi-simple quantum cohomoly

$\widetilde{X} := \mathrm{Bl}_{pt}(X)$ note that $H^*(\widetilde{X}) = H^*(X) \oplus \widetilde{H}^*(E)$ where $E \in \widetilde{X}$ is the exceptional divisor

Every reflection vector decomposes into two parts: a cohomology class in $H^*(X)$ and a cohomology class in $H^*(E)$. The second part is essentially independent of X. Therefore, in order to compute it, we can work with any target manifold X. The simplest possible choice is $X = \mathbb{P}^n$. The main goal of this thesis will be to determine the set of reflection vector for the blow-up of $\mathbb{P}^n$ at one point. Unfortunately, we could not achieve completely our goal. We still need to construct an integration cycle for a certain oscillatory integral corresponding to the structure sheaf $\mathcal{O}_E$ of the exceptional divisor. Nevertheless after a little bit of work, we should be able to overcome the difficulty.

Givental's mirror symmetry results for Fano-toric manifolds imply that there exists an isomorphism.

$$\mathcal{E} : \Lambda_{\mathbb{C}} \to \mathcal{K}^{\circ}(\widetilde{X}) \otimes \mathbb{C}$$

such that,

$$\int_{\Gamma} e^{f/z} \omega = (2\pi)^{\frac{n-1}{2}} (-z)^{\frac{n}{2}} (S(0,Q,z)(-z)^{\theta}(-z)^{\rho})\Psi_{\tau}(\mathcal{E}_{\Gamma}), 1)$$

where

$$\Lambda_{\mathbb{C}} := H_n((\mathbb{C}^*)^n, \mathrm{Re}(f(x,Q)) > M; \mathbb{C}) \cong \mathbb{C}^N,$$

where $M \gg 0$, ρ is given by classical cup product multiplication by $c_1(T\,\mathrm{Bl}(\mathbb{P}^n))$. The calibration $S(t,Q,z)$, the hodge grading operator θ, and the Givental mirror $((\mathbb{C}^*)^n, f, \omega)$ for $\mathrm{Bl}(\mathbb{P}^n)$ are defined in Chapter 2 Subsection 2.5, Section 4.

Our main result can be stated as follows.

Theorem 1.3. *(Theorem 3.1) Suppose that $Q_1 = e_1^{\tau_1}$, $Q_2 = e^{\tau_2}$ where τ_1, $\tau_2 \in \mathbb{R}$ Put $\Psi_{\tau} = e^{-\tau_1 p_1 - \tau_2 p_2}\Psi(E)$ if $z \in \mathbb{R}_{<0}$. Then*

$$\int\limits_{\mathbb{R}^n_{>0}} e^{f(x,\tau)z^{-1}} \omega = (2\pi)^{\frac{n-1}{2}} (-z)^{\frac{n}{2}} (S(0,Q,z)(-z)^{\theta}(-z)^{\rho})\Psi_{\tau}(\mathcal{O}), 1)$$

$\mathbb{R}^n_+ \in \Lambda$.

Theorem1.3 was proved also by Iritani [**16**]. We give a different proof which in some sense is simpler and we believe that our argument can be generalized for non-Fano toric manifolds. Let us denote by

$$\Lambda_{\mathbb{Z}} := H_n((\mathbb{C}^*)^n, \mathrm{Re}(f(x,Q)) > M; \mathbb{Z}) \cong \mathbb{Z}^N.$$

Using Theorem 1.3 Iritani proved (see [**16**]) that $\mathcal{E}$ induces an isomorphism

$$\mathcal{E}_{\Gamma} : \Lambda_{\mathbb{Z}} \longrightarrow K^0(X). \tag{1}$$

We would like to prove a stronger result. Namely, the lattice $\Lambda_{\mathbb{Z}}$ has a $\mathbb{Z}$-basis consisting of Lefschetz thimbles Γ_i $(1 \leq i \leq N)$ constructed via Morse theory. More precisely, Γ_i is the uion of the gradient trajectories of $\mathrm{Re}(f(z))$ flowing into the critical point corresponding to the critical value u_i. Our goal is to determine the images of Γ_i in $\mathcal{K}^0(X)$ via (1). The Lefschetz thimble Γ_i can be constructed as follows. Let us choose a path C_i from λ° to u_i in the λ-plane. Using the map

$$f : (\mathbb{C}^*)^n \to \mathbb{C}$$

we lift each C_i to a cycle $\Gamma_i \in \Lambda_{\mathbb{Z}}$. Using Morse theory, it can be proved that

$$\Lambda_{\mathbb{Z}} = \oplus_{i=1}^{N} \mathbb{Z} \cdot L_i.$$

Our goal is to prove that if we have a full exceptional collections $E_1, \cdots, E_N$ then we can find paths C_i $(1 \le i \le N)$ such that $E_i = \mathcal{E}_{L_i}$. Note that $\partial\Gamma_i \in H_{n-1}(f^{-1}(\lambda^\circ);\mathbb{Z})$ is a vanishing cycle (vanishes along the path C_i). Picard-Lefschetz theory [**1**] implies that $\Psi_\tau(E_i)$ is a reflection vector corresponding to the path C_i. Therefore, we can say that $\Lambda_{\mathbb{Z}}$ is the reflection lattice, i.e. lattice spanned over $\mathbb{Z}$ by the reflection vectors. Analytic continuations along $Q_1 = 0$ and $Q_2 = 0$ acts on the set $\{u_1(Q), \cdots, u_N(Q)\}$ of critical values of f by permutations. This action has two orbits $\{u_1(Q), \cdots, u_{n+1}(Q)\}$ and $\{u_{n+2}(Q), \cdots, u_N(Q)\}$. We prove that $\Gamma_1 = [\mathbb{R}^n_{>0}] \in \Lambda_{\mathbb{Z}}$ corresponds to $C_1 = [u_1, \lambda^\circ]$ and $\mathcal{E}_{\Gamma_1} = \mathcal{O}$(structure sheaf) It is easy to see that by using analytic continuation along $Q_1 = 0$ and $Q_2 = 0$ we can construct cycles Γ_i $(2 \le i \le n+1)$ corresponding to critical values u_i $(2 \le i \le n+1)$; Γ_i is obtained from Γ_1 by parallel transport along an appropriate chosen contour around $Q_1 Q_2 = 0$. We could not find cycles corresponding to the second orbit $\{u_{n+2}(Q), \cdots, u_N(Q)\}$ of the monodromy action. We expect that there is a cycle Γ_{n+2} corresponding to u_{n+2} such that $\mathcal{E}_{\Gamma_{n+2}} = \mathcal{O}_E$ (structure sheaf of the exceptional divisor E). Using monodromy transformations we can construct the remaining cycles Γ_i $(n+2 \le i \le N)$ from Γ_{n+2}. Therefore, what is left is to find a cycle Γ_{n+2} such that the identity in Theorem 1.3 holds for $\mathbb{R}^n_{>0}$ replaced by Γ_{n+2} and $\mathcal{O}$ replaced by $\mathcal{O}_E$.

CHAPTER 2

Background

1. Frobenius manifolds

Following Dubrovin [**7**], we recall the notion of a Frobenius manifold. Then we proceed by defining the so-called *second structure connection and reflection vectors* of a semi-simple Frobenius manifold.

1.1. Definition. Suppose that M is a complex manifold and $\mathcal{T}_M$ is the sheaf of holomorphic vector fields on M. The manifold M is equipped with the following structures:

(F1) A non-degenerate symmetric bilinear pairing

$$(\cdot,\cdot):\mathcal{T}_M\otimes\mathcal{T}_M\to\mathcal{O}_M.$$

(F2) A Frobenius multiplication: commutative associative multiplication

$$\cdot\bullet\cdot:\mathcal{T}_M\otimes\mathcal{T}_M\to\mathcal{T}_M,$$

such that $(v_1\bullet w,v_2)=(v_1,w\bullet v_2)\ \forall v_1,v_2,w\in\mathcal{T}_M$.

(F3) A unit vector field: global vector field $\mathbf{1}\in\mathcal{T}_M(M)$ such that

$$\mathbf{1}\bullet v=v,\quad \nabla^{\text{L.C.}}_v\mathbf{1}=0,\quad \forall v\in\mathcal{T}_M,$$

where $\nabla^{\text{L.C.}}$ is the Levi–Civita connection of the pairing $(\cdot,\cdot)$.

(F4) An Euler vector field $E\in\mathcal{T}_M(M)$ such that

$$E(v_1,v_2)-([E,v_1],v_2)-(v_1,[E,v_2])=(2-n)(v_1,v_2)$$

for some constant $n\in\mathbb{C}$.

Given the data (F1)-(F4), we define the so called *Dubrovin's connection* on the vector bundle $TM\times\mathbb{C}^*\to M\times\mathbb{C}^*$

$$\nabla_v:=\nabla^{\text{L.C.}}_v-z^{-1}v\bullet,\quad v\in\mathcal{T}_M,$$
$$\nabla_{\partial/\partial z}:=\frac{\partial}{\partial z}-z^{-1}\theta+z^{-2}E\bullet,$$

where z is the standard coordinate on $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$, where $v\bullet$ is an endomorphism of $\mathcal{T}_M$ defined by the Frobenius multiplication by the vector field v, and where $\theta:\mathcal{T}_M\to\mathcal{T}_M$ is an $\mathcal{O}_M$-modules morphism defined by

$$\theta(v):=\nabla^{\text{L.C.}}_v(E)-\Big(1-\frac{D}{2}\Big)v.$$

Definition 2.1. The data $((\cdot,\cdot),\bullet,\mathbf{1},E)$, satisfying the properties $(F1)-(F4)$, is said to be a *Frobenius structure* of conformal dimension n if the corresponding Dubrovin connection is flat.

Let us proceed with recalling the notion of 2nd structure connection and reflection vectors. We follow the exposition from [**19**]. We are going to work only with Frobenius manifolds satisfying the following 4 additional conditions:

(i) The tangent bundle TM is trivial and it admits a trivialization given by a frame of global flat vector fields.

(ii) Recall that the operator

$$\mathrm{ad}_E : \mathcal{T}_M \to \mathcal{T}_M, \quad v \mapsto [E, v]$$

preserves the space of flat vector fields. We require that the restriction of ad_E to the space of flat vector fields is a diagonalizable operator with rational eigenvalues.

(iii) The Frobenius manifold has a *calibration* (see Section 1.2).

(iv) The Frobenius manifold has a direct product decomposition $M = \mathbb{C} \times B$ such that if we denote by $t_1 : M \to \mathbb{C}$ the projection along B, then dt_1 is a flat 1-form and $\langle dt_1, \mathbf{1} \rangle = 1$.

Conditions (i)–(iv) are satisfied for all Frobenius manifolds constructed by quantum cohomology or by the primitive forms in singularity theory.

Let us fix a base point $t^\circ \in M$ and a basis $\{\phi_i\}_{i=1}^N$ of the reference tangent space $H := T_{t^\circ}M$. Furthermore, let $(t_1, \ldots, t_N)$ be a local flat coordinate system on an open neighborhood of t° such that $\partial/\partial t_i = \phi_i$ in H. The flat vector fields $\partial/\partial t_i$ $(1 \le i \le N)$ extend to global flat vector fields on M and provide a trivialization of the tangent bundle $TM \cong M \times H$. This allows us to identify the Frobenius multiplication $\bullet$ with a family of associative commutative multiplications $\bullet_t : H \otimes H \to H$ depending analytically on $t \in M$. Modifying our choice of $\{\phi_i\}_{i=1}^N$ and $\{t_i\}_{i=1}^N$ if necessary we may arrange that

$$E = \sum_{i=1}^{N} ((1 - d_i)t_i + r_i)\partial/\partial t_i,$$

where $\partial/\partial t_1$ coincides with the unit vector field $\mathbf{1}$ and the numbers

$$0 = d_1 \le d_2 \le \cdots \le d_N = n$$

are symmetric with respect to the middle of the interval $[0, n]$. The number D is known as the *conformal dimension* of M. The operator $\theta : \mathcal{T}_M \to \mathcal{T}_M$ defined above preserves the subspace of flat vector fields. It induces a linear operator on H, known to be skew symmetric with respect to the Frobenius pairing $(\, , \,)$. Following Givental, we refer to θ as the *Hodge grading operator*.

There are two flat connections that one can associate with the Frobenius structure. The first one is the *Dubrovin connection* – defined above. The Dubrovin connection in flat coordinates takes the following form:

$$\begin{aligned} \nabla_{\partial/\partial t_i} &= \frac{\partial}{\partial t_i} - z^{-1}\phi_i \,\bullet \\ \nabla_{\partial/\partial z} &= \frac{\partial}{\partial z} + z^{-1}\theta - z^{-2}E\bullet \end{aligned}$$

where z is the standard coordinate on $\mathbb{C}^* = \mathbb{C} - \{0\}$ and for $v \in \Gamma(M, \mathcal{T}_M)$ we denote by $v\bullet : H \to H$ the linear operator of Frobenius multiplication by v.

Our main interest is in the *2nd structure connection*

$$\begin{aligned} \nabla^{(n)}_{\partial/\partial t_i} &= \frac{\partial}{\partial t_i} + (\lambda - E\bullet_t)^{-1}(\phi_i \bullet_t)(\theta - n - 1/2) \\ \nabla^{(n)}_{\partial/\partial \lambda} &= \frac{\partial}{\partial \lambda} - (\lambda - E\bullet_t)^{-1}(\theta - n - 1/2), \end{aligned}$$

where $n \in \mathbb{C}$ is a complex parameter. This is a connection on the trivial bundle

$$(M \times \mathbb{C})' \times H \to (M \times \mathbb{C})',$$

where

$$(M\times\mathbb{C})' = \{(t,\lambda)\mid \det(\lambda - E\bullet_t)\neq 0\}.$$

The hypersurface $\det(\lambda - E\bullet_t)=0$ in $M\times\mathbb{C}$ is called the *discriminant*.

1.2. Period vectors. The definition of the period map depends on the choice of a *calibration* $S(t,z)$ of M. By definition (see [**11**]), the calibration is an operator series $S = 1+\sum_{k=1}^\infty S_k(t)z^{-k}$, $S_k\in \mathrm{End}(H)$, such that the Dubrovin's connection has a fundamental solution near $z=\infty$ of the form

$$S(t,z)z^{\theta}z^{-\rho},$$

where $\rho\in\mathrm{End}(H)$ is a nilpotent operator, $[\theta,\rho]=-\rho$, and the following symplectic condition holds

$$S(t,z)S(t,-z)^T = 1,$$

where ^{T} denotes transposition with respect to the Frobenius pairing.

Let us fix a reference point $(t^\circ,\lambda^\circ)\in(M\times\mathbb{C})'$ such that λ° is a sufficiently large real number. It is easy to check that the following functions provide a fundamental solution to the 2nd structure connection

$$I^{(n)}(t,\lambda)=\sum_{k=0}^{\infty}(-1)^k S_k(t)\widetilde{I}^{(n+k)}(\lambda),$$

where

$$\widetilde{I}^{(m)}(\lambda)=e^{-\rho\partial_\lambda\partial_m}\Big(\frac{\lambda^{\theta-m-\frac12}}{\Gamma(\theta-m+\frac12)}\Big).$$

The 2nd structure connection has a Fuchsian singularity at infinity, therefore the series $I^{(n)}(t,\lambda)$ is convergent for all (t,λ) sufficiently close to (t°,λ°). Using the differential equations we extend $I^{(n)}$ to a multi-valued analytic function on $(M\times\mathbb{C})'$. We define the following multi-valued functions taking values in H:

$$I_a^{(n)}(t,\lambda):=I^{(n)}(t,\lambda)a,\quad a\in H,\quad n\in\mathbb{Z}.$$

These functions will be called *period vectors*. Using analytic continuation we get a representation

$$\pi_1((M\times\mathbb{C})',(t^\circ,\lambda^\circ))\to \mathrm{GL}(H) \tag{2}$$

called the *monodromy representation* of the Frobenius manifold. The image W of the monodromy representation is called the *monodromy group*.

Under the semi-simplicity assumption, we may choose a generic reference point t° on M, such that the Frobenius multiplication $\bullet_{t^\circ}$ is semi-simple and the operator $E\bullet_{t^\circ}$ has N pairwise different eigenvalues u_i° $(1\le i\le N)$. The fundamental group $\pi_1((M\times\mathbb{C})',(t^\circ,\lambda^\circ))$ fits into the following exact sequence

$$\pi_1(F^\circ,\lambda^\circ)\xrightarrow{i_*}\pi_1((M\times\mathbb{C})',(t^\circ,\lambda^\circ))\xrightarrow{p_*}\pi_1(M,t^\circ)\longrightarrow 1, \tag{3}$$

where $p:(M\times\mathbb{C})'\to M$ is the projection on M, $F^\circ = p^{-1}(t^\circ)=\mathbb{C}\setminus\{u_1^\circ,\dots,u_N^\circ\}$ is the fiber over t°, and $i: F^\circ\to(M\times\mathbb{C})'$ is the natural inclusion. For a proof we refer to [22], Proposition 5.6.4 or [**21**], Lemma 1.5 C. Using the exact sequence (3) we get that the monodromy group W is generated by the monodromy transformations representing the lifts of the generators of $\pi_1(M,t^\circ)$ in $\pi_1((M\times\mathbb{C})',(t^\circ,\lambda^\circ))$ and the generators of $\pi_1(F^\circ,\lambda^\circ)$.

The image of $\pi_1(F^\circ, \lambda^\circ)$ under the monodromy representation is a reflection group that can be described as follows. Using the differential equations of the 2nd structure connection it is easy to prove that the pairing

$$(a|b) := (I_a^{(0)}(t,\lambda), (\lambda - E\bullet)I_b^{(0)}(t,\lambda)) \tag{4}$$

is independent of t and λ. This pairing is known as the *intersection pairing*. Suppose now that γ is a simple loop in F°, i.e., a loop that starts at λ°, approaches one of the punctures u_i° along a path γ' that ends at a point sufficiently close to u_i°, goes around u_i°, and finally returns back to λ° along γ'. By analyzing the second structure connection near $\lambda = u_i$ it is easy to see that up to a sign there exists a unique $a \in H$ such that $(a|a) = 2$ and the monodromy transformation of a along γ is $-a$. The monodromy transformation representing $\gamma \in \pi_1(F^\circ, \lambda^\circ)$ is the reflection defined by the following formula:

$$w_a(x) = x - (a|x)a. \tag{5}$$

Let us denote by R the set of all $a \in H$ as above determined by all possible choices of simple loops in F°. We refer to the elements of R as reflection vectors.

2. Quantum cohomology of the blow up

Let X be a smooth projective variety of dimension n.

2.1. Cones: ample, nef, and curve. Let us recall some standard facts about divisors and line bundles on X that are needed for the definition of quantum cohomology. The main reference for further details is [**17**].

Let us denote by $\mathrm{Div}(X)$ the group of Cartier divisors on X. If $D \in \mathrm{Div}(X)$, then there is an associated line bundle, i.e., locally free sheaf $\mathcal{O}_X(D)$, defined as follows: if $U \subset X$ is an open subset, such that, the Cartier divisor has a representative $f \in \Gamma(U, \mathcal{O}_X)$, then $\mathcal{O}_X(D)|_U = f^{-1}\mathcal{O}_X|_U$. By definition, $\mathcal{O}_X(D)$ is a subsheaf of the sheaf of meromorphic functions on X. Sometimes, if no confusion is likely to occure, we drop X and denote the structure sheaf $\mathcal{O}_X$ of X simply by $\mathcal{O}$.

Definition 2.2. If $D_1, D_2 \in \mathrm{Div}(X)$, then

a) We say that D_1 and D_2 are *linearly equivalent* and write $D_1 \equiv_{\mathrm{lin}} D_2$ if $\mathcal{O}(D_1 - D_2)$ is a trivial bundle.

b) We say that D_1 and D_2 are *numerically equivalent* and write $D_1 \equiv_{\mathrm{num}} D_2$ if

$$D_1 \cdot C := \int_{[C]} c_1(\mathcal{O}(D)) = \int_{[C]} c_1(\mathcal{O}(D_2)) =: D_2 \cdot C$$

for every irreducible curve C in X. □

Let us denote by $\mathrm{NS}^1(X) = \mathrm{Div}(X)/\mathrm{Num}(X)$, where $\mathrm{Num}(X)$ is the subgroup of $\mathrm{Div}(X)$ consisting of divisors numerically equivalent to 0. It is known that $\mathrm{NS}^1(X)$ is a free Abelian group of finite rank. The group $\mathrm{NS}^1(X)$ is known as the *Neron–Severi* group of X and its rank r is called the *Picard number* of X. Since, we assume that X is smooth, we have also the following cohomological interpretatoion: the map $D \mapsto c_1(\mathcal{O}(D))$ induses an isomorphism

$$\mathrm{NS}^1(X) \xrightarrow{\cong} H^2(X;\mathbb{Z})_{\mathrm{t.f.}} \cap H^{1,1}(X) ,$$

where for an Abelian group A, we denote by $A_{\mathrm{t.f.}}$ the torsion free part of A.

Definition 2.3. a) A line bundle L on X is said to be *very ample* if there is a closed embedding of X in $\mathbb{P}^N$ such that $L = \mathcal{O}_{\mathbb{P}^N}(1)|_X$. Furthermore, L is called *ample* if there exists an integer $m \in \mathbb{Z}_{>0}$, such that, $L^{\otimes m}$ is very ample.

b) A divisor $D \in \mathrm{Div}(X)$ is said to be ample (resp. very ample) if the corresponding line bundle $\mathcal{O}(D)$ is ample (resp. very ample). □

The following proposition is known as Cartan–Serre–Grothendieck theorem.

Proposition 2.4. *If X is a projective variety and L is a line bundle on X, then the following conditions are equivalent:*

(i) *L is ample.*

(ii) *If $\mathcal{F}$ is a coherent sheaf on X, then there exists an integer m_1 (depending on $\mathcal{F}$), such that, $H^i(X, \mathcal{F} \otimes L^m) = 0$ for all $i > 0$ and $m \geq m_1$.*

(iii) *If $\mathcal{F}$ is a coherent sheaf on X, then there exists an integer m_2 (depending on $\mathcal{F}$), such that, $\mathcal{F} \otimes L^{\otimes m}$ is generated by global sections for all $m \geq m_2$.*

(iv) *There exists an integer $m_3 \in \mathbb{Z}_{>0}$, such that, $L^{\otimes m}$ is very ample for every $m \geq m_3$.* □

If k is one of the fields $\mathbb{R}$ or $\mathbb{Q}$, then let us define $\mathrm{Div}_k(X) := \mathrm{Div}(X) \otimes_{\mathbb{Z}} k$ and $\mathrm{Num}_k(X) \subseteq \mathrm{Div}_k(X)$ to be the subvector space consisting of elements $\sum c_i A_i$, $c_i \in k$, $A_i \in \mathrm{Div}(X)$, such that, $\sum_i c_i (A_i \cdot C) = 0$ for all irreducible curves $C \subseteq X$. We have the following relation

$$\mathrm{NS}^1(X)_k := \mathrm{Div}_k(X)/\mathrm{Num}_k(X) \cong \mathrm{NS}^1(X) \otimes_{\mathbb{Z}} k.$$

A divisor $D \in \mathrm{Div}_k(X)$ is said to be ample if $D = \sum_i c_i A_i$ with $c_i > 0$ and $A_i \in \mathrm{Div}(X)$ is ample.

Proposition 2.5. *If $H \in \mathrm{Div}_k(X)$ is an ample divisor and $E \in \mathrm{Div}_k(X)$ is arbitrary, then $H + \epsilon E$ is ample for $0 < \epsilon \ll 1$.*

Let us recall that a divisor $D \in \mathrm{Div}_k(X)$ is called nef if $D \cdot C \geq 0$ for all irreducible curves $C \subseteq X$. A line bundle L is said to be nef if $\int_C c_1(L) \geq 0$ for all irreducible curves $C \subseteq X$. The following proposition is known as the Kleiman theorem.

Proposition 2.6. *If L is a nef line bundle, then*

$$\int_V c_1(L)^{\dim(V)} \geq 0$$

for every irreducible subvariety $V \subseteq X$.

Proposition 2.6 has the following corollary.

Corollary 2.7. *a) If $D \in \mathrm{Div}_{\mathbb{R}}(X)$ is nef and $H \in \mathrm{Div}_{\mathbb{R}}(X)$ is ample, then $D + \epsilon H$ is ample for every $\epsilon > 0$.*

b) If $D, H \in \mathrm{Div}_{\mathbb{R}}(X)$ and $D + \epsilon H$ is ample for every $0 < \epsilon \ll 1$, then D is nef.

c) Suppose that $H \in \mathrm{Div}_{\mathbb{R}}(X)$ is an ample divisor. Then a divisor $D \in \mathrm{Div}_{\mathbb{R}}(X)$ is ample if and only if there exists an $\epsilon > 0$, such that, $\frac{D \cdot C}{H \cdot C} \geq \epsilon$ for every irreducible curve $C \subseteq X$.

Under the canonical quotient map $\mathrm{Div}_{\mathbb{R}}(X) \longrightarrow \mathrm{NS}^1(X)_{\mathbb{R}}$, the set of all ample divisors in $\mathrm{Div}_{\mathbb{R}}(X)$ maps to a cone $\mathrm{Amp}(X) \subset \mathrm{NS}^1(X)_{\mathbb{R}}$, called the *ample cone* of X. Similarly, the image of the set of nef divisors is a cone $\mathrm{Nef}(X) \subseteq \mathrm{NS}^1(X)_{\mathbb{R}}$, called the *nef cone* of X. Using Corollary 2.7, we get that $\mathrm{Nef}(X)$ is a closed subset and $\mathrm{Amp}(X)$ is the interior of $\mathrm{Nef}(X)$, that is, a divisor on X is ample if and only if, it is in the interior of the nef cone.

The nef cone can be characterized also as the dual of the cone of curves. Let us denote by $\mathrm{N}_1(X)$ the quotient of the free Abelian group generated by the irreducible curves $C \subseteq X$ modulo numerical equivalence. Recall that $C' = \sum_i a_i' C_i'$ and $C'' = \sum_j a_j'' C_j''$ are *numerically equivalent* if $\int_{C'} c_1(L) = \int_{C''} c_1(L)$ for every line bundle L on X. There is a natural isomorphism

$$\mathrm{N}_1(X) \xrightarrow{\cong} H_2(X;\mathbb{Z})_{\mathrm{t.f.}} \cap \mathrm{P.D.}(H^{n-1,n-1}(X)) \,, \tag{6}$$

given by mapping an irreducible curve C to its homology class $[C] \in H_2(X;\mathbb{Z})$. Here $\mathrm{P.D.}: H^{2n-2}(X;\mathbb{C}) \to H_2(X;\mathbb{C})$ is the Poincare isomorphism. Let us denote by $\mathrm{NE}(X)$ the subset of $\mathrm{N}_1(X)$ consisting of elements that can be represented by $\sum_i a_i C_i$, where $C_i \subseteq X$ is an irreducible curve and $a_i \geq 0$ is a non-negative integer. The image of $\mathrm{NE}(X)$ in $H_2(X,\mathbb{Z})_{\mathrm{t.f.}}$ via (6) will be denoted by $\mathrm{Eff}(X)$ – the cone of *effective curve classes*. Similarly, by replacing integer with real coefficients, we define $\mathrm{N}_1(X)_{\mathbb{R}}$ and $\mathrm{NE}(X)_{\mathbb{R}}$. The set $\mathrm{NE}(X)_{\mathbb{R}}$ is a cone in $\mathrm{N}_1(X)_{\mathbb{R}}$, called the *curve cone* of X. It turns out that the closure of the curve cone is the dual of the nef cone, that is,

$$\overline{\mathrm{NE}}(X)_{\mathbb{R}} = \{\gamma \in \mathrm{N}_1(X)_{\mathbb{R}} \mid \delta \cdot \gamma \geq 0 \quad \forall \quad \delta \in \mathrm{Nef}(X)\}.$$

Lemma 2.8. *If X is a smooth projective variety, then there exists a set of ample divisors $D_1, \ldots, D_r$ whose numerical equivalence classes form a $\mathbb{Z}$-basis of* $\mathrm{NS}^1(X)$.

Proof. Suppose that $H \in \mathrm{Div}(X)$ is an ample divisor whose class in $\mathrm{NS}^1(X)$ is primitive, that is, $H = nH'$ for some $H' \in \mathrm{NS}^1(X)$ implies that $n = \pm 1$. We claim that $\mathrm{NS}^1(X)/\mathbb{Z}H$ is a free Abelian group of rank $r-1$. Indeed, we have to prove that the quotient is torsion free. Suppose that it is not. Then there exists $E \in \mathrm{NS}^1(X)$ and $m \in \mathbb{Z}$, such that, $mE = nH$ for some $n \in \mathbb{Z}$. Since $\mathrm{NS}^1(X)$ is torsion free, we may assume that m and n are relatively prime. Therefore, there exist k, l, such that, $km + ln = 1$ and we have

$$lmE = lnH = (1-km)H \quad \Rightarrow \quad H = m(lE + kH).$$

Since H is primitive, we get $m = \pm 1$, that is, $E \in \mathbb{Z}H$ – this proves that we can not have torsion elements in $\mathrm{NS}^1(X)/\mathbb{Z}H$.

Let us choose $E_1, \ldots, E_{r-1} \in \mathrm{NS}^1(X)$ that represent a $\mathbb{Z}$-basis of $\mathrm{NS}^1(X)/\mathbb{Z}H$. Let us choose $n \gg 0$ such that $E_i + nH$ is ample for all $1 \leq i \leq r-1$. It is easy to check that $D_i = E_i + nH$ $(1 \leq i \leq r-1)$, $D_r = H$ is an ample $\mathbb{Z}$-basis of $\mathrm{NS}^1(X)$. □

2.2. Cohomology of the blow up. Let us fix a point $x^\circ \in X$ and denote by $\widetilde{X}$ the blow up of X at the point x°. Let $\pi : \widetilde{X} \to X$ be the canonical projection map and $E := \pi^{-1}(x^\circ)$ the exceptional fiber. Clearly E is a Weyl divisor in $\widetilde{X}$ and hence a Cartier divisor because $\widetilde{X}$ is smooth. Let $e = c_1(\mathcal{O}(E)) = \mathrm{P.D.}(E)$. Using a Mayer–Vietories sequence argument, it is easy to prove the following two facts:

(1) The pullback map $\pi^* : \; H^*(X;\mathbb{C}) \longrightarrow H^*(\widetilde{X};\mathbb{C})$ is injective, so we can view the cohomology $H^*(X;\mathbb{C})$ as a subvector space of $H^*(\widetilde{X};\mathbb{C})$.

(2) We have a direct sum decomposition

$$H^*(\widetilde{X};\mathbb{C}) = H^*(X;\mathbb{C}) \bigoplus \bigoplus_{i=1}^{n-1} \mathbb{C}\, e^i.$$

The Poincare pairing of $\widetilde{X}$ can be computed as follows. Let us choose a basis ϕ_i $(1 \le i \le N)$ of $H^*(X;\mathbb{C})$, such that,

(i) $\phi_1 = 1$ and $\phi_N = \mathrm{P.D.}(x^\circ)$,

(ii) $\phi_{i+1} = c_1(\mathcal{O}(D_i))$ $(1 \le i \le r)$, where D_i $(1 \le i \le r)$ is an ample $\mathbb{Z}$-basis of $\mathrm{NS}^1(X)$ (see Lemma 2.8).

Lemma 2.9. *Let $(\ ,\ )^{\widetilde{X}}$ and $(\ ,\ )^X$ be the Poincare pairings on respectively $\widetilde{X}$ and X. Then we have*
a) $(\phi_i, \phi_j)^{\widetilde{X}} = (\phi_i, \phi_j)^X$ for all $1 \le i, j \le N$.
b) $(\phi_i, e^k)^{\widetilde{X}} = 0$ for $1 \le i \le N$ and $1 \le k \le n-1$.
c) $e^n = (-1)^{n-1}\phi_N$ and $(e^k, e^{n-k})^{\widetilde{X}} = (-1)^{n-1}$.

Proof. Parts a) and b) follow easily by the projection formula and Poincare duality. The second part of c) is a consequence of the first part, so we need only to prove that $e^n = (-1)^{n-1}\phi_N$. We have $e^n = c\phi_N$ for dimension reasons. Note that $E \cong \mathbb{P}^{n-1}$ and $\mathcal{O}(E)|_E = \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. Therefore, $e|_E = c_1(O(E)|_E) = -p$, where $p = c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(1))$ is the standard hyperplane class of $\mathbb{P}^{n-1}$. We get

$$c = \int_{[\widetilde{X}]} e^n = \int_{[E]} e^{n-1} = \int_{[\mathbb{P}^{n-1}]} (-p)^{n-1} = (-1)^{n-1}.$$

□

The ring structure of $H^*(\widetilde{X};\mathbb{C})$ with respect to the cup product is also easy to compute. We have

(1) $H^*(X;\mathbb{C})$ is a subring of $H^*(\widetilde{X};\mathbb{C})$.

(2) $\phi_i \cup e^k = 0$, $1 \le i \le N$, $1 \le k \le n-1$.

(3)

$$e^k \cup e^l = \begin{cases} e^{k+l} & \text{if } k+l < n, \\ (-1)^{n-1}\phi_N & \text{if } k+l = n, \\ 0 & \text{if } k+l > n. \end{cases}$$

Property (1) follows from the fact that pullback in cohomology is a ring homomorphism. The formulas in (3) follow from Lemma 2.9, part c). Finally, (2) follows from (1), (3) and Lemma 2.9, part b).

2.3. K-ring of the blow up. Let us compute the topological K-ring of X. We will be interested only in manifolds X, such that, the corresponding quantum cohomology is semi-simple. Such X are known to have cohomology classes of Hodge type (p,p) only. In particular, $K^1(X) \otimes \mathbb{Q} = 0$. To simplify the exposition, let us assume that $K^1(X) = 0$. In our arguments below we will have to work with non-compact manifolds. However, in all cases the non-compact manifolds are homotopy equivalent to finite CW-complexes, so we define the corresponding K-groups by taking the K-groups of the corresponding finite CW-complexes, i.e., in the case of non-compact manifolds we choose the homotopical version of topological K-theory.

Proposition 2.10. *a) The K-theoretic pullback $\pi^* : K^0(X) \to K^0(\widetilde{X})$ is injective.*
b) We have

$$K^0(\widetilde{X}) = K^0(X) \bigoplus \bigoplus_{j=1}^{n-1} \mathbb{Z}\mathcal{O}_E^j,$$

where $K^0(X)$ is viewed as a subring of $K^0(\widetilde{X})$ via the K-theoretic pullback π^* and $\mathcal{O}_E := \mathcal{O} - \mathcal{O}(-E)$ is the structure sheaf of the exceptional divisor.

Proof. Let $U \subset X$ be a small open neighborhood of the center of the blow up x° and $V := X \setminus \{x^\circ\}$. Note that $\{U, V\}$ is a covering of X. Put $\widetilde{U} = \pi^{-1}(U)$ and $\widetilde{V} := \pi^{-1}(V)$, then $\{\widetilde{U}, \widetilde{V}\}$ is a covering of $\widetilde{X}$. Let us compare the reduced K-theoretic Mayer–Vietories sequences of these two coverings. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{K}^{-1}(X) \arrow[r] \arrow[d] & \widetilde{K}^{-1}(V)\oplus\widetilde{K}^{-1}(U) \arrow[r] \arrow[d] & \widetilde{K}^{-1}(U\setminus x^\circ) \arrow[r] \arrow[d, "\cong"'] & \widetilde{K}^{0}(X) \arrow[r] \arrow[d] & \widetilde{K}^{0}(V)\oplus\widetilde{K}^{0}(U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{K}^{-1}(\widetilde{X}) \arrow[r] & \widetilde{K}^{-1}(\widetilde{V})\oplus\widetilde{K}^{-1}(\widetilde{U}) \arrow[r] & \widetilde{K}^{-1}(\widetilde{U}\setminus E) \arrow[r] & \widetilde{K}^{0}(\widetilde{X}) \arrow[r] & \widetilde{K}^{0}(\widetilde{V})\oplus\widetilde{K}^{0}(\widetilde{U}) \arrow[r] & 0
\end{tikzcd}$$

where the vertical arrows in the above diagram are induced by the K-theoretic pullback π^* and the vanishing $\widetilde{K}^{\mathrm{ev}}(U\setminus x^\circ) = \widetilde{K}^0(\widetilde{U}\setminus E) = 0$ follows from the fact that $\widetilde{U}\setminus E \cong U\setminus x^\circ$ is homotopic to $\mathbb{S}^{2n-1}$ – the $(2n-1)$-dimensional sphere. Note that $\widetilde{K}^{-1}(U) = \widetilde{K}^0(U) = 0$, because U is contractible and $\widetilde{K}^{-1}(\widetilde{U}) = 0$, because $\widetilde{U}$ is homotopy equivalent to $E \cong \mathbb{P}^{n-1}$. We get that the second vertical arrow is an isomorphism ($V \cong \widetilde{V}$) and hence, recalling the 5-lemma, we get $\widetilde{K}^{-1}(\widetilde{X}) = \widetilde{K}^{-1}(X) = 0$. A straightforward diagram chasing shows that the 4th vertical arrow is injective, i.e., we proved a).

Note that the above diagram yields the following short exact sequence

$$0 \longrightarrow \widetilde{K}^0(X) \xrightarrow{\pi^*} \widetilde{K}^0(\widetilde{X}) \xrightarrow{|_E} \widetilde{K}^0(\mathbb{P}^{n-1}) \longrightarrow 0\ , \tag{7}$$

where the map $|_E$ is the restriction to the exceptional divisor $E \cong \mathbb{P}^{n-1}$. The above exact sequence splits, because $\widetilde{K}^0(\mathbb{P}^{n-1}) \cong \mathbb{Z}^{n-1}$ is a free module. Note that $\mathcal{O}_E|_E = \mathcal{O}_{\mathbb{P}^{n-1}} - \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ is the generator of $\widetilde{K}^0(\mathbb{P}^{n-1})$, so part b) follows from the exactness of (7). □

Let us compute the K-theoretic product. Note that $\pi_*(\mathcal{O}_{\widetilde{X}}) = \mathcal{O}_X$. Therefore, $\pi_*\pi^*(F) = F$ for every $F \in K^0(X)$. Let us compute $\mathcal{O}_E \otimes \pi^* F$ for $F \in \widetilde{K}^0(X)$. The restriction of $\mathcal{O}_E \otimes \pi^* F$ to E is 0. Recalling the exact sequence (7) we get $\mathcal{O}_E \otimes \pi^* F = \pi^* G$ for some $G \in \widetilde{K}^0(X)$. Taking pushforward, we get

$$G = \pi_*(\mathcal{O}_E \otimes \pi^* F) = \pi_*(\mathcal{O}_E) \otimes F = \iota_{x^\circ}(\mathbb{C}) \otimes F = \mathrm{rk}(F)\iota_{x^\circ}(\mathbb{C}) = 0,$$

where $\iota_{x^\circ}(\mathbb{C})$ is the skyscraper sheaf on X and in the 3rd equality we used the exact sequence

$$0 \longrightarrow \mathcal{O}(-E) \longrightarrow \mathcal{O} \longrightarrow \iota_*(\mathcal{O}_{\mathbb{P}^{n-1}}) \longrightarrow 0,$$

where $\iota : \mathbb{P}^{n-1} \to \widetilde{X}$ is the embedding whose image is the exceptional divisor. This sequence implies $\mathcal{O}_E = \iota_* \mathcal{O}_{\mathbb{P}^{n-1}} \Rightarrow \pi_* \mathcal{O}_E = (\pi \circ \iota)_* \mathcal{O}_{\mathbb{P}^{n-1}} = \iota_{x^\circ}(\mathbb{C})$. We proved that

$$\mathcal{O}_E \otimes \pi^* F = 0, \quad \forall F \in \widetilde{K}^0(X).$$

It remains only to compute $\mathcal{O}_E^n$. The restriction of $\mathcal{O}_E^n$ to E is $(1 - \mathcal{O}_{\mathbb{P}^{n-1}}(-1))^n = 0$. Therefore, $\mathcal{O}_E^n = \pi^* F$. The Chern character $\mathrm{ch}(\mathcal{O}_E^n) = (1 - \exp(-c_1(\mathcal{O}(E))))^n = e^n = (-1)^{n-1}\phi_N$, where we used Lemma 2.9, part c). On the other hand, the Chern character of the skyscraper sheaf can be computed easily with the Grothendieck–Riemann–Roch formula. Namely, we have

$$\mathrm{ch}(\iota_*^\circ(\mathbb{C})) \cup \mathrm{td}(X) = \iota_*^\circ(\mathrm{ch}(\mathbb{C}) \cup \mathrm{td}(x^\circ)) = \iota_*^\circ(1) = \mathrm{P.D.}(x^\circ) = \phi_N,$$

where $\iota^\circ : x^\circ \to X$ is the natural inclusion of the point x°. The above formula implies $\mathrm{ch}(\iota_{x^\circ}(\mathbb{C})) = \phi_N$. Camparing with the formula for $\mathrm{ch}(\mathcal{O}_E^n)$, we get

$$\mathcal{O}_E^n = (-1)^{n-1}\, \iota_{x^\circ}(\mathbb{C}) \mod \ker(\mathrm{ch}).$$

Finally, let us finish this section by quoting the formula for the K-theoretic class of the tangent bundle (see [**8**], Lemma 15.4):

$$T\widetilde{X} = TX - n - 1 + n\mathcal{O}(-E) + \mathcal{O}(E). \tag{8}$$

2.4. Gromov–Witten theory. Let us recall some basics on Gromov–Witten (GW) theory. For further details we refer to [**18**]. The main object is the moduli space of stable maps $\overline{\mathcal{M}}_{g,k}(X,\beta)$, where g,k are non-negative integers and $\beta \in \mathrm{Eff}(X)$. By definition, a stable map consists of the following data $(\Sigma, z_1, \ldots, z_k, f)$:

(1) Σ is a Riemann surface with at most nodal singular points.
(2) $z_1,\ldots,z_k$ are *marked points*, that is, smooth pairwise-distinct points on Σ.
(3) $f : \Sigma \to X$ is a holomorphic map, such that, $f_*[\Sigma] = \beta$.
(4) The map is stable, i.e., the automorphism group of $(\Sigma, z_1, \ldots, z_k, f)$ is finite.

Two stable maps $(\Sigma, z_1, \ldots, z_k, f)$ and $(\Sigma', z_1', \ldots, z_k', f')$ are called equivalent if there exists a biholomorphism $\phi : \Sigma \to \Sigma'$, such that, $\phi(z_i) = z_i'$ and $f' \circ \phi = f$. The moduli space of equivalence classes of stable maps is known to be a proper Delign–Mumford stack with respect to the etale topology on the category of schemes (see [**3**]). The corresponding coarse moduli space $\overline{M}_{g,k}(X,\beta)$ has a structure of a projective variety, which however could be very singular. We have the following diagram:

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,k+1}(X,\beta) \arrow[r, "\mathrm{ev}_{k+1}"] \arrow[d, "\pi"'] & X \\
\overline{\mathcal{M}}_{g,k}(X,\beta) \arrow[r, "\mathrm{ev}_i"] \arrow[d, "\mathrm{ft}"'] & X \ (1 \le i \le k) \\
\overline{\mathcal{M}}_{g,k} &
\end{tikzcd}$$

where $\mathrm{ev}_i(\Sigma, z_1, \ldots, z_k, f) := f(z_i)$, π is the map forgetting the last marked point an contracting all unstable components, and ft is the map forgetting the holomorphic map f and contracting all unstable components. The moduli space has natural orbifold line bundles L_i $(1 \le i \le k)$ whose fiber at a point $(\Sigma, z_1, \ldots, z_k, f)$ is the cotangent line $T^*_{z_i}\Sigma$ equipped with the action of the automorphism group of $(\Sigma, z_1, \ldots, z_k, f)$. Let $\psi_i = c_1(L_i)$ be the first Chern class. The most involved construction in GW theory is the construction of the so called *virtual fundamental cycle*. The construction has as an input the complex $(R\pi_* \mathrm{ev}_{k+1}^* TX)^\vee$ which gives rise to a perfect obstruction theory on $\overline{\mathcal{M}}_{g,k}(X,\beta)$ relative to $\overline{\mathcal{M}}_{g,k}$ (see [**4**, **5**]) and yields a homology cycle in $\overline{M}_{g,k}(X,\beta)$ of complex dimension

$$3g - 3 + k + n(1-g) + \langle c_1(TX), \beta \rangle$$

known as the virtual fundamental cycle. GW invariants are by definition the following correlators:

$$\langle a_1 \psi_1^{l_1}, \ldots, a_k \psi^{l_k} \rangle_{g,k,\beta} = \int_{[\overline{M}_{g,k}(X,\beta)]^{\mathrm{virt}}} \mathrm{ev}_1^*(a_1) \cdots \mathrm{ev}_k^*(a_k) \psi_1^{l_1} \cdots \psi_k^{l_k},$$

where $a_1, \ldots, a_k \in H^*(X;\mathbb{C})$ and $l_1, \ldots, l_k$ are non-negative integers.

2.5. Quantum cohomology of X. Let q_i $(1 \le i \le r)$ be formal variables. If $\beta \in \mathrm{Eff}(X)$, then we put $q^\beta = q_1^{\langle \phi_2, \beta\rangle} \cdots q_r^{\langle \phi_{r+1},\beta\rangle}$. The group ring $\mathbb{C}[\mathrm{Eff}(X)]$ is called the *Novikov ring* of X and the variables q_i are called *Novikov variables*. Note that the Novikov variables determine an embedding of the Novikov ring into the ring of formal power series $\mathbb{C}[[q_1, \dots, q_r]]$.

Recall the basis ϕ_i $(1 \le i \le N)$ of $H^*(X;\mathbb{C})$. We will assume that the basis is homogeneous and let $t = (t_1, \dots, t_N)$ be the corresponding linear coordinates. The quantum cup product $\bullet_{t,q}$ of X is a deformation of the classical cup product defined by

$$(\phi_a \bullet_{t,q} \phi_b, \phi_c) := \langle \phi_a, \phi_b, \phi_c \rangle_{0,3}(t) = \sum_{m=0}^{\infty} \sum_{\beta \in \mathrm{Eff}(X)} \frac{q^\beta}{m!} \langle \phi_a, \phi_b, \phi_c, t, \dots, t\rangle_{0,3+m,\beta}.$$

Using string and divisor equation, we get that the structure constants of the quantum cup product, i.e., the 3-point genus-0 correlators in the above formula are independent of t_1 and are formal power series in the following variables:

$$q_1 e^{t_2}, \dots, q_r e^{t_r}, t_{r+1}, \dots, t_N.$$

Let us also denote the calibration

$$S(t,Q,z) = 1 + \sum_{k=1}^{\infty} S_k(t,Q) z^{-1}$$

where

$$(S_K \phi, \phi_b) = < \phi_a \psi^{k-1}, \phi_b >_{0,2} (t) = \sum_{d \in \mathrm{Eff}(X)} \sum_{l=0}^{\infty} \frac{Q^d}{l!} < \phi_a \psi^{k-1}, \phi_b, t, \cdots, t >_{0,l+2,d} .$$

We are going to consider only manifolds X, such that, the quantum cup product is analytic. More precisely, let us allow for the Novikov variables to take values $0 < |q_i| < 1$ $(1 \le i \le r)$. Then we will assume that there exists an $\epsilon > 0$, such that, the structure constants of the quantum cup product are convergent power series for all t satisfying

$$\mathrm{Re}(t_i) < \log \epsilon \quad (2 \le i \le r+1), \quad |t_j| < \epsilon \quad r+1 < j \le N. \tag{9}$$

The inequalities (9) define an open subset $M \subset H^*(X;\mathbb{C})$. The main fact about genus-0 GW invariants is that M has a Frobenius structure, such that, the Frobenius pairing is the Poincare pairing, the Frobenius multiplication is the quantum cup product, the unit $\mathbf{1} = \phi_1$, and the Euler vector field is

$$E = \sum_{i=1}^{N} (1-d_i) t_i \tfrac{\partial}{\partial t_i} + \sum_{j=2}^{r+1} (c_1(TX), \phi^j) \tfrac{\partial}{\partial t_j},$$

where d_i is the complex degree of ϕ_i, that is, $\phi_i \in H^{2d_i}(X;\mathbb{C})$ and ϕ^j $(1 \le j \le N)$ is the basis of $H^*(X;\mathbb{C})$ dual to ϕ_i $(1 \le i \le N)$ with respect to the Poincare pairing. Let us point out that under our assumption $K^1(X) = 0$ the cohomology groups $H^{\mathrm{odd}}(X;\mathbb{C}) = 0$. Otherwise, M has to be given the structure of a supermanifold (see [**18**]). The conformal dimension of M is $n = \dim_{\mathbb{C}}(X)$ and the Hodge grading operator takes the form

$$\theta(\phi_i) = \left(\tfrac{n}{2} - d_i\right) \phi_i, \quad 1 \le i \le N.$$

If the Frobenius manifold M corresponding to quantum cohomology is semi-simple, then there is a conjectural description of the set of reflection vectors, which can be viewed as part of Dubrovin's conjecture.

Let us give a precise statement. Let us denote by $D^b(X)$ the derived category of the category of bounded complexes of coherent sheaves on X, that is, the bounded derived category of X.

Definition 2.11. a) A sequence $(\mathcal{E}_1,\ldots,\mathcal{E}_N)$ of objects in $D^b(X)$ is called an *exceptional collection* if $RHom(\mathcal{E}_i,\mathcal{E}_j)=0$ for $i>j$ and $RHom(\mathcal{E}_i,\mathcal{E}_i)=\mathbb{C}[0]$.

b) An exceptional collection $(\mathcal{E}_1,\ldots,\mathcal{E}_N)$ is called *full exceptional collection* if the smallest subcategory of $D^b(X)$ that contains $\mathcal{E}_i$ $(1\le i\le N)$ and is closed under isomorphisms, shifts, and cones, is $D^b(X)$ itself.

Let us recall also Iritani's integral structure (see [**16**]):

$$\Psi_q : K^0(X)_{\text{t.f.}} \to H^*(X;\mathbb{C}) \tag{10}$$

defined by

$$\Psi_q(E)=(2\pi)^{\frac{1-n}{2}}\widehat{\Gamma}(X)\cup e^{-\sum_{i=1}^r p_i\log q_i}\cup(2\pi\mathbf{i})^{\deg}(\mathrm{ch}(E)),$$

where deg is the complex degree operator, that is, $\deg(\phi)=i\phi$ for $\phi\in H^{2i}(X;\mathbb{C})$, $\mathbf{i}:=\sqrt{-1}$, $n=\dim_{\mathbb{C}}(X)$, and $\widehat{\Gamma}(X)=\widehat{\Gamma}(TX)$ is the *Gamma*-class of X. Recall that for a vector bundle E with Chern roots $x_1,\ldots,x_r$ the Γ-class of E is defined by

$$\widehat{\Gamma}(E)=\prod_{i=1}^r\Gamma(1+x_i).$$

Conjecture 2.12. *If the Frobenius manifold M corresponding to quantum cohomology is semi-simple, then the image of Ψ_q in $H^*(X;\mathbb{C})$ coincides with the $\mathbb{Z}$-span of the set of all reflection vectors.*

The above conjecture is motivated by Iritani's results in [**16**], which give a confirmative answer for the case of weak Fano toric orbifolds. Moreover, motivated by Dubrovin's conjecture, it is natural to make the following stronger version of Conjecture 2.12.

Conjecture 2.13. *If $(\mathcal{E}_1,\ldots,\mathcal{E}_N)$ is a full exceptional collection in $D^b(X)$, then $\Psi_q([\mathcal{E}_i])$ is a reflection vector for all $1\le i\le N$.*

Let us make several remarks.

(1) Conjecture 2.13 implies Conjecture 2.12.
(2) Conjecture 2.13 gives us a method for computing all reflection vectors in quantum cohomology: the set of all reflection vectors coincides with the smallest set that contains $\alpha_i:=\Psi_q([\mathcal{E}_i])$ for $1\le i\le N$ and is closed under the action of the reflections w_{α_j} $(1\le j\le N)$ defined by (5).
(3) Recall that the Euler pairing is defined by

$$\chi(E,F)=\sum_{i=0}^n(-1)^i\dim\mathrm{Ext}^i(E,F),$$

for complex vector bundles E and F on X. The above definition induces a non-degenerate integral pairing on $K^0(X)_{\text{t.f.}}$. The intersection pairing (4) coincides with the symmetrization of the Euler pairing under the map Ψ_q, that is,

$$(\Psi_q(E)|\Psi_q(F))=\chi(E,F)+\chi(F,E),\quad \forall E,F\in K^0(X).$$

Our main goal is to prove that if Conjecture 2.13 is true for X, then it is true for the blow up $\widetilde{X}$.

2.6. Quantum cohomology of the blow up. Let us first compare the effective curve cones $\mathrm{Eff}(X)$ and $\mathrm{Eff}(\widetilde{X})$. We have an exact sequence

$$0 \longrightarrow H_2(\mathbb{P}^{n-1};\mathbb{Z}) \xrightarrow{\iota_*} H_2(\widetilde{X};\mathbb{Z}) \xrightarrow{\pi_*} H_2(X;\mathbb{Z}) \longrightarrow 0,$$

where $\iota : \mathbb{P}^{n-1} \to \widetilde{X}$ is the natural closed embedding of the exceptional divisor. The proof of the exactness is similar to the proof of (7). In particular, since the torsion free part of the above sequence splits, we get

$$H_2(\widetilde{X};\mathbb{Z})_{\text{t.f.}} = H_2(X;\mathbb{Z})_{\text{t.f.}} \oplus \mathbb{Z}\ell,$$

where $\ell \in H_2(E;\mathbb{Z})$ is the class of a line in the exceptional divisor. The cone of effective curve classes $\mathrm{Eff}(\widetilde{X}) \subset \mathrm{Eff}(X) \oplus \mathbb{Z}\ell$. The Novikov variables of the blow up will be fixed to be the Novikov variables of X and an extra variable corresponding to the line bundle $\mathcal{O}(E)$. In other words, for $\widetilde{\beta} = \beta + d\ell \in \mathrm{Eff}(\widetilde{X})$, put

$$q^{\widetilde{\beta}} = q^{\beta} q_{r+1}^{\langle c_1(O(E)),\widetilde{\beta}\rangle} = q_1^{\langle \phi_2,\beta\rangle} \cdots q_r^{\langle \phi_{r+1},\beta\rangle} q_{r+1}^{-d}$$

Note that $\mathcal{O}(E)$ is not an ample line bundle: for example, $\ell \cdot E = -1 < 0$. Our choice of q_{r+1} makes the structure constants formal Laurent (not power) series in q_{r+1}. Following Bayer (see [2]) we write $q_{r+1} = Q^{n-1}$ for some formal variable Q. Let us recall the basis ϕ_i $(1 \le i \le N)$ of $H^*(X;\mathbb{C})$. Put $\phi_{N+k} = e^k$ $(1 \le k \le n-1)$. Then ϕ_i $(1 \le i \le \widetilde{N} := N+n-1)$ is a basis of $H^*(\widetilde{X};\mathbb{C})$. Let $t = (t_1,\dots,t_{\widetilde{N}})$ be the corresponding linear coordinate system on $H^*(\widetilde{X};\mathbb{C})$. The structure constants of the quantum cohomology of $\widetilde{X}$ take the form

$$(\phi_a \bullet_{t,q} \phi_b, \phi_c) := \langle \phi_a, \phi_b, \phi_c \rangle_{0,3}(t) = \sum_{m=0}^{\infty} \sum_{\widetilde{\beta}=(\beta,d)} \frac{q^{\beta} Q^{-d(n-1)}}{m!} \langle \phi_a, \phi_b, \phi_c, t, \dots, t\rangle_{0,3+m,\widetilde{\beta}}.$$

3. Toric manifolds

First let us fix some basic notation.

(a) Matrix M: $M = (m_{ij})_{1\le i\le r\ 1\le j\le N}$, where $m_{ij} \in \mathbb{Z}$.
(b) Moment map: $\mu : \mathbb{C}^N \to \mathbb{R}^r$ defined by

$$\mu(z_1,\cdots,z_n) = (\sum_{j=1}^{N} m_{1j}|z_j|^2, \cdots, \sum_{j=1}^{N} m_{rj}|z_j|^2).$$

(c) Complex torus: $T_{\mathbb{C}} = (\mathbb{C}^*)^r$ acting on $\mathbb{C}^N$ by

$$t \cdot (z_1,\cdots,z_N) = \left(\prod_{i=1}^{r} t_i^{m_{i1}} z_1, \cdots, \prod_{i=1}^{r} t_i^{m_{iN}} z_N\right),$$

where $t \in T_{\mathbb{C}}$.

Definition 2.14. $\omega \in \mathbb{R}^r$ is called *regular value* if $\mu^{-1}(\omega)$ is a manifold. i.e $\forall z \in \mu^{-1}(\omega)$ $d\mu : T_z\mathbb{C}^N \to T_\omega \mathbb{R}^r$ is surjective.

Let us denote $\mathbb{R}^r_{reg} :=$ {the set of the regular values}. For $I = \{i_1,\cdots,i_s\} \subseteq \{1,\cdots,N\}$, define

$$C^I = \mathrm{Span}\{e_i\}_{i\in I} \quad \text{where } e_i = (0,\cdots,1,\cdots,0) \text{ only } i^{th} \text{ is } 1$$

A point ω is not a regular value, iff there exists $I = (i_1, \cdots, i_{r-1})$, such that $\omega \in \mu(\mathbb{C}^I)$, which means that $\mathbb{R}^r_{reg} = R^r \setminus \bigcup_I \mu(C^I)$. The connected components of $\mathbb{R}^r_{reg}$ will be called chambers. Let $K \subset \mathbb{R}^r_{reg}$ be one of the chambers. Then, the quotient

$$X_{M,K} := \mu^{-1}(K)/T_{\mathbb{C}}$$

is called a toric variety.

Let us define line bundles

$$L_i := \mu^{-1}(K) \times \mathbb{C}/T_{\mathbb{C}},$$

where

$$t \cdot [z, v] = [t \cdot z, t_i v] \ (v \in \mathbb{C}).$$

We have an $\mathbb{R}$–linear isomorphism

$$\begin{array}{ccc} \mathbb{R}^r & \cong & H^2(X;\mathbb{R}) \\ p_i & \longmapsto & c_1(L_i), \end{array}$$

where $p_i = (0, \cdots, 1, \cdots, 0)$ with 1 on the i-th place. Put $u_j = \sum_{i=1}^r p_i m_i j$.

Remark 2.15. There are line bundles

$$U_j \longmapsto X_{M,K} \ (i \le j \le N)$$
$$U_i = \mu^{-1}(K) \times \mathbb{C} \setminus T_{\mathbb{C}}$$

where $t \in T_{\mathbb{C}}$ acts by

$$t \cdot [z, v] = [t \cdot v, \prod_{i=1}^r t_i^{m_{ij}} v].$$

Note that $U_j = L_1^{m_{ij}} \otimes \cdots \otimes L_r^{m_{rj}}$. Therefore, $u_j = \sum_{i=1}^r m_{ij} c_1(L_i) = c_1(U_j)$.

Remark 2.16. $M = (m_{ij})_{1 \le i \le r,\ 1 \le j \le N}$ defines a map

$$\begin{array}{c} \mathbb{R}^n \to \mathbb{R}^r \\ e_j \longmapsto u_j = \sum_{i=1}^r p_i m_{ij}. \end{array}$$

If $I = (i_1, i_2, \cdots, i_k) \subset \{1, 2, \cdots, N\}$,then let us define

$$\sigma_I := \mathbb{R}_{\ge 0} u_{i_1} + \cdots + \mathbb{R}_{\ge 0} u_{i_k}.$$

Note that

$$\mathbb{R}^r_{reg} := \mathbb{R}^r \setminus \bigcup_{I:dim\ \mu(\mathbb{C}^I) = r-1} \mu(\mathbb{C}^I).$$

It is well know that

$$H^\bullet(X : \mathbb{R}) \cong \mathbb{R}[p_1, \cdots, p_r]/I_{M,K},$$

where $I_{M,K}$ is the ideal generated by the monomials $\prod_{J \in I^\circ} u_j$, $I \subset \{1, 2, \cdots, n\}$ such that $\sigma_I \not\supset K$, and $I^\circ = \{1, 2, \cdots, n\} \setminus I$.

REMARK 2.17. If $I' \subset I''$ and $\sigma_{I''} \not\supset K$, then $\sigma_{I'} \not\supset K$. We also have that, $\prod_{j\in(I'')^\circ} u_j$ divides $\prod_{j\in(I')^\circ} u_j$. Therefore, $\prod_{j\in(I'')^\circ} u_j \in I_{M,K}$ implies $\prod_{j\in(I')^\circ} u_j \in I_{M,K}$. Hence, $I_{M,K}$ is generated by $\prod_{j\in I^\circ} u_j$, where $I \subset \{1,2,\cdots,n\}$ is a maximal subset such that $\sigma_I \not\supset K$.

The toric variety $X_{M,K}$ is compact iff the system

$$\sum_{j=1}^{N} x_j u_j = 0$$
$$x_1,\cdots,x_N \geq 0$$

has only the trivial solution

$$x_1 = \cdots = x_r = 0.$$

It is known that $X_{M,K}$ is a manifold iff for every subset $J = (j_1,\cdots,j_r)$ such that $\sigma_J \supset K$, the determinant $\det(m_{ij})_{1\leq i\leq r,\ j\in J} = \pm 1$. For toric manifolds the map

$$p_i \longmapsto c_1(L_i)$$

induces an isomorphism

$$\mathbb{Z}^r \to H^2(X;\mathbb{Z})$$

$$\mathrm{Eff}(X_{M,K}) := K_{\mathbb{Z}}^{\vee} := \{d \in (\mathbb{Z}^R)^{\vee} | < d,\omega >\geq 0,\ \forall\omega \in K\}.$$

Under the isomorphism $\mathbb{R}^r \cong H^2(X_{M,K}\mathbb{R})$, the chamber K is identified with the kähler cone of $X_{M,K}$ and $K_{\mathbb{Z}}^{\vee} \subset H_2(X_{M,K};\mathbb{Z})$ is the cone of curves of $X_{M,K}$. Let $\{e_i\}_{i=1}^r \subset H_2(X;\mathbb{Z}) \cong (\mathbb{Z}^r)^{\vee}$ be the basis dual to $\{p_i\}_{i=1}^r$. Then $(\mathbb{Z}^r)^{\vee} \cong \mathbb{Z}^r$ and

$$\mathrm{Eff}(X_{M,K}) = \{d \in \mathbb{Z}^r | \sum_{i=1}^{r} d_i\omega_i \geq 0\ \forall\omega = (\omega_1,\cdots,\omega_r) \in K\}.$$

Let $Q = (Q_1,\cdots,Q_r)$ be the Novikov variables as usual,

$$Q^d := Q_1^{d_1}\cdots Q_r^{d_r},$$

for $d \in \mathrm{Eff}(X_{M,K})$. Then apply [**9**], the I-fuction of $X_{M,K}$ is defined by

$$I_{M,K}(Q,z) := \sum_{d\in\mathrm{Eff}(X_{M,K})} Q^d \prod_{j=1}^{N} \frac{\prod_{m=-\infty}^{0}(u_j + mz)}{\prod_{m=-\infty}^{<u_j,d>}(u_j + mz)}. \tag{11}$$

According to Givental (see [**16**] and [**6**]), if $X_{M,K}$ is a Fano toric manifold, that is,

$$c_1(TX) = \sum_{j=1}^{N} u_j \in K,$$

then $I_{M,K}(Q,-z) = S(0,Q,z)^{-1}\cdot 1$,

$$J_X(t,Q,z) = -z + t + \sum_{a=1}^{N}\sum_{u=0}^{\infty} < \phi_a\psi^k >_{0,1} (t)\phi^a(-z)^{-k-1} = -z\cdot S(t,Q,z)^{-1}\cdot 1.$$

Usually $J_X(t,Q,-z)$ is called J-function of X.

We are going to work with the toric variety corresponding to $M = \begin{pmatrix} 1 & \cdots & 1 & -1 & 0 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$ and $K = \mathbb{R}^2_{>0}$. We have that

$$X_{M,K} = (\mathbb{C}^n \backslash 0) \times (\mathbb{C} \backslash 0)/(\mathbb{C}^*)^2,$$

where the action of $t \in (\mathbb{C}^*)^2$ is given by

$$t \cdot (z, \lambda_1, \lambda_2) := (t \cdot z, t_1^{-1} t_2 \lambda_1, t_2 \lambda_2).$$

Let us consider the following diagram:

$$\begin{tikzcd} X_{M,K} \arrow[r, shift left, "\pi_{n-1}"] \arrow[d, "\pi_n"'] & \mathbb{P}^{n-1} \arrow[l, shift left, "j"] \\ \mathbb{P}^n & \end{tikzcd}$$

where the maps π_{n-1}, π_n and j are defined by

$$\begin{aligned} \pi_{n-1}(z, \lambda_1, \lambda_2) &:= [z_1 : \cdots : z_n], \\ \pi_n(z, \lambda_1, \lambda_2) &:= [\lambda_1 z_1 : \cdots : \lambda_1 z_n : \lambda_2], \\ j([z_1 : \cdots : z_n]) &= (z_1, \cdots, z_n, 0, 1). \end{aligned}$$

Put $E := j(\mathbb{P}^{n-1}) \subset X_{M,K}$. Note that

$$\pi_n(E) = [0 : \cdots : 0 : 1]$$

and that

$$X_{M,K} = \mathrm{Bl}_{[0:\cdots:0:1]}(\mathbb{P}^n).$$

Put

$$L_1 = \pi_{n-1}^*(\mathcal{O}_{\mathbb{P}^{n-1}}(1)), \quad L_2 = \pi_n^*(\mathcal{O}_{\mathbb{P}^n}(1)).$$

Recall that

$$\mathcal{O}_{\mathbb{P}^{n-1}}(1) = (\mathbb{C}^n \backslash 0) \times \mathbb{C}/\mathbb{C}^*,$$

we get that,

$$L_1 = (\mathbb{C}^n \backslash 0) \times (\mathbb{C}^2 \backslash 0) \times \mathbb{C}/(\mathbb{C}^*)^2,$$

where the action of $t \in (\mathbb{C}^*)^2$ is given by

$$t \cdot (z, \lambda_1, \lambda_2, \mu) = (t \cdot z, t_1^{-1} t_2 \lambda_1, t_2 \lambda_2, t_1 \mu).$$

In other words, L_1 is the line bundle corresponding to the character

$$\begin{aligned} \mathcal{X}_1 :\ T^2_{\mathbb{C}} &\longrightarrow \mathbb{C}^*, \\ \mathcal{X}_1(t_1, t_2) &= t_1. \end{aligned}$$

Similarly, L_2 is the line bundle corresponding to the character

$$\mathcal{X}_2 : T^2_{\mathbb{C}} \longrightarrow \mathbb{C}^*,$$
$$\mathcal{X}_2(t_1, t_2) = t_2.$$

We get $c_1(L_1) = p_1$ and $c_1(L_2) = p_2$.

Recall the Remark 2.16 and Remark 2.17. In our case I in Remark 2.17 are $I = \{1, 2, \cdots, n\}$ and $I = \{n+1, n+2\}$. Therefore,

$$H^\bullet(\mathrm{Bl}(\mathbb{P}^n)) = H^\bullet(X_{M,K}) = \mathbb{C}[p_1, p_2]/\langle p_2(p_2 - p_1) = 0\ ,\ p_1^n = 0\rangle.$$

Let e_1 = a class of line in $E = j(\mathbb{P}^{n-1})$, $e_2 = \pi_n^{-1}($ line in $\mathbb{P}^n$ avoiding $[0 : \cdots : 0 : 1]$, We can get,

$$\langle p_1, e_1\rangle = \langle c_1(L_1), e_1\rangle = \int_{\pi_{n-1}(e_1)} c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) = 1,$$

$$\langle p_1, e_2\rangle = \langle c_1(L_1), e_2\rangle = \int_{\pi_{n-1}(e_2)=\pi_{n-1}(e_1)} c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) = 1,$$

$$\langle p_2, e_1\rangle = \langle c_1(L_2), e_1\rangle = \int_{\pi_n(e_1)=0} c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) = 0,$$

$$\langle p_2, e_2\rangle = \langle c_1(L_2), e_2\rangle = \int_{\pi_n(e_2)=\widetilde{e_2}=\text{lin in }\mathbb{P}^n} c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) = 1.$$

Suppose that $\mathcal{O}(E) = L_1^a L_2^b$. We have that

$$\langle c_1(\mathcal{O}(E)), e_1\rangle = \int_{e_1} c_1(\mathcal{O}(E))|_E = \int_{e_1} c_1(\mathcal{O}_{\mathbb{P}^{n-1}}(-1)) = -1$$
$$\langle c_1(\mathcal{O}(E)), e_2\rangle = \int_{e_2} c_1(\mathcal{O}(E)) = 0$$

Which means that, $a = -1,\ a + b = 0$. Thus, $a = -1,\ b = 1$.Therefore, we have that $E = p_2 - p_1$.

Let us denote that $\Phi_{i,j}$ be the basis of $H^\bullet(\mathrm{Bl}(\mathbb{P}^n))$, where $0 < i \le n$, $j = 1, 2$ and $\Phi_{i,j} = p_1^{i-1} p_2^{j-1}$. For $\int_{\mathrm{Bl}(\mathbb{P}^n)} E^n = (-1)^{n-1}$, we have that

$$\int_{\mathrm{Bl}(\mathbb{P}^n)} E^n = \int_{\mathrm{Bl}(\mathbb{P}^n)} (-1)(p_2 - p_1)^{n-1} p_1 = \cdots = \int_{\mathrm{Bl}(\mathbb{P}^n)} (-1)^{n-1} p_1^{n-1}(p_2 - p1) = \int_{\mathrm{Bl}(\mathbb{P}^n)} (-1)^{n-1} p_1^{n-1} p_2$$

Which means $\int_{\mathrm{Bl}(\mathbb{P}^n)} p_1^{n-1} p_2 = 1$

$$\int_{\mathrm{Bl}(\mathbb{P}^n)} p_1^i p_2^j = \int_{\mathrm{Bl}(\mathbb{P}^n)} p_1^i (p_2 - p_1 + p_1)^{j-1} p_2 = \int_{\mathrm{Bl}(\mathbb{P}^n)} p_1^{i+j-1} p_2$$

Therefore if $i + j = n$, $\int_{\mathrm{Bl}(\mathbb{P})} p_1^i p_2^j = 1$

Let us specialize formula (11) to $\mathrm{Bl}(\mathbb{P}^n)$. Recall that

$$u_1 = \cdots = u_n = p_1$$
$$u_{n+1} = -p_1 + p_2$$
$$u_{n+2} = p_2$$

If $\langle u_i, d\rangle < 0$ for $i \le n$, then we will have p_1^n in the numerator, i.e. the terms for which $\langle u_1, d\rangle < 0$ vanish. In other words, $\langle u_i, d\rangle \ge 0$ for all $1 \le i \le n$. If $\langle u_{n+2}, d\rangle < 0$, we also have $\langle u_{n+1}, d\rangle = \langle u_{n+2}, d\rangle - \langle u_1, d\rangle < 0$, that is, we have $p_2(p_2-p_1)$ in the numerator. Hence, $\langle u_{n+2}, d\rangle \geq 0$. If we let $\langle u_i, d\rangle = D_1$, $(i \leq n)$, $\langle u_{n+2}, d\rangle = D_2$, then we have that $\langle u_{n+2}, d\rangle = D_2 - D_1$. Let us introduce the notation $\prod_{m=1}^{-n} \frac{1}{p_2-p_1+mz} = \prod_{m=-n+1}^{0} (p_2 - p_1 + mz)$ for $n > 0$ and $\prod_{i=1}^{0} \frac{1}{p_2-p_1+mz} = 1$. The I-function for the blow-up of $\mathbb{P}^n$ is

$$I_{M,K}(Q,z) = \sum_{D_1,D_2\geq 0} \frac{Q_1^{D_1} Q_2^{D_2}}{\prod_{m=1}^{D_1} (p_1+mz)^n \prod_{m=1}^{D_2} (p_2+mz) \prod_{m=1}^{D_2-D_1} (p_2-p_1+mz)}$$

4. Mirror of blow-up

According to Givental, the mirror of $\mathrm{Bl}(\mathbb{P}^n)$ is given by the restricion of $f(x) := x_1 + \cdots + x_{n+2}$ to the complex torus $x \in (\mathbb{C}^*)^{n+2} : \prod_{j=1}^{N} x_j^{m_{ij}} = Q_i$ $(i = 1, 2)$. Since $x_1 \cdots x_n x_{n+1}^{-1} = Q_1$, $x_{n+1}x_{n+2} = Q_2$. We have

$$f(x) = x_1 + \cdots + x_n + \frac{x_1 \cdots x_n}{Q_1} + \frac{Q_1 Q_2}{x_1 \cdots x_n}.$$

Put $\omega = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$. Then $\left((\mathbb{C}^*)^{n+2}, f, \omega\right)$ is a mirror model of $\mathrm{Bl}(\mathbb{P}^n)$ in the sense of Givental

If X is a Projective manifold, let us recall Definition 2.11

$$\Psi : K^{\circ}(X)/\text{torsion} \longrightarrow H^{\bullet}(X;\mathbb{C})$$

$$\Psi(E) = (2\pi)^{\frac{1-D}{2}} \hat{\Gamma}(X) \cup (2\pi i)^{\deg} \mathrm{ch}(E),$$

where $\hat{\Gamma}(X) = \Gamma(TX)$ and for a vector bundle E, Chern roots $x_1, \cdots .x_r$ we denote by

$$\hat{\Gamma}(E) = \prod \Gamma(1 + x_i)$$

its $\Gamma - class$.

In our case the $\hat{\Gamma}(\mathrm{Bl}(\mathbb{P}^n)) = \Gamma(1 + p_1)^n \Gamma(1 + p_2)\Gamma(1 + p_2 - p_1)$

5. J-function and quantum cohomology

Recall that Φ_a $(1 \leq a \leq N)$ are the basis of $H^*(X;\mathbb{C})$. For $1 \leq i \leq r$, $\Phi_{i+1} = c_1(\mathcal{O}(D_i))$, where D_i are ample $\mathbb{Z}$-basis of $\mathrm{NS}^1(X)$. Let us denote $\Phi_{i+1} = p_i$. We assume that $H^*(X;\mathbb{C})$ is generated as an algebra by $p_1, \cdots, p_r$. Using divisor equations, which are

$$z\frac{\partial}{\partial t_{i+1}} S(t,Q,z) = zQ_i \partial_{Q_i} S(t,Q,z) + S(t,Q,z) p_i \cup,$$

we have

$$\begin{aligned} z\frac{\partial}{\partial t_{i+1}} S^{-1}(t,Q,z) &= -S^{-1}(t,Q,z)\left(z\frac{\partial}{\partial t_{i+1}} S(t,Q,z)\right) S^{-1}(t,Q,z) \\ &= -S^{-1}(t,Q,z)\left(zQ_i\partial_{Q_i} S(t,Q,z) + S(t,Q,z)p_i\cup\right) S^{-1}(t,Q,z) \\ &= -S^{-1}(t,Q,z) zQ_i \partial_{Q_i} S(t,Q,z) S^{-1}(t,Q,z) - p_i \cup S^{-1}(t,Q,z) \\ &= zQ_i\partial_{Q_i} S^{-1}(t,Q,z) - p_i \cup S^{-1}(t,Q,z) \\ &= \left(zQ_i\partial_{Q_i} - p_i\cup\right) S^{-1}(t,Q,z). \end{aligned}$$

Recall that $J(t,Q,z) = -zS(t,Q,z)^{-1} \cdot 1$, where $S(0,Q,z) = \sum_{k=1}^{\infty} S_k(0,Q,z) z^{-k}$ is the calibration. We have

$$z\frac{\partial}{\partial t_{i+1}} J(t,Q,z) = \left(zQ_i\partial_{Q_i} - p_i\cup\right) J(t,Q,z).$$

For $1 \le a \le N$, we have

$$\begin{aligned} z\frac{\partial}{\partial t_a}J(t,Q,z) &= z(-z)\frac{\partial}{\partial t_a}\left(S^{-1}(t,Q,z)\cdot 1\right) \\ &= -z\left(-S^{-1}(t,Q,z)z\frac{\partial}{\partial t_a}S(t,Q,z)S^{-1}(t,Q,z)\right)\cdot 1 \\ &= -z\left(-S^{-1}(t,Q,z)(\Phi_a\bullet)\right)\cdot 1 \\ &= zS^{-1}(t,Q,z)\Phi_a\bullet. \end{aligned}$$

Theorem 2.18. *Given a polynomial $\mathcal{F}(\xi_1,\cdots,\xi_r)\in\mathbb{C}][\xi_1,\cdots,\xi_r]$, $\mathcal{F}(p_1\cup -zQ_1\partial_{Q_1},\cdots,p_r\cup -zQ_1\partial_{Q_r}) = 0$. Then*

$$\mathcal{F}(p_1\bullet,\cdots,p_r\bullet) = 0$$

Proof. We have

$$\frac{\partial}{\partial t}f(t) = \left(\frac{\partial}{\partial t}\circ f(t)\right)(1).$$

Then,

$$\left(\frac{\partial}{\partial t}\circ f(t)\right)g(t) = \frac{\partial f}{\partial t}\cdot g + f\cdot\frac{\partial g}{\partial t}.$$

Thus,

$$\frac{\partial}{\partial t}\circ f(t) = \frac{\partial f}{\partial t} + f\cdot\frac{\partial}{\partial t}.$$

Hence,

$$S\circ\frac{\partial}{\partial t_{i+1}}\circ S^{-1} = \frac{\partial}{\partial t_{i+1}} + S\left(\frac{\partial}{\partial t_{i+1}}S^{-1}\right) = \frac{\partial}{\partial t_{i+1}} + S\left(-S^{-1}\frac{\partial S}{\partial t_{i+1}}S^{-1}\right) = \frac{\partial}{\partial t_{i+1}} + \frac{\partial}{\partial t_{i+1}}S^{-1} = \frac{\partial}{\partial t_{i+1}} - \frac{1}{z}p_i\bullet.$$

Therefore,

$$0 = \mathcal{F}(p_1\bullet - z\frac{\partial}{\partial t_1},\cdots,p_2\bullet - z\frac{\partial}{\partial t_2})\cdot 1 = \mathcal{F}(p_1\bullet,\cdots,p_r\bullet)\cdot 1 + O(z).$$

Put $z = 0$, we have

$$\mathcal{F}(p_1\bullet,\cdots,p_r\bullet) = 0$$

□

Let us denote

$$\mathcal{A}_i = S(t,Q,z)\circ(p_i\cup -zQ_i\partial_{Q_i})\circ S^{-1}(t,Q,z) = p_i\bullet - zQ_i\partial_{Q_i},$$

Put

$$\Phi_a = \mathcal{F}_a(p_1,\cdots,p_r),\ 1\le a\le N,$$

where $\mathcal{F}_a$ are polynomials of p_i $(1\le i\le r)$. We have

$$-\frac{1}{z}\mathcal{F}_a\left(p_1 - zQ_1\partial_{Q_1},\cdots,p_r - zQ_r\partial_{Q_r}\right)J(0,Q,z) = S^{-1}(0,Q,z)\mathcal{F}_a(\mathcal{A}_1,\cdots,\mathcal{A}_r)\Phi_1$$

Let M be a $N\times N$ matrix whose a-th column is $-\frac{1}{z}\mathcal{F}_a\left(p_1 - zQ_1\partial_{Q_1},\cdots,p_1 - zQ_r\partial_{Q_r}\right)J(0,Q,z)$.

$$M_{ab} = -\frac{1}{z}\left(\Phi^b,\mathcal{F}_a\left(p_1 - zQ_1\partial_{Q_1},\cdots,p_r - zQ_r\partial_{Q_r}\right)J(0,Q,z)\right).$$

Let U be a $N\times N$ matrix whose a-th column is $\mathcal{F}(\mathcal{A}_1,\cdots,\mathcal{A}_r)\Phi_1$, that is

$$U_{ab}=\left(\Phi^b,\mathcal{F}_a(\mathcal{A}_1,\cdots,\mathcal{A}_r)\Phi_1\right).$$

We have $M(0,Q,z)=S^{-1}(0,Q,z)\cdots U(0,Q,z)$, where entries of $\left(S^{-1}(0,Q,z)-I\right)$ are $O(z^{-1})$ and entries of $U(0,Q,z)$ are in $\mathbb{C}[z]$. Therefore, $S^{-1}(0,Q,z)$ can be found via the Birkhoff factorization of $M(0,Q,z)$.

Remark 2.19. Let us denote $M=M_mz^m+M_{m-1}z^{m-1}+\cdots$, where M_mz^m is the highest order term of M. If $m=0$, then $M(0,Q,z)=S^{-1}(0,Q,z)\cdot M_0$.

If we know S(0,Q,z), using divisor equation, we have that

$$\begin{aligned}p_i\bullet&=zQ_i\frac{\partial}{\partial Q_i}S(t,Q,z)S^{-1}(t,Q,z)+S(t,Q,z)p_i\cup S^{-1}(t,Q,z)\\&=-S(t,Q,z)zQ_i\frac{\partial}{\partial Q_i}S^{-1}(t,Q,z)+S(t,Q,z)p_i\cup S^{-1}(t,Q,z)\\&=S(t,Q,z)\left(p_i\cup -zQ_i\frac{\partial}{\partial Q_i}\right)S^{-1}(t,Q,z).\end{aligned}$$

The equation above means that,

$$S^{-1}(t,Q,z)p_i\bullet=\left(p_i\cup -zQ_i\frac{\partial}{\partial Q_i}\right)S^{-1}(t,Q,z)$$

By comparing the coefficient of z^0, we have that

$$p_i\bullet=p_i\cup -zQ_i\partial_{Q_i}S_1(0,Q,z). \tag{12}$$

Recall that the basis of $H^\bullet(\mathrm{Bl}(\mathbb{P}^n))$ are $\Phi_{i,j}=p_1^{i-1}p_2^{j-1}$ $(0<i\le n,\ j=1,2)$. Since $H^\bullet(\mathrm{Bl}(\mathbb{P}^n))$ is generated as an algebra by p_1 and p_2, we can compute S from J-fuctioin. Then we can compute the quantum cup product by (12). For $1\le i\le n$, let us denote $\Phi_i=p_1^{i-1}$ and $\Phi_{i+n}=p_1^{i-1}p_2$. The S_1 we get by this definition is not the usual on. But we still get quantum product correctly.

Remark 2.20.

$$[z^{-1}]\frac{A}{a-z}=-[z^{-1}]\frac{A}{z}\frac{1}{1-\frac{a}{z}}=-[z^0]\frac{A}{1-\frac{a}{z}}=-A[z^0]\sum_{i=0}^{\infty}\left(\frac{a}{z}\right)^i=-A$$

Let $1\le a\le n$. There are two cases Φ_a and Φ_{a+n}. For Φ_a

$$\begin{aligned}M(\Phi_a)&=-\frac{1}{z}(p_1-zQ_1\partial_{Q_1})^{a-1}J(0,Q,z)\\&=(p_1-zQ_1\partial_{Q_1})^{a-1}\sum_{D_1,D_2\ge 0}\frac{Q_1^{D_1}Q_2^{D_2}}{\prod_{m=1}^{D_1}(p_1-mz)^n\prod_{m=1}^{D_2}(p_2-mz)\prod_{m=1}^{D_2-D_1}(p_2-p_1-mz)}\\&=\sum_{D_1>0,\ D_2\ge 0}\frac{(p_1-D_1z)^{a-1}Q_1^{D_1}Q_2^{D_2}}{\prod_{m=1}^{D_1}(p_1-mz)^n\prod_{m=1}^{D_2}(p_2-mz)\prod_{m=1}^{D_2-D_1}(p_2-p_1-mz)}+p_1^{a-1}\\&=\sum_{D_1>0,\ D_2\ge 0}\frac{(p_1-D_1z)^{a-1}Q_1^{D_1}Q_2^{D_2}}{\prod_{m=1}^{D_1}(p_1-mz)^n\prod_{m=1}^{D_2}(p_2-mz)\prod_{m=1}^{D_2-D_1}(p_2-p_1-mz)}+p_1^{a-1}.\end{aligned}$$

Let Ord be the order of the highest order term of the first part, where $D_1>0$ and $D_2\ge 0$. For $(D_2\ge D_1>0)$, we have $\mathrm{Ord}=a-1-D_1n-D_2-D_2+D_1=-D_1(n-1)-2D_2+a-1\le -2$. For $0\le D_2<D_1$, we have

$\mathrm{Ord} = a-1-D_1 n-D_2-1-D_2+D_1 = -D_1(n-1)-2D_2+a-2 \leq -1$. When $\mathrm{Ord} = -1$, we have $D_2 = 0,\ D_1 = 1$ and $a = n$. Then,

$$[z^{-1}]\frac{(p_1-D_1z)^{a-1}Q_1^{D_1}Q_2^{D_2}}{\prod\limits_{m=1}^{D_1}(p_1-mz)^n\prod\limits_{m=1}^{D_2}(p_2-mz)\prod\limits_{m=1}^{D_2-D_1}(p_2-p_1-mz)} = [z^{-1}]\frac{Q_1(p_2-p_1)}{(p_1-z)} = Q_1(p_2-p_1).$$

Thus,

$$M_{ab} = \begin{cases} \delta_{ab} + O(z^{-2}) & a < n \text{ or } b \neq 2, n+1, \\ Q_1 z^{-1} + O(z^{-2}) & a = n,\ b = 2, \\ -Q_1 z^{-1} + O(z^{-2}) & a = n,\ b = n+1. \end{cases}$$

For Φ_{a+n},

$$\begin{aligned} M(\Phi_{a+n}) =& -\frac{1}{z}(p_1 - zQ_1\partial_{Q_1})^{a-1}(p_2 - zQ_2\partial_{Q_2})J(0,Q,z) \\ =& (p_1 - zQ_1\partial_{Q_1})^{a-1}(p_2 - zQ_2\partial_{Q_2})\sum_{D_1,D_2\geq 0}\frac{Q_1^{D_1}Q_2^{D_2}}{\prod\limits_{m=1}^{D_1}(p_1-mz)^n\prod\limits_{m=1}^{D_2}(p_2-mz)\prod\limits_{m=1}^{D_2-D_1}(p_2-p_1-mz)} \\ =& \sum_{D_2>0}\frac{p_1^{a-1}(p_2-D_2z)Q_2^{D_2}}{\prod\limits_{m=1}^{D_2}(p_2-mz)\prod\limits_{m=1}^{D_2}(p_2-p_1-mz)} + \sum_{D_1>0}\frac{(p_1-D_1z)^{a-1}p_2Q_1^{D_1}\prod\limits_{m=1-D_1}^{0}(p_2-p_1-mz)}{\prod\limits_{m=1}^{D_1}(p_1-mz)^n} \\ &+ \sum_{D_1>0,\ D_2>0}\frac{(p_1-D_1z)^{a-1}(p_2-D_2z)Q_1^{D_1}Q_2^{D_2}}{\prod\limits_{m=1}^{D_1}(p_1-mz)^n\prod\limits_{m=1}^{D_2}(p_2-mz)\prod\limits_{m=1}^{D_2-D_1}(p_2-p_1-mz)} + p_1^{a-1}p_2, \end{aligned}$$

which has four parts. Let Ord_1 be the order of the highest order term of the first part, where $D_2 > 0$. We have $\mathrm{Ord}_1 = -2D_2 + 1 \leq -1$. When $\mathrm{Ord}_1 = -1$ we have $D_2 = 1$. Then,

$$[z^{-1}]\frac{p_1^{a-1}(p_2-D_2z)Q_2^{D_2}}{\prod\limits_{m=1}^{D_2}(p_2-mz)\prod\limits_{m=1}^{D_2}(p_2-p_1-mz)} = [z^{-1}]\frac{p_1^{a-1}Q_2}{(p_2-p_1-z)} = -p_1^{a-1}Q_2.$$

Let Ord_2 be the order of the highest order term of the second part, where $D_1 > 0$. We have $\mathrm{Ord}_2 = D_1 - 1 - D_1 n + a - 1 = D_1(n-1) + a - 2 \leq -1$. When $\mathrm{Ord}_2 = -1$, we have $D_1 = 1$ and $a = n$. Then,

$$[z^{-1}]\frac{(p_1-D_1z)^{n-1}p_2Q_1^{D_1}\prod\limits_{m=1-D_1}^{0}(p_2-p_1-mz)}{\prod\limits_{m=1}^{D_1}(p_1-mz)^n} = [z^{-1}]\frac{p_2Q_1}{(p_1-z)} = p_2(p_2-p_1)Q_1 = 0.$$

Let Ord_3 be the order of the highest order term of the third part, where $D_1 > 0$ and $D_2 > 0$.When $D_2 < D_1$ we have $\mathrm{Ord}_3 = a - D_1 n - D_2 + D_1 - D_2 - 1 < -1$. When $D_2 \geq D_1$ we have $\mathrm{Ord}_3 = a - D_1 n - D_2 - D_2 + D_1 = a - 2D_2 - D_1(n-1) \leq -1$. $\mathrm{Ord}_3 = -1$ when $D_1 = D_2 = -1$ and $a = n$. Then,

$$[z^{-1}]\frac{(p_1-D_1z)^{a-1}(p_2-D_2z)Q_1^{D_1}Q_2^{D_2}}{\prod\limits_{m=1}^{D_1}(p_1-mz)^n\prod\limits_{m=1}^{D_2}(p_2-mz)\prod\limits_{m=1}^{D_2-D_1}(p_2-p_1-mz)} = [z^{-1}]\frac{Q_1Q_2}{(p_1-z)} = -Q_1Q_2.$$

Thus,

$$M_{(a+n)b} = \begin{cases} \delta_{(a+n)b} - \delta_{ab}Q_2 & a < n \text{ or } b \neq 1, n, \\ -Q_1Q_2 & a = n,\ b = 1, \\ -Q_2 & a = n,\ b = n. \end{cases}$$

Hence, $M = I + O(z^{-1})$. We have $M_0 = I$. By Remark 2.19, we get $M(0, Q, z) = S^{-1} \cdot I = S^{-1}$. By (12), for the quantum product, we have

1) $p_1 \bullet \Phi_{i,1} = \Phi_{i+1,1}$, where $i \leq n-1$, $j = 1, 2$.
2) $p_2 \bullet \Phi_{i,1} = \Phi_{i,2}$, where $i \leq n$.
3) $p_2 \bullet \Phi_{i,2} = \Phi_{i+1,2} + Q_2\Phi_{i,1}$, where $i \leq n-1$.
4) $p_1 \bullet \Phi_{n,2} = Q_1 Q_2 \Phi_{1,1}$.
5) $p_1 \bullet \Phi_{n,1} = Q_1 \Phi_{1,2} - Q_1 \Phi_{2,1}$.
6) $p_2 \bullet \Phi_{n,2} = Q_2 \Phi_{n,1} + Q_1 Q_2 \Phi_{1,1}$.

CHAPTER 3

Main result

1. Theorem 1.3

In this section we will prove Theorem 1.3 by calculating both sides of the identity. For the right hand side we will prove a residue formula for the blow-up and then use this formula to prove that right hand side equal to a summation of Gamma-funcion. For the left hand we will list some basic properties of Gamma-function and then prove Lemma3.4, which will be useful in the proof that the residue of the LHS equal to the RHS. Then by some estimation we will finish the proof of Theorem1.3

THEOREM 3.1. *Suppose that $Q_1 = e_1^{\tau_1}$, $Q_2 = e^{\tau_2}$ Put $\Psi_\tau(E) = e^{-\tau_1 p_1 - \tau_2 p_2}\Psi(E)$ if $z \in \mathbb{R}_{<0}$, then*

$$\int\limits_{\mathbb{R}^n_{>0}} e^{f(x,\tau)z^{-1}}\omega = (2\pi)^{\frac{n-1}{2}}(-z)^{\frac{n}{2}}(S(0,Q,z)(-z)^{\theta}(-z)^{\rho})\Psi_\tau(\mathcal{O}),1)$$

Let us calculate RHS and LHS separately.

1.1. Right hand side.

$$\mathrm{RHS} = (2\pi)^{\frac{n-1}{2}}(-z)^{\frac{n}{2}}\left((-z)^{\theta}(-z)^{\rho}\cdot\Psi_\tau(\mathcal{O}), S(0,Q,z)^T\cdot 1\right)$$

we also have that

$$S(0,Q,z)^T\cdot 1 = S(0,Q,-z)^{-1}\cdot 1 = \frac{1}{z}J_X(0,Q,-z) = I(Q,z).$$

REMARK 3.2. since

$$(-z)^{-\theta}p_i = (-z)^{-\theta}p_i(-z)^{\theta}(-z)^{-\theta} = (-zp)(-z)^{-\theta} \text{ and } (-z)^{-\theta}\cdot 1 = (-z)^{-\frac{n}{2}}$$

we have that

$$(-z)^{\theta}S(0,Q,z)^{-1}\cdot 1 = \sum_{D_1,D_2\geq 0}\frac{Q_1^{D_1}Q_2^{D_2}}{\prod\limits_{m=1}^{D_1}(-zp_1+mz)^n\prod\limits_{m=1}^{D_2}(-zp_2+mz)\prod\limits_{m=1}^{D_2-D_1}(-zp_2+zp_1+mz)}(-z)^{-\frac{n}{2}}$$

$$\mathrm{RHS} = (2\pi)^{\frac{n-1}{2}}\sum_{D_1,D_2=0}^{\infty}\frac{(\Psi_\tau(0),(-z)^{\rho}(Q_1(-z)^{-(n-1)})^{D_1}(Q_2(-z)^{-2})^{D_2})}{\prod\limits_{m=1}^{D_2-D_1}(p_2-p_1-m)\prod\limits_{m=1}^{D_1}(p_1-m)^n\prod\limits_{m=1}^{D_2}(p_2-m)},$$

by $\Gamma_{bl} = \Gamma(1+p_1)^n\Gamma(1+p_2)\Gamma(1+p_2-p_1)$,

$$\mathrm{RHS} = \sum_{D_1,D_2=0}^{\infty}\left(\Gamma(1+p_1)^n\Gamma(1+p_2)\Gamma(1+p_2-p_1), \frac{e^{-p_1\tau_1-p_2\tau_2}(-z)^{\rho}\left(e^{\tau_1}(-z)^{-(n-1)}\right)^{D_1}\left(e^{\tau_2}\left(z_z^{-2}\right)\right)^{D_2}}{\prod\limits_{m=1}^{D_2-D_1}(p_2-p_1-m)\prod\limits_{m=1}^{D_1}(p_1-m)^n\prod\limits_{m=1}^{D_2}(p_2-m)}\right)$$

$$= \sum_{D_1,D_2=0}^{\infty}\left(p_1(\Gamma(p_1-D_1))^n(p_2\Gamma(p_2-D_2))((p_2-p_1)\Gamma(p_2-p_1-D_2+D_1)), e^{g}\right)$$

Where, $g = -(p_1 - D_1)(\tau_1 - (n-1)\log(-z)) - (p_2 - D_2)(\tau_2 - 2\log(-z))$ and $E = p_2 - p_1$,
we have that $\int E^n = (-1)^{n-1}$, $(p_2 - p_1)^n = (p_2 - p_1)^{n-1}(p_2 - p_1) = (-1)^{n-1} p_1^{n-1}(p_2 - p_1)$and $p_1^{n-1} p_2 = 1$
$H^\circ(Bl(\mathbb{P}^n)) = \mathbb{C}[p_1, p_2]/\langle p_2(p_2 - p_1) = 0.\ p_1^n = 0\rangle$, Poincare pairing

$$\left(p_1^i, p_2^j\right) = \int_{\mathrm{Bl}} p_1^i p_2^j = \begin{cases} 1, \text{ if } 0 \le i < n, i + j = n \\ 0, \text{ othermise} \end{cases}$$

Therefore, if

$$f(p_1, p_2) = \sum_{i,j=0}^{\infty} f_{ij} p_1^i p_2^j \in \mathbb{C}[p_1, p_2],$$

then

$$\int_{(Bl)} f(p_1, p_2) = \sum_{i=0}^{n-1} f_{i,n-i}$$

Then we have that

$$\mathrm{Res}_{p_1=0} \mathrm{Res}_{p_2=p_1} \frac{f(p_1, p_2)}{p_1^n (p_2 - p_1) p_2} = \mathrm{Res}_{p_1=0} \quad \frac{f(p_1, p_1)}{p_1^{n-1}} = \sum_{i=0}^{n} f_{i,n-i}$$

also

$$\mathrm{Res}_{p_1=0} \mathrm{Res}_{p_2=0} \frac{f(p_1, p_2)}{p_1^n (p_2 - p_1) p_2} = \mathrm{Res}_{p_1=0} \frac{f(p_1, 0)}{p_1^n \cdot (-p_1)} = -\mathrm{Res}_{p_1=0} \frac{f(p_1, 0)}{p_1^{n+1}} = -f_{n,0}$$

Adding up the two residues we get

$$\left(\mathrm{Res}\ {}_{p_1=0}\mathrm{Res}\ {}_{p_2=p_1} + \mathrm{Res}_{p_1=0} \mathrm{Res}_{p_2=0} \right) \frac{f(p_1, p_2)}{p_1^n (p_2 - p_1) p_2} = \sum_{i=0}^{n-1} f_{i,n-i} = \int_{Bl} f(p, p_2)$$

We proved the following residue formula:

$$\int_{[Bl(\mathbb{P}^n)]} f(p_1, p_2) = \left(\mathrm{Res}_{p_1=0} \mathrm{Res}_{p_2=p_1} + \mathrm{Res}_{p_1=0} \mathrm{Res}_{p_2=0} \right) \frac{f(p_1, p_2)}{p_1^n (p_2 - p_1) p_2}.$$

Thus,

$$\text{(13)} \qquad \mathrm{RHS} = \sum_{D_1=0}^{\infty} \sum_{D_2=0}^{\infty} \mathrm{Res}_{p_1=-D_1}(\mathrm{Res}_{p_2=p_1-D_2} + \mathrm{Res}_{p_2=-D_2}) \Gamma(p_1))^n \Gamma(p_2) \Gamma(p_2 - p_1) e^{-p_1\tau_1 - p_2\tau_2} dp_1 dp_2$$

1.2. Left hand side.

$$\mathrm{LHS} = \int e^{\left(x_1 + \cdots + x_n + e^{-\tau_1} x_1 \cdots x_n + \frac{e^{\tau_1 + \tau_2}}{x_1 \cdots x_n}\right) z^{-1}} \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$

First we let $x_i = -ze^{y_i}$, then we let $e^{-\tilde{\tau}_1} = (-z)^{n-1} e^{-\tau_1}$ and $e^{\tilde{\tau}_2} = \frac{e^{\tau_1 + \tau_2}}{(-z)^{n+1}} \cdot e^{-\widetilde{\tau_1}} = \frac{e^{\tau_2}}{z^2}$. Using the substitution $x_i = -ze^{y_i}$ $(1 \le i \le n)$, we get that :

$$\mathrm{LHS} = \int \exp\left(-(e^{y_1} + \cdots e^{y_n}) - e^{-\tilde{\tau}_1 + y_1 + \cdots + y_n} - e^{\tilde{\tau}_1 + \tilde{\tau}_2 - (y_1 + \cdots y_n)} \right) dy.$$

Let us define

$$I(t_1, t_2) := \int \exp\left(-(e^{y_1} + \cdots e^{y_n}) - e^{-t_1 + y_1 + \cdots + y_n} - e^{t_1 + t_2 - (y_1 + \cdots y_n)} \right) dy,$$

$$I(\varepsilon_1, \varepsilon_2, t_1, t_2) := e^{\varepsilon_1 t_1 + \varepsilon_2 t_2} I(t_1, t_2).$$

Fourier inversion formula yields

$$I(\varepsilon_1,\varepsilon_2,t_1,t_2)=\left(\frac{1}{2\pi}\right)^2\lim_{A_1,A_2\to+\infty}\int_{-A_2}^{A_2}\int_{-A_1}^{A_1}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t_1-s_1)\xi_1+i(t_2-s_2)\xi_2}I(\varepsilon_1,\varepsilon_2,s_1,s_2)ds_1ds_2d\xi_1d\xi_2$$

Thus,

$$I(t_1,t_2)=\left(\frac{1}{2\pi}\right)^2\lim_{A_1,A_2\to+\infty}\int_{-A_2}^{A_2}\int_{-A_1}^{A_1}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(t_1-s_1)\xi_1+\varepsilon_1(s_1-t_1)+i(t_2-s_2)\xi_2+\varepsilon_2(s_2-t_2)}I(s_1,s_2)ds_1ds_2d\xi_1d\xi_2.$$

Put $p_1=\varepsilon_1-i\xi_1$, $p_2=\varepsilon_2-i\xi_2$, we have $d\xi_1=idp_1$, $d\xi_2=idp_2$. Thus,

$$I(t_1,t_2)=\left(\frac{1}{2\pi}\right)^2\int_{\varepsilon_2-i\infty}^{\varepsilon_2+i\infty}\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty} e^{-p_2t_2-p_1t_1}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{p_1s_1+p_2s_2}I(s_1,s_2)ds_1ds_2dp_1dp_2,$$

where

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} e^{p_1s_1+p_2s_2}I(s_1,s_2)ds_1ds_2\\ =&\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^n}\exp\Big(p_1s_1+p_2s_2-(e^{y_1}+\cdots e^{y_n})-e^{-s_1+y_1+\cdots+y_n}-e^{s_1+s_2-(y_1+\cdots y_n)}\Big)dyds_1ds_2.\end{aligned}$$

Let us do the substitution $s_1=y_1+\cdots+y_{n+1}$, $s_2=y_{n+2}-y_{n+1}$, and recall that $\int_{\mathbb{R}}\exp(py-e^y)dy=\int_{\mathbb{R}}e^{-t}t^{p-1}dt=\Gamma(p)$. Then we get

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} e^{p_1s_1+p_2s_2}I(s_1,s_2)ds_1ds_2\\ =&\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}^n} exp\,(p_1(y_1+\cdots+y_n)+(p_1-p_2)y_{n-1}+p_2y_{n+2}-e^{y_1}+\cdots+e^{y_{n+2}})\,dy_1\cdots dy_{n+2}\\ =&\Gamma(p_1)^n\Gamma(p_2-p_1)\Gamma(p_2)\end{aligned}$$

$$I(t_1,t_2)=\left(\frac{1}{2\pi}\right)^2\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty}\int_{\varepsilon_2-i\infty}^{\varepsilon_2+i\infty} e^{-p_2t_2-p_1t_1}\Gamma(p_2)\Gamma(p_2-p_1)\Gamma(p_1)^n dp_2dp_1$$

In order to transform the LHS in the form of residue, we need some estimates for the Γ-function.

Proposition 3.3. *We have*

1) $\Gamma(z+1)=z\Gamma(z)$.
2) $|\Gamma(bi)|^2=\frac{\pi}{b\sinh(\pi b)}$.
3) $|\Gamma(-N+bi)|^2=\frac{\pi}{b\sinh(\pi b)}\prod_{k=1}^N\frac{1}{k^2+b^2}$, $N\in\mathbb{N}$.
4) $|\Gamma(a+bi)|^2=|\Gamma(a)|^2\prod_{k=1}^\infty\frac{(a+k)^2}{b^2+(a+k)^2}$.
5) $|\Gamma(a+bi)|\le|\Gamma(a)|$.
6) $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}$.
7) $|\Gamma(1+N+bi)|^2=\frac{\pi b}{\sinh(\pi b)}\prod_{k=1}^N(k^2+b^2)$ $n\in\mathbb{N}$.

Proof. The above properties are well known, so we omit the proof. □

LEMMA 3.4. *For all $a, b \in \mathbb{R}$, $|b| \geq 1$, let us define $m \in \mathbb{Z}$, such that, $-m < a \leq -m+1$. Then, $|\Gamma(a+bi)| \leq C|\Gamma(-m+1+bi)|$, where $C = 2$.*

PROOF. For $a_2 > a_1 \geq 2$ we have that $\Gamma(a_2+bi)| \geq |\Gamma(a_1+bi)|$. For $2 > a > 1$, $\Gamma(a) < \Gamma(1) = \Gamma(2)$. Thus, we can check the lemma easily when $a \geq 1$.

Suppose that $a < 1$. To begin with, note $b \neq 0$, $|\Gamma(2+bi)|^2 = |\Gamma(-m+1+bi)|^2 \prod_{j=-1}^{m-1}(j^2+b^2)$ by Proposition 3.3 1). Similarly, using Proposition3.3 1), for $-m < a < -m+1$, let $\alpha = a+m$ we have that

$$|\Gamma(a+bi)|^2 = |\Gamma(\alpha+1+bi)|^2 \prod_{j=0}^{m} \frac{1}{(\alpha-j)^2+b^2} = |\Gamma(\alpha+1)|^2 \left(\prod_{k=1}^{\infty} \frac{(\alpha+1+k)^2}{b^2+(\alpha+1+k)^2}\right)\left(\prod_{j=0}^{m} \frac{1}{(\alpha-j)^2+b^2}\right)$$

Since $0 < \alpha \leq 1$, we get

$$\begin{aligned}|\Gamma(a+bi)|^2 &\leq |\Gamma(2)|^2 \left(\prod_{k=1}^{\infty} \frac{(2+k)^2}{b^2+(2+k)^2}\right)\left(\prod_{j=0}^{m-1} \frac{1}{j^2+b^2}\right)\frac{1}{b^2} \\ &= |\Gamma(1-m+bi)|^2 \frac{(b+1)^2}{b^2}.\end{aligned}$$

Therefore, $|\Gamma(a+bi)| \leq C|\Gamma(-m+1+bi)|$ where $C = 2$, for $-m < a \leq -m+1$ when $|b| \geq 1$.

□

REMARK 3.5. Using Proposition 3.3 2), 3) and 5), for integer $N < 0$ and $|b| \geq 1$ we have that,

$$|\Gamma(-N+bi)|^2 = \frac{\pi}{b\sinh(\pi b)} \prod_{k=1}^{N} \frac{1}{k^2+b^2} < \frac{\pi}{b\sinh(\pi b)} = |\Gamma(bi)|^2 \leq b^2|\Gamma(bi)|^2 = |\Gamma(1+bi)|^2.$$

Next, for all $a, b \in \mathbb{R}$, $a \leq 1$ and $|b| \geq 1$, such that, $-m < a \leq -m+1$ ($m \in \mathbb{Z}$), using Lemma 3.4, we get

$$|\Gamma(a+bi)|^2 \leq 4|\Gamma(-m+bi)|^2 \leq 4|\Gamma(1+bi)|^2.$$

Then, let us consider the derivative of $|\Gamma(1+bi)|^2$,

$$\frac{d}{db}\left(\frac{b\pi}{\sinh(\pi b)}\right) = \frac{\pi\sinh(\pi b) - \pi^2 b\cosh(\pi b)}{\sinh^2(\pi b)}.$$

When $b > 1$,

$$\frac{\pi\sinh(\pi b) - \pi^2 b\cosh(\pi b)}{\sinh^2(\pi b)} \leq \frac{\pi(\sinh(\pi b) - \cosh(\pi b))}{\sinh^2(\pi b)} = \frac{-\pi e^{-b}}{\sinh^2(\pi b)} < 0.$$

Finally, since $|\Gamma(1+bi)| = |\Gamma(1-bi)|$, for $|b| \geq 1$, we get

$$|\Gamma(1+bi)|^2 \leq |\Gamma(1+i)|^2.$$

We would like to prove that if $p_1 \in \varepsilon_1 + i\mathbb{R}$, then

$$\int_{\varepsilon_2+i\mathbb{R}} e^{-p_2t_2}\Gamma(p_1)^n\Gamma(p_2-p_1)\Gamma(p_2)dp_2 = \lim_{N\to\infty}\int_{L_2(N)} e^{-p_2t_2}\Gamma(p_1)^n\Gamma(p_2-p_1)\Gamma(p_2)dp_2,$$

where $L_2(N)$ is the contour consisting of $L_{20} = (\varepsilon_2 - Ni, \varepsilon_2 + Ni)$, $L_{21} = (\varepsilon_2 + Ni, -N+\delta+Ni)$, $L_{22} = (-N+\delta+Ni, -N+\delta-Ni)$ and $L_{23} = (-N+\delta-Ni, \varepsilon_2-Ni)$, and δ is a real number such that $-\frac{7}{8} \leq \delta \leq -\frac{1}{8}$. For every $\varepsilon_1 \in (0,1)$, there exists a δ satisfying the condition above such that $\frac{1}{4} < |\varepsilon_1 - \delta - 1| < \frac{3}{4}$. We will prove that $\lim_{N\to\infty}\int_{L_{2i}} e^{-p_2t_2}\Gamma(p_1)^n\Gamma(p_2-p_1)\Gamma(p_2)dp_2 = 0$ ($i = 1,2,3$). Suppose that N is so big that $|\mathrm{Im}\, p_1| < N-1$. Let us estimate the integrals using Lemma 3.4

First let us consider the integral on $(\varepsilon_2 Ni, -N+\delta+Ni)$ and $(-N+\delta-Ni, \varepsilon_2-Ni)$. The estimations are similar on those two parts. Let us consider only the integral on $(\varepsilon_2+Ni, -N+\delta+Ni)$. Put

$$J_{21} = \int_{Ni}^{-N+\delta+Ni} e^{-p_2 t_2} \Gamma(p_2)\Gamma(p_2 - p_1) dp_2$$

Let $p_1 = a + bi$. We have $a = \varepsilon_1 < \varepsilon_2$ and $|b| < N - 1$ according to the choice of N from above. Suppose that p_2 belongs to L_{21}. We estimate the integral in two steps.

i) When $|t_2| < \frac{\pi}{2}$, we have

$$|\Gamma(p_2)\Gamma(p_2 - p_1)| \leq 4|\Gamma(1 + Ni)\Gamma(1 + i)|,$$

where we need Remark 3.5 and Lemma 3.4. By Proposition 3.3, as t_2 is a real number, we get

$$|J_{21}| \leq 4(N+2)\left|N\Gamma(Ni)\Gamma(1+i)e^{(N+1)|t_2|}\right| \leq 4(N+2)\left|N\left(\frac{\pi}{N\sinh(\pi N)}\right)^{\frac{1}{2}}\Gamma(1+i)e^{(N+1)|t_2|}\right| = \mathrm{O}\left(\frac{N^2}{e^{(\frac{\pi}{2}-|t_2|)N}}\right)$$

obviously $|J_{21}| \to 0$ when $N \to +\infty$

ii) When $|t_2| \geq \frac{\pi}{2}$.

Let us divide the J_{21} into two parts by dividing the integration contour with point $A = Ni - \frac{2\pi-1}{4|t_2|}N$. Note that since $|t_2| \geq \frac{\pi}{2}$, $\frac{2\pi-1}{4|t_2|} < 1$. We get

$$J_{21} = \int_{\varepsilon_2+Ni}^{A} e^{-p_2 t_2}\Gamma(p_2)\Gamma(p_2 - p_1)dp_2 + \int_{A}^{-N+\delta+Ni} e^{-p_2 t_2}\Gamma(p_2)\Gamma(p_2 - p_1)dp_2.$$

Let us consider the first integral, which we denote by J_{211}. Since $\mathrm{Re}(p_2) \in [A, \varepsilon_2])$, we get

$$\left|e^{-p_2 t_2}\Gamma(p_2)\Gamma(p_2 - p_1)dp_2\right| \leq e^{\frac{2\pi-1}{4}N}|\Gamma(p_2)\Gamma(p_2 - p_1)|.$$

Using the Lemma 3.4 and Proposition 3.3 we have that

$$\begin{aligned}
|J_{211}| \leq & 4\left(\frac{2\pi-1}{4|t_2|}N + 1\right)e^{\frac{2\pi-1}{4}N}|N\Gamma(Ni)\Gamma(1+i)| \\
\leq & \left(\frac{2\pi-1}{|t_2|}N + 1\right)e^{\frac{2\pi-1}{4}N}\left|\left(\frac{N\pi}{\sinh(\pi N)}\right)^{\frac{1}{2}}\right| \\
= & \mathrm{O}\left(\frac{N^{\frac{3}{2}}e^{\frac{2\pi-1}{4}N}}{e^{\frac{\pi N}{2}}}\right).
\end{aligned}$$

Obviously, $|J_{211}| \to 0$ when $N \to +\infty$.

Then, let us consider the second integral, which we denote by J_{212}. We can let N big enough such that $\varepsilon_2 - a - \frac{2\pi-1}{4|t_2|}N < 0$ and let $B = \lfloor \frac{2\pi-1}{4|t_2|}N \rfloor$. By Lemma 3.4 and Remark 3.5, we have

$$|J_{212}| = \left|\int_{A}^{-N+\delta+Ni} e^{-p_2 t_2}\Gamma(p_2)\Gamma(p_2 - p_1)dp_2\right| \leq 4Ne^{(N+1)|t_2|}|\Gamma(1 - B + Ni)\Gamma(1 - B + i)|.$$

Using Proposition 3.3 we get

$$\begin{aligned}
|J_{212}| \leq & 4Ne^{(N+1)|t_2|}|\Gamma(1 - B + Ni)\Gamma(1 - B + i)| \\
= & 4Ne^{(N+1)|t_2|}\left(\frac{\pi}{N\sinh(\pi N)}\left(\prod_{k=1}^{B-1}(k^2+N^2)^{-1}\right)\left(\frac{\pi}{\sinh(\pi)}\prod_{k=1}^{-1+B}\frac{1}{k^2+1}\right)\right)^{\frac{1}{2}}
\end{aligned}$$

obviously $|J_{212}| \to 0$ when $N \to +\infty$.

Then let us consider the integral on $(N+\delta+Ni,-N+\delta-Ni)$. By using Proposition 3.3

$$
\begin{aligned}
|J_{22}| &\leq 2Ne^{(N+1)|t_2|}|(\Gamma(-N+\delta)\Gamma(-N+\delta-a))| \\
&\leq 2Ne^{(N+1)|t_2|}|\frac{1}{N!}\Gamma(\delta)|\frac{1}{\lfloor N-\delta-1+a\rfloor!}\Gamma(\delta-\varepsilon_1)| \\
&= \mathrm{O}\left(\frac{Ne^{t_2N}}{(N!)(\lfloor N-\delta-1+a\rfloor!)}\right).
\end{aligned}
$$

Obviously, $|J_{211}|\to 0$ when $N\to+\infty$. Therefore,

$$\lim_{N\to+\infty}\int_{L_2(N)} e^{-p_2t_2}\Gamma(p_1)^n\Gamma(p_2-p_1)\Gamma(p_2)dp_2=\int_{\varepsilon_2+\mathbb{R}i} e^{-p_2t_2}\Gamma(p_1)^n\Gamma(p_2-p_1)\Gamma(p_2)dp_2.$$

On the other hand,

$$
\begin{aligned}
&\int_{L_2(N)} e^{-p_2t_2}\Gamma(p_1)^n\Gamma(p_2-p_1)\Gamma(p_2)dp_2 \\
=&\left(\sum_{m=0}^{N}\mathrm{Res}_{p_2=-m}+\sum_{m=0}^{N\,\text{or}\,N+1}\mathrm{Res}_{p_2=p_1-m}\right)e^{-p_2t_2-p_1t_1}\Gamma(p_2)\Gamma(p_2-p_1)\Gamma(p_1)^n dp_2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{\varepsilon_2-i\infty}^{\varepsilon_2+i\infty} e^{-p_2t_2-p_1t_1}\Gamma(p_2)\Gamma(p_2-p_1)\Gamma(p_1)^n dp_2 \\
=&\sum_{m=0}^{\infty}\left(\mathrm{Res}_{p_2=-m}+\mathrm{Res}_{p_2=p_1-m}\right)e^{-p_2t_2-p_1t_1}\Gamma(p_2)\Gamma(p_2-p_1)\Gamma(p_1)^n dp_2
\end{aligned}
$$

$$I(t_1,t_2)=\left(\frac{1}{2\pi}\right)\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}e^{-p_1t_1}\Gamma(p_1)^n\left(e^{(-j+p_1)t_2}\Gamma(-j+p_1)+e^{-jt_2}\Gamma(-j-p_1)\right)dp_1$$

By Proposition 3.3, when $\mathrm{Re}\,p_1=\varepsilon_1$ for all j.

$$
\begin{aligned}
&\left|e^{-p_1t_1}\Gamma(p_1)^n\left(e^{(-j+p_1)t_2}\Gamma(-j+p_1)+e^{-jt_2}\Gamma(-j-p_1)\right)\right| \\
\leq&\left|e^{-\varepsilon_1t_1}\Gamma(\varepsilon_1)^n\left(e^{(-j+\varepsilon_1)t_2}\Gamma(-j+\varepsilon_1)+e^{-jt_2}\Gamma(-j-\varepsilon_1)\right)\right| \\
=&\left|e^{-\varepsilon_1t_1}\Gamma(\varepsilon_1)^n\left(\frac{e^{(-j+\varepsilon_1)t_2}}{\prod_{k=1}^{j}(-k+\varepsilon_1)}\Gamma(\varepsilon_1)+\frac{e^{-jt_2}}{\prod_{k=1}^{j}(-k-\varepsilon_1)}\Gamma(-\varepsilon_1)\right)\right| \\
\leq&\left|e^{-\varepsilon_1t_1}\Gamma(\varepsilon_1)^n\left(\frac{e^{(-j+\varepsilon_1)t_2}}{(j-1)!(1-\varepsilon_1)}\Gamma(\varepsilon_1)+\frac{e^{-jt_2}}{j!}\Gamma(-\varepsilon_1)\right)\right|=\mathrm{O}\left(\frac{e^{-jt_2}}{(j-1)!}\right)+\mathrm{O}\left(\frac{e^{-jt_2}}{j!}\right).
\end{aligned}
$$

This means that the function of p_1 in the integral is uniformly absolutely-convergent when $\mathrm{Re}\,p_1=\varepsilon_1$. Therefore, the order of summation $\sum_{j=0}^{\infty}$ and integration $\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty}$ is interchangeable.

Let us consider $\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty} e^{-p_1(t_1+t_2)-jt_2}\Gamma(p_1)^n\Gamma(-j+p_1)dp_1$ and $\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty} e^{-p_1t_1-jt_2}\Gamma(p_1)^n\Gamma(-j-p_1)dp_1$ ($j\in\mathbb{N}^*$, $n\geq 2$, $\varepsilon_1<1$).

Let $L_1(N)$ ($N\geq 10$) be the contour consisting of $L_{10}=(\varepsilon_1-Ni,\varepsilon_1+Ni)$, $L_{11}=(\varepsilon_1+Ni,-N+\delta+Ni)$, $L_{12}=(-N+\delta+Ni,-N+\delta-Ni)$ and $L_{13}=(-N+\delta-Ni,\varepsilon_1-Ni)$. We have δ is a real number such that $-\frac{3}{4}\leq\delta\leq-\frac{1}{4}$.

Firstly, let us consider the first integral. By Proporsition 3.3 $|\Gamma(-j+p_1)| = |\Gamma(p_1)\prod_{k=1}^{j}(-k+p_1)^{-1}| \leq |\frac{1}{j!}|\Gamma(p_1)|$, when $p_1 \in L_1$. Therefore, we get that $|e^{-p_1(t_1+t_2)-jt_2}\Gamma(p_1)^n\Gamma(-j+p_1)| \leq |e^{-p_1(t_1+t_2)}\Gamma(p_1)^{n+1}\frac{e^{-jt_2}}{(j-1)!(1-\varepsilon_1)}|$. Thus, we can get that $\lim_{N\to\infty}\int_{L_{1k}} e^{-p_1(t_1+t_2)-jt_2}\Gamma(p_1)^n\Gamma(-k+p_1)dp_1 = 0$ $(j=1,2,3)$ by the similar way we did on L_2. Which means

$$(14)\qquad \left(\frac{1}{2\pi}\right)\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty} e^{-p_1(t_1+t_2)-jt_2}\Gamma(p_1)^n\Gamma(-j+p_1)dp_1 = \sum_{m=0}^{\infty}\mathrm{Res}_{p_1=-m}\, e^{-p_1(t_1+t_2)-jt_2}\Gamma(p_1)^n\Gamma(-j+p_1)dp_1.$$

Remark 3.6. Because $\lim_{j\to\infty}\frac{e^{-jt_2}}{(j-1)!(1-\varepsilon_1)} = 0$, there is a $M(t_2,\varepsilon_1)$ such that $\forall j \in \mathbb{N}^*$ $\frac{e^{-jt_2}}{(j-1)!(1-\varepsilon_1)} \leq M(t_2,\varepsilon_1)$.Which means $|e^{-p_1(t_1+t_2)-jt_2}\Gamma(p_1)^n\Gamma(-j+p_1)| \leq |e^{-p_1(t_1+t_2)}\Gamma(p_1)^{n+1}M(t_2,\varepsilon_1)|$. Hence the speed of the integral convergent to residue does not depend on j.

Then let us consider $\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty} e^{-p_1t_1-jt_2}\Gamma(p_1)^n\Gamma(-j-p_1)dp_1$. By Proposition 3.3, 1) and 6), we have that $|\Gamma(p_1)^n\Gamma(-j-p_1)| = |\Gamma(p_1)^{n-1}\frac{\pi}{\sin(\pi p_1)}\prod_{k=1}^{j-1}(-k-p_1)^{-1}|$. When $p_1 \in L_{1k}(k=1,3)$, $|\Gamma(p_1)^n\Gamma(-j-p_1)| \leq |\Gamma(p_1)^{n-1}\frac{\pi}{N^{j-1}\sin(\pi p_1)}|$, and $|\sin(\pi p_1)| \geq \pi$ because of $N \geq 10$. Thus, $|\Gamma(p_1)^n\Gamma(-j-p_1)| \leq \Gamma(p_1)^{n-1}(n \geq 2)$. Then, by the similar way we have done on L_2, we have that $\lim_{N\to\infty}\int_{L_{1k}} e^{-p_1t_1-jt_2}\Gamma(p_1)^n\Gamma(-j-p_1)dp_1 = 0$ $(k=1,3)$. When $p_1 \in L_{12}$, $|\sin(p_1)| \geq \sin\left(\frac{\pi}{4}\right)$ and $\prod_{k=1}^{j-1}|-k-p_1|^{-1} \leq 16$, because of $-\frac{3}{4} \leq \delta \leq -\frac{1}{4}$. Therefore $|\Gamma(p_1)^n\Gamma(-j-p_1)| \leq \frac{16\pi}{\sin\left(\frac{1}{4}\pi\right)}\Gamma(p_1)^{n-1}(n \geq 2)$. Hence

$$(15)\qquad \left(\frac{1}{2\pi}\right)\int_{\varepsilon_1-i\infty}^{\varepsilon_1+i\infty} e^{-p_1t_1-jt_2}\Gamma(p_1)^n\Gamma(-j-p_1)dp_1 = \sum_{m=0}^{\infty}\mathrm{Res}_{p_1=-m}\, e^{-p_1t_1-jt_2}\Gamma(p_1)^n\Gamma(-j-p_1)dp_1$$

Remark 3.7. The speed of this integral convergent to residue is not depend on j because the inequality we used to estimate the integral does not include j in it.

By (14) ,(15) ,Remark 3.6, Remark 3.7 and $\sum_{j=0}^{\infty}\left|\frac{1}{j!}\right| < \infty$, we get

$$(16)\qquad I(t_1,t_2) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\mathrm{Res}_{p_1=-j}\left(\mathrm{Res}_{p_2=-k}+\mathrm{Res}_{p_2=-k+p_1}\right)e^{-p_2t_2-p_1t_1}\Gamma(p_2)\Gamma(p_2-p_1)\Gamma(p_1)^n dp_2dp_1$$

By(13), (16) we proved Theorem 3.1.

2. Lemma 3.8

Let us denote that $Q_1 := e^{\tau_1} \in \mathbb{R} > 0$, $Q_2 := e^{\tau_2} \in \mathbb{R} > 0$. The function $g_{\tau_1,\tau_2}(x) := f(x,\tau_1,\tau_2) = x_1 + \cdots + x_n + \frac{x_1\cdots x_n}{Q_1} + \frac{Q_1Q_2}{x_1\cdots x_n}$ defines a real-valued function on $\mathbb{R}^n_{>0}$ with minimal value $u(\tau_1,\tau_2)$ achieved at a unique critical point $\nu = (\nu_1,\cdots,\nu_n)$. Put $\alpha_\lambda = \{x \in \mathbb{R}^n_{>0} | g_{\tau_1,\tau_2}(x) \leq \lambda\}$. For all $m \in Q$ let us define

$$I^{(-m)}(\tau_1,\tau_2,\lambda) := \int_{\alpha_\lambda}\frac{(\lambda - f(x,\tau_1,\tau_2))^{m+\frac{1}{2}}}{\Gamma\left(m+\frac{1}{2}\right)}\omega.$$

Lemma 3.8. *If λ is sufficient close to $u(\tau_1,\tau_2)$, then*

$$I^{(-m)}(\tau_1,\tau_2,\lambda) = (\lambda - u(\tau_1,\tau_2))^{\frac{n-1}{2}+m}(c_0(\tau_1,\tau_2) + c_1(\tau_1,\tau_2)(\lambda - u(\tau_1,\tau_2)) + \cdots).$$

Proof. Let $\alpha_{\tau_1,\tau_2,\mu} = \{x \in \mathbb{R}^n_{>0} | f(x,\tau_1,\tau_2) = \mu\}$, we have that $I^{(-m)} = \int_{u(\tau_1,\tau_2)}^{\lambda}\frac{(\lambda-\mu)^{m-\frac{1}{2}}}{\Gamma\left(m+\frac{1}{2}\right)}\mathrm{d}\mu\int_{\alpha_{\tau_1,\tau_2,\mu}}\frac{\omega}{\mathrm{df}}$.

Let us define $A := \{x \in \mathbb{R}^n_{>0} | \mu \le f(x,\tau_1,\tau_2) \le \mu+\varepsilon\}$. The function f induces a locally trivial smooth fibration $f : A \to [0,\varepsilon]$. This can be proved by using gradient flow for real part of f. In particular we have a diffeomorphism $A \cong [0,\varepsilon] \times \alpha_{\tau_1,\tau_2,\mu}$. There exists η such that

$$\omega = \mathrm{df} \wedge \eta,$$

where η can be viewed as a holomorphic form in a neighborhood of $\alpha_{\tau_1,\tau_2,\mu}$ in $(\mathbb{C}^*)^n$.

Then we have that

$$\int_{\partial A} \mathrm{d}^{-1}\omega \underset{\text{Stokes'}}{=} \int_A \omega \underset{\text{Fubini's}}{=} \int_\mu^{\mu+\varepsilon} \int_{\alpha_{\tau_1,\tau_2,\mu}} \eta \, \mathrm{df},$$

which means that

$$\partial_\varepsilon \int_{\partial A} \mathrm{d}^{-1}\omega = \int_{\alpha_{\tau_1,\tau_2,\mu}} \frac{\omega}{\mathrm{df}}.$$

On the other hand, we have that

$$\partial_\varepsilon \int_{\partial A} \mathrm{d}^{-1}\omega = \partial_\mu \int_{\alpha_{\tau_1,\tau_2,\mu}} \mathrm{d}^{-1}\omega.$$

Therefore,

$$\begin{aligned} I^{(-m)} &= \int_{u(\tau_1,\tau_2)}^{\lambda} \mathrm{d}\mu \int_{\alpha_{\tau_1,\tau_2,\mu}} \frac{(\lambda-\mu)^{m-\frac{1}{2}}}{\Gamma\left(m+\frac{1}{2}\right)} \frac{\omega}{\mathrm{df}} \\ &= \int_{u(\tau_1,\tau_2)}^{\lambda} \frac{(\lambda-\mu)^{m-\frac{1}{2}}}{\Gamma\left(m+\frac{1}{2}\right)} \partial_\mu \int_{\alpha_{\tau_1,\tau_2,\mu}} \mathrm{d}^{-1}\omega \, \mathrm{d}\mu \end{aligned}$$

Then let us integrate by parts, we get

$$\begin{aligned} I^{(-m)} &= \int_{u(\tau_1,\tau_2)}^{\lambda} \frac{(\lambda-\mu)^{m-\frac{1}{2}}}{\Gamma\left(m+\frac{1}{2}\right)} \partial_\mu \int_{\alpha_{\tau_1,\tau_2,\mu}} d^{-1}\omega \, \mathrm{d}\mu \\ &= \int_{u(\tau_1,\tau_2)}^{\lambda} \frac{(\lambda-\mu)^{m-\frac{1}{2}}}{\Gamma\left(m+\frac{1}{2}\right)} \int_{\alpha_{\tau_1,\tau_2,\mu}} d^{-1}\omega \, \mathrm{d}\mu \Bigg|_{u(\tau_1,\tau_2)}^{\lambda} + \int_{u(\tau_1,\tau_2)}^{\lambda} \frac{(\lambda-\mu)^{m-\frac{3}{2}}}{\Gamma\left(m-\frac{1}{2}\right)} \left(\int_{\alpha_{\tau_1,\tau_2,\mu}} \omega \right) \mathrm{d}\mu. \end{aligned}$$

Note that when $\mu = \lambda$, $\mu - \lambda = 0$. When $\mu = u(\tau_1,\tau_2)$, $\int_{\alpha_{\tau_1,\tau_2,\mu}} \mathrm{d}^{-1}\omega = 0$. Thus,

$$I^{(-m)} = \int_{u(\tau_1,\tau_2)}^{\lambda} \frac{(\lambda-\mu)^{m-\frac{3}{2}}}{\Gamma\left(m-\frac{1}{2}\right)} \left(\int_{\alpha_{\tau_1,\tau_2,\mu}} \omega \right) \mathrm{d}\mu$$

If x is a critical point, then we have that $x_i \frac{\partial f}{\partial x_i} = 0$, which means, for all i, $x_i + \frac{x_1 \cdots x_n}{Q_1} - \frac{Q_1 Q_2}{x_1 \cdots x_n} = 0$. Therefore, for each critical point there exist a number t such that $v_i = t$ for all i and $t + \frac{t^n}{Q_1} - \frac{Q_1 Q_2}{t^n} = 0$. Let us calculate the second derivative of f.

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \begin{cases} \frac{t^{n-2}}{Q_1} + \frac{Q_1 Q_2}{t^{n+2}} & \text{if } i \neq j, \\ \frac{2 Q_1 Q_2}{t^{n+2}} & \text{if } i = j. \end{cases}$$

We also know that

$$\frac{t^{n-2}}{Q_1}+\frac{Q_1Q_2}{t^{n+2}}-\frac{2Q_1Q_2}{t^{n+2}}=\frac{t^{n-2}}{Q_1}-\frac{Q_1Q_2}{t^{n+2}}=-\frac{1}{t^2}\neq 0$$

Therefore, f is a Morse function.

If λ is sufficient close to $u(\tau_1,\tau_2)$, then by the complex version of Morse Lemma [**20**] there exit coordinates $z_1,\cdots,z_n$ such that

$$x_1+\cdots+x_n+\frac{x_1\cdots x_n}{e^{\tau_1}}+\frac{e^{\tau_1+\tau_2}}{x_1\cdots x_n}=u(\tau_1,\tau_2)+z_1^2+\cdots+z_n^2,$$

where $x_i=\nu_i+g_i(z_1,\cdots,z_n)$, where $g_i(z_1,\cdots,z_n)$ is at least linear in $z_1,\cdots,z_n$, and we have that $\omega=\frac{\mathrm{dx_1}}{x_1}\cdots\frac{\mathrm{dx_n}}{x_n}=(\tilde{c}_0(\tau_1,\tau_2)+\mathrm{o}(z_1,\cdots,z_n))$

Let $z_i=(\mu-u(\tau_1,\tau_2))^{\frac{1}{2}}\tilde{z}_i$,$(i=1,\cdots,n)$

$$\int_{\alpha_\mu}\omega=\int_{z_1^2+\cdots+z_n^2\leq\mu-u(\tau_1,\tau_2)}(\tilde{c}_0(\tau_1,\tau_2)+\tilde{c}_1(\tau_1,\tau_2)+\cdots)\,\mathrm{dz}=\int_{|\tilde{z}|\leq 1}\sum_{k=0}^{\infty}\tilde{c}_k(\tau_1,\tau_2,\tilde{z})(\mu-u(\tau_1,\tau_2))^{\frac{n+k}{2}}\,\mathrm{d}\tilde{z},$$

where $\tilde{c}_k$ is homogeneous of degree k in z.

Note that

$$\begin{aligned}\int_{|\tilde{z}|\leq 1}\tilde{c}_k(\tau_1,\tau_2,\tilde{z})(\mu-u(\tau_1,\tau_2))^{\frac{n+k}{2}}\,\mathrm{d}\tilde{z}&=\int_{|\tilde{z}|\leq 1}\tilde{c}_k(\tau_1,\tau_2,-\tilde{z})(\mu-u(\tau_1,\tau_2))^{\frac{n+k}{2}}\,\mathrm{d}\tilde{z}\\&=(-1)^k\int_{|\tilde{z}|\leq 1}\tilde{c}_k(\tau_1,\tau_2,\tilde{z})(\mu-u(\tau_1,\tau_2))^{\frac{n+k}{2}}\,\mathrm{d}\tilde{z}\end{aligned}$$

which means k should be even.

Hence,

$$\int_{\alpha_\mu}\omega=(\mu-u(\tau_1,\tau_2))^{\frac{n}{2}}\left(\tilde{c}_0\,\mathrm{Vol}(|\tilde{z}|\leq 1)\sum\int_{|\tilde{z}|\leq 1}\tilde{c}_{2k}(\tau_1,\tau_2,\tilde{z})\,\mathrm{d}\,\tilde{z}(\mu-u(\tau_1,\tau_2))^k\right)$$

Let $\mu=u(\tau_1,\tau_2)+x(\lambda-u(\tau_1,\tau_2))$, which means $\lambda-\mu=(\lambda-u(\tau_1,\tau_2))(1-x)$ Therefore

$$\begin{aligned}\int\frac{(\lambda-\mu)^{m-\frac{3}{2}}}{\Gamma\left(m-\frac{1}{2}\right)}(\mu-u(\tau_1,\tau_2))^{\frac{n}{2}+k}\,\mathrm{d}\,\mu&=\int_0^1\frac{(1-x)^{m-\frac{3}{2}}}{\Gamma\left(m-\frac{1}{2}\right)}x^{\frac{n}{2}+k}(\lambda-\mu)^{m-\frac{3}{2}+\frac{n}{2}+k+1}\,\mathrm{d}\,x\\&=(\lambda-u(\tau_1,\tau_2))^{m+k+\frac{n-1}{2}}\frac{\mathcal{B}(m-\frac{1}{2},\frac{n}{2}+k+1)}{\Gamma(m-\frac{1}{2})}\\&=(\lambda-u(\tau_1,\tau_2))^{m+\frac{n-1}{2}}c_k'(\lambda-u(\tau_1,\tau_2))^k,\end{aligned}$$

which means

$$I^{(-m)}(\tau_1,\tau_2,\lambda)=(\lambda-u(\tau_1,\tau_2))^{\frac{n-1}{2}+m}(c_0(\tau_1,\tau_2)+c_1(\tau_1,\tau_2)(\lambda-u(\tau_1,\tau_2))+\cdots).$$

□

3. Lemma 3.9

Lemma 3.9. *We have*

$$\int_{u(\tau_1,\tau_2)}^{\infty}e^{\frac{\lambda}{z}}I^{(-m)}(\tau_1,\tau_2,\lambda)\,\mathrm{d}\lambda=(-z)^{m+\frac{1}{2}}\int_{\mathbb{R}_{>0}^n}e^{\frac{f(x,\tau_1,\tau_2)}{z}}\omega,$$

where $\tau_1,\ \tau_2\in\mathbb{R}$, $z\in\mathbb{R}_{<0}$

PROOF. Since

$$I^{(-m)}(\tau_1,\tau_2,\lambda)=\int\limits_{u(\tau_1,\tau_2)}^{\lambda}\int\limits_{\alpha_{\tau_1,\tau_2,\mu}}\frac{(\lambda-\mu)^{m-\frac{1}{2}}}{\Gamma\left(m+\frac{1}{2}\right)}\frac{\omega}{\mathrm{df}}\,\mathrm{d}\mu,$$

Let us calculate the LHS

$$\begin{aligned}\mathrm{LHS}&=\int\limits_{u(\tau_1,\tau_2)}^{\infty}\int\limits_{u(\tau_1,\tau_2)}^{\lambda}\int\limits_{\alpha_{\tau_1,\tau_2,\mu}}e^{\frac{\lambda}{z}}\frac{(\lambda-\mu)^{m-\frac{1}{2}}}{\Gamma\left(m+\frac{1}{2}\right)}\frac{\omega}{\mathrm{df}}\,\mathrm{d}\mu\,\mathrm{d}\lambda\\&=\int\limits_{u(\tau_1,\tau_2)}^{\infty}\mathrm{d}\mu\left(\int\limits_{\mu}^{\infty}e^{\frac{\lambda-\mu}{z}}\frac{(\lambda-\mu)^{m-\frac{1}{2}}}{\Gamma\left(m+\frac{1}{2}\right)}\,\mathrm{d}\lambda\right)e^{\frac{\mu}{z}}\int\limits_{\alpha_{\tau_1,\tau_2,\mu}}\frac{\omega}{\mathrm{df}}.\end{aligned}$$

Note that

$$\int_{\mu}^{\infty}e^{\frac{\lambda-\mu}{z}}(\lambda-\mu)^{m-\frac{1}{2}}\,\mathrm{d}\lambda=(-z)^{m+\frac{1}{2}}\int_{0}^{\infty}e^{\frac{\lambda-\mu}{z}}\left(\frac{(\lambda-\mu)}{-z}\right)^{m-\frac{1}{2}}\mathrm{d}\left(\frac{\lambda\text{-}\mu}{\text{-z}}\right)=(-z)^{m+\frac{1}{2}}\Gamma(m+\frac{1}{2}),$$

we have

$$\mathrm{LHS}=(-z)^{m+\frac{1}{2}}\int\limits_{u(\tau_1,\tau_2)}^{\infty}e^{\frac{\mu}{z}}\int\limits_{\alpha_{\tau_1,\tau_2,\mu}}\frac{\omega}{\mathrm{df}}\,\mathrm{d}\mu=(-z)^{m+\frac{1}{2}}\int_{\mathbb{R}^n_{>0}}e^{\frac{f(x,\tau_1,\tau_2)}{z}}\omega.$$

□

4. Theorem 3.10

THEOREM 3.10. *Exists $E_0\in H^*(X;\mathbb{C})$ independent of τ_1, τ_2 and λ such that*

$$(I_E^{(-m-1)},\Phi_{i,j})=(-\partial_{\tau_1})^{i-1}(-\partial_{\tau_2})^{j-1}I^{(-m-i-j+1+\frac{n}{2})}(\tau,\lambda)$$

where $\tau=(\tau_1,\tau_2)$, $E=e^{-\tau_1p_1-\tau_2p_2}E_0$ and $Q=(Q_1,Q_2)=(e^{\tau_1},e^{\tau_1})$

PROOF. By definition

$$I^{(m)}(t,Q,\lambda)=\sum_{k=0}^{\infty}(-1)^kS_k(t,Q)\widetilde{I}^{(m+k)}(\lambda),$$

where $\widetilde{I}^{(n)}(\lambda)=e^{\rho\partial_\lambda\partial_n}\frac{\lambda^{\theta-n-\frac{1}{2}}}{\Gamma(\theta-n+\frac{1}{2})}$ Since I is a solution to the second structure connection, we have ($t=t_2p_1+t_3p_2$)

$$\begin{aligned}&\frac{\partial}{\partial t_2}I^{(m)}=-p_1\bullet\partial_\lambda I^{(m)}\\&\frac{\partial}{\partial t_3}I^{(m)}=-p_2\bullet\partial_\lambda I^{(m)}\\&\left(\lambda\partial_\lambda+(n-1)\frac{\partial}{\partial t_2}+2\frac{\partial}{\partial t_3}\right)I^{(m)}=(\theta-m-\frac{1}{2})I^{(m)}.\end{aligned}$$

By divisor equation

$$\frac{\partial}{\partial t_{j+1}}S=Q_i\partial_{Q_j}S+S\frac{p_j\cup}{z}\quad(j=1,2),$$

which means

$$Q_i\partial_{Q_j}S_{k+1}=p_j\bullet S_k-S_kp_j\cup.$$

Therefore,

$$
\begin{aligned}
Q_i\partial_{Q_j} I^{(m)} &= \sum_{k=0}^{\infty}(-1)^{k+1}(p_j \bullet S_k - S_k p_j\cup)\widetilde{I}^{(m+k+1)}(\lambda) \\
&= -p_j \cdot I^{(m+1)} + \sum_{k=0}^{\infty}(-1)^k S_k p_j \cup e^{\rho\partial_\lambda\partial_n}\frac{\lambda^{\theta-m-k-1-\frac{1}{2}}}{\Gamma(\theta-m-k-1+\frac{1}{2})}
\end{aligned}
$$

Because $p_i\cup\theta = (\theta+1)p_i\cup$ we have that

$$
\begin{aligned}
p_j \cup e^{\rho\partial_\lambda\partial_n}\frac{\lambda^{\theta-m-k-1-\frac{1}{2}}}{\Gamma(\theta-m-k-1+\frac{1}{2})} &= e^{\rho\partial_\lambda\partial_n}\frac{\lambda^{\theta+1-m-k-\frac{1}{2}}}{\Gamma(\theta+1-m-k+\frac{1}{2})}p_j\cup \\
&= \widetilde{I}^{(m+k})p_j\cup
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sum_{k=0}^{\infty}(-1)^k S_k p_j \cup e^{\rho\partial_\lambda\partial_n}\frac{\lambda^{\theta-m-k-1-\frac{1}{2}}}{\Gamma(\theta-m-k-1+\frac{1}{2})} &= \sum_{k=0}^{\infty}(-1)^k S_k \widetilde{I}^{(m)}p_j\cup \\
&= I^{(m)}p_j\cup,
\end{aligned}
$$

which means

$$
Q_j\partial_{Q_j} I^{(m)} \quad = -p_j \bullet \partial_\lambda I^{(m)} + I(m)p_j\cup.
$$

Hence,

$$
\partial_{\tau_j} I^{(m)}(t,Q,\lambda) = \sum_{i=1,2}\frac{\mathrm{dQ_i}}{\mathrm{d\tau_j}}\partial_{Q_i} = Q_j\partial_{Q_j} I^{(m)}(t,Q,\lambda),
$$

therefore

$$
\partial_{\tau_j} I^{(m)}(t,Q,\lambda) = -p_j \bullet \partial_\lambda I^{(m)} + I(m)p_j\cup.
$$

Let us compute the equations for $I_E^{(m)}$.

$$
\begin{aligned}
\partial_{\tau_j} I_E^{(m)} &= (\partial_{\tau_j} I^{(m)})E + I^{(m)}(\partial_{\tau_j}E) \\
&= (-p_j \bullet \partial_\lambda I^{(m)} + I(m)p_j\cup)E + I^{(m)}(-p_j\cup E) \\
&= -p_j \bullet \partial_\lambda I_E^{(m)}
\end{aligned}
$$

Then,

$$
\begin{aligned}
\left(\lambda\partial_\lambda + (n-1)\frac{\partial}{\partial\tau_1} + 2\frac{\partial}{\partial\tau_2}\right)I_E^{(m)} &= \lambda\partial_\lambda I_E^{(m)} + (-(n-1)p_1 \bullet -2p_2\bullet)\,\partial_\lambda I_E^{(m)} \\
&= (\lambda - (n-1)p_1 \bullet -2p_2\bullet)\,\partial_\lambda I_E^{(m)} \\
&= \left(\lambda\partial_\lambda + (n-1)\frac{\partial}{\partial t_1} + 2\frac{\partial}{\partial t_2}\right)I^{(m)}E \\
&= \left(\theta - m - \frac{1}{2}\right)I_E^{(m)}.
\end{aligned}
$$

Therefore the equations for $I_E^{(m)}$ are

$$
\begin{aligned}
&\partial_{\tau_j} I_E^{(m)} = -p_j \bullet \partial_\lambda I_E^{(m)} \\
&\left(\lambda\partial_\lambda + (n-1)\frac{\partial}{\partial\tau_1} + 2\frac{\partial}{\partial\tau_2}\right)I_E^{(m)} = \left(\theta - m - \frac{1}{2}\right)I_E^{(m)}
\end{aligned}
$$

Let $L(\tau,\lambda)\in H$ be such that

$$\Big(L(\tau,\lambda),\Phi_{(i,j)}\Big)=(\partial_{\tau_1})^{i-1}(\partial_{\tau_2})^{j-1}I^{(-m-i-j+1+\frac{n}{2})}(\tau,\lambda).$$

Let us check that $L(\tau,\lambda)$ solves the above system of equations. First, by defination,

$$\alpha_{Q,\lambda}=\{x\in\mathbb{R}^n_{>0}|x_1+\cdots+x_n+\frac{x_1\cdots x_n}{Q_1}+\frac{Q_1Q_2}{x_1\cdots x_n}\leq\lambda\}.$$

We can see that $\alpha_{(c^{n-1}Q_1,c^2Q_2,c\lambda)}=c\alpha_{(Q_1,Q_2,\lambda)}$, which means that

$$I^{(-m)}(c^{n-1}Q_1,c^2Q_2,c\lambda)=c^{m-\frac{1}{2}}I^{(-m)}(Q_1,Q_2,\lambda).$$

Let us differentiate the both side with c and set c to be 1, we get that

$$\Big(\lambda\partial_\lambda+(n-1)Q_1\partial_{Q_1}+2Q_2\partial_{Q_2}\Big)I^{(-m)}=(m-\frac{1}{2})I^{(-m)}.$$

Then let us check that L follows the last equation,

$$\begin{aligned}&\Big(\lambda\partial_\lambda+(n-1)Q_1\partial_{Q_1}+2Q_2\partial_{Q_2}\Big)(L(\tau,\lambda),\Phi_{i,j})\\=&\Big(\lambda\partial_\lambda+(n-1)Q_1\partial_{Q_1}+2Q_2\partial_{Q_2}\Big)(-\partial_{\tau_1})^{i-1}(-\partial_{\tau_2})^{j-1}I^{(-m-i-j+1+\frac{n}{2})}(\tau,\lambda)\\=&(-m-i-j+\frac{n}{2}-\frac{3}{2})(L(\tau,\lambda),\Phi_{i,j}).\end{aligned}$$

On the other hand,

$$\theta(\Phi_{i,j})=(\frac{n}{2}-i-j+2)\Phi_{i,j},$$

which means that

$$\begin{aligned}\Big(\Big(\theta+m+\frac{1}{2}\Big)L,\Phi_{i,j}\Big)&=\Big(L,\Big(-\theta+m+\frac{1}{2}\Big)\Phi_{i,j}\Big)\\&=\Big(L,\Big(-\frac{n}{2}+i+j-2+m+\frac{1}{2}\Big)\Phi_{i,j}\Big)\\&=(-m-i-j+\frac{n}{2}-\frac{3}{2})(L(\tau,\lambda),\Phi_{i,j})\\&=\Big(\lambda\partial_\lambda+(n-1)Q_1\partial_{Q_1}+2Q_2\partial_{Q_2}\Big)(L(\tau,\lambda),\Phi_{i,j}).\end{aligned}$$

Then let us check L follows $\partial_{\tau_k}I^{(m)}_E=-p_k\bullet\partial_\lambda I^{(m)}_E$. Recalling Chapter 2 Section 5, there are 6 cases we need to prove. First, let us list the 6 cases.

1) $p_1\bullet\Phi_{i,1}=\Phi_{i+1,1}$, where $k=1$, $i\leq n-1$, $j=1,2$.
2) $p_2\bullet\Phi_{i,1}=\Phi_{i,2}$, where $k=2$, $i\leq n$, $j=1$.
3) $p_2\bullet\Phi_{i,2}=\Phi_{i+1,2}+Q_2\Phi_{i,1}$, where $k=2$, $i\leq n-1$, $j=2$.
4) $p_1\bullet\Phi_{n,2}=Q_1Q_2\Phi_{1,1}$, where $k=1$, $i=n$, $j=2$.
5) $p_1\bullet\Phi_{n,1}=Q_1\Phi_{1,2}-Q_1\Phi_{2,1}$, where $k=1$, $i=n$, $j=1$.
6) $p_2\bullet\Phi_{n,2}=Q_2\Phi_{n,1}+Q_1Q_2\Phi_{1,1}$, where $k=2$, $i=n$, $j=2$.

In case 1),

$$\begin{aligned}\partial_{\tau_1}(L,\Phi_{i,j})&=\partial_{\tau_1}(-\partial_{\tau_1})^{i-1}(-\partial_{\tau_2})^{j-1}I^{(-m-i-j+1+\frac{n}{2})}(\tau,\lambda)\\&=-(-\partial_{\tau_1})^{i}(-\partial_{\tau_2})^{j-1}I^{(-m-i-j+1+\frac{n}{2})}(\tau,\lambda)\\&=-(-\partial_{\tau_1})^{i}(-\partial_{\tau_2})^{j-1}\partial_\lambda I^{(-m-i-j+\frac{n}{2})}(\tau,\lambda)\\&=-\partial_\lambda(L,\Phi_{i+1,j}).\end{aligned}$$

In case 2),

$$\begin{aligned}\partial_{\tau_2}(L,\Phi_{i,j}) &= \partial_{\tau_2}(-\partial_{\tau_1})^{i-1}(-\partial_{\tau_2})^{j-1}I^{(-m-i-j+1+\frac{n}{2})}(\tau,\lambda)\\ &= -(-\partial_{\tau_1})^{i-1}(-\partial_{\tau_2})^{j}I^{(-m-i-j+1+\frac{n}{2})}(\tau,\lambda)\\ &= -(-\partial_{\tau_1})^{i-1}(-\partial_{\tau_2})^{j}\partial_\lambda I^{(-m-i-j+\frac{n}{2})}(\tau,\lambda)\\ &= -\partial_\lambda(L,\Phi_{i,j+1}).\end{aligned}$$

In case 3), let us denote $-l := -m-i-1+\frac{n}{2}$. Let us calculate $\partial_{\tau_2}(L,\Phi_{i,2})$,

$$\begin{aligned}\partial_{\tau_2}(L,\Phi_{i,2}) &= \partial_{\tau_2}(-\partial_{\tau_1})^{i-1}(-\partial_{\tau_2})I^{(-l)}(\tau,\lambda)\\ &= -(-\partial_{\tau_1})^{i-1}(-\partial_{\tau_2})^2\int\frac{(\lambda-f)^{l-\frac{1}{2}}}{\Gamma(l+\frac{1}{2})}\omega\\ &= -(-\partial_{\tau_1})^{i-1}\int\left(\frac{(\lambda-f)^{l-\frac{5}{2}}}{\Gamma(l-\frac{3}{2})}\left(\frac{Q_1Q_2}{x_1\cdots x_n}\right)^2-\frac{(\lambda-f)^{l-\frac{3}{2}}}{\Gamma(l-\frac{1}{2})}\left(\frac{Q_1Q_2}{x_1\cdots x_n}\right)\right).\end{aligned}$$

On the other hand, let us calculate $\partial_\lambda(L,\Phi_{i+1,2})$ and $\partial_\lambda(L,Q_2\Phi_{i,1})$,

$$\begin{aligned}\partial_\lambda(L,\Phi_{i+1,2}) &= \partial_\lambda(-\partial_{\tau_1})^{i}(-\partial_{\tau_2})I^{(-l-1)}(\tau,\lambda)\\ &= (-\partial_{\tau_1})^{i-1}(-\partial_{\tau_1})(-\partial_{\tau_2})I^{(-l)}(\tau,\lambda)\\ &= (-\partial_{\tau_1})^{i-1}(\partial_{\tau_1})(\partial_{\tau_2})I^{(-l)}\int\frac{(\lambda-f)^{l-\frac{1}{2}}}{\Gamma(l+\frac{1}{2})}\omega\\ &= (-\partial_{\tau_1})^{i-1}\int\left(\frac{(\lambda-f)^{l-\frac{5}{2}}}{\Gamma(l-\frac{3}{2})}\left(\frac{Q_1Q_2}{x_1\cdots x_n}\right)^2-\frac{(\lambda-f)^{l-\frac{5}{2}}}{\Gamma(l-\frac{3}{2})}Q_2-\frac{(\lambda-f)^{l-\frac{3}{2}}}{\Gamma(l-\frac{1}{2})}\left(\frac{Q_1Q_2}{x_1\cdots x_n}\right)\right).\end{aligned}$$

And

$$\begin{aligned}\partial_\lambda(L,Q_2\Phi_{i,1}) =& Q_2\partial_\lambda(-\partial_{\tau_1})^{i-1}I^{(-l+1)}(\tau,\lambda)\\ =& Q_2(-\partial_{\tau_1})^{i-1}I^{(-l+2)}(\tau,\lambda)\\ =& Q_2(-\partial_{\tau_1})^{i-1}\int\frac{(\lambda-f)^{l-\frac{5}{2}}}{\Gamma(l-\frac{3}{2})}\omega\end{aligned}$$

.

Thus,

$$-\partial_\lambda(L,\Phi_{i+1,2}+Q_2\Phi_{i,1}) = -(-\partial_{\tau_1})^{i-1}\int\left(\frac{(\lambda-f)^{l-\frac{5}{2}}}{\Gamma(l-\frac{3}{2})}\left(\frac{Q_1Q_2}{x_1\cdots x_n}\right)^2-\frac{(\lambda-f)^{l-\frac{3}{2}}}{\Gamma(l-\frac{1}{2})}\left(\frac{Q_1Q_2}{x_1\cdots x_n}\right)\right).$$

Therefore $\partial_{\tau_2}(L,\Phi_{i,2}) = -\partial_\lambda(L,\Phi_{i+1,2}+Q_2\Phi_{i,1})$.

In case 4), let us denote $-l := -m-n-1+\frac{n}{2}$. Let us calculate $\partial_{\tau_1}(L,\Phi_{n,2})$,

$$\begin{aligned}\partial_{\tau_1}(L,\Phi_{n,2}) =& (\partial_{\tau_2})(-\partial_{\tau_1})^{n}I^{(-l)}(\tau,\lambda)\\ =& (\partial_{\tau_2})(-\partial_{\tau_1})^{n-1}\int\frac{(\lambda-f)^{l-\frac{3}{2}}}{\Gamma(l-\frac{1}{2})}\left(\frac{Q_1Q_2}{x_1\cdots x_n}-\frac{x_1\cdots x_n}{Q_1}\right)\omega\\ =& (\partial_{\tau_2})(-\partial_{\tau_1})^{n-1}\partial_\lambda\int\frac{(\lambda-f)^{l-\frac{1}{2}}}{\Gamma(l+\frac{1}{2})}\left(\frac{Q_1Q_2}{x_1\cdots x_n}-\frac{x_1\cdots x_n}{Q_1}\right)\omega.\end{aligned}$$

We also have that,

$$d(\lambda-f)\wedge\frac{\mathrm{dx_2}}{x_2}\wedge\cdots\wedge\frac{\mathrm{dx_n}}{x_n} = \left(-x_1+\frac{Q_1Q_2}{x_1\cdots x_n}-\frac{x_1\cdots x_n}{Q_1}\right)\omega.$$

Since $\lambda - f$ vanishes on boundary of α,

$$\int (\lambda - f)^{m-\frac{1}{2}} d(\lambda - f) \wedge \frac{\mathrm{dx_2}}{x_2} \wedge \cdots \wedge \frac{\mathrm{dx_n}}{x_n} = 0.$$

Which means

$$\partial_{\tau_1}(L, \Phi_{n,2}) = (\partial_{\tau_2})(-\partial_{\tau_1})^{n-1}\partial_\lambda \int \frac{(\lambda - f)^{l-\frac{1}{2}}}{\Gamma(l+\frac{1}{2})} x_1 \omega$$

Suppose that for some s, such that $s < n$, we have that

$$\partial_{\tau_1}(L, \Phi_{n,2}) = (\partial_{\tau_2})(-\partial_{\tau_1})^{n-s}(\partial_\lambda)^s \int \frac{(\lambda - f)^{l-\frac{1}{2}}}{\Gamma(l+\frac{1}{2})} \prod_{r=1}^{s} x_r \omega.$$

Then

$$\begin{aligned}
\partial_{\tau_1}(L, \Phi_{n,2}) =&(\partial_{\tau_2})(-\partial_{\tau_1})^{n-s}(\partial_\lambda)^s \int \frac{(\lambda - f)^{l-\frac{1}{2}}}{\Gamma(l+\frac{1}{2})} \prod_{r=1}^{s} x_r \omega \\
=&(\partial_{\tau_2})(-\partial_{\tau_1})^{n-s-1}(\partial_\lambda)^s \int \frac{(\lambda - f)^{l-\frac{3}{2}}}{\Gamma(l-\frac{1}{2})} \left(\frac{Q_1 Q_2}{x_1 \cdots x_n} - \frac{x_1 \cdots x_n}{Q_1}\right) \prod_{r=1}^{s} x_r \omega \\
=&(\partial_{\tau_2})(-\partial_{\tau_1})^{n-s-1}(\partial_\lambda)^{s+1} \int \frac{(\lambda - f)^{l-\frac{1}{2}}}{\Gamma(l+\frac{1}{2})} \left(\frac{Q_1 Q_2}{x_1 \cdots x_n} - \frac{x_1 \cdots x_n}{Q_1}\right) \prod_{r=1}^{s} x_r \omega
\end{aligned}$$

We also have that,

$$d(\lambda - f)\frac{\mathrm{dx_1}}{x_1} \wedge \frac{\mathrm{dx_s}}{x_s} \wedge \cdots \wedge \frac{\mathrm{dx_{s+2}}}{x_{s+2}} \wedge \cdots \wedge \frac{\mathrm{dx_n}}{x_n} = (-1)^{s-1}\left(-x_{s+1} + \frac{Q_1 Q_2}{x_1 \cdots x_n} - \frac{x_1 \cdots x_n}{Q_1}\right)\omega.$$

Since $\lambda - f$ vanishes on boundary of α,

$$\int (\lambda - f)^{m-\frac{1}{2}} d(\lambda - f)\frac{\mathrm{dx_1}}{x_1} \wedge \frac{\mathrm{dx_s}}{x_s} \wedge \cdots \wedge \frac{\mathrm{dx_{s+2}}}{x_{s+2}} \wedge \cdots \wedge \frac{\mathrm{dx_n}}{x_n} = 0.$$

Which means

$$\partial_{\tau_1}(L, \Phi_{n,2}) = \partial_{\tau_1}(L, \Phi_{n,2}) = (\partial_{\tau_2})(-\partial_{\tau_1})^{n-s-1}(\partial_\lambda)^{s+1} \int \frac{(\lambda - f)^{l-\frac{1}{2}}}{\Gamma(l+\frac{1}{2})} \prod_{r=1}^{s+1} x_r \omega.$$

Therefore,

$$\partial_{\tau_1}(L, \Phi_{n,2}) = \partial_{\tau_1}(L, \Phi_{n,2}) = (\partial_{\tau_2})(\partial_\lambda)^n \int \frac{(\lambda - f)^{l-\frac{1}{2}}}{\Gamma(l+\frac{1}{2})} \prod_{r=1}^{n} x_r \omega = -Q_1 Q_2 (\partial_\lambda)^{n+1} I^{(-l)}.$$

On the other hand

$$-\partial_\lambda Q_1 Q_2 (L, \Phi_{1,1}) = -\partial_\lambda Q_1 Q_2 I^{(-l+n)} = -Q_1 Q_2 (\partial_\lambda)^{n+1} I^{(-l)} = \partial_{\tau_1}(L, \Phi_{n,2})$$

In case 5), let us denote $l := -m - n + \frac{n}{2}$

$$\begin{aligned}
\partial_{\tau_1}(L, \Phi_{n,1}) =&\partial_{\tau_1}(-\partial_{\tau_1})^{n-1} I^{(-l)}(\tau, \lambda) \\
=&-(-\partial_{\tau_1})^n I^{(-l)}(\tau, \lambda)
\end{aligned}$$

by the same way we used in case 4) we get that

$$\partial_{\tau_1}(L, \Phi_{n,1}) = -(\partial_\lambda)^n \int \frac{(\lambda - f)^{l-\frac{1}{2}}}{\Gamma(l+\frac{1}{2})} \prod_{r=1}^{n} x_r \omega.$$

On the other hand,

$$\begin{aligned} -\partial_\lambda(L, Q_1\Phi_{1,2} - Q_1\Phi_{2,1}) &= -Q_1\partial_\lambda(\partial_{\tau_1} - \partial_{\tau_2})I^{-l+n-2} \\ &= -Q_1(\partial_\lambda)^{n-1}(\partial_{\tau_1} - \partial_{\tau_2})\int \frac{(\lambda-f)^{l-\frac{1}{2}}}{\Gamma(l+\frac{1}{2})}x_1\cdots x_n\omega. \\ &= -(\partial_\lambda)^{n-1}\int \frac{(\lambda-f)^{l-\frac{3}{2}}}{\Gamma(l-\frac{1}{2})}\omega. \\ &= -(\partial_\lambda)^n I^{(-l)}. \end{aligned}$$

Therefore

$$\partial_{\tau_1}(L, \Phi_{n,1}) = -\partial_\lambda(L, Q_1\Phi_{1,2} - Q_1\Phi_{2,1})$$

In case 6), let us denote $l := -m-n-1+\frac{n}{2}$. We have that

$$\partial_{\tau_2}(L, \Phi_{n,2}) = \partial_{\tau_2}(-\partial_{\tau_1})^{n-1}(-\partial_{\tau_2})I^{(-l)}(\tau,\lambda).$$

On the other hand

$$\begin{aligned} (-\partial_\lambda)(L, Q_2\Phi_{n,1} + Q_1Q_2\Phi_{1,1}) &= (-\partial_\lambda)\Big(Q_2(-\partial_{\tau_1})^{n-1}I^{(-l+1)} + Q_1Q_2I^{(-l+n)}\Big) \\ &= (-\partial_\lambda)\Big(Q_2(-\partial_{\tau_1})^{n-1}I^{(-l+1)} + Q_1Q_2(\partial_\lambda)^nI^{(-l)}\Big) \end{aligned}$$

By 4) we have that

$$(-\partial_\lambda)\Big(Q_1Q_2(\partial_\lambda)^nI^{(-l)}\Big) = (\partial_{\tau_2})(-\partial_{\tau_1})^nI^{(-l)}(\tau,\lambda)$$

Therefore,

$$(-\partial_\lambda)(L, Q_2\Phi_{n,1} + Q_1Q_2\Phi_{1,1}) = (-\partial_\lambda)Q_2(-\partial_{\tau_1})^{n-1}I^{(-l+1)} + (\partial_{\tau_2})(-\partial_{\tau_1})^nI^{(-l)}(\tau,\lambda)$$

Then by 3) we have that

$$\partial_{\tau_2}(L, \Phi_{n,2}) = (-\partial_\lambda)(L, Q_2\Phi_{n,1} + Q_1Q_2\Phi_{1,1})$$

□

$$\begin{aligned} \int_{u(\tau)}^{\infty} e^{\frac{\lambda}{z}}\Big(I_E^{-m-1}(0,Q,\lambda), \Phi_{i,j}\Big)\mathrm{d}\lambda &= (-\partial_{\tau_1}^{i-1})(-\partial_{\tau_2}^{j-1})\int_{u(\tau)}^{\infty} e^{\frac{\lambda}{z}}\Big(I^{-m-i-j+1+\frac{n}{2}}\Big)\mathrm{d}\lambda \\ &\overset{\text{by Lemma 3.9}}{=} (-\partial_{\tau_1}^{i-1})(-\partial_{\tau_2}^{j-1})\int_{\mathbb{R}^n_{>0}} e^{\frac{f(x,\tau)}{z}}\omega \\ &\overset{\text{by Theorem 3.1}}{=} (-\partial_{\tau_1})^{i-1}(-\partial_{\tau_2})^{j-1}(-z)^{m+i+j-1+\frac{1-n}{2}}(2\pi)^{\frac{n-1}{2}}(-z)^{\frac{n}{2}}\Big(S(0,Q,z)(-z)^{\theta}(-z)^{\rho}\Psi_\tau(\mathcal{O}), 1\Big) \\ &= (2\pi)^{\frac{n-1}{2}}(-z)^{m+\frac{3}{2}}(z\partial_{\tau_1})^{i-1}(z\partial_{\tau_2})^{j-1}\Big(S(0,Q,z)(-z)^{\theta}(-z)^{\rho}e^{-\tau_1p_1-\tau_2p_2}\Psi(\mathcal{O}), 1\Big) \\ &= (2\pi)^{\frac{n-1}{2}}(-z)^{m+\frac{3}{2}}(z\partial_{\tau_1})^{i-1}(z\partial_{\tau_2})^{j-1}\Big(S(0,Q,z)e^{\frac{\tau_1p_1+\tau_2p_2}{z}}(-z)^{\theta}(-z)^{\rho}\Psi(\mathcal{O}), 1\Big) \end{aligned}$$

By divisor equation

$$(z\partial_{\tau_2})^{j-1}(S(0,Q,z)e^{\frac{\tau_1p_1+\tau_2p_2}{z}}, 1) = (P_1^{i-1}P_2^{j-1}\cdot S(0,Q,z)e^{\frac{\tau_1p_1+\tau_2p_2}{z}}, 1).$$

Thus,

$$\int_{u(\tau)}^{\infty} e^{\frac{\lambda}{z}}\left(I_E^{-m-1}(0,Q,\lambda),\Phi_{i,j}\right)\mathrm{d}\lambda = (2\pi)^{\frac{n-1}{2}}(-z)^{m+\frac{3}{2}}\left(S(0,Q,z)e^{\frac{\tau_1 p_1+\tau_2 p_2}{z}}(-z)^{\theta}(-z)^{\rho}\Psi(\mathcal{O}),\Phi_{i,j}\right),$$

which means

$$\int_{u(\tau)}^{\infty} e^{\frac{\lambda}{z}} I_E^{-m-1}(0,Q,\lambda) = (2\pi)^{\frac{n-1}{2}} S(0,Q,z)e^{\frac{+\tau_1 p_1+\tau_2 p_2}{z}}(-z)^{\theta+m+\frac{3}{2}}(-z)^{\rho}\Psi(\mathcal{O})).$$

Let us consider that the RHS and LHS as the polynomial of τ_1 and τ_2, whose coefficients of $\tau_1^0\tau_2^0$ on two sides are equal.

$$\int_{u(\tau)}^{\infty} e^{\frac{\lambda}{z}}\sum_{k=0}^{\infty}(-1)^k S_k(0,Q)\widetilde{I}^{(-m-1+k)}(\lambda)\,\mathrm{d}\lambda\, E_0 = (2\pi)^{\frac{n-1}{2}} S(0,Q,z)(-z)^{\theta+m+\frac{3}{2}}(-z)^{\rho}\Psi(\mathcal{O}))$$

Let Q goes to 0 we have that

$$\int_{u(\tau)}^{\infty} e^{\frac{\lambda}{z}}\widetilde{I}^{-m-1}(\lambda)\,\mathrm{d}\lambda\, E_0 = (2\pi)^{\frac{n-1}{2}}(-z)^{\theta+m+\frac{3}{2}}(-z)^{\rho}\Psi(\mathcal{O}))$$

Let z be -1

$$\int_{u(\tau)}^{\infty} e^{-\lambda}e^{\partial_\lambda \partial_m}\frac{(\lambda)^{\theta+m+\frac{3}{2}}}{\Gamma(\theta+m+\frac{5}{2})}\,\mathrm{d}\lambda\, E_0 = (2\pi)^{\frac{n-1}{2}}\Psi(\mathcal{O})).$$

We also have that

$$\int_{u(\tau)}^{\infty} e^{-\lambda}e^{\partial_\lambda \partial_m}\frac{(\lambda)^{\theta+m+\frac{3}{2}}}{\Gamma(\theta+m+\frac{5}{2})}\,\mathrm{d}\lambda = 1$$

Which means that

$$E_0 = (2\pi)^{\frac{n-1}{2}}\Psi(\mathcal{O}))$$

THEOREM 3.11. $E_0 = (2\pi)^{\frac{n-1}{2}}\Psi(\mathcal{O}))$.

5. $\Psi_\tau(\mathcal{O})$ is a reflection vector

In this section we will prove that $\Psi_\tau(\mathcal{O})$ is reflection vector. According to Theorem 3.11 and Lemma 3.8, the analytic continuation of $I_E^{(-m-1)}(Q,\lambda)$ along simple loop around the real critical value $u(\tau_1,\tau_2)$ is $I_{-E}^{(-m-1)}(Q,\lambda)$. Therefore, E is proportional to a reflection vector. Since $E = (2\pi)^{\frac{n-1}{2}}\Psi(\mathcal{O}))$, in order to prove that $\Psi_\tau(\mathcal{O})$ is a reflection vector, we need only to check that $(\Psi_\tau(\mathcal{O})|\Psi_\tau(\mathcal{O})) = 2$.

Firstly,

$$\begin{aligned}
\langle\Psi_\tau(\mathcal{O}),\Psi_\tau(\mathcal{O})\rangle &= (2\pi)^{1-n}\frac{1}{2\pi}\left(\Psi_\tau(\mathcal{O}),e^{\pi i\theta}e^{\pi i\rho}\Psi_\tau(\mathcal{O})\right)\\
&= (2\pi)^{-n}\Big(e^{-p_1\tau_1-p_2\tau_2}\Gamma(1+p_1)^n\Gamma(1+p_2)\Gamma(1+p_2-p_1),\\
&\quad e^{-\pi i((n-1)p_1+2p_2)}e^{-p_1\tau_1-p_2\tau_2}\Gamma(1-p_1)^n\Gamma(1-p_2)\Gamma(1-p_2+p_1)e^{\pi i\theta}\Big)\\
&= (-2\pi i)^{-n}\int_{\mathrm{Bl}(\mathbb{P}^n)}\left(\frac{2\pi p_1}{e^{2\pi i p_1\pi}-1}\right)^n\left(\frac{2\pi p_2}{e^{2\pi i p_2\pi}-1}\right)\left(\frac{2\pi(p_2-p_1)}{e^{2\pi i(p_2-p_1)\pi}-1}\right)
\end{aligned}$$

Let us compute the integral, suppose $\varepsilon_1 < \varepsilon_2$, we have that

$$\begin{aligned}
&\mathrm{Res}_{p_2=0}\left(\mathrm{Res}_{p_1=0}\,\frac{f(p_1,p_2)}{p_1^n(p_2-p_1)p_2}\right)\\
=&\frac{1}{(2\pi i)^2}\int_{|p_2|=\varepsilon_2}\int_{|p_1|=\varepsilon_1}\frac{f(p_1,p_2)}{p_1^n(p_2-p_1)p_2}\\
=&\frac{1}{(2\pi i)^2}\int_{|p_1|=\varepsilon_1}\int_{|p_2|=\varepsilon_2}\frac{f(p_1,p_2)}{p_1^n(p_2-p_1)p_2}\\
=&\mathrm{Res}_{p_1=0}(\mathrm{Res}_{p_2=0}+\mathrm{Res}_{p_2=p_1})\frac{f(p_1,p_2)}{p_1^n(p_2-p_1)p_2}\\
=&\int_{\mathrm{Bl}(\mathbb{P}^n)} f(p_1,p_2).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{\mathrm{Bl}(\mathbb{P}^n)}\left(\frac{2\pi i p_1}{e^{2\pi i p_1\pi}-1}\right)^n\left(\frac{2\pi i p_2}{e^{2\pi i p_2\pi}-1}\right)\left(\frac{2\pi i(p_2-p_1)}{e^{2\pi i(p_2-p_1)\pi}-1}\right)\mathrm{dp}_1\,\mathrm{dp}_2\\
=&\mathrm{Res}_{p_2=0}\left(\mathrm{Res}_{p_1=0}\left(\frac{2\pi i}{e^{2\pi i p_1\pi}-1}\right)^n\left(\frac{2\pi i}{e^{2\pi i p_2\pi}-1}\right)\left(\frac{2\pi i}{e^{2\pi i(p_2-p_1)\pi}-1}\right)\mathrm{dp}_1\,\mathrm{dp}_2\right).
\end{aligned}$$

Let $e^{2\pi i p_1}-1=x$, we have that $dp_1=\frac{\mathrm{dx}}{2\pi(x+1)}$, which means that,

$$\begin{aligned}
&\mathrm{Res}_{p_2=0}\left(\mathrm{Res}_{p_1=0}\left(\frac{2\pi i}{e^{2\pi i p_1\pi}-1}\right)^n\left(\frac{2\pi i}{e^{2\pi i p_2\pi}-1}\right)\left(\frac{2\pi i}{e^{2\pi i(p_2-p_1)\pi}-1}\right)\mathrm{dp}_1\,\mathrm{dp}_2\right)\\
=&\mathrm{Res}_{p_2=0}\left(\mathrm{Res}_{p_1=0}\left(\frac{2\pi i}{x}\right)^n\left(\frac{2\pi i}{e^{2\pi i p_2\pi}-1}\right)\left(\frac{2\pi i(x+1)}{e^{2\pi i p_2\pi}-x-1}\right)\frac{1}{2\pi i(x+1)}\,\mathrm{dx}\,\mathrm{dp}_2\right)\\
=&\mathrm{Res}_{p_2=0}\left(\mathrm{Res}_{p_1=0}\left(\frac{2\pi i}{x}\right)^n\left(\frac{2\pi i}{e^{2\pi i p_2\pi}-1}\right)\left(\frac{1}{(e^{2\pi i p_2\pi}-1)-x}\right)\mathrm{dx}\,\mathrm{dp}_2\right).
\end{aligned}$$

Since $\varepsilon_1 < \varepsilon_2$, let $e^{2\pi i p_2}-1=y$ we get

$$\begin{aligned}
&\mathrm{Res}_{p_2=0}\left(\mathrm{Res}_{p_1=0}\left(\frac{2\pi i}{x}\right)^n\left(\frac{2\pi i}{e^{2\pi i p_2\pi}-1}\right)\left(\frac{1}{(e^{2\pi i p_2\pi}-1)-x}\right)\mathrm{dx}\,\mathrm{dp}_2\right)\\
=&\mathrm{Res}_{p_2=0}\left(\mathrm{Res}_{p_1=0}\left(\frac{2\pi i}{x}\right)^n\left(\frac{2\pi i}{y}\right)\left(\frac{1}{y-x}\right)\mathrm{dx}\,\mathrm{dp}_2\right)\\
=&\mathrm{Res}_{p_2=0}\left(\mathrm{Res}_{p_1=0}\left(\frac{2\pi i}{x}\right)^n\left(\frac{2\pi i}{y}\right)\left(\frac{1}{y}\left(\sum_{j=0}^{\infty}\left(\frac{x}{y}\right)^j\right)\right)\mathrm{dx}\,\mathrm{dp}_2\right)\\
=&\mathrm{Res}_{p_2=0}\left(\frac{2\pi}{y}\right)^{n+1}\frac{1}{y+1}\,\mathrm{dy}\\
=&(-2\pi i)^n.
\end{aligned}$$

Thus,

$$(\Psi_\tau(\mathcal{O})|\Psi_\tau(\mathcal{O})) = 2\langle\Psi_\tau(\mathcal{O}),\Psi_\tau(\mathcal{O})\rangle = 2.$$

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