Contents

Chapter 1

Background

In this chapter, we will introduce some general background for the purpose of narration of next several chapters in which we will deal with more specific situation.

1.1 Frobenius manifolds

1.1.1 Definition

There are many ways to introduce Frobenius manifolds. Here, it is convenient to choose a set of axioms. The general reference for more details is [Man99]. Our definition is equivalent to (Definition 1.2 in [Dub96]). Let M be a complex manifold and denote by $\mathcal{T}_M$ the sheaf of holomorphic vector fields on M. One may assume that M is equipped with the following structures

1. Each tangent space $T_tM, t \in M$, is equipped with the structure of a *Frobenius algebra* depending holomorphically on t. In other words, we have a commutative associative multiplication $\bullet_t$ and symmetric non-degenerate bi-linear pairing $(\ \ ,\ \ )_t$ satisfying the Frobenius property

$$(v_1 \bullet_t w, v_2) = (v_1, w \bullet_t v_2), \qquad v_1, v_2, w \in T_tM$$

   The pointwise multiplication $\bullet_t$ defines a multiplication $\bullet$ in $\mathcal{T}_M$, i.e., an $\mathcal{O}_M$-bilinear map

$$\mathcal{T}_M \otimes \mathcal{T}_M \to \mathcal{T}_M, \qquad v_1 \otimes v_2 \mapsto v_1 \bullet v_2.$$

   The pairing $(\ \ ,\ \ )_t$ determines a $\mathcal{O}_M$-bilinear pairing

$$(\ \ ,\ \ ) : \mathcal{T}_M \otimes \mathcal{T}_M \to \mathcal{O}_M.$$

2. There exists a global vector field $e \in \mathcal{T}_M$, called *unit vector field*, such that

$$\nabla^{\text{L.C.}}_v e = 0, \qquad e \bullet v = v, \qquad \forall v \in \mathcal{T}_M,$$

   where $\nabla^{\text{L.C.}}$ is the Levi-Civita connection on $\mathcal{T}_M$ corresponding to the bi-linear pairing $(\ \ ,\ \ )$.

3. There exists a global vector field $E \in \mathcal{T}_M$, called *Euler vector field*, such that

$$E(v_1, v_2) - ([E, v_1], v_2) - (v_1, [E, v_2]) = (2 - D)(v_1, v_2),$$

   for all $v_1, v_2 \in \mathcal{T}_M$ and for some constant $D \in \mathbb{C}$.

The above data allows us to define the so-called *structure connection* ∇ on the vector bundle $\mathrm{pr}_M^* TM \to M \times \mathbb{C}^*$, where

$$\mathrm{pr}_M : M \times \mathbb{C}^* \to M, \qquad (t, z) \mapsto t$$

is the projection map. Namely,

$$\begin{aligned} \nabla_v &:= \nabla_v^{\text{L.C.}} - z^{-1} v\bullet, \qquad v \in \mathcal{T}_M \\ \nabla_{\partial/\partial z} &:= \frac{\partial}{\partial z} - z^{-1}\theta + z^{-2} E\bullet, \end{aligned}$$

$v\bullet$ and $E\bullet$ are $\mathcal{O}_M$-linear maps $\mathcal{T}_M \to \mathcal{T}_M$ corresponding to the Frobenius multiplication by respectively v and E. The $\mathcal{O}_M$-linear map $\theta : \mathcal{T}_M \to \mathcal{T}_M$ is defined by

$$\theta(v) := \nabla_v^{\text{L.C.}} E - (1 - D/2)v.$$

The operator θ is sometimes called *Hodge grading operator*. Let us point out that the term $(1 - D/2)v$ in the definition of $\theta(v)$ is inserted so that θ becomes skew-symmetric with respect to the Frobenius pairing

$$(\theta(v_1), v_2) + (v_1, \theta(v_2)) = 0, \qquad v_1, v_2 \in \mathcal{T}_M.$$

Definition 1.1 The data $((\quad,\quad), \bullet, e, E)$ satisfying the conditions (1), (2) and (3) from above is said to be a *Frobenius structure* on M of *conformal dimension* D if the structure connection ∇ is flat.

Let us state the following properties without proof. Actually, the proof is straightforward argument in Riemannian Geometry. And we will do the same to the propositions and theorems in this chapter of Background.

Proposition 1.2 *Suppose that* $(M, (,), \bullet, e, E)$ *is a Frobenius structure. Then*

1. *The Levi-Civita connection* $\nabla^{L.C.}$ *is flat.*

2. *Let* $t = (t_1, \ldots, t_N)$ *be* $\nabla^{L.C.}$*-flat coordinates defined on a contractible open subset* $U \subset M$*. There exists a holomorphic function* $F \in \mathcal{O}_M(U)$*, such that*

$$(\partial/\partial t_a \bullet \partial/\partial t_b, \partial/\partial t_c) = \frac{\partial^3 F}{\partial t_a \partial t_b \partial t_c}$$

and

$$EF = (3 - D)F + H,$$

where H *is a polynomial in* $t_1, \ldots, t_N$ *of degree at most* 2*.*

3. *The Hodge grading operator is covariantly constant:* $\nabla^{L.C.}\theta = 0$*. In particular, in flat coordinates* $t = (t_1, \ldots, t_N)$ *the matrix* $(\theta_{ab})_{a,b=1}^N$ *of* θ *defined by*

$$\theta(\partial/\partial t_b) = \sum_{a=1}^{N} \theta_{ab} \partial/\partial t_a$$

is constant.

4. *The following identity holds*

$$[E, v \bullet w] - [E, v] \bullet w - v \bullet [E, w] = v \bullet w, \qquad v, w \in \mathcal{T}_M$$

1.1.2 Semi-simple Frobenius manifolds

Definition 1.3 A Frobenius manifold $(M, (\quad,\quad), \bullet, e, E)$ is said to be *semi-simple* if there are local coordinates $u = (u_1, \ldots, u_N)$ defined in a neighborhood of some point on M such that

$$\partial/\partial u_i \bullet \partial/\partial u_j = \delta_{ij} \partial/\partial u_j, \qquad 1 \leqslant i, j \leqslant N.$$

The coordinates u_i are called *canonical coordinates*.

As we will see now, canonical coordinates are unique up to permutation and constant shifts. To avoid cumbersome notation we put $\partial_{u_i} := \partial/\partial u_i$.

Proposition 1.4 *Let $u = (u_1, \ldots, u_N)$ be canonical coordinates defined on some open subset $U \subset M$. Then*

1. *The Frobenius pairing takes the form*

$$(\partial_{u_i}, \partial_{u_j}) = \delta_{ij}\eta_j(u), \qquad 1 \leqslant i, j \leqslant N,$$

where $\eta_j \in \mathcal{O}_M(U)$ and $\eta_j(u) \neq 0$ for all $u \in U$.

2. *The unit vector field takes the form $e = \sum_{i=1}^N \partial_{u_i}$.*

3. *The 1-form $\sum_{i=1}^N \eta_i(u)du_i$ is closed.*

4. *There are constants $c_i (1 \leqslant i \leqslant N)$ such that*

$$E = \sum_{i=1}^N (u_i + c_i)\partial_{u_i}.$$

The last part of the above proposition shows that in every canonical coordinate system up to some constant shifts the canonical coordinates coincide with the eigenvalues of the operator $E\bullet$. Therefore, up to constant shifts and permutations the canonical coordinates are uniquely determined. From now on we will work only with canonical coordinates such that

$$E = \sum_{i=1}^N u_i \partial_{u_i}.$$

The question that we would like to answer now is the following. Let us assume that U is an open subset of the universal cover T of Z_N and $\sum_{i=1}^N \eta_i(u)du_i$ is a closed 1-form on U. The tangent bundle of T and hence of U as well is trivial, because T is a contractible Stein manifold, so according to the Grauert-Oka principle every holomorphic vector bundle on T is trivial. Alternatively, we can prove that $\mathcal{T}_T$ is a free $\mathcal{O}_T$-module by using that the vector fields ∂_{u_i} of the configuration space Z_N lift naturally to vector fields on T and provide a global trivialization of $\mathcal{T}_T$. Using the 1-form we define a pairing

$$(\partial_{u_i}, \partial_{u_j}) = \delta_{ij}\eta_j(u).$$

Let us also define multiplication

$$\partial_{u_i} \bullet \partial_{u_j} = \delta_{ij}\partial_{u_j}$$

and vector fields

$$e = \sum_{i=1}^N \partial_{\partial_{u_i}}, \qquad E = \sum_{i=1}^N u_i \partial_{u_i}.$$

The problem then is to classify all 1-forms $\sum_{i=1}^N \eta_i(u)du_i$ such that the above data determines a Frobenius structure on U. The answer is given by the following theorem.

Theorem 1.5 *The closed 1-form $\sum_{i=1}^N \eta_i(u)du_i$ determines a Frobenius structure on U of conformal dimension D if and only if the following conditions are satisfied*

1. *$\eta_i(u) \neq 0$ for all i and for all $u \in U$.*

2. *$e\eta_i(u) = 0$ for all i.*

3. *$E\eta_i(u) = -D\eta_i(u)$.*

4. *For all $k \neq i \neq j \neq k$ we have*

$$\frac{\partial \eta_{ij}}{\partial u_k} = \frac{1}{2}\left(\frac{\eta_{ij}\eta_{kj}}{\eta_j} + \frac{\eta_{jk}\eta_{ik}}{\eta_k} + \frac{\eta_{ki}\eta_{ji}}{\eta_i}\right),$$

where $\eta_{ab}(u) := \partial_{u_a}\eta_b(u)$.

1.1.3 The second structure connection

Let U be a contractible open subset of the configuration space

$$Z_N = \{u \in \mathbb{C}^N : u_i \neq u_j \text{ for } i \neq j\}.$$

And we fix a point $u^\circ \in Z_N$. Suppose that U is equipped with a semi-simple Frobenius structure $((\quad,\quad), \bullet, e, E)$. Put $H = T_{u^\circ}U$ and let us trivialize the tangent bundle

$$TU \cong U \times H \cong U \times \mathbb{C}^N$$

using the Levi-Civita connection. In other words, we fix a basis $\{\phi_a\}_{a=1}^N$ of H and let $\partial_{t_a} \in \mathcal{T}_U$ be the flat vector field on U obtained by parallel transport with respect to the Levi-Civita connection. Then the isomorphisms (1.1) are given by the maps

$$(u, v) \in TU \mapsto (u, v_1\phi_1 + \cdots + v_N\phi_N) \in U \times H \mapsto (u, v_1, \ldots, v_N) \in U \times \mathbb{C}^N, \qquad (1.1)$$

where $v \in T_uU$ and $v =: v_1\partial_{t_1} + \cdots + v_N\partial_{t_N}$. The isomorphism (1.1) identifies the structure connection of the Frobenius structure with the flat connection on the trivial bundle

$$(U \times \mathbb{C}^*) \times \mathbb{C}^N \to U \times \mathbb{C}^*$$

defined by

$$\begin{aligned} \nabla_{\partial_{u_i}} &= \partial_{u_i} - z^{-1}P_i(u), \qquad 1 \leqslant i \leqslant N, \\ \nabla_{\partial_z} &= \partial_z - z^{-1}\theta + z^{-2}\mathcal{E}(u), \end{aligned}$$

where $P_i : U \to \mathfrak{gl}(\mathbb{C}^N)$ is a holomorphic map whose (a, b)-entry $P_{iab}(u)$ is defined by the identity

$$\partial_{u_i} \bullet \partial_{t_b} = \sum_{a=1}^N P_{iab}(u)\partial_{t_a},$$

$\mathcal{E} = \sum_{i=1}^N u_iP_i(u)$, and θ is a constant matrix whose (a, b)-entry θ_{ab} is defined by

$$\theta(\partial_{t_b}) = [\partial_{t_b}, E] - (1 - D/2)\partial_{t_b} =: \sum_{a=1}^N \theta_{ab}\partial_{t_a}.$$

In order to justify the definition of the second structure connection we make the following heuristic argument. Suppose that the structure connection has a solution

$$J : U \times \mathbb{C}^* \to \mathbb{C}^N$$

given by a Laplace transform

$$J(u, z) = \frac{(-z)^{n-\frac{1}{2}}}{\sqrt{2\pi}} \int_\Gamma e^{\lambda/z} I^{(n)}(u, \lambda) d\lambda$$

along an appropriate contour $\Gamma \subset \mathbb{C}$ of some $\mathbb{C}^N$-valued function $I^{(n)}(u, \lambda)$ holomorphic for all $(u, \lambda) \in U \times \Gamma$. Here $n \in \mathbb{C}$ is an arbitrary number. Assuming that the Laplace transform works, we would get that $J(u, z)$ is a solution to the structure connection if and only if $I^{(n)}(u, \lambda)$ is a solution to the following connection

$$\begin{aligned} \nabla^{(n)}_{\partial_{u_i}} &= \partial_{u_i} + (\lambda - \mathcal{E})^{-1}P_i(u)(\theta - n - 1/2), \qquad 1 \leqslant i \leqslant N, \\ \nabla^{(n)}_{\partial_\lambda} &= \partial_\lambda - (\lambda - \mathcal{E})^{-1}(\theta - n - 1/2). \end{aligned}$$

This is a connection on

$$(U \times \mathbb{C})' \times \mathbb{C}^N \to (U \times \mathbb{C})',$$

where

$$(U \times \mathbb{C})' = \{(u,\lambda) \in U \times \mathbb{C} | \det(\lambda - \mathcal{E}) \neq 0\}.$$

Proposition 1.6 *The connection $\nabla^{(n)}$ is flat for all $n \in \mathbb{C}$.*

Lemma 1.7 *Let $\tilde{\Psi}$ be the matrix whose (a,i)-entry is given by $\tilde{\Psi}_{ai} = \partial t_a / \partial u_i$. Then*

$$\tilde{\Psi}^{-1} P_i \tilde{\Psi} = E_{ii}, \qquad \tilde{\Psi}^{-1} \mathcal{E} \tilde{\Psi} = diag(u_1, \dots, u_N),$$

where E_{ii} is the matrix whose entry in position (i,i) is 1 and all other entries are 0.

Lemma 1.8 *Let $n \in \mathbb{C}$ be arbitrary. Then the matrix-valued functions*

$$A_i^{(n)}(u) := P_i(u)(\theta - n - 1/2), \qquad 1 \leqslant i \leqslant N,$$

satisfy the Schlesinger equations.

Proof Using Lemma 1.7 we get

$$(\lambda - \mathcal{E})^{-1} P_i (\theta - n - \frac{1}{2}) = \frac{A_i^{(n)}(u)}{\lambda - u_i}.$$

Therefore,

$$\nabla^{(n)}_{\partial_{u_i}} = \partial_{u_i} + \frac{A_i^{(n)}(u)}{\lambda - u_i}, \qquad 1 \leqslant i \leqslant N, \tag{1.2}$$

$$\nabla^{(n)}_{\partial_\lambda} = \partial_\lambda - \sum_{i=1}^{N} \frac{A_i^{(n)}(u)}{\lambda - u_i}. \tag{1.3}$$

It remains only to recall Proposition 1.6. □

1.2 Calibration

Let us fix any point $t^\circ \in M$. We will do something similar to the previous subsection. We fix a basis $\{\phi_a\}_{a=1}^N$ of $H := T_{t^\circ} M$ and let $\partial_{t_a} \in \mathcal{T}_M$ be the flat vector field obtained by parallel transport with respect to the Levi-Civita connection. We will get a simply connected flat coordinate (V,t), where V is a simply connected neighborhood of t° extended by the parallel transport. Then the isomorphisms (1.4) are given by the maps

$$(t,v) \in TV \mapsto (t, v_1\phi_1 + \cdots + v_N\phi_N) \in V \times H \mapsto (t, v_1, \dots, v_N) \in V \times \mathbb{C}^N, \tag{1.4}$$

where $v \in T_t V$ and $v =: v_1 \partial_{t_1} + \cdots + v_N \partial_{t_N}$. The isomorphism (1.4) identifies the structure connection of the Frobenius structure with the flat connection on the trivial bundle

$$(V \times \mathbb{C}^*) \times \mathbb{C}^N \to V \times \mathbb{C}^*$$

defined by

$$\nabla_{\partial_{t_i}} = \partial_{t_i} - z^{-1} A_i(t), \qquad 1 \leqslant i \leqslant N, \tag{1.5}$$

$$\nabla_{\partial_z} = \partial_z - z^{-1}\theta + z^{-2} E \bullet (t), \tag{1.6}$$

where $A_i : V \to \mathfrak{gl}(\mathbb{C}^N)$ is a holomorphic map whose (a,b)-entry $A_{iab}(u)$ is defined by the identity

$$\partial_{t_i} \bullet \partial_{t_b} = \sum_{a=1}^{N} A_{iab}(t) \partial_{t_a},$$

$E\bullet : V \to \mathfrak{gl}(\mathbb{C}^N)$ is derived from Frobenius multiplication by Euler vector field E, and θ is the constant matrix as before.

1.2.1 Definition and existence of calibration

We are going to prove that (1.6) admits an isomonodromic family of weak Levelt's solutions, i.e., near $z = \infty$ the system (1.5)-(1.6) admits a fundamental solution of the form

$$\Phi(t,z) = S(t,z) z^{\delta} z^{\nu},$$

where the matrices $S(t,z) = S_0 + S_1(t)z^{-1} + S_2(t)z^{-2} \ldots$ with S_0 constant (independent of t and z) invertible matrix, δ is a diagonalizable constant matrix and ν is a nilpotent constant matrix. Moreover, we will prove that there exists a fundamental solution such that $S_0 = 1$.

Substituting the fundamental series $\Phi(t,z)$ in (1.5) and comparing the coefficients in front of powers of z, we get that

$$\partial_{t_i} S_k = A_i S_{k-1}, \qquad \forall 1 \leqslant i \leqslant N, \qquad k \in \mathbb{Z}_{>0}. \tag{1.7}$$

Since structure connection is flat, concretely, $[\nabla_{\partial_{t_i}}, \nabla_{\partial_{t_j}}] = 0, \forall 1 \leqslant i,j \leqslant N$, 1-form $\sum_{i=1}^{N} A_i S_{k-1} dt_i$ is closed. As V is simply connected, we can integrate the 1-form and find that

$$S_k(t) = S_k(t^\circ) + \int_{t^\circ}^{t} \sum_{i=1}^{N} A_i S_{k-1} dt_i. \tag{1.8}$$

Therefore, it is sufficient to determine $S_k(t)$ for a fixed $t = t^\circ$. For neighborhood V, the values of $S_k(t)$ are determined from the flatness of structure connection according to formula (1.8).

Next, let us solve (1.6) at $t = t^\circ$. It is convenient to introduce the following notation. Let $\mathrm{spec}(\delta)$ be the set of eigenvalues of the operator

$$\mathrm{ad}_\delta : \mathfrak{gl}(H) \to \mathfrak{gl}(H), X \to [\delta, X].$$

Let us denote by $\mathfrak{gl}_a(H)$ the eigensubspace of ad_δ with eigenvalue a. Then we have a direct sum decomposition of vector spaces

$$\mathfrak{gl}(H) = \bigoplus_{a \in \mathrm{spec}(\delta)} \mathfrak{gl}_a(H).$$

Let us denote by $X_{[a]}$ the projection of X on $\mathfrak{gl}_a(H)$. The matrices S, δ, and ν are identified with elements of $\mathfrak{gl}(H)$ via the basis $\{\phi_i\}_{i=1}^{N} \subset H$ that we fixed above.

Substituting the fundamental series $\Phi(t,z)$ in (1.6) and comparing the coefficients in front of powers of z, we get that $\nu_{[-l]} = 0$ if $l \notin \mathbb{Z}_{\geqslant 0}$ and that

$$\theta = \delta + \nu_{[0]}, \tag{1.9}$$

$$k S_k + [\theta, S_k] = E \bullet S_{k-1} + \sum_{l=1}^{k} S_{k-l} \nu_{[-l]}, \qquad k > 0. \tag{1.10}$$

(1.9) uniquely determines δ and $\nu_{[0]}$: δ is diagonizable. $\nu_{[0]}$ is nilpotent and $[\delta, \nu_{[0]}] = 0$. So δ and $\nu_{[0]}$ are uniquely determined by the Jordan-Chevalley decomposition.

For (1.10), the left hand side is

$$(k + \mathrm{ad}_\delta + \mathrm{ad}_{\nu_{[0]}}) S_k = \sum_{a \in \mathrm{spec}(\delta)} (k + a + \mathrm{ad}_{\nu_{[0]}}) (S_k)_{[a]}$$

where summation is finite since the matrix vector space has finite dimension. Note that $\mathrm{ad}_{\nu_{[0]}}$ preserves the eigenspace of ad_δ since $[\delta, \nu_{[0]}] = 0$, we have

$$(k + a + \mathrm{ad}_{\nu_{[0]}})(S_k)_{[a]} = (E \bullet S_{k-1})_{[a]} + \sum_{l=1}^{k} (S_{k-l})_{[a+l]}\, \nu_{[-l]} \tag{1.11}$$

If $k + a \neq 0$, then

$$
\begin{aligned}
(k + a + \mathrm{ad}_{\nu_{[0]}})(S_k)_{[a]} &= (E \bullet S_{k-1})_{[a]} + \sum_{l=1}^{k-1} (S_{k-l})_{[a+l]} \nu_{[-l]} + (S_0)_{[a+k]} \nu_{[-k]} \\
&= (E \bullet S_{k-1})_{[a]} + \sum_{l=1}^{k-1} (S_{k-l})_{[a+l]} \nu_{[-l]}
\end{aligned}
$$

and the operator $(k + a + \mathrm{ad}_{\nu_{[0]}})$ is invertible.

$$
(k + a + \mathrm{ad}_{\nu_{[0]}})^{-1} = \frac{1}{k+a} \sum_{i=0}^{\infty} \left(-\frac{\mathrm{ad}_{\nu_{[0]}}}{k+a} \right)^i,
$$

where the summation over i is actually finite since $\nu_{[0]}$ is nilpotent and then operator $\mathrm{ad}_{\nu_{[0]}}$ is nilpotent as well. Hence, $(S_k)_{[a]}$ can be determined by $S_{k-l}, \nu_{[l]}, l = 1, 2, \ldots, k-1$.

If $k + a = 0$, then

$$
\mathrm{ad}_{\nu_{[0]}}((S_k)_{[-k]}) = (E \bullet S_{k-1})_{[-k]} + \nu_{[-k]} + \sum_{l=1}^{k-1} (S_{k-l})_{[a+l]} \nu_{[-l]}.
$$

There will be ambiguity in the choice of $(S_k)_{[-k]}$ since the operator $\mathrm{ad}_{\nu_{[0]}}$ is non-invertible and $\nu_{[-k]}$ has not been determined. Actually, the situation is somewhat the other way round. We may choose $(S_k)_{[-k]} \in \mathfrak{gl}_{-k}(H)$ arbitrarily to determined $\nu_{[-k]}$.

Proposition 1.9 *$S_k(t), k = 1, 2, \ldots$ determined by (1.8) do satisfy (1.10) for all $t \in V$ and thus $\nabla_{\partial_z} \Phi(t, z) = 0$ holds not only at $t = t^\circ$ but also on the neighborhood V.*

Proof Let us prove it by induction. Since θ is a constant matrix,

$$
\begin{aligned}
kS_k(t) + [\theta, S_k(t)] =& kS_k(t^\circ) + [\theta, S_k(t^\circ)] + \int_{t^\circ}^{t} \sum_{i=1}^{N} (kA_i S_{k-1} + \mathrm{ad}_\theta(A_i S_{k-1})) dt_i \\
=& kS_k(t^\circ) + [\theta, S_k(t^\circ)] + \int_{t^\circ}^{t} \sum_{i=1}^{N} (kA_i S_{k-1} + \mathrm{ad}_\theta(A_i) S_{k-1} + A_i \mathrm{ad}_\theta(S_{k-1})) dt_i
\end{aligned}
$$

Note that, by the flatness of structure connection, we have $[\nabla_{\partial_{t_i}}, \nabla_{\partial_z}] = 0, \forall 1 \leqslant i \leqslant N$ which yields relation $A_i + [\theta, A_i] = \partial_{t_i}(E\bullet), \forall 1 \leqslant i \leqslant N$. Thus,

$$
\begin{aligned}
kS_k(t) + [\theta, S_k(t)] =& kS_k(t^\circ) + [\theta, S_k(t^\circ)] + \int_{t^\circ}^{t} \sum_{i=1}^{N} (\partial_{t_i}(E\bullet) S_{k-1} + A_i(k - 1 + \mathrm{ad}_\theta)(S_{k-1})) dt_i \\
=& kS_k(t^\circ) + [\theta, S_k(t^\circ)] + (E \bullet S_{k-1})(t) - (E \bullet S_{k-1})(t^\circ) \\
&+ \int_{t^\circ}^{t} \sum_{i=1}^{N} (-E \bullet \partial_{t_i}(S_{k-1}) + A_i(k - 1 + \mathrm{ad}_\theta)(S_{k-1})) dt_i
\end{aligned}
$$

When $k = 1$, the integral vanishes. We have

$$
S_1(t) + [\theta, S_1(t)] - (E\bullet)(t) = S_1(t^\circ) + [\theta, S_1(t^\circ)] - (E\bullet)(t^\circ) = \nu_{[-1]}.
$$

Our inductive hypothesis is

$$
nS_n(t) + [\theta, S_n(t)] = (E \bullet S_{n-1})(t) + \sum_{l=1}^{n} S_{n-l}(t) \nu_{[-l]}, \qquad \forall t \in V
$$

holds for $n = k-1$. When $n = k$,

$$\begin{aligned}
kS_k(t) + [\theta, S_k(t)] =& kS_k(t^\circ) + [\theta, S_k(t^\circ)] + (E \bullet S_{k-1})(t) - (E \bullet S_{k-1})(t^\circ) \\
&+ \int_{t^\circ}^{t} \sum_{i=1}^{N} (-E \bullet \partial_{t_i}(S_{k-1}) + A_i(k-1+\mathrm{ad}_\theta)(S_{k-1}))dt_i \\
=& kS_k(t^\circ) + [\theta, S_k(t^\circ)] + (E \bullet S_{k-1})(t) - (E \bullet S_{k-1})(t^\circ) \\
&+ \int_{t^\circ}^{t} \sum_{i=1}^{N} (-E \bullet A_i(S_{k-2}) + A_i(E \bullet S_{k-2} + \sum_{l=1}^{k-1} S_{k-1-l}\nu_{[-l]}))dt_i
\end{aligned}$$

Therefore,

$$\begin{aligned}
& kS_k(t) + [\theta, S_k(t)] - (E \bullet S_{k-1})(t) \\
=& kS_k(t^\circ) + [\theta, S_k(t^\circ)] - (E \bullet S_{k-1})(t^\circ) + \int_{t^\circ}^{t} \sum_{i=1}^{N} \sum_{l=1}^{k-1} \partial_{t_i}(S_{k-l})\nu_{[-l]}dt_i \\
=& kS_k(t^\circ) + [\theta, S_k(t^\circ)] - (E \bullet S_{k-1})(t^\circ) + \sum_{l=1}^{k} (S_{k-l}(t) - S_{k-l}(t^\circ))\nu_{[-l]}
\end{aligned}$$

We finished the induction step. □

We will see that the arbitrariness of $(S_k)_{[-k]}, -k \in \mathrm{spec}(\delta)$ will be reduced if the weak Levelt solution satifies the symplectic condition $S(t,-z)^T S(t,z) = 1$ in the following proposition.

Proposition 1.10 *There exists a weak Levelt solution such that*

$$S(t,-z)^T S(t,z) = 1,$$

where ^{T} *is transposition with respect to the Frobenius pairing on* $H = T_{u^\circ}U$.

Proposition 1.10 is known if θ is diagonalizable (see [Dub99]). In fact, the polynomiality of the primitive form might be sufficient to prove that θ is daigonalizable. However, at this point this is unknown. Let us modify the argument from [Dub99] in order to cover the case of θ non-diagonalizable.

Proof Let us first point out that the projection $X \to X_{[a]}$ commutes with transposition, i.e., $(X_{[a]})^T = (X^T)_{[a]}$ for all $X \in GL(H)$ and $a \in \mathrm{spec}(\delta)$. This follows from the skew-symmetry of $\theta = \delta + \nu_{[0]}$ and uniqueness of the Jordan-Chevalley decomposition. Namely, δ and $\nu_{[0]}$ are skew-symmetric as well, and thus $(\mathrm{ad}_\delta(X))^T = \mathrm{ad}_\delta(X^T)$. So $^T : \mathfrak{gl}_a(H) \to \mathfrak{gl}_a(H)$ is a linear automorphism for all $a \in \mathrm{spec}(\delta)$. Our claim follows easily. In the rest of the proof we put $X_{[a]}^T := (X_{[a]})^T = (X^T)_{[a]}$. And we shall also use $(\mathrm{ad}_{\nu_{[0]}}(X))^T = \mathrm{ad}_{\nu_{[0]}}(X^T)$.

In order to prove the proposition, we will show that one can choose $(S_k)_{[-k]}$ (and thus $\nu_{[-k]}$) for $-k \in \mathrm{spec}(\delta)$ in such a way that

$$\sum_{i=0}^{k} (-1)^i S_{k-i}^T S_i = 0, \qquad \forall k \in \mathbb{Z}_{>0}.$$

Let us show the above equation by induction. When $k = 1$,

$$(1 + a + \mathrm{ad}_{\nu_{[0]}})(S_1)_{[a]} = (E \bullet S_0)_{[a]} + (S_0)_{[a+1]}\,\nu_{[-1]} = (E\bullet)_{[a]} + (\mathrm{id}_H)_{[a+1]}\nu_{[-1]}.$$

If $1 + a \neq 0$, then

$$(1 + a + \mathrm{ad}_{\nu_{[0]}})(S_1)_{[a]} = (E\bullet)_{[a]}$$

By taking transposition,

$$(1 + a + \mathrm{ad}_{\nu_{[0]}})(S_1^T)_{[a]} = (E\bullet)_{[a]}^T.$$

Since the matrix for the operator $E\bullet$ is symmetric with respect to the Frobenius transposition ^{T},

$$(1+a+\mathrm{ad}_{\nu_{[0]}})(S_1^T - S_1)_{[a]} = (E\bullet)^T_{[a]} - (E\bullet)_{[a]} = 0$$

Note that $(1+a+\mathrm{ad}_{\nu_{[0]}})$ is invertible in the case $1+a \neq 0$, we obtain

$$(S_1^T - S_1)_{[a]} = 0, \qquad \forall a \neq -1$$

If $a=-1$, then

$$\nu_{[-1]} = \mathrm{ad}_{\nu_{[0]}}(S_1)_{[-1]} - (E\bullet)_{[-1]},$$

where we fix $(S_1)_{[-1]} \in \mathfrak{gl}_{[-1]}(H)$ to be arbitrary solution to $(S_1^T - S_1)_{[-1]} = 0$.

Our inductive hypothesis is that $\left(\sum_{i=0}^{n}(-1)^i S_{n-i}^T S_i\right)_{[a]} = 0, \forall a \neq -n$ holds for all $n = 1, 2, \ldots, k-1$.

For all $a \neq -k$,

$$\begin{aligned}
&(k+a+\mathrm{ad}_{\nu_{[0]}})\left(\sum_{i=0}^{k}(-1)^i S_{k-i}^T S_i\right)_{[a]}\\
=&(k+a+\mathrm{ad}_{\nu_{[0]}})\left(S_k^T + (-1)^k S_k + \sum_{i=1}^{k-1}(-1)^i S_{k-i}^T S_i\right)_{[a]}\\
=&(k+a+\mathrm{ad}_{\nu_{[0]}})\left((S_k^T)_{[a]} + (-1)^k (S_k)_{[a]} + \sum_{i=1}^{k-1}(-1)^i \sum_{l\in\mathrm{spec}(\delta)} (S_{k-i}^T)_{[l]}(S_i)_{[a-l]}\right)\\
=&\left((E\bullet S_{k-1})^T_{[a]} + \sum_{j=1}^{k-1}\nu_{[-j]}^T\left(S_{k-j}\right)^T_{[a+j]}\right) + (-1)^k\left((E\bullet S_{k-1})_{[a]} + \sum_{j=1}^{k-1}\left(S_{k-j}\right)_{[a+j]}\nu_{[-j]}\right)\\
&+\sum_{i=1}^{k-1}(-1)^i\sum_{l\in\mathrm{spec}(\delta)}\left(((k-i+l+\mathrm{ad}_{\nu_{[0]}})(S_{k-i}^T)_{[l]})(S_i)_{[a-l]} + (S_{k-i}^T)_{[l]}(i+a-l+\mathrm{ad}_{\nu_{[0]}})(S_i)_{[a-l]}\right)\\
=&\left((E\bullet S_{k-1})^T_{[a]} + \sum_{j=1}^{k-1}\nu_{[-j]}^T\left(S_{k-j}\right)^T_{[a+j]}\right) + (-1)^k\left((E\bullet S_{k-1})_{[a]} + \sum_{j=1}^{k-1}\left(S_{k-j}\right)_{[a+j]}\nu_{[-j]}\right)\\
&+\sum_{i=1}^{k-1}(-1)^i\left(\left(\sum_{a-l\in\mathrm{spec}(\delta)}(((k-i+l+\mathrm{ad}_{\nu_{[0]}})(S_{k-i}^T)_{[l]})(S_i)_{[a-l]}\right) + \left(\sum_{l\in\mathrm{spec}(\delta)}(S_{k-i}^T)_{[l]}(i+a-l+\mathrm{ad}_{\nu_{[0]}})(S_i)_{[a-l]}\right)\right)\\
=&\left((E\bullet S_{k-1})^T_{[a]} + \sum_{j=1}^{k-1}\nu_{[-j]}^T\left(S_{k-j}\right)^T_{[a+j]}\right) + (-1)^k\left((E\bullet S_{k-1})_{[a]} + \sum_{j=1}^{k-1}\left(S_{k-j}\right)_{[a+j]}\nu_{[-j]}\right)\\
&+\sum_{i=1}^{k-1}(-1)^i\sum_{a-l\in\mathrm{spec}(\delta)}\left((E\bullet S_{k-i-1})^T_{[l]} + \sum_{j=1}^{k-i}\nu_{[-j]}^T\left(S_{k-i-j}\right)^T_{[j+l]}\right)(S_i)_{[a-l]}\\
&+\sum_{i=1}^{k-1}(-1)^i\sum_{l\in\mathrm{spec}(\delta)}(S_{k-i}^T)_{[l]}\left((E\bullet S_{i-1})_{[a-l]} + \sum_{j=1}^{i}\left(S_{i-j}\right)_{[a-l+j]}\nu_{[-j]}\right)
\end{aligned}$$

Let us check that the terms involving $E\bullet$ cancel out

$$\begin{aligned}
&(E\bullet S_{k-1})^T_{[a]} + \sum_{i=1}^{k-1}(-1)^i \sum_{a-l\in\mathrm{spec}(\delta)} (E\bullet S_{k-i-1})^T_{[l]}(S_i)_{[a-l]} \\
&+ (-1)^k(E\bullet S_{k-1})_{[a]} + \sum_{i=1}^{k-1}(-1)^i \sum_{l\in\mathrm{spec}(\delta)} (S^T_{k-i})_{[l]}(E\bullet S_{i-1})_{[a-l]} \\
=& \sum_{i=0}^{k-1}(-1)^i \sum_{a-l\in\mathrm{spec}(\delta)} (E\bullet S_{k-i-1})^T_{[l]}(S_i)_{[a-l]} + \sum_{i=1}^{k}(-1)^i \sum_{l\in\mathrm{spec}(\delta)} (S^T_{k-i})_{[l]}(E\bullet S_{i-1})_{[a-l]} \\
=& \sum_{i=0}^{k-1}(-1)^i \sum_{a-l\in\mathrm{spec}(\delta)} (E\bullet S_{k-i-1})^T_{[l]}(S_i)_{[a-l]} - \sum_{i=0}^{k-1}(-1)^i \sum_{l\in\mathrm{spec}(\delta)} (S^T_{k-i-1})_{[l]}(E\bullet S_i)_{[a-l]} \\
=& \sum_{i=0}^{k-1}(-1)^i \left((E\bullet S_{k-i-1})^T S_i\right)_{[a]} - \sum_{i=0}^{k-1}(-1)^i \left(S^T_{k-i-1}(E\bullet S_i)\right)_{[a]} \\
=& \sum_{i=0}^{k-1}(-1)^i \left((E\bullet S_{k-i-1})^T S_i - S^T_{k-i-1}(E\bullet S_i)\right)_{[a]} = 0.
\end{aligned}$$

Again, in the last step, we use that the matrix $E\bullet$ is symmetric with respect to the Frobenius transposition ^{T}.

Next, let us see the term involving ν^T

$$\begin{aligned}
&\sum_{j=1}^{k-1} \nu^T_{[-j]}\left(S_{k-j}\right)^T_{[a+j]} + \sum_{i=1}^{k-1}(-1)^i \sum_{a-l\in\mathrm{spec}(\delta)} \sum_{j=1}^{k-i} \nu^T_{[-j]}\left(S_{k-i-j}\right)^T_{[j+l]}(S_i)_{[a-l]} \\
=& \sum_{j=1}^{k-1} \nu^T_{[-j]}\left(S_{k-j}\right)^T_{[a+j]} + \sum_{j=1}^{k-1} \nu^T_{[-j]} \sum_{i=1}^{k-j}(-1)^i \sum_{a-l\in\mathrm{spec}(\delta)} \left(S_{k-i-j}\right)^T_{[j+l]}(S_i)_{[a-l]} \\
=& \sum_{j=1}^{k-1} \nu^T_{[-j]}\left(\left(S_{k-j}\right)^T_{[a+j]} + \sum_{i=1}^{k-j}(-1)^i \sum_{a-l\in\mathrm{spec}(\delta)} \left(S_{k-i-j}\right)^T_{[j+l]}(S_i)_{[a-l]}\right) \\
=& \sum_{j=1}^{k-1} \nu^T_{[-j]}\left(\sum_{i=0}^{k-j}(-1)^i \sum_{a-l\in\mathrm{spec}(\delta)} \left(S_{k-i-j}\right)^T_{[j+l]}(S_i)_{[a-l]}\right) \\
=& \sum_{j=1}^{k-1} \nu^T_{[-j]}\left(\sum_{i=0}^{k-j}(-1)^i \left(S^T_{k-i-j}S_i\right)_{[a+j]}\right) = 0.
\end{aligned}$$

In the last step, we use the induction hypothesis taking $n = k - j$ and the condition that $a + j \neq -n = j - k$ is also satisfied for the hypothesis.

The calculation for the term with ν is the same. Therefore,

$$(k + a + \mathrm{ad}_{\nu_{[0]}})\left(\sum_{i=0}^{k}(-1)^i S^T_{k-i}S_i\right)_{[a]} = 0$$

Since operator $(k + a + \mathrm{ad}_{\nu_{[0]}})$ is invertible for all $a \neq -k$, we have

$$\left(\sum_{i=0}^{k}(-1)^i S^T_{k-i}S_i\right)_{[a]} = 0, \qquad \forall a \neq -k$$

Setting $(S_k)_{[-k]} = -\frac{1}{2}\sum_{i=1}^{k-1}(-1)^{k-i}(S^T_{k-i}S_i)_{[-k]} + B_k$, where $B_k \in \mathfrak{gl}_{-k}(H)$ is arbitrary satisfying

$$B^T_k = -(-1)^k B_k, \tag{1.12}$$

as we mentioned, the arbitrariness of $(S_k)_{[-k]} \in \mathfrak{gl}_{-k}(H)$ is reduced. Combining the cases $a \neq -k$ and $a = -k$, we get

$$\sum_{i=0}^{k}(-1)^i S_{k-i}^T S_i = 0.$$

Namely, we finished the induction step.

Since the symplectic condition $S(t,-z)^T S(t,z) = 1$ is equivalent to

$$\sum_{i=0}^{k}(-1)^i S_{k-i}^T S_i = 0, \qquad \forall k \in \mathbb{Z}_{>0},$$

the proposition is proved. □

Then, it is the time to show the definition of calibrations (see [Giv01])

Definition 1.11 The operator series $S(t,z)$ satisfying the conditions of Proposition 1.10 are called *calibrations*, i.e., these are operator series of the form $1 + S_1(t)z^{-1} + \ldots$ such that $S(t,-z)^T S(t,z) = 1$.

Remark 1.12 The series $S(t,z)$ is convergent and it defines an analytic function for all (t,z) sufficiently close to (t°, ∞). Indeed, the series $S(t^\circ, z)$ is a solution to a differential equation that has regular singularity at $z = \infty$. Therefore, it is convergent and it defines an analytic function near $z = \infty$ (see [Ince, Section ???][Inc44]). The series $S(t,z)$ is a solution to a system of holomorphic differential equations in t depending holomorphically on the parameter z near $z = \infty$ and the initial condition at $t = t^\circ$ is also holomorphic at $z = \infty$. Therefore, $S(t,z)$ must be holomorphic as a function in (t,z) (see [Arnold, Theorem ???][Arn89]). Moreover, since $S(t,z)$ for $(t,z) \in M \times \mathbb{C}^*$ is a solution to a system of linear holomorphic differential equations, the calibration $S(t,z)$ can be extended analytically along any path in $M \times \mathbb{C}^*$ that starts at the point (t°, ∞).

1.2.2 Uniqueness of calibration

Lemma 1.13 *Let $\beta_1, \ldots, \beta_m \in \mathbb{R} \setminus \{0\}$ where they are pairwise distict and $C_1, \ldots, C_m \in \mathfrak{gl}(H)$. If limit $\lim_{t\to+\infty}(\sum_{i=1}^m C_i e^{\beta_i t\sqrt{-1}})$ exists, then $C_1 = \cdots = C_m = 0$.*

Proof Denote $\sum_{i=1}^m C_i e^{\beta_i t\sqrt{-1}}$ by $L(t)$. Pick an arbitrary number Δt from $\mathbb{R} \setminus \bigcup_{1\leqslant i<j\leqslant m} \frac{2\pi}{\beta_i-\beta_j}\mathbb{Q}$. Then

$$\sum_{i=1}^{m} C_i e^{\beta_i(t+j\Delta t)\sqrt{-1}} = L(t+j\Delta t), \qquad j = 0,1,\ldots,m-1,$$

which can be written in the form of a Vandermonde matrix acting on the vector

$$(C_1 e^{\beta_1 t\sqrt{-1}}, \ldots, C_m e^{\beta_m t\sqrt{-1}})^T.$$

The way of choosing Δt make sure that the determinant of Vandermonde matrix does not vanish. Thus, for any $i \in \{1,2,\ldots,m\}$, $C_i e^{\beta_i t\sqrt{-1}}$ is the linear combination of $L(t), L(t+\Delta t), L(t+(m-1)\Delta t)$ given by the inverse of the Vandermonde matrix. Since $\lim_{t\to+\infty} L(t)$ exists, $\lim_{t\to+\infty} C_i e^{\beta_i t\sqrt{-1}}$ exists as well. The only possibility is that $C_1 = C_2 = \cdots = C_m = 0$ □

Theorem 1.14 *Let us fix a calibration $S(t,z)$ which determines ν and let $\mathscr{S}$ be the set of all calibrations and $\mathscr{N}$ be the set of the nilpotent constant matrices determined by calibrations. Define a map in the following way*

$$G \to \mathscr{S} \times \mathscr{N}$$
$$C(z) \mapsto (S(t,z)C(z), C(1)^{-1}\nu C(1)) =: (S'(t,z), \nu')$$

where $G := \{C(z) = 1 + \sum_{m=1}^\infty C_m z^{-m} | C_m \in \mathfrak{gl}_{-m}(H), C(-z)^T C(z) = 1\}$. Then the map is one to one correspondence.

Proof First, let us check the well-definedness. The fundamental solution

$$S(t,z)z^{\delta}z^{\nu}C(1) = S(t,z)z^{\delta}C(1)z^{C(1)^{-1}\nu C(1)} = S(t,z)C(z)z^{\delta}z^{C(1)^{-1}\nu C(1)}$$

where $S(t,z)C(z)$ satisfies the symplectic condition if $S(t,z)$ and $C(z)$ do.

Comparing the coefficient of $z^{-1}, z^{-2}, \dots$, it is clear that the map is injective.

Next, our goal is to show the map is surjective.

Assume that there are two fundamental solutions and a constant invertible matrix C, such that

$$S'(t,z)z^{\delta}z^{\nu'} = S(t,z)z^{\delta}z^{\nu}C, \qquad S'(t,z), S(t,z) \in \mathscr{S} \quad \nu', \nu \in \mathscr{N}$$

We have

$$S(t,z)^{-1}S'(t,z) = z^{\delta}z^{\nu}Cz^{-\nu'}z^{-\delta} = z^{N(z)}C(z)z^{-N'(z)}$$

where $N(z) := z^{\delta}\nu z^{-\delta} = \nu_{[0]} + \sum_{l=1}^{\infty}\nu_{[-l]}z^{-l}$, $N'(z) := z^{\delta}\nu' z^{-\delta} = \nu_{[0]} + \sum_{l=1}^{\infty}\nu'_{[-l]}z^{-l}$ and $C(z) := z^{\delta}Cz^{-\delta}$. Then

$$S(t,z)^{-1}S'(t,z) = e^{N(z)\log z}C(z)e^{-N'(z)\log z} = C(z) + (N(z)C(z) - C(z)N'(z))\log z + \dots$$

Since $N(z)$ and $N'(z)$ are nilpotent matrices, the right hand side is a polynomial of $\log z$, denoting the degree of the polynomial by n. We claim that $C(z) \in 1 + \mathfrak{gl}(H)[z^{-1}]z^{-1}$. Let us move z around ∞ for one loop. Since $S(t,z), S'(t,z), N(z), N'(z)$ and $C(z)$ are series of z^{-1}, they remain the value after moving around the loop. We have

$$\sum_{i=1}^{n} P_i(z)((\log z + 2\pi\sqrt{-1})^i - (\log z)^i) = 0, \tag{1.13}$$

where $P_1(z) = N(z)C(z) - C(z)N'(z), \dots$. One may prove that given (1.13) holds for arbitrary $P_i(z) \in \mathfrak{gl}(H)[z^{-1}]$, then $P_1(z) = P_2(z) = \cdots = P_n(z) = 0$ by induction. Therefore, we have that $C(z) = S(t,z)^{-1}S'(t,z)$ is symplectic and that $N(z)C(z) = C(z)N'(z)$ and thus $\nu' = C(1)^{-1}\nu C(1)$. And that the map is surjective is proved.

Finally, We will prove the claim that $C(z) \in 1 + \mathfrak{gl}(H)[z^{-1}]z^{-1}$.

$$C(z) := z^{\delta}Cz^{-\delta} = C_{[0]} + \sum_{a \in \mathrm{spec}(\delta)\setminus\{0\}} C_{[a]}z^{a}, \qquad \tilde{m} := \max\{\mathrm{Re}(a) | a \in \mathrm{spec}(\delta)\setminus\{0\}, C_{[a]} \neq 0\}$$

While $\lim_{z\to+\infty} C(z) = \lim_{z\to+\infty} e^{-N(z)\log z}S(t,z)^{-1}S'(t,z)e^{N'(z)\log z} = 1$. Then if $\tilde{m} > 0$, $\lim_{z\to+\infty}\sum_{a\in\mathrm{spec}(\delta)} C_{[a]}z^{a-\tilde{m}} = 0$, which leads that $C_{[\tilde{a}]} = 0, \forall \tilde{a} \in \{a \in \mathrm{spec}(\delta)\setminus\{0\} | C_{[a]} \neq 0, \mathrm{Re}(a) = \tilde{m}\}$ via Lemma 1.13. If $\tilde{m} = 0$,

$$\lim_{z\to+\infty}(C_{[0]} + \sum_{a:\mathrm{Re}(a)=0, a\neq 0} C_{[a]}z^{a} + \sum_{a:\mathrm{Re}(a)<0} C_{[a]}z^{a}) = 1.$$

Similarly, $C_{[\tilde{a}]} = 0, \forall \tilde{a} \in \{a \in \mathrm{spec}(\delta)\setminus\{0\} | C_{[a]} \neq 0, \mathrm{Re}(a) = \tilde{m}\}$ via Lemma 1.13. These two contradictions give us $\tilde{m} < 0$ and that $C_{[0]} = 1$.

Let us move z around ∞ for one loop for the formula $S(t,z)^{-1}S'(t,z) = e^{N(z)\log z}C(z)e^{-N'(z)\log z}$ as we did. After multiplying $e^{-N(z)\log z}$ and $e^{N'(z)\log z}$ from left and right, respectively, we get

$$\sum_{a\in\mathrm{spec}(\delta)} e^{2\pi\sqrt{-1}N(z)}C_{[a]}z^{a}e^{2\pi\sqrt{-1}a}e^{-2\pi\sqrt{-1}N'(z)} = \sum_{a\in\mathrm{spec}(\delta)} C_{[a]}z^{a}.$$

If $\{a \in \mathrm{spec}(\delta)\setminus\mathbb{Z} | C_{[a]} \neq 0\}$ is not empty, denote $m' := \max\{\mathrm{Re}(a) | a \in \mathrm{spec}(\delta)\setminus\mathbb{Z}, C_{[a]} \neq 0\}$.

$$e^{2\pi\sqrt{-1}N(z)}\left(\sum_{a\in\mathbb{Z}} C_{[a]}z^{a} + \sum_{a\notin\mathbb{Z}, \mathrm{Re}(a)=m'} C_{[a]}z^{a}e^{2\pi\sqrt{-1}a} + \sum_{a\notin\mathbb{Z}, \mathrm{Re}(a)<m'} C_{[a]}z^{a}e^{2\pi\sqrt{-1}a}\right)e^{-2\pi\sqrt{-1}N'(z)}$$
$$= \sum_{a\in\mathbb{Z}} C_{[a]}z^{a} + \sum_{a\notin\mathbb{Z}, \mathrm{Re}(a)=m'} C_{[a]}z^{a} + \sum_{a\notin\mathbb{Z}, \mathrm{Re}(a)<m'} C_{[a]}z^{a}.$$

Comparing the coefficient of $a \in \mathrm{spec}(\delta) \setminus \mathbb{Z}, \mathrm{Re}(a) = m'$, we have

$$e^{2\pi\sqrt{-1}\nu_{[0]}} C_{[a]} e^{2\pi\sqrt{-1}a} e^{-2\pi\sqrt{-1}\nu_{[0]}} = C_{[a]}.$$

Due to the fact that $\nu_{[0]}$ is constant nilpotent matrix, we obtain $e^{2\pi\sqrt{-1}a} = 1$, which contradicts to $a \notin \mathbb{Z}$. Therefore $\{a \in \mathrm{spec}(\delta) \setminus \mathbb{Z} | C_{[a]} \neq 0\}$ is empty. Combining the previous argument, the claim is proved. □

1.3 Period vectors

The definition of the period map depends on the choice of a calibration of M. So we will use the notation of the previous section. Let us fix a reference point $(t^\circ, \lambda^\circ) \in (M \times \mathbb{C})' := \{(t, \lambda) | \det(\lambda - E\bullet_t) \neq 0\}$ such that λ° is a sufficiently large real number.

Proposition 1.15 *The following functions provide a fundamental solution to the 2nd structure connection*

$$I^{(n)}(t, \lambda) = \sum_{k=0}^{\infty} (-1)^k S_k(t) \tilde{I}^{(n+k)}(\lambda),$$

where

$$\tilde{I}^{(m)}(\lambda) = e^{-\sum_{l=0}^{\infty} \nu_{[-l]} (-\partial_\lambda)^l \partial_m} \left(\frac{\lambda^{\delta - m - \frac{1}{2}}}{\Gamma(\delta - m + \frac{1}{2})} \right).$$

Proof First, let us show that $\nabla^{(n)}_{\partial_\lambda} I^{(n)} = 0$.

$$\begin{aligned}(\lambda - E\bullet)\nabla^{(n)}_{\partial_\lambda} I^{(n)} =& (\lambda - E\bullet)\partial_\lambda I^{(n)} - (\theta - n - \frac{1}{2}) I^{(n)} \\ =& \sum_{k=0}^{\infty} (-1)^k S_k(t) \lambda \partial_\lambda \tilde{I}^{(n+k)}(\lambda) - \sum_{k=0}^{\infty} (-1)^k E \bullet \partial_\lambda S_k(t) \tilde{I}^{(n+k)}(\lambda) \\ & - \sum_{k=0}^{\infty} (-1)^k (\theta - n - \frac{1}{2}) S_k(t) \tilde{I}^{(n+k)}(\lambda)\end{aligned}$$

We may apply (1.10) to the last line and it will be

$$-(\theta - n - \frac{1}{2})\tilde{I}^{(n)}(\lambda) - \sum_{k=1}^{\infty} (-1)^k \left(S_k(t)(\theta - n - k - \frac{1}{2}) + E \bullet S_{k-1}(t) + \sum_{l=1}^{k} S_{k-l}(t) \nu_{[-l]} \right) \tilde{I}^{(n+k)}(\lambda).$$

Let us rearrange these two summation and shift indices of S such that, in the summation over k, there is only S_k and we will get

$$-\sum_{k=0}^{\infty} (-1)^k \left(S_k(\delta - n - k - \frac{1}{2}) \tilde{I}^{(n+k)} - E \bullet S_k \tilde{I}^{(n+k+1)} + \sum_{l=0}^{\infty} (-1)^l S_k \nu_{[-l]} \tilde{I}^{(n+k+l)} \right).$$

Note that $\partial_\lambda \tilde{I}^{(n+k)} = \tilde{I}^{(n+k+1)}$, the two terms with $E\bullet$ will cancel out each other. Similarly, $\tilde{I}^{(n+k+l)} = (\partial_\lambda)^l \tilde{I}^{(n+k)}$ and then

$$(\lambda - E\bullet)\nabla^{(n)}_{\partial_\lambda} I^{(n)} = \sum_{k=0}^{\infty} (-1)^k S_k(t) \left(\lambda \partial_\lambda - (\delta - n - k - \frac{1}{2}) - \sum_{l=0}^{\infty} \nu_{[-l]} (-\partial_\lambda)^l \right) \tilde{I}^{(n+k)}(\lambda).$$

Next we will show that $\left(\lambda \partial_\lambda - (\delta - m - \frac{1}{2}) - \sum_{l=0}^{\infty} \nu_{[-l]} (-\partial_\lambda)^l \right) \tilde{I}^{(m)}(\lambda) = 0$. An observation is that $(\lambda \partial_\lambda - (\delta - m - \frac{1}{2})) \left(\frac{\lambda^{\delta - m - \frac{1}{2}}}{\Gamma(\delta - m + \frac{1}{2})} \right) = 0$. So we want to commute $\lambda \partial_\lambda - (\delta - m - \frac{1}{2})$

with $-\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m$. The calculation process is the following

$$(\lambda\partial_\lambda-(\delta-m-\frac{1}{2}))(-\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m)=-\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m(\lambda\partial_\lambda-l)+\sum_{l=0}^{\infty}\nu_{[-l]}(\delta-l)(-\partial_\lambda)^l\partial_m$$
$$-\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l(\partial_m m-1)-\frac{1}{2}\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m.$$

The two term with $-l$ will cancel out each other.

$$(\lambda\partial_\lambda-(\delta-m-\frac{1}{2}))(-\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m)=-\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m(\lambda\partial_\lambda-(\delta-m-\frac{1}{2}))+\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l.$$

And

$$(\lambda\partial_\lambda-(\delta-m-\frac{1}{2}))e^{-\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m}=e^{-\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m}(\lambda\partial_\lambda-(\delta-m-\frac{1}{2}))+\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l e^{-\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m},$$

which yields $\left(\lambda\partial_\lambda-(\delta-m-\frac{1}{2})\right)\tilde{I}^{(m)}(\lambda)=\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\tilde{I}^{(m)}(\lambda)$. Since $\lambda-E\bullet$ is invertible on $(M\times\mathbb{C})'$, we finished the proof of $\nabla^{(n)}_{\partial_\lambda}I^{(n)}=0$.

Finally, let us show that $\nabla^{(n)}_{\partial_{t_i}}I^{(n)}=0$ with the help of $(\lambda-E\bullet)\nabla^{(n)}_{\partial_\lambda}I^{(n)}=0$. Let us consider

$$(\lambda-E\bullet)\nabla^{(n)}_{\partial_{t_i}}I^{(n)}=(\lambda-E\bullet)\partial_{t_i}I^{(n)}+\phi_i\bullet(\theta-n-\frac{1}{2})I^{(n)}.$$

In virtue of $(\lambda-E\bullet)\nabla^{(n)}_{\partial_\lambda}I^{(n)}=0$, i.e., $(\theta-n-\frac{1}{2})I^{(n)}=(\lambda-E\bullet)\partial_\lambda I^{(n)}$, we have

$$(\lambda-E\bullet)\nabla^{(n)}_{\partial_{t_i}}I^{(n)}=(\lambda-E\bullet)\partial_{t_i}I^{(n)}+\phi_i\bullet(\lambda-E\bullet)\partial_\lambda I^{(n)}=(\lambda-E\bullet)(\partial_{t_i}+\phi_i\bullet\partial_\lambda)I^{(n)}.$$

Again, we get $\nabla^{(n)}_{\partial_{t_i}}I^{(n)}=(\partial_{t_i}+\phi_i\bullet\partial_\lambda)I^{(n)}$. Then, recalling (1.7), we have

$$\nabla^{(n)}_{\partial_{t_i}}I^{(n)}=\sum_{k=0}^{\infty}(-1)^k\partial_{t_i}S_k(t)\tilde{I}^{(n+k)}(\lambda)+\sum_{k=0}^{\infty}(-1)^kA_i(t)S_k(t)\tilde{I}^{(n+k+1)}(\lambda)$$
$$=\sum_{k=0}^{\infty}(-1)^k\partial_{t_i}S_k(t)\tilde{I}^{(n+k)}(\lambda)-\sum_{k=0}^{\infty}(-1)^{k+1}\partial_{t_i}S_{k+1}(t)\tilde{I}^{(n+k+1)}(\lambda)=0.$$

Hence we finished the proof. □

The 2nd structure connection has a Fuchsian singularity at infinity, therefore the series $I^{(n)}(t,\lambda)$ is convergent for all (t,λ) sufficiently close to (t°,λ°). Using the differential equations we extend $I^{(n)}$ to a multi-valued analytic function on $(M\times\mathbb{C})'$. We define the following multi-valued functions taking values in H:

$$I^{(n)}_a(t,\lambda):=I^{(n)}(t,\lambda)a,\qquad a\in H,\qquad n\in\mathbb{C}$$

These functions will be called *period vectors*. Using analytic continuation we get a representation

$$\pi_1((M\times\mathbb{C})',(t^\circ,\lambda^\circ))\to\mathrm{GL}(H)$$

called the *monodromy representation* of the Frobenius manifold. The image W of the monodromy representation is called the *monodromy group*.

Under the semi-simplicity assumption, we may choose a generic reference point t° on M, such that the Frobenius multiplication $\bullet_{t^\circ}$ is semi-simple and the operator $E\bullet_{t^\circ}$ has N pairwise different eigenvalues $u_i^\circ(1\leqslant i\leqslant N)$. The fundamental group $\pi_1((M\times\mathbb{C})',(t^\circ,\lambda^\circ))$

fits into the following exact sequence

$$\pi_1(F^\circ,\lambda^\circ) \xrightarrow{i_*} \pi_1((M\times\mathbb{C})',(t^\circ,\lambda^\circ)) \to \pi_1(M,t^\circ) \to 1 \tag{1.14}$$

where $p:(M\times\mathbb{C})' \to M$ is the projection on M, $F^\circ = p^{-1}(t^\circ) = C\{u_1^\circ,\dots,u_N^\circ\}$ is the fiber over t°, and $i: F^\circ \to (M\times\mathbb{C})'$ is the natural inclusion. For a proof we refer to [Shi], Proposition 5.6.4 or [Nor83], Lemma 1.5 C. Using the exact sequence (1.14) we get that the monodromy group W is generated by the monodromy transformations representing the lifts of the generators of $\pi_1(M,t^\circ)$ in $\pi_1((M\times\mathbb{C})',(t^\circ,\lambda^\circ))$ and the generators of $\pi_1(F^\circ,\lambda^\circ)$.

The image of $\pi_1(F^\circ,\lambda^\circ)$ under the monodromy representation is a reflection group that can be described as follows. Using the differential equations of the 2nd structure connection it is easy to prove that the pairing

$$(a|b) := (I_a^{(0)}(t,\lambda),(\lambda - E\bullet)I_b^{(0)}(t,\lambda))$$

is independent of t and λ. This pairing is known as the *intersection pairing*. Suppose now that γ is a simple loop in F°, i.e., a loop that starts at λ°, approaches one of the punctures u_i° along a path γ' that ends at a point sufficiently close to u_i°, goes around u_i°, and finally returns back to λ° along γ'. By analyzing the second structure connection near $\lambda = u_i$ it is easy to see that up to a sign there exists a unique $a\in H$ such that $(a|a)=2$ and the monodromy transformation of a along γ is $-a$. The monodromy transformation representing $\gamma\in\pi_1(F^\circ,\lambda^\circ)$ is the reflection defined by the following formula:

$$w_a(x) = x - (a|x)a.$$

Let us denote by R the set of all $a\in H$ as above determined by all possible choices of simple loops in F°. We refer to the elements of R as reflection vectors.

1.3.1 Reflection vectors (Vanishing cycles)

In this section, we shall assume that $n\in\mathbb{Z}$. In the definition of the set R of reflection vectors that we gave just now, we fixed a semi-simple point $t^\circ\in M$ and moved λ in $\mathbb{C}-\{u_1^\circ,\dots,u_N^\circ\}$. On a neighborhood of t°, the semi-simplicity assumption and that $E\bullet_t$ has N pairwise different eigenvalues u_i still hold. Next we will find a fundamental solution $Y^{(i)}(u,\lambda)$ to differential equation $\nabla^{(n)}_{\partial/\partial\lambda}Y^{(i)}(u,\lambda)=0$ near $\lambda=u_i$. Let us denote by $y^{(j)}$ the jth of column vectors of the matrix $Y^{(i)}(u,\lambda)$,

$$y^{(j)} = (\lambda-u_i)^\alpha\Big(y_0^{(j)} + \sum_{k=1}^{\infty} y_k^{(j)}(\lambda-u_i)^k\Big),$$

where $y_0^{(j)}, y_k^{(j)}$ are column vectors depending on u. Let us see the coefficient of $(\lambda-u_i)^{\alpha-1}$ in the equation $\nabla^{(n)}_{\partial/\partial\lambda}y^{(j)}=0$ recalling (1.3) and then we have,

$$A_i^{(n)}(u)y_0^{(j)} = \alpha y_0^{(j)}$$

According to the definition of $A_i^{(n)}(u)$ and Lemma 1.7,

$$E_{ii}\tilde{\Psi}^{-1}(\theta - n - \frac{1}{2})\tilde{\Psi}(\tilde{\Psi}^{-1}y_0^{(j)}) = \alpha\tilde{\Psi}^{-1}y_0^{(j)}.$$

By direct calculation,

$$\det\left(\alpha - E_{ii}\tilde{\Psi}^{-1}(\theta - n - \frac{1}{2})\tilde{\Psi}\right) = \alpha^{N-1}\Big(\alpha - \left(\tilde{\Psi}^{-1}(\theta-n-\frac{1}{2})\tilde{\Psi}\right)_{ii}\Big).$$

Lemma 1.16 *Let η be diag$\{\eta_1,\ldots,\eta_N\}$. Then $(\tilde{\Psi}^{-1}\theta\tilde{\Psi})^T\eta = -\eta(\tilde{\Psi}^{-1}\theta\tilde{\Psi})$, where ^{T} represents the standard matrix transposition, and thus $\left(\tilde{\Psi}^{-1}(\theta-n-\frac{1}{2})\tilde{\Psi}\right)_{ii} = -n-\frac{1}{2}$.*

Proof Note that θ is skew symmetric with respect to the Frobenius pairing, i.e.,

$$(\theta(\partial_{u_i}),\partial_{u_j}) = -(\partial_{u_i},\theta(\partial_{u_j})),$$

where

$$\theta\partial_{u_i} = \sum_{a=1}^{N}\frac{\partial t_a}{\partial u_i}\theta(\partial_{t_a}) = \sum_{a,b=1}^{N}\frac{\partial t_a}{\partial u_i}\theta_{ba}\partial_{t_b} = \sum_{a,b,j=1}^{N}\tilde{\Psi}_{ai}\theta_{ba}(\tilde{\Psi}^{-1})_{jb}\partial_{u_j}.$$

Thus, $(\theta(\partial_{u_i}),\partial_{u_j}) = \sum_{a,b=1}^{N}\tilde{\Psi}_{ai}\theta_{ba}(\tilde{\Psi}^{-1})_{jb}\eta_j = \eta_j(\tilde{\Psi}^{-1}\theta\tilde{\Psi})_{ji}$. Similarly, $(\partial_{u_i},\theta(\partial_{u_j})) = \eta_i(\tilde{\Psi}^{-1}\theta\tilde{\Psi})_{ij}$. Therefore, $\eta_j(\tilde{\Psi}^{-1}\theta\tilde{\Psi})_{ji} = -\eta_i(\tilde{\Psi}^{-1}\theta\tilde{\Psi})_{ij}, 1\leqslant i,j\leqslant N$, and thus $(\tilde{\Psi}^{-1}\theta\tilde{\Psi})_{ii}=0$. □

We have eigenvalue $\alpha = 0$ and $\alpha = -n-\frac{1}{2}$ of $A_i^{(n)}(u)$. Since $\left(\tilde{\Psi}^{-1}(\theta-n-\frac{1}{2})\tilde{\Psi}\right)_{ii} = -n-\frac{1}{2}\neq 0$, we have $E_{ii}\tilde{\Psi}^{-1}(\theta-n-\frac{1}{2})\tilde{\Psi}$ and thus $A_i^{(n)}(u)$ are diagonalizable, namely, the dimension of eigenspace for eigenvalue $\alpha=0$ is $N-1$ and that for eigenvalue $\alpha=-n-\frac{1}{2}$ is 1. Let $y_0^{(j)}$ be the eigenvector of $A_i^{(n)}(u)$ for eigenvalue $\alpha=0$ if $j\neq i$ and let $y_0^{(i)}$ be that for eigenvalue $\alpha=-n-\frac{1}{2}$. We will see that $y^{(j)}$ is uniquely determined by $y_0^{(j)}$ for all $1\leqslant j\leqslant N$. For the case $j\neq i$, let us see the coefficient of $(\lambda-u_i)^{k+\alpha-1}$ in the equation $\nabla^{(n)}_{\partial/\partial\lambda}y^{(j)}=0$ recalling (1.3) and then we have,

$$\left(k-A_i^{(n)}(u)\right)y_k^{(j)} - \sum_{s\neq i}\left(\frac{A_s^{(n)}(u)}{\lambda-u_s}y^{(j)}\right)_{(\lambda-u_i)^{k-1}} = 0,$$

where $\frac{1}{\lambda-u_s} = \frac{1}{u_i-u_s}\frac{1}{1-\frac{\lambda-u_i}{u_s-u_i}} = -\sum_{l'=0}^{\infty}\frac{(\lambda-u_i)^{l'}}{(u_s-u_i)^{l'+1}}$. Then the above equation can be converted into

$$\left(k-A_i^{(n)}(u)\right)y_k^{(j)} = -\sum_{l'+l''=k-1,s\neq i,1\leqslant s\leqslant N}\frac{A_s^{(n)}(u)}{(u_s-u_i)^{l'+1}}y_{l''}^{(j)}$$

Since $\det\left(k-A_i^{(n)}(u)\right)\neq 0, \forall k\in\mathbb{Z}_{\geqslant 1}$, we can determine $y_k^{(j)}$ by $y_{l''}^{(j)}, l''=0,1,\ldots,k-1$. The same argument holds for the case $j=i$. Thus, we get that $Y^{(i)}(u,\lambda) = [y^{(1)}(u,\lambda),\ldots,y^{(N)}(u,\lambda)]$ is a fundamental solution for $\nabla^{(n)}_{\partial/\partial\lambda}Y^{(i)}(u,\lambda)=0$ near $\lambda=u_i$.

As $I^{(n)}(u,\lambda)$ is also a fundamental solution to $\nabla^{(n)}_{\partial/\partial\lambda}I^{(n)}(u,\lambda) = 0$, there is a matrix $C^{(i)}(u)$ depending on u such that $I^{(n)}(u,\lambda) = Y^{(i)}(u,\lambda)C^{(i)}(u), \forall 1\leqslant i\leqslant N$. Let us denote by φ_i the reflection vector with respect to eigenvalue u_i. The equation $I^{(n)}_{\varphi_i}(u,\lambda) = Y^{(i)}(u,\lambda)C^{(i)}(u)\varphi_i$ will be transformed into the following equation via analytical continuation around $\lambda=u_i$,

$$-I^{(n)}_{\varphi_i}(u,\lambda) = Y^{(i)}(u,\lambda)\mathrm{diag}\{1,\ldots,1,-1,1,\ldots,1\}C^{(i)}(u)\varphi_i,$$

where the -1 in the diagonal matrix is on the ith position. Then we have

$$-C^{(i)}(u)\varphi_i = \mathrm{diag}\{1,\ldots,1,-1,1,\ldots,1\}C^{(i)}(u)\varphi_i,$$

which means that $\varphi_i = \kappa_i C^{(i)}(u)^{-1}e_i, \quad \kappa_i\in\mathbb{C}$. Here e_i is is the column vector of which the ith entry is 1 and the rest are zeros. And one may solve κ_i via the equation $(\varphi_i|\varphi_i) = \kappa_i^2(C^{(i)}(u)^{-1}e_i|C^{(i)}(u)^{-1}e_i) = 2$.

Let us show that $(C^{(i)}(u)^{-1}e_i|C^{(i)}(u)^{-1}e_i)\neq 0$. Analytical continuation of the constant $(I_a^{(n)},(\lambda-E\bullet)I_b^{(-n)})$ around $\lambda=u_i$ is $(I^{(n)}_{w_i^{(n)}(a)},(\lambda-E\bullet)I^{(-n)}_{w_i^{(-n)}(b)})$, where $a,b\in T_{t^\circ}M$ and $w_i^{(n)}$

is monodromy of n around $\lambda = u_i$. As we discussed, $I^{(n)}_{w_i^{(n)}(a)} = I^{(n)}(\alpha - 2(C_n^{(i)}(u))^{-1}E_{ii}C_n^{(i)}(u)a)$. Thus, $(I_a^{(n)}, (\lambda - E\bullet)I_b^{(-n)}) = (I^{(n)}_{w_i^{(n)}(a)}, (\lambda - E\bullet)I^{(-n)}_{w_i^{(-n)}(b)})$ gives us

$$\begin{aligned} &-2(I^{(n)}(C_n^{(i)}(u))^{-1}E_{ii}C_n^{(i)}(u)a, (\lambda - E\bullet)I_b^{(-n)}) - 2(I_a^{(n)}, (\lambda - E\bullet)I^{(-n)}(C_{-n}^{(i)}(u))^{-1}E_{ii}C_{-n}^{(i)}(u)b) \\ &+ 4(I^{(n)}(C_n^{(i)}(u))^{-1}E_{ii}C_n^{(i)}(u)a, (\lambda - E\bullet)I^{(-n)}(C_{-n}^{(i)}(u))^{-1}E_{ii}C_{-n}^{(i)}(u)b) = 0 \end{aligned}$$

Take $a = (C_n^{(i)}(u))^{-1}e_i$ and $b = (C_{-n}^{(i)}(u))^{-1}e_j$, when $j \neq i$, $-2(I^{(n)}(C_n^{(i)}(u))^{-1}E_{ii}C_n^{(i)}(u)a, (\lambda - E\bullet)I_b^{(-n)}) = 0$, i.e., $(I_a^{(n)}, (\lambda - E\bullet)I_b^{(-n)}) = 0$. Then, since $(\cdot, \cdot)$ and $(\lambda - E\bullet)$ are non-degenerate, when $j = i$,

$$(I^{(n)}(C_n^{(i)}(u))^{-1}e_i, (\lambda - E\bullet)I^{(-n)}(C_{-n}^{(i)}(u))^{-1}e_j) \neq 0.$$

Letting $n = 0$, $(C^{(i)}(u)^{-1}e_i | C^{(i)}(u)^{-1}e_i) \neq 0$.

1.3.2 The ring of modular functions

Our main interest is in the period map

$$Z : ((M \times \mathbb{C})')^{\sim} \to H^*, \qquad (t, \lambda) \to Z(t, \lambda)$$

where $((M \times \mathbb{C})')^{\sim}$ is the universal cover of $(M \times \mathbb{C})'$ and $Z(t, \lambda) \in H^*$ is defined by

$$\langle Z(t, \lambda), \alpha \rangle := Z_\alpha(t, \lambda) = (I_\alpha^{(-1)}(t, \lambda), 1).$$

The flow of the unit vector field $\mathbf{1}$ defines a free action of $\mathbb{C}$ on M

$$\mathbb{C} \times M \to M, \qquad (x, t) \mapsto t + x\mathbf{1}.$$

Let us identify the orbit space $B := M/\mathbb{C}$ with the submanifold $\{t_1 = 0\} \subset M$. Then we have an isomorphism

$$\mathbb{C} \times B \simeq M, \qquad (x, t) \mapsto t + x\mathbf{1}.$$

The period map has the following translation symmetry

$$Z(t, \lambda) = Z(t - \lambda\mathbf{1}, 0).$$

Therefore, we will restrict our analysis to the case $t_1 = 0$, i.e., we will assume that $t \in B$ and that the period map is defined on the universal cover of

$$X := (B \times \mathbb{C})' = \{(t, \lambda) \in B \times \mathbb{C} | \det(\lambda - E\bullet) \neq 0\}.$$

Let us denote by $\Omega \subset H^*$ the image of the period map Z. This is a W-invariant subset which will be called the *period domain*. In general very little is known about such period domains. For example it would be interesting to classify semi-simple Frobenius manifolds such that the action of W on Ω is properly discontinuous and the quotient $[\Omega/W]$ is an orbifold whose coarse moduli space is isomorphic to the Frobenius manifold M. Furthermore, we would like to introduce the ring of *modular functions*

$$\mathcal{M}(\Omega, W) := \{f \in \Gamma(\Omega, \mathcal{O}_{H^*})^W | f \circ Z \in \mathcal{O}(B \times \mathbb{C})\},$$

where $\Gamma(\Omega, \mathcal{O}_{H^*})^W$ is the ring of W-invariant holomorphic functions in Ω. Note that in general if $f \in \Gamma(\Omega, \mathcal{O}_{H^*})^W$ is an arbitrary function, then the composition $f \circ Z$ defines a holomorphic function on $(B \times \mathbb{C})'$. The condition in the above definition requires that $f \circ Z$ extends analytically across the discriminant.

Definition 1.17 The period map Z is said to be *invertible* if there exists a set of modular functions $f_i \in \mathcal{M}(\Omega, W)(1 \leqslant i \leqslant N)$ such that the set of holomorphic functions $f_i \circ Z(1 \leqslant$

$i \leqslant N$) is a coordinate system on $B \times \mathbb{C}$. A set of such modular functions $\{f_i\}_{i=1}^N$ is called the *inverse of the period map*.

There are two reasons why we are interested in finding the inverse of the period map. The first one is related to the discussion above. We expect that if the period map is invertible then the corresponding modular functions f_i will give a complete set of recursion relations, which would allow us to determine the genus-0 total descendant potential in terms of the monodromy data of the Frobenius manifold via an explicit recursion. The second reason is related to the problem of uniformizing a semi-simple Frobenius manifold. We expect that semi-simple Frobenius manifolds relevant in the study of mirror symmetry are quotients of a simply connected domain by a discrete group. At this point we can only speculate, but we believe that the problem of uniformizing the Frobenius manifold corresponding to the quantum cohomology of some smooth projective variety X is related to the problem of constructing the manifold of stability conditions of the bounded derived category $D^b(\mathrm{Coh}X)$.

Chapter 2

Classification of semi-simple Frobenius manifold M

In this chapter, let M be a semi-simple Frobenius manifold in the form of $\mathbb{C} \times B$ where B is a Riemann surface. And we assume that $x := (x_1, x_2) \in \mathbb{C} \times B =: M$ and $\frac{\partial}{\partial x_1} \in \mathcal{T}_M(M)$ is the unit vector field of Frobenius manifold M. We shall first show that if we have a local biholomorphism from one complex manifold to a Frobenius manifold then we can get a Frobenius structure on the complex manifold by taking the pullback of the Frobenius structure on the image. Then we will give the definition of *maximal Frobenius manifold* based on the previous statement. Finally, we shall classify M, in the sense that, for all M, its Frobenius structure is a pullback of that of one of three types of *maximal Frobenius manifolds*. The main theorem of this chapter is placed at the end.

Proposition 2.1 (Pullback of Frobenius structure by local biholomorphism) *Let $\phi : X \to Y$ be a local biholomorphism, namely, a holomorphism with non-degenerate tangent map, where X is a complex manifold and Y is a Frobenius manifold with Frobenius structure $(g_Y(\cdot,\cdot), \bullet_Y, E_Y, e_Y)$ and* $\dim_{\mathbb{C}}(X) = \dim_{\mathbb{C}}(Y)$. *Define $\forall v_1, v_2 \in T_x X$,*

$$\begin{aligned} g_{X,x}(v_1, v_2) :=& g_{Y,\phi(x)}(d_x\phi(v_1), d_x\phi(v_2)), \\ v_1 \bullet_{X,x} v_2 :=& (d_x\phi)^{-1}(d_x\phi(v_1) \bullet_{Y,\phi(x)} d_x\phi(v_2)), \\ E_{X,x} :=& (d_x\phi)^{-1}(E_{Y,\phi(x)}), \\ e_{X,x} :=& (d_x\phi)^{-1}(e_{Y,\phi(x)}). \end{aligned}$$

Then $(g_X(\cdot,\cdot), \bullet_X, E_X, e_X)$ is a Frobenius structure on X.

Proof Since ϕ is a local biholomorphism, for any $x \in X$, there is an open neighborhood U_x of x such that $\phi|_{U_x}$ is a biholomorphism. Besides, $d_x\phi\colon T_xX \to T_{\phi(x)}Y$ is an isomorphism. Thus, we have a structure on each point of U_x defined in the proposition and that $\phi|_{U_x}$ is biholomorphism guarantees the structure on U_x is Frobenius structure. Thanks to the arbitrariness of $x \in X$, $(g_X(\cdot,\cdot), \bullet_X, E_X, e_X)$ is a Frobenius structure on X. □

If ϕ is a surjective local biholomorphism and Y is a semi-simple Frobenius manifold, then let $y \in Y$ be a semi-simple point and $x \in \phi^{-1}(y)$, then $\phi|_{U_x}$'s being biholomorphism ensure that x is a semisimple point as well.

Remark 2.2 The pullback of semi-simple Frobenius structure via a surjective local biholomorphism is also a semi-simple Frobenius structure.

Definition 2.3 (maximal Frobenius manifold) Frobenius manifold X is called *maximal* if its Frobenius structure is not a pullback of that on another Frobenius manifold.

For any $M := \mathbb{C} \times B$ that we assumed at the beginning of the chapter, Let us take its universal cover

$$\tilde{M} := \mathbb{C} \times \tilde{B} \xrightarrow{(\mathrm{id}_{\mathbb{C}}, \pi)} \mathbb{C} \times B$$

where $\tilde{B}$ is the univeral cover of B and $B = (\tilde{B}/\Gamma)$. Here Γ is a discrete subgroup of $\mathrm{Aut}(\tilde{B})$ such that B is a Riemann surface. And we have $\pi \circ \gamma = \pi, \forall \gamma \in \Gamma$. Besides, the uniformization theorem tells us that $\tilde{B}$ is comformally equivalent to one of three Riemann surfaces: the upper half plane $\mathbb{H}$, the complex plane $\mathbb{C}$ and the Riemann sphere $\mathbb{CP}^1$. Meanwhile since we can extend the flat coordinate on the simply connected $\tilde{M}$ and $\mathbb{C} \times \mathbb{CP}^1$ has no trivial tangent bundle. $\tilde{B}$ must be $\mathbb{H}$ or $\mathbb{C}$ only.

Apparently, the univeral covering map $(\mathrm{id}_{\mathbb{C}}, \pi)$ is a surjective local biholomorphism. Due to Remark 2.2, $\tilde{M}$ is a semi-simple Frobenius manifold as well by the pullback of $(\mathrm{id}_{\mathbb{C}}, \pi)$. Let us study its cannonical coordinates.

2.1 Canonical coordinates on $\tilde{M}$

Choose a simply connected open subset U of $\tilde{M}$ that admits canonical coordiates $u = (u_1, u_2)$. There is a transform of coordinates, i.e. local biholomorphism $(u_1(\tilde{x}_1, \tilde{x}_2), u_2(\tilde{x}_1, \tilde{x}_2))$, where $\tilde{x}_1 \in \mathbb{C}$ and $\tilde{x}_2 \in \tilde{B} = \mathbb{H}$ or $\mathbb{C}$ and thus $(\tilde{x}_1, \tilde{x}_2)$ are global coordinates. According to chain rule, we have

$$\frac{\partial}{\partial \tilde{x}_1} = \frac{\partial u_1}{\partial \tilde{x}_1}\frac{\partial}{\partial u_1} + \frac{\partial u_2}{\partial \tilde{x}_1}\frac{\partial}{\partial u_2}$$

Note that $\frac{\partial}{\partial \tilde{x}_1}$ is the unit vector field and u is canonical coordinates. We have

$$\frac{\partial u_1}{\partial \tilde{x}_1} = \frac{\partial u_2}{\partial \tilde{x}_1} \equiv 1.$$

Thus, we may assume

$$u_1 = \tilde{x}_1 + g_1(\tilde{x}_2)$$
$$u_2 = \tilde{x}_1 + g_2(\tilde{x}_2)$$

where $g_1, g_2 \in \mathcal{O}_{\tilde{M}}(U)$, $g_1 - g_2$ nowhere vanishes and $\frac{\partial}{\partial \tilde{x}_1}(g_1 - g_2) = 0$. Since $\frac{\partial}{\partial \tilde{x}_2} = \frac{\partial g_1}{\partial \tilde{x}_2}\partial_{u_1} + \frac{\partial g_2}{\partial \tilde{x}_2}\partial_{u_2}$,

$$\begin{aligned}\frac{\partial}{\partial \tilde{x}_2} \bullet \frac{\partial}{\partial \tilde{x}_2} &= \left(\frac{\partial g_1}{\partial \tilde{x}_2}\right)^2 \partial_{u_1} + \left(\frac{\partial g_2}{\partial \tilde{x}_2}\right)^2 \partial_{u_2} \\ &= \tilde{a}\frac{\partial}{\partial \tilde{x}_1} + \tilde{b}\frac{\partial}{\partial \tilde{x}_2} = \tilde{a}(\partial_{u_1} + \partial_{u_2}) + \tilde{b}\left(\frac{\partial g_1}{\partial \tilde{x}_2}\partial_{u_1} + \frac{\partial g_2}{\partial \tilde{x}_2}\partial_{u_2}\right)\end{aligned}$$

where $\tilde{a}$ and $\tilde{b}$ are holomorphic function on $\tilde{M}$. By solving the following equation for $\tilde{b}$,

$$\left(\frac{\partial g_1}{\partial \tilde{x}_2}\right)^2 = \tilde{a} + \tilde{b}\frac{\partial g_1}{\partial \tilde{x}_2},$$
$$\left(\frac{\partial g_2}{\partial \tilde{x}_2}\right)^2 = \tilde{a} + \tilde{b}\frac{\partial g_2}{\partial \tilde{x}_2},$$

we get $\tilde{b} = \frac{\partial}{\partial \tilde{x}_2}(g_1 + g_2) \in \mathcal{O}(\tilde{M})$, which implies $g_1 + g_2 \in \mathcal{O}(\tilde{M})$ as $\tilde{M}$ is simply connected.

Taking the following biholomorphism from $\tilde{M}$ to itself

$$\begin{aligned} x_1 &= \tilde{x}_1 + \frac{g_1 + g_2}{2} \\ x_2 &= \tilde{x}_2, \end{aligned}$$

then canonical coordinates will be

$$u_1 = x_1 - \frac{g_2 - g_1}{2}$$
$$u_2 = x_1 + \frac{g_2 - g_1}{2}$$

Setting $g(x_2) := \frac{g_2 - g_1}{2} \in \mathcal{O}_{\tilde{M}}(U)$, finally, canonical coordinates is in the form

$$u_1 = x_1 - g(x_2) \tag{2.1}$$
$$u_2 = x_1 + g(x_2). \tag{2.2}$$

We will call it canonical choice of coordinate in Theorem 2.4

Recall the notation about semi-simple Frobenius manifolds in (1.1.2), we have canonical cooridinates

$$(\partial_{u_i}, \partial_{u_j}) = \delta_{ij}\eta_i(u), \partial_{u_i} := \frac{\partial}{\partial u_i}, \qquad i, j = 1, 2$$

And by Proposition 1.4 and Theorem 1.5, noting that $N = 2$, (4) of Theorem 1.5 is trivial, we obtain

$$(\partial_{u_1} + \partial_{u_2})\eta_i = 0 \tag{2.3}$$
$$(u_1\partial_{u_1} + u_2\partial_{u_2})\eta_i = -D\eta_i \tag{2.4}$$

where D is conformal dimension. Then we get

$$\frac{\partial_{u_1}\eta_i}{\eta_i} du_1 = \frac{-D}{u_2 - u_1} d(-u_1)$$
$$\frac{\partial_{u_2}\eta_i}{\eta_i} du_2 = \frac{-D}{u_2 - u_1} d(u_2)$$

Summing up the above equation, we get

$$\frac{d\eta_i}{\eta_i} = \frac{-D}{u_2 - u_1} d(u_2 - u_1) = \frac{-D}{g(x_2)} dg(x_2)$$

Integrate the differential equation related to η_i (Here we use the condition that U is simply connected). And since $\frac{\partial \eta_i}{\partial x_1} = (\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2})\eta_i = 0$, we get

$$\eta_i = c_i e^{-D \ln g(x_2)} = c_i g^{-D}, \qquad c_i \in \mathbb{C}^*$$

By Proposition 1.4, we have $\partial_{u_1}\eta_2 = \partial_{u_2}\eta_1$. Take this relation into (2.3), we have $\partial_{u_1}(\eta_1 + \eta_2) = \partial_{u_2}(\eta_1 + \eta_2) = 0$. Thus, $\eta_1 + \eta_2$ is a complex constant.

Similarly, $\partial_{x_i} := \frac{\partial}{\partial x_i}$. Frobenius pairing is

$$(\partial_{x_1}, \partial_{x_1}) = \eta_1 + \eta_2 = (c_1 + c_2)g^{-D}$$
$$(\partial_{x_1}, \partial_{x_2}) = (c_2 - c_1)g' g^{-D}$$
$$(\partial_{x_2}, \partial_{x_2}) = (c_1 + c_2)(g')^2 g^{-D}$$

where $g' := \frac{\partial g}{\partial x_2}$. The determinant of the Frobenius pairing matrix is $4c_1c_2(g'g^{-D})^2$. Thanks to the non-degeneracy of Frobenius pairing, $(g'g^{-D})^2 \in \mathcal{O}_B(B)$ has no zeros.

Euler vector field and Frobenius pairing are

$$E = x_1\partial_{x_1} + \frac{g}{g'}\partial_{x_2}$$
$$\partial_{x_2} \bullet \partial_{x_2} = (g')^2\partial_{x_1}$$

2.2 $D \neq 0$ case

Since $D \neq 0$, $c_1 + c_2 = 0$ and denote $2c_2 = c \neq 0$.

Theorem 2.4 *Let $((\cdot,\cdot), \bullet, e = \partial_{x_1}, E)$ semi-simple Frobenius structure on $\tilde{M}$. Then there is a non-zero complex constant c and a canonical choice of coordinates (x_1, x_2) such that*

1. *If $\frac{g}{g'} \in \mathcal{O}(\tilde{B})$ has no zero, then the Frobenius structure has the form*

$$(\partial_{x_i}, \partial_{x_j}) = \delta_{i+j,3} \cdot c \cdot G' e^{(1-D)G}, \tag{2.5a}$$
$$\partial_{x_2} \bullet \partial_{x_2} = (G')^2 e^{2G} \partial_{x_1}, \tag{2.5b}$$
$$E = x_1 \partial_{x_1} + \frac{1}{G'} \partial_{x_2}, \tag{2.5c}$$

where $G \in \mathcal{O}(\tilde{B})$ and G' has no zero.

2. *If $\frac{g}{g'} \in \mathcal{O}(\tilde{B})$ has at least one zero, then $D = 1 - \frac{2}{q}$ for some $q \in \mathbb{Z}_{\geqslant 2}$*

$$(\partial_{x_i}, \partial_{x_j}) = \delta_{i+j,3} \cdot c \cdot \frac{q}{2} H'(x_2), \tag{2.6a}$$
$$\partial_{x_2} \bullet \partial_{x_2} = \frac{q^2}{4} (H'(x_2))^2 (H(x_2))^{q-2} \partial_{x_1}, \tag{2.6b}$$
$$E = x_1 \partial_{x_1} + \frac{2}{q} \frac{H(x_2)}{H'(x_2)} \partial_{x_2}, \tag{2.6c}$$

where $H \in \mathcal{O}(\tilde{B})$ and H' has no zero.

Proof The existence of the canonical choice of coordinates (x_1, x_2) was shown in the previous section.

1. Due to the assumption, $\frac{g'}{g} = (\log g)' \in \mathcal{O}_{\tilde{B}}(\tilde{B})$. Since $\tilde{B}$ is simply connected,

$$g(x_2) = g(x_2^\circ) \exp\left(\int_{x_2^\circ}^{x_2} (\log g(\tilde{x}_2))' d\tilde{x}_2 \right) \in \mathcal{O}_{\tilde{B}}(\tilde{B}).$$

And then we have a holomorphic function G on $\tilde{B}$, such that $g = e^G$ and G' nowhere vanishes. It is easy to calculate that the Frobenius structure can be rewritten in terms of G in the way that we stated in the proposition.

2. First, $D \neq 1$, otherwise the assumption contradicts to $(\partial_{x_1}, \partial_{x_2}) = cg'g^{-D} \in \mathcal{O}(\tilde{M})$. We have $g'g^{-D} \cdot \frac{g}{g'} = g^{1-D} \in \mathcal{O}(\tilde{B})$ and that $(\partial_{x_1}, \partial_{x_2}) = \frac{c}{1-D} \frac{\partial}{\partial x_2}(g^{1-D}) \in \mathcal{O}(\tilde{B})$ has no zeros. On the other hand, $\frac{g'}{g}$ is a meromorphic function on B. Supposing $\frac{g}{g'} \in \mathcal{O}(\tilde{B})$ has a zero at $x_2 = a \in \tilde{B}$, let us take its Laurent series near $x_2 = a$,

$$\frac{g'}{g} = \sum_{i=1}^{m} \frac{c_i}{(x_2 - a)^i} + O((x_2 - a)^0).$$

Then

$$H(x_2) := e^{(1-D)\int \frac{g'}{g} dx_2} = (x_2 - a)^{(1-D)c_1} e^{\sum_{i=2}^{m} \frac{c_i(1-D)}{1-i}(x_2-a)^{1-i} + O((x_2-a)^0)} = g^{1-D} \in \mathcal{O}(\tilde{B}).$$

Therefore, $c_i = 0$ for $i > 1$. And that $H'(x_2) = \frac{\partial}{\partial x_2}(g^{1-D}) \in \mathcal{O}(\tilde{B})$ has no zeros yields

$$(1 - D)c_1 = 1.$$

Furthermore, $(\frac{g'}{g})^2 \cdot e^{2\int \frac{g'}{g} dx_2} = (g')^2 \in \mathcal{O}(\tilde{B})$, then $\frac{1}{(x_2-a)^2}(x_2 - a)^{2c_1} \in \mathcal{O}(\tilde{B})$, i.e., $2c_1 \in \mathbb{Z}_{\geqslant 2}$. Hence, $D = 1 - \frac{2}{q}$ for some $q \in \mathbb{Z}_{\geqslant 2}$. Again, it is easy to calculate that the Frobenius structure can be rewritten in terms of G in the way that we stated in the proposition. □

2.2.1 $\frac{g}{g'} \in \mathcal{O}(\tilde{B})$ has no zero

An observation is that if we define $\phi : \tilde{M} \to \mathbb{C}^2$ in the way that $(x_1, x_2) \mapsto (\hat{x}_1 = x_1, \hat{x}_2 = G(x_2))$, then ϕ is a local biholomorphism due to $G'(x_2) \neq 0, \forall x_2 \in \tilde{B}$.

Consider the 2-dimensional configuration space $Z_2 := \{(\hat{u}_1, \hat{u}_2) \in \mathbb{C}^2 | u_1 \neq u_2\}$ and the Frobenius structure is that in Proposition 1.4, i.e., $(\hat{u}_1, \hat{u}_2)$ are canonical coordinates. Consider the universal cover $\mathbb{C}^2 = \{(\hat{x}_1 = \frac{\hat{u}_1+\hat{u}_2}{2}, \hat{x}_2) \in \mathbb{C}^2 | e^{\hat{x}_2} = \frac{\hat{u}_2-\hat{u}_1}{2}\}$ of Z_2, and the pullback of Frobenius structure from Z_2 to $\mathbb{C}^2$ is

$$\begin{aligned} e =&\partial_{\hat{x}_1}, \\ (\partial_{\hat{x}_i}, \partial_{\hat{x}_j}) =&\delta_{i+j,3} \cdot \hat{c} \cdot e^{(1-D)\hat{x}_2}, \\ \partial_{\hat{x}_2} \bullet \partial_{\hat{x}_2} =&e^{2\hat{x}_2} \partial_{\hat{x}_1}, \\ E =&\hat{x}_1 \partial_{\hat{x}_1} + \partial_{\hat{x}_2}. \end{aligned} \tag{2.7}$$

Corollary 2.5 *If $((\cdot,\cdot), \bullet, e = \partial_{x_1})$ is a semi-simple Frobenius structure on $\tilde{M}$, then $\exists G \in \mathcal{O}(\tilde{B})$ and $G'(x_2) \neq 0, \quad \forall x_2 \in \tilde{B}$ such that the Frobenius structure is a pullback of the Frobenius structure on $\mathbb{C}^2 := \{(\hat{x}_1, \hat{x}_2) \in \mathbb{C}^2\}$ via the map $\phi : \tilde{M} \to \mathbb{C}^2, (x_1, x_2) \mapsto (\hat{x}_1 = x_1, \hat{x}_2 = G(x_2))$.*

Proof We only need to choose $\hat{c} = c$. Calculating the tangent map of ϕ and comparing (2.5) and (2.7), one can find the corollary. □

Next, recall that $M = \mathbb{C} \times (\tilde{B}/\Gamma)$ where $\tilde{B}$ is $\mathbb{C}$ or $\mathbb{H}$ and Γ is a discrete subgroup of $\mathrm{Aut}(\tilde{B})$. π the universal covering map from $\tilde{B}$ to $\tilde{B}/\Gamma$, and then $\forall \gamma \in \Gamma, \pi \circ \gamma = \pi$. And we pullbacked the semi-simple Frobenius structure on M to get that on $\tilde{M}$. Due to $\pi \circ \gamma = \pi$, $(\mathrm{id}_{\mathbb{C}}, \gamma)$ gives us an automorphism of Frobenius structure on $\tilde{M}$. The tangent map of $(\mathrm{id}_{\mathbb{C}}, \gamma)$ and (2.5c) yield $G'(x_2) = \frac{\partial}{\partial x_2} G(\gamma(x_2))$. And the tangent map and (2.5b) give us

$$G(\gamma(x_2)) = G(x_2) + k\pi\sqrt{-1}, \qquad k \in \mathbb{Z}.$$

If $D = 1$, we cannot get more equation with respect to G from (2.5a). we have following commutative diagram of Frobenius structures, where the Frobenius structure on $\mathbb{C} \times \mathbb{C}^*$ will be explained afterwards

$$\begin{array}{ccc} \tilde{M} := \mathbb{C} \times \tilde{B} & \xrightarrow{\phi := (\mathrm{id}_{\mathbb{C}}, G)} & \mathbb{C}^2 \\ {\scriptstyle (\mathrm{id}_{\mathbb{C}}, \pi)} \downarrow & & \downarrow {\scriptstyle (\mathrm{id}_{\mathbb{C}}, e^{2\hat{x}_2})} \\ M := \mathbb{C} \times B & \dashrightarrow & \mathbb{C} \times \mathbb{C}^* \end{array}$$

And if $D \neq 1$, then the tangent map of $(\mathrm{id}_{\mathbb{C}}, \gamma)$ and (2.5a) lead to

$$G(\gamma(x_2)) = G(x_2) + \frac{2k'}{1-D}\pi\sqrt{-1}, \qquad k' \in \mathbb{Z}.$$

Hence, if $D \in \mathbb{C} \setminus \mathbb{Q}$, we have $G \circ \gamma = G, \forall \gamma \in \Gamma$. Consequently, we have following commutative diagram of Frobenius structures which gives us the first class of Semi-simple Frobenius manifolds

$$\begin{array}{ccc} \tilde{M} := \mathbb{C} \times \tilde{B} & \xrightarrow{\phi := (\mathrm{id}_{\mathbb{C}}, G)} & \mathbb{C}^2 \\ {\scriptstyle (\mathrm{id}_{\mathbb{C}}, \pi)} \downarrow & \nearrow & \\ M := \mathbb{C} \times B & & \end{array}$$

If $D \in \mathbb{Q} \setminus \{1\}$, then $k = \frac{2k'}{1-D}$, i.e., $D = 1 - \frac{2k'}{k} = 1 - \frac{2p}{q}$, where p and $q > 0$ are relatively prime. And $k' = p \cdot m$ and $k = q \cdot m$ for $m \in \mathbb{Z}$, where m depends on γ while p and q only depend on D. Therefore, we have following commutative diagram of Frobenius structures

$$\begin{array}{ccc} \tilde{M} := \mathbb{C}\times\tilde{B} & \xrightarrow{\phi := (\mathrm{id}_{\mathbb{C}}, G)} & \mathbb{C}^2 \\ {\scriptstyle (\mathrm{id}_{\mathbb{C}},\pi)}\Big\downarrow & & \Big\downarrow{\scriptstyle (\mathrm{id}_{\mathbb{C}}, e^{\frac{2}{q}\hat{x}_2})} \\ M := \mathbb{C}\times B & \dashrightarrow & \mathbb{C}\times\mathbb{C}^* \end{array}$$

When $p = 0$, set $q = 1$. Then the case $D = 1$ is included. Let me point out that D is odd number if and only if $q = 1$ and that D is even number if and only if $q = 2$.

Again, consider the 2-dimensional configuration space $Z_2 := \{(\hat{u}_1, \hat{u}_2) \in \mathbb{C}^2 | \hat{u}_1 \neq \hat{u}_2\}$ and the Frobenius structure is that in Proposition 1.4, i.e., $(\hat{u}_1, \hat{u}_2)$ are canonical coordinates. Consider a finite cover $\mathbb{C}\times\mathbb{C}^* = \left\{ \left(\bar{x}_1 = \frac{\hat{u}_1+\hat{u}_2}{2}, \bar{x}_2 = \left(\frac{\hat{u}_2-\hat{u}_1}{2}\right)^{\frac{2}{q}} \right) \in \mathbb{C}^2 \middle| \hat{u}_1 \neq \hat{u}_2 \right\}$ of Z_2, and the pullback of Frobenius structure from Z_2 to $\mathbb{C}\times\mathbb{C}^*$ is

$$\begin{aligned} e =&\partial_{\bar{x}_1}, \\ (\partial_{\bar{x}_i}, \partial_{\bar{x}_j}) =&\delta_{i+j,3}\cdot c\cdot \frac{q}{2}\bar{x}_2^{p-1}, \\ \partial_{\bar{x}_2}\bullet\partial_{\bar{x}_2} =&\frac{q^2}{4}\bar{x}_2^{q-2}\partial_{\bar{x}_1}, \\ E =&\bar{x}_1\partial_{\bar{x}_1} + \frac{2}{q}\bar{x}_2\partial_{\bar{x}_2}. \end{aligned} \tag{2.8}$$

Remark 2.6 If $p = 1$ and $q \geqslant 2$, then the Frobenius structure extends to $\mathbb{C}^2$

2.2.2 $\frac{g}{g'} \in \mathcal{O}(\tilde{B})$ has at least one zero

Similarly, if we define $\phi : \tilde{M} \to \mathbb{C}^2$ in the way that $(x_1, x_2) \mapsto (\bar{x}_1 = x_1, \bar{x}_2 = H(x_2))$, then ϕ is a local biholomorphism due to $H'(x_2) \neq 0, \forall x_2 \in \tilde{B}$. Note that, in this case, the condition of the previous remark holds, we will see that here the Frobenius structure on $\mathbb{C}^2$ was extended from that on $\mathbb{C}\times\mathbb{C}^*$. And that is why we use $(\bar{x}_1, \bar{x}_2)$ instead of $(\hat{x}_1, \hat{x}_2)$ for $\mathbb{C}^2$ here. According to the same reason stated in the previous case, the tangent map of $(\mathrm{id}_{\mathbb{C}}, \gamma)$, (2.6a) and (2.6c) yield that H is Γ-invariant. Furthermore, calculting the tangent map of ϕ and comparing (2.6) and (2.8) for $p = 1$ and $q \in \mathbb{Z}_{\geqslant 2}$, we have the pullback of the Frobenius structure on $\mathbb{C}^2$ via ϕ is that on $\tilde{M}$. Therefore, we have following commutative diagram of Frobenius structures

$$\begin{array}{ccc} \tilde{M} := \mathbb{C}\times\tilde{B} & & \\ {\scriptstyle (\mathrm{id}_{\mathbb{C}},\pi)}\Big\downarrow & \searrow^{\phi := (\mathrm{id}_{\mathbb{C}}, H)} & \\ M := \mathbb{C}\times B & \dashrightarrow & \mathbb{C}^2 \end{array}$$

To conclude this section or chapter?, we state the following theorem.

Theorem 2.7 (Classification of M) *Considering $D \neq 0$ and the setting of semi-simple Frobenius manifold stated at the beginning of this chapter, there are three types of maximal semi-simple Frobenius manifold.*

1. *If $D \in \mathbb{C}\setminus\mathbb{Q}$, Frobenius manifold $\mathbb{C}^2$ with Frobenius structure (2.7)*
2. *If $D = 1 - \frac{2}{q}$ for some $q \in \mathbb{Z}_{\geqslant 3}$, Frobenius manifold $\mathbb{C}^2$ with Frobenius structure (2.8) with $p = 1$.*
3. *Otherwise, i.e., $D = 1 - \frac{2p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$ are relatively prime satisfying that $p \neq 1$ or $q = 1$, Frobenius manifold $\mathbb{C}\times\mathbb{C}^*$ with Frobenius structure (2.8).*

Here, in the first and the third cases, every point is semi-simple, while in the second case, semi-simple point set is $\mathbb{C} \times \mathbb{C}^* \subset \mathbb{C}^2$*. And any semi-simple Frobenius structure that we consider in this chapter is a pullback of one of the three types.*

Chapter 3

Computation of period map on maximal semi-simple Frobenius manifold

3.1 Flat coordinates (t_1, t_2)

For both Frobenius structures (2.7) and (2.8), though the expression of t_2 in terms of $\hat{x}_2$ and $\bar{x}_2$ are different, the expression of t_2 in terms of $\hat{u}_1$ and $\hat{u}_2$ are the same since both of Frobenius structures are derived from that on the 2-dimensional configuration space Z_2. For the simplicity of notation, in this chapter, we will use u_1 and u_2 instead of $\hat{u}_1$ and $\hat{u}_2$ in the previous chapter. We may choose flat coordinates as the following,

$$\begin{aligned} t_1 =& \frac{u_1 + u_2}{2} \\ t_2 =& \begin{cases} \frac{1}{1-D}\left(\left(\frac{u_2-u_1}{2}\right)^{1-D} - 1\right) & \text{if } D \in \mathbb{C} \setminus \{0,1\} \\ \log\left(\frac{u_2-u_1}{2}\right) & \text{if } D = 1 \end{cases} \end{aligned}$$

And under this setting, the expression of t_2 for $D = 1$ is the limit of that of t_2 for $D \to 1$.

Then we consider $\tilde{\Psi}$ which was introduced in Lemma 1.7

$$\begin{aligned} \tilde{\Psi} =& \begin{pmatrix} \frac{\partial t_1}{\partial u_1} & \frac{\partial t_1}{\partial u_2} \\ \frac{\partial t_2}{\partial u_1} & \frac{\partial t_2}{\partial u_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2}\left(\frac{u_2-u_1}{2}\right)^{-D} & \frac{1}{2}\left(\frac{u_2-u_1}{2}\right)^{-D} \end{pmatrix} \\ =& \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2}\left(\frac{u_2-u_1}{2}\right)^{-D} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \end{aligned}$$

and

$$\tilde{\Psi}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2\left(\frac{u_2-u_1}{2}\right)^{D} \end{pmatrix}$$

According to Lemma 7.2,

$$\begin{aligned} P_1 =& \tilde{\Psi} E_{11} \tilde{\Psi}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -\left(\frac{u_2-u_1}{2}\right)^{D} \\ -\left(\frac{u_2-u_1}{2}\right)^{-D} & 1 \end{pmatrix}, \\ P_2 =& \frac{1}{2} \begin{pmatrix} 1 & \left(\frac{u_2-u_1}{2}\right)^{D} \\ \left(\frac{u_2-u_1}{2}\right)^{-D} & 1 \end{pmatrix}. \end{aligned}$$

Let us find the constant matrix for θ under the bases of $\partial_{t_1}, \partial_{t_2}$. By direct calculation, the Euler vector field $E = t_1\partial_{t_1} + ((1-D)t_2 + 1)\partial_{t_2}$ By definition and the flatness of $\partial_{t_1}, \partial_{t_2}$,

$$\theta(\partial_{t_1}) = \nabla^{L.C.}_{\partial_{t_1}} E - (1-\frac{D}{2})\partial_{t_1} = [\partial_{t_1}, E] - (1-\frac{D}{2})\partial_{t_1} = \frac{D}{2}\partial_{t_1}$$

where $\nabla^{L.C.}$ is Levi-Civita connection.

Similarly, $\theta(\partial_{t_2}) = [\partial_{t_2}, E] - (1-\frac{D}{2})\partial_{t_2} = (1-D-1+\frac{D}{2})\partial_{t_2} = -\frac{D}{2}\partial_{t_2}$. Therefore, $\theta = \begin{pmatrix} \frac{D}{2} & 0 \\ 0 & -\frac{D}{2} \end{pmatrix}$ under the bases of $\partial_{t_1}, \partial_{t_2}$.

Under the bases of $\partial_{t_1}, \partial_{t_2}$, denoting $L^+ := \frac{D}{2} - n - \frac{1}{2}, L^- := -\frac{D}{2} - n - \frac{1}{2}$

$$A_1^{(n)}(u) = P_1(\theta - n - \frac{1}{2}) = \frac{1}{2}\begin{pmatrix} 1 & -\left(\frac{u_2-u_1}{2}\right)^D \\ -\left(\frac{u_2-u_1}{2}\right)^{-D} & 1 \end{pmatrix}\begin{pmatrix} L^+ & 0 \\ 0 & L^- \end{pmatrix}$$

$$A_2^{(n)}(u) = P_2(\theta - n - \frac{1}{2}) = \frac{1}{2}\begin{pmatrix} 1 & \left(\frac{u_2-u_1}{2}\right)^D \\ \left(\frac{u_2-u_1}{2}\right)^{-D} & 1 \end{pmatrix}\begin{pmatrix} L^+ & 0 \\ 0 & L^- \end{pmatrix}$$

3.2 Second structure connection

Recalling the proof of Lemma 1.8, the second structure connection $\nabla^{(n)}_{\partial_\lambda} = \partial_\lambda - \frac{A_1^{(n)}(u)}{\lambda - u_1} - \frac{A_2^{(n)}(u)}{\lambda - u_2}$. For any fixed $u = (u_1, u_2)$, we would like to find a vector field $X = \begin{pmatrix} X_1(u,\lambda) \\ X_2(u,\lambda) \end{pmatrix}$ (under the bases of $\partial_{t_1}, \partial_{t_2}$) such that $\nabla^{(n)}_{\partial_\lambda} X = 0$.

By $x := \frac{\lambda - u_1}{u_2 - u_1}$, we transform $\nabla^{(n)}_{\partial_\lambda} X = 0$ into

$$\partial_x \begin{pmatrix} X_1(u,x) \\ X_2(u,x) \end{pmatrix} = \left(\frac{A_1^{(n)}(u)}{x} + \frac{A_2^{(n)}(u)}{x-1}\right)\begin{pmatrix} X_1(u,x) \\ X_2(u,x) \end{pmatrix}$$

Taking $T\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ where $T = \begin{pmatrix} 2L^- + 2 & 1-2x \\ 0 & -\left(\frac{u_2-u_1}{2}\right)^{-D} \end{pmatrix}$, $T^{-1} = \begin{pmatrix} (2L^-+2)^{-1} & \frac{2x-1}{(2L^-+2)\tilde{c}} \\ 0 & -\left(\frac{u_2-u_1}{2}\right)^{D} \end{pmatrix}$,

$$\partial_x \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = T^{-1}\left(\left(\frac{A_1^{(n)}(u)}{x} + \frac{A_2^{(n)}(u)}{x-1}\right)T - (\partial_x T)\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

After some calculation, we get

$$\partial_x \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ \left(n^2 - \left(\frac{D-1}{2}\right)^2\right)\left(\frac{1}{x} - \frac{1}{x-1}\right)y_1 - \left(n+\frac{1}{2}\right)\left(\frac{1}{x} + \frac{1}{x-1}\right)y_2 \end{pmatrix}$$

Then we have a second-order ODE, in particular, a hypergeometric equation

$$x(1-x)\frac{d^2y_1}{dx^2} - \left(n+\frac{1}{2}\right)(2x-1)\frac{dy_1}{dx} - \left(n^2 - \left(\frac{D-1}{2}\right)^2\right)y_1 = 0,$$

where the parameters of the hypergeometric equation as follows,

$$\begin{aligned} a =& n - \frac{D-1}{2} \\ b =& n + \frac{D-1}{2} \\ c =& n + \frac{1}{2} \end{aligned}$$

We will see later on that we are interested in the case λ near ∞.

If $a - b$ is not an integer, i,e. D is not an integer, then there are two independent solutions to the hypergeometric equations near ∞.

$$\begin{aligned} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} &= \begin{pmatrix} x^{-a}F(a, 1+a-c; 1+a-b; x^{-1}) & x^{-b}F(b, 1+b-c; 1+b-a; x^{-1}) \\ \frac{d}{dx}(x^{-a}F(a, 1+a-c; 1+a-b; x^{-1})) & \frac{d}{dx}(x^{-b}F(b, 1+b-c; 1+b-a; x^{-1})) \end{pmatrix} \\ &= \begin{pmatrix} x^{-a}F(a, 1+a-c; 1+a-b; x^{-1}) & x^{-b}F(b, 1+b-c; 1+b-a; x^{-1}) \\ -ax^{-a-1}F(a+1, 1+a-c; 1+a-b; x^{-1}) & -bx^{-b-1}F(b+1, 1+b-c; 1+b-a; x^{-1}) \end{pmatrix} \end{aligned}$$

If $a - b$ is an integer, i,e. D is an integer, then there is logarithmic singularity at $\lambda = \infty$. There are three cases.

Let us define

$$\tilde{\Phi}(x) := F(a, 1+a-c; 1; x^{-1}) \log(x^{-1}) + \sum_{m=1}^{\infty} \tilde{A}_m \tilde{B}_m x^{-m},$$

where

$$\tilde{A}_m = \frac{(a)_m (1+a-c)_m}{(m!)^2}.$$

Here $(q)_m$ is the (rising) Pochhammer symbol, which is defined by:

$$(q)_m = \begin{cases} 1 & m = 0 \\ q \cdot (q+1) \cdot \dots \cdot (q+m-1) & m \in \mathbb{Z}_{>0} \end{cases}$$

$\tilde{A}_m$ is the coefficient of x^{-m} in $F(a, 1+a-c; 1; x^{-1})$ and

$$\begin{aligned} \tilde{B}_m =& \frac{1}{a} + \frac{1}{a+1} + \dots + \frac{1}{a+m-1} + \frac{1}{1+a-c} + \frac{1}{1+a-c+1} + \dots + \frac{1}{1+a-c+m-1} \\ & - 2\left(1 + \frac{1}{2} + \dots + \frac{1}{m}\right). \end{aligned}$$

The convention for $\tilde{B}_0$ is 0 for later purpose. The power series $\sum_{m=1}^{\infty} \tilde{A}_m \tilde{B}_m x^{-m}$ converges in $|x^{-1}| < 1$. Here, we assumed that neither a nor $1 + a - c$ is an integer less than 1. If either a or $1 + a - c$ is an integer less than 1 we set

$$\tilde{B}_m = 0, m > \tilde{N},$$

where $\tilde{N}$ is the minimum of m such that

$$(a+m)(1+a-c+m) = 0.$$

In this case $\sum_{m=1}^{\infty} \tilde{A}_m \tilde{B}_m x^{-m}$ is a polynomial of x^{-1}.

The first case is $a = b$, i.e., $D = 1$. The fundamental solution

$$\begin{aligned} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} &= \begin{pmatrix} x^{-a}\tilde{\Phi}(x) & x^{-a}F(a, 1+a-c; 1+a-b; x^{-1}) \\ \frac{d}{dx}(x^{-a}\tilde{\Phi}(x)) & \frac{d}{dx}(x^{-a}F(a, 1+a-c; 1+a-b; x^{-1})) \end{pmatrix} \\ &= \begin{pmatrix} x^{-a}\tilde{\Phi}(x) & x^{-a}F(a, 1+a-c; 1+a-b; x^{-1}) \\ \frac{d}{dx}(x^{-a}\tilde{\Phi}(x)) & -ax^{-a-1}F(a+1, 1+a-c; 1+a-b; x^{-1}) \end{pmatrix}. \end{aligned}$$

The second case is $a < b$, i.e, $D \in \mathbb{Z}_{>1}$. The fundamental solution

$$\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} x^{-b}\frac{d^{D-1}}{d(x^{-1})^{D-1}}\tilde{\Phi}(x) & x^{-b}F(b,1+b-c;1+b-a;x^{-1}) \\ \frac{d}{dx}\left(x^{-b}\frac{d^{D-1}}{d(x^{-1})^{D-1}}\tilde{\Phi}(x)\right) & \frac{d}{dx}(x^{-b}F(b,1+b-c;1+b-a;x^{-1})) \end{pmatrix}$$
$$= \begin{pmatrix} x^{-b}\frac{d^{D-1}}{d(x^{-1})^{D-1}}\tilde{\Phi}(x) & x^{-b}F(b,1+b-c;1+b-a;x^{-1}) \\ \frac{d}{dx}\left(x^{-b}\frac{d^{D-1}}{d(x^{-1})^{D-1}}\tilde{\Phi}(x)\right) & -bx^{-b-1}F(b+1,1+b-c;1+b-a;x^{-1}) \end{pmatrix}.$$

And by the sufficient and necessary conditions on the page 11 of [Mat85], we have if $D \in (2\mathbb{Z})_{>1}$ or $D \in (2\mathbb{Z}+1)_{>3}, n = -1$ then

$$x^{-b}\frac{d^{D-1}}{d(x^{-1})^{D-1}}\tilde{\Phi}(x) = x^{-a}\sum_{m=0}^{D-2}\frac{(a)_m(1+a-c)_m}{m!\cdot(2-D)_m}x^{-m} = x^{-a}F(a,1+a-c;1+a-b;x^{-1})$$

The third case is $a > b$, i.e., $D \in \mathbb{Z}_{<1}$. The fundamental solution

$$\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} x^{-a}F(a,1+a-c;1+a-b;x^{-1}) & x^{-a}(1-x^{-1})^{c-a-b}\frac{d^{1-D}}{d(x^{-1})^{1-D}}\left((1-x^{-1})^{2a-c}\tilde{\Phi}(x)\right) \\ \frac{d}{dx}(x^{-a}F(a,1+a-c;1+a-b;x^{-1})) & \frac{d}{dx}\left(x^{-a}(1-x^{-1})^{c-a-b}\frac{d^{1-D}}{d(x^{-1})^{1-D}}\left((1-x^{-1})^{2a-c}\tilde{\Phi}(x)\right)\right) \end{pmatrix}$$
$$= \begin{pmatrix} x^{-a}F(a,1+a-c;1+a-b;x^{-1}) & x^{-a}(1-x^{-1})^{c-a-b}\frac{d^{1-D}}{d(x^{-1})^{1-D}}\left((1-x^{-1})^{2a-c}\tilde{\Phi}(x)\right) \\ -ax^{-a-1}F(a+1,1+a-c;1+a-b;x^{-1}) & \frac{d}{dx}\left(x^{-a}(1-x^{-1})^{c-a-b}\frac{d^{1-D}}{d(x^{-1})^{1-D}}\left((1-x^{-1})^{2a-c}\tilde{\Phi}(x)\right)\right) \end{pmatrix}.$$

Via similar reference, we have if $D \in (2\mathbb{Z})_{<1}$ or $D \in (2\mathbb{Z}+1)_{<-1}, n = -1$then

$$\begin{aligned} & x^{-a}(1-x^{-1})^{c-a-b}\frac{d^{1-D}}{d(x^{-1})^{1-D}}\left((1-x^{-1})^{2a-c}\tilde{\Phi}(x)\right) \\ =& x^{-b}\sum_{m=0}^{-D}\frac{(a+D-1)_m(1+a-c+D-1)_m}{m!\cdot(D)_m}x^{-m} = x^{-b}F(b,1+b-c;1+b-a;x^{-1}) \end{aligned}$$

Thus we can merge the case $D \in \mathbb{C}\setminus\mathbb{Z}$, $D \in (2\mathbb{Z})_{>1}$ and $D \in (2\mathbb{Z})_{<1}$ into the case $D \in \mathbb{C}\setminus(2\mathbb{Z}+1)$. And we will see the similar fusion in the next section. These two fusions will simplify the later arguement and save calculation process of intersection pairing.

For all $D \in \mathbb{C}$, we set

$$T\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} =: \begin{pmatrix} \Phi_1 & \Phi_2 \end{pmatrix} =: \Phi,$$

where $\begin{pmatrix} y_{11} \\ y_{21} \end{pmatrix}$ and $\begin{pmatrix} y_{12} \\ y_{22} \end{pmatrix}$ play the role of $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, and Φ_1 and Φ_2 play the role of $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ as before.

3.3 Calibration

Let us recall the section about calibration in Chapter 1. Under the basis $(\partial_{t_1}, \partial_{t_2})$, since $\theta = \begin{pmatrix} \frac{D}{2} & 0 \\ 0 & -\frac{D}{2} \end{pmatrix}$ is a diagonal matrix, $\theta = \delta$, $\nu_{[0]} = 0$

If $D = 0$, then $\theta = \delta = 0, \nu = \nu_{[0]} = 0, kS_k = E \bullet S_{k-1}, k \in \mathbb{Z}_+$, which gives us that all the S_k's are uniquely determined.

If $D \in \mathbb{C} \setminus \{0\}$, then $\mathrm{spec}(\delta) = \{-D, 0, D\}$ and

$$\mathfrak{gl}_D(H) = \left\{ X \in \mathfrak{gl}(H) \middle| X = \begin{pmatrix} 0 & x_{12} \\ 0 & 0 \end{pmatrix}, x_{12} \in \mathbb{C} \right\}$$
$$\mathfrak{gl}_0(H) = \left\{ X \in \mathfrak{gl}(H) \middle| X = \begin{pmatrix} x_{11} & 0 \\ 0 & x_{22} \end{pmatrix}, x_{11}, x_{22} \in \mathbb{C} \right\}$$
$$\mathfrak{gl}_{-D}(H) = \left\{ X \in \mathfrak{gl}(H) \middle| X = \begin{pmatrix} 0 & 0 \\ x_{21} & 0 \end{pmatrix}, x_{21} \in \mathbb{C} \right\},$$

which means $\nu = \nu_{[-|D|]}$.

The recursion relation will be simplified as following, for $D \in \mathbb{Z} \setminus \{0\}$

$$(k + a)(S_k)_{[a]} = \left(E \bullet S_{k-1} + S_{k-|D|}\nu_{[-|D|]} \right)_{[a]}, \qquad a = -|D|, 0, |D|, \tag{3.1}$$

where $S_{k-|D|} = 0$ if $k < |D|$. When $k < |D|$, S_k is uniquely determined by recursion. When $k = |D|$, the ambiguity of choosing $(S_k)_{[-|D|]}$ emerges. But $\nu = (E \bullet S_{|D|-1})_{[-|D|]}$ is determined uniquely

Note that under the basis $(\partial_{t_1}, \partial_{t_2})$, the transposition with respect to Frobenius pairing $(X_{ij})^T_{2\times 2} = (X_{3-j,3-i})_{2\times 2}, \forall X \in \mathfrak{gl}(H)$.

If D is odd, the condition that $(-1)^{-|D|}C^T_{|D|} = (-1)^{-|D|}C_{|D|} = -C_{|D|}$ is satisfied, recalling the notation and definition in Theorem 1.14 on page 11. And (1.12) holds automatically. Here $k = |D|$. Essentially, these two fact are the same.

When $D = 1$, under the basis $(\partial_{t_1}, \partial_{t_2})$, $E\bullet$ corresponds to the matrix $\begin{pmatrix} t_1 & e^{2t_2} \\ 1 & t_1 \end{pmatrix}$. The recursion relation (1.11) yields $0 = \begin{pmatrix} t_1 & e^{2t_2} \\ 1 & t_1 \end{pmatrix}_{[-1]} + \nu_{[-1]}$, i.e. $\nu = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$.

When D is an odd number different from 1, under the basis $(\partial_{t_1}, \partial_{t_2})$, $E\bullet$ corresponds to the matrix $\begin{pmatrix} t_1 & ((1-D)t_2+1)^{\frac{1+D}{1-D}} \\ (1-D)t_2+1 & t_1 \end{pmatrix}$. The recursion relation (1.11) gives us

$$\nu_{[-|D|]} = -\left(E \bullet (|D| - 1 + \mathrm{ad}_\delta)^{-1} \circ E \bullet (|D| - 2 + \mathrm{ad}_\delta)^{-1} \circ E \bullet \ldots (1 + \mathrm{ad}_\delta)^{-1} \circ E \bullet 1 \right)_{[-|D|]}.$$

Since ν is a constant matrix, we may let $t_1 = t_2 = 0$, i.e. $E\bullet = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Direct calculation yields the nonzero entry in ν is

$$-\frac{(-1)^{\frac{|D|-1}{2}}}{2^{|D|-1}\left(\left(\frac{|D|-1}{2}\right)!\right)^2}$$

And the expression with factorial the nonzero entry includes the case $D = 1$. Note that, under the basis $(\partial_{t_1}, \partial_{t_2})$, the equation

$$E\bullet = \begin{pmatrix} \frac{u_1+u_2}{2} & \left(\frac{u_2-u_1}{2}\right)^{1+D} \\ \left(\frac{u_2-u_1}{2}\right)^{1-D} & \frac{u_1+u_2}{2} \end{pmatrix} \tag{3.2}$$

holds for any $D \in (2\mathbb{Z} + 1)$.

If D is even, the condition that $(-1)^{-|D|}C^T_{|D|} = (-1)^{-|D|}C_{|D|} = -C_{|D|}$ gives us $C_{|D|} = 0$. Similarly, the condition that $B^T_k = -(-1)^k B_k$ yields $B_k = 0$, where $k = |D|$. That is to say, in this case, calibration $S(t, z)$ is determined uniquely. With similar argument in the case that D is odd, we have $\nu = 0$. And thus we may include the case $D = 0$ into this case.

If $D \in \mathbb{C} \setminus \mathbb{Z}$, then the fact that $\nu_{[-l]} = 0, \forall l \in \mathbb{C} \setminus \mathbb{Z}_{\geqslant 0}$ means $\nu = \nu_{[-|D|]} = 0$. And the recursion relation will be simplified further as following,

$$(k+a)(S_k)_{[a]} = (E \bullet S_{k-1})_{[a]}, \qquad a = -D, 0, D,$$

and thus calibration $S(t,z)$ is uniquely determined by recursion.

The last two cases, i.e., that D is even and that $D \in \mathbb{C} \setminus \mathbb{Z}$ are very similar. We will deal with these two cases together.

3.4 The period map

In the rest of this chapter, we will consider the case $D \in \mathbb{C} \setminus (2\mathbb{Z}+1)$. Proposition 1.15 tells us that, in this case,

$$I^{(n)}(t,\lambda) = \sum_{k=0}^{\infty} (-1)^k S_k(t) \frac{\lambda^{\theta-(n+k)-\frac{1}{2}}}{\Gamma(\theta-(n+k)+\frac{1}{2})}$$

provides a fundamental solution to the 2nd structure connection. Thus, recalling Φ at the end of second structure connection, we have a $C(t,n) \in GL(\mathbb{C}^2)$ such that $I^{(n)}(t,\lambda) = \Phi(t,\lambda)C(t,n)$. Taking a sufficiently large circle centered at zero on the complex plane of λ, let λ move counterclockwise for one round, i.e. $I^{(n)}(t, \lambda e^{2\pi\sqrt{-1}}) = \Phi(t, \lambda e^{2\pi\sqrt{-1}})C(t,n)$. The LHS is

$$\begin{aligned}
&\sum_{k=0}^{\infty} (-1)^k S_k(t) \frac{\lambda^{\theta-(n+k)-\frac{1}{2}} e^{2\pi(\theta-n-\frac{1}{2})\sqrt{-1}}}{\Gamma(\theta-(n+k)+\frac{1}{2})} \\
=& I^{(n)}(t,\lambda) \begin{pmatrix} e^{2\pi(\frac{D-1}{2}-n)\sqrt{-1}} & 0 \\ 0 & e^{2\pi(-\frac{D+1}{2}-n)\sqrt{-1}} \end{pmatrix} \\
=& \Phi(t,\lambda)C(t,n) \begin{pmatrix} e^{2\pi(\frac{D-1}{2}-n)\sqrt{-1}} & 0 \\ 0 & e^{2\pi(-\frac{D+1}{2}-n)\sqrt{-1}} \end{pmatrix}
\end{aligned}$$

The RHS is $\Phi(t,\lambda) \begin{pmatrix} e^{-2\pi a\sqrt{-1}} & 0 \\ 0 & e^{-2\pi b\sqrt{-1}} \end{pmatrix} C(t,n)$. Since $\Phi(t,\lambda) \in GL(\mathbb{C}^2)$ and LHS=RHS, we have

$$\begin{pmatrix} e^{-2\pi a\sqrt{-1}} & 0 \\ 0 & e^{-2\pi b\sqrt{-1}} \end{pmatrix} C(t,n) = C(t,n) \begin{pmatrix} e^{2\pi(\frac{D-1}{2}-n)\sqrt{-1}} & 0 \\ 0 & e^{2\pi(-\frac{D+1}{2}-n)\sqrt{-1}} \end{pmatrix},$$

which leads to $C(t,n) = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$. After calculation for the coefficient of the leading term of λ, we will find that

$$\begin{aligned}
c_1 =& \frac{(u_2-u_1)^{-n+\frac{D}{2}-\frac{1}{2}}}{2(1-D)\Gamma(\frac{D}{2}-n+\frac{1}{2})} \\
c_2 =& -\frac{(u_2-u_1)^{-n-\frac{D}{2}-\frac{1}{2}}}{\Gamma(-\frac{D}{2}-n+\frac{3}{2})} \left(\frac{u_2-u_1}{2}\right)^D
\end{aligned}$$

Since the intersection pairing is indpendent of t and λ, let $t_1 = \lambda = 1, t_2 = 0$, then $u_2 - u_1 = x^{-1} = 2, -(\frac{u_2-u_1}{2})^{-D} = -1$. Noting that $(1+a-c) : (1+a-b) = (1+b-c) : (1+b-a) = 1 : 2$, we can use a formula for $x^{-a}F(a, 1+a-c; 1+a-b; x^{-1}), x^{-b}F(b, 1+b-c; 1+b-a; x^{-1})$ and their differential. By direct calculation, the matrix for intersection pairing is

$$
\begin{aligned}
&\begin{pmatrix} (\partial_{t_1}|\partial_{t_1}) & (\partial_{t_1}|\partial_{t_2}) \\ (\partial_{t_1}|\partial_{t_2}) & (\partial_{t_2}|\partial_{t_2}) \end{pmatrix} \\
=& -c(C(t,n))^t \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}^t T^t T_{-n} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}_{-n} C(t,-n) \\
=& -c(C(t,n))^t \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}^t \begin{pmatrix} 4\left((\frac{1-D}{2})^2 - n^2\right) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}_{-n} C(t,-n) \\
=& -c(C(t,n))^t \begin{pmatrix} 0 & 4(n+\frac{1-D}{2})(1-D) \\ 4(-n+\frac{1-D}{2})(1-D) & 0 \end{pmatrix} C(t,-n) \\
=& \frac{c}{\pi} \begin{pmatrix} 0 & \sin\left(\pi\left(n+\frac{1-D}{2}\right)\right) \\ \sin\left(\pi\left(-n+\frac{1-D}{2}\right)\right) & 0 \end{pmatrix}
\end{aligned}
$$

where the lower index $-n$ means replacing n in the matrix by $-n$.

For the period map,

$$
\langle Z(t,\lambda), \partial_{t_1}\rangle := (I^{(-1)}_{\partial_{t_1}}(t,\lambda), \partial_{t_1}) = cI^{(-1)}_{21}(t,\lambda) = acc_1 \left(\frac{u_2-u_1}{2}\right)^{-D} x^{-a-1} F(a+1, 1+a-c; 1+a-b; x^{-1})
$$

$$
\langle Z(t,\lambda), \partial_{t_2}\rangle := (I^{(-1)}_{\partial_{t_2}}(t,\lambda), \partial_{t_1}) = cI^{(-1)}_{22}(t,\lambda) = bcc_2 \left(\frac{u_2-u_1}{2}\right)^{-D} x^{-b-1} F(b+1, 1+b-c; 1+b-a; x^{-1})
$$

Note that we have explicit functions for $I^{(-1)}_{21}(t,\lambda)$ and $I^{(-1)}_{22}(t,\lambda)$ as well. Then, by direct calculation,

$$
\begin{aligned}
\langle Z(t,\lambda), \partial_{t_1}\rangle =& \frac{2^{\frac{1-D}{2}} c}{(D-1)\Gamma\left(\frac{1+D}{2}\right)} \left(x^{\frac{1}{2}} + (x-1)^{\frac{1}{2}}\right)^{D-1} \left(\frac{u_2-u_1}{2}\right)^{\frac{1-D}{2}} \\
\langle Z(t,\lambda), \partial_{t_2}\rangle =& \frac{2^{-\frac{1-D}{2}} c}{\Gamma\left(\frac{1-D}{2}+1\right)} \left(x^{\frac{1}{2}} + (x-1)^{\frac{1}{2}}\right)^{1-D} \left(\frac{u_2-u_1}{2}\right)^{\frac{1-D}{2}},
\end{aligned}
$$

which satisfies the translation symmetry $Z(t,\lambda) = Z(t-\lambda\partial_{t_1}, 0)$.

3.5 Reflection vectors (Vanishing cycle) and Monodromy group

Here we follow the notation and the idea of computation in 1.3.1. i.e. Reflection vectors $\varphi_\pm$ are equal to $(C^{(i)}(u))^{-1}e_i (i=1,2)$ up to $\kappa_i(u) \in \mathbb{C}^*$, respectively. Note that $C^{(i)}(u)$ may depend on $n \in \mathbb{Z}$, but in the following calculation, we will choose $n=-1$ and thus omit the index for n. Since $I^{(-1)}(u,\lambda) = Y^{(i)}(u,\lambda)C^{(i)}(u)$ where $Y^{(i)}(u,\lambda)$ are fundamental solutions for $\nabla_{\partial/\partial\lambda}$ near $\lambda = u_i$, we will compare the coefficients of order 0 and $-n-\frac{1}{2} = \frac{1}{2}$ (leading and subleading terms) of the lower rows of the matrices to get $C^{(i)}(u) := \begin{pmatrix} c^{(i)}_{11} & c^{(i)}_{12} \\ c^{(i)}_{21} & c^{(i)}_{22} \end{pmatrix}$.

For $i=2$, the coefficients are in the following matrix

$$
\tilde{\Psi} \begin{pmatrix} 1 & 0 \\ (n+\frac{1}{2})^{-1} & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2}\left(\frac{u_2-u_1}{2}\right)^{-D} & \frac{1}{2}\left(\frac{u_2-u_1}{2}\right)^{-D} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2}\left(\frac{u_2-u_1}{2}\right)^{-D} & \frac{1}{2}\left(\frac{u_2-u_1}{2}\right)^{-D} \end{pmatrix}
$$

where the first column stands for the coefficients of $(\lambda-u_2)^0$ and the second column for that of $(\lambda-u_2)^{\frac{1}{2}}$. Moving x from a number $\gg 0$ to 1 along real axis, we will take binomial expansion near $x = 1$

$$\begin{aligned}\left(x^{\frac{1}{2}}+(x-1)^{\frac{1}{2}}\right)^{D-1} &= \left((x-1+1)^{\frac{1}{2}}+(x-1)^{\frac{1}{2}}\right)^{D-1} \\ &= \left(1+(x-1)^{\frac{1}{2}}+\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}(x-1)^k\right)^{D-1} \\ &=1+\sum_{k'=1}^{\infty}\binom{D-1}{k'}\left((x-1)^{\frac{1}{2}}+\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}(x-1)^k\right)^{k'} \\ &=1+(D-1)(x-1)^{\frac{1}{2}}+O(x-1)\end{aligned}$$

Similarly, $\left(x^{\frac{1}{2}}+(x-1)^{\frac{1}{2}}\right)^{1-D} = 1+(1-D)(x-1)^{\frac{1}{2}}+O(x-1)$

According to $I^{(-1)}(u,\lambda) = Y^{(i)}(u,\lambda)C^{(i)}(u)$, we have an equation for the coefficient of $(x-1)^0$ in $I^{(-1)}_{21}$ which is

$$\frac{2^{\frac{1-D}{2}}}{(D-1)\Gamma\left(\frac{1+D}{2}\right)}\left(\frac{u_2-u_1}{2}\right)^{\frac{1-D}{2}} = -\frac{3}{2}\left(\frac{u_2-u_1}{2}\right)^{-D}c^{(2)}_{11}.$$

Then, we have

$$c^{(2)}_{11} = \frac{2^{1-D}}{3(1-D)\Gamma\left(\frac{1+D}{2}\right)}(u_2-u_1)^{\frac{1+D}{2}}.$$

And we have an equation for that of $(x-1)^{\frac{1}{2}}$ in $I^{(-1)}_{21}$ (note that at this time we have more coefficient $(u_2-u_1)^{\frac{1}{2}}$ on the RHS to convert $(\lambda-u_2)^{\frac{1}{2}}$ into $(x-1)^{\frac{1}{2}}$)

$$\frac{(D-1)2^{\frac{1-D}{2}}}{(D-1)\Gamma\left(\frac{1+D}{2}\right)}\left(\frac{u_2-u_1}{2}\right)^{\frac{1-D}{2}} = \frac{(u_2-u_1)^{\frac{1}{2}}}{2}\left(\frac{u_2-u_1}{2}\right)^{-D}c^{(2)}_{21}.$$

Then we have

$$c^{(2)}_{21} = \frac{2^{1-D}}{\Gamma\left(\frac{1+D}{2}\right)}(u_2-u_1)^{\frac{D}{2}}.$$

We do the same things for those in $I^{(-1)}_{22}$. So we have

$$c^{(2)}_{12} = -\frac{(u_2-u_1)^{\frac{1+D}{2}}}{3\Gamma(\frac{1-D}{2}+1)}, \qquad c^{(2)}_{22} = \frac{(1-D)(u_2-u_1)^{\frac{D}{2}}}{\Gamma(\frac{1-D}{2}+1)},$$

and

$$\det(C^{(2)}(u)) = c^{(2)}_{11}c^{(2)}_{22} - c^{(2)}_{12}c^{(2)}_{21} = \frac{2^{2-D}(u_2-u_1)^{D+\frac{1}{2}}}{3\Gamma(\frac{1-D}{2}+1)\Gamma(\frac{1+D}{2})}.$$

Then

$$(C^{(2)}(u))^{-1}e_2 = \begin{pmatrix}\frac{-c^{(2)}_{12}}{\det(C^{(2)}(u))} \\ \frac{c^{(2)}_{11}}{\det(C^{(2)}(u))}\end{pmatrix} = \begin{pmatrix}2^{D-2}\Gamma\left(\frac{1+D}{2}\right)(u_2-u_1)^{-\frac{D}{2}} \\ 2^{-2}\Gamma\left(\frac{1-D}{2}\right)(u_2-u_1)^{-\frac{D}{2}}\end{pmatrix}$$

And we have following equation on intersection pairing to solve κ_2,

$$\begin{aligned}(\varphi_+|\varphi_+) = 2 &=2\kappa_2^2\frac{-c^{(2)}_{12}c^{(2)}_{11}}{\left(\det(C^{(2)}(u))\right)^2}(\partial_{t_1}|\partial_{t_2}) \\ &=2\kappa_2^2 2^{D-4}\Gamma\left(\frac{1+D}{2}\right)\Gamma\left(\frac{1-D}{2}\right)(u_2-u_1)^{-D}(\partial_{t_1}|\partial_{t_2})\end{aligned}$$

$$\kappa_2^2 = \frac{2^{4-D}(u_2-u_1)^D}{\Gamma\left(\frac{1+D}{2}\right)\Gamma\left(\frac{1-D}{2}\right)}\left(\frac{c}{\Gamma(\frac{1+D}{2})\Gamma(\frac{1-D}{2})}\right)^{-1} = \frac{2^{4-D}(u_2-u_1)^D}{c}$$

Therefore,

$$\begin{aligned}\varphi_+ =& \frac{2^{2-\frac{D}{2}}(u_2-u_1)^{\frac{D}{2}}}{\sqrt{c}}\left(2^{D-2}\Gamma\left(\frac{1+D}{2}\right)(u_2-u_1)^{-\frac{D}{2}}\partial_{t_1} + 2^{-2}\Gamma\left(\frac{1-D}{2}\right)(u_2-u_1)^{-\frac{D}{2}}\partial_{t_2}\right)\\ =& \frac{1}{\sqrt{c}}\left(2^{\frac{D}{2}}\Gamma\left(\frac{1+D}{2}\right)\partial_{t_1} + 2^{-\frac{D}{2}}\Gamma\left(\frac{1-D}{2}\right)\partial_{t_2}\right)\end{aligned}$$

Denote $v_1 := -c^{-\frac{1}{2}}2^{\frac{D}{2}}\Gamma\left(\frac{1+D}{2}\right)\partial_{t_1}$, $\quad v_2 := c^{-\frac{1}{2}}2^{-\frac{D}{2}}\Gamma\left(\frac{1-D}{2}\right)\partial_{t_2}$. Then

$$\varphi_+ = -v_1 + v_2, \qquad (v_i|v_j) = -\delta_{i+j,3}, \qquad i,j \in \{1,2\}$$

For $i = 1$, the coefficients are in the following matrix

$$\tilde{\Psi}\begin{pmatrix}1 & (n+\frac{1}{2})^{-1}\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2}\left(\frac{u_2-u_1}{2}\right)^{-D} & \frac{1}{2}\left(\frac{u_2-u_1}{2}\right)^{-D}\end{pmatrix}\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2}\left(\frac{u_2-u_1}{2}\right)^{-D} & \frac{3}{2}\left(\frac{u_2-u_1}{2}\right)^{-D}\end{pmatrix}$$

where the first column stands for the coefficients of $(\lambda-u_1)^{\frac{1}{2}}$ and the second column for that of $(\lambda-u_1)^0$.

Moving x from a number $\gg 0$ to its opposite number counterclockwise along the upper circumference centered at $x=0$ and then to $x=0$ along the negative half real axis, we will take binomial expansion near $x=0$

$$\begin{aligned}\left(x^{\frac{1}{2}} + (x-1)^{\frac{1}{2}}\right)^{D-1} =& e^{\frac{D-1}{2}\pi\sqrt{-1}}\left((e^{-\pi\sqrt{-1}}x)^{\frac{1}{2}} + (1-x)^{\frac{1}{2}}\right)^{D-1}\\ =& e^{\frac{D-1}{2}\pi\sqrt{-1}}\left(1 + (e^{-\pi\sqrt{-1}}x)^{\frac{1}{2}} + \sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}(-x)^k\right)^{D-1}\\ =& e^{\frac{D-1}{2}\pi\sqrt{-1}}\left(1 + \sum_{k'=1}^{\infty}\binom{D-1}{k'}\left((e^{-\pi\sqrt{-1}}x)^{\frac{1}{2}} + \sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}(-x)^k\right)^{k'}\right)\\ =& e^{\frac{D-1}{2}\pi\sqrt{-1}} + e^{(\frac{D}{2}-1)\pi\sqrt{-1}}(D-1)x^{\frac{1}{2}} + O(x),\end{aligned}$$

where in the last equation we move from 0^- to 0^+ along the upper half circumference centered at $x=0$ clockwise and thus we get the coefficient $e^{(\frac{D}{2}-1)\pi\sqrt{-1}}(D-1)$ for $x^{\frac{1}{2}}$.

Similarly, $\left(x^{\frac{1}{2}} + (x-1)^{\frac{1}{2}}\right)^{1-D} = e^{\frac{1-D}{2}\pi\sqrt{-1}} + e^{-\frac{D}{2}\pi\sqrt{-1}}(1-D)x^{\frac{1}{2}} + O(x)$.

Again, according to $I^{(-1)}(u,\lambda) = Y^{(i)}(u,\lambda)C^{(i)}(u)$, we have an equation for the coefficient of $x^{\frac{1}{2}}$ in $I_{21}^{(-1)}$ (note that at this time we have more coefficient $(u_2-u_1)^{\frac{1}{2}}$ on the RHS to convert $(\lambda-u_1)^{\frac{1}{2}}$ into $x^{\frac{1}{2}}$) which is

$$\frac{e^{(\frac{D}{2}-1)\pi\sqrt{-1}}(D-1)2^{\frac{1-D}{2}}}{(D-1)\Gamma\left(\frac{1+D}{2}\right)}\left(\frac{u_2-u_1}{2}\right)^{\frac{1-D}{2}} = -\frac{(u_2-u_1)^{\frac{1}{2}}}{2}\left(\frac{u_2-u_1}{2}\right)^{-D}c_{11}^{(1)}.$$

Then, we have $c_{11}^{(1)} = -\frac{e^{(\frac{D}{2}-1)\pi\sqrt{-1}}2^{1-D}}{\Gamma(\frac{1+D}{2})}(u_2-u_1)^{\frac{D}{2}}$.

And we have an equation for that of x^0 in $I_{21}^{(-1)}$

$$\frac{e^{\frac{D-1}{2}\pi\sqrt{-1}}2^{\frac{1-D}{2}}}{(D-1)\Gamma\left(\frac{1+D}{2}\right)}\left(\frac{u_2-u_1}{2}\right)^{\frac{1-D}{2}} = \frac{3}{2}\left(\frac{u_2-u_1}{2}\right)^{-D}c_{21}^{(1)}.$$

Then we have $c^{(1)}_{21} = \frac{e^{\frac{D-1}{2}\pi\sqrt{-1}}2^{1-D}}{3(D-1)\Gamma(\frac{1+D}{2})}(u_2-u_1)^{\frac{1+D}{2}}$.

We do the same things for those in $I^{(-1)}_{22}$. So we have

$$c^{(1)}_{12} = -\frac{e^{-\frac{D}{2}\pi\sqrt{-1}}(1-D)}{\Gamma\left(\frac{1-D}{2}+1\right)}(u_2-u_1)^{\frac{D}{2}}, \qquad c^{(1)}_{22} = \frac{e^{\frac{1-D}{2}\pi\sqrt{-1}}}{3\Gamma\left(\frac{1-D}{2}+1\right)}(u_2-u_1)^{\frac{1+D}{2}}$$

and

$$\det(C^{(1)}(u)) = c^{(1)}_{11}c^{(1)}_{22} - c^{(1)}_{12}c^{(1)}_{21} = -\frac{e^{-\frac{1}{2}\pi\sqrt{-1}}2^{2-D}}{3\Gamma\left(\frac{1-D}{2}+1\right)\Gamma\left(\frac{1+D}{2}\right)}(u_2-u_1)^{D+\frac{1}{2}}.$$

Then

$$(C^{(1)}(u))^{-1}e_1 = \begin{pmatrix} \frac{c^{(1)}_{22}}{\det(C^{(1)}(u))} \\ \frac{-c^{(1)}_{21}}{\det(C^{(1)}(u))} \end{pmatrix} = \begin{pmatrix} -e^{(-\frac{D}{2}+1)\pi\sqrt{-1}}2^{D-2}\Gamma\left(\frac{1+D}{2}\right)(u_2-u_1)^{-\frac{D}{2}} \\ -e^{\frac{D}{2}\pi\sqrt{-1}}2^{-2}\Gamma\left(\frac{1-D}{2}\right)(u_2-u_1)^{-\frac{D}{2}} \end{pmatrix}.$$

And we have following equation on intersection pairing to solve κ_1,

$$\begin{aligned}(\varphi_-|\varphi_-) = 2 =& 2\kappa_1^2\frac{-c^{(1)}_{21}c^{(1)}_{22}}{\left(\det(C^{(1)}(u))\right)^2}(\partial_{t_1}|\partial_{t_2})\\ =& 2\kappa_1^2 e^{\pi\sqrt{-1}}2^{D-4}\Gamma\left(\frac{1+D}{2}\right)\Gamma\left(\frac{1-D}{2}\right)(u_2-u_1)^{-D}(\partial_{t_1}|\partial_{t_2})\end{aligned}$$

$$\kappa_1^2 = \frac{e^{-\pi\sqrt{-1}}2^{4-D}(u_2-u_1)^D}{\Gamma\left(\frac{1+D}{2}\right)\Gamma\left(\frac{1-D}{2}\right)}\left(\frac{c}{\Gamma(\frac{1+D}{2})\Gamma(\frac{1-D}{2})}\right)^{-1} = \frac{e^{-\pi\sqrt{-1}}2^{4-D}(u_2-u_1)^D}{c}$$

Therefore,

$$\begin{aligned}\varphi_- =& \frac{e^{-\frac{1}{2}\pi\sqrt{-1}}2^{2-\frac{D}{2}}(u_2-u_1)^{\frac{D}{2}}}{\sqrt{c}}(e^{(-\frac{D}{2}+1)\pi\sqrt{-1}}2^{D-2}\Gamma\left(\frac{1+D}{2}\right)(u_2-u_1)^{-\frac{D}{2}}\partial_{t_1}\\ &+e^{\frac{D}{2}\pi\sqrt{-1}}2^{-2}\Gamma\left(\frac{1-D}{2}\right)(u_2-u_1)^{-\frac{D}{2}}\partial_{t_2})\\ =&\frac{1}{\sqrt{c}}\left(e^{-\frac{D-1}{2}\pi\sqrt{-1}}2^{\frac{D}{2}}\Gamma\left(\frac{1+D}{2}\right)\partial_{t_1} + e^{\frac{D-1}{2}\pi\sqrt{-1}}2^{-\frac{D}{2}}\Gamma\left(\frac{1-D}{2}\right)\partial_{t_2}\right)\\ =&-e^{-\frac{D-1}{2}\pi\sqrt{-1}}v_1 + e^{\frac{D-1}{2}\pi\sqrt{-1}}v_2\end{aligned}$$

And we have $(\varphi_+|\varphi_-) = -e^{-\frac{D-1}{2}\pi\sqrt{-1}} - e^{\frac{D-1}{2}\pi\sqrt{-1}}$.
$w_{\varphi_+}, w_{\varphi_-} \in GL(H)$ are the generators of the monodromy group W, where $w_{\varphi_\pm}(\alpha) = \alpha - (\alpha|\varphi_\pm)\varphi_\pm$. The corresponding matrices under the basis of (v_1, v_2) are $w_{\varphi_\pm} = \begin{pmatrix} 0 & e^{(\frac{1-D}{2}\mp\frac{1-D}{2})\pi\sqrt{-1}} \\ e^{(\frac{D-1}{2}\mp\frac{D-1}{2})\pi\sqrt{-1}} & 0\end{pmatrix}$.

Let us check the result for $D=\frac{1}{3}$, i.e,

$$w_{\varphi_+} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad w_{\varphi_-} = \begin{pmatrix} 0 & e^{\frac{2}{3}\pi\sqrt{-1}} \\ e^{-\frac{2}{3}\pi\sqrt{-1}} & 0 \end{pmatrix}$$

$$w_{\varphi_+}\circ w_{\varphi_-} = \begin{pmatrix} e^{-\frac{2}{3}\pi\sqrt{-1}} & 0 \\ 0 & e^{\frac{2}{3}\pi\sqrt{-1}} \end{pmatrix}, \qquad w_{\varphi_+}\circ w_{\varphi_-} = \begin{pmatrix} e^{\frac{2}{3}\pi\sqrt{-1}} & 0 \\ 0 & e^{-\frac{2}{3}\pi\sqrt{-1}} \end{pmatrix}$$

$$w_{\varphi_-}\circ w_{\varphi_+}\circ w_{\varphi_-} = \begin{pmatrix} 0 & e^{\frac{4}{3}\pi\sqrt{-1}} \\ e^{-\frac{4}{3}\pi\sqrt{-1}} & 0 \end{pmatrix}$$

$$w_{\varphi_+} \circ w_{\varphi_-} \circ w_{\varphi_+} = \begin{pmatrix} 0 & e^{-\frac{2}{3}\pi\sqrt{-1}} \\ e^{\frac{2}{3}\pi\sqrt{-1}} & 0 \end{pmatrix}$$

Hence $w_{\varphi_-} \circ w_{\varphi_+} \circ w_{\varphi_-} = w_{\varphi_+} \circ w_{\varphi_-} \circ w_{\varphi_+}$. Therefore, when $D = \frac{1}{3}$, the monodromy group is isomorphic to permutation group S_3.

In general, $w_{\varphi_+} \circ w_{\varphi_+} = w_{\varphi_-} \circ w_{\varphi_-}$ is the identity element (matrix).

$$w_{\varphi_+} \circ w_{\varphi_-} = \begin{pmatrix} e^{(D-1)\pi\sqrt{-1}} & 0 \\ 0 & e^{(1-D)\pi\sqrt{-1}} \end{pmatrix} = (w_{\varphi_-} \circ w_{\varphi_+})^{-1}$$

Therefore, every element in the free group generated by w_{φ_+} and w_{φ_-} (i.e. the monodromy group W) can be reduced to the form in which w_{φ_+} and w_{φ_-} appear alternatively, which is attributed to the power $\frac{1}{2}$ on x and $(x-1)$. And we can see that the monodromy group is generated by $w_{\varphi_+} \circ w_{\varphi_-}$ and w_{φ_+} as well.

3.6 The image and inverse of the period map based on the classification

Recalling the section of the period map, let

$$\begin{aligned} z_1 :=& \langle Z(t,\lambda), v_1\rangle = \langle Z(t,\lambda), -c^{-\frac{1}{2}} 2^{\frac{D}{2}} \Gamma\left(\frac{1+D}{2}\right) \partial_{t_1}\rangle \\ =& \frac{\sqrt{2c}}{1-D}\left(x^{\frac{1}{2}} + (x-1)^{\frac{1}{2}}\right)^{D-1} \left(\frac{u_2-u_1}{2}\right)^{\frac{1-D}{2}} \\ z_2 :=& \langle Z(t,\lambda), v_2\rangle = \langle Z(t,\lambda), c^{-\frac{1}{2}} 2^{-\frac{D}{2}} \Gamma\left(\frac{1-D}{2}\right) \partial_{t_2}\rangle \\ =& \frac{\sqrt{2c}}{1-D}\left(x^{\frac{1}{2}} + (x-1)^{\frac{1}{2}}\right)^{1-D} \left(\frac{u_2-u_1}{2}\right)^{\frac{1-D}{2}}, \end{aligned}$$

Then, recalling subsection 1.3.2 the orbit space $B := M/\mathbb{C}$ with the submanifold $\{t_1 = 0\} \subset M$ which coincides with notation in Chapter 2, now let us consider the period map on $((B \times \mathbb{C})')^{\sim}$.

As for the first case in Theroem 2.7,

$$\begin{aligned} z_1 =& 2^{\frac{1-D}{2}} \frac{\sqrt{2c}}{1-D}\left((\lambda - e^{\hat{x}_2})^{\frac{1}{2}} + (\lambda + e^{\hat{x}_2})^{\frac{1}{2}}\right)^{D-1} e^{(1-D)\hat{x}_2} \in \mathbb{C}^* \\ z_2 =& 2^{\frac{D-1}{2}} \frac{\sqrt{2c}}{1-D}\left((\lambda - e^{\hat{x}_2})^{\frac{1}{2}} + (\lambda + e^{\hat{x}_2})^{\frac{1}{2}}\right)^{1-D} \in \mathbb{C}^*, \end{aligned}$$

Then,

$$z_1 z_2 = \frac{2ce^{(1-D)\hat{x}_2}}{(1-D)^2} \neq 0.$$

On the other hand, when we scale λ, u_1 and u_2 by a nonzero complex constant, we will find that z_1 and z_2 will be scaled by the constant to the power of $\frac{1-D}{2}$, which implies that the image $\Omega \subset H^*$ of the period map Z is the $\mathbb{C}^*$-fiber of a domain in $\mathbb{P}^1$. Consider $\frac{z_2}{z_1}$ (or $\frac{z_1}{z_2}$)

$$\frac{z_2}{z_1} = \left(x^{\frac{1}{2}} + (x-1)^{\frac{1}{2}}\right)^{2(1-D)} = \left(2x - 1 + 2x^{\frac{1}{2}}(x-1)^{\frac{1}{2}}\right)^{1-D}$$

Note that $\det(\lambda - E\bullet) \neq 0$ yields $x \neq 0$ and $x \neq 1$. The problem of figuring Ω out is reduced to that of $\{2x - 1 + 2x^{\frac{1}{2}}(x-1)^{\frac{1}{2}} | x \in \mathbb{C} \setminus \{0,1\}\} = \mathbb{C} \setminus \{0, \pm 1\}$. In conclusion,

$$\Omega = \left\{ (z_1, z_2) \in (\mathbb{C}^*)^2 \middle| z_1 z_2 = \frac{2ce^{(1-D)\hat{x}_2}}{(1-D)^2}, \hat{x}_2 \in \mathbb{C} \right\} - \bigcup_{m \in \mathbb{Z}} V_m$$

where $V_m := \left\{(z_1, z_2) \in (\mathbb{C}^*)^2 \middle| \frac{z_2}{z_1} = e^{(1-D)m\pi\sqrt{-1}}\right\}$.

As for the third case in Theroem 2.7, since $1 - D \in \mathbb{Q} \setminus (2\mathbb{Z})$, $q \neq 1$, i.e., $q \in \mathbb{Z}_{\geqslant 2}$ and then $p \in \mathbb{Z}_{\neq 1}$.

$$\begin{aligned} z_1 &= 2^{\frac{1-D}{2}} \frac{\sqrt{2c}}{1-D} \left((\lambda - \bar{x}_2^{\frac{q}{2}})^{\frac{1}{2}} + (\lambda + \bar{x}_2^{\frac{q}{2}})^{\frac{1}{2}} \right)^{D-1} \bar{x}_2^{p} \in \mathbb{C}^* \\ z_2 &= 2^{\frac{D-1}{2}} \frac{\sqrt{2c}}{1-D} \left((\lambda - \bar{x}_2^{\frac{q}{2}})^{\frac{1}{2}} + (\lambda + \bar{x}_2^{\frac{q}{2}})^{\frac{1}{2}} \right)^{1-D} \in \mathbb{C}^*, \end{aligned}$$

Similar discussion yields

$$\Omega = \left\{ (z_1, z_2) \in (\mathbb{C}^*)^2 \middle| z_1 z_2 = \frac{2c\bar{x}_2^{p}}{(1-D)^2}, \bar{x}_2 \in \mathbb{C}^* \right\} - \bigcup_{m \in \mathbb{Z}} V_m$$

where $V_m := \left\{(z_1, z_2) \in (\mathbb{C}^*)^2 \middle| \frac{z_2}{z_1} = e^{\frac{2p}{q} m\pi\sqrt{-1}}\right\}$.

As for the second case in Theroem 2.7,

$$\begin{aligned} z_1 &= 2^{\frac{1-D}{2}} \frac{\sqrt{2c}}{1-D} \left((\lambda - \bar{x}_2^{\frac{q}{2}})^{\frac{1}{2}} + (\lambda + \bar{x}_2^{\frac{q}{2}})^{\frac{1}{2}} \right)^{D-1} \bar{x}_2 \in \mathbb{C} \\ z_2 &= 2^{\frac{D-1}{2}} \frac{\sqrt{2c}}{1-D} \left((\lambda - \bar{x}_2^{\frac{q}{2}})^{\frac{1}{2}} + (\lambda + \bar{x}_2^{\frac{q}{2}})^{\frac{1}{2}} \right)^{1-D} \in \mathbb{C}^*, \end{aligned}$$

Similar discussion yields

$$\Omega = \left\{ (z_1, z_2) \in \mathbb{C} \times \mathbb{C}^* \middle| z_1 z_2 = \frac{2c\bar{x}_2}{(1-D)^2}, \bar{x}_2 \in \mathbb{C} \right\} - \bigcup_{m \in \mathbb{Z}} V_m$$

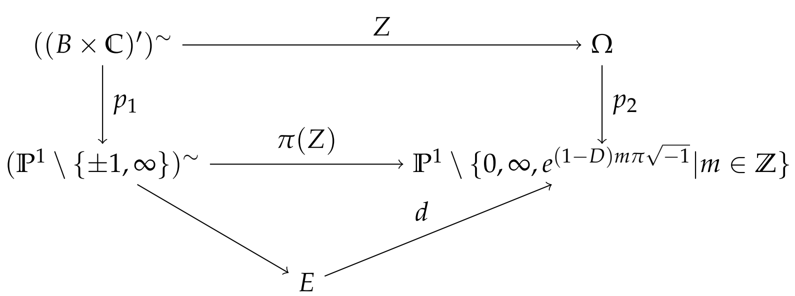
where $V_m := \left\{(z_1, z_2) \in \mathbb{C} \times \mathbb{C}^* \middle| \frac{z_2}{z_1} = e^{\frac{2}{q} m\pi\sqrt{-1}}\right\}$.

The period map (under the basis (v_1, v_2)) can be expressed as a row matrix $\begin{pmatrix} z_1(t,\lambda) & z_2(t,\lambda) \end{pmatrix}$.

$$\begin{aligned} \begin{pmatrix} z_1(t,\lambda) & z_2(t,\lambda) \end{pmatrix} w_{\varphi_+} \circ w_{\varphi_-} &= \begin{pmatrix} z_1 e^{(D-1)\pi\sqrt{-1}} & z_2 e^{(1-D)\pi\sqrt{-1}} \end{pmatrix} \\ \begin{pmatrix} z_1(t,\lambda) & z_2(t,\lambda) \end{pmatrix} w_{\varphi_+} &= \begin{pmatrix} z_2 & z_1 \end{pmatrix} \end{aligned}$$

If $1 - D \in \mathbb{Q} \setminus (2\mathbb{Z} \cup \{1\})$ and, recalling the notation of the previous chapter, $1 - D = \frac{2p}{q}$ where $q \in \mathbb{Z}_{>1}, p \in \mathbb{Z}$ except $q = 2p = 2$ and $\gcd(p, q) = 1$, then $(w_{\varphi_+} \circ w_{\varphi_-})^q = (w_{\varphi_-} \circ w_{\varphi_+})^q$ is the identity element of the monodromy group. Note that q is the smallest positive integer to get the identity element. Thus, if $1 - D \in \mathbb{Q} \setminus 2\mathbb{Z}$, then the monodromy group is isomorphic to dihedral group D_q, and if $1 - D \in \mathbb{C} \setminus \mathbb{Q}$ then the monodromy group is isomorphic to infinite dihedral group Dih_∞.

Let us move on to the inverse of the period map and first deal with the third case in Theroem 2.7 using the notation of subsection 1.3.2



where

$$p_1(\bar{x}_2,\lambda) := \frac{\lambda}{\bar{x}_2^{\frac{q}{2}}} =: 2x-1, \qquad p_2(z_1,z_2) := \frac{z_2}{z_1}$$

and

$$\pi(Z)(\frac{\lambda}{\bar{x}_2^{\frac{q}{2}}}) := \left(2x-1+2x^{\frac{1}{2}}(x-1)^{\frac{1}{2}}\right)^{1-D}.$$

Furthermore, (E,d) is monodromy cover ($\begin{cases} |p| & \text{if } q=2 \\ 2|p| & \text{if } q\in\mathbb{Z}_{>2}\end{cases}$-fold due to power $1-D$ of $\pi(Z)$) of $\mathbb{P}^1\setminus\{0,\infty,e^{\frac{2p}{q}m\pi\sqrt{-1}}|m\in\mathbb{Z}\}$ And we can invert $\pi(Z)$ by $j(z) := \frac{1}{2}(z^{\frac{1}{1-D}}+z^{-\frac{1}{1-D}})$, i.e. $(j\circ\pi(Z))(2x-1) = 2x-1$. Obviously, j is invariant under monodromy. And I would like to say that the power $\frac{1}{1-D}$ in $j(z)$ means j lifts $z(2x-1)$ to one of $\begin{cases} |p| & \text{if } q=2 \\ 2|p| & \text{if } q\in\mathbb{Z}_{>2}\end{cases}$ sheets of E and we set the sheet which $z(2x-1)$ is lifted into to be the sheet where $2x-1$ is located.

Thus we have following

$$\begin{aligned} f_1 =& \frac{(1-D)^2}{2c}z_1z_2 \in \Gamma(\Omega,\mathcal{O}_{H^*})^W \\ f_2 =& \frac{1}{2}(\left(\frac{z_2}{z_1}\right)^{\frac{1}{1-D}} + \left(\frac{z_2}{z_1}\right)^{-\frac{1}{1-D}})(\frac{(1-D)^2}{2c}z_1z_2)^{\frac{1}{1-D}} \\ =& \frac{1}{2}(\frac{(1-D)^2}{2c})^{\frac{1}{1-D}}(z_1^{\frac{2}{1-D}}+z_2^{\frac{2}{1-D}}) = \frac{1}{2}(\frac{(1-D)^2}{2c})^{\frac{q}{2p}}(z_1^{\frac{q}{p}}+z_2^{\frac{q}{p}}) \end{aligned} \tag{3.3}$$

where $f_1\in\mathcal{M}(\Omega,W)$ since $f_1\circ Z$ extends analytically across the discriminant, i.e. $f_1\circ Z\in\mathcal{O}(B\times\mathbb{C})$, and $f_2\circ Z\in\mathcal{O}(B\times\mathbb{C})$ but f_2 is $|p|$-value function though f_2 is W-invariant. In fact, $f_1\circ Z = \bar{x}_2^p$ and $f_2\circ Z = \lambda$

The condition that $|p| = 1$ in the third case of Theroem 2.7 means $p=-1$, then $\frac{1}{f_1}\in\Gamma(\Omega,\mathcal{O}_{H^*})^W$ and $\frac{1}{f_1}\circ Z = \bar{x}_2$. Then $\{\frac{1}{f_1},f_2\}$ is the inverse of the period map Z.

In the second case of Theroem 2.7, $p=1$. One can check that $\{f_1,f_2\}$ defined as above is the inverse of the period map Z.

In the first case of Theroem 2.7, the diagram is the same as that of the third case, but where $p_1(\bar{x}_2,\lambda) := \frac{\lambda}{e^{\hat{x}_2}} =: 2x-1$ and (E,d) is monodromy cover of infinite fold. Then $\frac{1}{1-D}\log f_1\circ Z = \hat{x}_2$ and $f_2\circ Z=\lambda$ but $\frac{1}{1-D}\log f_1$ and f_2 are not holomorphic function on Ω.

Theorem 3.1 *Providing conformal dimension $D\in\mathbb{C}\setminus((2\mathbb{Z}+1)\cup\{0\})$, the period map Z is invertible if and only if D is in one of the following two cases*

1. *$D = 1-\frac{2}{q}$ for some $q\in\mathbb{Z}_{\geqslant 3}$, and in this case $\{f_1,f_2\}$ is the inverse of the period map Z.*

2. *$D = 1+\frac{2}{q}$ for some $q\in\mathbb{Z}_{\geqslant 2}$, and in this case $\{\frac{1}{f_1},f_2\}$ is the inverse of the period map Z.*

where (3.3) gives the explicit formula by setting $p=1$ and $p=-1$ for the first and the second cases, respectively.

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