# 博士論文

論文題目　Integral Structures in the Local Algebra of a Singularity

（特異点の局所代数の整構造について）

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# Contents

# Chapter 1

# Introduction

Let $f \in \mathbb{C}[x_1, x_2, x_3]$ be a weighted homogeneous polynomial representing the germ of a simple singularity of type A, D, or E. Let $f^T \in \mathbb{C}[x_1, x_2, x_3]$ be the corresponding Berglund–Hübsch dual of $f$ (see Section 4.1.1). Fan–Jarvis–Ruan proved in [16], using also results of Givental–Milanov [22] and Frenkel–Givental–Milanov [18], that the generating function of Fan–Jarvis–Ruan–Witten (FJRW) invariants of $f^T$ can be identified with a tau-function of a specific Kac–Wakimoto hierarchy. The identification however involves rescaling the dynamical variables of the Kac–Wakimoto hierarchy and the precise values of the rescaling constants were left unknown. One application of the results in this thesis is to obtain explicit formulas for the rescaling constants. Such an explicit identification is needed if one is interested in constructing a matrix model for the FJRW invariants of $f^T$, similar to the Kontsevich's matrix model in [30]. We are not going to compute the rescaling coefficients in this thesis. The computation is straightforward and it should probably be done only when needed. Let us try to explain instead why this small technical detail leads to a very interesting problem in singularity theory.

Let us recall that for any singularity $f$ there is a natural way to construct a semi-simple Frobenius structure on the space of miniversal deformations of $f$ (see [24]). The construction depends on the choice of a *primitive form* in the sense of Saito [43] and it essentially coincides with what Saito called *flat structure*. On the other hand, motivated by Gromov–Witten theory, Givental introduced the notion of a *total descendent potential* for every semi-simple Frobenius manifold (see [21, 20]). Givental conjectured [21] and Teleman proved [49] that if the Frobenius structure corresponding to the quantum cohomology of a compact Kähler manifold $X$ is semi-simple, then his definition coincides with the generating function of Gromov–Witten invariants of $X$. Let us return to our settings, i.e., the case of a simple weighted homogeneous singularity $f$ on 3 variables. The standard holomorphic volume form $dx_1 \wedge dx_2 \wedge dx_3$ is primitive. Therefore, following Givental, we can define total descendent potential. The latter will be called, the total descendent potential of $f$. Fan–Jarvis–Ruan proved in [16] that the generating function of FJRW invariants of $f^T$ coincides with the *total descendant potential* of $f$. Furthermore, Givental–Milanov [22] and Frenkel–Givental–Milanov [18] proved that the total descendant potential of $f$ is a tau-function of the principal Kac–Wakimoto hierarchy of the same type A, D, or E as the singularity $f$. Finally, the outcome of the above work is that the generating function of FJRW invariants of $f^T$ is a tau-function of an appropriate Kac–Wakimoto hierarchy. However, there is still a small gap in this statement. Namely, while the state space of FJRW theory is identified explicitly with the Milnor ring of the singularity (see [16]), the identification of the Milnor ring and the Cartan subalgebra of the corresponding simple Lie algebra is given by a period map and it is not explicit. In order to obtain an explicit identification, we need to determine the image of the root lattice in the Milnor ring of the singularity. This is exactly the problem that we want to solve in this thesis.

It turns out that our answer can be stated quite elegantly via relative K-theory. The idea to look for such a description comes from the work of Iritani [27], Chiodo–Iritani–Ruan [11], and Chiodo–Nagel [12]. More precisely, Iritani was able to prove in [27] that the Milnor lattice of the mirror of a Fano toric orbifold $X$ can be identified with the topological K-ring $K^0(X)$. The identification uses a period map to embed the Milnor lattice in $H^*(X;\mathbb{C})$ and a certain $\Gamma$-class modification of the Chern character map to embed $K^0(X)$ in $H^*(X;\mathbb{C})$. The lattice in $H^*(X;\mathbb{C})$, obtained either as the image of the Milnor lattice via the period map or as the image of $K^0(X)$ via the $\Gamma$-class modification of the Chern character map, is known as $\Gamma$-integral structure in quantum cohomology. Isolated singularities are almost never mirror models of a manifold. Nevertheless, Chiodo–Iritani–Ruan have proposed an analogue of the $\Gamma$-integral structure for singularities of Fermat type. The analogue of $H^*(X;\mathbb{C})$ is played by the Milnor ring $H_f$, while $K^0(X)$ is replaced with an appropriate category of equivariant matrix factorizations of $f$. Finally, Chiodo–Nagel were able to find an isomorphism between $H_f$ and an appropriate relative orbifold cohomology group. Since, the Chern character gives an isomorphism between cohomology and K-theory and the Grothendieck group of the category of matrix factorization also has the flavor of a topological K-ring, after expecting more carefully the constructions in [11] and [12], we see that there is a natural candidate for a $\Gamma$-integral structure for Fermat type singularities. After several trial and errors we were able to find the correct topological $K$-ring and the correct modification of the Chern character map. Consequently, the main theorem (Theorem 3.3) of this thesis the following

**Theorem 1.1** There exists a linear isomorphism

$$\mathrm{mir}:\ H_f \longrightarrow H^*_{\mathrm{orb}}([\mathbb{C}^3/G^T],[V_1^T/G^T];\mathbb{C})\ ,$$

such that, the map

$$\mathrm{mir}^{-1}\circ \mathrm{ch}_\Gamma:\ K^0_{\mathrm{orb}}([\mathbb{C}^3/G^T],[V_1^T/G^T]) \xrightarrow{\cong} \Psi(H_2(f^{-1}(1);\mathbb{Z}))$$

is an isomorphism of Abelian groups.

where $H_f$ is the Milnor ring of the singularity and $\Psi$ is the period map. The theorem identifies the Milnor lattice $\Psi(H_2(f^{-1}(1);\mathbb{Z})) \subset H_f$ with the image of relative K-group via the $\Gamma$-class modification of the Chern character map.

Moreover our proposal makes sense not only for Fermat type polynomials, but more generally for an arbitrary *invertible polynomial*. Nevertheless, let us return to our current settings of simple singularities. We believe that our results can be generalized to all invertible polynomials. In Chapter 4, we will give some progress that we have already made, which is the Seifert form of the basis of middle homology group.

Yet another series of works are worth mentioning. They mainly focus on the conjecture that for an invertible polynomial $f \in \mathbb{C}[x_1,\dots,x_n] =: S$, there is an equivalence of triangulated categories between triangulated category $\mathrm{HMF}_S^{L_f}(f)$ of $L_f$-graded matrix factorizations of $f$ and derived directed Fukaya category of $f^T$ [15], [48]. Futaki–Ueda proved homological mirror symmetry for Lefschetz fibrations obtained as Sebastiani–Thom sums of polynomials of types A or D in [19] and produced a more intricate formula for the group action on both sides of the mirror correspondence. Based on this conjecture, it is natural to expect the existence of a full exceptional collection in $\mathrm{HMF}_S^{L_f}(f)$. Hirano-Ouchi [25] prove this for $f$ being of chain type by their semi-orthogonal decomposition theorem and an induction on the number of variables in $f$. In the updated version, they further explicitly constructed a full strong exceptional collection. Aramaki–Takahashi [1] also gave a construction of a full exceptional collection and its Euler matrix was given. Then a work which is highly related to the thesis is due to Otani–Takahashi [41]. In their paper, the main theorem is that there is a mirror isomorphism between Gamma integral structure on the full exceptional collection of Aramaki–Takahashi [1] and integral structure on the Milnor ring defined by the image of $\Pi$ from middle homology group. The result of Chapter 2 was considered as ADE cases of their result. Cycles of middle homology group of Milnor fiber of a chain type polynomial were constructed inductively. We will calculate the Seifert form of this basis of cycles in Chapter 3.

For the calculation of equivariant topological K-ring, one of the methods is to figure out the $G$-CW complex which is a $G$-deformation retract of the Milnor fiber. Ruddat–Sibilla-Treumann–Zaslow [42] tell us that Log geometry can be applied to find a deformation retract of smooth affine hypersurfaces. Some works are worth doing to get the equivariant case, i.e., combinatorial description for certain $G$-cellular decomposition of the Milnor fiber.

## 1.1 Organization of the thesis

In Chapter 2, we first give some preliminaries on equivariant topological K-theory, which is useful for Section 3.3. Section 2.2 (Frobenius manifolds) and Section 2.3 (Calibration) is not used in the latter chapters; however, they are helpful for explaining reflection vectors and integral structure. The section of integral structure also introduces the motivation and application of the result of this thesis.

In Chapter 3, we compute the image of the Milnor lattice of an ADE singularity under a period map. We also prove that the Milnor lattice can be identified with an appropriate relative K-group defined through the Berglund-Huebsch dual of the corresponding singularity. In the last chapter, we figure out the image (Proposition 4.14) of the Milnor lattice of the singularity of an invertible polynomial of chain type using the basis of middle homology constructed by Otani-Takahashi [41]. We give the Seifert form of the basis as well. Some conjectures on equivariant K-theory are raised.

# Chapter 2

# Background

In this chapter, we will introduce some general background for the purpose of narration of next several chapters in which we will deal with more specific situation.

## 2.1 Equivariant K-theory

The purpose of this section is to set down the basic facts about equivariant K-theory. Ordinary K-theory was invented by M.F. Atiyah, and most of the results are due to him – see [4], [7], [6], [5]. The equivariant version of K-theory was introduced by G. Seagel [46] following ideas of Atiyah. Let us outline the main steps in the construction of equivariant K-theory. Our exposition follows closely [46].

### 2.1.1 $G$-vector bundles

Let us fix a topological group $G$. A *$G$-space* is a topological space $X$ together with a continuous map $G \times X \to X$ satisfying the associativity condition $g_1(g_2x) = (g_1g_2)x$ for all $g_1, g_2 \in X$ and $x \in X$ and the condition $ex = x$ for all $x \in X$ and $e$ is the identity element of $G$.

A *$G$-map* between two $G$-spaces is a continuous map which commutes with the action of $G$. More generally, if $X$ is a $G$-space and $Y$ is an $H$-space and $\theta : H \to G$ is a continuous group homomorphism, we say that $f : Y \to X$ is a $\theta$-equivariant map if it is continuous and if $f(hy) = \theta(h)f(y)$ for all $h \in H, y \in Y$

**Definition 2.1** Let $X$ be a $G$-space.

a) A $G$-space $E$ together with a $G$-map $p : E \to X$ is said to be a *$G$-equivariant* vector bundle or simply a *$G$-vector bundle* on $X$ if the following two conditions hold:

(i) $p : E \to X$ is a complex vector bundle on $X$.

(ii) for every $g \in G$ and $x \in X$, the group action $g : E_x \to E_{gx}$ is a linear map, where $E_x := p^{-1}(x)$ denotes the fiber of $E$ at $x$.

b) If $E$ and $F$ are $G$-vector bundles on $X$, then a *$G$-equivariant* morphism or simply a *$G$-homomorphism* $\phi : E \to F$ is a map which is both a vector bundle homomorphism and a $G$-map.

Let us introduce the following notation. If $M$ is a finite-dimensional complex representation of $G$ and $X$ is a given $G$-space, then $\underline{M} := X \times M$ has a natural structure of a $G$-vector bundle on $X$, that is, $\underline{M}$ is a trivial vector bundle on $X$ equipped with the diagonal action of $G$: $g \cdot (x, \lambda) = (gx, g\lambda)$. If $E$ and $F$ are two $G$-vector bundles on $X$ then the vector bundles $E \oplus F$, $E \otimes F$, and $\mathrm{Hom}(E, F)$ have a natural structure of $G$-vector bundles, that is, the $G$-action is defined as follows:

$$g \cdot (\lambda + \mu) := g\lambda + g\mu, \quad g \cdot (\lambda \otimes \mu) := (g\lambda) \otimes (g\mu), \quad g \cdot f(\lambda) := gf(g^{-1}\lambda),$$

where $\lambda \in E_x$, $\mu \in F_x$, and $f \in \mathrm{Hom}(E_x, F_x)$. Finally, if $\phi : Y \to X$ is a $G$-map of $G$-spaces, and $E$ is a vector bundle on $X$, then the pullback vector bundle $\phi^* E$ has a natural structure of a $G$-vector bundle. More generally, if $Y$ is an $H$-space, $X$ a $G$-space, $\alpha : H \to G$ a homomorphism, and $\phi : Y \to X$ such that $\phi(hy) = \alpha(h)\phi(y)$, then $\phi^* E$ is an $H$-vector bundle on $Y$. If $i : Y \to X$ is the inclusion of a subspace, $i^* E$ is often written $E|_Y$.

### 2.1.2 Partition of unity

**Definition 2.2** A topological space $X$ is said to be *locally compact* if for every $x \in X$ there exists an open subset $U \subset X$, such that, $x \in U$ and the closure $\overline{U}$ of $U$ in $X$ is compact.

The proof of the following lemma can be found in [28].

**Lemma 2.3** If $X$ is locally compact topological space and $K \subset U$ where $K$ and $U$ are respectively a compact and an open subset of $X$, then there exists an open subset $V$ of $X$, such that, the closure $\overline{V}$ of $V$ in $X$ is compact and

$$K \subset V \subset \overline{V} \subset U.$$

Suppose that $f : X \to \mathbb{R}$ is a continuous function. Let us introduce the following notation: $K \prec f$ stands for

(i) $K$ is compact,

(ii) $f$ has compact support and $0 \leq f(x) \leq 1$ for all $x \in X$

(iii) $f(x) = 1$ for all $x \in K$,

and $f \prec U$ stands for

(i) $U$ is open,

(ii) $f$ has compact support and $0 \leq f(x) \leq 1$ for all $x \in X$

(iii) $\mathrm{supp}(f) \subset U$.

The following result, known also as Urysohn's lemma, is well known.

**Lemma 2.4** Suppose that $X$ is a locally compact Hausdorff topological space, $K$ is a compact subset of $X$, and $U$ is an open neighborhood of $K$, that is, $U$ is open and $K \subset U$. Then, there exists a continuous function $f : X \to \mathbb{R}$ with compact support, such that, $K \prec f \prec U$.

**Theorem 2.5 (Partition of Unity)** Suppose that $X$ is a locally compact Hausdorff space, $K$ is a compact subset, and $\{U_i\}_{i=1}^n$ is a finite open covering of $K$. Then, there exists $h_i \prec U_i$, such that,

$$h_1(x) + h_2(x) + \cdots + h_n(x) = 1 \quad \forall x \in K.$$

**Proof** The argument follows the exposition of Rudin. If $x \in K$, then $x \in U_i$ for some $i$. Recalling Lemma 2.3 we get that there exists an open neighborhood $W_x$ of $x$, such that, $W_x \subset \overline{W}_x \subset U_i$ where the closure $\overline{W}_x$ is compact. Since $K$ is compact, we can find finitely many points $x_1, \ldots, x_m$, such that, the open subsets $W_j := W_{x_j}$ $(1 \le j \le m)$ form an open covering of $K$. Let $H_i$ be the union of all $\overline{W}_j$, such that, $\overline{W}_j \subset U_i$. Note that $H_i$ is compact and that $H_i \subset U_i$. By Urysohn's lemma we get that there exists a continuous function $g_i : X \to \mathbb{R}$, such that, $H_i \prec g_i \prec U_i$. It is straightforward to check that the functions

$$\begin{aligned} h_1 &:= g_1 \\ h_2 &:= (1-g_1)g_2 \\ &\vdots \\ h_n &:= (1-g_1)\cdots(1-g_{n-1})g_n \end{aligned}$$

have all required properties. □

The above theorem has the following corollary. If $X$ is a compact Hausdorff space and $\{U_i\}_{i\in I}$ is an arbitrary open covering of $X$, then there exists a set of continuous functions $f_i : X \to \mathbb{R}$, such that,

(i) $f_i \prec U_i$,

(ii) $\forall x \in X$ the set $\{i \in I \mid f_i(x) \neq 0\}$ is finite,

(iii) $\sum_{i\in I} f_i(x) = 1 \; \forall x \in X$.

A set of continuous functions $\{f_i\}_{i\in I}$ satisfying the above properties is said to be a *partition of unity* subordinate to the open covering $\{U_i\}_{i\in I}$.

Finally, let us recall the *Tietze extension theorem*. Recall that a topological space $X$ is said to be *normal* if every two disjoint closed subsets $A$ and $B$ of $X$ have open neighborhoods $A \subset U$ and $B \subset V$, such that, $U \cap V = \emptyset$.

**Theorem 2.6 (Tietze Extension Theorem)** Suppose that $X$ is a normal topological space, $A \subset X$ is a closed subset, and $f : A \to \mathbb{R}$ is a continuous function. Then, $f$ can be extended continuously on the entire space $X$, that is, there exists a continuous function $F : X \to \mathbb{R}$, such that, $F|_A = f$. Moreover, if $f$ is bounded on $A$, then there exists a continuous extension $F$, such that,

$$|F(x)| \le \sup\{|f(y)| \mid y \in A\}, \quad \forall x \in X.$$

Using Lemma 2.3 it is easy to prove that every compact Hausdorff space is normal. In particular, Tietze extension theorem applies to compact Hausdorff spaces.

### 2.1.3 Equivariant K-Theory

We now come to the definition of the ring $K_G(X)$. The definition makes sense for arbitrary $G$-spaces $X$. However, the resulting groups are known to have nice properties only if $X$ is a compact Hausdorff space. Let us assume that $G$ is a compact Lie group and that $X$ is a compact Hausdorff $G$-space.

**Proposition 2.7** If $E$ is a $G$-vector bundle on $X$ and $A \subset X$ is a closed $G$-subspace, then every continuous $G$-equivariant section of $E|_A$ extends to a $G$-equivariant section of $E$.

**Proof** The standard argument from the non-equivariant case (see [7], Lemma 1.1) works in the equivariant settings too. Let us outline the main steps. Suppose that $s : A \to E$ is a $G$-equivariant section. Let $\{U_i\}_{i=1}^m$ be an open covering of $A$, such that, $E|_{U_i} \cong U_i \times \mathbb{C}^r$ is trivial. Under the trivializations, we have $s(a) = (a, s_i(a))$ for some continuous functions $s_i : U_i \cap A \to \mathbb{C}^r$. Recalling the Tietze extension theorem, we get that $s|_{U_i \cap A}$ extends to a continuous section $\widetilde{s_i} \in \Gamma(U_i, E)$. Let $h_i$ $(1 \le i \le m)$ be a partition of unity subordinate to the covering $\{U_i\}_{i=1}^m$, that is, $h_i \prec U_i$ and $\sum_{i=1}^m h_i(x) = 1$ for all $x \in A$. Put $t(x) := \sum_{i=1}^m s_i(x)h_i(x)$. Note that $t$ is a global section of $E$ because $\mathrm{supp}(h_i) \subset U_i$ and that $t(x) = s(x)$ for all $x \in A$, that is, $t$ is an extension of $s$. Finally, in order to construct a $G$-equivariant extension, we need just to take the average

$$\widetilde{t}(x) := \int_{g \in G} g^{-1} t(gx) d\mu(g),$$

where $d\mu(g)$ is the Haar measure on $G$. Clearly, $\widetilde{t}$ is the required extension. □

Proposition 2.7 has two important corollaries.

**Corollary 2.8** Suppose that $E$ and $F$ are $G$-vector bundles on $X$, $A \subset X$ is a closed $G$-subspace of $X$, and $f : E|_A \to F|_A$ is a morphism of $G$-vector bundles. Then

a) The morphism $f$ extends to a morphism of $G$-vector bundles $\widetilde{f} : E \to F$.

b) If $f$ is an isomorphism, then for every extension $\widetilde{f}$ as in a), there exists a $G$-equivariant open neighborhood $U$ of $A$ in $X$, such that, $\widetilde{f}|_U : E|_U \cong F|_U$ is an isomorphism.

**Definition 2.9** Suppose that $X$ and $Y$ are two $G$-spaces and $\varphi_i : Y \to X$ $(i = 0, 1)$ are two $G$-equivariant continuous maps. We say that $\varphi_0$ and $\varphi_1$ are $G$-homotopic if there exists a continuous map $H : [0, 1] \times Y \to X$, such that,

(i) $H(t, gy) = gH(t, y)$ for all $(t, y) \in [0, 1] \times Y$ and $g \in G$,

(ii) $H(0, y) = \varphi_0(y)$ and $H(1, y) = \varphi_1(y)$ for all $y \in Y$.

**Corollary 2.10** If $Y$ is a compact $G$-space and $\varphi_i : Y \to X$ $(i = 0, 1)$ are $G$-equivariant $G$-homotopic maps, then $\varphi_0^* E \cong \varphi_1^* E$ for every $G$-vector bundle $E$ on $X$.

The proofs of both Corollaries are straightforward generalizations of the well known proofs from the non-equivariant case (see [7], Lemma 1.2 and Proposition 1.3).

The set of isomorphism classes of $G$-vector bundles on $X$ forms an abelian semigroup under $\oplus$. The associated abelian group is called $K_G(X)$: its elements are formal differences $E_0 - E_1$ of $G$-vector bundles on X, modulo the equivalence relation $E_0 - E_1 = E_0' - E_1' \Leftrightarrow E_0 \oplus E_1' \oplus F \cong E_0' \oplus E_1 \oplus F$ for some $G$-vector bundle $F$ on $X$. The tensor product of $G$-vector bundles induces the structure of a commutative ring in $K_G(X)$.

**Definition 2.11** Suppose that $X$ and $Y$ are $G$-spaces. A continuous $G$-map $f : X \to Y$ is said to be a *$G$-homotopy equivalence* if there exists a continuous $G$-map $g : Y \to X$, such that, $f \circ g$ is $G$-homotopy equivalent to the identity map $\mathrm{id}_Y$ and $g \circ f$ is $G$-homotopy equivalent to the identity map $\mathrm{id}_X$.

Recalling Corollary 2.10, we get that the isomorphism class of the ring $K_G(X)$ depends only on the $G$-homotopy type of $X$. Namely, note that if $\phi : Y \to X$ is a $G$-map of compact $G$-spaces, then the pullback functor $E \mapsto \phi^* E$ induces a morphism of rings $\phi^* : K_G(X) \to K_G(Y)$. If $\phi$ is a homotopy equivalence, then $\phi^*$ is an isomorphism. Let us list several cases in which the equivariant K-ring can be computed explicitly in terms of the representation ring $R(G)$ of the compact Lie group $G$ and the non-equivariant $K$-ring $K(X)$.

1. If $G$ is the trivial group, then $K_G(X) = K(X)$.
2. If $X$ is a point, then $K_G(\mathrm{pt}) = R(G)$. In particular, since we always have a contraction $G$-map $X \to \mathrm{pt}$, the ring $K_G(X)$ is a $R(G)$-module. The map $R(G) \to K_G(X)$ is induced by $M \mapsto \underline{M}$.
3. $K_G(G/H) \cong R(H)$ when $H$ is a closed subgroup of $G$.
4. If $G$ acts freely on $X$, then the pullback via the quotient map $\mathrm{pr} : X \to X/G$ induces an isomorphism $K_G(X) \cong K(X/G)$.

The following case will be quite useful for our purposes. Suppose that $\alpha : G \to H$ is a surjective homomorphism of compact Lie groups and that $X$ is a compact Hausdorff $H$-space. Then we have a homomorphism $K_H(X) \to K_G(X)$ which gives an $H$-vector bundle the $G$-action induced from the homomorphism $\alpha$. Combining this with the natural map $R(G) \to K_G(X)$ we have a morphism of rings $\mu : R(G) \otimes_{R(H)} K_H(X) \to K_G(X)$. The case when $H = 1$ was considered by Segal – see [46], Proposition 2.2. He proved that $\mu$ is an isomorphism. Unfortunately, Segal's argument is hard to generalize if $H \neq 1$, that is, we could not prove that $\mu$ is an isomorphism for $H \neq 1$. However, if we make an extra assumption that $G$ is a finite abelian group, then we have the following result.

**Proposition 2.12** If $\alpha : G \to H$ is a surjective homomorphism of finite abelian groups and $X$ is a compact Hausdorff $H$-space, then the natural map

$$\mu : R(G) \otimes_{R(H)} K_H(X) \to K_G(X)$$

is an isomorphism of rings.

**Proof** The idea is to construct an inverse to $\mu$. In order to do this we have to prepare some notation and recall some facts from the representation theory of finite abelian groups. Let $A := \mathrm{Ker}(\alpha)$. Recall that the irreducible representations of $A$ are one dimensional and that they are parameterized by the elements of the character group $\widehat{A} := \mathrm{Hom}(A, \mathbb{C}^*)$: $\lambda \in \widehat{A}$ corresponds to the $A$-module $\mathbb{C}_\lambda := \mathbb{C}$ where the action of $a \in A$ is given by multiplication by $\lambda(a)$. Suppose that $E$ is a $G$-vector bundle, then each fiber $E_x$ of $E$ is an $A$-module and by taking the decomposition of each fiber into sum of irreducible $A$-modules we get the vector bundle decomposition $E = \oplus_{\lambda \in \widehat{A}} E_\lambda$, where

$$E_\lambda := \{v \in E \mid a \cdot v = \lambda(a)v \ \forall a \in A\}.$$

Since the group $G$ is abelian, we get that $E_\lambda$ is $G$-invariant, that is, $E_\lambda$ is a $G$-vector sub-bundle of $E$.

By assumption, we have an exact sequence

$$1 \longrightarrow A \xrightarrow{i} G \xrightarrow{\alpha} H \longrightarrow 1\,,$$

where $i$ is the inclusion map. Applying the functor $\mathrm{Hom}(\ ,\mathbb{C}^*)$ we get an exact sequence for the character groups

$$1 \longrightarrow \widehat{H} \xrightarrow{\widehat{\alpha}} \widehat{G} \xrightarrow{\widehat{i}} \widehat{A} \longrightarrow 1\,.$$

Let us denote by $L_\lambda := \underline{\mathbb{C}}_\lambda \in K_G(X)$ the $G$-line bundle on $X$ corresponding to the character $\lambda \in \widehat{G}$. Note that for every $\lambda \in \widehat{G}$ the $G$-vector bundle $L_\lambda^{-1} \otimes E_{\widehat{i}(\lambda)}$ has a trivial $A$-action. Therefore, the vector bundle has an induced structure of $G/A \cong H$-vector bundle. It is straightforward to check that the inverse of $\mu$ is given by the following formula:

$$\nu(E) := \sum_{[\lambda]\in\widehat{G}/\widehat{H}} \mathbb{C}_\lambda \otimes_{R(H)} (L_\lambda^{-1} \otimes E_{\widehat{i}(\lambda)}),$$

where the sum is over a set of $\lambda \in \widehat{G}$ whose coset classes $\lambda\widehat{H}$ exhaust the elements of the quotient $\widehat{G}/\widehat{H} \cong \widehat{A}$. □

Our next goal is to construct the higher equivariant K-groups and the long exact sequence of a pair. To begin with, we need the following proposition.

**Proposition 2.13** If $E$ is a $G$-vector bundle on $X$, then there is a $G$-module $M$ and a $G$-vector bundle $E^\perp$ such that $E \oplus E^\perp \cong \underline{M}$.

The proof of Proposition 2.13 is rather complicated (it relies on the Peter-Weyl theorem) – see [46], Proposition 2.4. Two $G$-vector bundles $E$, $E'$ on $X$ are called *stably equivalent* if there exist $G$-modules $M$, $M'$ such that $E \oplus \underline{M} \cong E' \oplus \underline{M'}$. Proposition 2.13 implies that the stable equivalence classes of $G$-vector bundles on $X$ form an abelian group under $\oplus$. This group is called $\widetilde{K}_G(X)$; it can be identified naturally with a quotient group of $K_G(X)$, that is, we have an exact sequence of $R(G)$-modules

$$0 \to R(G) \to K_G(X) \to \widetilde{K}_G(X) \to 0,$$

where the map $R(G) \to K_G(X)$ is $M \mapsto \underline{M}$.

Suppose that $X$ is a $G$-space with a base point 0, such that, $g0 = 0$ for all $g \in G$. Let us denote by $\mathrm{C}\,X$ the *reduced* cone on $X$, that is,

$$\mathrm{C}\,X := X \times [0,1]/(X \times 0) \cup (0 \times [0,1]),$$

where $[0,1]$ is the unit interval in $\mathbb{R}$. If $i_1 : X \to Y_1$, $i_2 : X \to Y_2$ are two inclusions of compact $G$-spaces with base point, then we denote by $Y_1 \sqcup_X Y_2$ the topological space obtained from the topological sum $Y_1 \sqcup Y_2$ by identifying $i_1(x)$ with $i_2(x)$ for each $x \in X$. There is an obvious embedding of $X$ in $\mathrm{C}\,X$, and $\mathrm{C}\,X \sqcup_X \mathrm{C}\,X$ is called the *reduced suspension* of $X$, and written $\mathrm{S}\,X$.

**Proposition 2.14** If $X$ is a compact $G$-space with base point, and $A$ is a closed $G$-subspace (with the same base point), then the sequence

$$\widetilde{K}_G(X \sqcup_A \mathrm{C}\,A) \to \widetilde{K}_G(X) \to \widetilde{K}_G(A)$$

is exact, where both maps are induced by the natural restriction maps.

The proof is straightforward (see also [46], Proposition 2.6). Proposition 2.14 can be viewed as a formula for the kernel of the restriction map $\widetilde{K}_G(X) \to \widetilde{K}_G(A)$. Using this proposition we get that the kernel of the restriction map $\widetilde{K}_G(X \sqcup_A \mathrm{C}\,A) \to \widetilde{K}_G(X)$ is precisely $\widetilde{K}_G(X \sqcup_A \mathrm{C}\,A \sqcup_X \mathrm{C}\,X) = \widetilde{K}_G(\mathrm{C}\,X \sqcup_A \mathrm{C}\,A)$.

**Proposition 2.15** The inclusion map $\mathrm{S}\,A \to \mathrm{C}\,X \sqcup_A \mathrm{C}\,A$ is a $G$-homotopy equivalence.

**Proof** We follow the ideas explained in [17], Sections 5.5 and 5.6. Let us introduce the notion of a *G-Borsuk pair*. A pair $(X, A)$ of a $G$-space $X$ and a closed $G$-subspace $A$ is called a $G$-Borsuk pair if for every other $G$-space $Y$, every continuous $G$-map $F : X \to Y$, and every $G$-homotopy $f_s : A \to Y$ $(0 \le s \le 1)$ such that $f_0 = F|_A$, there exists a $G$-homotopy $F_s : X \to Y$ $(0 \le s \le 1)$ such that $F_0 = F$ and $F_s|_A = f_s$. It can be checked that $(\mathrm{C}\,X \sqcup_A \mathrm{C}\,A, \mathrm{C}\,X)$ is a $G$-Borsuk pair. Indeed, in the notation of the definition of a $G$-Borsuk pair, given a $G$-homotopy $f_s : \mathrm{C}\,X \to Y$ and a $G$-map $F : \mathrm{C}\,X \sqcup_A \mathrm{C}\,A \to Y$ such that $F|_{\mathrm{C}\,X} = f_0$, the required homotopy $F_s$ can be constructed as follows

$$F_s(x, t') = f_s(x, t'), \quad \forall [x, t'] \in \mathrm{C}\,X$$

and

$$F_s(a, t'') = \begin{cases} F(a, (1+s)t'') & \text{if } t'' \le \frac{1}{1+s} \\ f_{(1+s)t''-1}(a, 1) & \text{if } t'' \ge \frac{1}{1+s} \end{cases}, \quad \forall [a, t''] \in \mathrm{C}\,A,$$

where we denoted by $[x, t''] \in \mathrm{C}\,X$ the point obtained from $(x, t'') \in X \times [0, 1]$ via the quotient map. In particular, when $Y = \mathrm{C}\,X \sqcup_A \mathrm{C}\,A$, $F = \mathrm{id}$, and

$$f_s(x, t') = [x, (1-s)t'] \in \mathrm{C}\,X \subset \mathrm{C}\,X \sqcup_A \mathrm{C}\,A,$$

our construction yields a homotopy $F_s : \mathrm{C}\,X \sqcup_A \mathrm{C}\,A \to \mathrm{C}\,X \sqcup_A \mathrm{C}\,A$ such that $F_0 = \mathrm{id}$ while the image of $F_1$ is in $\mathrm{S}\,A$. It remains only to check that $F_1|_{\mathrm{S}\,A}$ is $G$-homotopic to the identity map in $\mathrm{S}\,A$ which is obvious because $F_s$ maps $\mathrm{S}\,A$ into $\mathrm{S}\,A$ so for a homotopy we can simply take $F_s|_{\mathrm{S}\,A}$. □

**Remark 2.16** Similarly, one can prove that $(\mathrm{C}\,A, A)$ is a $G$-Borsuk pair and deduce that the quotient map

$$X \sqcup_A \mathrm{C}\,A \to X \sqcup_A \mathrm{C}\,A / \mathrm{C}\,A = X/A$$

is a $G$-homotopy equivalence.

We get the following sequence

$$\widetilde{K}_G(\mathrm{S}\,X) \to \widetilde{K}_G(\mathrm{S}\,A) \to \widetilde{K}_G(X \sqcup_A \mathrm{C}\,A) \to \widetilde{K}_G(X) \to \widetilde{K}_G(A) \tag{2.1}$$

which is exact except possibly at the 2nd term. However $\mathrm{S}\,X$ is $G$-homotopy equivalent to

$$\mathrm{C}(X \sqcup_A \mathrm{C}\,A) \sqcup_{X \sqcup_A \mathrm{C}\,A} (\mathrm{C}\,X \sqcup_A \mathrm{C}\,A).$$

Therefore, thanks to Proposition 2.14, the sequence 2.1 is exact.

**Definition 2.17** If $X$ is a compact $G$-space with base point, and $A$ is a closed $G$-subspace, define (for any $q \in \mathbb{N}$)

$$\widetilde{K}_G^{-q}(X) := \widetilde{K}_G(\mathrm{S}^q\,X),$$
$$\widetilde{K}_G^{-q}(X, A) := \widetilde{K}_G(\mathrm{S}^q(X \sqcup_A \mathrm{C}\,A)),$$

where $\mathrm{S}^q\,X$ is the $q$-fold suspension of $X$ defined recursively by $\mathrm{S}^q\,X := \mathrm{S}(\mathrm{S}^{q-1}\,X)$ for $q > 0$ and $\mathrm{S}^0\,X := X$.

Since we have $\mathrm{S}^q(X \sqcup_A \mathrm{C}\,A) = \mathrm{S}^q\,X \sqcup_{\mathrm{S}^q\,A} \mathrm{C}\,\mathrm{S}^q\,A$, by iterating the sequence 2.1 we get an exact sequence infinite to the left

$$\cdots \to \widetilde{K}_G^{-q}(X, A) \to \widetilde{K}_G^{-q}(X) \to \widetilde{K}_G^{-q}(A) \to \widetilde{K}_G^{-q+1}(X, A) \to \dots$$
$$\to \widetilde{K}_G(X, A) \to \widetilde{K}_G(X) \to \widetilde{K}_G(A).$$

The standard way to extend K-theory to non-compact spaces is to consider the one-point compactification. The resulting theory is known as K-theory with compact support. In the $G$-equivariant settings this approach works too. If $X$ is a locally compact $G$-space which is not compact, let $X^+$ denote its one-point compactification, a compact $G$-space with base point. If $X$ is already compact, define $X^+ = X \sqcup \{0\}$, the sum of $X$ and a base point.

**Definition 2.18** If $X$ is a locally compact $G$-space, and $A$ is a closed subspace, define $K_G^{-q}(X)_{\text{cpt}} := \widetilde{K}_G^{-q}(X^+)$ and $K_G^{-q}(X, A)_{\text{cpt}} = \widetilde{K}_G^{-q}(X^+, A^+)$.

**Remark 2.19** We have $K_G^{-q}(X, \emptyset)_{\text{cpt}} = K_G^{-q}(X)_{\text{cpt}}$ and $K_G^{-q}(X)_{\text{cpt}} = K_G^0(X \times \mathbb{R}^q)_{\text{cpt}}$ and $K_G^{-q}(X, A)_{\text{cpt}} = K_G^0(X \times \mathbb{R}^q, A \times \mathbb{R}^q)_{\text{cpt}}$ for any locally compact $G$-space $X$ and closed $G$-subspace $A$.

We have the following excision theorem (see [46], Proposition 2.9).

**Proposition 2.20** If $A$ is a closed $G$-subspace of a locally compact $G$-space $X$, then the natural map

$$K_G^{-q}(X - A)_{\text{cpt}} \to K_G^{-q}(X, A)_{\text{cpt}}$$

is an isomorphism.

Let us discuss the case when $X$ is a compact Hausdorff $G$-space without a base point. In this case the $G$-equivariant $K$-group can be defined vie the *non-reduced* suspension. The latter is defined by

$$\Sigma X := \text{Con}(X) \sqcup_X \text{Con}(X),$$

where $\text{Con}(X) := X \times [0,1]/X \times \{0\}$ is the cone of $X$. To begin with, note that for any $G$-subset $A \subset X$ we have the following $G$-homotopy equivalence:

$$X^+ \sqcup_{A^+} \text{C}\, A^+ \simeq X \sqcup_A \text{Con}(A),$$

where if $A = \emptyset$, then we define $\text{Con}(\emptyset) := \text{pt}$. The RHS is a $G$-space with a base point the vertex of the cone $\text{Con}(A)$. For a contractible space the reduced and the non-reduced suspension are homotopy equivalent. Therefore,

$$S(X \sqcup_A \text{Con}(A)) = \Sigma X \sqcup_{\Sigma A} S\,\text{Con}(A) \simeq \Sigma X \sqcup_{\Sigma A} \Sigma\,\text{Con}(A) = \Sigma(X \sqcup_A \text{Con}(A)) \simeq \Sigma(X/A).$$

Note that since the operations Con and $\Sigma$ commute we also have that

$$S(X \sqcup_A \text{Con}(A)) \simeq \Sigma X \sqcup_{\Sigma A} \text{Con}(\Sigma A).$$

Therefore,

$$K_G^{-1}(X, A) = \widetilde{K}_G(\Sigma(X/A)) = \widetilde{K}_G(\Sigma X \sqcup_{\Sigma A} \text{Con}(\Sigma A)) = K_G(\Sigma X, \Sigma A).$$

Similarly, for all $q \geq 0$ we have the following formulas:

$$K_G^{-q}(X, A) = \widetilde{K}_G(\Sigma^q(X/A)) = K_G(\Sigma^q X, \Sigma^q A).$$

### 2.1.4 Complexes of $G$-vector bundles

In order to complete the construction of $G$-equivariant $K$-theory, we still need to explain how to extend the long exact sequence infinitely to the right. This is done by the Thom isomorphism theorem which in particular yields the periodicity isomorphism $K_G^{q+2}(X,A)_{\mathrm{cpt}} \cong K_G^q(X,A)_{\mathrm{cpt}}$. In order to establish the Thom isomorphism, it is important to obtain an equivalent description of the relative $K$-group in terms of complexes of vector bundles. We will do this following the ideas of Segal (see [46]).

Suppose that $X$ is a locally compact, paracompact, Hausdorff $G$-space, where $G$ is a compact Lie group. A complex of $G$-vector bundles on $X$ is a sequence $E^i$ ($i \in \mathbb{Z}$) of $G$-vector bundles and a sequence $d_E^i : E^i \to E^{i+1}$ of morphisms of $G$-vector bundles, such that, $d_E^{i+1} \circ d_E^i = 0$ for all $i$. We usually denote the complex simply by $E^\bullet$ and the differential by $d_E$ or simply by $d$ if no confusion is likely to occur. The complex $E^\bullet$ is called *finite* (or *bounded*) if $E^i \neq 0$ only for finitely many $i$. A morphism $f : E^\bullet \to F^\bullet$ between two complexes of $G$-vector bundles is a sequence $f^i : E^i \to F^i$ of morphisms of $G$-vector bundles compatible with the differentials $d_E$ and $d_F$ of the two complexes: $d_F \circ f^i = f^{i+1} \circ d_E$. Two morphisms $f_1, f_2 : E^\bullet \to F^\bullet$ of complexes of $G$-vector bundles are said to be $G$-*homotopic*, denoted by $f_1 \simeq f_2$, if there exists a sequence of morphisms $h^i : E^i \to F^{i-1}$ of $G$-vector bundles, such that, $f_1^i - f_2^i = h^{i+1} \circ d_E^i + d_F^{i-1} \circ h^i$. Two complexes $E^\bullet$ and $F^\bullet$ of $G$-vector bundles are said to be $G$-homotopy equivalent if there are morphisms, called homotopy equivalences, $f : E^\bullet \to F^\bullet$ and $g : F^\bullet \to E^\bullet$, such that, $g \circ f \simeq \mathrm{id}_E$ and $f \circ g \simeq \mathrm{id}_F$.

**Definition 2.21** Suppose that $E^\bullet$ is a complex of $G$-vector bundles. The set of points

$$\mathrm{supp}(E^\bullet) := \{x \in X \mid H^i(E_x^\bullet) \neq 0 \text{ for some } i\}$$

is called the support of $E^\bullet$.

A complex whose cohomology vanishes is also called *exact* or *acyclic*. The support consists of the points $x$ at which the complex fails to be exact. Note that the support of a bounded complex $E^\bullet$ is a closed subset. Indeed, the dimension of the subspace $\mathrm{Im}(d_x^{i-1})$ of $E_x^i$ as a function of $x$ can only increase if we vary $x$ nearby, while the value of $\mathrm{Ker}(d_x^i)$ can only decrease if we vary $x$ nearby. In particular, if $H^i(E_x) = 0$, then $H^i(E_{x'}) = 0$ for all $x'$ sufficiently close to $x$. Therefore, the complement of the support is an open subset.

Suppose now that $A \subset X$ is a closed subset. Let us denote by $C_G(X,A)$ the category of bounded complexes of $G$-vector bundles whose support is contained in $X \setminus A$ and the morphisms are just morphisms of complexes of $G$-vector bundles. Let $D_G(X,A)$ be the homotopy category of $C_G(X,A)$, that is, the objects are the same as the objects of $C_G(X,A)$ but the morphisms between two complexes $E^\bullet$ and $F^\bullet$ are given by the $[E^\bullet, F^\bullet] :=$ homotopy classes of morphisms $E^\bullet \to F^\bullet$. In particular, two objects $E^\bullet$ and $F^\bullet$ in $D_G(X,A)$ are isomorphic, denoted by $E^\bullet \simeq F^\bullet$, if there exists a homotopy equivalence $f : E^\bullet \to F^\bullet$. Following Segal (see [46]), let us define the equivalence relation $\sim$ in $D_G(X,A)$: we say that two complexes $E_0^\bullet$ and $E_1^\bullet$ of $D_G(X,A)$ are *equivalent*, and we write $E_0^\bullet \sim E_1^\bullet$, if there is an object $E^\bullet$ of $D_G(X \times [0,1], A \times [0,1])$ and homotopy equivalences $E_0^\bullet \simeq E^\bullet|_{X \times \{0\}}$ and $E_1^\bullet \simeq E^\bullet|_{X \times \{1\}}$.

**Remark 2.22** Segal used the word "homotopic" for the equivalence relation $\sim$. However, nowadays the notion of homotopy equivalence between two complexes is quite standard. The homotopy equivalence relation is contained in $\sim$, that is, $E^\bullet \simeq F^\bullet$ implies $E^\bullet \sim F^\bullet$. In order to avoid confusion we would not give a special name to the equivalence relation $\sim$.

Similarly, we define the categories $C_G(X,A)_{\mathrm{cpt}}$ and its homotopy category $D_G(X,A)_{\mathrm{cpt}}$ in which the complexes are required to have *compact* support contained in $X \setminus A$. Put

$$Q_G(X,A) := D_G(X,A)/\sim \quad \text{and} \quad Q_G(X,A)_{\mathrm{cpt}} := D_G(X,A)_{\mathrm{cpt}}/\sim .$$

If $X$ is compact, then $Q_G(X,A) = Q_G(X,A)_{\mathrm{cpt}}$. Note that $Q_G(X,A)$ and $Q_G(X,A)_{\mathrm{cpt}}$ are abelian semi-groups with respect to the operation $\oplus$: direct sum of complexes. Recall that $D_G(X,A)$ is a triangulated category (see [28]). We would like to prove that every triangle in $D_G(X,A)$ gives rise to a relation in $Q_G(X,A)$. In fact, it can be proved that these are all relations and therefore the natural map which assigns to the homotopy class of a complex $E^\bullet \in D_G(X,A)$ its equivalence class in $Q_G(X,A)$ induces an isomorphism between the Grothendieck group of the triangulated category $D_G(X,A)$ and $Q_G(X,A)$. Let us recall the notion of a *triangle* in $D_G(X,A)$ (see [28]).

**Definition 2.23** A sequence of complexes of $G$-vector bundles

$$P^\bullet \xrightarrow{\alpha} Q^\bullet \xrightarrow{\beta} R^\bullet$$

is said to be *split exact* if for every $i$ there exists an isomorphism $Q^i \cong P^i \oplus R^i$, such that, $\alpha(x) = (x,0)$ and $\beta(x,y) = y$.

Given a split exact sequence the differential $d_Q$, under the splitting, takes the form

$$d_Q^i(x,y) = (a_{11}^i(x) + a_{12}^i(y), a_{21}^i(x) + a_{22}^i(y)), \quad (x,y) \in P^i \oplus R^i,$$

where $a_{11}^i : P^i \to P^{i+1}$, $a_{12}^i : R^i \to P^{i+1}$, $a_{21}^i : R^i \to P^{i+1}$ and $a_{22}^i : R^i \to R^{i+1}$. The compatibility of $d_Q$ with $\alpha$ and $\beta$ implies that $a_{11}^i = d_P^i$, $a_{21}^i = 0$ and $a_{22}^i = d_R^i$. The condition $d_Q^2 = 0$ implies that $a_{12}^i$ defines a morphism of complexes of $G$-vector bundles $a_{12} : R \to P[1]$ where $P[n]$ denotes the complex with $P^i[n] := P^{i+n}$ and $d_{P[n]}^i := (-1)^n d_P^{i+n}$. The homotopy class $h \in [R, P[1]]$ of the map $a_{12}$ is independent of the choice of the splitting and it is called the *homotopy invariant* of the split exact sequence.

**Definition 2.24** A sequence in $C_G(X,A)$ of the form

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{f[1]} A^\bullet[1] \tag{2.2}$$

is said to be a *triangle* if there exists a split exact sequence $P \to Q \to R$ and a homotopy commutative diagram

$$\begin{array}{ccccccc} A^\bullet & \xrightarrow{f} & B^\bullet & \xrightarrow{g} & C^\bullet & \xrightarrow{f[1]} & A^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P^\bullet & \longrightarrow & Q^\bullet & \longrightarrow & R^\bullet & \xrightarrow{h} & P^\bullet[1] \end{array},$$

in which the vertical arrows are homotopy equivalences and $h$ is the homotopy invariant of the split exact sequence.

Suppose now that we have a triangle (2.2). We claim that $[A^\bullet] - [B^\bullet] + [C^\bullet] = 0$, where $[\ ]$ denotes the equivalence class of a complex in $Q_G(X,A)$. Let $P^\bullet$, $Q^\bullet$, and $R^\bullet$ be as in Definition 2.24. By definition, we have: $[A^\bullet] = [P^\bullet]$,

$[B^\bullet] = [Q^\bullet]$, and $[C^\bullet] = [R^\bullet]$. Let us consider the family of complexes $Q_t^\bullet$ $(0 \le t \le 1)$ defined by

$$Q_t^i := Q^i \cong P^i \oplus R^i, \quad d^i_{Q_t}(x,y) := (d^i_P(x) + th(y), d^i_R(y)).$$

Note that $Q^\bullet_{t=1} = Q^\bullet$ while $Q^\bullet_{t=0} = P^\bullet \oplus R^\bullet$. This proves that $Q^\bullet \sim P^\bullet \oplus R^\bullet$ and our claim follows. Let us check that the semi-group $Q_G(X,A)$ is an abelian group. Let us recall the *mapping cone* complex $\mathrm{Con}^\bullet(f)$ of a morphism $f : E^\bullet \to F^\bullet$:

$$\mathrm{Con}^i(f) := E^{i+1} \oplus F^i, \quad d^i(x,y) := (-d_E^{i+1}(x), -f^{i+1}(x) + d^i_F(y)).$$

It is a standard fact (see [28]) that the following sequence

$$E^\bullet \xrightarrow{f} F^\bullet \longrightarrow \mathrm{Con}^\bullet(f) \longrightarrow E^\bullet[1]$$

is a triangle. We get the following relation in $Q_G(X,A)$:

$$[F^\bullet] = [E^\bullet] + [\mathrm{Con}^\bullet(f)].$$

In particular, since for $f = 0$, $\mathrm{Con}^\bullet(f) = E^\bullet[1] \oplus F^\bullet$, we get that the shift functor $E^\bullet \mapsto E^\bullet[1]$ induces the inverse in $Q_G(X,A)$ and hence $Q_G(X,A)$ is a group. Furthermore, if $E^\bullet$ and $F^\bullet$ are complexes on $X$ one can form their tensor product $E^\bullet \otimes F^\bullet$, with $(E^\bullet \otimes F^\bullet)^k = \bigoplus_{p+q=k} E^p \otimes F^q$. One has $\mathrm{supp}(E^\bullet \otimes F^\bullet) = \mathrm{supp}(E^\bullet) \cap \mathrm{supp}(F^\bullet)$. The tensor product of complexes induces a homomorphism

$$Q_G(X,A)_{\mathrm{cpt}} \otimes Q_G(X,A)_{\mathrm{cpt}} \to Q_G(X,A)_{\mathrm{cpt}}.$$

The above product is associative and it turns $Q_G(X,A)_{\mathrm{cpt}}$ into a commutative ring.

Our next goal is to prove that the abelian group $Q_G(X,A)$ is generated by two term complexes of the form $E \to \underline{M}$ where $E$ is a $G$-vector bundle and $M$ is a finite-dimensional $G$-module. We need to recall two standard results. A morphism $f : E^\bullet \to F^\bullet$ is said to be a *quasi-isomorphism* if the induced maps on cohomology $H^i(E_x) \to H^i(F_x)$ are isomorphisms for all $i$ and for all $x \in X$. In that case we will also say that the complexes $E^\bullet$ and $F^\bullet$ are quasi-isomorphic. We have the following proposition.

**Proposition 2.25** A morphism $f : E^\bullet \to F^\bullet$ between two complexes of $G$-vector bundles is a quasi-isomorphism if and only if it is a homotopy equivalence.

The second result is that acyclic complexes split in the following sense.

**Proposition 2.26** If a complex $E^\bullet$ of $G$-vector bundles is acyclic, then there exists a complex $Z^\bullet$ of $G$-vector bundles and isomorphisms $E^i \cong Z^{i+1} \oplus Z^i$, such that, the differential $d^i_E(x,y) = (0,x)$ for $x \in Z^{i+1}$ and $y \in Z^i$.

Suppose now that $E^\bullet$ is an arbitrary complex in $C_G(X,A)$. A complex is said to be *elementary* if it has only two non-zero terms in degrees $i$ and $i+1$ for some $i$ which are identical and the differential $d^i$ is the identity map (see [46], Appendix). Adding an elementary complex to $E^\bullet$ does not change the homotopy class of $E^\bullet$ – thanks to Proposition 2.25. Using Proposition 2.13 we get that by adding elementary complexes to $E^\bullet$ we can arrange that $E^\bullet$ has the form $E^i = 0$ for $i < a$ and $E^i = \underline{M}_i$ for all $i > a$, that is, only $E^a$ is possibly a non-trivial bundle. Let $Z^i$ be the $G$-vector bundles on $A$ that provide the splitting of $E^\bullet|_A$ as in Proposition 2.26. It is easy to check by decreasing induction on $i$ that all $Z^i$ are stably trivial $G$-vector bundles on $A$. Therefore, by adding to $E^\bullet$ elementary complexes of the form $\underline{N} \to \underline{N}$ we may arrange that all $Z^i = \underline{N_i}$ are trivial bundles. Let us look at the complex $E^\bullet|_A$. We have $E^i|_A \cong A \times (N_{i+1} \oplus N_i)$ and the differential $d^i_E(x,y) = (0,x)$ for $(x,y) \in N_{i+1} \oplus N_i$ is constant with respect to $A$. Therefore, by replacing $A$ with $X$ we obtain a complex $F^i := X \times (N_{i+1} \oplus N_i)$.

**Lemma 2.27** The isomorphism $f : E^\bullet|_A \to F^\bullet|_A$ extends to a morphism of complexes $f : E^\bullet \to F^\bullet$.

**Proof** We have $f^i = (f^i_1, f^i_0)$ where $f^i_1 : E^i|_A \to A \times N_{i+1}$ and $f^i_0 : E^i|_A \to A \times N_i$. The compatibility with the differentials of $E^\bullet$ and $F^\bullet$ is equivalent to the identities

$$f^i_1 = f^{i+1}_0 \circ d^i_E \quad \text{and} \quad f^{i+1}_1 \circ d^i_E = 0.$$

Let $n$ be the maximal non-trivial degree of $E^\bullet$, that is, $E^i = 0$ for $i > n$. Since $0 = E^{n+1}|_A = A \times (N_{n+2} \oplus N_{n+1})$, we get that $N_i = 0$ for all $i > n$. Therefore, $E^n|_A \cong A \times N_n$, $f^n_1 = 0$, $f^n_0 : E^n|_A \cong A \times N_n$ is an isomorphism. Let us extend $f^n_0$ arbitrary to a morphism $E^n \to F^n = X \times N_n$ and keep $f^n_1 = 0$. Suppose that we have determined the extensions $f^j_1 : E^j \to X \times N_{j+1}$ and $f^j_0 \to X \times N_j$ for all $i < j \leq n$. Note that the extension of $f^i_1 = f^{i+1}_0 \circ d^i_E$ is uniquely determined from $f^{i+1}$ and it automatically satisfies $f^i_1 \circ d^{i-1}_E = 0$. According to Corollary 2.8, the isomorphism $f^i_0 : E^i|_A \to A \times N_i$ can be extended to a morphism $E^i \to X \times N_i$. Let us pick one of these extensions. The constructed sequence of $G$-vector bundle morphisms $f^i_1 : E^i \to X \times N_{i+1}$ and $f^i_0 : E^i \to X \times N_i$ give a sequence of maps $f^i = (f^i_1, f^i_0) : E^i \to X \times (N_{i+1} \oplus N_i) = F^i$ which by construction is compatible with the differentials of $E^\bullet$ and $F^\bullet$ and therefore it defines a morphism of complexes $f : E^\bullet \to F^\bullet$. □

**Lemma 2.28** Suppose that $f : E^\bullet \to F^\bullet$ is a morphism in $C_G(X, A)$, such that, $f|_A : E^\bullet|_A \to F^\bullet|_A$ is an isomorphism, then

$$[E^\bullet] = [F^\bullet] + \sum_{i \in \mathbb{Z}} (-1)^i [\, E^i \xrightarrow{f^i} F^i \,]$$

where [ ] denotes the equivalence class in $Q_G(X, A)$.

**Proof** We already know that $[F^\bullet] - [E^\bullet]$ coincides with the equivalence class of the mapping cone complex $\mathrm{Con}^\bullet(f)$. For simplicity, assume that $E^i = F^i = 0$ for $i < 0$. Let $\tau_{\geq 1}(E^\bullet)$ be the complex which in degree $\leq 0$ is 0 while in degree $i \geq 1$ coincides with $E^i$. Similarly, let $\tau_{\geq 1}(f) : \tau_{\geq 1}(E^\bullet) \to \tau_{\geq 1}(F^\bullet)$ be the morphism which in degree $i \leq 0$ is 0 and in degree $i \geq 1$ is $f^i$. Note that we have a commutative diagram

$$\begin{array}{ccccccc}
E^0 & \xrightarrow{f^0} & F^0 & & & & \\
\downarrow{\scriptstyle d_E} & & \downarrow{\scriptstyle (0,-d_F)} & & & & \\
E^1 & \xrightarrow[(-d_E,-f^1)]{} & E^2 \oplus F^1 & \longrightarrow & E^3 \oplus F^2 & \longrightarrow & \cdots
\end{array}$$

where the lower raw is the mapping cone complex of $\tau_{\geq 1}(f)$. Let us interpret the above diagram as a morphism $\varphi$ between the two term complex on the upper row and the complex on the lower raw. It is straightforward to check that the mapping cone of $\varphi$ is a translation of the mapping cone of $f$, that is,

$$\mathrm{Con}^\bullet(\varphi) = \mathrm{Con}^\bullet(f)[-1].$$

The triangle corresponding to the mapping cone of $\varphi$ yields the following relation in $Q_G(X,A)$:

$$[\ E^0 \xrightarrow{f^0} F^0\ ] = [\mathrm{Con}^\bullet(\tau_{\geq 1}(f))] - [\mathrm{Con}^\bullet(f)[-1]] = [\mathrm{Con}^\bullet(\tau_{\geq 1}(f))] + [\mathrm{Con}^\bullet(f)]$$

The formula in the lemma follows easily by induction on the length of $\mathrm{Con}^\bullet(f)$. □

Let us apply Lemma 2.27 to construct an extension $f : E^\bullet \to F^\bullet$ in our settings. Note that by definition the complex $F^\bullet$ is a sum of elementary complexes $\Rightarrow [F^\bullet] = 0$. Therefore, the formula in Lemma 2.28 gives the required decomposition of $[E^\bullet]$ as a sum of two term complexes.

Finally, let us compare the abelian group $Q_G(X,A)$ with the relative $K$-ring $K_G(X,A)$. Note that there is a natural map

$$K_G(X,A)_{\mathrm{cpt}} \to Q_G(X,A)_{\mathrm{cpt}} \tag{2.3}$$

defined as follows. By definition $K_G(X,A) = \widetilde{K}(X^+ \sqcup_{A^+} \mathrm{C}\,A^+)$. A $G$-vector bundle on $X^+ \sqcup_{A^+} \mathrm{C}\,A^+$ restricts to a $G$-vector bundle $E$ on $X^+$ and to a trivial $G$-vector bundle $\underline{M}$ on $\mathrm{C}\,A^+$ where $M$ is a finite dimensional $G$-module. Moreover, we have an isomorphism $\phi : E|_A \to A \times M$. Using the Tietze extension theorem (see Corollary 2.8) we can extend $\phi$ to a morphism $\widetilde{\phi} : E \to X \times M$. The map (2.3) is given by

$$(E, M, \phi) \mapsto\ [E \xrightarrow{\widetilde{\phi}} X \times M]$$

The following result is due to Segal (see [46], Proposition A.I).

**Proposition 2.29** The map (2.3) is an isomorphism of abelian groups.

The group $Q_G(X,A)$ is a homotopy invariant in the sense explained below. In particular, even if $X$ is non-compact as long as the $G$-pair $(X,A)$ has the $G$-homotopy type of a $G$-pair $(Y,B)$ with $Y$ compact, we can make use of the properties of the K-rings established for compact spaces.

**Definition 2.30** a) A $G$-pair is a pair $(X,A)$ of a $G$-space $X$ and a $G$-subspace $A$.

b) A $G$-map between two $G$-pairs, denoted by $f : (X,A) \to (Y,B)$ is a continuous $G$-map $f : X \to Y$, such that, $f(A) \subset B$.

c) A $G$-homotopy between two $G$-maps $f_0, f_1 : (X,A) \to (Y,B)$ is a continuous $G$-map $\varphi : (X \times [0,1], A \times [0,1]) \to (Y,B)$, such that, $\varphi|_{X\times 0} = f_0$ and $\varphi|_{X\times 1} = f_1$. If a homotopy exists, then we say that $f_0$ and $f_1$ are homotopic and write $f_0 \simeq f_1$.

d) A $G$-map $f : (X,A) \to (Y,B)$ between two $G$-pairs is said to be a homotopy equivalence if there exists a $G$-map $g : (Y,B) \to (X,A)$, such that, $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Suppose that $f : (X,A) \to (Y,B)$ is a $G$-map between two $G$-pairs, where $X$ and $Y$ are locally compact, paracompact, Hausdorff $G$-spaces, $G$ is a compact Lie group and $A \subset X$ and $B \subset Y$ are closed subsets. The pullback operation of vector bundles induces a homeomorphism of rings $f^* : Q_G(Y,B) \to Q_G(X,A)$. Note that if two maps $f_0, f_1 : (X,A) \to (Y,B)$ are $G$-homotopic, then $f_0^* = f_1^*$. Indeed, suppose that $E^\bullet$ is a complex of $G$-vector bundles. We have to prove that $f_0^* E^\bullet \sim f_1^* E^\bullet$. Let $\varphi : (X \times [0,1], A \times [0,1]) \to (Y,B)$ be the homotopy between $f_0$ and $f_1$. Note that the support of $\varphi^* E^\bullet$ is contained in $X \times [0,1] \setminus A \times [0,1]$ and $\varphi^* E^\bullet|_{X\times\{0\}} = f_0^* E^\bullet$ and $\varphi^* E^\bullet|_{X\times\{1\}} = f_1^* E^\bullet$. In particular, if $f : (X,A) \to (Y,B)$ is a $G$-homotopy equivalence, then $f^* : Q_G(Y,B) \to Q_G(X,A)$ is an isomorphism of rings.

### 2.1.5 The Thom homomorphism and periodicity

Our next goal is to define the Thom homomorphism for $K_G$ and to recall the Thom isomorphism theorem. First observe that if $E$ is a $G$-vector bundle on $X$ and $s$ is an equivariant section of $E$ one can form the *Koszul complex*

$$\cdots \to 0 \to \mathbb{C} \xrightarrow{d} \wedge^1 E \xrightarrow{d} \wedge^2 E \xrightarrow{d} \dots$$

where $d$ is defined by $d(\xi) := \xi \wedge s(x)$ for $\xi \in \wedge^i E_x$. This complex is acyclic at all points $x$ at which $s(x) \neq 0$, so its support is the set of zeros of $s$.

Now, if $p : E \to X$ is the projection, the bundle $p^*E$ on $E$ has a natural section which is the diagonal map $\delta : E \to E \times_X E = p^*E$. This section $\delta$ vanishes precisely on the zero-section of $E$. Following Segal, let us denote by $\wedge_E^\bullet$ the Koszul complex on $E$ formed from $p^*E$ and $\delta$. If $F^\bullet$ is a complex with compact support on $X$ then $p^*F^\bullet$ is a complex on $E$ with support $p^{-1}(\mathrm{supp}(F^\bullet))$, and $\wedge_E^\bullet \otimes p^*F^\bullet$ is a complex with compact support on $E$. The assignment $F^\bullet \mapsto \wedge_E^\bullet \otimes p^*F^\bullet$ induces an additive homomorphism $\phi_* : K_G(X)_{\mathrm{cpt}} \to K_G(E)_{\mathrm{cpt}}$ which is called the *Thom homomorphism*. If $X$ is compact, then $\wedge_E^\bullet$ has compact support and it defines the *Thom class* $\phi_*(1) = \lambda_E$ in $K_G(E)_{\mathrm{cpt}}$. Finally, replacing $X$ and $E$ by $X \times \mathbb{R}^q$ and $E \times \mathbb{R}^q$ we get a Thom homomorphism $\phi_* : K_G^{-q}(X) \to K_G^{-q}(E)$ for each $q \in \mathbb{N}$. We have the following important isomorphism (see [46], Proposition 3.2).

**Proposition 2.31** The Thom homomorphism $\phi_* : K_G^*(X)_{\mathrm{cpt}} \to K_G^*(E)_{\mathrm{cpt}}$ is an isomorphism for any $G$-vector bundle $E$ on a locally compact $G$-space $X$.

Applying 2.31 to the trivial bundle $\underline{\mathbb{C}}$, since $K_G^{-q-2}(X)_{\mathrm{cpt}} = K_G^{-q}(X \times \mathbb{C})_{\mathrm{cpt}}$, we get

**Proposition 2.32** The Thom homomorphism corresponding to the trivial $G$-vector bundle $X \times \mathbb{C} \to X$ defines an isomorphism $K_G^{-q}(X)_{\mathrm{cpt}} \cong K_G^{-q-2}(X)_{\mathrm{cpt}}$.

Proposition 2.32 suggests that one should define $K_G^q(X)_{\mathrm{cpt}}$ for positive $q$ as $K_G^{q-2n}(X)_{\mathrm{cpt}}$, where $n \geqslant q/2$. Then one has cohomological exact sequences extending infinitely in both directions. Equivalently, we have the following exact cyclic sequence

$$\begin{array}{ccccc}
K_G^{-1}(X,A)_{\mathrm{cpt}} & \longrightarrow & K_G^{-1}(X)_{\mathrm{cpt}} & \longrightarrow & K_G^{-1}(A)_{\mathrm{cpt}} \\
{\scriptstyle\delta_0}\uparrow & & & & \downarrow{\scriptstyle\delta_{-1}} \\
K_G^{0}(A)_{\mathrm{cpt}} & \longleftarrow & K_G^{0}(X)_{\mathrm{cpt}} & \longleftarrow & K_G^{0}(X,A)_{\mathrm{cpt}}
\end{array}$$

where the map $\delta_0$ is the composition of the periodicity isomorphism $K_G^0(A)_{\mathrm{cpt}} \cong K_G^{-2}(A)_{\mathrm{cpt}}$ and the boundary morphism $K_G^{-2}(A)_{\mathrm{cpt}} \to K_G^{-1}(X,A)_{\mathrm{cpt}}$. The boundary morphism $\delta_{-1}$ can be described as follows. An element in $K_G^{-1}(A)_{\mathrm{cpt}}$ is represented by a $G$-vector bundle on $\Sigma(A^+)$. Such a vector bundle is obtained by gluing two copies of a trivial bundle C $A^+ \times M$ along $A^+$ where $M$ is a finite-dimensional $G$-module. The gluing isomorphism $\phi$ : $A^+ \times M \to A^+ \times M$ can be extended to a morphism $X^+ \times M \to X^+ \times M$ thaks to the Tietze extension theorem. The resulting two term complex represents an element of $Q_G(X^+, A^+) \cong K_G(X^+, A^+) = K_G(X,A)_{\mathrm{cpt}}$ which coincides with the image of $\delta_{-1}$. Finally, if the $G$-pair $(X,A)$ has the homotopy type of a $G$-pair $(Y,B)$ with $Y$

compact, then the above cyclic sequence remain exact after removing the index $cpt$ from the K-groups. Note that in general the groups $K_G^{-i}(X,A)_{\text{cpt}}$ and

$$K_G^{-i}(X,A) = \widetilde{K}_G(\Sigma^i(X/A)) = Q_G(\Sigma^i X, \Sigma^i A)$$

are very different.

## 2.2 Frobenius manifolds

### 2.2.1 Definition

There are many ways to introduce Frobenius manifolds. Here, it is convenient to choose a set of axioms. The general reference for more details is [13] and [33]. Our definition is equivalent to (Definition 1.2 in [13]). Let $M$ be a complex manifold and denote by $\mathcal{T}_M$ the sheaf of holomorphic vector fields on $M$. Let us assume that $M$ is equipped with the following structures

1. Each tangent space $T_tM, t \in M$, is equipped with the structure of a *Frobenius algebra* depending holomorphically on $t$. In other words, we have a commutative associative multiplication $\bullet_t$ and symmetric non-degenerate bi-linear pairing $(\quad,\quad)_t$ satisfying the Frobenius property

$$(v_1 \bullet_t w, v_2) = (v_1, w \bullet_t v_2), \qquad v_1, v_2, w \in T_tM$$

The pointwise multiplication $\bullet_t$ defines a multiplication $\bullet$ in $\mathcal{T}_M$, i.e., an $\mathcal{O}_M$-bilinear map

$$\mathcal{T}_M \otimes \mathcal{T}_M \to \mathcal{T}_M, \qquad v_1 \otimes v_2 \mapsto v_1 \bullet v_2.$$

The pairing $(\quad,\quad)_t$ determines a $\mathcal{O}_M$-bilinear pairing

$$(\quad,\quad) : \mathcal{T}_M \otimes \mathcal{T}_M \to \mathcal{O}_M.$$

2. There exists a global vector field $e \in \mathcal{T}_M$, called *unit vector field*, such that

$$\nabla_v^{\text{L.C.}} e = 0, \qquad e \bullet v = v, \qquad \forall v \in \mathcal{T}_M,$$

where $\nabla^{\text{L.C.}}$ is the Levi-Civita connection on $\mathcal{T}_M$ corresponding to the bi-linear pairing $(\quad,\quad)$.

3. There exists a global vector field $E \in \mathcal{T}_M$ , called *Euler vector field*, such that

$$E(v_1, v_2) - ([E, v_1], v_2) - (v_1, [E, v_2]) = (2 - D)(v_1, v_2),$$

for all $v_1, v_2 \in \mathcal{T}_M$ and for some constant $D \in \mathbb{C}$.

The above data allows us to define the so-called *structure connection* $\nabla$ on the vector bundle $\text{pr}_M^* TM \to M \times \mathbb{C}^*$, where

$$\text{pr}_M : M \times \mathbb{C}^* \to M, \qquad (t, z) \mapsto t$$

is the projection map. Namely,

$$\nabla_v := \nabla_v^{\text{L.C.}} - z^{-1} v\bullet, \qquad v \in \mathcal{T}_M$$
$$\nabla_{\partial/\partial z} := \frac{\partial}{\partial z} - z^{-1}\theta + z^{-2} E\bullet,$$

$v\bullet$ and $E\bullet$ are $\mathcal{O}_M$-linear maps $\mathcal{T}_M \to \mathcal{T}_M$ corresponding to the Frobenius multiplication by respectively $v$ and $E$. The $\mathcal{O}_M$-linear map $\theta : \mathcal{T}_M \to \mathcal{T}_M$ is defined by

$$\theta(v) := \nabla_v^{\text{L.C.}} E - (1 - D/2)v.$$

The operator $\theta$ is sometimes called *Hodge grading operator*. Let us point out that the term $(1 - D/2)v$ in the definition of $\theta(v)$ is inserted so that $\theta$ becomes skew-symmetric with respect to the Frobenius pairing

$$(\theta(v_1), v_2) + (v_1, \theta(v_2)) = 0, \qquad v_1, v_2 \in \mathcal{T}_M.$$

**Definition 2.33** The data $((\quad,\quad), \bullet, e, E)$ satisfying the conditions (1), (2) and (3) from above is said to be a *Frobenius structure* on $M$ of *conformal dimension* $D$ if the structure connection $\nabla$ is flat.

Let us state the following properties without proof. Actually, the proof is straightforward argument in Riemannian Geometry. And we will do the same to the propositions and theorems in this chapter of Background.

**Proposition 2.34** Suppose that $(M, (,), \bullet, e, E)$ is a Frobenius structure. Then

1. The Levi-Civita connection $\nabla^{\text{L.C.}}$ is flat.

2. Let $t = (t_1, \ldots, t_N)$ be $\nabla^{\text{L.C.}}$-flat coordinates defined on a contractible open subset $U \subset M$. There exists a holomorphic function $F \in \mathcal{O}_M(U)$, such that

   $$(\partial/\partial t_a \bullet \partial/\partial t_b, \partial/\partial t_c) = \frac{\partial^3 F}{\partial t_a \partial t_b \partial t_c}$$

   and

   $$EF = (3 - D)F + H,$$

   where $H$ is a polynomial in $t_1, \ldots, t_N$ of degree at most 2.

3. The Hodge grading operator is covariantly constant: $\nabla^{\text{L.C.}}\theta = 0$. In particular, in flat coordinates $t = (t_1, \ldots, t_N)$ the matrix $(\theta_{ab})_{a,b=1}^N$ of $\theta$ defined by

   $$\theta(\partial/\partial t_b) = \sum_{a=1}^N \theta_{ab} \partial/\partial t_a$$

   is constant.

4. The following identity holds

   $$[E, v \bullet w] - [E, v] \bullet w - v \bullet [E, w] = v \bullet w, \qquad v, w \in \mathcal{T}_M.$$

### 2.2.2 Semi-simple Frobenius manifolds

**Definition 2.35** A Frobenius manifold $(M, (\quad,\quad), \bullet, e, E)$ is said to be *semi-simple* if there are local coordinates $u = (u_1, \ldots, u_N)$ defined in a neighborhood of some point on M such that

$$\partial/\partial u_i \bullet \partial/\partial u_j = \delta_{ij}\partial/\partial u_j, \qquad 1 \leqslant i, j \leqslant N.$$

The coordinates $u_i$ are called *canonical coordinates*.

As we will see now, canonical coordinates are unique up to permutation and constant shifts. To avoid cumbersome notation we put $\partial_{u_i} := \partial/\partial u_i$.

**Proposition 2.36** Let $u = (u_1, \ldots, u_N)$ be canonical coordinates defined on some open subset $U \subset M$. Then

1. The Frobenius pairing takes the form

$$(\partial_{u_i}, \partial_{u_j}) = \delta_{ij}\eta_j(u), \qquad 1 \leqslant i, j \leqslant N,$$

   where $\eta_j \in \mathcal{O}_M(U)$ and $\eta_j(u) \neq 0$ for all $u \in U$.

2. The unit vector field takes the form $e = \sum_{i=1}^N \partial_{u_i}$.

3. The 1-form $\sum_{i=1}^N \eta_i(u) du_i$ is closed.

4. There are constants $c_i (1 \leqslant i \leqslant N)$ such that

$$E = \sum_{i=1}^N (u_i + c_i)\partial_{u_i}.$$

The last part of the above proposition shows that in every canonical coordinate system up to some constant shifts the canonical coordinates coincide with the eigenvalues of the operator $E\bullet$. Therefore, up to constant shifts and permutations the canonical coordinates are uniquely determined. From now on we will work only with canonical coordinates such that

$$E = \sum_{i=1}^N u_i \partial_{u_i}.$$

The question that we would like to answer now is the following. Let us assume that $U$ is an open subset of the universal cover $T$ of $Z_N$ and $\sum_{i=1}^N \eta_i(u) du_i$ is a closed 1-form on $U$. The tangent bundle of $T$ and hence of $U$ as well is trivial, because $T$ is a contractible Stein manifold, so according to the Grauert-Oka principle every holomorphic vector bundle on $T$ is trivial. Alternatively, we can prove that $\mathcal{T}_T$ is a free $\mathcal{O}_T$-module by using that the vector fields $\partial_{u_i}$ of the configuration space $Z_N$ lift naturally to vector fields on $T$ and provide a global trivialization of $\mathcal{T}_T$. Using the 1-form we define a pairing

$$(\partial_{u_i}, \partial_{u_j}) = \delta_{ij}\eta_j(u).$$

Let us also define multiplication

$$\partial_{u_i} \bullet \partial_{u_j} = \delta_{ij}\partial_{u_j}$$

and vector fields

$$e = \sum_{i=1}^{N} \partial_{u_i}, \qquad E = \sum_{i=1}^{N} u_i \partial_{u_i}.$$

The problem then is to classify all 1-forms $\sum_{i=1}^{N} \eta_i(u) du_i$ such that the above data determines a Frobenius structure on $U$. The answer is given by the following theorem.

**Theorem 2.37** The closed 1-form $\sum_{i=1}^{N} \eta_i(u) du_i$ determines a Frobenius structure on U of conformal dimension $D$ if and only if the following conditions are satisfied

1. $\eta_i(u) \neq 0$ for all $i$ and for all $u \in U$.
2. $e\eta_i(u) = 0$ for all $i$.
3. $E\eta_i(u) = -D\eta_i(u)$.
4. For all $k \neq i \neq j \neq k$ we have

$$\frac{\partial \eta_{ij}}{\partial u_k} = \frac{1}{2}\left(\frac{\eta_{ij}\eta_{kj}}{\eta_j} + \frac{\eta_{jk}\eta_{ik}}{\eta_k} + \frac{\eta_{ki}\eta_{ji}}{\eta_i}\right),$$

   where $\eta_{ab}(u) := \partial_{u_a}\eta_b(u)$.

### 2.2.3 The second structure connection

In order to justify the definition of the second structure connection we make the following heuristic argument. Suppose that the structure connection has a solution

$$J : M \times \mathbb{C}^* \to \mathbb{C}^N$$

given by a Laplace transform

$$J(t,z) = \frac{(-z)^{n-\frac{1}{2}}}{\sqrt{2\pi}} \int_{\Gamma} e^{\lambda/z} I^{(n)}(t,\lambda) d\lambda$$

along an appropriate contour $\Gamma \subset \mathbb{C}$ of some $\mathbb{C}^N$-valued function $I^{(n)}(t,\lambda)$ holomorphic for all $(t,\lambda) \in M \times \Gamma$. Here $n \in \mathbb{C}$ is an arbitrary number. Assuming that the Laplace transform works, we would get that $J(t,z)$ is a solution to the structure connection if and only if $I^{(n)}(t,\lambda)$ is a solution to the following connection

$$\begin{aligned} \nabla^{(n)}_{\partial_{t_i}} &= \partial_{t_i} + (\lambda - E\bullet)^{-1}(\partial_{t_i}\bullet)(\theta - n - 1/2), \qquad 1 \leqslant i \leqslant N, \\ \nabla^{(n)}_{\partial_\lambda} &= \partial_\lambda - (\lambda - E\bullet)^{-1}(\theta - n - 1/2). \end{aligned}$$

This is a connection on the vector bundle

$$\mathrm{pr}^* TM \to (M \times \mathbb{C})'$$

where

$$(M \times \mathbb{C})' = \{(t,\lambda) \in M \times \mathbb{C} \,|\, \det(\lambda - E\bullet_t) \neq 0\}$$

and

$$\mathrm{pr} : (M \times \mathbb{C})' \to M, \qquad (x,\lambda) \mapsto x$$

**Proposition 2.38** The connection $\nabla^{(n)}$ is flat for all $n \in \mathbb{C}$.

Let $U$ be a contractible open subset of the configuration space

$$Z_N = \{u \in \mathbb{C}^N : u_i \neq u_j \text{ for } i \neq j\}.$$

And we fix a point $u^\circ \in Z_N$. Suppose that $U$ is equipped with a semi-simple Frobenius structure $((\quad,\quad), \bullet, e, E)$. Put $H = T_{u^\circ}U$ and let us trivialize the tangent bundle

$$TU \cong U \times H \cong U \times \mathbb{C}^N \tag{2.4}$$

using the Levi-Civita connection. In other words, we fix a basis $\{\phi_a\}_{a=1}^N$ of $H$ and let $\partial_{t_a} \in \mathcal{T}_U$ be the flat vector field on U obtained by parallel transport with respect to the Levi-Civita connection. Then the isomorphisms (2.4) are given by the maps

$$(u, v) \in TU \mapsto (u, v_1\phi_1 + \cdots + v_N\phi_N) \in U \times H \mapsto (u, v_1, \ldots, v_N) \in U \times \mathbb{C}^N,$$

where $v \in T_uU$ and $v =: v_1\partial_{t_1} + \cdots + v_N\partial_{t_N}$. The isomorphism (2.4) identifies the structure connection of the Frobenius structure with the flat connection on the trivial bundle

$$(U \times \mathbb{C}^*) \times \mathbb{C}^N \to U \times \mathbb{C}^*$$

defined by

$$\begin{aligned} \nabla_{\partial_{u_i}} &= \partial_{u_i} - z^{-1}P_i(u), \qquad 1 \leqslant i \leqslant N, \\ \nabla_{\partial_z} &= \partial_z - z^{-1}\theta + z^{-2}\mathcal{E}(u), \end{aligned}$$

where $P_i : U \to \mathfrak{gl}(\mathbb{C}^N)$ is a holomorphic map whose $(a, b)$-entry $P_{iab}(u)$ is defined by the identity

$$\partial_{u_i} \bullet \partial_{t_b} = \sum_{a=1}^{N} P_{iab}(u)\partial_{t_a},$$

$\mathcal{E} = \sum_{i=1}^N u_i P_i(u)$, and $\theta$ is a constant matrix whose $(a, b)$-entry $\theta_{ab}$ is defined by

$$\theta(\partial_{t_b}) = [\partial_{t_b}, E] - (1 - D/2)\partial_{t_b} =: \sum_{a=1}^{N} \theta_{ab}\partial_{t_a}.$$

**Lemma 2.39** Let $\tilde{\Psi}$ be the matrix whose $(a, i)$-entry is given by $\tilde{\Psi}_{ai} = \partial t_a/\partial u_i$. Then

$$\tilde{\Psi}^{-1}P_i\tilde{\Psi} = E_{ii}, \qquad \tilde{\Psi}^{-1}\mathcal{E}\tilde{\Psi} = \operatorname{diag}(u_1, \ldots, u_N),$$

where $E_{ii}$ is the matrix whose entry in position $(i, i)$ is 1 and all other entries are 0.

**Lemma 2.40** Let $n \in \mathbb{C}$ be arbitrary. Then the matrix-valued functions

$$A_i^{(n)}(u) := P_i(u)(\theta - n - 1/2), \qquad 1 \leqslant i \leqslant N,$$

satisfy the Schlesinger equations.

**Proof** Using Lemma 2.39 we get

$$(\lambda - \mathcal{E})^{-1} P_i(\theta - n - \frac{1}{2}) = \frac{A_i^{(n)}(u)}{\lambda - u_i}.$$

Therefore,

$$\nabla^{(n)}_{\partial_{u_i}} = \partial_{u_i} + \frac{A_i^{(n)}(u)}{\lambda - u_i}, \qquad 1 \leqslant i \leqslant N, \tag{2.5}$$

$$\nabla^{(n)}_{\partial_\lambda} = \partial_\lambda - \sum_{i=1}^{N} \frac{A_i^{(n)}(u)}{\lambda - u_i}. \tag{2.6}$$

It remains only to recall Proposition 2.38. □

## 2.3 Calibration

Let us fix any point $t^\circ \in M$. We will do something similar to the previous subsection. We fix a basis $\{\phi_a\}_{a=1}^N$ of $H := T_{t^\circ} M$ and let $\partial_{t_a} \in \mathcal{T}_M$ be the flat vector field obtained by parallel transport with respect to the Levi-Civita connection. We will get a simply connected flat coordinate $(V, t)$, where $V$ is a simply connected neighborhood of $t^\circ$ extended by the parallel transport. Then the isomorphisms (2.7) are given by the maps

$$(t, v) \in TV \mapsto (t, v_1\phi_1 + \cdots + v_N\phi_N) \in V \times H \mapsto (t, v_1, \ldots, v_N) \in V \times \mathbb{C}^N, \tag{2.7}$$

where $v \in T_t V$ and $v =: v_1 \partial_{t_1} + \cdots + v_N \partial_{t_N}$. The isomorphism (2.7) identifies the structure connection of the Frobenius structure with the flat connection on the trivial bundle

$$(V \times \mathbb{C}^*) \times \mathbb{C}^N \to V \times \mathbb{C}^*$$

defined by

$$\nabla_{\partial_{t_i}} = \partial_{t_i} - z^{-1} A_i(t), \qquad 1 \leqslant i \leqslant N, \tag{2.8}$$

$$\nabla_{\partial_z} = \partial_z - z^{-1}\theta + z^{-2} E \bullet (t), \tag{2.9}$$

where $A_i : V \to \mathfrak{gl}(\mathbb{C}^N)$ is a holomorphic map whose $(a, b)$-entry $A_{iab}(u)$ is defined by the identity

$$\partial_{t_i} \bullet \partial_{t_b} = \sum_{a=1}^{N} A_{iab}(t) \partial_{t_a},$$

$E\bullet : V \to \mathfrak{gl}(\mathbb{C}^N)$ is derived from Frobenius multiplication by Euler vector field $E$, and $\theta$ is the constant matrix as before.

### 2.3.1 Definition and existence of calibration

We are going to prove that (2.9) admits an isomonodromic family of weak Levelt's solutions, i.e., near $z = \infty$ the system (2.8)-(2.9) admits a fundamental solution of the form

$$\Phi(t,z) = S(t,z)z^{\delta}z^{\nu},$$

where the matrices $S(t,z) = S_0 + S_1(t)z^{-1} + S_2(t)z^{-2}\ldots$ with $S_0$ constant (independent of $t$ and $z$) invertible matrix, $\delta$ is a diagonalizable constant matrix and $\nu$ is a nilpotent constant matrix. Moreover, we will prove that there exists a fundamental solution such that $S_0 = 1$.

Substituting the fundamental series $\Phi(t,z)$ in (2.8) and comparing the coefficients in front of powers of $z$, we get that

$$\partial_{t_i} S_k = A_i S_{k-1}, \qquad \forall 1 \leqslant i \leqslant N, \qquad k \in \mathbb{Z}_{>0}. \tag{2.10}$$

Since structure connection is flat, concretely, $[\nabla_{\partial_{t_i}}, \nabla_{\partial_{t_j}}] = 0, \forall 1 \leqslant i,j \leqslant N$, 1-form $\sum_{i=1}^{N} A_i S_{k-1} dt_i$ is closed. As $V$ is simply connected, we can integrate the 1-form and find that

$$S_k(t) = S_k(t^\circ) + \int_{t^\circ}^{t} \sum_{i=1}^{N} A_i S_{k-1} dt_i. \tag{2.11}$$

Therefore, it is sufficient to determine $S_k(t)$ for a fixed $t = t^\circ$. For neighborhood $V$, the values of $S_k(t)$ are determined from the flatness of structure connection according to formula (2.11).

Next, let us solve (2.9) at $t = t^\circ$. It is convenient to introduce the following notation. Let $\operatorname{spec}(\delta)$ be the set of eigenvalues of the operator

$$\operatorname{ad}_\delta : \mathfrak{gl}(H) \to \mathfrak{gl}(H), X \to [\delta, X].$$

Let us denote by $\mathfrak{gl}_a(H)$ the eigensubspace of $\operatorname{ad}_\delta$ with eigenvalue $a$. Then we have a direct sum decomposition of vector spaces

$$\mathfrak{gl}(H) = \bigoplus_{a \in \operatorname{spec}(\delta)} \mathfrak{gl}_a(H).$$

Let us denote by $X_{[a]}$ the projection of $X$ on $\mathfrak{gl}_a(H)$. The matrices $S$, $\delta$, and $\nu$ are identified with elements of $\mathfrak{gl}(H)$ via the basis $\{\phi_i\}_{i=1}^{N} \subset H$ that we fixed above.

Substituting the fundamental series $\Phi(t,z)$ in (2.9) and comparing the coefficients in front of powers of $z$, we get that $\nu_{[-l]} = 0$ if $l \notin \mathbb{Z}_{\geqslant 0}$ and that

$$\theta = \delta + \nu_{[0]}, \tag{2.12}$$

$$kS_k + [\theta, S_k] = E \bullet S_{k-1} + \sum_{l=1}^{k} S_{k-l}\nu_{[-l]}, \qquad k > 0. \tag{2.13}$$

(2.12) uniquely determines $\delta$ and $\nu_{[0]}$: $\delta$ is diagonizable. $\nu_{[0]}$ is nilpotent and $[\delta, \nu_{[0]}] = 0$. So $\delta$ and $\nu_{[0]}$ are uniquely determined by the Jordan-Chevalley decomposition.

For (2.13), the left hand side is

$$(k + \mathrm{ad}_\delta + \mathrm{ad}_{\nu_{[0]}})S_k = \sum_{a\in \mathrm{spec}(\delta)} (k + a + \mathrm{ad}_{\nu_{[0]}})(S_k)_{[a]}$$

where summation is finite since the matrix vector space has finite dimension. Note that $\mathrm{ad}_{\nu_{[0]}}$ preserves the eigenspace of $\mathrm{ad}_\delta$ since $[\delta, \nu_{[0]}] = 0$, we have

$$(k + a + \mathrm{ad}_{\nu_{[0]}})(S_k)_{[a]} = (E \bullet S_{k-1})_{[a]} + \sum_{l=1}^{k} (S_{k-l})_{[a+l]}\, \nu_{[-l]} \tag{2.14}$$

If $k + a \neq 0$, then

$$\begin{aligned}(k + a + \mathrm{ad}_{\nu_{[0]}})(S_k)_{[a]} &= (E \bullet S_{k-1})_{[a]} + \sum_{l=1}^{k-1} (S_{k-l})_{[a+l]}\, \nu_{[-l]} + (S_0)_{[a+k]}\, \nu_{[-k]}\\ &= (E \bullet S_{k-1})_{[a]} + \sum_{l=1}^{k-1} (S_{k-l})_{[a+l]}\, \nu_{[-l]}\end{aligned}$$

and the operator $(k + a + \mathrm{ad}_{\nu_{[0]}})$ is invertible.

$$(k + a + \mathrm{ad}_{\nu_{[0]}})^{-1} = \frac{1}{k+a} \sum_{i=0}^{\infty} \left( -\frac{\mathrm{ad}_{\nu_{[0]}}}{k+a} \right)^i,$$

where the summation over $i$ is actually finite since $\nu_{[0]}$ is nilpotent and then operator $\mathrm{ad}_{\nu_{[0]}}$ is nilpotent as well. Hence, $(S_k)_{[a]}$ can be determined by $S_{k-l}, \nu_{[-l]}, l = 1, 2, \ldots, k-1$.

If $k + a = 0$, then

$$\mathrm{ad}_{\nu_{[0]}}((S_k)_{[-k]}) = (E \bullet S_{k-1})_{[-k]} + \nu_{[-k]} + \sum_{l=1}^{k-1} (S_{k-l})_{[a+l]}\, \nu_{[-l]}. \tag{2.15}$$

There will be ambiguity in the choice of $(S_k)_{[-k]}$ since the operator $\mathrm{ad}_{\nu_{[0]}}$ is non-invertible and $\nu_{[-k]}$ has not been determined. Actually, the situation is somewhat the other way around. We choose $(S_k)_{[-k]} \in \mathfrak{gl}_{-k}(H)$ arbitrarily and use equation (2.15) to determined $\nu_{[-k]}$.

**Proposition 2.41** $S_k(t), k = 1, 2, \ldots$ determined by (2.11) do satisfy (2.13) for all $t \in V$ and thus $\nabla_{\partial_z}\Phi(t, z) = 0$ holds not only at $t = t^\circ$ but also on the neighborhood $V$.

**Proof** Let us prove it by induction. Since $\theta$ is a constant matrix,

$$\begin{aligned}kS_k(t) + [\theta, S_k(t)] =& kS_k(t^\circ) + [\theta, S_k(t^\circ)] + \int_{t^\circ}^{t} \sum_{i=1}^{N} (kA_i S_{k-1} + \mathrm{ad}_\theta(A_i S_{k-1}))dt_i\\ =& kS_k(t^\circ) + [\theta, S_k(t^\circ)] + \int_{t^\circ}^{t} \sum_{i=1}^{N} (kA_i S_{k-1} + \mathrm{ad}_\theta(A_i) S_{k-1} + A_i \mathrm{ad}_\theta(S_{k-1}))dt_i\end{aligned}$$

Note that, by the flatness of structure connection, we have $[\nabla_{\partial_{t_i}}, \nabla_{\partial_z}] = 0, \forall 1 \leqslant i \leqslant N$ which yields relation $A_i + [\theta, A_i] = \partial_{t_i}(E\bullet), \forall 1 \leqslant i \leqslant N$. Thus,

$$\begin{aligned}
kS_k(t) + [\theta, S_k(t)] =& kS_k(t^\circ) + [\theta, S_k(t^\circ)] + \int_{t^\circ}^{t} \sum_{i=1}^{N} (\partial_{t_i}(E\bullet)S_{k-1} + A_i(k-1+\mathrm{ad}_\theta)(S_{k-1}))dt_i \\
=& kS_k(t^\circ) + [\theta, S_k(t^\circ)] + (E \bullet S_{k-1})(t) - (E \bullet S_{k-1})(t^\circ) \\
&+ \int_{t^\circ}^{t} \sum_{i=1}^{N} (-E \bullet \partial_{t_i}(S_{k-1}) + A_i(k-1+\mathrm{ad}_\theta)(S_{k-1}))dt_i
\end{aligned}$$

When $k = 1$, the integral vanishes. We have

$$S_1(t) + [\theta, S_1(t)] - (E\bullet)(t) = S_1(t^\circ) + [\theta, S_1(t^\circ)] - (E\bullet)(t^\circ) = \nu_{[-1]}.$$

Our inductive hypothesis is

$$nS_n(t) + [\theta, S_n(t)] = (E \bullet S_{n-1})(t) + \sum_{l=1}^{n} S_{n-l}(t)\nu_{[-l]}, \qquad \forall t \in V$$

holds for $n = k-1$. When $n = k$,

$$\begin{aligned}
kS_k(t) + [\theta, S_k(t)] =& kS_k(t^\circ) + [\theta, S_k(t^\circ)] + (E \bullet S_{k-1})(t) - (E \bullet S_{k-1})(t^\circ) \\
&+ \int_{t^\circ}^{t} \sum_{i=1}^{N} (-E \bullet \partial_{t_i}(S_{k-1}) + A_i(k-1+\mathrm{ad}_\theta)(S_{k-1}))dt_i \\
=& kS_k(t^\circ) + [\theta, S_k(t^\circ)] + (E \bullet S_{k-1})(t) - (E \bullet S_{k-1})(t^\circ) \\
&+ \int_{t^\circ}^{t} \sum_{i=1}^{N} (-E \bullet A_i(S_{k-2}) + A_i(E \bullet S_{k-2} + \sum_{l=1}^{k-1} S_{k-1-l}\nu_{[-l]}))dt_i
\end{aligned}$$

Therefore,

$$\begin{aligned}
& kS_k(t) + [\theta, S_k(t)] - (E \bullet S_{k-1})(t) \\
=& kS_k(t^\circ) + [\theta, S_k(t^\circ)] - (E \bullet S_{k-1})(t^\circ) + \int_{t^\circ}^{t} \sum_{i=1}^{N} \sum_{l=1}^{k-1} \partial_{t_i}(S_{k-l})\nu_{[-l]} dt_i \\
=& kS_k(t^\circ) + [\theta, S_k(t^\circ)] - (E \bullet S_{k-1})(t^\circ) + \sum_{l=1}^{k} (S_{k-l}(t) - S_{k-l}(t^\circ))\nu_{[-l]}
\end{aligned}$$

We finished the induction step. □

We will see that the arbitrariness of $(S_k)_{[-k]}, -k \in \mathrm{spec}(\delta)$ will be reduced if the weak Levelt solution satifies the symplectic condition $S(t,-z)^T S(t,z) = 1$ in the following proposition.

**Proposition 2.42** There exists a weak Levelt solution such that

$$S(t,-z)^T S(t,z) = 1,$$

where $^T$ is transposition with respect to the Frobenius pairing on $H = T_{u^\circ}U$.

Proposition 2.42 is known if $\theta$ is diagonalizable (see [14]). Let us modify the argument from [14] in order to cover the case of $\theta$ non-diagonalizable.

**Proof** The proof is in my master thesis. □

Let us denote $\mathscr{N} := \bigoplus_{m=0}^{\infty} \mathfrak{gl}_{-m}(H)$. Then, it is the time to show the definition of calibrations (see [20])

**Definition 2.43** An operator series of the form $S(t,z) = 1 + S_1(t)z^{-1} + \dots$ is said to be a *calibration* if there is $\nu \in \mathscr{N}$ such that

1. $S(t,z)z^{\delta}z^{\nu}$ is a solution to $\nabla$. And
2. $S(t,-z)^T S(t,z) = 1$.

Note that $\nu$ is unique.

Let us discuss the analytic property of $S(t,z)$. Let us first fix $t \in M$. Since $\nabla_{\partial/\partial z}$ has a regular singularity at $z = \infty$, irregular singularity at $z = 0$, and no other singularity, we get that $S(t,z)$ is analytic for all $z \in \mathbb{P}^1 \setminus \{0\}$, where $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ is the extended complex plane. (see [26])

The connection operators $\nabla_{\partial/\partial t_i}$, $(1 \leqslant i \leqslant N)$ depend analytically on $(t,z) \in M \times (\mathbb{P}^1 \setminus \{0\})$. Then we obtain that $S(t,z)$ can be extended analytically along any path in $M \times (\mathbb{P}^1 \setminus \{0\})$ (see [2]), that is, $S(t,z)$ is a multivalued analytic function on $M \times (\mathbb{P}^1 \setminus \{0\})$.

In particular, following Givental (see [20]), let us introduce the holomorphic twisted loop group $\mathscr{L}\mathrm{GL}^{(2)}(H) := \{A(z) \in \mathrm{Hol}(\mathbb{C}^*, \mathrm{GL}(H)) | A^T(-z)A(z) = 1\}$ then the calibration $S(t,z) \in \mathscr{L}\mathrm{GL}^{(2)}(H)$ for all $t \in M$.

### 2.3.2 Uniqueness of calibration

**Lemma 2.44** Let $\beta_1, \dots, \beta_m \in \mathbb{R} \setminus \{0\}$ where they are pairwise distict and $C_1, \dots, C_m \in \mathfrak{gl}(H)$. If limit $\lim_{t \to +\infty}(\sum_{i=1}^m C_i e^{\beta_i t\sqrt{-1}})$ exists, then $C_1 = \dots = C_m = 0$.

**Proof** Denote $\sum_{i=1}^m C_i e^{\beta_i t\sqrt{-1}}$ by $L(t)$. Pick an arbitrary number $\Delta t$ from $\mathbb{R} \setminus \bigcup_{1 \leqslant i < j \leqslant m} \frac{2\pi}{\beta_i - \beta_j}\mathbb{Q}$. Then

$$\sum_{i=1}^m C_i e^{\beta_i (t + j\Delta t)\sqrt{-1}} = L(t + j\Delta t), \qquad j = 0, 1, \dots, m-1,$$

which can be written in the form of a Vandermonde matrix acting on the vector

$$(C_1 e^{\beta_1 t\sqrt{-1}}, \dots, C_m e^{\beta_m t\sqrt{-1}})^T.$$

The way of choosing $\Delta t$ make sure that the determinant of Vandermonde matrix does not vanish. Thus, for any $i \in \{1, 2, \dots, m\}$, $C_i e^{\beta_i t\sqrt{-1}}$ is the linear combination of $L(t), L(t+\Delta t), L(t + (m-1)\Delta t)$ given by the inverse of the Vandermonde matrix. Since $\lim_{t \to +\infty} L(t)$ exists, $\lim_{t \to +\infty} C_i e^{\beta_i t\sqrt{-1}}$ exists as well. The only possibility is that $C_1 = C_2 = \dots = C_m = 0$ □

We need to define a group $G$ and its action on $\mathscr{L}\mathrm{GL}^{(2)}(H) \times \mathscr{N}$ which will facilitate the narration of the next theorem.

$$G := \left\{ C(z) = 1 + \sum_{m=1}^{\infty} C_m z^{-m} \middle| C_m \in \mathfrak{gl}_{-m}(H), C(-z)^T C(z) = 1 \right\}$$

and its action is

$$(C(z); A(z), \nu) \mapsto (A(z)C(z), C(1)^{-1}\nu C(1)).$$

**Theorem 2.45** Let us shrink the set on which the group $G$ acts into the set of all pairs $(S(t,z), \nu)$ consisting of a calibration and a corresponding nilpotent operator $\nu$. Then the group $G$ acts faithfully and transitively on the set after shrinking.

**Proof** The proof is in my master thesis. □

## 2.4 Period vectors

The definition of the period map depends on the choice of a calibration of $M$. So we will use the notation of the previous section. Let us fix a reference point $(t^\circ, \lambda^\circ) \in (M \times \mathbb{C})' := \{(t, \lambda) | \det(\lambda - E\bullet_t) \neq 0\}$ such that $\lambda^\circ$ is a sufficiently large real number.

**Proposition 2.46** The following functions provide a fundamental solution to the 2nd structure connection

$$I^{(n)}(t, \lambda) = \sum_{k=0}^{\infty} (-1)^k S_k(t) \tilde{I}^{(n+k)}(\lambda),$$

where

$$\tilde{I}^{(m)}(\lambda) = e^{-\sum_{l=0}^{\infty} \nu_{[-l]}(-\partial_\lambda)^l \partial_m} \left( \frac{\lambda^{\delta - m - \frac{1}{2}}}{\Gamma(\delta - m + \frac{1}{2})} \right).$$

**Proof** First, let us show that $\nabla^{(n)}_{\partial_\lambda} I^{(n)} = 0$.

$$\begin{aligned}
(\lambda - E\bullet)\nabla^{(n)}_{\partial_\lambda} I^{(n)} =& (\lambda - E\bullet)\partial_\lambda I^{(n)} - (\theta - n - \frac{1}{2}) I^{(n)} \\
=& \sum_{k=0}^{\infty} (-1)^k S_k(t) \lambda \partial_\lambda \tilde{I}^{(n+k)}(\lambda) - \sum_{k=0}^{\infty} (-1)^k E \bullet \partial_\lambda S_k(t) \tilde{I}^{(n+k)}(\lambda) \\
& - \sum_{k=0}^{\infty} (-1)^k (\theta - n - \frac{1}{2}) S_k(t) \tilde{I}^{(n+k)}(\lambda)
\end{aligned}$$

We may apply (2.13) to the last line and it will be

$$-(\theta - n - \frac{1}{2})\tilde{I}^{(n)}(\lambda) - \sum_{k=1}^{\infty} (-1)^k \left( S_k(t)(\theta - n - k - \frac{1}{2}) + E \bullet S_{k-1}(t) + \sum_{l=1}^{k} S_{k-l}(t)\nu_{[-l]} \right) \tilde{I}^{(n+k)}(\lambda).$$

Let us rearrange these two summation and shift indices of $S$ such that, in the summation over $k$, there is only $S_k$ and we will get

$$-\sum_{k=0}^{\infty} (-1)^k \left( S_k(\delta - n - k - \frac{1}{2})\tilde{I}^{(n+k)} - E \bullet S_k \tilde{I}^{(n+k+1)} + \sum_{l=0}^{\infty} (-1)^l S_k \nu_{[-l]} \tilde{I}^{(n+k+l)} \right).$$

Note that $\partial_\lambda \tilde{I}^{(n+k)} = \tilde{I}^{(n+k+1)}$, the two terms with $E\bullet$ will cancel out each other. Similarly, $\tilde{I}^{(n+k+l)} = (\partial_\lambda)^l \tilde{I}^{(n+k)}$ and then

$$(\lambda - E\bullet)\nabla^{(n)}_{\partial_\lambda} I^{(n)} = \sum_{k=0}^{\infty}(-1)^k S_k(t)\left(\lambda\partial_\lambda - (\delta - n - k - \frac{1}{2}) - \sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\right)\tilde{I}^{(n+k)}(\lambda).$$

Next we will show that $\left(\lambda\partial_\lambda - (\delta - m - \frac{1}{2}) - \sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\right)\tilde{I}^{(m)}(\lambda) = 0$. An observation is that $(\lambda\partial_\lambda - (\delta - m - \frac{1}{2}))\left(\frac{\lambda^{\delta-m-\frac{1}{2}}}{\Gamma(\delta-m+\frac{1}{2})}\right) = 0$. So we want to commute $\lambda\partial_\lambda - (\delta - m - \frac{1}{2})$ with $-\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m$. The calculation process is the following

$$\begin{aligned}(\lambda\partial_\lambda - (\delta - m - \frac{1}{2}))(-\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m) = &-\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m(\lambda\partial_\lambda - l) + \sum_{l=0}^{\infty}\nu_{[-l]}(\delta - l)(-\partial_\lambda)^l\partial_m \\ &-\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l(\partial_m m - 1) - \frac{1}{2}\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m.\end{aligned}$$

The two term with $-l$ will cancel out each other.

$$(\lambda\partial_\lambda - (\delta - m - \frac{1}{2}))(-\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m) = -\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m(\lambda\partial_\lambda - (\delta - m - \frac{1}{2})) + \sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l.$$

And

$$(\lambda\partial_\lambda - (\delta - m - \frac{1}{2}))e^{-\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m} = e^{-\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m}(\lambda\partial_\lambda - (\delta - m - \frac{1}{2})) + \sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l e^{-\sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\partial_m},$$

which yields $\left(\lambda\partial_\lambda - (\delta - m - \frac{1}{2})\right)\tilde{I}^{(m)}(\lambda) = \sum_{l=0}^{\infty}\nu_{[-l]}(-\partial_\lambda)^l\tilde{I}^{(m)}(\lambda)$. Since $\lambda - E\bullet$ is invertible on $(M \times \mathbb{C})'$, we finished the proof of $\nabla^{(n)}_{\partial_\lambda} I^{(n)} = 0$.

Finally, let us show that $\nabla^{(n)}_{\partial_{t_i}} I^{(n)} = 0$ with the help of $(\lambda - E\bullet)\nabla^{(n)}_{\partial_\lambda} I^{(n)} = 0$. Let us consider

$$(\lambda - E\bullet)\nabla^{(n)}_{\partial_{t_i}} I^{(n)} = (\lambda - E\bullet)\partial_{t_i} I^{(n)} + \phi_i \bullet (\theta - n - \frac{1}{2})I^{(n)}.$$

In virtue of $(\lambda - E\bullet)\nabla^{(n)}_{\partial_\lambda} I^{(n)} = 0$, i.e., $(\theta - n - \frac{1}{2})I^{(n)} = (\lambda - E\bullet)\partial_\lambda I^{(n)}$, we have

$$(\lambda - E\bullet)\nabla^{(n)}_{\partial_{t_i}} I^{(n)} = (\lambda - E\bullet)\partial_{t_i} I^{(n)} + \phi_i \bullet (\lambda - E\bullet)\partial_\lambda I^{(n)} = (\lambda - E\bullet)(\partial_{t_i} + \phi_i \bullet \partial_\lambda)I^{(n)}.$$

Again, we get $\nabla^{(n)}_{\partial_{t_i}} I^{(n)} = (\partial_{t_i} + \phi_i \bullet \partial_\lambda)I^{(n)}$. Then, recalling (2.10), we have

$$\begin{aligned}\nabla^{(n)}_{\partial_{t_i}} I^{(n)} &= \sum_{k=0}^{\infty}(-1)^k\partial_{t_i} S_k(t)\tilde{I}^{(n+k)}(\lambda) + \sum_{k=0}^{\infty}(-1)^k A_i(t)S_k(t)\tilde{I}^{(n+k+1)}(\lambda) \\ &= \sum_{k=0}^{\infty}(-1)^k\partial_{t_i} S_k(t)\tilde{I}^{(n+k)}(\lambda) - \sum_{k=0}^{\infty}(-1)^{k+1}\partial_{t_i} S_{k+1}(t)\tilde{I}^{(n+k+1)}(\lambda) = 0.\end{aligned}$$

Hence we finished the proof. □

The 2nd structure connection has a Fuchsian singularity at infinity, therefore the series $I^{(n)}(t,\lambda)$ is convergent for all $(t,\lambda)$ sufficiently close to $(t^\circ,\lambda^\circ)$. Using the differential equations we extend $I^{(n)}$ to a multi-valued analytic function on $(M\times\mathbb{C})'$. We define the following multi-valued functions taking values in $H$:

$$I_a^{(n)}(t,\lambda) := I^{(n)}(t,\lambda)a, \qquad a\in H, \qquad n\in\mathbb{C}$$

These functions will be called *period vectors*. Using analytic continuation we get a representation

$$\pi_1((M\times\mathbb{C})',(t^\circ,\lambda^\circ)) \to \mathrm{GL}(H)$$

called the *monodromy representation* of the Frobenius manifold. The image $W$ of the monodromy representation is called the *monodromy group*.

Under the semi-simplicity assumption, we may choose a generic reference point $t^\circ$ on $M$, such that the Frobenius multiplication $\bullet_{t^\circ}$ is semi-simple and the operator $E\bullet_{t^\circ}$ has $N$ pairwise different eigenvalues $u_i^\circ(1\leqslant i\leqslant N)$. The fundamental group $\pi_1((M\times\mathbb{C})',(t^\circ,\lambda^\circ))$ fits into the following exact sequence

$$\pi_1(F^\circ,\lambda^\circ)\xrightarrow{i_*}\pi_1((M\times\mathbb{C})',(t^\circ,\lambda^\circ))\to\pi_1(M,t^\circ)\to 1 \tag{2.16}$$

where $p:(M\times\mathbb{C})'\to M$ is the projection on $M$, $F^\circ = p^{-1}(t^\circ) = C\{u_1^\circ,\ldots,u_N^\circ\}$ is the fiber over $t^\circ$, and $i: F^\circ\to(M\times\mathbb{C})'$ is the natural inclusion. For a proof we refer to [47], Proposition 5.6.4 or [40], Lemma 1.5 C. Using the exact sequence (2.16) we get that the monodromy group $W$ is generated by the monodromy transformations representing the lifts of the generators of $\pi_1(M,t^\circ)$ in $\pi_1((M\times\mathbb{C})',(t^\circ,\lambda^\circ))$ and the generators of $\pi_1(F^\circ,\lambda^\circ)$.

The image of $\pi_1(F^\circ,\lambda^\circ)$ under the monodromy representation is a reflection group that can be described as follows. Using the differential equations of the 2nd structure connection it is easy to prove that the pairing

$$(a|b) := (I_a^{(0)}(t,\lambda),(\lambda-E\bullet)I_b^{(0)}(t,\lambda))$$

is independent of $t$ and $\lambda$. This pairing is known as the *intersection pairing*. Suppose now that $\gamma$ is a simple loop in $F^\circ$, i.e., a loop that starts at $\lambda^\circ$, approaches one of the punctures $u_i^\circ$ along a path $\gamma'$ that ends at a point sufficiently close to $u_i^\circ$, goes around $u_i^\circ$, and finally returns back to $\lambda^\circ$ along $\gamma'$. By analyzing the second structure connection near $\lambda=u_i$ it is easy to see that up to a sign there exists a unique $a\in H$ such that $(a|a)=2$ and the monodromy transformation of $a$ along $\gamma$ is $-a$. The monodromy transformation representing $\gamma\in\pi_1(F^\circ,\lambda^\circ)$ is the reflection defined by the following formula:

$$w_a(x) = x-(a|x)a.$$

Let us denote by $R$ the set of all $a\in H$ as above determined by all possible choices of simple loops in $F^\circ$. We refer to the elements of $R$ as reflection vectors.

### 2.4.1 Reflection vectors (Vanishing cycles)

In this section, we shall assume that $n\in\mathbb{Z}$. In the definition of the set $R$ of reflection vectors that we gave just now, we fixed a semi-simple point $t^\circ\in M$ and moved $\lambda$ in $\mathbb{C}-\{u_1^\circ,\ldots,u_N^\circ\}$. On a neighborhood of $t^\circ$, the semi-simplicity assumption and that $E\bullet_t$ has $N$ pairwise different eigenvalues $u_i$ still hold. Next we will find a fundamental solution $Y^{(i)}(u,\lambda)$ to differential equation $\nabla^{(n)}_{\partial/\partial\lambda}Y^{(i)}(u,\lambda)=0$ near $\lambda=u_i$.

**Proposition 2.47** 1. In a neighborhood of $\lambda = u_i$, the differential equation $\nabla^{(n)}_{\partial/\partial\lambda} y = 0$ has a $(N-1)$-dimensional space of holomorphic solutions.

2. In a neighborhood of $\lambda = u_i$, the differential equation $\nabla^{(n)}_{\partial/\partial\lambda} y = 0$ has a unique solution of the form

$$y^{(i)}(\lambda) = \sqrt{2\pi}\frac{(\lambda-u_i)^{-n-\frac{1}{2}}}{\Gamma(-n+\frac{1}{2})}(\frac{1}{\sqrt{\eta_i}}\frac{\partial}{\partial u_i} + O(\lambda-u_i)).$$

**Proof** Let us denote by $y^{(j)}$ the $j$th of column vectors of the matrix $Y^{(i)}(u,\lambda)$,

$$y^{(j)} = (\lambda-u_i)^{\alpha}(y_0^{(j)} + \sum_{k=1}^{\infty} y_k^{(j)}(\lambda-u_i)^k),$$

where $y_0^{(j)}, y_k^{(j)}$ are column vectors depending on $u$. Let us see the coefficient of $(\lambda-u_i)^{\alpha-1}$ in the equation $\nabla^{(n)}_{\partial/\partial\lambda} y^{(j)} = 0$ recalling (2.6) and then we have,

$$A_i^{(n)}(u) y_0^{(j)} = \alpha y_0^{(j)}$$

According to the definition of $A_i^{(n)}(u)$ and Lemma 2.39,

$$E_{ii}\tilde{\Psi}^{-1}(\theta - n - \frac{1}{2})\tilde{\Psi}(\tilde{\Psi}^{-1}y_0^{(j)}) = \alpha\tilde{\Psi}^{-1}y_0^{(j)}.$$

By direct calculation,

$$\det\left(\alpha - E_{ii}\tilde{\Psi}^{-1}(\theta - n - \frac{1}{2})\tilde{\Psi}\right) = \alpha^{N-1}(\alpha - \left(\tilde{\Psi}^{-1}(\theta - n - \frac{1}{2})\tilde{\Psi}\right)_{ii}).$$

**Lemma 2.48** Let $\eta$ be $\mathrm{diag}\{\eta_1,\ldots,\eta_N\}$. Then $(\tilde{\Psi}^{-1}\theta\tilde{\Psi})^T\eta = -\eta(\tilde{\Psi}^{-1}\theta\tilde{\Psi})$, where $^T$ represents the standard matrix transposition, and thus $\left(\tilde{\Psi}^{-1}(\theta - n - \frac{1}{2})\tilde{\Psi}\right)_{ii} = -n - \frac{1}{2}$.

**Proof** Note that $\theta$ is skew symmetric with respect to the Frobenius pairing, i.e.,

$$(\theta(\partial_{u_i}), \partial_{u_j}) = -(\partial_{u_i}, \theta(\partial_{u_j})),$$

where

$$\theta\partial_{u_i} = \sum_{a=1}^{N}\frac{\partial t_a}{\partial u_i}\theta(\partial_{t_a}) = \sum_{a,b=1}^{N}\frac{\partial t_a}{\partial u_i}\theta_{ba}\partial_{t_b} = \sum_{a,b,j=1}^{N}\tilde{\Psi}_{ai}\theta_{ba}(\tilde{\Psi}^{-1})_{jb}\partial_{u_j}.$$

Thus, $(\theta(\partial_{u_i}), \partial_{u_j}) = \sum_{a,b=1}^{N}\tilde{\Psi}_{ai}\theta_{ba}(\tilde{\Psi}^{-1})_{jb}\eta_j = \eta_j(\tilde{\Psi}^{-1}\theta\tilde{\Psi})_{ji}$. Similarly, $(\partial_{u_i}, \theta(\partial_{u_j})) = \eta_i(\tilde{\Psi}^{-1}\theta\tilde{\Psi})_{ij}$. Therefore, $\eta_j(\tilde{\Psi}^{-1}\theta\tilde{\Psi})_{ji} = -\eta_i(\tilde{\Psi}^{-1}\theta\tilde{\Psi})_{ij}, 1 \leqslant i,j \leqslant N$, and thus $(\tilde{\Psi}^{-1}\theta\tilde{\Psi})_{ii} = 0$. □

Let us return to the proof of Proposition 2.47 have eigenvalue $\alpha = 0$ and $\alpha = -n - \frac{1}{2}$ of $A_i^{(n)}(u)$. Since $\left(\tilde{\Psi}^{-1}(\theta - n - \frac{1}{2})\tilde{\Psi}\right)_{ii} = -n - \frac{1}{2} \neq 0$, we have $E_{ii}\tilde{\Psi}^{-1}(\theta - n - \frac{1}{2})\tilde{\Psi}$ and thus $A_i^{(n)}(u)$ are diagonalizable, namely, the dimension of eigenspace for eigenvalue $\alpha = 0$ is $N-1$ and that for eigenvalue $\alpha = -n - \frac{1}{2}$ is 1. Let $y_0^{(j)}$ be the eigenvector of $A_i^{(n)}(u)$ for eigenvalue $\alpha = 0$ if $j \neq i$ and let $y_0^{(i)}$ be that for eigenvalue $\alpha = -n - \frac{1}{2}$. We will see that $y^{(j)}$ is uniquely determined by $y_0^{(j)}$ for all $1 \leqslant j \leqslant N$. For the case $j \neq i$, let us see the coefficient of $(\lambda - u_i)^{k+\alpha-1}$ in the equation $\nabla^{(n)}_{\partial/\partial\lambda} y^{(j)} = 0$ recalling (2.6) and then we have,

$$\left(k - A_i^{(n)}(u)\right) y_k^{(j)} - \sum_{s\neq i} \left( \frac{A_s^{(n)}(u)}{\lambda - u_s} y^{(j)} \right)_{(\lambda-u_i)^{k-1}} = 0,$$

where $\frac{1}{\lambda-u_s} = \frac{1}{u_i-u_s}\frac{1}{1-\frac{\lambda-u_i}{u_s-u_i}} = -\sum_{l'=0}^{\infty} \frac{(\lambda-u_i)^{l'}}{(u_s-u_i)^{l'+1}}$. Then the above equation can be converted into

$$\left(k - A_i^{(n)}(u)\right) y_k^{(j)} = - \sum_{l'+l''=k-1, s\neq i, 1\leqslant s\leqslant N} \frac{A_s^{(n)}(u)}{(u_s - u_i)^{l'+1}} y_{l''}^{(j)}$$

Since $\det\left(k - A_i^{(n)}(u)\right) \neq 0, \forall k \in \mathbb{Z}_{\geqslant 1}$, we can determine $y_k^{(j)}$ by $y_{l''}^{(j)}, l'' = 0, 1, \ldots, k-1$. The first part of the proposition was proved. The same argument holds for the case $j = i$. Thus, we get that $Y^{(i)}(u,\lambda) = [y^{(1)}(u,\lambda), \ldots, y^{(N)}(u,\lambda)]$ is a fundamental solution for $\nabla^{(n)}_{\partial/\partial\lambda} Y^{(i)}(u,\lambda) = 0$ near $\lambda = u_i$. The first part of the proposition was proved.

Note that $y_0^{(i)}$ is proportional to $\partial_{u_i}$. This is because $e_i$ is an eigenvector with eigenvalue $-n-\frac{1}{2}$ of $E_{ii}\tilde{\Psi}^{-1}(\theta - n - \frac{1}{2})\tilde{\Psi}$, i.e., $\tilde{\Psi}e_i = \partial_{u_i}$ is an eigenvector with eigenvalue $-n-\frac{1}{2}$ of $A^{(n)} = \tilde{\Psi}E_{ii}\tilde{\Psi}^{-1}(\theta - n - \frac{1}{2})$. By choosing the coefficient of $\partial_{u_i}$ for the later purpose, we get the second part of the proposition. □

**Definition 2.49** $\varphi \in H = T_{t^\circ}M$ is called a reflection vector if there is a path from $(t^\circ, \lambda^\circ)$ to $(t, u_i(t))$ avoiding the discriminant, such that,

$$I^{(n)}(t,\lambda)\varphi = \sqrt{2\pi}\frac{(\lambda - u_i)^{-n-\frac{1}{2}}}{\Gamma(-n+\frac{1}{2})}\left(\frac{1}{\sqrt{\eta_i}}\frac{\partial}{\partial u_i} + O(\lambda - u_i)\right),$$

where gamma function comes from $\partial_\lambda I^{(n)} = I^{(n+1)}$ and the rest part of the coefficient is due to the intersection pairing $(\varphi|\varphi) = 2$.

As $I^{(n)}(u,\lambda)$ is also a fundamental solution to $\nabla^{(n)}_{\partial/\partial\lambda} I^{(n)}(u,\lambda) = 0$, there is a matrix $C^{(i)}(u)$ depending on $u$ such that $I^{(n)}(u,\lambda) = Y^{(i)}(u,\lambda)C^{(i)}(u), \forall 1 \leqslant i \leqslant N$.

**Corollary 2.50** $\varphi_i := (C^{(i)}(u))^{-1}e_i$ is the reflection vector. Therefore for every reference path from $(t^\circ, \lambda^\circ)$ to $(t, u_i(t))$ avoiding the discriminant, there exists a corresponding reflection vector.

### 2.4.2 Integral structure

Suppose that $M$ is a semi-simple Frobenius manifold and $t^\circ$ is a semi-simple point. We will be interested only in a small open neighborhood of $t^\circ$ in which the tangent bundle $TM$ can be trivialized by a frame of flat vector fields $\phi_a = \partial/\partial t_a$ $(1 \leq a \leq N)$. The vector space $H := T_{t^\circ}M$ is identified with the space of flat vector fields via the flat Levi–Civita connection of the Frobenius pairing. We would like to introduce the $\mathbb{Z}$-submodule $\Lambda$ of $H$ generated by all reflection vectors. Let us make the following assumption about the Frobenius manifold:

(i) The grading operator $\theta$ is diagonalizable with rational eigenvalues.

(ii) There exists a calibration $S(t,z)$ with a corresponding nilpotent operator $\rho$, such that, $[\theta,\rho]=-\rho$.

All examples of Frobenius manifolds coming from quantum cohomology and singularity theory satisfy these conditions. Let us fix a calibration $S(t,z)=1+S_1(t)z^{-1}+S_2(t)z^{-2}+\cdots$, $S_k(t)\in \mathrm{End}(H)$ for which condition (ii) above holds.

**Lemma 2.51** Let $e$ be the unit vector field of the Frobenius manifold. Put $S_1(t)e=\sum_{a=1}^N \tau_a(t)\phi_a$. Then the coefficients $\tau_a(t)$ $(1\le a\le N)$ form a flat coordinate system in a neighborhood of $t^\circ$.

**Proof** Suppose that $(t_1,\ldots,t_N)$ is a flat coordinate system such that $\phi_a=\partial/\partial t_a$ and $\phi_1=e$ is the unit vector field. We have $z\partial_{t_a}S(t,z)=\phi_a\bullet S(t,z)$. Comparing the coefficients in front of $z^0$ we get $\partial_{t_a}S_1(t)=\Omega_a(t)$, where $\Omega_a(t)\in\mathrm{End}(H)$ is the operator of Frobenius multiplication by $\phi_a$. Since $\Omega_a(t)e=\phi_a$ we get $\frac{\partial\tau_b}{\partial t_a}=\delta_{a,b}\Rightarrow \tau_a(t)=t_a+c_a$ where $c_a$ are constants independent of $t$, that is, $\tau_a(t)$ are also flat coordinates. □

Therefore, after fixing a basis $\{\phi_a\}_{a=1}^N$ for the space $H$ of flat vector fields and a calibration $S(t,z)$, there is a canonical choice of flat coordinates $(t_1,\ldots,t_N)$, such that, $\phi_a=\partial/\partial t_a$ $(1\le a\le N)$ and $S_1(t)e=t_1\phi_1+\cdots+t_N\phi_N$.

Let us introduce the following pairing:

$$\langle a,b\rangle := \frac{1}{2\pi}\,(a,e^{\pi \mathbf{i}\theta}e^{\pi\mathbf{i}\rho}b),\quad a,b\in H.$$

We will refer to it as the *Euler* pairing. The following proposition is proved in [37].

**Proposition 2.52** The interesection pairing is a symmetrization of the Euler pairing, that is, $(a|b):=\langle a,b\rangle+\langle b,a\rangle$.

**Definition 2.53** We say that the Frobenius manifold has an *integral structure* if the $\mathbb{Z}$-submodule $\Lambda$ of $H$ generated by the reflection vectors is a free $\mathbb{Z}$-module of rank $N$ and the Euler pairing is integral on $\Lambda$, that is, $\langle a,b\rangle\in\mathbb{Z}$ for all $a,b\in\Lambda$.

It is expected that the Frobenius manifold underlying the quantum cohomology of a smooth projective variety $X$ has an integral structure given by the image of the topological K-ring $K^0(X)$ via an appropriate modification of the Chern character map (see [27]). Under the identification of $\Lambda\cong K^0(X)$, the pairing $\langle\ ,\ \rangle$ coincides with the Euler pairing in $K$-theory which explains the name of $\langle\ ,\ \rangle$ given above.

In singularity theory, the space of flat vector fields is naturally identified with the local algebra $H_f$ of some singularity $f$. Given a primitive form in the sense of K. Saito, we can construct a Frobenius manifold structure on the space of miniversal deformations of $f$. Moreover, it is known that the Frobenius structure has an integral structure with the lattice $\Lambda$ isomorphic to the Milnor lattice of the singularity $f$ and the set of reflection vectors is identified with the set of vanishing cycles. The main motivation for this thesis is to find out if the embedding of the Milnor lattice in the local algebra $H_f$ has an explicit description similar to the one in quantum cohomology suggested by Iritani. Such a description is important for the applications to integrable systems. Let us try to elaborate on this statement.

Suppose that our Frobenius manifold has an integral structure. Let us recall the notion of a lattice vertex algebra $V_\Lambda$ (see [29] for some background). As a vector space

$$V_\Lambda := \mathrm{Sym}(H[s^{-1}]s^{-1})\otimes\mathbb{C}[\Lambda],$$

where $\mathbb{C}[\Lambda]$ is the twisted group algebra of the lattice $\Lambda$, that is, formal linear combinations of $e^\alpha$ ($\alpha \in \Lambda$) with multiplication

$$e^\alpha\, e^\beta = e^{\pi \mathbf{i} \langle \alpha,\beta\rangle} e^{\alpha+\beta}.$$

Put $a_n := as^n$ for $a \in H$ and $n \in \mathbb{Z}$. Let us recall that the vector space $H[s,s^{-1}] \oplus \mathbb{C}K$ has a structure of a Heisenberg Lie algebra with Lie bracket defined by

$$[a_m, b_n] = m\delta_{m,-n}(a|b)K, \quad a,b \in H, \quad m,n \in \mathbb{Z}.$$

The symmetric algebra $\mathrm{Sym}(H[s^{-1}]s^{-1})$ is an irreducible highest weight representation for the Heisenberg Lie algebra where the operators $a_m$ with $m < 0$ act as multiplication operators, the action of $a_n$ with ($n \geq 0$) is uniquely determined by the commutation relations in the Heisenberg Lie algebra and $a_n 1 := 0$, and the central element $K$ acts by 1. The Fock space representation is extended to a representation on $V_\Lambda$, such that,

$$a_n e^\beta = \delta_{n,0}(a|\beta)e^\beta, \quad n \geq 0, \quad a \in H.$$

A key structure in the lattice vertex algebra $V_\Lambda$ is the *state-field correspondence*: it is a map which to each vector $v \in V_\Lambda$, also called *state*, associates a formal series $Y(v,\zeta) := \sum_{n\in\mathbb{Z}} v_{(n)}\zeta^{-n-1}$ with coefficients $v_{(n)} \in \mathrm{End}(V_\Lambda)$. The series $Y(v,\zeta)$ is also called a *field* which means that for each $w \in V_\Lambda$, the Laurent series $Y(v,\zeta)w \in V_\Lambda((\zeta))$ has a finite order pole at $\zeta = 0$. The coefficients $v_{(n)}$ are also known as the modes of $v$. It is known that the state-field correspondence for $V_\Lambda$ is uniquely determined by the definitions

$$Y(a_{-1},\zeta) = \sum_{n\in\mathbb{Z}} a_n \zeta^{-n-1}, \quad a \in H \tag{2.17}$$

$$Y(e^\alpha,\zeta) = e^\alpha \zeta^{\alpha_0} \exp\Big(\sum_{n<0} \alpha_n \frac{\zeta^{-n}}{-n}\Big) \exp\Big(\sum_{n>0} \alpha_n \frac{\zeta^{-n}}{-n}\Big), \tag{2.18}$$

where $\alpha \in \Lambda$ and $\zeta^{\alpha_0} e^\beta = \zeta^{(\alpha|\beta)} e^\beta$, and the following Operator Product Expansion formula

$$Y(a_{(m)}b,\zeta) = \frac{1}{k!}\partial^k_{\zeta_1} \Big( (\zeta_1-\zeta)^{m+1+k} Y(a,\zeta_1)Y(b,\zeta)\Big)\Big|_{\zeta_1=\zeta}, \tag{2.19}$$

where $k$ is chosen so big that the the product of the fields $Y(a,\zeta_1)$ and $Y(b,\zeta)$ has a pole along $\zeta_1 = \zeta$ of order at most $m+k+1$.

**Remark 2.54** If the vertex algebra comes from a quantum field theory in the sense of Wightman, then the composition of any two fields $Y(a,\zeta_1)Y(b,\zeta_2)c$ is a symmetric (with respect to permutations of the pairs $(a,\zeta_1)$ and $(b,\zeta_2)$) meromorphic function in $(\zeta_1,\zeta_2) \in \mathbb{D}^2$ with a finite order pole along the diagonal $\zeta_1 = \zeta_2$. In the mathematical reformulation of Wightman's theory, the fields $Y(a,\zeta)$ are expanded into Laurent series at $\zeta = 0$ and the finite order pole condition is replaced with locality: that is there exists $N > 0$, such that, $(\zeta_1-\zeta_2)^N[Y(a,\zeta_1), Y(b,\zeta_2)] = 0$.

Let us recall yet another Fock space associated to the Frobenius manifold which is part of the so-called Givental' quantization formalism (see [20]). Let $\mathcal{H} := H((z^{-1}))$ be the symplectic loop space of Givental, that is, the vector space of formal Laurent series in $z$ with coefficients in $H$ equipped with the symplectic pairing

$$\Omega(f(z), g(z)) = \mathrm{Res}_{z=0}(f(-z), g(z))dz,$$

where the residue is interpreted formally as the coefficient in front of $dz/z$. The symplectic vector space has a natural polzarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ where $\mathcal{H}_+ = H[z]$ and $\mathcal{H}_- = H[\![z^{-1}]\!]z^{-1}$ are Lagrangian vector subspaces. Let us introduce the Fock space

$$\widehat{\mathcal{O}}_{\mathcal{H}_+,-z} := \mathbb{C}_\hbar[\![q_0, q_1 + e, q_2, \ldots]\!],$$

where $\mathbb{C}_\hbar = \mathbb{C}((\hbar))$ ($\hbar$ is a formal variable), $q_k = (q_{k,1}, \ldots, q_{k,N})$ are formal vector variables, and $q_1 + e = (q_{1,1} + 1, q_{1,2}, \ldots, q_{1,N})$ is the so-called *dilaton* shift. We think secretly of the Fock space as the completion of the space of germs of holomorphic functions on $\mathcal{H}_+$ at the point $-ez$. The Fock space is a representation of the Heisenberg Lie algebra $H[z, z^{-1}] \oplus \mathbb{C}$ with Lie bracket

$$[f(z), g(z)] = \Omega(f(z), g(z)), \quad f(z), g(z) \in H[z, z^{-1}].$$

More precisely, let $\phi^i$ $(1 \le i \le N)$ be the basis of $H$ dual to $\{\phi_i\}$ $(1 \le i \le N)$. The representation of the Heisenberg Lie algebra is given by the following quantization rules:

$$\phi^i(-z)^{-k-1} \mapsto (\phi^i(-z)^{-k-1})\widehat{\ } := q_{k,i}/\sqrt{\hbar}, \quad \phi_i z^k \mapsto (\phi_i z^k)\widehat{\ } := -\sqrt{\hbar}\frac{\partial}{\partial q_{k,i}}.$$

Following Bakalov–Milanov (see [9]), we would like to define a representation of $V_\Lambda$ on $\widehat{\mathcal{O}}_{\mathcal{H}_+,-z}$, that is, a state-field correspondence $Y_t^M(v, \lambda)$ which for each fixed $v \in V_\Lambda$ is a multi-valued analytic function in $(t, \lambda) \in M \times \mathbb{C} \setminus \mathrm{discr}$ with values $\mathrm{End}(\widehat{\mathcal{O}}_{\mathcal{H}_+,-z})$. There is a subtlety in the construction however. The fields $Y_t(v, \lambda)$ act only on a certain subsapce of the Fock space consisting of the so-called *tame series*. To define the latter, let us introduce the notation

$$(\mathbf{q} + ez)^m := (q_{1,1} + 1)^{m_{1,1}} \prod_{(k,i) \neq (1,1)} q_{k,i}^{m_{k,i}}$$

where $m = (m_{k,i})$ is a sequence of non-negative integers, such that, only finitely many of them are non-zero. A monomial $\hbar^{g-1}(\mathbf{q} + e)^m$ is said to be *tame* if

$$\sum_{k,i} k m_{k,i} \le 3g - 3 + \sum_{k,i} m_{k,i}.$$

The subspace of the Fock space $\widehat{\mathcal{O}}_{\mathcal{H}_+,-z}$ consisting of formal power series involving only tame monomials is denoted by $\widehat{\mathcal{O}}^{\mathrm{tame}}_{\mathcal{H}_+,-z}$. Note that the vertex operators

$$\Gamma^\alpha(t, \lambda) := \exp\Big(\sum_{k=0}^\infty \sum_{i=1}^n (I_\alpha^{(-k-1)}(t, \lambda), \phi_i)\frac{q_{k,i}}{\sqrt{\hbar}}\Big) \exp\Big(\sum_{k=0}^\infty \sum_{i=1}^n (-1)^{k+1}(I_\alpha^{(k)}(t, \lambda), \phi^i)\sqrt{\hbar}\frac{\partial}{\partial q_{k,i}}\Big)$$

act on elements of the tame Fock space $\widehat{\mathcal{O}}^{\mathrm{tame}}_{\mathcal{H}_+,-z}$ and produce formal power series in $\mathbf{q} + e$ whose coefficients are formal Laurent series in $\sqrt{\hbar}$ whose coefficients are multi-valued analytic functions on $M \times \mathbb{C} \setminus \mathrm{discr}$. The state-field correspondence for the representation of $V_\Lambda$ is defined by $Y_t(\mathbf{1}, \lambda) := 1$,

$$Y_t(a_{-1}, \lambda) = \sum_{k=0}^\infty \sum_{i=1}^N (-1)^{k+1}(I_a^{(k+1)}(t, \lambda), \phi^i)\sqrt{\hbar}\frac{\partial}{\partial q_{k,i}} + \sum_{k=0}^\infty \sum_{i=1}^N (I_a^{(-k)}(t, \lambda), \phi_i)\frac{q_{k,i}}{\sqrt{\hbar}},$$

and

$$Y_t(e^\alpha,\lambda) = b_\alpha(t,\lambda)\Gamma^\alpha(t,\lambda),$$

where $\mathbf{1} = 1\otimes e^0$ is the vacuum of $V_\Lambda$, $a\in H$, and $\alpha\in\Lambda$. The scalar-valued functions $b_\alpha(t,\lambda)$ should be chosen appropriately so that certain locality and conformal invariance properties hold. Their precise value would not be important to us. The remarkable fact discovered by Bakalov–Milanov (see [9], Proposition 3.2) is that the Operator Product Expansion formula given in the form (2.19) remains the same for the representation. In other words, the states of $V_\Lambda$ should be represented by fields such that

$$Y_t(a_{(m)}b,\lambda) = \frac{1}{k!}\partial^k_{\lambda_1}\left((\lambda_1-\lambda)^{m+1+k}Y_t(a,\lambda_1)Y_t(b,\lambda)\right)\Big|_{\lambda_1=\lambda}, \tag{2.20}$$

where again $k$ should be chosen sufficiently large so that the pole of the composition of the two fields at $\lambda_1=\lambda$ is canceled. It is clear that the OPE formulas (2.20) determines uniquely the state field correspondence $Y_t(\ ,\lambda)$ from the fields that we already defined. Moreover, the resulting fields $Y_t(a,\lambda)$ act on tame elements and produce formal power series in $\mathbf{q}+e$ whose coefficients are Laurent series in $\sqrt{\hbar}$ whose coefficients are multi-valued analytic functions on $M\times\mathbb{C}\setminus$ discr.

The relation to integrable systems is the following. Suppose that we can find an element $\omega\in V_\Lambda\otimes V_\Lambda$, such that,

$$(e^\alpha_{(0)}\otimes 1 + 1\otimes e^\alpha_{(0)})\omega = 0, \quad \forall\alpha\in\mathcal{R} \tag{2.21}$$

where $\mathcal{R}\subset\Lambda$ denotes the set of reflection vectors and $e^\alpha_{(0)}$ are the 0-modes of $e^\alpha$. Then we would like to consider the set of all elements $\mathcal{A}(\hbar,\mathbf{q})$ of the tame Fock space satisfying the condition that $Y_t(\omega,\lambda)\mathcal{A}(\hbar,\mathbf{q}')\mathcal{A}(\hbar,\mathbf{q}'')$ is regular in $\lambda$, that is, the expansion into a formal power series in $\mathbf{q}'+e$ and $\mathbf{q}''+e$ yields a formal power series whose coefficients are formal Laurent series in $\sqrt{\hbar}$ whose coefficients are *polynomials* in $\lambda$. The polynomiality condition means that all coefficients in front of negative powers of $\lambda$, in the Laurent series expansion near $\lambda=\infty$, must vanish. This is equivalent to an infinite system of quadratic equations for the coefficients of $\mathcal{A}(\hbar,\mathbf{q})$ which we expect to be equivalent to an integrable hierarchy of Hirota Quadratic Equations. Moreover, the higher genus reconstruction of Givental, yields the so-called *total ancestor potential* of the Frobenius manifold. One can prove that the total ancestor potential is a solution to the Hirota Quadratic Equations.

The problem is whether a state $\omega$ satisfying the equations (2.21) exists. The answer is known to be positive only for the root lattices of type $ADE$. If $\Lambda$ is a root lattice of type $ADE$ and $\mathcal{R}$ is the corresponding root system, then the Casimir operator

$$\omega := \sum_{\alpha\in\mathcal{R}} e^\alpha\otimes e^{-\alpha} + \frac{1}{2}\sum_{i=1}^N (a_i s^{-1})\otimes(b_i s^{-1}),$$

satisfies the equations (2.21), where $\{a_i\}_{i=1}^N$ and $\{b_i\}_{i=1}^N$ are bases of $H$ dual with respect to the intersection pairing $(\ |\ )$. The corresponding Hirota Quadratic Equations can be identified with a Kac–Wakimoto hierarchy – see [22, 18].

In order to motivate the problem in the current thesis. Let us suppose that we know how to find an element $\omega$ satisfying (2.21). The next step would be to investigate the Hirota Quadratic equations defined by the bi-linear operator $Y_t(\omega,\lambda)$. The question then is: are there explicit formulas for the period vectors $I^{(n)}_\alpha(t,\lambda)$ when $\alpha\in\Lambda$? In fact, we have an obvious relation $I^{(n+1)}_\alpha(t,\lambda) = \partial_\lambda I^{(n)}_\alpha(t,\lambda)$, so in some sense it is sufficient to find such formulas for one of the periods. There is a further simplification which comes from the fact that the Hirota Quadratic Equations can be conjugated by the calibration $S(t,z)$ which has the following effect

$$Y_t(\omega,\lambda)\,(\widehat{S}\otimes\widehat{S}) = (\widehat{S}\otimes\widehat{S})\,\widetilde{Y}(\omega,\lambda),$$

where $\widehat{S}$ means the quantized action of the calibration on the Fock space in the sense of Givental (see [20]) and $\widetilde{Y}(\ ,\lambda)$ is the state-field correspondence defined in the same way as $Y_t(\ ,\lambda)$ except that instead of the periods $I^{(n)}_\alpha(t,\lambda)$ we use the *calibrated periods* $\widetilde{I}^{(n)}_\alpha(\lambda)$ defined in Proposition 2.46. The calibrated periods are given by explicit formulas as long as we know $\alpha$, that is, for the applications to integrable systems we need to know how the reflection lattice $\Lambda$ is embedded into the space $H$ of flat vector fields. In the case of quantum cohomology there is a nice conjectural answer which can be worked out from the work of Iritani. Our expectation is that a similar answer should be available in singularity theory as well. Form the results of this thesis one can easily formulate a conjecture for all weighted-homogeneous singularities corresponding to the so-called invertible polynomials. It would be interesting to generalize our work to all weighted-homogeneous singularities. This however seems to require new ideas.

## 2.5 Weighted Homogeneous Singularities

A polynomial $f = f(x_1,\dots,x_n) \in \mathbb{C}[x_1,\dots,x_n]$ is called a *weighted homogeneous* polynomial if there are positive integers $w_1,\dots,w_n$ and $d$ such that $f(\lambda^{w_1}x_1,\dots,\lambda^{w_n}x_n) = \lambda^d f(x_1,\dots,x_n)$ for all $\lambda\in\mathbb{C}^*$.

**Definition 2.55** A polynomial $f\in\mathbb{C}[x_1,\dots,x_n]$ is called *invertible* if the following conditions are satisfied.

- The number of variables coincides with the number of monomials in $f$:

  $$f(x_1,\dots,x_n) = \sum_{i=1}^n c_i \prod_{j=1}^n x_j^{\mathbb{E}_{ij}}$$

  for some coefficients $c_i\in\mathbb{C}^*$ and non-negative integers $\mathbb{E}_{ij}$ for $i,j = 1,\dots,n$.

- The weights $q_i$ of $x_i$ are uniquely determined by the condition that $f$ has weighted degree 1. We can write the $q_i$ as

  $$q_i = \frac{w_i}{d} \quad \text{with} \quad \gcd(d,w_1,\dots,w_n) = 1,$$

  which is equivalent to the condition that the matrix $\mathbb{E} := (\mathbb{E}_{ij})$ is invertible over $\mathbb{Q}$.

The fact that a polynomial $f$ is invertible implies that $f$ is weighted homogeneous.

A weighted homogeneous polynomial $f$ is called *non-degenerate* if it has at most an isolated critical point at the origin in $\mathbb{C}^n$, i.e., the system of equations $\{\frac{\partial f}{\partial x_i} = 0\}$ has a unique solution at the origin. Let $f\in\mathbb{C}[x_1,\dots,x_n]$ be a non-degenerated, weighted homogeneous polynomial. We define its *maximal group of diagonal symmetries* to be

$$G_f = \{(\lambda_1,\dots,\lambda_n)\in(\mathbb{C}^*)^n | f(\lambda_1 x_1,\dots,\lambda_n x_n) = f(x_1,\dots,x_n)\}\,.$$

In such a polynomial the $c_i$ can be absorbed by rescaling the variables, so we will always assume from now on that $c_i = 1$.

The *Jacobian ideal* $\mathrm{Jac}(f)$ of a non-degenerate polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is the ideal generated by the partial derivatives:

$$\mathrm{Jac}(f) := \left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

and the *Milnor ring* $H_f$ (also called the local algebra) is $H_f := \mathbb{C}[x_1, \dots, x_n] / \mathrm{Jac}(f)$.

We also introduce the following complex vector space $\Omega_f$. Let $\Omega^p(\mathbb{C}^n)$ be the complex vector space of holomorphic $p$-forms of $\mathbb{C}^n$. Consider the complex vector space

$$\Omega_f := \Omega^n(\mathbb{C}^n)/df \wedge \Omega^{n-1}(\mathbb{C}^n).$$

Note that $\Omega_f$ is naturally a free $H_f$-module of rank one, namely, by using the standard volume form $d\mathbf{x} := dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ we have the following isomorphism

$$H_f \xrightarrow{\cong} \Omega_f, \quad [\phi(\mathbf{x})] \mapsto [\phi(\mathbf{x}) d\mathbf{x}].$$

One can define the weighted degree of monomials $\mathbf{x}^{\mathbf{k}} := x_1^{k_1} \dots x_n^{k_n}$ and monomial volume form $\mathbf{x}^{\mathbf{k}} d\mathbf{x} := x_1^{k_1} \dots x_n^{k_n} dx_1 \wedge \cdots \wedge dx_n$ in the following way.

$$\mathrm{wt}(\mathbf{x}^{\mathbf{k}}) = \sum_{i=1}^{n} k_i q_i, \qquad \mathrm{wt}(\mathbf{x}^{\mathbf{k}} d\mathbf{x}) = \sum_{i=1}^{n} (k_i + 1) q_i.$$

**Proposition 2.56** (cf. [23]). Define a $\mathbb{C}$-bilinear form $K_f^{(0)} : \Omega_f \times \Omega_f \to \mathbb{C}$ by

$$K_f^{(0)}([\phi_1(\mathbf{x})d\mathbf{x}], [\phi_2(\mathbf{x})d\mathbf{x}]) := \mathrm{Res}_{x=0} \frac{\phi_1(\mathbf{x})\phi_2(\mathbf{x})d\mathbf{x}}{\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2} \cdots \frac{\partial f}{\partial x_n}} = \frac{1}{(2\pi\sqrt{-1})^n} \int_{\gamma_\epsilon} \frac{\phi_1(\mathbf{x})\phi_2(\mathbf{x})d\mathbf{x}}{\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2} \cdots \frac{\partial f}{\partial x_n}},$$

where $\gamma_\epsilon$ is a sufficiently small cycle around the unique critical point $\mathbf{x} = 0$ defined by $|\frac{\partial f}{\partial x_1}| = \cdots = |\frac{\partial f}{\partial x_n}| = \epsilon$. Then, the bilinear form $K_f^{(0)}$ on $\Omega_f$ is non-degenerate. We call it residue pairing on $\Omega_f$.

Let us denote by $(\ ,\ )$ the *residue pairing* on $H_f$ corresponding to the standard volume form $d\mathbf{x}$, that is,

$$([\phi_1(x)], [\phi_2(x)]) := \mathrm{Res}_{x=0} \frac{\phi_1(\mathbf{x})\phi_2(\mathbf{x})d\mathbf{x}}{\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2} \cdots \frac{\partial f}{\partial x_n}}, \quad \phi_1, \phi_2 \in \mathbb{C}[x_1, \dots, x_n].$$

The Milnor ring is a finite-dimensional $\mathbb{C}$-vector space, graded by the weighted degree of the monomials. The subspaces of highest weighted degree is one-dimensional, spanned by the Hessian determinant $\mathrm{hess}(f)$, and has weighted degree equal to the *central charge*

$$D = \sum_{i=1}^{n} (1 - 2q_i).$$

Let $\theta : \Omega_f \to \Omega_f$ (resp. $\theta : H_f \to H_f$) be a linear operator defined by

$$\theta(\phi(\mathbf{x})d\mathbf{x}) := \left( \frac{n}{2} - \mathrm{wt}(\phi(\mathbf{x})d\mathbf{x}) \right) \phi(\mathbf{x})d\mathbf{x}$$

and

$$\theta(\phi(\mathbf{x})) := \left(\frac{D}{2} - \mathrm{wt}(\phi(\mathbf{x}))\right)\phi(\mathbf{x})$$

where $\phi(\mathbf{x})$ is a weighted homogeneous polynomial.

The residue pairing $(\ ,\ )$ makes $H_{f_n}$ into a Frobenius algebra; that is, for every $a, b, c \in H_{f_n}$ we have

$$(a \cdot b, c) = (a, b \cdot c).$$

The residue pairing in $H_{f_n}$ can be computed as

$$(a, b) = \mu \frac{ab}{\mathrm{hess}(f)}, \tag{2.22}$$

by which we mean

$$ab = \mu^{-1}\,\mathrm{hess}(f)(a, b) + \text{lower-degree terms},$$

where $\mu$ is the dimension of the vector space $H_f$.

### 2.5.1 Opposite subspaces

Motivated by the miniversal deformation of the singularity, we introduce twisted de Rham cohomology and higher-residue pairing. First, we define twisted de Rham complexes

$$\left(\Omega^\bullet(\mathbb{C}^n)[\![w]\!]w^{-k}, d_f\right), \qquad k \in \mathbb{Z}$$

and

$$\left(\Omega^\bullet(\mathbb{C}^n)((w)), d_f\right),$$

where the differential $d_f := wd + df\wedge$. One may prove that the cohomologies of these complexes are concentrated in degree $n+1$. Therefore, let us introduce the following twisted de Rham cohomology groups:

$$\widehat{\mathcal{H}}_f^{(k)} := H^n\left(\Omega^\bullet(\mathbb{C}^n)[\![w]\!]w^{-k}, d_f\right) = \Omega^n(\mathbb{C}^n)[\![w]\!]w^{-k}/(wd + df\wedge)\Omega^{n-1}(\mathbb{C}^n)[\![w]\!]w^{-k}$$

and

$$\widehat{\mathcal{H}}_f := H^n\left(\Omega^\bullet(\mathbb{C}^n)((w)), d_f\right) = \Omega^n(\mathbb{C}^n)((w))/(wd + df\wedge)\Omega^{n-1}(\mathbb{C}^n)((w)).$$

Let us introduce the following involution $*$ on $p \in \mathbb{C}((w))$:

$$p = \sum_{k\in\mathbb{Z}} p_k w^k \mapsto p^* := \sum_{k\in\mathbb{Z}} p_k(-w)^k.$$

A $\mathbb{C}$-bilinear pairing

$$K_f : \widehat{\mathcal{H}}_f \times \widehat{\mathcal{H}}_f \to \mathbb{C}((w))$$

is said to be a *higher residue pairing* if the following properties are satisfied:

1. For all $\omega_1, \omega_2 \in \widehat{\mathcal{H}}_f$,

$$K_f(\omega_1, \omega_2) = (-1)^n K_f(\omega_2, \omega_1)^*.$$

2. For all $p \in \mathbb{C}((w))$ and for all $\omega_1, \omega_2 \in \widehat{\mathcal{H}}_f$,

$$pK_f(\omega_1, \omega_2) = K_f(p\omega_1, \omega_2) = k_f(\omega_1, p^*\omega_2).$$

3. If $\omega_1, \omega_2 \in \widehat{\mathcal{H}}_f^{(0)}$, then

$$K_f(\omega_1, \omega_2) \in \mathbb{C}[\![w]\!]w^n$$

and the following diagram is commutative

$$\begin{array}{ccc} \widehat{\mathcal{H}}_f^{(0)} \times \widehat{\mathcal{H}}_f^{(0)} & \xrightarrow{K_f} & \mathbb{C}[\![w]\!]w^n \\ {\scriptstyle r^{(0)} \times r^{(0)}} \downarrow & & \downarrow {\scriptstyle \text{mod } \mathbb{C}[\![w]\!]w^{n+1}} \\ \Omega_f \times \Omega_f & \xrightarrow{K_f^{(0)}} & \mathbb{C} \end{array}$$

where $r^{(0)} : \widehat{\mathcal{H}}_f^{(0)} \to \widehat{\mathcal{H}}_f^{(0)} / w\widehat{\mathcal{H}}_f^{(0)} = \Omega_f$ is the natural quotient map.

4. The following version of the Leibnitz rule holds:

$$\partial_w K_f(\omega_1, \omega_2) = K_f(\nabla_{\partial/\partial w}(\omega_1), \omega_2) - K_f(\omega_1, \nabla_{\partial/\partial w}(\omega_2)),$$

for all $\omega_1, \omega_2 \in \widehat{\mathcal{H}}_f$.

A theorem says that such higher residue pairing exists. We have the following formal power series

$$K_f(\omega_1, \omega_2) := \sum_{p=0}^{\infty} K_f^{(p)}(\omega_1, \omega_2) w^{p+n},$$

for all $\omega_1, \omega_2 \in \widehat{\mathcal{H}}_f^{(0)}$.

**Definition 2.57** A subspace $P \subset \widehat{\mathcal{H}}_f$ is said to be a homogeneous opposite subspace if

1. $P$ is Lagrangian with respect to

$$\Omega(\phi_1, \phi_2) := \mathrm{Res}_{w=0} K_f(\phi_1, \phi_2) w^{-n} dw$$

2. $\widehat{\mathcal{H}}_f = \widehat{\mathcal{H}}_f^{(0)} \oplus P$

3. $w^{-1}P \subset P$

4. (homogeneity) $w\nabla_{\partial/\partial w}(P) \subset P$

**Proposition 2.58** If $P$ is a homogeneous opposite subspace, then

1. The quotient map $r^{(0)}$

$$\widehat{\mathcal{H}}_f^{(0)} \to \widehat{\mathcal{H}}_f^{(0)} / w\widehat{\mathcal{H}}_f^{(0)} = \Omega_f$$

induces an isomorphism

$$\widehat{\mathcal{H}}_f^{(0)} \cap wP \overset{\cong}{\to} \Omega_f.$$

Let

$$\sigma : \Omega_f \to \widehat{\mathcal{H}}_f^{(0)} \cap wP \subset \widehat{\mathcal{H}}_f^{(0)}$$

be the corresponding inverse.

2. $K_f^{(p)}(\phi_1, \phi_2) = 0$ for all $p > 0, \phi_1, \phi_2 \in \widehat{\mathcal{H}}_f^{(0)} \cap wP$.

**Proof** 1. First, let us prove that the map is injective. Suppose that $\phi \in \widehat{\mathcal{H}}_f^{(0)} \cap wP$ is mapped to 0 in $\Omega_f$, that is, $\phi \in w\widehat{\mathcal{H}}_f^{(0)}$. It follows that $\phi \in w\left(\widehat{\mathcal{H}}_f^{(0)} \cap P\right) = 0$. For the surjectivity, we need to prove that for a given $\phi \in \widehat{\mathcal{H}}_f^{(0)}$ there exists $\psi \in \widehat{\mathcal{H}}_f^{(0)}$, such that, $\phi + w\psi \in \widehat{\mathcal{H}}_f^{(0)} \cap wP$ . Using condition (2) from the definition of an opposite subspace (see Definition 2.57), we get that $w^{-1}\phi = \psi_1 + \psi_2$, for some $\psi_1 \in \widehat{\mathcal{H}}_f^{(0)}$ and $\psi_2 \in P$. Note that $\psi = -\psi_1$ has the required property.

2. Suppose that $\phi_1, \phi_2 \in \widehat{\mathcal{H}}_f^{(0)} \cap wP$. Note that

$$K_f^{(p)}(\phi_1, \phi_2) = -\operatorname{Res}_{w=0} K_f^{(p)}(w^{-p}\phi_1, w^{-1}\phi_2)w^{-n}dw = -\Omega(w^{-p}\phi_1, w^{-1}\phi_2).$$

Since $\phi_1, \phi_2 \in wP$ and $w^{-1}P \subset P$, we get $w^{-l}\phi_1 \in P$ and $w^{-1}\phi_2 \in P$. The vanishing claim follows from the fact that $P$ is a Lagrangian subspace. □

**Remark 2.59** A basis of $\widehat{\mathcal{H}}_f^{(0)} \cap wP$ is usually called a *good basis* of $\widehat{\mathcal{H}}_f^{(0)}$, i.e., if $\omega_1, \ldots, \omega_\mu \subset \widehat{\mathcal{H}}_f^{(0)} \cap wP$ is a basis, then

$$\widehat{\mathcal{H}}_f^{(0)} = \bigoplus_{i=1}^{\mu} \mathbb{C}[\![w]\!]\omega_i, \qquad P = \bigoplus_{i=1}^{\mu} \mathbb{C}[w^{-1}]w^{-1}\omega_i.$$

### 2.5.2 Steenbrink's Hodge filtration

Using the Hodge structure on $H^{n-1}(f^{-1}(1), \mathbb{C})$ we will prove the existence of a homogeneous opposite subspace.

Let us define the following complex vector bundle

$$\bigcup_{\lambda \in \mathbb{C}^*} H^{n-1}(f^{-1}(\lambda), \mathbb{C}) \to \mathbb{C}^*$$

with fiber $H^{n-1}(f^{-1}(\lambda), \mathbb{C})$. The complex vector bundle is called *vanishing cohomology bundle* and denoted by $H^{n-1}$.

Put $\mathfrak{h} := H^{n-1}(f^{-1}(1), \mathbb{C})$ for the fiber of $H^{n-1}$ at $\lambda = 1$. Parallel transport along the unit circle $|\lambda| = 1$ in counter clockwise direction defines a linear operator $M \in \operatorname{End}(\mathfrak{h})$ which we called the classical monodromy operator. A linear operator $\mathcal{N}$ with eigenvalues in $(-1, 0]$ can be defined by $M = e^{-2\pi \mathbf{i} \mathcal{N}}$.

**Remark 2.60** (cf. Lemma 9.4 of [38])We have homoemorphism between fibers $f^{-1}(\lambda)$ and $f^{-1}(\lambda \cdot e^{\mathbf{i}\theta})$ given by

$$f^{-1}(\lambda) \ni \mathbf{x} = (x_1, \ldots, x_n) \mapsto (e^{\mathbf{i}\theta q_1}x_1, \ldots, e^{\mathbf{i}\theta q_n}x_n).$$

Recall that weighted homogeneity

$$e^{\mathbf{i}\theta}f(x_1,\ldots,x_n)=f(e^{\mathbf{i}\theta q_1}x_1,\ldots,e^{\mathbf{i}\theta q_n}x_n).$$

We can define a global section of $H^{n-1}$ which will be called *elementary section*. If $A\in\mathfrak{h}$, then

$$\lambda^{\mathcal{N}}A\in H^{n-1}(f^{-1}(\lambda);\mathbb{C})\cong\mathfrak{h},$$

is a global section. This is due to the fact that if we analytic continuate $\lambda$ around $\lambda=0$:

$$\lambda^{\mathcal{N}}A\mapsto\lambda^{\mathcal{N}}e^{2\pi\mathbf{i}\mathcal{N}}M(A)=\lambda^{\mathcal{N}}A.$$

Given a holomorphic form $\omega\in\Omega^n(\mathbb{C}^n)$ let us recall the so-called *geometric sections* of the vanishing cohomology bundle

$$s(\omega,\lambda):=\int\frac{\omega}{df}\in H^{n-1}(f^{-1}(\lambda);\mathbb{C}),$$

where $\omega/df$ denotes a holomorphic $n-1$-form $\eta$ defined in a tubular neighborhood of $f^{-1}(\lambda)$, such that, $\omega=df\wedge\eta$. The choice of $\eta$ is not unique, but its restriction to $f^{-1}(\lambda)$ is uniquely determined. Note that if $\omega=\mathbf{x}^{\mathbf{k}}d\mathbf{x}$, then

$$s(\omega,\lambda)=\lambda^{\mathrm{wt}(\omega)-1}s(\omega,1),\tag{2.23}$$

since

$$\int\frac{\omega}{df}=\int_{\mathbf{x}\in f^{-1}(\lambda)}\frac{\mathbf{x}^{\mathbf{k}}d\mathbf{x}}{df(\mathbf{x})}=\lambda^{\mathrm{wt}(\omega)-1}\int_{\mathbf{y}\in f^{-1}(1)}\frac{\mathbf{y}^{\mathbf{k}}d\mathbf{y}}{df(\mathbf{y})}$$

where $x_i=\lambda^{q_i}y_i$.

Let us recall the following defintion

**Definition 2.61** Suppose that $H$ is a complex vector space equipped with a real structure $H_{\mathbb{R}}$. A *Polarized Hodge Structure* on $H$ of weight $r$ is the data of a decreasing filtration $F^p(p\in\mathbb{Z})$ of $H$

$$F^p=0\quad\text{for } p\gg0,\quad F^{p+1}\subseteq F^p,\quad F^p=H\quad\text{for } p\ll0$$

and a real $(-1)^r$-symmetric form $S$, that is, $S(x,y)=(-1)^rS(y,x)$, such that,

1. $H=F^p\oplus\overline{F^{r-p+1}}$ for all $p\in\mathbb{Z}$,
2. $S(F^p,F^{r-p+1})=0$, for all $p$,
3. $\mathbf{i}^{r^2+2r-2p}S(x,\overline{x})>0$ for all $x\in F^p\cap\overline{F^{r-p}}\setminus\{0\}$.

**Proposition 2.62** We define the following vector subspace for $p\in\mathbb{Z}$

$$F^p\mathfrak{h}:=\{A\in\mathfrak{h}|\exists\omega=\sum_i\omega_i,\quad \omega_i\text{ are weighted-homogeneous forms such that } A=s(\omega,1)\text{ and }\mathrm{wt}(\omega_i)\leq n-p\}.$$

Then

1. $F^p\mathfrak{h} \subseteq F^{p-1}\mathfrak{h}$, i.e., it is a decreasing filtration.

2. The filtration $F^p\mathfrak{h}$ is $M$-invariant.

**Proof** 1. If $A = s(\omega, 1)$ for $\omega$ weighted homogeneous, we define the form

$$\widetilde{\omega} = df \wedge d^{-1}\omega,$$

where $d^{-1}\omega$ denotes any homogeneous form $\eta$ such that $d\eta = \omega$. Then

$$s(\widetilde{\omega}, \lambda) = \int d^{-1}\omega.$$

Take partial derivative with respect to $\lambda$,

$$\partial_\lambda s(\widetilde{\omega}, \lambda) = \nabla_{\partial/\partial\lambda} \int d^{-1}\omega = \int \frac{\omega}{df} = s(\omega, \lambda).$$

It is easy to see that $\widetilde{\omega}$ is homogeneous form of weight $\mathrm{wt}(\omega) + 1$. According to (2.23),

$$s(\widetilde{\omega}, \lambda) = \lambda^{\mathrm{wt}(\widetilde{\omega})-1} s(\widetilde{\omega}, 1) = \lambda^{\mathrm{wt}(\omega)} s(\widetilde{\omega}, 1)$$

Then,

$$\lambda^{\mathrm{wt}(\omega)-1} s(\omega, 1) = s(\omega, \lambda) = \partial_\lambda \left( \lambda^{\mathrm{wt}(\omega)} s(\widetilde{\omega}, 1) \right) = \mathrm{wt}(\omega) \lambda^{\mathrm{wt}(\omega)-1} s(\widetilde{\omega}, 1).$$

To get $A$, we put $\lambda = 1$:

$$A = s(\omega, 1) = \mathrm{wt}(\omega) s(\widetilde{\omega}, 1)$$

If $A \in F^p\mathfrak{h}$, then $\mathrm{wt}(\omega) \leqslant n - p$. Therefore, $\mathrm{wt}(\widetilde{\omega}) \leqslant n - (p-1)$, which implies $A \in F^{p-1}\mathfrak{h}$.

2. Let $A = s(\omega, 1)$ and $\omega = \sum_i \omega_i$ where $\omega_i$ are weighted homogeneous form of weight smaller or equal than $n - p$. Denote $A_i := s(\omega_i, 1) \in F^p\mathfrak{h}$. Analytic continuation around $\lambda = 0$ along $|\lambda| = 1$ yields

$$M(A_i) = e^{-2\pi \mathbf{i}\, \mathrm{wt}(\omega_i)} A_i.$$

According to the definition of $A, A_i$, we have $A = \sum_i A_i$. Thus,

$$F^p\mathfrak{h} = \bigoplus_{s \in \mathbb{S}} F^p\mathfrak{h} \cap \mathfrak{h}_s,$$

where $\mathfrak{h}_s := \mathrm{Ker}(M - s\,\mathrm{id})$ and $\mathbb{S}$ denotes the set o fcomplex numbers with absolute value 1. □

From the proof of 2. of 2.62, we define $F^p\mathfrak{h}_s = F^p\mathfrak{h} \cap \mathfrak{h}_s$. We have

$$0 = F^n\mathfrak{h}_s \subseteq F^{n-1}\mathfrak{h}_s \subseteq \cdots \subseteq F^0\mathfrak{h}_s = \mathfrak{h}_s$$

where $s \in \mathbb{S}$.

Furthermore, denote $\mathfrak{h}_{\neq 1} := \bigoplus_{s \neq 1} \mathfrak{h}_s$. For $\mathfrak{h}$, we have

$$\mathfrak{h} = H^{n-1}(f^{-1}(1), \mathbb{C}) \supset H^{n-1}(f^{-1}(1), \mathbb{R}) =: \mathfrak{h}_{\mathbb{R}},$$

where $\mathfrak{h}_{\mathbb{R}}$ is real structure. Recall Definition 2.61, we can find that $\{F^p\mathfrak{h}_{\neq 1}\}_{p\in\mathbb{Z}}$ is a Hodge filtration of weight $n-1$, i.e., $\mathfrak{h}_{\neq 1} = F^p\mathfrak{h}_{\neq 1} \oplus \overline{F^{n-p}\mathfrak{h}_{\neq 1}}$. While, $\{F^p\mathfrak{h}_1\}_{p\in\mathbb{Z}}$ is a Hodge filtration of weight $n$, i.e., $\mathfrak{h}_1 = F^p\mathfrak{h}_1 \oplus \overline{F^{n-p+1}\mathfrak{h}_1}$.

Let us define *Polarizing form* $S$ as follows,

$$S : \mathfrak{h}_{\mathbb{Z}} \times \mathfrak{h}_{\mathbb{Z}} \to \mathbb{Z}$$

$$S(A,B) := \begin{cases} (-1)^{\frac{(n-1)(n-2)}{2}} L(A,(M-\mathrm{id})^{-1}B) & \text{if } A,B \in \mathfrak{h}_{\neq 1} \\ -(-1)^{\frac{(n-1)(n-2)}{2}} L(A,B) & \text{if } A,B \in \mathfrak{h}_1 \end{cases}$$

where $L$ is the Seifert form.

For the above two filtrations (restricted to $\mathfrak{h}_{\mathbb{Z}}$) and Polarizing form we have the following conclusion, $S(F^p\mathfrak{h}_{\neq 1}, F^{n-p}\mathfrak{h}_{\neq 1}) = 0$ and $S(F^p\mathfrak{h}_1, F^{n-p+1}\mathfrak{h}_1) = 0$for all $p \in \mathbb{Z}$, equivalently,

$$S(F^p\mathfrak{h}_{\neq 1}, F^q\mathfrak{h}_{\neq 1}) = 0 \qquad \text{if } p+q \geqslant n$$

and

$$S(F^p\mathfrak{h}_1, F^q\mathfrak{h}_1) = 0 \qquad \text{if } p+q \geqslant n+1.$$

**Remark 2.63** If $\omega = \mathbf{x}^{\mathbf{k}}d\mathbf{x}$ is homogeneous, then

$$S(\omega, 1) \in F^{n+1-\lceil \mathrm{wt}(\omega) \rceil}\mathfrak{h}$$

where $\lceil a \rceil$ is the ceiling of $a$, i.e., the smallest integer that is bigger or equal than $a$.

**Definition 2.64** An increasing filtration $\{U_p\mathfrak{h}\}_{p\in\mathbb{Z}}$, (i.e. $U_p\mathfrak{h} \subset U_{p+1}\mathfrak{h}$) is said to be an opposite filtration to $\{F^p\mathfrak{h}\}$ if

1. $U_p\mathfrak{h}$ is $M$-invariant and we have similar decomposition:

$$U_p\mathfrak{h} = \bigoplus_{s\in S} U_p\mathfrak{h}_s,$$

where $U_p\mathfrak{h}_s := U_p\mathfrak{h} \cap \mathfrak{h}_s$.

2. The filtration is finite:

$$U_p\mathfrak{h} = \begin{cases} 0 & \text{for } p \ll 0 \\ \mathfrak{h} & \text{for } p \gg 0 \end{cases}.$$

3. We have $\mathfrak{h} = \bigoplus_{p\in\mathbb{Z}} F^p\mathfrak{h} \cap U_p\mathfrak{h}$.

4. $S(U_p\mathfrak{h}_{\neq 1}, U_q\mathfrak{h}_{\neq 1}) = 0$ if $p+q < n-1$.
$S(U_p\mathfrak{h}_1, U_q\mathfrak{h}_1) = 0$ if $p+q < n$.

Note that $U_p\mathfrak{h}_{\neq 1} := \overline{F^{n-1-p}\mathfrak{h}_{\neq 1}}$ and $U_p\mathfrak{h}_1 := \overline{F^{n-p}\mathfrak{h}_1}$ is an opposite filtration.

The idea to use opposite filtrations to construct opposite subspaces is due to M. Saito [44] (see also [24], Theorem 7.16).

### 2.5.3 Opposite filtrations and opposite subspaces

Let us recall the twisted de Rham cohomology groups $\widehat{\mathcal{H}}_f, \widehat{\mathcal{H}}_f^{(0)}$ equipped with the higher residue pairing $K_f$. We are going to construct an embedding

$$\psi : H^n(f^{-1}(1), \mathbb{C}) \to \widehat{\mathcal{H}}_f^{(0)}.$$

If $A \in F^p\mathfrak{h}$, then there is $\omega = \sum_i \omega_i$ such that $A = s(\omega, 1)$, where $\omega_i$ is weighted homogeneous and $\mathrm{wt}(\omega_i) \leqslant n - p$. Then $\psi$ is defined by

$$\psi(A) = \sum_i (-w)^{n - \lceil \mathrm{wt}(\omega_i) \rceil} [\omega_i].$$

**Proposition 2.65** The higher residue pairing $K_f^{(m)}(\psi(A_1), \psi(A_2))$ could be non-zero only if $A_1, A_2 \in \mathfrak{h}_{\neq 1}$ and $m = n$, or $A_1, A_2 \in \mathfrak{h}_1$ and $m = n + 1$. The following formula holds:

$$K_f^{(m)}(\psi(A_1), \psi(A_2)) = \frac{1}{(2\pi \mathbf{i})^m} S(A_1, A_2),$$

where $m = n$ in the first case and $m = n + 1$ in the second case.

**Proposition 2.66** If $U_p\mathfrak{h} (p \in \mathbb{Z})$ is an opposite filtration, then the subspace

$$P := \mathrm{Span}_{\mathbb{C}}\{\psi(A) w^{-p-k-1} | p \in \mathbb{Z}, A \in F^p\mathfrak{h} \cap U_p\mathfrak{h}, k \geqslant 0\}$$

is a homogeneous opposite subspace.

To construct a good basis Remark 2.59, let $A_i \in F^{p_i}\mathfrak{h} \cap U_{p_i}\mathfrak{h}, \quad (1 \leqslant i \leqslant \mu)$ be a basis of $\mathfrak{h}$. One may assume that $A_i$ are eigenvectors of $M$:

$$M(A_i) = e^{-2\pi \mathbf{i} \alpha_i} A_i, \qquad -1 < \alpha_i \leqslant 0.$$

Then the embedding $\psi$ yields

$$\psi(A_i) = (-w)^{p_i} [\omega_i]$$

where $\omega_i$ is a homogeneous form of weight $n - p_i + \alpha_i$. Thus, we can say that the cohomology classes $[\omega_i], \quad (1 \leqslant i \leqslant \mu)$ form a good basis of $\widehat{\mathcal{H}}_f^{(0)} \cap wP$.

We want to define the linear map

$$\Pi : H_{n-1}(f^{-1}(1), \mathbb{C}) \to \Omega_f \cong H_f$$

by

$$K_f^{(0)} \left( \frac{1}{\Gamma(\theta + l + \frac{1}{2})} \Pi(\gamma), \phi \right) := \frac{1}{(2\pi)^l} \int_\gamma \frac{\omega}{df}$$

where $\gamma \in H_{n-1}(f^{-1}(1), \mathbb{C}), \phi \in \Omega_f$ and $\omega \in \Omega^n(\mathbb{C}^n)$ is a holomorphic form satisfying:

1. $[\omega] \in \widehat{\mathcal{H}}_f^{(0)} \cap wP$.
2. $[\![\omega]\!] = \phi$.

# Chapter 3

# ADE singularity

## 3.1 Introduction

### 3.1.1 Simple singularities

Let us give a precise statement of the problem that we want to solve. Let $f(x_1, x_2, x_3) = g(x_1, x_2) + x_3^2$, where $g$ is one of the polynomials listed in the following table:

| Type | $A_N$ | $D_N$ | $E_6$ | $E_7$ | $E_8$ |
|---|---|---|---|---|---|
| $g$ | $x_1^{N+1} + x_2^2$ | $x_1^2 x_2 + x_2^{N-1}$ | $x_1^3 + x_2^4$ | $x_1^3 + x_1 x_2^3$ | $x_1^3 + x_2^5$ |

The polynomial $f$ represents the germ of a simple singularity at $x = 0$. Let $H_f := \mathbb{C}[x_1, x_2, x_3]/(f_{x_1}, f_{x_2}, f_{x_3})$ be the *Milnor ring* of $f$, where $f_{x_i} := \frac{\partial f}{\partial x_i}$. Let us denote by $(\ ,\ )$ the *residue pairing* on $H_f$ corresponding to the standard volume form $\omega = dx_1 \wedge dx_2 \wedge dx_3$, that is,

$$(\phi_1(x), \phi_2(x)) := \mathrm{Res}_{x=0} \frac{\phi_1(x)\phi_2(x)\omega}{f_{x_1} f_{x_2} f_{x_3}} .$$

The hypersurfaces $V_\lambda = \{x \in \mathbb{C}^3 \mid f(x) = \lambda\}$ for $\lambda \neq 0$ are non-singular and their union has a structure of a smooth fibration on $\mathbb{C} \setminus \{0\}$ known as the *Milnor fibration*. Let us fix a reference point $\lambda = 1$ and consider the middle homology group $H_2(V_1; \mathbb{Z})$, known also as the *Milnor lattice*. Our interest is in the period vectors $I^{(-1)}_\alpha(\lambda) \in H_f$ defined by

$$(I^{(-1)}_\alpha(\lambda), \phi_i) := \frac{1}{2\pi} \int_{\alpha_\lambda} \phi_i(x) \frac{\omega}{df},$$

where $\alpha \in H_2(V_1; \mathbb{C})$, $\phi_i(x)$ $(1 \leq i \leq N)$ is a set of polynomials representing a basis of $H_f$, $\alpha_\lambda \in H_2(V_\lambda; \mathbb{C})$ is obtained from $\alpha$ via a parallel transport along some reference path, and $\frac{\omega}{df}$ is the so-called *Gelfand–Leray* form (see [3]). Alternatively, we can view each period vector as a multivalued analytic function $I^{(-1)}_\alpha : \mathbb{C} \setminus \{0\} \to H_f$.

Let us assign degree $c_i \in \mathbb{Q}_{>0}$ to $x_i$ $(1 \leq i \leq 3)$, such that, the polynomial $f$ has degree 1. Then the Milnor ring becomes a graded ring. The highest possible degree of a homogeneous element in $H_f$ is $D = \sum_{i=1}^3 (1 - 2c_i) = 1 - \frac{2}{h}$, where $h$ is the Coxeter number of the corresponding root system. Put $\theta := \frac{D}{2} - \deg$, where $\deg : H_f \to H_f$ is the linear operator uniquely determined by the following condition: if $\phi$ is a weighted homogeneous element of degree $d$, then $\deg(\phi) = d\phi$. For homogeneity reasons, the period vectors have the form

$$I^{(-1)}_{\alpha}(\lambda) = \frac{\lambda^{\theta+1/2}}{\Gamma(\theta+3/2)}\,\Psi(\alpha), \tag{3.1}$$

where $\Psi : H_2(V_1;\mathbb{C}) \to H_f$ is a linear isomorphism. Our goal is to compute the image of the Milnor lattice $H_2(V_1;\mathbb{Z})$ via the map $\Psi$. The solution to this problem is given in Section 4.2. Explicit formulas for the image of the Milnor lattice via the map $\Psi$ are given in Sections 3.2.3–3.2.7. The main feature of our answer is that it involves various $\Gamma$-constants and roots of unity. The second goal of this thesis is to show that although the formulas look cumbersome, in fact there is an interesting structure behind them.

### 3.1.2 K-theoretic interpretation of the Milnor lattice

The polynomials $f$ corresponding to a simple singularity are invertible polynomials in the sense of [10] (see also [31]). Each polynomial is uniquely determined by a $3\times 3$ matrix $A = (a_{ij})_{1\le i,j\le 3}$ with non-negative integer coefficients, such that,

$$f(x) = \sum_{i=1}^{3} x_1^{a_{i1}} x_2^{a_{i2}} x_3^{a_{i3}}.$$

Following Fan–Jarvis–Ruan (see [16]) we consider also the Berglund–Hübsch dual polynomial

$$f^T(x) = \sum_{i=1}^{3} x_1^{a_{1i}} x_2^{a_{2i}} x_3^{a_{3i}}.$$

Let $G^T$ be the group of diagonal symmetries of $f^T$, that is,

$$G^T := \{t \in (\mathbb{C}^*)^3 \mid t_1^{a_{1i}} t_2^{a_{2i}} t_3^{a_{3i}} = 1\ \forall i\}.$$

Let $a^{ij}$ $(1 \le i,j \le 3)$ be the entries of the inverse matrix $A^{-1}$. The group $G^T$ is generated by the following elements

$$\overline{\rho}_i = (e^{2\pi \mathbf{i} a^{i1}}, e^{2\pi \mathbf{i} a^{i2}}, e^{2\pi \mathbf{i} a^{i3}}), \quad 1 \le i \le 3.$$

Finally, let $V_1^T = \{x \in \mathbb{C}^3 \mid f^T(x) = 1\}$. Our main interest is in the topological relative K-theoretic orbifold group

$$K^0_{\mathrm{orb}}([\mathbb{C}^3/G^T],[V_1^T/G^T]) := K^0_{G^T}(\mathbb{C}^3, V_1^T).$$

In general, there is no satisfactory definition of K-theory for non-compact spaces. However, in our case the pair $(\mathbb{C}^3, V_1^T)$ is $G^T$-equivariantly homotopic to a pair of finite $CW$ complexes, so we may think of $(\mathbb{C}^3, V_1^T)$ as a $G^T$-equivariant pair of finite $CW$-complexes. We refer to [46] for some background on equivariant topological K-theory.

Motivated by Iritani's $\Gamma$-integral structure in quantum cohomology (see [27]), we will now construct a linear map

$$\mathrm{ch}_\Gamma :\ K^0_{\mathrm{orb}}([\mathbb{C}^3/G^T],[V_1^T/G^T]) \otimes \mathbb{C} \longrightarrow H_{\mathrm{orb}}([\mathbb{C}^3/G^T],[V_1^T/G^T];\mathbb{C})\ , \tag{3.2}$$

which is a certain $\Gamma$-class modification of the orbifold Chern character map. For a $G^T$-equivariant space $X$ and $g \in G^T$, let us denote by $\mathrm{Fix}_g(X) := \{x \in X \mid gx = x\}$ the set of fixed points. The elements in the relative $K$-group will be identified with isomorphism classes $[E \to F]$ of two-term complexes $E \xrightarrow{d} F$ of $G^T$-equivariant vector bundles, such that, the differential $d$ is a morphism of $G^T$-equivariant vector bundles and $d|_{V_1^T} : E_{V_1^T} \to F_{V_1^T}$ is an isomorphism. Note that for $g \in G^T$, the restriction of a vector bundle $E|_{\mathrm{Fix}_g(\mathbb{C}^3)}$ decomposes as a direct sum of eigen-subbundles $E_\zeta$ and that the restriction to $\mathrm{Fix}_g(\mathbb{C}^3)$ of every two term complex $E \xrightarrow{d} F$ decomposes as a direct sum of two term subcomplexes $E_\zeta \xrightarrow{d_\zeta} F_\zeta$, where $d_\zeta = d|_{E_\zeta}$. We have the following well known decomposition (e.g. see [8], Theorem 2):

$$\mathrm{Tr}:\ K^0_{G^T}(\mathbb{C}^3, V_1^T) \otimes \mathbb{C} \xrightarrow{\cong} \oplus_{g \in G^T} \left[K^0(\mathrm{Fix}_g(\mathbb{C}^3), \mathrm{Fix}_g(V_1^T)) \otimes \mathbb{C}\right]^{G^T},$$

where $[\,]^{G^T}$ denotes the $G^T$-invariant part and the morphism Tr is defined by

$$\mathrm{Tr}([E \to F]) = \bigoplus_{g \in G^T} \bigoplus_{\zeta \in \mathbb{C}^*} \zeta [E_\zeta \to F_\zeta].$$

**Remark 3.1** The above decomposition is proved in [8] in the case of absolute K-theory. However, using the long exact sequence of a pair, it is straightforward to extend the result to relative K-theory as well.

The standard Chern character map gives an isomorphism

$$\mathrm{ch}:\ K^0(\mathrm{Fix}_g(\mathbb{C}^3), \mathrm{Fix}_g(V_1^T)) \otimes \mathbb{C} \longrightarrow H^{\mathrm{ev}}(\mathrm{Fix}_g(\mathbb{C}^3), \mathrm{Fix}_g(V_1^T); \mathbb{C}) .$$

Finally, if $G$ is a finite group acting on a smooth manifold $M$, such that the quotient groupoid $[M/G]$ is an effective orbifold, then $H^*(M/G; \mathbb{C}) \cong [H^*(M; \mathbb{C})]^G$. Indeed, for a finite group $G$ the operation taking $G$-invariants is an exact functor from the category of $G$-vector spaces to the category of vector spaces. Therefore

$$H^i(M/G; \mathbb{C}) \cong H^i([\Gamma(M, \mathcal{A}^*_M)]^G) = [H^i(M, \mathcal{A}^*_M)]^G \cong [H^i(M; \mathbb{C})]^G,$$

where $\mathcal{A}^*_M$ is the sheaf of smooth differential forms on $M$ with complex coefficients, the first isomorphism is Satake's de Rham theorem for orbifolds (see [45]), and the last one is the de Rham's theorem for the manifold $M$. Using the long exact sequence of a pair, we get also that $H^i(M/G, N/G; \mathbb{C}) \cong [H^i(M, N; \mathbb{C})]^G$ for any $G$-invariant submanifold $N \subset M$. On the other hand, by definition,

$$H^*_{\mathrm{orb}}([\mathbb{C}^3/G^T], [V_1^T/G^T]; k) = \bigoplus_{g \in G^T} H^*(\mathrm{Fix}_g(\mathbb{C}^3)/G^T, \mathrm{Fix}_g(V_1^T)/G^T; k), \quad k = \mathbb{Q}, \mathbb{R}, \mathbb{C}.$$

Therefore, the composition $\widetilde{\mathrm{ch}} := \mathrm{ch} \circ \mathrm{Tr}$ defines a ring homomorphism

$$\widetilde{\mathrm{ch}}:\ K^0_{\mathrm{orb}}([\mathbb{C}^3/G^T], [V_1^T/G^T]) \otimes \mathbb{C} \longrightarrow H^{\mathrm{ev}}_{\mathrm{orb}}([\mathbb{C}^3/G^T], [V_1^T/G^T]; \mathbb{C})$$

which is the orbifold version of the Chern character map. Clearly $\widetilde{\mathrm{ch}}$ is an isomorphism over $\mathbb{C}$.

**Remark 3.2** Orbifold cohomology $H^*_{\mathrm{orb}}$ has two natural gradings – standard topological degree grading coming from the topological space underlying the orbit space and Chen–Ruan grading. In this thesis we work with the topological grading and the topological cup product.

Let us recall also the definition of the $\Gamma$-class. If $E \in K^0_{\mathrm{orb}}([\mathbb{C}^3/G^T]) := K^0_{G^T}(\mathbb{C}^3)$ is an orbifold vector bundle and $\mathrm{Tr}(E) = \sum_g \sum_\zeta \zeta E_\zeta$, then each eigenvalue $\zeta = e^{2\pi \mathbf{i}\alpha}$, where $0 \leq \alpha < 1$ is a rational number and we define

$$\widehat{\Gamma}(E) = \sum_g \prod_{\zeta = e^{2\pi \mathbf{i}\alpha}} \prod_{i=1}^{\mathrm{rk}(E_\zeta)} \Gamma(1 - \alpha + \delta_{\zeta,i}) \quad \in \quad H^{\mathrm{ev}}_{\mathrm{orb}}([\mathbb{C}^3/G^T]),$$

where $\delta_{\zeta,i}$ $(1 \leq i \leq \mathrm{rk}(E_\zeta))$ are the Chern roots of the vector bundle $E_\zeta$. If $E = [T\mathbb{C}^3/G^T]$ is the orbifold tangent bundle, then the $\Gamma$-class is denoted by $\widehat{\Gamma}([\mathbb{C}^3/G^T])$. The map (3.2) is defined by the following formula:

$$\mathrm{ch}_\Gamma([E \to F]) := \frac{1}{2\pi} \widehat{\Gamma}([\mathbb{C}^3/G^T]) \cup (2\pi \mathbf{i})^{\deg_{\mathbb{C}}} \iota^* \widetilde{\mathrm{ch}}([E \to F]),$$

where $\deg_{\mathbb{C}}(\phi) = i\phi$ for $\phi \in H^{2i}_{\mathrm{orb}}([\mathbb{C}^3/G^T], [V_1^T/G^T]; \mathbb{C})$ and $\iota^*$ is an involution in orbifold cohomology that exchanges the direct summands corresponding to $g$ and $g^{-1}$. Note that the definition of $\iota^*$ makes sense because $\mathrm{Fix}_g = \mathrm{Fix}_{g^{-1}}$.

**Theorem 3.3** There exists a linear isomorphism

$$\mathrm{mir}: \ H_f \longrightarrow H^*_{\mathrm{orb}}([\mathbb{C}^3/G^T], [V_1^T/G^T]; \mathbb{C}),$$

such that, the map

$$\mathrm{mir}^{-1} \circ \mathrm{ch}_\Gamma : \ K^0_{\mathrm{orb}}([\mathbb{C}^3/G^T], [V_1^T/G^T]) \xrightarrow{\cong} \Psi(H_2(f^{-1}(1); \mathbb{Z}))$$

is an isomorphism of Abelian groups.

Unfortunately we do not have a conceptual definition of the map mir. Our definition is on a case by case basis. We expect that $H^*_{\mathrm{orb}}([\mathbb{C}^3/G^T], [V_1^T/G^T]; \mathbb{C})$ has a natural identification with the state space of FJRW-theory under which mir is identified with the mirror map of Fan–Jarvis–Ruan (see [16]). Let us point out also that in all cases the following two properties are satisfied:

1. If $x_1^{m_1} x_2^{m_2} x_3^{m_3}$ is a homogeneous monomial representing a vector in $H_f$, then its image under mir is in the twisted sector corresponding to $g = \overline{\rho}_1^{m_1+1} \overline{\rho}_2^{m_2+1} \overline{\rho}_3^{m_3+1}$.

2. The map mir is defined over $\mathbb{Q}$, that is, mir provides an isomorphism

$$\mathbb{Q}[x_1, x_2, x_3]/(f_{x_1}, f_{x_2}, f_{x_3}) \cong H^*_{\mathrm{orb}}([\mathbb{C}^3/G^T], [V_1^T/G^T]; \mathbb{Q}).$$

## 3.2 Period map image of the Milnor lattice

### 3.2.1 Suspension isomorphism in vanishing homology

We will reduce the problem of computing periods of the hypersurface $V_\lambda$ to computing periods of the Riemann surfaces

$$M_\mu := \{(x_1, x_2) \in \mathbb{C}^2 \mid g(x_1, x_2) = \mu\}.$$

Consider the map $V_\lambda \to \mathbb{C}$, $(x_1, x_2, x_3) \mapsto g(x_1, x_2)$. The fibers of this map are given by

$$V_{\lambda,\mu} := M_\mu \times \{-\sqrt{\lambda - \mu}, \sqrt{\lambda - \mu}\}.$$

Suppose now that $A \in H_1(M_\lambda; \mathbb{Z})$ is any cycle. The following two maps

$$\phi_\pm : A \times [0,1] \to V_\lambda, \quad (x_1, x_2, t) \mapsto (t^{c_1} x_1, t^{c_2} x_2, \pm\sqrt{\lambda(1-t)})$$

have images that fit together and give a two-dimensional cycle $\alpha \in V_\lambda$, that is, $\alpha = \Sigma A$ is the suspension of the cycle $A$. It is known that the above suspension operation $\Sigma : H_1(M_\lambda; \mathbb{Z}) \to H_2(V_\lambda; \mathbb{Z})$ is an isomorphism (see [3], Theorem 2.9).

Note that we may choose the basis of $H_f$ to be such that $\phi_i = \phi_i(x_1, x_2)$ does not depend on $x_3$. Then the integral

$$\frac{1}{2\pi} \int_{\alpha_\lambda} \phi_i \frac{\omega}{df} = \frac{1}{2\pi} \partial_\lambda \int_{\alpha_\lambda} d^{-1}(\phi_i \omega) = \frac{1}{2\pi} \partial_\lambda \int_{\alpha_\lambda} x_3 \phi_i(x_1, x_2) dx_1 \wedge dx_2,$$

where in the first equality we used the Stoke's theorem (see [3], Lemma 7.2). Using Fubini's theorem (see [3], Lemma 7.2), we have

$$\int_{\alpha_\lambda} x_3 \phi_i(x_1, x_2) dx_1 \wedge dx_2 = \int_0^\lambda (\lambda - \mu)^{1/2} \int_{A_\mu} \frac{\phi_i(x_1, x_2) dx_1 dx_2}{dg} d\mu - \int_\lambda^0 (-(\lambda - \mu)^{1/2}) \int_{A_\mu} \frac{\phi_i(x_1, x_2) dx_1 dx_2}{dg} d\mu,$$

where the first integral represents integrating over $\phi_+(A \times [0,1])$, the second one over $\phi_-(A \times [0,1])$, and $A_\mu \in H_1(M_\mu)$ for $\mu = \lambda t$ is obtained from $A$ via the rescaling $(x_1, x_2) \mapsto (t^{c_1} x_1, t^{c_2} x_2)$. We get

$$\frac{1}{2\pi} \int_{\alpha_\lambda} \phi_i \frac{\omega}{df} = \frac{1}{\pi} \partial_\lambda \int_0^\lambda (\lambda - \mu)^{1/2} \int_{A_\mu} \frac{\phi_i(x_1, x_2) dx_1 dx_2}{dg} d\mu. \tag{3.3}$$

The image of the Milnor lattice $H_2(V_1; \mathbb{Z})$ will be computed with formula (3.3).

### 3.2.2 Simple singularities and root systems

Let us first recall several well known facts about simple singularities, which will be needed in our computation (see [3], Theorem 3.14). The analytic continuation of $I_\alpha^{(-1)}(\lambda)$ along a loop around $\lambda = 0$ yields $I_{\sigma(\alpha)}^{(-1)}(\lambda)$, where $\sigma : H_2(V_1; \mathbb{Z}) \to H_2(V_1; \mathbb{Z})$ is the so-called *classical monodromy* operator. Recalling the definition of $\Psi$ (see formula (3.1)), we get the following relation:

$$\Psi(\sigma(\alpha)) = -e^{2\pi \mathbf{i}\, \theta}\, \Psi(\alpha), \tag{3.4}$$

where $\mathbf{i} := \sqrt{-1}$. In particular, knowing the image of one cycle $\alpha$ allows us to find the image of the entire $\sigma$-orbit of $\alpha$.

Let us define

$$(\alpha|\beta) := \lambda(I_\alpha^{(0)}(\lambda), I_\beta^{(0)}(\lambda)),$$

where $I_\alpha^{(0)}(\lambda) := \partial_\lambda I_\alpha^{(-1)}(\lambda)$. It is straightforward to check that

$$(\alpha|\beta) = \frac{1}{\pi}(\Psi(\alpha), \cos(\pi\theta)\Psi(\beta)). \tag{3.5}$$

It is known that $(\alpha|\beta) = -\alpha \circ \beta$, where $\circ$ is the intersection pairing (see [22, 43]). In particular, the form $(\ |\ )$ takes integer values on the Milnor lattice.

Finally, let us also recall that we have the following remarkable facts (see [3], Theorem 3.14):

1. The set of vanishing cycles of the singularity $f$ coincides with the set of all $\alpha \in H_2(V_1;\mathbb{Z})$ such that $(\alpha|\alpha) = 2$.

2. The triple (Milnor lattice, set of vanishing cycles, pairing $(\ |\ )$) form a root system of the same type as the type of the singularity $f$, that is, the set of vanishing cycles corresponds to the roots, the Milnor lattice corresponds to the root lattice, and $(\ |\ )$ corresponds to the invariant bilinear form.

3. The classical monodromy corresponds to a Coxeter transformation.

### 3.2.3 $A_N$-singularity

Let us fix the following basis of $H_f$:

$$\phi_i = x_1^{i-1}, \quad 1 \leqslant i \leqslant N.$$

The residue pairing takes the form

$$(\phi_i, \phi_j) = \frac{1}{4h}\delta_{i+j,h}, \qquad 1 \leqslant i, j \leqslant N,$$

where $h = N+1$ is the Coxeter number. The Riemann surface $M_\mu$ for $\mu \neq 0$ is a non-singular curve in $\mathbb{C}^2$ defined by the equation $x_1^{N+1} + x_2^2 = \mu$. The projection $(x_1, x_2) \mapsto x_1$ defines a degree 2 branched covering $M_\mu \to \mathbb{C}$, with branching points $x_{1,k} = \mu^{\frac{1}{N+1}}\eta_{N+1}^k$ ($k \in \mathbb{Z}_{N+1}$), where $\eta_{N+1} := e^{2\pi\mathbf{i}/(N+1)}$ and $\mathbb{Z}_{N+1} := \mathbb{Z}/(N+1)\mathbb{Z}$.

Let us construct a basis of $H_1(M_\mu;\mathbb{Z}) \cong \mathbb{Z}^N$. Cycles on $M_\mu$ can be visualized easily via their projections on the $x_1$-plane $\mathbb{C}$. Let $L_k$ ($k \in \mathbb{Z}_{N+1}$) be the line segment $[0, x_{1,k}]$ (in the $x_1$-plane). Let $A'_k$ be a loop in the $x_1$-plane that starts at $x_1 = 0$, it goes along the line segment $L_k$, just before hitting the branched point $x_{1,k}$ it makes a small loop $C_k$ counterclockwise around $x_{1,k}$, it returns back to the starting point along the line segment $L_k$, and then the loop continues to travel in a similar fashion along $L_{k+1}$ except that this time we make a small loop $C_{k+1}^{-1}$ clockwise around $x_{1,k+1}$. In other words $A'_k = L_{k+1}^{-1} \circ C_{k+1}^{-1} \circ L_{k+1} \circ L_k^{-1} \circ C_k \circ L_k$. Note that $A'_k$ lifts to two loops $A_{k,a}, a \in \mathbb{Z}_2$ on $M_\mu$, where the starting point of $A'_k$ lifts to $x_{2,k,a} := \mu^{\frac{1}{2}}(-1)^a$. The cycles $A_{k,a}$ satisfy the following relations $A_{k,0} = -A_{k,1}$ and $\sum_{k=0}^N A_{k,a} = 0$. Let us asume $a \in \mathbb{Z}_2 \setminus \{0\}$ and $k \in \mathbb{Z}_{N+1} \setminus \{0\}$, then we get $N$ loops whose homology classes, as we will see later on, represent a basis of $H_1(M_\mu;\mathbb{Z})$.

Let us compute the periods of the holomorphic forms

$$\phi_i(x_1,x_2)\frac{dx_1dx_2}{dg} = -\frac{x_1^{i-1}dx_1}{2x_2}$$

along the cycles $A_{k,a}$. The paths $L_k$ and $C_k$ can be parametrized as follows:

$$L_k: \quad x_1 = \eta_{N+1}^k \mu^{\frac{1}{N+1}} t^{\frac{1}{N+1}}, \quad 0 \leqslant t \leqslant \left(1 - \frac{\epsilon}{\mu^{\frac{1}{N+1}}}\right)^{N+1},$$
$$C_k: \quad x_1 = \eta_{N+1}^k \mu^{\frac{1}{N+1}} + \epsilon e^{\mathbf{i}\theta} \quad \frac{2k-N-1}{N+1}\pi \leqslant \theta \leqslant \frac{2k+N+1}{N+1}\pi.$$

The integrals along the lifts of $C_k$ contribute to the period integral terms of order $O(\epsilon^{\frac{1}{2}})$. These terms vanish in the limit $\epsilon \to 0$. The periods that we want to compute are independent of $\epsilon$ for homotopy reasons. Therefore, by passing to the limit $\epsilon \to 0$ we get

$$\int_{A_{k,a}} \phi_i(x_1,x_2)\frac{dx_1dx_2}{dg} = (1-(-1))(\int_{L_{k,a}} - \int_{L_{k+1,a}})\frac{-x_1^{i-1}dx_1}{2x_2} = (\int_{L_{k+1,a}} - \int_{L_{k,a}})\frac{x_1^{i-1}dx_1}{x_2}.$$

The integrals along $L_{k,a}$ can be expressed in terms of Euler's Beta function $B(a,b) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$,

$$\int_{L_{k,a}} \frac{x_1^{i-1}dx_1}{x_2} = (-1)^a \int_0^1 \frac{\eta_{N+1}^{ki}\mu^{\frac{i}{N+1}} t^{\frac{i}{N+1}-1}dt}{(N+1)\mu^{\frac{1}{2}}(1-t)^{\frac{1}{2}}} = (-1)^a \frac{\eta_{N+1}^{ki}\mu^{\frac{i}{N+1}-\frac{1}{2}}}{N+1} B(\frac{i}{N+1},\frac{1}{2}).$$

Let $\alpha_{k,a} = \Sigma A_{k,a}$ be the suspension. Recalling formula (3.3) and using that

$$\int_0^\lambda (\lambda-\mu)^{1/2}\mu^a d\mu = \lambda^{a+3/2}B(a+1,3/2), \tag{3.6}$$

we get

$$\begin{aligned}(I^{(-1)}_{\alpha_{k,a}}(\lambda),\phi_i) :=& \frac{1}{2\pi}\int_{\alpha_{k,a}} \phi_i \frac{\omega}{df} = \frac{1}{\pi}\partial_\lambda \int_0^\lambda (\lambda-\mu)^{\frac{1}{2}} \int_{A_{k,a}} \phi_i(x_1,x_2)\frac{dx_1dx_2}{dg} d\mu \\ =& (-1)^a \frac{\eta_{N+1}^{ki}(\eta_{N+1}^i - 1)}{2i}\lambda^{\frac{i}{N+1}}.\end{aligned}$$

Recalling the formulas for the residue pairing we get

$$I^{(-1)}_{\alpha_{k,a}}(\lambda) = 4h\sum_{i=1}^N (I^{(-1)}_{\alpha_{k,a}}(\lambda),\phi_{h-i})\phi_i.$$

By definition $\theta(\phi_i) = (\frac{1}{2} - \frac{i}{N+1})\phi_i$. Therefore, using (3.1), we get

$$\Psi(\alpha_{k,a}) = (-1)^a 2\sum_{i=1}^N \eta_{N+1}^{-ki}(\eta_{N+1}^{-i} - 1)\Gamma(1-\frac{i}{N+1})\phi_i \tag{3.7}$$

Let us point out that formula (3.4) yields the following formulas for the classical monodromy operator

$$\sigma(\alpha_{k,a}) = -\alpha_{k+1,a+1} = \alpha_{k+1,a} \quad k \in \mathbb{Z}_{N+1}, a \in \mathbb{Z}_2.$$

The intersection pairing takes the form

$$\begin{aligned}(\alpha_{k,1}|\alpha_{l,1}) =& \frac{1}{\pi}(\Psi(\alpha_{k,1}), \cos(\pi\theta)\Psi(\alpha_{l,1})) \\ =& \frac{2}{h}\sum_{i=1}^{N} \eta_{N+1}^{(l-k)i}(1-\cos(\frac{2i\pi}{h})) = 2\delta_{k,l} - \delta_{l-k,1} - \delta_{l-k,N},\end{aligned}$$

where $k, l \in \mathbb{Z}_{N+1}$ and the Kronecker delta is also on $\mathbb{Z}_{N+1}$. Note that $(\alpha_{k,1}|\alpha_{k,1}) = 2$, so $\alpha_{k,1}$ is a vanishing cycle. The determinant of the intersection pairing in the basis $\{\alpha_{k,1}\}$, that is, the determinant of the matrix $(\alpha_{k,1}|\alpha_{l,1})_{k,l=1}^{N}$ is $N+1$, which coincides with the determinant of the Cartan matrix of the simple Lie algebra of type $A_N$. Therefore, $\{\alpha_{k,1}\}$ is a $\mathbb{Z}$-basis of the root lattice, that is, $H_2(V_\lambda; \mathbb{Z})$ and hence their images $\Psi(\alpha_{k,1})$ (see formula (3.7)) give a basis for the image of the Milnor lattice in $H_f$.

### 3.2.4 $D_N$-singularity

Let us fix the following basis of $H_f$:

$$\phi_i(x_1, x_2) = \begin{cases} x_2^{i-1} & \text{if } 1 \leq i \leq N-1, \\ 2x_1 & \text{if } i = N. \end{cases}$$

The residue pairing takes the form

$$(\phi_i, \phi_j) = \frac{1}{2h}\delta_{i+j,N} \quad (1 \leq i, j \leq N-1), \quad (\phi_i, \phi_N) = -\delta_{i,N} \quad (1 \leq i \leq N),$$

where $h = 2N-2$ is the Coxeter number. The Riemann surface $M_\mu$ for $\mu \neq 0$ is a non-singular curve in $\mathbb{C}^2$ defined by the equation $x_1^2 x_2 + x_2^{N-1} = \mu$. The projection $(x_1, x_2) \mapsto x_2$ defines a degree 2 branched covering $M_\mu \to \mathbb{C}^*$, with branching points $x_{2,k} = \mu^{\frac{1}{N-1}}\eta^{2k}$ $(1 \leq k \leq N-1)$, where $\eta = e^{2\pi \mathbf{i}/h}$. Let $A_k'$ be a simple loop in $\mathbb{C}^*$ around the line segment $L_k := [0, x_{2,k}]$, that is, $A_k'$ is a loop starting at a point on the line segment $L_k$ sufficiently close to 0, it goes along the line segment $L_k$, just before hitting the branch point $x_{2,k}$ it makes a small loop $C_k$ around it and it returns back to the starting point along the line segment $L_k$, and finally it makes a small loop $C_0$ around 0. Clearly, the loop $A_k' = C_0 \circ L_k^{-1} \circ C_k \circ L_k$ lifts to a loop $A_k$ in $M_\mu$. Let us compute the periods of the holomorphic forms

$$\phi_i(x_1, x_2)\frac{dx_1 dx_2}{dg} = \begin{cases} \frac{x_2^{i-1}dx_2}{2x_1x_2} & \text{if } 1 \leq i \leq N-1, \\ \frac{dx_2}{x_2} & \text{if } i = N, \end{cases}$$

along the cycle $A_k$. If $i = N$, then the period integral is just $2\pi\mathbf{i}$. Suppose that $1 \le i \le N-1$. Let us parametrize $A'_k$ as follows:

$$\begin{aligned} C_0 :& \quad x_2 = \epsilon e^{\mathbf{i}\theta} \quad (0 \le \theta \le 2\pi), \\ L_k :& \quad x_2 = \mu^{\frac{1}{N-1}} \eta^{2k} t \quad (\epsilon \mu^{-\frac{1}{N-1}} \le t \le 1 - \epsilon \mu^{-\frac{1}{N-1}}), \\ C_k :& \quad x_2 = \mu^{\frac{1}{N-1}} \eta^{2k} + \epsilon e^{\mathbf{i}\theta} \quad (0 \le \theta \le 2\pi). \end{aligned}$$

The integrals along the lifts of $C_0$ and $C_k$ contribute to the period integral terms of orders respectively $O(\epsilon^{i-1/2})$ and $O(\epsilon^{1/2})$. These terms vanish in the limit $\epsilon \to 0$. The two lifts of $L_k$, before and after going around the branch point $x_{2,k}$, have parametrizations, such that,

$$x_2 = \mu^{\frac{1}{N-1}} \eta^{2k} t, \quad x_1 x_2 = \sqrt{(\mu - x_2^{N-1}) x_2} = \eta^k \mu^{1/2+1/h} (1 - t^{N-1})^{1/2} t^{1/2}$$

where $t$ varies from 0 to 1, and

$$x_2 = \mu^{\frac{1}{N-1}} \eta^{2k} t, \quad x_1 x_2 = -\sqrt{(\mu - x_2^{N-1}) x_2} = -\eta^k \mu^{1/2+1/h} (1 - t^{N-1})^{1/2} t^{1/2}$$

where $t$ varies from 1 to 0. Now it is clear that the period integral, after passing to the limit $\epsilon \to 0$, takes the form

$$\int_{A_k} \phi_i \frac{dx_1 dx_2}{dg} = \mu^{\frac{m_i}{h} - \frac{1}{2}} \eta^{m_i k} \int_0^1 t^{i-\frac{3}{2}} (1 - t^{N-1})^{-1/2} dt,$$

where $m_i := 2i - 1$ $(1 \le i \le N-1)$. The above integral can be computed as follows,

$$\int_0^1 t^{i-3/2} (1 - t^{N-1})^{-1/2} dt = \frac{1}{N-1} \int_0^1 s^{(2i-1)\frac{1}{h} - 1} (1 - s)^{-\frac{1}{2}} ds = \frac{1}{N-1} B\left(\frac{m_i}{h}, \frac{1}{2}\right).$$

We get the following formulas:

$$\int_{A_k} \phi_i \frac{dx_1 dx_2}{dg} = \begin{cases} \frac{1}{N-1} \mu^{\frac{m_i}{h} - \frac{1}{2}} \eta^{m_i k} B(\frac{m_i}{h}, \frac{1}{2}), & \text{if } 1 \le i \le N-1, \\ 2\pi\mathbf{i}, & \text{if } i = N. \end{cases}$$

Let $\alpha_k = \Sigma A_k$ be the suspension. Recalling formula (3.3) and using (3.6), we get

$$\frac{1}{2\pi} \int_{\alpha_k} \phi_i \frac{\omega}{df} = \begin{cases} \frac{1}{h} \eta^{m_i k} \frac{\lambda^{m_i/h}}{m_i/h}, & \text{if } 1 \le i \le N-1, \\ 2\mathbf{i}\lambda^{1/2}, & \text{if } i = N. \end{cases}$$

Therefore,

$$I_{\alpha_k}^{(-1)}(\lambda) = 2 \sum_{i=1}^{N-1} \eta^{m_i k} \frac{\lambda^{m_i/h}}{m_i/h} \phi_{N-i} - 2\mathbf{i}\lambda^{1/2} \phi_N.$$

Note that $\theta(\phi_i) = \left(\frac{m_{N-i}}{h} - \frac{1}{2}\right)\phi_i$ for $1 \leq i \leq N-1$ and $\theta(\phi_N) = 0$. Therefore

$$\Psi(\alpha_k) = 2\sum_{i=1}^{N-1} \eta^{m_i k}\Gamma(m_i/h)\phi_{N-i} - \mathbf{i}\Gamma(m_N/h)\phi_N, \tag{3.8}$$

where $m_N = N-1$.

**Remark 3.4** The numbers $\frac{m_i}{h} = \frac{2i-1}{h}$ $(1 \leq i \leq N-1)$, $\frac{m_N}{h} = \frac{1}{2}$ are the Coxeter exponents.

Put

$$v_k = 2\sum_{i=1}^{N-1} \eta^{m_i k}\Gamma(m_i/h)\phi_{N-i}, \quad 1 \leq k \leq N-1,$$

and $v_N = \mathbf{i}\Gamma(m_N/h)\phi_N$.

**Proposition 3.5** The image of the Milnor lattice under the map $\Psi$ is the lattice in $H_f$ with $\mathbb{Z}$-basis

$$\beta_1 = v_1 - v_2, \ldots, \beta_{N-1} = v_{N-1} - v_N,\ \beta_N = v_{N-1} + v_N.$$

**Proof** Using formula (3.5), it is straightforward to check that $\{v_i\}_{1\leq i\leq N}$ is an orthonormal basis of $H_f$ with respect to the intersection pairing, that is, $(v_i|v_j) = \delta_{i,j}$. We have $\Psi(\alpha_k) = v_k - v_N$ and $\Psi(\sigma\alpha_k) - \Psi(\alpha_{k+1}) = 2v_N$. Therefore, $\beta_i$ belongs to the image of the Milnor lattice. On the other hand, since $(\beta_i|\beta_i) = 2$, we get that $\beta_i$ is the image of a vanishing cycle. Recalling the root system interpretation of the set of vanishing cycles, we get that $\beta_i$ $(1 \leq i \leq N)$ are simple roots and that the corresponding Dynkin diagram is the Dynkin diagram of type $D_N$. Since the Milnor lattice is spanned by the set of vanishing cycles, the claim of the proposition follows.

### 3.2.5 $E_6$-singularity

Let us fix the following basis of $H_f$ :

$$\phi_i = \begin{cases} x_2^{i-1} & \text{if } 1 \leqslant i \leqslant 3, \\ x_1 x_2^{i-4} & \text{if } 4 \leqslant i \leqslant 6. \end{cases}$$

The residue pairing takes the form

$$(\phi_i, \phi_j) = \frac{1}{2h}\delta_{i+j,7}, \quad (1 \leqslant i, j \leqslant 6),$$

where $h = 12$ is the Coxeter number. The Riemann surface $M_\mu$ for $\mu \neq 0$ is a non-singular curve in $\mathbb{C}^2$ defined by the equation $x_1^3 + x_2^4 = \mu$. The projection $(x_1, x_2) \mapsto x_2$ defines a degree 3 branched covering $M_\mu \to \mathbb{C}$, with branching points $x_{2,k} = \mu^{\frac{1}{4}}\mathbf{i}^k, k \in \mathbb{Z}_4$.

Let $L_k$ $(k \in \mathbb{Z}_4)$ be the line segment $[0, \mu^{\frac{1}{4}}\mathbf{i}^k]$. Let $A'_k$ be a loop in the $x_2$-plane $\mathbb{C}$ going around the branch points $x_{2,k}$ and $x_{2,k+1}$ in the following way: the loop starts at 0, it goes along the line segment $L_k$, just before hitting the branch point $x_{2,k}$ it makes a small loop $C_k$ counterclockwise around $x_{2,k}$, it returns back to the starting point along $L_k$; then the loop travels in a similar fashion along $L_{k+1}$ except that this time we make a small loop $C_{k+1}^{-1}$ in *clockwise* direction around $x_{2,k+1}$. Clearly, the loop $A'_k = L_{k+1}^{-1} \circ C_{k+1}^{-1} \circ L_{k+1} \circ L_k^{-1} \circ C_k \circ L_k$ lifts to three loops $A_{k,a}, a \in \mathbb{Z}_3$ in $M_\mu$, depending on how we choose the lift of the base point, i.e., the $x_1$-coordinate of the lift of the base point of $A'_k$ could take the following values: $x_{1,a} = \mu^{\frac{1}{3}}\eta_3^a$, where $\eta_3 := e^{\frac{2\pi \mathbf{i}}{3}}$.

Let us consider the loops $A_{k,a}$ with $a \in \mathbb{Z}_3 \setminus \{0\}$ and $k \in \mathbb{Z}_4 \setminus \{0\}$. Let us compute the periods of the holomorphic forms

$$\phi_i(x_1,x_2)\frac{dx_1 dx_2}{dg} = \begin{cases} \frac{x_2^{i-1}dx_2}{3x_1^2} & \text{if } 1 \leqslant i \leqslant 3, \\ \frac{x_2^{i-4}dx_2}{3x_1} & \text{if } 4 \leqslant i \leqslant 6, \end{cases}$$

along the cycles $A_{k,a}$. As a byproduct of our computation we will get that the homology classes of these 6 loops form a basis of $H_1(M_\mu;\mathbb{Z})$. Let us parametrize $A'_k$ as follows:

$$L_k: \quad x_2 = \mathbf{i}^k \mu^{\frac{1}{4}} t^{\frac{1}{4}}, \quad 0 \leqslant t \leqslant \left(1 - \frac{\epsilon}{\mu^{\frac{1}{4}}}\right)^4,$$
$$C_k: \quad x_2 = \mathbf{i}^k \mu^{\frac{1}{4}} + \epsilon e^{\mathbf{i}\theta} \quad \frac{k-2}{2}\pi \leqslant \theta \leqslant \frac{k+2}{2}\pi.$$

The integrals along the lifts of $C_k$ contribute to the period integral terms of orders $\begin{cases} O(\epsilon^{\frac{1}{3}}) & \text{if } 1 \leqslant i \leqslant 3 \\ O(\epsilon^{\frac{2}{3}}) & \text{if } 4 \leqslant i \leqslant 6 \end{cases}$. These terms vanish in the limit $\epsilon \to 0$. Therefore, under this limit, the periods of the holomorphic forms

$$\int_{A_{k,a}} \phi_i(x_1,x_2)\frac{dx_1 dx_2}{dg} = \begin{cases} (1-\eta_3^{-2})(\int_{L_{k,a}} - \int_{L_{k+1,a}})\frac{\phi_i dx_2}{3x_1^2} & \text{if } 1 \leqslant i \leqslant 3, \\ (1-\eta_3^{-1})(\int_{L_{k,a}} - \int_{L_{k+1,a}})\frac{\phi_i dx_2}{3x_1^2} & \text{if } 4 \leqslant i \leqslant 6, \end{cases}$$

where $\eta_3 := e^{\frac{2\pi \mathbf{i}}{3}}$ and the integral

$$\int_{L_{k,a}} \frac{\phi_i dx_2}{3x_1^2} = \begin{cases} \int_0^1 \frac{\mathbf{i}^{ki}\mu^{\frac{i}{4}} t^{\frac{i}{4}-1} dt}{12\mu^{\frac{2}{3}}(1-t)^{\frac{2}{3}}\eta_3^{2a}} = \frac{\mathbf{i}^{ki}}{12}\eta_3^a \mu^{\frac{i}{4}-\frac{2}{3}} B(\frac{i}{4},\frac{1}{3}) & \text{if } 1 \leqslant i \leqslant 3, \\ \int_0^1 \frac{\mathbf{i}^{k(i-3)}\mu^{\frac{i-3}{4}} t^{\frac{i-3}{4}-1} dt}{12\mu^{\frac{1}{3}}(1-t)^{\frac{1}{3}}\eta_3^{a}} = \frac{\mathbf{i}^{k(i-3)}}{12}\eta_3^{2a} \mu^{\frac{i-3}{4}-\frac{1}{3}} B(\frac{i-3}{4},\frac{2}{3}) & \text{if } 4 \leqslant i \leqslant 6. \end{cases}$$

Then,

$$\int_{A_{k,a}} \phi_i(x_1,x_2)\frac{dx_1 dx_2}{dg} = \begin{cases} (1-\eta_3^{-2})(1-\mathbf{i}^i)\frac{\mathbf{i}^{ki}}{12}\eta_3^a \mu^{\frac{i}{4}-\frac{2}{3}} B(\frac{i}{4},\frac{1}{3}) & \text{if } 1 \leqslant i \leqslant 3, \\ (1-\eta_3^{-1})(1-\mathbf{i}^{i-3})\frac{\mathbf{i}^{k(i-3)}}{12}\eta_3^{2a} \mu^{\frac{i-3}{4}-\frac{1}{3}} B(\frac{i-3}{4},\frac{2}{3}) & \text{if } 4 \leqslant i \leqslant 6. \end{cases}$$

Let $\alpha_{k,a} = \Sigma A_{k,a}$ be the suspension. Recalling formula (3.3) and using (3.6)

$$\begin{aligned}(I^{(-1)}_{\alpha_{k,a}}(\lambda),\phi_i) :=& \frac{1}{2\pi}\int_{\alpha_{k,a}} \phi_i \frac{\omega}{df} = \frac{1}{\pi}\partial_\lambda \int_0^\lambda (\lambda-\mu)^{\frac{1}{2}} \int_{A_{k,a}} \phi_i(x_1,x_2)\frac{dx_1 dx_2}{dg} d\mu \\ =& \begin{cases} \frac{\sqrt{3}}{12\pi}\eta_3^a e^{-\frac{\pi}{6}\mathbf{i}} \lambda^{\frac{i}{4}-\frac{1}{6}} \frac{\Gamma(\frac{3}{2})\Gamma(\frac{i}{4})\Gamma(\frac{1}{3})}{\Gamma(\frac{i}{4}+\frac{5}{6})} \mathbf{i}^{ki}(1-\mathbf{i}^i) & \text{if } 1 \leqslant i \leqslant 3, \\ \frac{\sqrt{3}}{12\pi}\eta_3^{2a} e^{\frac{\pi}{6}\mathbf{i}} \lambda^{\frac{i-3}{4}+\frac{1}{6}} \frac{\Gamma(\frac{3}{2})\Gamma(\frac{i-3}{4})\Gamma(\frac{2}{3})}{\Gamma(\frac{i-3}{4}+\frac{7}{6})} \mathbf{i}^{k(i-3)}(1-\mathbf{i}^{i-3}) & \text{if } 4 \leqslant i \leqslant 6. \end{cases}\end{aligned}$$

Recalling the formulas for the residue pairing in the basis $\{\phi_i\}$ we get

$$I^{(-1)}_{\alpha_{k,a}}(\lambda) = 2h\sum_{i=1}^{6}(I^{(-1)}_{\alpha_{k,a}}(\lambda),\phi_{7-i})\phi_i.$$

Recalling formula (3.1) and using that by definition $\theta(\phi_i) = \begin{cases} (\frac{2}{3} - \frac{i}{4})\phi_i & \text{if } 1 \leqslant i \leqslant 3 \\ (\frac{1}{3} - \frac{i-3}{4})\phi_i & \text{if } 4 \leqslant i \leqslant 6 \end{cases}$, we get

$$\begin{aligned} \Psi(\alpha_{k,a}) =& \sqrt{\frac{3}{\pi}} \sum_{i=1}^{3} e^{\frac{\pi}{6}\mathbf{i}} \eta_3^{2a} \Gamma(1 - \frac{i}{4}) \Gamma(\frac{2}{3}) \mathbf{i}^{-ki} (1 - \mathbf{i}^{-i}) \phi_i + \\ & \sqrt{\frac{3}{\pi}} \sum_{i=4}^{6} e^{-\frac{\pi}{6}\mathbf{i}} \eta_3^{a} \Gamma(1 - \frac{i-3}{4}) \Gamma(\frac{1}{3}) \mathbf{i}^{k(3-i)} (1 - \mathbf{i}^{3-i}) \phi_i. \end{aligned} \tag{3.9}$$

Let us also point out that by using formula (3.4), we get the following formulas for the classical monodromy operator:

$$\sigma(\alpha_{k,a}) = -\alpha_{k+1,a+1} \quad k \in \mathbb{Z}_4, a \in \mathbb{Z}_3.$$

Recalling formula (3.5), we get that the intersection pairing

$$\begin{aligned} (\alpha_{k,a}|\alpha_{l,b}) =& \frac{1}{\pi} (\Psi(\alpha_{k,a}), \cos(\pi\theta) \Psi(\alpha_{l,b})) \\ =& \frac{1}{8} \sum_{i=1}^{3} (\eta_3^{b-a} \mathbf{i}^{(l-k)i} + \eta_3^{a-b} \mathbf{i}^{(k-l)i}) \frac{\cos((\frac{i}{4} - \frac{2}{3})\pi)}{\sin(\frac{i}{4}\pi) \sin(\frac{\pi}{3})} (2 - \mathbf{i}^{-i} - \mathbf{i}^{i}) \\ =& \sum_{i=1}^{3} \cos(\frac{2}{3}(b-a)\pi + \frac{i}{2}(l-k)\pi) \frac{\cos((\frac{i}{4} - \frac{2}{3})\pi)}{\sin(\frac{\pi}{3})} \sin(\frac{i}{4}\pi). \end{aligned}$$

Let us identify $\mathbb{Z}_3 \setminus \{0\} = \{1,2\}$ and $\mathbb{Z}_4 \setminus \{0\} = \{1,2,3\}$. Every $1 \le a' \le 6$ can be written uniquely in the form $a' = 3(a-1) + k$, where $1 \le a \le 2$ and $1 \le k \le 3$. Let us define $\alpha_{a'} := \alpha_{k,a}$. The intersection pairings $(\alpha_{a'}|\alpha_{b'})$ are straightforward to compute using the formula from above. We get that $(\alpha_{a'}|\alpha_{b'})$ coincides with the $(a', b')$-entry of the following matrix:

$$\begin{pmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ -1 & 2 & -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 0 & 1 & -1 \\ -1 & 1 & 0 & 2 & -1 & 0 \\ 0 & -1 & 1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{pmatrix}.$$

The above matrix has determinant 3. Therefore, the set $\{\alpha_{k,a} \mid 1 \le a \le 2, \ 1 \le k \le 3\}$ is a set of linearly independent vanishing cycles. Since the set of all vanishing cycles is a root system of type $E_6$ and the determinant of the Cartan matrix of a root system of $E_6$ is also 3, we get that the $\{\alpha_{k,a}\}$ is a set of simple roots. In particular, it is a $\mathbb{Z}$-basis of the Milnor lattice.

### 3.2.6 $E_7$-singularity

Let us fix the following basis of $H_f$ :

$$\phi_i = \begin{cases} x_1^{i-1} & \text{if } 1 \leqslant i \leqslant 3, \\ x_2 x_1^{i-4} & \text{if } 4 \leqslant i \leqslant 6, \\ x_2^2 & \text{if } i = 7. \end{cases}$$

The residue pairing takes the form

$$(\phi_i, \phi_j) = \begin{cases} \frac{1}{h}\delta_{i+j,7} & (1 \leqslant i, j \leqslant 6), \\ -\frac{1}{6} & i = j = 7, \\ 0 & \text{otherwise}, \end{cases}$$

where $h = 18$ is the Coxeter number. The Riemann surface $M_\mu$ for $\mu \neq 0$ is a non-singular curve in $\mathbb{C}^2$ defined by the equation $x_1^3 + x_1 x_2^3 = \mu$. The projection $(x_1, x_2) \mapsto x_1$ defines a degree 3 branched covering $M_\mu \to \mathbb{C}^*$, with branching points $x_{1,k} = \mu^{\frac{1}{3}} \eta_3^k (0 \leqslant k \leqslant 2)$, where $\eta_3 = e^{\frac{2}{3}\pi \mathbf{i}}$.

The method of constructing loops in $M_\mu$ is similar to that of $D_N$-singularity. Let $A'_k$ be a simple loop in $\mathbb{C}^*$ around the line segment $L_k := [0, x_{1,k}]$, that is, $A'_k$ is a loop starting at a point $\tilde{\epsilon}\eta_3^k$ $(0 < \tilde{\epsilon} \ll 1)$ on the line segment $L_k$ sufficiently close to 0, it goes along the line segment $L_k$, just before hitting the branch point $x_{1,k}$ it makes a small loop $C_k$ counterclockwise around it, it returns back to the starting point along the line segment $L_k$, and finally it makes a small loop $C_0$ counterclockwise around 0. Clearly, the loop $A'_k = C_0 \circ L_k^{-1} \circ C_k \circ L_k$ lifts to a loop $A_{k,a}, \quad a = 0, 1, 2$ in $M_\mu$, where $a$ indicates the lift of the base point, that is, the base point is lifted to $\left(x_{1,k,a} = \tilde{\epsilon}\eta_3^k, x_{2,k,a} = \left(\frac{\mu - \tilde{\epsilon}^3}{\tilde{\epsilon}}\right)^{\frac{1}{3}} \eta_3^{a - \frac{k}{3}}\right)$. We will compute the period integrals along $A_{k,a}$ for $0 \leq k, a \leq 2$. As a biproduct of our computation we will get that the following set of 7 loops $\{A_{0,1}, A_{1,1}, A_{2,1}, A_{0,2}, A_{1,2}, A_{2,2}, A_{0,0}\}$ represents a basis of $H_1(M_\mu; \mathbb{Z})$.

Let us compute the periods of the holomorphic forms

$$\phi_i(x_1, x_2) \frac{dx_1 dx_2}{dg} = \begin{cases} -\frac{x_1^{i-2} dx_1}{3x_2^2} & \text{if } 1 \leq i \leq 3, \\ -\frac{x_1^{i-5}}{3x_2} dx_1 & \text{if } 4 \leq i \leq 6, \\ -\frac{dx_1}{3x_1} & \text{if } i = 7. \end{cases}$$

along the cycle $A_{k,a}$. Let us parametrize $A'_k$ as follows:

$$\begin{aligned} C_0: \quad & x_1 = \tilde{\epsilon} e^{\mathbf{i}\theta}, \quad \frac{2k}{3}\pi \leq \theta \leq \frac{2k+6}{3}\pi, \\ L_k: \quad & x_1 = \eta_3^k \mu^{\frac{1}{3}} t^{\frac{1}{3}}, \quad \frac{\epsilon^3}{\mu} \leqslant t \leqslant \left(1 - \frac{\epsilon}{\mu^{\frac{1}{3}}}\right)^3, \\ C_k: \quad & x_1 = \eta_3^k \mu^{\frac{1}{3}} + \epsilon e^{\mathbf{i}\theta}, \quad \frac{2k-3}{3}\pi \leqslant \theta \leqslant \frac{2k+3}{3}\pi. \end{aligned}$$

The integrals along the lifts of $C_0$ and $C_k$ contribute to the period integral terms of orders respectively

$$\begin{cases} O(\tilde{\epsilon}^{i-1+\frac{2}{3}}) \text{ and } O(\epsilon^{1-\frac{2}{3}}) & \text{if } 1 \leq i \leq 3, \\ O(\tilde{\epsilon}^{i-4+\frac{1}{3}}) \text{ and } O(\epsilon^{1-\frac{1}{3}}) & \text{if } 4 \leq i \leq 6, \\ O(\tilde{\epsilon}^0) \text{ and } O(\epsilon^1) & \text{if } i = 7. \end{cases}$$

In the limit $\tilde{\epsilon}, \epsilon \to 0$ all integrals along the loops $C_0$ and $C_k$ vanish except for the integral along $C_0$ when $i = 7$. The latter however is straightforward to compute. Therefore, after passing to the limit $\epsilon, \tilde{\epsilon} \to 0$, we get

$$\int_{A_{k,a}} \phi_i(x_1, x_2) \frac{dx_1 dx_2}{dg} = \begin{cases} (1 - \eta_3^{-2}) \int_{L_{k,a}} \frac{-\phi_i dx_1}{3x_1 x_2^2} & \text{if } 1 \le i \le 3, \\ (1 - \eta_3^{-1}) \int_{L_{k,a}} \frac{-\phi_i dx_1}{3x_1 x_2^2} & \text{if } 4 \le i \le 6, \\ -\frac{2\pi \mathbf{i}}{3} & \text{if } i = 7, \end{cases}$$

where $\eta_3 := e^{\frac{2\pi \mathbf{i}}{3}}$ and the integral

$$\int_{L_{k,a}} \frac{-\phi_i dx_1}{3x_1 x_2^2} = \begin{cases} \int_0^1 \frac{-\eta_3^{k(i-\frac{1}{3})-2a} \mu^{\frac{1}{3}(i-\frac{7}{3})} t^{\frac{1}{3}(i-\frac{10}{3})} dt}{9(1-t)^{\frac{2}{3}}} = -\frac{\eta_3^{k(i-\frac{1}{3})-2a} \mu^{\frac{1}{3}(i-\frac{7}{3})}}{9} B(\frac{i}{3} - \frac{1}{9}, \frac{1}{3}) & \text{if } 1 \leqslant i \leqslant 3, \\ \int_0^1 \frac{-\eta_3^{k(i-4+\frac{1}{3})-a} \mu^{\frac{1}{3}(i-4-\frac{2}{3})} t^{\frac{1}{3}(i-4-\frac{8}{3})} dt}{9(1-t)^{\frac{1}{3}}} = -\frac{\eta_3^{k(i-4+\frac{1}{3})-a} \mu^{\frac{1}{3}(i-4-\frac{2}{3})}}{9} B(\frac{i-4}{3} + \frac{1}{9}, \frac{2}{3}) & \text{if } 4 \leqslant i \leqslant 6. \end{cases}$$

Let $\alpha_{k,a} = \Sigma A_{k,a}$ be the suspension. Recalling formula (3.3) and using (3.6)

$$\begin{aligned} (I^{(-1)}_{\alpha_{k,a}}(\lambda), \phi_i) :=& \frac{1}{2\pi} \int_{\alpha_{k,a}} \phi_i \frac{\omega}{df} = \frac{1}{\pi} \partial_\lambda \int_0^\lambda (\lambda - \mu)^{\frac{1}{2}} \int_{A_{k,a}} \phi_i(x_1, x_2) \frac{dx_1 dx_2}{dg} d\mu \\ =& \begin{cases} -\frac{\sqrt{3}}{9\pi} \eta_3^{k(i-\frac{1}{3})-2a} e^{-\frac{\pi}{6}\mathbf{i}} \lambda^{\frac{i}{3} - \frac{7}{9} + \frac{1}{2}} \frac{\Gamma(\frac{3}{2})\Gamma(\frac{i}{3} - \frac{1}{9})\Gamma(\frac{1}{3})}{\Gamma(\frac{i}{3} - \frac{7}{9} + \frac{3}{2})} & \text{if } 1 \leqslant i \leqslant 3, \\ -\frac{\sqrt{3}}{9\pi} \eta_3^{k(i-4+\frac{1}{3})-a} e^{\frac{\pi}{6}\mathbf{i}} \lambda^{\frac{i-4}{3} - \frac{2}{9} + \frac{1}{2}} \frac{\Gamma(\frac{3}{2})\Gamma(\frac{i-4}{3} + \frac{1}{9})\Gamma(\frac{2}{3})}{\Gamma(\frac{i-4}{3} + \frac{7}{9} + \frac{1}{2})} & \text{if } 4 \leqslant i \leqslant 6, \\ -\frac{2}{3}\mathbf{i}\lambda^{\frac{1}{2}} & \text{if } i = 7, \end{cases} \end{aligned}$$

Recalling the formulas for the residue pairing in the basis $\{\phi_i\}$ we get

$$I^{(-1)}_{\alpha_{k,a}}(\lambda) = h \sum_{i=1}^{6} (I^{(-1)}_{\alpha_{k,a}}(\lambda), \phi_{7-i}) \phi_i + 4\mathbf{i}\lambda^{\frac{1}{2}} \phi_7.$$

Recalling formula (3.1) and using that

$$\theta(\phi_i) = \begin{cases} (\frac{4}{9} - \frac{i-1}{3})\phi_i & \text{if } 1 \leqslant i \leqslant 3, \\ (\frac{2}{9} - \frac{i-4}{3})\phi_i & \text{if } 4 \leqslant i \leqslant 6, \\ 0 & \text{if } i = 7, \end{cases}$$

we get

$$\begin{aligned} \Psi(\alpha_{k,a}) =& -\sqrt{\frac{3}{\pi}} \sum_{i=1}^{3} e^{\frac{\pi}{6}\mathbf{i}} \eta_3^{k(\frac{1}{3}-i)-a} \Gamma(\frac{3-i}{3} + \frac{1}{9}) \Gamma(\frac{2}{3}) \phi_i \\ & - \sqrt{\frac{3}{\pi}} \sum_{i=4}^{6} e^{-\frac{\pi}{6}\mathbf{i}} \eta_3^{k(4-i-\frac{1}{3})-2a} \Gamma(\frac{7-i}{3} - \frac{1}{9}) \Gamma(\frac{1}{3}) \phi_i + 2\sqrt{-\pi}\phi_7. \end{aligned} \tag{3.10}$$

Recalling formula (3.4), we get the following formulas for the classical monodromy operator:

$$\sigma(\alpha_{k,a}) = -\alpha_{k+1,a+1} \quad k, a \in \mathbb{Z}_3.$$

Using formula (3.5), we get that the intersection pairing

$$\begin{aligned}(\alpha_{k,a}|\alpha_{l,b}) =& \frac{1}{\pi}(\Psi(\alpha_{k,a}), \cos(\pi\theta)\Psi(\alpha_{l,b})) \\ =& \frac{2}{3} + \frac{1}{6}\sum_{i=1}^{3}(\eta_3^{b-a+(l-k)(i-\frac{1}{3})} + \eta_3^{a-b+(k-l)(i-\frac{1}{3})})\frac{\cos((\frac{i}{3}-\frac{7}{9})\pi)}{\sin((\frac{i}{3}-\frac{1}{9})\pi)\sin(\frac{\pi}{3})} \\ =& \frac{2}{3} + \frac{1}{3}\sum_{i=1}^{3}\cos(\frac{2\pi}{3}(a-b+(k-l)(i-\frac{1}{3})))\frac{\cos((\frac{i}{3}-\frac{7}{9})\pi)}{\sin((\frac{i}{3}-\frac{1}{9})\pi)\sin(\frac{\pi}{3})}.\end{aligned}$$

Let us identify $\mathbb{Z}_3 = \{0, 1, 2\}$. Every $1 \le a' \le 7$ can be written uniquely in the form $a' = 3(a-1) + k + 1$, $1 \le a \le 3$, $0 \le k \le 2$. Put $\alpha_{a'} := \alpha_{k,\bar{a}}$, where $0 \le \bar{a} \le 2$ is the remainder of $a$ modulo 3. Using the above formula, we get that the intersection pairing in the basis $\{\alpha_{a'}\}_{1 \le a' \le 7}$ takes the form

$$\begin{pmatrix} 2 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

The above matrix has determinant 2. Since the determinant of the Cartan matrix of the root system of type $E_7$ is also 2, the conclusion is the same as in the case of $E_6$-singularity, that is, the cycles $(\alpha_1, \dots, \alpha_7) = (\alpha_{0,1}, \alpha_{1,1}, \alpha_{2,1}, \alpha_{0,2}, \alpha_{1,2}, \alpha_{2,2}, \alpha_{0,0})$ form a $\mathbb{Z}$-basis of the Milnor lattice and hence their images under $\Psi$, computed by formula (3.10), give a $\mathbb{Z}$-basis for the image of the Milnor lattice in $H_f$.

### 3.2.7 $E_8$-singularity

Let us fix the following basis of $H_f$ :

$$\phi_i = \begin{cases} x_2^{i-1} & \text{if } 1 \leqslant i \leqslant 4, \\ x_1 x_2^{i-5} & \text{if } 5 \leqslant i \leqslant 8. \end{cases}$$

The residue pairing takes the form

$$(\phi_i, \phi_j) = \frac{1}{h}\delta_{i+j,9}, \quad (1 \leqslant i, j \leqslant 8),$$

where $h = 30$ is the Coxeter number. The Riemann surface $M_\mu$ for $\mu \neq 0$ is a non-singular curve in $\mathbb{C}^2$ defined by equation $x_1^3 + x_2^5 = \mu$. The projection $(x_1, x_2) \mapsto x_2$ defines a degree 3 branched covering $M_\mu \to \mathbb{C}$, with branching points $x_{2,k} = \mu^{\frac{1}{5}}\eta_5^k, k \in \mathbb{Z}_5$, where $\eta_5 = e^{\frac{2}{5}\pi\mathbf{i}}$.

The method for constructing loops in $M_\mu$ is almost the same as that for $E_6$-singularity. Let us omit the similar narration, i.e., we define the loops $A_{k,a}$ in the same way, except that now $a \in \mathbb{Z}_3 \setminus \{0\}$ and $k \in \mathbb{Z}_5 \setminus \{0\}$. We will see that the homology classes of these 8 loops form a $\mathbb{Z}$-basis of $H_1(M_\mu;\mathbb{Z})$. Let us compute the periods of the holomorphic forms

$$\phi_i(x_1,x_2)\frac{dx_1dx_2}{dg}=\begin{cases}\frac{x_2^{i-1}dx_2}{3x_1^2} & \text{if } 1\leqslant i\leqslant 4,\\ \frac{x_2^{i-5}dx_2}{3x_1} & \text{if } 5\leqslant i\leqslant 8,\end{cases}$$

along the cycle $A_{k,a}$. Let us parametrize $A'_k$ as follows:

$$L_k:\quad x_2=\eta_5^k\mu^{\frac{1}{5}}t^{\frac{1}{5}}\quad (0\leqslant t\leqslant \left(1-\frac{\epsilon}{\mu^{\frac{1}{5}}}\right)^5),$$
$$C_k:\quad x_2=\eta_5^k\mu^{\frac{1}{5}}+\epsilon e^{\mathbf{i}\theta}\quad (\frac{2k-5}{5}\pi\leqslant\theta\leqslant\frac{2k+5}{5}\pi).$$

The integrals along the lifts of $C_k$ contribute to the period integral terms of orders $\begin{cases}O(\epsilon^{\frac{1}{3}}) & \text{if } 1\leqslant i\leqslant 3\\ O(\epsilon^{\frac{2}{3}}) & \text{if } 4\leqslant i\leqslant 6\end{cases}$. These terms vanish in the limit $\epsilon\to 0$. Therefore, under this limit, the periods of the holomorphic forms

$$\int_{A_{k,a}}\phi_i(x_1,x_2)\frac{dx_1dx_2}{dg}=\begin{cases}(1-\eta_3^{-2})(\int_{L_{k,a}}-\int_{L_{k+1,a}})\frac{\phi_i dx_2}{3x_1^2} & \text{if } 1\leqslant i\leqslant 4,\\ (1-\eta_3^{-1})(\int_{L_{k,a}}-\int_{L_{k+1,a}})\frac{\phi_i dx_2}{3x_1^2} & \text{if } 5\leqslant i\leqslant 8\end{cases},$$

where $\eta_3:=e^{\frac{2\pi\mathbf{i}}{3}}$ and the integral

$$\int_{L_{k,a}}\frac{\phi_i dx_2}{3x_1^2}=\begin{cases}\int_0^1\frac{\eta_5^{ki}\mu^{\frac{i}{5}}t^{\frac{i}{5}-1}dt}{15\mu^{\frac{2}{3}}(1-t)^{\frac{2}{3}}\eta_3^{2a}}=\frac{\eta_5^{ki}}{15}\eta_3^a\mu^{\frac{i}{5}-\frac{2}{3}}B(\frac{i}{5},\frac{1}{3}) & \text{if } 1\leqslant i\leqslant 4,\\ \int_0^1\frac{\eta_5^{k(i-4)}\mu^{\frac{i-4}{5}}t^{\frac{i-4}{5}-1}dt}{15\mu^{\frac{1}{3}}(1-t)^{\frac{1}{3}}\eta_3^a}=\frac{\eta_5^{k(i-4)}}{15}\eta_3^{2a}\mu^{\frac{i-4}{5}-\frac{1}{3}}B(\frac{i-4}{5},\frac{2}{3}) & \text{if } 5\leqslant i\leqslant 8.\end{cases}$$

Then,

$$\int_{A_{k,a}}\phi_i(x_1,x_2)\frac{dx_1dx_2}{dg}=\begin{cases}(1-\eta_3^{-2})(1-\eta_5^i)\frac{\eta_5^{ki}}{15}\eta_3^a\mu^{\frac{i}{5}-\frac{2}{3}}B(\frac{i}{5},\frac{1}{3}) & \text{if } 1\leqslant i\leqslant 4,\\ (1-\eta_3^{-1})(1-\eta_5^{i-4})\frac{\eta_5^{k(i-4)}}{15}\eta_3^{2a}\mu^{\frac{i-4}{5}-\frac{1}{3}}B(\frac{i-4}{5},\frac{2}{3}) & \text{if } 5\leqslant i\leqslant 8.\end{cases}$$

Let $\alpha_{k,a}=\Sigma A_{k,a}$ be the suspension. Recalling formula (3.3) and using (3.6)

$$\begin{aligned}(I^{(-1)}_{\alpha_{k,a}}(\lambda),\phi_i):=&\frac{1}{2\pi}\int_{\alpha_{k,a}}\phi_i\frac{\omega}{df}=\frac{1}{\pi}\partial_\lambda\int_0^\lambda(\lambda-\mu)^{\frac{1}{2}}\int_{A_{k,a}}\phi_i(x_1,x_2)\frac{dx_1dx_2}{dg}d\mu\\ =&\begin{cases}\frac{\sqrt{3}}{15\pi}\eta_3^a e^{-\frac{\pi}{6}\mathbf{i}}\lambda^{\frac{i}{5}-\frac{1}{6}}\frac{\Gamma(\frac{3}{2})\Gamma(\frac{i}{5})\Gamma(\frac{1}{3})}{\Gamma(\frac{i}{5}+\frac{5}{6})}\eta_5^{ki}(1-\eta_5^i) & \text{if } 1\leqslant i\leqslant 4,\\ \frac{\sqrt{3}}{15\pi}\eta_3^{2a}e^{\frac{\pi}{6}\mathbf{i}}\lambda^{\frac{i-4}{5}+\frac{1}{6}}\frac{\Gamma(\frac{3}{2})\Gamma(\frac{i-4}{5})\Gamma(\frac{2}{3})}{\Gamma(\frac{i-4}{5}+\frac{7}{6})}\eta_5^{k(i-4)}(1-\eta_5^{i-4}) & \text{if } 5\leqslant i\leqslant 8.\end{cases}\end{aligned}$$

Recalling the formulas for residue pairing, we get

$$I^{(-1)}_{\alpha_{k,a}}(\lambda)=h\sum_{i=1}^{8}(I^{(-1)}_{\alpha_{k,a}}(\lambda),\phi_{9-i})\phi_i.$$

By definition,

$$\theta(\phi_i) = \begin{cases} (\frac{2}{3} - \frac{i}{5})\phi_i, & \text{if } 1 \leqslant i \leqslant 4, \\ (\frac{1}{3} - \frac{i-4}{5})\phi_i, & \text{if } 5 \leqslant i \leqslant 8. \end{cases}$$

Therefore, recalling formula (3.1), we get

$$\begin{aligned} \Psi(\alpha_{k,a}) =& \sqrt{\frac{3}{\pi}} \sum_{i=1}^{4} e^{\frac{\pi}{6}\mathbf{i}} \eta_3^{2a} \Gamma(1 - \frac{i}{5}) \Gamma(\frac{2}{3}) \eta_5^{-ki} (1 - \eta_5^{-i}) \phi_i \\ &+ \sqrt{\frac{3}{\pi}} \sum_{i=5}^{8} e^{-\frac{\pi}{6}\mathbf{i}} \eta_3^{a} \Gamma(1 - \frac{i-4}{5}) \Gamma(\frac{1}{3}) \eta_5^{k(4-i)} (1 - \eta_5^{4-i}) \phi_i. \end{aligned} \tag{3.11}$$

Recalling formula (3.4), we get that the following formulas for the classical monodromy operator:

$$\sigma(\alpha_{k,a}) = -\alpha_{k+1,a+1} \quad k \in \mathbb{Z}_5, a \in \mathbb{Z}_3.$$

Recalling formula (3.5), the intersection pairing

$$\begin{aligned} (\alpha_{k,a}|\alpha_{l,b}) =& \frac{1}{\pi}(\Psi(\alpha_{k,a}), \cos(\pi\theta)\Psi(\alpha_{l,b})) \\ =& \frac{1}{10} \sum_{i=1}^{4} (\eta_3^{b-a} \eta_5^{(l-k)i} + \eta_3^{a-b} \eta_5^{(k-l)i}) \frac{\cos((\frac{i}{5} - \frac{2}{3})\pi)}{\sin(\frac{i}{5}\pi)\sin(\frac{\pi}{3})} (2 - \eta_5^{-i} - \eta_5^{i}) \\ =& \frac{4}{5} \sum_{i=1}^{4} \cos(\frac{2}{3}(b-a)\pi + \frac{2i}{5}(l-k)\pi) \frac{\cos((\frac{i}{5} - \frac{2}{3})\pi)}{\sin(\frac{\pi}{3})} \sin(\frac{i}{5}\pi) \end{aligned}$$

Let us identify $\mathbb{Z}_3 \setminus \{0\} = \{1,2\}$ and $\mathbb{Z}_5 \setminus \{0\} = \{1,2,3,4\}$. Every $1 \leq a' \leq 8$ can be written uniquely in the form $a' = 4(a-1) + k$, where $1 \leq a \leq 2$ and $1 \leq k \leq 4$. Put $\alpha_{a'} := \alpha_{k,a}$. Then the intersection matrix $(\alpha_{a'}|\alpha_{b'})$ takes the following form:

$$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

The above matrix has determinant 1. Since the determinant of the Cartan matrix of the root system of type $E_8$ is also 1, the conclusion is the same as in the previous cases.

## 3.3 K-theoretic interpretation

The goal of this section is to prove Theorem 3.3

### 3.3.1 Fermat cases

Let us compute explicitly the map $\mathrm{ch}_\Gamma$ for $f(x) = f^T(x) = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$ with $a_3 = 2$. In fact, our computation works for arbitrary $a_3$ as well, except for one small technical detail, that is, we will prove that the group $K^{-1}_{G^T}(V_1^T)$ is torsion free. This fact should be true for any positive integer $a_3$, but the argument that we give works only if $a_3 = 2$. The group

$$G^T = \{g = (g_1, g_2, g_3) \in (\mathbb{C}^*) \mid g_1^{a_1} = g_2^{a_2} = g_3^{a_3} = 1\}.$$

If $g \in G^T$ is such that $I = \{i \mid g_i = 1\}$ is a non-empty set, then it is easy to see that the map $x = (x_1, x_2, x_3) \mapsto (x_i^{a_i})_{i\in I}$ induces isomorphisms $\mathrm{Fix}_g(\mathbb{C}^3)/G^T \cong \mathbb{C}^I$ and $\mathrm{Fix}_g(V_1^T)/G^T \cong H_I$, where $H_I \subset \mathbb{C}^I$ is the hyperplane $\sum_{i\in I} y_i = 1$. Since the pair $(\mathbb{C}^I, H_I)$ is contractible the groups

$$K^0(\mathrm{Fix}_g(\mathbb{C}^3)/G^T, \mathrm{Fix}_g(V_1^T)/G^T) = H^*(\mathrm{Fix}_g(\mathbb{C}^3)/G^T, \mathrm{Fix}_g(V_1^T)/G^T) = 0.$$

If $g \in G^T$ is such that $g_i \neq 1$ for all $i$, then $\mathrm{Fix}_g(\mathbb{C}^3) = \{0\}$ and $\mathrm{Fix}_g(V_1^T) = \emptyset$. Note that the number of such $g$ is $N = (a_1 - 1)(a_2 - 1)(a_3 - 1)$, that is, the multiplicity of the singularity corresponding to the polynomial $f$.

**Lemma 3.6** The group $K^{-1}_{G^T}(V_1^T)$ is torsion free.

Let us postpone the proof of this lemma until Section 3.3.4. Note that $K^{-1}_{G^T}(V_1^T) \otimes \mathbb{C} = 0$. Therefore, according to the above Lemma 3.6, we have $K^{-1}_{G^T}(V_1^T) = 0$. The long exact sequence of the pair $(\mathbb{C}^3, V_1^T)$ yields the following exact sequence

$$0 \longrightarrow K^0_{G^T}(\mathbb{C}^3, V_1^T) \longrightarrow K^0_{G^T}(\mathbb{C}^3) \longrightarrow K^0_{G^T}(V_1^T)\ .$$

On the other hand, $K^0_{G^T}(\mathbb{C}^3)$ coincides with the representation ring of $G^T$, that is,

$$K^0_{G^T}(\mathbb{C}^3) = \mathbb{Z}[L_1, L_2, L_3]/(L_1^{a_1} - 1, L_2^{a_2} - 1, L_3^{a_3} - 1),$$

where $L_i = \mathbb{C}^3 \times \mathbb{C}$ is the trivial bundle with $G^T$-action $g \cdot (x, \lambda) := (gx, g_i\lambda)$. Note that $T\mathbb{C}^3 \cong L_1 + L_2 + L_3$ in the category of $G^T$-equivariant bundles. We claim that

$$K^0_{G^T}(\mathbb{C}^3, V_1^T) = (L_1 - 1)(L_2 - 1)(L_3 - 1)\mathbb{Z}[L_1, L_2, L_3]/(L_1^{a_1} - 1, L_2^{a_2} - 1, L_3^{a_3} - 1). \tag{3.12}$$

Indeed, note that $s_i(x) = (x, f_{x_i})$ is a $G^T$-equivariant section of $L_i^{-1}$. The Koszul complex corresponding to the sequence $(s_1, s_2, s_3)$ has the form

$$L_1L_2L_3 \longrightarrow \bigoplus_{1\leq i<j\leq 3} L_iL_j \longrightarrow \bigoplus_{1\leq i\leq 3} L_i \longrightarrow \underline{\mathbb{C}}\ ,$$

where $\underline{\mathbb{C}}$ is the trivial bundle with trivial $G^T$-action. The sequence $(s_1, s_2, s_3)$ is regular, so the corresponding Koszul complex is a resolution of the structure sheaf of the zero locus $\{s_1 = s_2 = s_3 = 0\}$. The zero locus is $\{0\}$ and since $0 \notin V_1^T$ the restriction of the Koszul complex to $V_1^T$ is exact, i.e., the Koszul complex represents an element of $K^0_{G^T}(\mathbb{C}^3, V_1^T)$. This proves that the RHS of (3.12) is a $\mathbb{Z}$-submodule of the LHS. Note that both the LHS and the RHS of (3.12) are free $\mathbb{Z}$-modules of rank $N$. Therefore, the quotient of LHS by RHS is a finite Abelian group. In order to prove that the quotient is 0, it is sufficient to prove that if $g \in K^0_{G^T}(\mathbb{C}^3)$ and $mg$ belongs to the RHS of (3.12) for some integer $m$, then $g$ belongs to the RHS of (3.12) too. The proof is straightforward so we leave it as an exercise.

Let us fix the following basis of $K^0_{G^T}(\mathbb{C}^3, V_1^T)$:

$$A_{m_1,m_2,m_3} := L_1^{m_1}L_2^{m_2}L_3^{m_3}(L_1-1)(L_2-1)(L_3-1), \quad 0 \le m_i \le a_i - 2.$$

Let $e_{k_1,k_2,k_3} = 1 \in H^0(\mathrm{Fix}_g(\mathbb{C}^3)/G, \mathrm{Fix}_g(V_1^T)/G)$, where $g = (e^{2\pi \mathbf{i} k_1/a_1}, e^{2\pi \mathbf{i} k_2/a_2}, e^{2\pi \mathbf{i} k_3/a_3})$. We get

$$\mathrm{ch}_\Gamma(A_{m_1,m_2,m_3}) = \frac{1}{2\pi}\sum_{k_1=1}^{a_1-1}\sum_{k_2=1}^{a_2-1}\sum_{k_3=1}^{a_3-1}\prod_{i=1}^{3}\Big(\Gamma\Big(1-\tfrac{k_i}{a_i}\Big)\, e^{-2\pi \mathbf{i}\, k_i m_i/a_i}\,\Big(e^{-2\pi \mathbf{i} k_i/a_i}-1\Big)\Big)\; e_{k_1,k_2,k_3},$$

where the ingredients of the above formula are computed as follows. Since $\mathrm{Fix}_g(\mathbb{C}^3) = \{0\}$ and the action of $g$ on $L_i|_{\mathrm{Fix}_g(\mathbb{C}^3)}$ is given by multiplication by $e^{2\pi \mathbf{i}\, k_i/a_i}$ we get

$$\widehat{\Gamma}(L_i)|_{\mathrm{Fix}_g(\mathbb{C}^3)/G^T} = \Gamma\Big(1-\tfrac{k_i}{a_i}\Big) \quad \text{and} \quad \iota^*\widetilde{\mathrm{ch}}(L_i)|_{\mathrm{Fix}_g(\mathbb{C}^3)/G^T} = \widetilde{\mathrm{ch}}(L_i^{-1})|_{\mathrm{Fix}_g(\mathbb{C}^3)/G^T} = e^{-2\pi \mathbf{i}\, k_i/a_i},$$

where we used that $\iota^*(L_i) = L_i^{-1}$. The orbifold tangent bundle $[T\mathbb{C}^3/G^T] = L_1 + L_2 + L_3$ so its $\Gamma$-class is $\prod_{i=1}^3 \widehat{\Gamma}(L_i)$, while $\iota^*\widetilde{\mathrm{ch}}|_{\mathrm{Fix}_g(\mathbb{C}^3)/G^T}$ is a ring homomorphism, so the computation of its value on $A_{m_1,m_2,m_3}$ amounts to the substitution $L_i \mapsto e^{-2\pi \mathbf{i}\, k_i/a_i}$. Let us specialize the above formula to the cases of $A_N$, $E_6$, and $E_8$ singularities. In the first case $a_1 = N+1$, $a_2 = a_3 = 2$. The above formula takes the form

$$\mathrm{ch}_\Gamma(A_{m,0,0}) = 2\sum_{k=1}^{N}\eta^{-km}(\eta^{-k}-1)\Gamma\Big(1-\tfrac{k}{N+1}\Big)e_{k,1,1}.$$

Comparing with (3.7), we get that if we define $\mathrm{mir}(\phi_i) = e_{i,1,1}$ $(1 \le i \le N)$, then the images of $\Psi$ and $\mathrm{ch}_\Gamma$ will coincide. The vanishing cycle $\alpha_{k,a}$ corresponds to $(-1)^a A_{k,0,0}$.

For the case of $E_6$ we have $a_1 = 3$, $a_2 = 4$, $a_3 = 2$. The formula takes the form

$$\mathrm{ch}_\Gamma(A_{m_1,m_2,0}) = -\frac{1}{\sqrt{\pi}}\sum_{k_1=1}^{2}\sum_{k_2=1}^{3}\Gamma\Big(1-\tfrac{k_1}{3}\Big)\Gamma\Big(1-\tfrac{k_2}{4}\Big)\eta_3^{-k_1m_1}\eta_4^{-k_2m_2}(\eta_3^{-k_1}-1)(\eta_4^{-k_2}-1)e_{k_1,k_2,1},$$

where $\eta_3 = e^{2\pi\mathbf{i}/3}$ and $\eta_4 = e^{2\pi\mathbf{i}/4} = \mathbf{i}$. Note that $\eta_3^{-1} - 1 = -\sqrt{3}e^{\pi\mathbf{i}/6}$ and $\eta_3^{-2} - 1 = -\sqrt{3}e^{-\pi\mathbf{i}/6}$. Comparing with (3.9) we get that if we define

$$\mathrm{mir}(\phi_i) = \begin{cases} e_{1,i,1}, & \text{for } 1 \le i \le 3, \\ e_{2,i-3,1}, & \text{for } 4 \le i \le 6, \end{cases}$$

then the images of $\Psi$ and $\mathrm{ch}_\Gamma$ will coincide. The vanishing cycle $\alpha_{k,a}$ corresponds to $-A_{a,k,0}$.

Suppose now that the singularity is of type $E_8$, that is, $a_1 = 3$, $a_2 = 5$, and $a_3 = 2$. The formula takes the form

$$\mathrm{ch}_\Gamma(A_{m_1,m_2,0}) = -\frac{1}{\sqrt{\pi}}\sum_{k_1=1}^{2}\sum_{k_2=1}^{4}\Gamma\Big(1-\tfrac{k_1}{3}\Big)\Gamma\Big(1-\tfrac{k_2}{5}\Big)\eta_3^{-k_1m_1}\eta_5^{-k_2m_2}(\eta_3^{-k_1}-1)(\eta_5^{-k_2}-1)e_{k_1,k_2,1},$$

where $\eta_3 = e^{2\pi\mathbf{i}/3}$ and $\eta_5 = e^{2\pi\mathbf{i}/5}$. Comparing with formula (3.11) we get that if we define

$$\operatorname{mir}(\phi_i) = \begin{cases} e_{1,i,1}, & \text{for } 1 \le i \le 4, \\ e_{2,i-4,1}, & \text{for } 5 \le i \le 8, \end{cases}$$

then the images of $\Psi$ and $\mathrm{ch}_\Gamma$ will coincide. The vanishing cycle $\alpha_{k,a}$ corresponds to $-A_{a,k,0}$.

### 3.3.2 Suspension isomorphism

Suppose that $X$ is a finite CW-complex equipped with an action of a finite (or more generally compact Lie) group $G$. Let $\mu_2 = \{\pm 1\}$ and

$$\Sigma X = {}_{X\times\{-1\}}\backslash X\times[-1,1]/_{X\times\{1\}}$$

be the suspension of $X$, where the double quotient simply means that the quotient is taken in two steps: first by, say, $X \times \{-1\}$ and then by $X \times \{1\}$. Note that $G \times \mu_2$ acts naturally on $\Sigma X$ via $(g,\epsilon)\cdot(x,t) = (gx, \epsilon t)$ and that the 0-dimensional sphere $\mathbb{S}^0 := \Sigma X - X \times (-1,1)$ is a $G \times \mu_2$-equivariant subcomplex of $\Sigma X$.

Let $L = [-1,1] \times \mathbb{C}$ be the trivial $\mu_2$-equivariant line bundle on the interval $I := [-1,1]$, where the representation of $\mu_2$ on $\mathbb{C}$ is given by $\epsilon \cdot \lambda = \epsilon\lambda$. It is an easy and amusing exercise to check that $K^0_{\mu_2}(I, \partial I) = \mathbb{Z}\,\ell$, where $\ell$ is the relative K-theoretic class of the complex $L \xrightarrow{t} \underline{\mathbb{C}}$, where $\underline{\mathbb{C}} = I \times \mathbb{C}$ is the trivial $\mu_2$-equivariant line bundle corresponding to the trivial representation of $\mu_2$ on $\mathbb{C}$ and the map is induced by $(t,\lambda) \mapsto (t, t\lambda)$.

**Lemma 3.7** The exterior tensor product by $\ell$ induces an isomorphism $K^i_G(X) \cong K^i_{G\times\mu_2}(\Sigma X, \mathbb{S}^0)$.

**Proof** By definition

$$K^i_{G\times\mu_2}(\Sigma X, \mathbb{S}^0) = \widetilde{K}^i_{G\times\mu_2}(\Sigma X/\mathbb{S}^0) = K^i_{G\times\mu_2}(X\times I, X\times\partial I) \cong K^i_G(X)\otimes K^0_{\mu_2}(I,\partial I),$$

where we used that $K^i_{\mu_2}(I,\partial I)$ is isomorphic to $\mathbb{Z}$ for $i$ even and 0 for $i$ odd, so the last isomorphism is given by the equivariant Künneth formula (see [39]).

Suppose now that $Y \subset X$ is a $G$-invariant CW-subcomplex of $X$. Using the long exact sequence of the triple $\mathbb{S}^0 \subset \Sigma Y \subset \Sigma X$ and Lemma 3.7, it is straightforward to prove the following corollary.

**Corollary 3.8** The exterior tensor product by $\ell$ induces an isomorphism

$$K^i_G(X,Y) \cong K^i_{G\times\mu_2}(\Sigma X, \Sigma Y).$$

### 3.3.3 The relative K-ring for $D_N$-singularity

Let us return to the settings of $D_N$-singularity. We have $f^T(x) = x_1^2 + x_1 x_2^{N-1} + x_3^2$. The group of diagonal symmetries of $f^T$ is

$$G^T = \{t \in (\mathbb{C}^*)^3 \mid t_1^2 = t_1 t_2^{N-1} = t_3^2 = 1\}.$$

Let $L_i = \mathbb{C}^3 \times \mathbb{C}$ be the $G^T$-equivariant line bundle for which the action of $G^T$ on $\mathbb{C}$ is given by the character $G^T \to \mathbb{C}^*$, $(t_1, t_2, t_3) \mapsto t_i$. Let us introduce the following $N$ complexes of $G^T$-equivariant vector bundles on $\mathbb{C}^3$ :

$$E_i^\bullet : \quad L_1 L_2^{i-1} L_3 \xrightarrow{d_0} L_1 L_2^{i-1} \oplus L_2^{i-1} L_3 \xrightarrow{d_1} L_2^{i-1}\,, \quad 1 \le i \le N-1,$$

where the differentials are defined by $d_0(x, \lambda) = (x, -x_3\lambda, x_1\lambda)$ and $d_1(x, \lambda_1, \lambda_3) = (x, x_1\lambda_1 + x_3\lambda_3)$,

$$E_N^\bullet : \quad L_3 \xrightarrow{d_0} \underline{\mathbb{C}} \oplus L_3 \xrightarrow{d_1} \underline{\mathbb{C}}\,,$$

where $d_0(x, \lambda) = (x, -x_3\lambda, x_1^2\lambda)$ and $d_1(x, \lambda_1, \lambda_3) = (x, x_1^2\lambda_1 + x_3\lambda_3)$.

**Proposition 3.9** The relative $K$-ring $K^0_{G^T}(\mathbb{C}^3, V^T) \cong \mathbb{Z}^N$ and the complexes $E_i^\bullet$ $(1 \le i \le N)$ represent a $\mathbb{Z}$-basis.

**Proof** Note that the complex $E_i^\bullet$ $(1 \le i \le N-1)$ is a tensor product of $L_2^{i-1}$, $L_1 \xrightarrow{x_1} \underline{\mathbb{C}}$, and $L_3 \xrightarrow{x_3} \underline{\mathbb{C}}$ and that the complex $E_N^\bullet$ is a tensor product of $\underline{\mathbb{C}} \xrightarrow{x_1^2} \underline{\mathbb{C}}$ and $L_3 \xrightarrow{x_3} \underline{\mathbb{C}}$. On the other hand, we have $G^T = A \times {}^{-}{}_2$, where $A = \{t \in (\mathbb{C}^*)^2 \mid t_1^2 = t_1 t_2^{N-1} = 1\}$. Recalling Corollary 3.8 we get $K^0_{G^T}(\mathbb{C}^3, V^T) \cong K^0_A(\mathbb{C}^2, M)$, where $M = \{x \in \mathbb{C}^2 \mid x_1^2 + x_1 x_2^{N-1} = 1\}$. Slightly abusing the notation we denote by $L_1$ and $L_2$ the restriction of the vector bundles $L_1$ and $L_2$ to $\mathbb{C}^2$. Note that the operation tensor product by the complex $L_3 \xrightarrow{x_3} \underline{\mathbb{C}}$ is precisely the exterior tensor product by the complex $\ell$ in the suspension isomorphism from Corollary 3.8. Therefore, it is sufficient to prove that the complexes $L_1 L_2^{i-1} \xrightarrow{x_1} L_2^{i-1}$ $(1 \le i \le N-1)$ and $\underline{\mathbb{C}} \xrightarrow{x_1^2} \underline{\mathbb{C}}$ represent a $\mathbb{Z}$-basis of $K^0_A(\mathbb{C}^2, M)$.

The long exact sequence of the pair $(\mathbb{C}^2, M)$ yields the following exact sequence:

$$0 \longrightarrow K_A^{-1}(M) \xrightarrow{\delta} K_A^0(\mathbb{C}^2, M) \xrightarrow{\rho} K_A^0(\mathbb{C}^2) \longrightarrow K_A^0(M)\,,$$

where we used that $K_A^{-1}(\mathbb{C}^2) = 0$. We have

$$K_A^0(\mathbb{C}^2) = \mathbb{Z}[L_1, L_2]/\langle L_1^2 - 1, L_1 L_2^{N-1} - 1\rangle,$$

where the RHS is the representation ring of $A$. Just like in the Fermat cases it is easy to prove that the image of $\rho$ coincides with the ideal $(L_1 - 1)K_A^0(\mathbb{C}^2)$. Note that $\mathrm{Im}(\rho) \cong \mathbb{Z}^{N-1}$ and that $\rho(E_i^\bullet) = L_2^{i-1}(L_1 - 1)$ $(1 \le i \le N-1)$ is a $\mathbb{Z}$-basis. It remains only to prove that $K_A^{-1}(M) \cong \mathbb{Z}$ and that $\mathrm{Im}(\delta)$ is generated as a $\mathbb{Z}$-module by the complex $E_N^\bullet$.

Let us first prove that $K_A^{-1}(M) \cong \mathbb{Z}$. Let $\pi : M \to \mathbb{C}^*$ be the map $(x_1, x_2) \mapsto x_1^2$. The map $\pi$ is a branched covering with only one branch point, that is, $1 \in \mathbb{C}^*$. The corresponding ramification points are $R = \{(-1, 0), (1, 0)\}$. Note that $R$ is an $A$-invariant subset. The idea is to use the long exact sequence of the pair $(M, M \setminus R)$. The action of $A$ on $M \setminus R$ is free, so we have

$$K_A^i(M \setminus R) = K^i((M \setminus R)/A) = K^i(\mathbb{C} \setminus \{0, 1\}).$$

Therefore $K_A^0(M \setminus R) \cong \mathbb{Z}$ and $K_A^{-1}(M \setminus R) \cong \mathbb{Z}^2$. The groups $K^i(M, M \setminus R)$ are also easy to compute. Let $U \subset M$ be a small $A$-invariant open neighborhood of $R$. Then by excision $K_A^i(M, M \setminus R) = K_A^i(U, U \setminus R)$. Note that the open neighborhood $U$ can be identified with an open neighborhood of $R$ in the normal bundle $\nu_R$ to $R$ in $M$. Indeed, the normal bundle is trivial $\nu_R = R \times \mathbb{C}$ and a point in $(x_1, x_2) \in U$ satisfies $x_1 = \frac{1}{2}(-x_2^{N-1} \pm \sqrt{x_2^{2N-2}+4})$, so the map $U \to \nu_R$, $(x_1, x_2) \mapsto ((\pm 1, 0), x_2)$ identifies $U$ with an open neighborhood of the zero section $R$ in $\nu_R$. Clearly, the pullback of $\nu_R$ to $U$ is $L_2$ and the Thom class of $\nu_R$ is represented as an element of $K^0_A(\nu_R) = K^0_A(U, U \setminus R)$ by the complex $\underline{\mathbb{C}} \xrightarrow{x_2} L_2$ . According to Thom isomorphism $K^i_A(U, U \setminus R) \cong K^i_A(R)$. Note that $R$ is an $A$-orbit, that is, $R = A/B$, where $B$ is the cyclic subgroup of $A$ generated by $(1, \eta^2)$. Therefore, $K^{-1}_A(R) = 0$ and $K^0_A(R)$ coincides with the representation ring of $B$. Since the Thom isomorphism is given by tensor product with the Thom class, we get

$$K^0_A(M, M \setminus R) = \bigoplus_{i=1}^{N-1} \mathbb{Z}\,[\, L_2^{i-1} \xrightarrow{x_2} L_2^i \,]\ .$$

The long exact sequence of the pair $(M, M \setminus R)$ takes the form

$$0 \longrightarrow K^{-1}_A(M) \longrightarrow K^{-1}_A(M \setminus R) \xrightarrow{\delta} K^0_A(M, M \setminus R)\ .$$

We already proved that $K^{-1}_A(M \setminus R) \cong K^{-1}(\mathbb{C}^2 \setminus \{0,1\}) \cong \mathbb{Z}^2$. We will make use of the following explicit interpretation of the $K$-group $K^{-1}_A(\ )$. By definition, for any finite CW-complex $X$, we have $K^{-1}_A(X) = \widetilde{K}^0_A(\Sigma(X \sqcup \mathrm{pt}))$. Since the complement of $X$ in $\Sigma(X \sqcup \mathrm{pt})$ is contractible, we can think of an element of $K^{-1}_A(X)$ as a representation of $A$ on some vector space $\mathbb{C}^r$ and an $A$-equivariant isomorphism $\phi : X \times \mathbb{C}^r \to X \times \mathbb{C}^r$, that is, an $A$-equivariant morphism $X \to \mathrm{GL}_r(\mathbb{C})$. In our case the elements of $K^{-1}_A(M \setminus R)$ are obtained by pullback from $K^{-1}(\mathbb{C} \setminus \{0,1\})$. The latter is generated by two elements that correspond, in the way described above, to the two maps $\mathbb{C} \setminus \{0,1\} \to \mathbb{C}^*$, $t \mapsto t$ and $t \mapsto 1-t$. Therefore, the group $K^{-1}_A(M \setminus R)$ is generated by the two elements that correspond to the two maps $M \setminus R \to \mathbb{C}^*$ defined by $(x_1, x_2) \mapsto x_1^2$ and $(x_1, x_2) \mapsto 1 - x_1^2 = x_1 x_2^{N-1}$. The connecting morphism $\delta$ can be described as follows. Given an $A$-equivariant isomorphism $\phi : (M \setminus R) \times \mathbb{C}^r \to (M \setminus R) \times \mathbb{C}^r$, then let us pick an extension to a vector bundle morphism $\widetilde{\phi} : M \times \mathbb{C}^r \to M \times \mathbb{C}^r$. The resulting complex clearly represents an element of $K^0_A(M, M \setminus R)$ and that is what $\delta(\phi)$ is. The extensions in our case are straightforward to construct. We get that

$$\mathrm{Im}(\delta) = \mathbb{Z}\,[\, \underline{\mathbb{C}} \xrightarrow{x_1^2} \underline{\mathbb{C}} \,] + \mathbb{Z}\,[\, \underline{\mathbb{C}} \xrightarrow{x_1 x_2^{N-1}} \underline{\mathbb{C}} \,].$$

Note however, that $x_1 \neq 0$ so $x_1^2$ defines an isomorphism, i.e., the first complex is 0 in $K^0_A(M, M \setminus R)$. In particular, the kernel of the connecting homomorphism $\delta$ is $\cong \mathbb{Z}$ and it is generated by the element in $K^{-1}_A(M \setminus R)$ corresponding to the map $M \setminus R \to \mathbb{C}^*$, $(x_1, x_2) \to x_1^2$. This map extends to $M$, so we get that $K^{-1}_A(M) \cong \mathbb{Z}$ with generator corresponding to the map $M \to \mathbb{C}^*$, $(x_1, x_2) \mapsto x_1^2$. Returning to the long exact sequence of the pair $(\mathbb{C}^2, M)$, we get that the connecting morphism $K^{-1}_A(M) \to K^0_A(\mathbb{C}^2, M)$ maps the generator of $K^{-1}_A(M)$ to the complex $\underline{\mathbb{C}} \xrightarrow{x_1^2} \underline{\mathbb{C}}$ . This completes the proof of the proposition.

### 3.3.4 Proof of Lemma 3.6

We follow the same strategy as in the proof of Proposition 3.9. Let us denote by $M \subset \mathbb{C}^2$ the Riemann surface defined by $x_1^{a_1} + x_2^{a_2} = 1$. Let $A = \{t \in (\mathbb{C}^*)^2 \mid t_1^{a_1} = t_2^{a_2} = 1\}$. Since $K^{-1}_{G^T}(V_1^T) \cong K^{-1}_A(M)$, we get that it is sufficient to prove that $K^{-1}_A(M)$ is torsion free. Let $\pi : M \to \mathbb{C}$ be the map $(x_1, x_2) \mapsto x_1^{a_1}$. The map $\pi$ is a branched covering with only one branching point, that is, $1 \in \mathbb{C}$. The corresponding ramification points are $R = \{(\xi, 0) \mid \xi^{a_1} = 1\}$. The torsion freeness can be deduced easily from the long exact sequence of the pair $(M, M \setminus R)$. The action of $A$ on $M \setminus R$ is free, so we have

$$K_A^{-1}(M \setminus R) = K^{-1}((M \setminus R)/A) = K^{-1}(\mathbb{C} \setminus \{1\}) \cong \mathbb{Z}.$$

Using the Thom isomorphism for the normal bundle to $R$ in $M$, we get $K_A^{-1}(M, M \setminus R) = K_A^{-1}(R)$. On the other hand, note that $R$ is the orbit of $A$ through the point $(1,0) \in M$, we get $R = A/B$, where $B \subset A$ is the cyclic subgroup generated by $(1, \eta_{a_2})$, $\eta_{a_2} = e^{2\pi \mathbf{i}/a_2}$. Therefore, $K_A^{-1}(R) = K_A^{-1}(A/B) = K^{-1}(B) = 0$. Recalling the long exact sequence of the pair $(M, M \setminus R)$, we get

$$0 \longrightarrow K_A^{-1}(M) \longrightarrow K_A^{-1}(M \setminus R) \xrightarrow{\delta} K_A^0(M, M \setminus R) \tag{3.13}$$

We get that $K_A^{-1}(M)$ can be embedded as a subgroup of $K_A^{-1}(M \setminus R) \cong \mathbb{Z}$. The latter is torsion free, so $K_A^{-1}(M)$ must be also torsion free.

**Remark 3.10** The above argument can be continued to give a direct proof of the fact that $K_A^{-1}(M) = 0$. Namely, using the Thom isomorphism, we can prove that the group $K_A^0(M, M \setminus R)$ is a free Abelian group of rank $a_2$ and that the complexes $[L_2^{i-1} \xrightarrow{x_2} L_2^i]$ $(1 \le i \le a_2)$ represent a $\mathbb{Z}$-basis. Moreover, the image of the connecting morphism $\delta$ in (3.13) can be computed explicitly as well, that is, it coincides with the sum of the above complexes. In particular, we get that $\delta$ is an injective map, and hence $K_A^{-1}(M) = 0$.

**Remark 3.11** The long exact sequences of the pairs $(M, M \setminus R)$ and $(\mathbb{C}^2, M)$ can be computed explicitly, that is, both the groups and the differentials can be determined. This allows us to give an alternative proof of formula (3.12). We leave the details to the interested reader.

### 3.3.5 The relative K-ring for $E_7$-singularity

The argument from the previous section works also for $E_7$-singularity. Let us only state the result. The proof is completely analogous.

We have $f^T(x) = x_1^3 x_2 + x_2^3 + x_3^2$ The group of diagonal symmetries of $f^T$ is

$$G^T = \{t \in (\mathbb{C}^*)^3 \mid t_1^3 t_2 = t_2^3 = t_3^2 = 1\}.$$

Let $L_i = \mathbb{C}^3 \times \mathbb{C}$ be the $G^T$-equivariant line bundle for which the action of $G^T$ on $\mathbb{C}$ is given by the character $G^T \to \mathbb{C}^*$, $(t_1, t_2, t_3) \mapsto t_i$. Let us introduce the following 7 complexes of $G^T$-equivariant vector bundles on $\mathbb{C}^3$ :

$$E_i^\bullet : \quad L_1^{i-1} L_3^{-1} \xrightarrow{d_0} L_1^{i-1} \oplus L_1^{i-1} L_2 L_3^{-1} \xrightarrow{d_1} L_1^{i-1} L_2, \quad 1 \le i \le 6,$$

where the differentials are defined by $d_0(x, \lambda) = (x, -x_3\lambda, x_2\lambda)$ and $d_1(x, \lambda_2, \lambda_3) = (x, x_2\lambda_2 + x_3\lambda_3)$,

$$E_7^\bullet : \quad L_3 \xrightarrow{d_0} \underline{\mathbb{C}} \oplus L_3 \xrightarrow{d_1} \underline{\mathbb{C}} \,,$$

where $d_0(x, \lambda) = (x, -x_3\lambda, x_2^3\lambda)$ and $d_1(x, \lambda_2, \lambda_3) = (x, x_2^3\lambda_2 + x_3\lambda_3)$.

**Proposition 3.12** Let $V^T = \{x \in \mathbb{C}^3 \mid f^T(x) = 1\}$. The relative $K$-ring $K^0_{G^T}(\mathbb{C}^3, V^T) \cong \mathbb{Z}^7$ and the complexes $E_i^\bullet$ $(1 \le i \le 7)$ represent a $\mathbb{Z}$-basis.

### 3.3.6 Γ-integral structure for $D_N$-singularity

Let us compute $\mathrm{ch}_\Gamma(E_i^\bullet)$ for $1 \le i \le N$. After a straightforward computation we get that the relative cohomology group $H(\mathrm{Fix}_g(\mathbb{C}^3), \mathrm{Fix}_g(V^T))^{G^T}$ is not zero only in the following two cases: 1) $g = (g_1, g_2, g_3)$ with $g_i \neq 1$ for all $i$ and 2) $g = (1, 1, -1)$. For the first case, there are $N-1$ elements, that is, $g = (-1, \eta^{2i-1}, -1)$ $(1 \le i \le N-1)$ and the fixed point subsets are $\mathrm{Fix}_g(\mathbb{C}^3) = \{0\}$ and $\mathrm{Fix}_g(V^T) = \emptyset$. Therefore, $H(\mathrm{Fix}_g(\mathbb{C}^3), \mathrm{Fix}_g(V^T); \mathbb{C})^{G^T} \cong \mathbb{C}$ is non-trivial only in degree 0 and we denote by $e_i := 1$ the unit of the cohomology group. For the second case, $\mathrm{Fix}_g(\mathbb{C}^3) = \mathbb{C}^2$ and $\mathrm{Fix}_g(V^T) = M = \{x_1^2 + x_1 x_2^{N-1} = 1\}$. The relative cohomology group $H^i(\mathbb{C}^2, M; \mathbb{C})^{G^T} \cong H^{i-1}(M/G^T; \mathbb{C})$ for $i > 0$ and $= 0$ for $i = 0$. As we already explained above $M/G^T = \mathbb{C}^*$, so the relative cohomology is non-zero only in degree 2, i.e., for $i = 2$. Since $M$ is a Stein manifold, we can describe the relative cohomology in terms of the holomorphic de Rham complexes on $\mathbb{C}^2$ and $M$. Namely, consider the complex of Abelian groups

$$\Gamma(\mathbb{C}^2, \Omega^\bullet_{\mathbb{C}^2})^{G^T} \oplus \Gamma(M, \Omega^{\bullet-1}_M)^{G^T}, \quad d(\omega, \alpha) = (d\omega, \omega|_M - d\alpha). \tag{3.14}$$

A closed form $(\omega, \alpha)$ in degree $i$, that is, $d(\omega, \alpha) = 0$, defines naturally a linear functional on the space of dimension $i$ relative chains $\gamma \subset \mathbb{C}^2$ with $\partial\gamma \subset M$, that is,

$$\gamma \mapsto \int_\gamma \omega - \int_{\partial\gamma} \alpha.$$

Using the de Rham theorem for $\mathbb{C}^2$ and $M$, it is easy to prove that the above map induces an isomorphism between the $i$-th cohomology of the complex (3.14) and $H^i(\mathbb{C}^2, M; \mathbb{C})^{G^T}$. Let us denote by $e_N \in H^2(\mathbb{C}^2, M; \mathbb{C})$ the cohomology class corresponding to the form $(0, -\frac{1}{2\pi \mathrm{i}} dx_1/x_1)$.

Suppose now that $g = (-1, \eta^{2a-1}, -1)$, $1 \le a \le N-1$. Let us compute the component of $\mathrm{ch}_\Gamma(E_i^\bullet)$ for $1 \le i \le N-1$ in $H^0(\mathrm{Fix}_g(\mathbb{C}^3), \mathrm{Fix}_g(V^T); \mathbb{C})^{G^T}$. Note that in this case we have an isomorphism $K^0(\mathrm{Fix}_g(\mathbb{C}^3), \mathrm{Fix}_g(V^T)) \cong K^0(\mathrm{Fix}_g(\mathbb{C}^3))$. The image of $\iota^* \,\mathrm{Tr}(E_i^\bullet)$ is

$$\eta^{-(2a-1)(i-1)} L_2^{i-1}(L_1 L_3 - (-L_3 - L_1) + \mathbb{C}) = 4\eta^{-(2a-1)(i-1)}\mathbb{C},$$

where again we abused the notation by denoting by $L_i$ the restriction of $L_i$ to $\mathrm{Fix}_g(\mathbb{C}^3) = \{0\}$. The component of the Γ-class is

$$\Gamma(L_1 + L_2 + L_3) = \Gamma(1 - 1/2)\Gamma(1 - (2a-1)/h)\Gamma(1 - 1/2) e_a,$$

where $h := 2N - 2$. Therefore, the component of $\mathrm{ch}_\Gamma(E_i^\bullet)$ is

$$2\eta^{-(2a-1)(i-1)}\Gamma(1 - m_a/h) e_a.$$

The component of $\mathrm{ch}_\Gamma(E_N)$ is clearly 0, because the image of the complex $E_N^\bullet$ in $K^0(\mathrm{Fix}_g(\mathbb{C}^3))$ is 0.

Suppose that $g = (1,1,-1)$. Let us compute the component of $\mathrm{ch}_\Gamma(E_i^\bullet)$ in $H^2(\mathrm{Fix}_g(\mathbb{C}^3), \mathrm{Fix}_g(V^T); \mathbb{C})^{G^T} = H^2(\mathbb{C}^2, M)^{G^T} = \mathbb{C}\, e_N$. By definition the component of $\iota^* \operatorname{Tr}(E_i^\bullet)$ in $K^0(\mathbb{C}^2, M)^{G^T}$ is

$$-L_2^{i-1}[\, L_1L_3 \xrightarrow{-x_1} L_3 \longrightarrow 0\,] + L_2^{i-1}[\, 0 \longrightarrow L_1 \xrightarrow{x_1} \underline{\mathbb{C}}\,],$$

where the above complexes are concentrated in degrees 0, 1, and 2 and the vector bundles $L_i$ ($1 \le i \le 3$) are trivial line bundles on $\mathbb{C}^2$. The second complex, as an element of $K^0(\mathbb{C}^2, M)$, is equivalent to the two-term complex $\underline{\mathbb{C}} \xrightarrow{\overline{x}_1} \underline{\mathbb{C}}$ . Therefore, the component of $\iota^* \operatorname{Tr}(E_i^\bullet)$ ($1 \le i \le N-1$) takes the form

$$-[\, \underline{\mathbb{C}} \xrightarrow{x_1} \underline{\mathbb{C}}\,] + [\, \underline{\mathbb{C}} \xrightarrow{d} \underline{\mathbb{C}}\,].$$

In order to compute the Chern character of the above complexes, we use the following commutative diagram:

$$\begin{array}{ccc} \widetilde{K}^0(\Sigma M) \cong \widetilde{K}^{-1}(M) & \xrightarrow{\cong} & K^0(\mathbb{C}^2, M) \ , \\ \Big\downarrow \mathrm{ch} & & \Big\downarrow \mathrm{ch} \\ H^2(\Sigma M) \cong H^1(M) & \xrightarrow{\delta} & H^2(\mathbb{C}^2, M) \end{array}$$

where the horizontal arrows come from the long exact sequence of the pair $(\mathbb{C}^2, M)$ and the vertical arrows are isomorphisms. Under the isomorphism $\widetilde{K}^0(\Sigma M) \cong K^0(\mathbb{C}^2, M)$, the complex $\underline{\mathbb{C}} \xrightarrow{x_1} \underline{\mathbb{C}}$ corresponds to $P-1$, where $P$ is a line bundle on $\Sigma M$ obtained by gluing two trivial line bundles along $M$ using the gluing function $M \to \mathbb{C}^*$, $(x_1, x_2) \mapsto x_1$. The first Chern class of $P$ is easy to compute. If $\gamma$ is a closed loop in $M$ representing a cohomology class in $H_1(M)$, then $\Sigma\gamma$ is a sphere in $H_2(\Sigma M)$ and hence $P|_{\Sigma\gamma}$ is a line bundle on the sphere obtained from gluing two trivial line bundles on the two hemi-spheres along the equator $\gamma$ using the map $\gamma \to \mathbb{C}^*$, $(x_1, x_2) \to x_1$. By definition $\langle c_1(P), \Sigma\gamma\rangle$ coincides with the degree of the map $\gamma \to \mathbb{S}^1$, $(x_1, x_2) \mapsto x_1/|x_1|$, that is,

$$\langle c_1(P), \Sigma\gamma\rangle = \frac{1}{2\pi\mathbf{i}} \int_\gamma \frac{dx_1}{x_1}.$$

In other words, under the suspension isomorphism, $c_1(P)$ coincides with the de Rham cohomology class of the form $\frac{1}{2\pi\mathbf{i}} dx_1/x_1$. Recalling the de Rham model for the relative cohomology group $H^2(\mathbb{C}^2, M)$, we get that $\delta(c_1(P)) = e_N$. Note that $c_1(P) = \mathrm{ch}(P-1)$, so $\mathrm{ch}(\ \underline{\mathbb{C}} \xrightarrow{x_1} \underline{\mathbb{C}}\ ) = e_N$. The vector bundle corresponding to the other complex $\underline{\mathbb{C}} \xrightarrow{\overline{x}_1} \underline{\mathbb{C}}$ is $P^{-1}$, so we get

$$\mathrm{ch}(\iota^* \operatorname{Tr}(E_i^\bullet)) = -2e_N.$$

Hence

$$\mathrm{ch}_\Gamma(E_i^\bullet)_g = \frac{1}{2\pi}\Gamma(1/2)(2\pi\mathbf{i})(-2e_N) = -2\mathbf{i}\Gamma(1/2)e_N,$$

where the index $g$ is to remind us that this is the component corresponding to the fixed point set of $g = (1, 1, -1)$. The computation of $\mathrm{ch}_\Gamma(E^\bullet_N)$ is the same, except that everywhere we have to replace the vector bundle $P$ with $P^2$, so

$$\mathrm{ch}_\Gamma(E^\bullet_N)_g = \frac{1}{2\pi}\Gamma(1/2)(2\pi\mathbf{i})(-4e_N) = -4\mathbf{i}\Gamma(1/2)e_N.$$

Combining our computations we get the following result:

$$\mathrm{ch}_\Gamma(E^\bullet_i) = 2\sum_{a=1}^{N-1}\eta^{-(2a-1)(i-1)}\Gamma(1-m_a/h)e_a - 2\mathbf{i}\Gamma(1/2)e_N$$

and $\mathrm{ch}_\Gamma(E^\bullet_N) = -4\mathbf{i}\Gamma(1/2)e_N$. Comparing with formula (3.8) we get that if we define $\mathrm{mir}(\phi_i) = e_i$ for $1 \le i \le N-1$ and $\mathrm{mir}(\phi_N) = 2e_N$, then the statement of Theorem 3.3 will hold. The vanishing cycle $\alpha_k$ $(1 \le k \le N-1)$ corresponds to the relative $K$-theoretic class of the complex $E^\bullet_k$.

### 3.3.7 $\Gamma$-integral structure for $E_7$-singularity

The computation in this case is similar to the case of $D_N$-singularity. Let us sketch only the main steps and leave the details as an exercise. The goal is to compute $\mathrm{ch}_\Gamma(E^\bullet_l)$ for $1 \le l \le 7$. After a straightforward computation we get that the relative cohomology group $H(\mathrm{Fix}_g(\mathbb{C}^3), \mathrm{Fix}_g(V^T))^{G^T}$ is not zero only in the following two cases: 1) $g = (g_1, g_2, g_3)$ with $g_i \ne 1$ for all $i$ and 2) $g = (1, 1, -1)$. Put $\eta = e^{2\pi\mathbf{i}/9}$ and $\eta_3 = e^{2\pi\mathbf{i}/3}$. For the first case, there are 6 elements, that is, $g = (\eta^{3i-r}, \eta_3^r, -1)$ $(1 \le i \le 3, 1 \le r \le 2)$ and the fixed-point subsets are $\mathrm{Fix}_g(\mathbb{C}^3) = \{0\}$ and $\mathrm{Fix}_g(V^T) = \emptyset$. Therefore, $H(\mathrm{Fix}_g(\mathbb{C}^3), \mathrm{Fix}_g(V^T); \mathbb{C})^{G^T} \cong \mathbb{C}$ is non-trivial only in degree 0 and we denote by $e_{3i-r} \in H(\mathrm{Fix}_g(\mathbb{C}^3), \mathrm{Fix}_g(V^T); \mathbb{C})^{G^T}$ the unit of the cohomology group. For the second case, $\mathrm{Fix}_g(\mathbb{C}^3) = \mathbb{C}^2$ and $\mathrm{Fix}_g(V^T) = M = \{x_1^3x_2 + x_2^3 = 1\}$. The relative cohomology group $H^i(\mathbb{C}^2, M; \mathbb{C})^{G^T} \cong H^{i-1}(M/G^T; \mathbb{C})$ for $i > 0$ and $= 0$ for $i = 0$. Just like in the $D_N$-case, we have $M/G^T = \mathbb{C}^*$, so the relative cohomology is non-zero only in degree 2, i.e., for $i = 2$. Let us denote by $e_7 \in H^2(\mathbb{C}^2, M)^{G^T}$ the cohomology class corresponding to the differential form $(0, -\frac{1}{2\pi i}dx_2/x_2)$.

Suppose that $g = (\eta^{3i-r}, \eta_3^r, -1)$. We have

$$\iota^*\,\mathrm{Tr}(E^\bullet_l)_g = -2(1-\eta_3^{-r})\eta^{-(3i-r)(l-1)}\mathbb{C} \quad \in K^0(\mathrm{Fix}_g(\mathbb{C}^3))$$

and the component of the $\Gamma$-class of $[T\mathbb{C}^3/G^T]$ in $H(\mathrm{Fix}_g(\mathbb{C}^3)/G^T)$ is

$$\Gamma(1-(3i-r)/9)\Gamma(1-r/3)\Gamma(1/2).$$

Therefore

$$\mathrm{ch}_\Gamma(E^\bullet_l)_g = -\frac{1}{\sqrt{\pi}}(1-\eta_3^{-r})\eta^{-(3i-r)(l-1)}\Gamma(1-(3i-r)/9)\Gamma(1-r/3)e_{3i-r}, \quad 1 \le l \le 6.$$

Note that $\mathrm{ch}_\Gamma(E^\bullet_7)_g = 0$.

Suppose now that $g = (1, 1, -1)$. Then we have

$$\iota^*\,\mathrm{Tr}(E^\bullet_l)_g = -[\,\underline{\mathbb{C}} \xrightarrow{x_2} \underline{\mathbb{C}} \longrightarrow 0\,] + [\,0 \longrightarrow \underline{\mathbb{C}} \xrightarrow{x_2} \underline{\mathbb{C}}\,]. \tag{3.15}$$

Under the isomorphism $\widetilde{K}^0(\Sigma M) \cong K^0(\mathbb{C}^2, M)$, the complex $[\,\underline{\mathbb{C}} \xrightarrow{x_2} \underline{\mathbb{C}}\,]$ corresponds to $P-1$, where $P$ is a vector bundle on $\Sigma M$ obtained by gluing two trivial line bundles along $M$ with gluing function $M \to \mathbb{C}^*$, $(x_1, x_2) \mapsto x_2$. Under the suspension isomorphism $H^2(\Sigma M) \cong H^1(M)$, we have that $\mathrm{ch}(P-1) = c_1(P)$ is the cohomology class corresponding to the form $\frac{1}{2\pi\mathbf{i}} dx_2/x_2$. The latter, under the boundary isomorphism $H^1(M) \to H^2(\mathbb{C}^2, M)$ is mapped precisely to $e_7$, that is, $\mathrm{ch}([\,\underline{\mathbb{C}} \xrightarrow{x_2} \underline{\mathbb{C}}\,]) = e_7$. The Chern character of the second complex in (3.15) is $-e_7$, so we get $\mathrm{ch}(\iota^* \operatorname{Tr}(E_l^\bullet))_g = -2e_7$. Hence

$$\mathrm{ch}_\Gamma(E_l^\bullet)_g = \frac{1}{2\pi}\Gamma(1/2)(2\pi\mathbf{i})(-2e_7) = -2\mathbf{i}\sqrt{\pi}e_7.$$

The computation of $\mathrm{ch}_\Gamma(E_7^\bullet)_g$ is the same as above except that we have to replace the bundle $P$ with $P^3$, that is, we get $\mathrm{ch}_\Gamma(E_7^\bullet)_g = -6\mathbf{i}\sqrt{\pi}e_7$.

Collecting the results of our computations we get

$$\mathrm{ch}_\Gamma(E_l^\bullet) = -\sum_{r=1}^{2}\sum_{i=1}^{3} \frac{1}{\sqrt{\pi}} (1-\eta_3^{-r})\eta^{-(3i-r)(l-1)}\Gamma(1-(3i-r)/9)\Gamma(1-r/3)e_{3i-r} - 2\mathbf{i}\sqrt{\pi}e_7$$

for $1 \le l \le 6$ and $\mathrm{ch}_\Gamma(E_7^\bullet) = -6\mathbf{i}\sqrt{\pi}e_7$. Let us compare the above formula with (3.10). Note that $1-\eta_3^{-1} = \sqrt{3}e^{\pi\mathbf{i}/6}$ and $1-\eta_3^{-2} = \sqrt{3}e^{-\pi\mathbf{i}/6}$. We would like to find $k, a \in \mathbb{Z}_3$, such that, $\mathrm{mir} \circ \Psi(\alpha_{k,a}) = \mathrm{ch}_\Gamma(E_l^\bullet)$. Let us write $l-1 = 3m+k$ for $0 \le k \le 2$, $0 \le m \le 1$. Then the above formula will hold if we choose $a = -m$ and define

$$\mathrm{mir}(\phi_i) = e_{3i-1} \quad (1 \le i \le 3), \quad \mathrm{mir}(\phi_{j+3}) = e_{3j-2} \quad (1 \le j \le 3), \quad \mathrm{mir}(\phi_7) = -e_7.$$

Note that $\mathrm{mir} \circ \Psi(\alpha_{k,0} + \alpha_{k,1} + \alpha_{k,2}) = \mathrm{ch}_\Gamma(E_7^\bullet)$. Using these formulas, we get immediately that the maps $\mathrm{mir} \circ \Psi$ and $\mathrm{ch}_\Gamma$ identify the Milnor lattice $H_2(f^{-1}(1);\mathbb{Z})$ with the relative K-ring $K^0(\mathbb{C}^3, V^T)$. This completes the proof of Theorem 3.3.

# Chapter 4

# Chain type singularity

## 4.1 Chain type singularity

Following the notions introduced at the beginning of Section 2.5, we would like to introduce the following proposition.

**Proposition 4.1** (cf. [32]) Any non-degenerate, invertible polynomial can be written as a ThomSebastiani sum (or decoupled sum) $f = f_1 \oplus \cdots \oplus f_p$ of invertible ones (in groups of different variables) $f_\nu, \quad \nu = 1, \ldots p$ of the following types

1. chain type $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}, \quad m \geq 1$, when $m = 1$, it is also Fermat type,
2. loop type $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}x_1, \quad m \geq 2.$

We also assume that $a_i \geq 2$, so that there are no terms of the form $x_ix_j$.

In this thesis, we will work with chain type polynomial or its modified version for even variables case. The modification is due to the upper index of the period vectors being integer.

Let $f_n \in \mathbb{C}[x_1, \ldots, x_{2\lfloor \frac{n}{2} \rfloor + 1}]$ be the following,

$$f_n = \begin{cases} x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}, & n \quad \text{is odd,} \\ x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n} + x_{n+1}^2, & n \quad \text{is even.} \end{cases} \tag{4.1}$$

The hypersurfaces $V_\lambda = \{x \in \mathbb{C}^{2\lfloor \frac{n}{2} \rfloor + 1} \mid f_n(x) = \lambda\}$ for $\lambda \neq 0$ are non-singular and their union has a structure of a smooth fibration on $\mathbb{C} \setminus \{0\}$ known as the *Milnor fibration*. Let us fix a reference point $\lambda = 1$ and consider the middle homology group $H_{2\lfloor \frac{n}{2} \rfloor}(V_1; \mathbb{Z})$, known also as the *Milnor lattice*. One may refer to Section 2.5.3 and Section 2.4.2 to find that our interest is in the period vectors $I_\alpha^{(-\lfloor \frac{n}{2} \rfloor)}(\lambda) \in H_{f_n}$ defined by

$$(I_\alpha^{(-\lfloor \frac{n}{2} \rfloor)}(\lambda), \phi_i) := (2\pi)^{-\lfloor \frac{n}{2} \rfloor} \int_{\alpha_\lambda} \phi_i(x) \frac{\omega}{df},$$

where $\alpha \in H_{2\lfloor \frac{n}{2} \rfloor}(V_1; \mathbb{C})$, $\phi_i(x)$ $(1 \leq i \leq N)$ is a set of polynomials representing a basis of $H_{f_n}$, $\alpha_\lambda \in H_{2\lfloor \frac{n}{2} \rfloor}(V_\lambda; \mathbb{C})$ is obtained from $\alpha$ via a parallel transport along some reference path, and $\frac{\omega}{df}$ is the so-called *Gelfand–Leray* form (see [3]). Alternatively, we can view each period vector as a multivalued analytic function $I_\alpha^{(-\lfloor \frac{n}{2} \rfloor)} : \mathbb{C} \setminus \{0\} \to H_{f_n}$.

For homogeneity reasons, the period vectors have the form

$$I_{\alpha}^{(-\lfloor \frac{n}{2} \rfloor)}(\lambda) = \frac{\lambda^{\theta - 1/2 + \lfloor \frac{n}{2} \rfloor}}{\Gamma(\theta + 1/2 + \lfloor \frac{n}{2} \rfloor)} \Psi(\alpha), \tag{4.2}$$

where $\Psi : H_{2\lfloor \frac{n}{2} \rfloor}(V_1;\mathbb{C}) \to H_{f_n}$ is a linear isomorphism. Our goal is to compute the image of the Milnor lattice $H_{2\lfloor \frac{n}{2} \rfloor}(V_1;\mathbb{Z})$ via the map $\Psi$. The solution to this problem is given in Section 4.2. Explicit formulas for the image of the Milnor lattice via the map $\Psi$ are given in Proposition 4.14. The main feature of our answer is that it involves various $\Gamma$-constants and roots of unity using the basis of middle homology constructed by Otani-Takahashi [41]. We give the Seifert form of the basis as well.

### 4.1.1 K-theoretic interpretation of the Milnor lattice

The modified chain type polynomial $f_n$ corresponds to a $(2\lfloor \frac{n}{2} \rfloor + 1) \times (2\lfloor \frac{n}{2} \rfloor + 1)$ matrix $A = (a_{ij})_{1 \le i,j \le 2\lfloor \frac{n}{2} \rfloor + 1}$ with non-negative integer coefficients, such that,

$$f_n(x_1, \dots, x_{2\lfloor \frac{n}{2} \rfloor + 1}) = \sum_{i=1}^{2\lfloor \frac{n}{2} \rfloor + 1} \prod_{j=1}^{2\lfloor \frac{n}{2} \rfloor + 1} x_j^{a_{ij}},$$

namely, when $n$ is odd, $A = (a_{ij})_{1 \le i,j \le n} = (a_i \delta_{i,j} + \delta_{i+1,j})_{1 \le i,j \le n}$.

when $n$ is even,

$$A = (a_{ij})_{1 \le i,j \le n+1} = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & \vdots \\ 0 & \cdots & \cdots & 0 & a_n & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 2 \end{pmatrix}.$$

Following Fan–Jarvis–Ruan (see [16]) we consider also the Berglund–Hübsch dual polynomial

$$f_n^T(x_1, \dots, x_{2\lfloor \frac{n}{2} \rfloor + 1}) = \sum_{i=1}^{2\lfloor \frac{n}{2} \rfloor + 1} \prod_{j=1}^{2\lfloor \frac{n}{2} \rfloor + 1} x_j^{a_{ji}},$$

Let $G_{f_n^T}$ be the maximal group of diagonal symmetries of $f_n^T$, that is,

$$G_{f_n^T} := \left\{ t \in (\mathbb{C}^*)^{2\lfloor \frac{n}{2} \rfloor + 1} \;\middle|\; \prod_{j=1}^{2\lfloor \frac{n}{2} \rfloor + 1} t_j^{a_{ji}} = 1 \quad \forall i \right\}.$$

Finally, let $V_{f_n^T=1} = \{x \in \mathbb{C}^{2\lfloor \frac{n}{2} \rfloor + 1} \mid f_n^T(x) = 1\}$. Our main interest is in the topological relative K-theoretic orbifold group

$$K^0_{\mathrm{orb}}([\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor + 1}/G_{f_n^T}], [V_{f_n^T=1}/G_{f_n^T}]) := K^0_{G_{f_n^T}}(\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor + 1}, V_{f_n^T=1}).$$

In general, there is no satisfactory definition of K-theory for non-compact spaces. However, in our case the pair $(\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}, V_{f_n^T=1})$ is $G_{f_n^T}$-equivariantly homotopic to a pair of finite $CW$ complexes, so we may think of $(\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}, V_{f_n^T=1})$ as a $G_{f_n^T}$-equivariant pair of finite $CW$-complexes. We refer to [46] for some background on equivariant topological K-theory.

Motivated by Iritani's Γ-integral structure in quantum cohomology (see [27]), we will now construct a linear map

$$\mathrm{ch}_\Gamma : \ K^0_{\mathrm{orb}}([\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}/G_{f_n^T}],[V_{f_n^T=1}/G_{f_n^T}]) \otimes \mathbb{C} \longrightarrow H_{\mathrm{orb}}([\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}/G_{f_n^T}],[V_{f_n^T=1}/G_{f_n^T}];\mathbb{C}) \ , \tag{4.3}$$

which is a certain Γ-class modification of the orbifold Chern character map. For a $G_{f_n^T}$-equivariant space $X$ and $g \in G_{f_n^T}$, let us denote by $\mathrm{Fix}_g(X) := \{x \in X \mid gx = x\}$ the set of fixed points. The elements in the relative $K$-group will be identified with isomorphism classes $[E \to F]$ of two-term complexes $E \xrightarrow{d} F$ of $G_{f_n^T}$-equivariant vector bundles, such that, the differential $d$ is a morphism of $G_{f_n^T}$-equivariant vector bundles and $d|_{V_{f_n^T=1}} : E|_{V_{f_n^T=1}} \to F|_{V_{f_n^T=1}}$ is an isomorphism. Note that for $g \in G_{f_n^T}$, the restriction of a vector bundle $E|_{\mathrm{Fix}_g(\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1})}$ decomposes as a direct sum of eigen-subbundles $E_\zeta$ and that the restriction to $\mathrm{Fix}_g(\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1})$ of every two term complex $E \xrightarrow{d} F$ decomposes as a direct sum of two term subcomplexes $E_\zeta \xrightarrow{d_\zeta} F_\zeta$, where $d_\zeta = d|_{E_\zeta}$. We have the following well known decomposition (e.g. see [8], Theorem 2):

$$\mathrm{Tr} : \ K^0_{G_{f_n^T}}(\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}, V_{f_n^T=1}) \otimes \mathbb{C} \xrightarrow{\cong} \bigoplus_{g \in G_{f_n^T}} \left[K^0(\mathrm{Fix}_g(\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}), \mathrm{Fix}_g(V_{f_n^T=1})) \otimes \mathbb{C}\right]^{G_{f_n^T}} \ ,$$

where $[\,]^{G_{f_n^T}}$ denotes the $G_{f_n^T}$-invariant part and the morphism Tr is defined by

$$\mathrm{Tr}([E \to F]) = \bigoplus_{g \in G_{f_n^T}} \bigoplus_{\zeta \in \mathbb{C}^*} \zeta [E_\zeta \to F_\zeta].$$

**Remark 4.2** The above decomposition is proved in [8] in the case of absolute K-theory. However, using the long exact sequence of a pair, it is straightforward to extend the result to relative K-theory as well.

The standard Chern character map gives an isomorphism

$$\mathrm{ch} : \ K^0(\mathrm{Fix}_g(\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}), \mathrm{Fix}_g(V_{f_n^T=1})) \otimes \mathbb{C} \longrightarrow H^{\mathrm{ev}}(\mathrm{Fix}_g(\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}), \mathrm{Fix}_g(V_{f_n^T=1});\mathbb{C}) \ .$$

Finally, if $G$ is a finite group acting on a smooth manifold $M$, such that the quotient groupoid $[M/G]$ is an effective orbifold, then $H^*(M/G;\mathbb{C}) \cong [H^*(M;\mathbb{C})]^G$. Indeed, for a finite group $G$ the operation taking $G$-invariants is an exact functor from the category of $G$-vector spaces to the category of vector spaces. Therefore

$$H^i(M/G;\mathbb{C}) \cong H^i([\Gamma(M,\mathcal{A}^*_M)]^G) = [H^i(M,\mathcal{A}^*_M)]^G \cong [H^i(M;\mathbb{C})]^G,$$

where $\mathcal{A}^*_M$ is the sheaf of smooth differential forms on $M$ with complex coefficients, the first isomorphism is Satake's de Rham theorem for orbifolds (see [45]), and the last one is the de Rham's theorem for the manifold $M$. Using the long exact sequence of a pair, we get also that $H^i(M/G, N/G;\mathbb{C}) \cong [H^i(M,N;\mathbb{C})]^G$ for any $G$-invariant submanifold

$N \subset M$. On the other hand, by definition,

$$H^*_{\mathrm{orb}}([\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}/G_{f_n^T}],[V_{f_n^T=1}/G_{f_n^T}];k) = \bigoplus_{g \in G_{f_n^T}} H^*(\mathrm{Fix}_g(\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1})/G_{f_n^T}, \mathrm{Fix}_g(V_{f_n^T=1})/G_{f_n^T};k),$$
$$k = \mathbb{Q},\mathbb{R},\mathbb{C}.$$

Therefore, the composition $\widetilde{\mathrm{ch}} := \mathrm{ch} \circ \mathrm{Tr}$ defines a ring homomorphism

$$\widetilde{\mathrm{ch}}: \ K^0_{\mathrm{orb}}([\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}/G_{f_n^T}],[V_{f_n^T=1}/G_{f_n^T}]) \otimes \mathbb{C} \longrightarrow H^{\mathrm{ev}}_{\mathrm{orb}}([\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}/G_{f_n^T}],[V_{f_n^T=1}/G_{f_n^T}];\mathbb{C})$$

which is the orbifold version of the Chern character map. Clearly $\widetilde{\mathrm{ch}}$ is an isomorphism over $\mathbb{C}$.

**Remark 4.3** Orbifold cohomology $H^*_{\mathrm{orb}}$ has two natural gradings – standard topological degree grading coming from the topological space underlying the orbit space and Chen–Ruan grading. In this paper we work with the topological grading and the topological cup product.

Let us recall also the definition of the $\Gamma$-class. If $E \in K^0_{\mathrm{orb}}([\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}/G_{f_n^T}]) := K^0_{G_{f_n^T}}(\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1})$ is an orbifold vector bundle and $\mathrm{Tr}(E) = \sum_g \sum_\zeta \zeta E_\zeta$, then each eigenvalue $\zeta = e^{2\pi \mathbf{i} \alpha}$, where $0 \le \alpha < 1$ is a rational number and we define

$$\widehat{\Gamma}(E) = \sum_g \prod_{\zeta = e^{2\pi\mathbf{i}\alpha}} \prod_{i=1}^{\mathrm{rk}(E_\zeta)} \Gamma(1 - \alpha + \delta_{\zeta,i}) \quad \in \quad H^{\mathrm{ev}}_{\mathrm{orb}}([\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}/G_{f_n^T}]),$$

where $\delta_{\zeta,i}$ ($1 \le i \le \mathrm{rk}(E_\zeta)$) are the Chern roots of the vector bundle $E_\zeta$. If $E = [T\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}/G_{f_n^T}]$ is the orbifold tangent bundle, then the $\Gamma$-class is denoted by $\widehat{\Gamma}([\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}/G_{f_n^T}])$. The map (**??**) is defined by the following formula:

$$\mathrm{ch}_\Gamma([E \to F]) := \frac{1}{(2\pi)^{\lfloor \frac{n}{2} \rfloor}} \widehat{\Gamma}([\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}/G_{f_n^T}]) \cup (2\pi\mathbf{i})^{\deg_{\mathbb{C}}} \iota^* \widetilde{\mathrm{ch}}([E \to F]),$$

where $\deg_{\mathbb{C}}(\phi) = i\phi$ for $\phi \in H^{2i}_{\mathrm{orb}}([\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}/G_{f_n^T}],[V_{f_n^T=1}/G_{f_n^T}];\mathbb{C})$ and $\iota^*$ is an involution in orbifold cohomology that exchanges the direct summands corresponding to $g$ and $g^{-1}$. Note that the definition of $\iota^*$ makes sense because $\mathrm{Fix}_g = \mathrm{Fix}_{g^{-1}}$.

**Conjecture 4.4** There exists a linear isomorphism

$$\mathrm{mir}: \ H_{f_n} \longrightarrow H^*_{\mathrm{orb}}([\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}/G_{f_n^T}],[V_{f_n^T=1}/G_{f_n^T}];\mathbb{C}) \ ,$$

such that, the map

$$\mathrm{mir}^{-1} \circ \mathrm{ch}_\Gamma: \ K^0_{\mathrm{orb}}([\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}/G_{f_n^T}],[V_{f_n^T=1}/G_{f_n^T}]) \xrightarrow{\cong} \Psi(H_{2\lfloor \frac{n}{2} \rfloor}(f_n^{-1}(1);\mathbb{Z}))$$

is an isomorphism of Abelian groups.

Unfortunately we do not have a conceptual definition of the map mir. Our definition is on a case by case basis. We expect that $H^*_{\mathrm{orb}}([\mathbb{C}^{2\lfloor \frac{n}{2} \rfloor+1}/G_{f_n^T}],[V_{f_n^T=1}/G_{f_n^T}];\mathbb{C})$ has a natural identification with the state space of FJRW-theory under which mir is identified with the mirror map of Fan–Jarvis–Ruan (see [16]).

## 4.2 Period map image of the Milnor lattice

Let us start with formulating the maximal group of diagonal symmetries, $\mathbb{C}$-basis of Milnor ring, etc., of chain type invertible polynomial $f_n$ defined in (4.2), explicitly.

We denote by $G_{f_n}$ the maximal group of diagonal symmetries of $f_n$, that is,

$$G_{f_n} := \left\{ t \in (\mathbb{C}^*)^{2\lfloor \frac{n}{2} \rfloor + 1} \,\middle|\, \prod_{j=1}^{2\lfloor \frac{n}{2} \rfloor + 1} t_j^{a_{ij}} = 1 \quad \forall i \right\}.$$

It is not hard to find the following proposition

**Proposition 4.5** Each element $g \in G_{f_n}$ has a unique expression of the form $g = (\mathbf{e}[\alpha_1], \dots, \mathbf{e}[\alpha_n])$ with $0 \leqslant \alpha_i < 1$, where $\mathbf{e}[\alpha] := \exp(2\pi\sqrt{-1}\alpha)$.

More concretely, set $d_0 := 1$ and $d_i := a_1 \cdots a_i$ for $i = 1, \dots, n$. When $n$ is odd, then Then $G_{f_n}$ is a cyclic group whose generator is

$$\left( \mathbf{e}\left[(-1)^n \frac{1}{d_n}\right], \dots, \mathbf{e}\left[(-1)^{n+i-1} \frac{d_{i-1}}{d_n}\right], \dots, \mathbf{e}\left[-\frac{d_{n-1}}{d_n}\right] \right).$$

While, when $n$ is even, then $G_{f_n}$ is the direct sum of two cyclic groups whose generators are

$$\left( \mathbf{e}\left[(-1)^n \frac{1}{d_n}\right], \dots, \mathbf{e}\left[(-1)^{n+i-1} \frac{d_{i-1}}{d_n}\right], \dots, \mathbf{e}\left[-\frac{d_{n-1}}{d_n}\right], 1 \right) \quad \text{and} \quad (1, \dots, 1, -1).$$

**Definition 4.6** For each non-negative integer $n$, define sets $B'_{f_n|_s}$, $(s = 0, 1, 2, \dots, \lfloor \frac{n}{2} \rfloor)$, $B_{f_n}$ of monomials in $\mathbb{C}[x_1, \dots, x_n]$ as follows: Let $B'_{f_0|_0} := \{1\}$, $B'_{f_1|_0} = \{x_1^{k_1} | 0 \le k_1 \le a_1 - 2\}$ and if $n \ge 2$,

$$B'_{f_n|_s} := \left\{ x_1^{a_1-1} x_3^{a_3-1} \cdots x_{2s-1}^{a_{2s-1}-1} x_{2s+1}^{k_{2s+1}} x_{2s+2}^{k_{2s+2}} \cdots x_n^{k_n} \,\middle|\, \begin{cases} 0 \le k_{2s+1} \le a_{2s+1} - 2, \\ 0 \le k_{2s+i} \le a_{2s+i} - 1 \quad (i = 2, \cdots, n - 2s) \end{cases} \right\}.$$

Let

$$B_{f_n} := \bigcup_{s=0}^{\lfloor \frac{n}{2} \rfloor} B'_{f_n|_s}.$$

And we can identify the above set of monomials with the following set,

$$\widetilde{B}'_{f_n|_s} := \left\{ \mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n \,\middle|\, x^{\mathbf{k}} \in B'_{f_n|_s} \right\}.$$

Similarly,

$$\widetilde{B}_{f_n} := \bigcup_{s=0}^{\lfloor \frac{n}{2} \rfloor} \widetilde{B}'_{f_n|_s}.$$

**Proposition 4.7** The set $B_{f_n}$ defines a $\mathbb{C}$-basis of the Milnor ring $H_{f_n}$. Namely, we have $H_{f_n} = \langle [\phi^{(n)}(\mathbf{x})] | \phi^{(n)}(\mathbf{x}) \in B_{f_n} \rangle_{\mathbb{C}}$.

**Remark 4.8** For $n$ even, since $\frac{\partial f_n}{\partial x_{n+1}} = 2x_{n+1}$, we merge the cases where $n$ is even and odd.

Define a positive integer $\mu_n$ by $\mu_n := \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \frac{d_n}{d_{2s}} - \frac{d_n}{d_{2s+1}}$ (where $\frac{d_n}{d_{n+1}} = 0$) $= \sum_{i=0}^{n} (-1)^i \frac{d_n}{d_i}$.

**Corollary 4.9** The Milnor number $\mu_{f_n} = \dim_{\mathbb{C}} H_{f_n}$ is given by $\mu_n$.

Denote by $\bar{A}$ the invertible matrix associated to $\begin{cases} f_n & n \text{ is odd} \\ f_n - x_{n+1}^2 & n \text{ is even} \end{cases} \in \mathbb{C}[x_1, \dots, x_n]$, which is given by

$$\bar{A} = (a_{ij})_{1\le i,j\le n} = (a_i\delta_{i,j} + \delta_{i+1,j})_{1\le i,j\le n} = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & a_n \end{pmatrix}.$$

**Definition 4.10** For each $\mathbf{k} = (k_1, \dots, k_n)$, define rational numbers $\omega_{\mathbf{k},1}^{(n)}, \dots, \omega_{\mathbf{k},n}^{(n)}$ by

$$(\omega_{\mathbf{k},1}^{(n)}, \dots, \omega_{\mathbf{k},n}^{(n)}) := (k_1+1, \dots, k_n+1)\bar{A}^{-\mathrm{T}}.$$

In particular, $\omega_{\mathbf{0},i}^{(n)}$ are nothing but the weight of $x_i$. Therefore, the central charge or conformal dimension is $\sum_{i=1}^{n}(1 - 2\omega_{\mathbf{0},i}^{(n)})$

Consider the following number set,

$$I_i := \begin{cases} \{a \in d_i\mathbb{Z}/d_n\mathbb{Z} | d_{i+1} \nmid a\} & i = 0, 1, \dots, n-1 \\ \{0\} := d_n\mathbb{Z}/d_n\mathbb{Z} & i = n \end{cases}$$

For $a \in I_k$ and $i = 1, 2, \dots, n$, set

$$\omega_{a,i}^{(n)} = \frac{(-1)^{n-i}a \mod d_i}{d_i},$$

apparently, $\omega_{d_n - a,i}^{(n)} = \begin{cases} 0 = \omega_{a,i}^{(n)} & i = 1, \dots, k \\ 1 - \omega_{a,i}^{(n)} & i = k+1, \dots, n \end{cases}.$

**Proposition 4.11** The map

$$\psi : \widetilde{B}'_{f_n|_s} := \left\{ \mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n \middle| x^{\mathbf{k}} \in B'_{f_n|_s} \right\} \xrightarrow{\cong} I_{2s}$$

defined by

$$\psi(\mathbf{k}) = \psi(k_1, \dots, k_n) := \sum_{l=1}^{n} (-1)^{n-l} d_{l-1}(k_l + 1)$$

is a bijection of sets of $\frac{d_n}{d_{2s}} - \frac{d_n}{d_{2s+1}}$ elements and satisfies $\omega_{\psi(\mathbf{k}),i}^{(n)} = \begin{cases} 0 & i = 1, \dots, 2s \\ \omega_{\mathbf{k},i}^{(n)} & i = 2s+1, \dots, n \end{cases}.$

**Proposition 4.12** For each $\mathbf{k} \in \widetilde{B}'_{f_n|_s}$, set its dual $\mathbf{k}^* \in \widetilde{B}'_{f_n|_s}$ such that $(k_{2s+1}+k^*_{2s+1},\dots,k_n+k^*_n) = (a_{2s+1}-2, a_{2s+2}-1, a_{2s+3}-1,\dots,a_n-1)$, which is equivalent to $\psi(\mathbf{k})+\psi(\mathbf{k}^*) = d_n$ and therefore $\omega^{(n)}_{\mathbf{k}^*,i} = \begin{cases} 1 & i=1,3,5,\dots,2s-1 \\ 0 & i=2,4,6,\dots,2s \\ 1-\omega^{(n)}_{\mathbf{k},i} & i=2s+1,\dots,n \end{cases}$

Furthermore, we have recursion formula $\omega^{(n)}_{\mathbf{k},i-1} = k_i+1-a_i\omega^{(n)}_{\mathbf{k},i}, \quad i=2s+1,\dots,n$, which will be useful in the calculation afterwards.

**Proposition 4.13** For $\mathbf{k}=(k_1,\dots,k_n), \mathbf{l}=(l_1,\dots,l_n)$ such that $\mathbf{x}^{\mathbf{k}} := x_1^{k_1}\cdots x_n^{k_n}, \mathbf{x}^{\mathbf{l}} := x_1^{l_1}\cdots x_n^{l_n} \in B_{f_n}$, we have the following relation for the residue pairing.

$$\eta_{\mathbf{kl}} := (\mathbf{x}^{\mathbf{k}},\mathbf{x}^{\mathbf{l}}) := \mathrm{Res}_{x=0}\frac{\mathbf{x}^{\mathbf{k}+\mathbf{l}}d\mathbf{x}}{2^{\frac{1+(-1)^n}{2}}\prod_{i=1}^n\frac{\partial f_n}{\partial x_i}} = \begin{cases} 2^{-\frac{1+(-1)^n}{2}}\frac{(-1)^s}{a_1a_3\cdots a_{2s-1}}\cdot\frac{d_{2s}}{d_n} & \mathbf{k},\mathbf{l}\in\widetilde{B}'_{f_n|_s} \text{ and } \mathbf{l}=\mathbf{k}^* \\ 0 & \text{otherwise} \end{cases}.$$

**Proof** The result can be also calculated by (2.22) straightforwardly.

We denote its inverse by $\eta^{\mathbf{kl}}$, i.e., $\sum_{\mathbf{l}\in B_{f_n}}\eta^{\mathbf{kl}}\eta_{\mathbf{lj}} = \delta_{\mathbf{kj}}$, then

$$\eta^{\mathbf{kl}} = \begin{cases} 2^{\frac{1+(-1)^n}{2}}(-1)^s\cdot a_1a_3\cdots a_{2s-1}\cdot\frac{d_n}{d_{2s}} & \mathbf{k},\mathbf{l}\in\widetilde{B}'_{f_n|_s} \text{ and } \mathbf{l}=\mathbf{k}^* \\ 0 & \text{otherwise} \end{cases}$$

We construct vanishing cycle $\alpha_1^{(n)} \in H_{2\lfloor\frac{n}{2}\rfloor}(f_n^{-1}(1);\mathbb{Z})$ inductively according to [41] (section 5, with opposite orientation and with modification for $n$ being even) and generate the rest vanishing cycles $\alpha_j^{(n)} \in H_{2\lfloor\frac{n}{2}\rfloor}(f_n^{-1}(1);\mathbb{Z}), j=2,\dots,\mu_n$ by the following $G_{f_n}$-action

$$(x_1,\dots,x_i,\dots,x_n)\mapsto\left(\mathbf{e}\left[(-1)^n\frac{j-1}{d_n}\right]\cdot x_1,\dots,\mathbf{e}\left[(-1)^{n+i-1}\frac{(j-1)d_{i-1}}{d_n}\right]\cdot x_i,\dots,\mathbf{e}\left[-\frac{(j-1)d_{n-1}}{d_n}\right]\cdot x_n\right).$$

when $n$ is odd. While, when $n$ is even, the action on the first $n$ components are the same as that of the case $n$ is odd, and the last component remains.

If $n$ is odd, the image of the vanishing cycles $\alpha_j^{(n)} \in H_{2\lfloor\frac{n}{2}\rfloor}(f_n^{-1}(1);\mathbb{Z}), j=1,\dots,\mu_n$ via the map $\Psi$ is

$$\begin{aligned}\Psi(\alpha_j^{(n)}) &= \sum_{\mathbf{k},\mathbf{l}\in\widetilde{B}_{f_n}}(\Psi(\alpha_j^{(n)}),\mathbf{x}^{\mathbf{l}})\eta^{\mathbf{lk}}\mathbf{x}^{\mathbf{k}} = \sum_{\mathbf{k}\in B_{f_n}}(\Psi(\alpha_j^{(n)}),\mathbf{x}^{\mathbf{k}^*})\eta^{\mathbf{k}^*\mathbf{k}}\mathbf{x}^{\mathbf{k}} \\ &= \sum_{\mathbf{k}\in\widetilde{B}_{f_n}}\eta^{\mathbf{k}^*\mathbf{k}}\mathbf{x}^{\mathbf{k}}\int_0^\infty e^{-\lambda}(I^{(-\frac{n-1}{2})}_{\alpha_j^{(n)}}(\lambda),\mathbf{x}^{\mathbf{k}^*})d\lambda \\ &= (2\pi)^{-\frac{n-1}{2}}\sum_{\mathbf{k}\in\widetilde{B}_{f_n}}\eta^{\mathbf{k}^*\mathbf{k}}\mathbf{x}^{\mathbf{k}}\int_0^\infty e^{-\lambda}\int_{(\alpha_\lambda)_j^{(n)}}\mathbf{x}^{\mathbf{k}^*}\frac{d\mathbf{x}}{df_n}d\lambda \\ &= (2\pi)^{-\frac{n-1}{2}}\sum_{\mathbf{k}\in\widetilde{B}_{f_n}}\eta^{\mathbf{k}^*\mathbf{k}}\mathbf{x}^{\mathbf{k}}\int_{\Gamma_j^{(n)}}e^{-f_n}\mathbf{x}^{\mathbf{k}^*}d\mathbf{x},\end{aligned}$$

where $\Gamma_j^{(n)}$ is $\alpha_j^{(n)}$'s corresponding class in $H_n(\mathbb{C}^n, \mathrm{Re}(f_n) \gg 0; \mathbb{Z})$. According to section 5 of [41], when $\mathbf{x}^{\mathbf{k}^*} \in B'_{f_n|_s}$,

$$\int_{\Gamma_j^{(n)}} e^{-f_n} \mathbf{x}^{\mathbf{k}^*} d\mathbf{x} = \mathbf{e}\left[-(j-1)\omega_{\mathbf{k},n}^{(n)}\right] \int_{\Gamma_1^{(n)}} e^{-f_n} \mathbf{x}^{\mathbf{k}^*} d\mathbf{x}$$

$$=(\eta^{\mathbf{k}^*\mathbf{k}})^{-1} \mathbf{e}\left[-(j-1)\omega_{\mathbf{k},n}^{(n)}\right] \cdot (2\pi\mathbf{i})^s \prod_{j=s}^{\lceil \frac{n}{2} \rceil -1} \left(\mathbf{e}\left[\omega_{\mathbf{k},2j+1}^{(n)}\right] - 1\right) \cdot \prod_{k=2s+1}^{n} \Gamma\left(\omega_{\mathbf{k}^*,k}^{(n)}\right).$$

If $n$ is even, let us calculate $(I_{\alpha_j^{(n)}}^{(-\lfloor \frac{n}{2} \rfloor)}(\lambda), \mathbf{x}^{\mathbf{k}^*})$ first. Recall (3.3)

$$\begin{aligned}
(I_{\alpha_j^{(n)}}^{(-\lfloor \frac{n}{2} \rfloor)}(\lambda), \mathbf{x}^{\mathbf{k}^*}) &= (2\pi)^{-\lfloor \frac{n}{2} \rfloor} \int_{(\alpha_\lambda)_j^{(n)}} \mathbf{x}^{\mathbf{k}^*} \frac{dx_1 \ldots dx_{n+1}}{df_n} \\
&= \frac{2}{(2\pi)^{\lfloor \frac{n}{2} \rfloor}} \partial_\lambda \int_0^\lambda (\lambda-\mu)^{\frac{1}{2}} \int_{A_\mu} \mathbf{x}^{\mathbf{k}^*} \frac{dx_1 \ldots dx_n}{d(f_n - x_{n+1}^2)} d\mu \\
&= \frac{2}{(2\pi)^{\lfloor \frac{n}{2} \rfloor}} \partial_\lambda \int_0^\lambda (\lambda-\mu)^{\frac{1}{2}} \frac{\mu^{\theta+\frac{n}{2}-1}}{\Gamma(\theta+\frac{n}{2})} \int_{\Gamma_j^{(n)}} e^{-(f_n - x_{n+1}^2)} \mathbf{x}^{\mathbf{k}^*} d\mathbf{x} d\mu \\
&= \frac{\sqrt{\pi}}{(2\pi)^{\lfloor \frac{n}{2} \rfloor}} \frac{\lambda^{\theta+\frac{n}{2}-\frac{1}{2}}}{\Gamma(\theta+\frac{n}{2}+\frac{1}{2})} \int_{\Gamma_j^{(n)}} e^{-(f_n - x_{n+1}^2)} \mathbf{x}^{\mathbf{k}^*} d\mathbf{x},
\end{aligned}$$

where $\Gamma_j^{(n)}$ is $\alpha_j^{(n)}$'s corresponding class in $H_n(\mathbb{C}^n, \mathrm{Re}(f_n) \gg 0; \mathbb{Z})$, up to a suspension. Again, according to section 5 of [41], when $\mathbf{x}^{\mathbf{k}^*} \in B'_{f_n|_s}$,

$$\int_{\Gamma_j^{(n)}} e^{-(f_n - x_{n+1}^2)} \mathbf{x}^{\mathbf{k}^*} d\mathbf{x} = \mathbf{e}\left[-(j-1)\omega_{\mathbf{k},n}^{(n)}\right] \int_{\Gamma_1^{(n)}} e^{-(f_n - x_{n+1}^2)} \mathbf{x}^{\mathbf{k}^*} d\mathbf{x}$$

$$=2(\eta^{\mathbf{k}^*\mathbf{k}})^{-1} \mathbf{e}\left[-(j-1)\omega_{\mathbf{k},n}^{(n)}\right] \cdot (2\pi\mathbf{i})^s \prod_{j=s}^{\lceil \frac{n}{2} \rceil -1} \left(\mathbf{e}\left[\omega_{\mathbf{k},2j+1}^{(n)}\right] - 1\right) \cdot \prod_{k=2s+1}^{n} \Gamma\left(\omega_{\mathbf{k}^*,k}^{(n)}\right),$$

where we need to mulitply by 2 since the residue pairing changes. Combining these two cases, we have the following

**Proposition 4.14** The image of the vanishing cycles $\alpha_j^{(n)} \in H_{2\lfloor \frac{n}{2} \rfloor}(f_n^{-1}(1); \mathbb{Z}), j = 1, \ldots, \mu_n$ via the map $\Psi$ is

$$\Psi(\alpha_j^{(n)}) = \frac{(2\sqrt{\pi})^{\frac{1+(-1)^n}{2}}}{(2\pi)^{\lfloor \frac{n}{2} \rfloor}} \sum_{\mathbf{k} \in \widetilde{B}_{f_n}} \mathbf{x}^{\mathbf{k}} \mathbf{e}\left[-(j-1)\omega_{\mathbf{k},n}^{(n)}\right] \cdot (2\pi\mathbf{i})^s \prod_{j=s}^{\lceil \frac{n}{2} \rceil -1} \left(\mathbf{e}\left[\omega_{\mathbf{k},2j+1}^{(n)}\right] - 1\right) \cdot \prod_{k=2s+1}^{n} \Gamma\left(\omega_{\mathbf{k}^*,k}^{(n)}\right)$$

Let us calculate Seifert form $\langle \alpha_a, \alpha_b \rangle := \frac{1}{2\pi}(e^{\pi\sqrt{-1}\theta}\Psi(\alpha_a), \Psi(\alpha_b)), \theta(\mathbf{x}^{\mathbf{k}}) = \left(\frac{n}{2} - \sum_{i=1}^{n} \omega_{\mathbf{k},i}^{(n)}\right)\mathbf{x}^{\mathbf{k}} = \left(\frac{n}{2} - s - \sum_{i=2s+1}^{n} \omega_{\mathbf{k},i}^{(n)}\right)\mathbf{x}^{\mathbf{k}}$ to show that the vanishing cycles $\alpha_j^{(n)} \in H_{2\lfloor\frac{n}{2}\rfloor}(f_n^{-1}(1);\mathbb{Z}), j = 1, \ldots, \mu_n$ give us a $\mathbb{Z}$-basis of Milnor lattice.

$$
\begin{aligned}
&\langle \alpha_a, \alpha_b \rangle \\
=&\frac{(4\pi)^{\frac{1+(-1)^n}{2}}}{(2\pi)^{2\lfloor\frac{n}{2}\rfloor+1}} \sum_{s=0}^{\lfloor\frac{n}{2}\rfloor} \sum_{\mathbf{k}\in\widetilde{B}'_{f_n|s}} \eta_{\mathbf{k}^*\mathbf{k}} \cdot \mathbf{e}\left[\frac{n}{4} - \frac{s}{2} + (b-a)\omega_{\mathbf{k},n}^{(n)} - \frac{1}{2}\sum_{i=2s+1}^{n} \omega_{\mathbf{k},i}^{(n)}\right] \cdot \\
&\cdot (2\pi\mathbf{i})^{2s} \prod_{j=s}^{\lceil\frac{n}{2}\rceil-1} \left(2 - \mathbf{e}\left[\omega_{\mathbf{k},2j+1}^{(n)}\right] - \mathbf{e}\left[-\omega_{\mathbf{k},2j+1}^{(n)}\right]\right) \cdot \prod_{k=2s+1}^{n} \Gamma\left(1-\omega_{\mathbf{k},k}^{(n)}\right)\Gamma\left(\omega_{\mathbf{k},k}^{(n)}\right) \\
=&2\mathbf{i}^n \sum_{s=0}^{\lfloor\frac{n}{2}\rfloor} \sum_{\mathbf{k}\in\widetilde{B}'_{f_n|s}} \eta_{\mathbf{k}^*\mathbf{k}} \cdot \mathbf{e}\left[(b-a)\omega_{\mathbf{k},n}^{(n)} - \frac{1}{2}\sum_{i=2s+1}^{n} \omega_{\mathbf{k},i}^{(n)}\right] \cdot \\
&\cdot \prod_{j=s}^{\lceil\frac{n}{2}\rceil-1} \sin^2(\pi\omega_{\mathbf{k},2j+1}^{(n)}) \cdot \prod_{k=2s+1}^{n} \frac{1}{\sin(\pi\omega_{\mathbf{k},k}^{(n)})}
\end{aligned}
$$

For clarity, we consider the following factor in the summation, where $l \in \{s, s+1, s+2, \ldots, \lceil\frac{n}{2}\rceil - 2\}$,

$$
\begin{aligned}
&\mathbf{e}\left[-\frac{1}{2}\omega_{\mathbf{k},2l+1}^{(n)} - \frac{1}{2}\omega_{\mathbf{k},2l+2}^{(n)}\right] \cdot \sin(\pi\omega_{\mathbf{k},2l+1}^{(n)}) \cdot \frac{1}{\sin(\pi\omega_{\mathbf{k},2l+2}^{(n)})} \\
=&\mathbf{e}\left[-\frac{1}{2}\omega_{\mathbf{k},2l+1}^{(n)} - \frac{1}{2}\omega_{\mathbf{k},2l+2}^{(n)}\right] \cdot \frac{\mathbf{e}\left[\frac{1}{2}\omega_{\mathbf{k},2l+1}^{(n)}\right] - \mathbf{e}\left[-\frac{1}{2}\omega_{\mathbf{k},2l+1}^{(n)}\right]}{2\mathbf{i}} \cdot \frac{2\mathbf{i}}{\mathbf{e}\left[\frac{1}{2}\omega_{\mathbf{k},2l+2}^{(n)}\right] - \mathbf{e}\left[-\frac{1}{2}\omega_{\mathbf{k},2l+2}^{(n)}\right]} \\
=&-\frac{1-\mathbf{e}\left[-\omega_{\mathbf{k},2l+1}^{(n)}\right]}{1-\mathbf{e}\left[\omega_{\mathbf{k},2l+2}^{(n)}\right]} = -\frac{1-\mathbf{e}\left[a_{2l+2}\omega_{\mathbf{k},2l+2}^{(n)} - k_{2l+2} - 1\right]}{1-\mathbf{e}\left[\omega_{\mathbf{k},2l+2}^{(n)}\right]} = -\frac{1-\mathbf{e}\left[a_{2l+2}\omega_{\mathbf{k},2l+2}^{(n)}\right]}{1-\mathbf{e}\left[\omega_{\mathbf{k},2l+2}^{(n)}\right]} \\
=&-\sum_{m=0}^{a_{2l+2}-1} \mathbf{e}\left[m\omega_{\mathbf{k},2l+2}^{(n)}\right].
\end{aligned}
$$

For the last factor containing $(b-a)$ in the summation, when $n$ is even and $2s < n$, under the same procedure, we obtain,

$$
\begin{aligned}
&\mathbf{e}\left[(b-a)\omega_{\mathbf{k},n}^{(n)} - \frac{1}{2}\omega_{\mathbf{k},n-1}^{(n)} - \frac{1}{2}\omega_{\mathbf{k},n}^{(n)}\right] \cdot \sin(\pi\omega_{\mathbf{k},n-1}^{(n)}) \cdot \frac{1}{\sin(\pi\omega_{\mathbf{k},n}^{(n)})} \\
=&-\sum_{m=0}^{a_n-1} \mathbf{e}\left[(b-a+m)\omega_{\mathbf{k},n}^{(n)}\right],
\end{aligned}
$$

when $n = 2s$ we do not need to do anything, since $\omega_{\mathbf{k},n}^{(n)} = 0$ and $\mathbf{e}\left[(b-a)\omega_{\mathbf{k},n}^{(n)}\right] = 1$,
while $n$ is odd, we get

$$\begin{aligned}
&\mathbf{e}\left[(b-a)\omega_{\mathbf{k},n}^{(n)} - \frac{1}{2}\omega_{\mathbf{k},n}^{(n)}\right] \cdot \sin(\pi\omega_{\mathbf{k},n}^{(n)}) \\
=&\mathbf{e}\left[(b-a)\omega_{\mathbf{k},n}^{(n)} - \frac{1}{2}\omega_{\mathbf{k},n}^{(n)}\right] \cdot \frac{\mathbf{e}\left[\frac{1}{2}\omega_{\mathbf{k},n}^{(n)}\right] - \mathbf{e}\left[-\frac{1}{2}\omega_{\mathbf{k},n}^{(n)}\right]}{2\mathbf{i}} \\
=&-\frac{\mathbf{i}}{2}\left(\mathbf{e}\left[(b-a)\omega_{\mathbf{k},n}^{(n)}\right] - \mathbf{e}\left[(b-a-1)\omega_{\mathbf{k},n}^{(n)}\right]\right).
\end{aligned}$$

To simplify the result of Seifert form, we start from decomposing the summation $\sum_{\mathbf{k}\in\widetilde{B}'_{fn|s}}$ into $\sum_{\text{the first } n-1 \text{ components of } \mathbf{k} \text{ ranges in } \widetilde{B}'_{fn|s}}$ and $\sum_{k_n} = \begin{cases} \sum_{k_n=0}^{a_n-1}, & 2s+1<n \\ \sum_{k_n=0}^{a_n-2}, & 2s+1=n \\ k_n=0, & 2s=n \end{cases}$. According to the definition $\omega_{\mathbf{k},i}^{(n)}$, it depends only on the $(2s+1)$-th to $i$-th component of $\mathbf{k}$. Thus, the summation $\sum_{\mathbf{k}\in\widetilde{B}'_{fn|s}}$ could be transformed into the following form,

$$\sum_{\text{the first } n-1 \text{ components of } \mathbf{k} \text{ ranges in } \widetilde{B}'_{fn|s}} \left(\prod_{l=s}^{\lceil\frac{n}{2}\rceil-2} \text{the factors with } l\right) \sum_{k_n} \text{the last factor}$$

Then we exchange the order of summations $\sum_{k_n}$ and $\sum_{m=0}^{a_n-1}$, when $n$ is even and $2s<n$. Hence, when $n=2s$, $\sum_{k_n}$ the last factor $=1$.

When $2s+1<n$ and $n$ is even,

$$\begin{aligned}
\sum_{k_n} \text{the last factor} =& -a_n \sum_{m=0}^{a_n-1} \delta_{m,a-b \bmod a_n} \mathbf{e}\left[\frac{-1}{a_n}(b-a+m)\omega_{\mathbf{k},n-1}^{(n)}\right] \\
=& -a_n \mathbf{e}\left[\left\lfloor\frac{-1}{a_n}(b-a)\right\rfloor \omega_{\mathbf{k},n-1}^{(n)}\right].
\end{aligned}$$

When $2s+1=n$,

$$\sum_{k_n} \text{the last factor} = -\frac{\mathbf{i}}{2}a_n(\delta_{0,b-a \bmod a_n} - \delta_{0,b-a-1 \bmod a_n}).$$

When $2s+1<n$ and $n$ is odd, $\sum_{k_n}$ the last factor

$$= -\frac{\mathbf{i}}{2}a_n\left(\delta_{0,b-a \bmod a_n}\mathbf{e}\left[\frac{-1}{a_n}(b-a)\omega_{\mathbf{k},n-1}^{(n)}\right] - \delta_{0,b-a-1 \bmod a_n}\mathbf{e}\left[\frac{-1}{a_n}(b-a-1)\omega_{\mathbf{k},n-1}^{(n)}\right]\right).$$

The case that $2s+1=n$ can be merged into the last case. Here the result that Kronecker deltas are zero is from the sum of $a_n$(at most)-th root of unity (up to a certain rotation $\mathbf{e}\left[\frac{-1}{a_n}(b-a+m)\omega_{\mathbf{k},n-1}^{(n)}\right]$ around zero for each of them) is zero.

We can do almost the same but relatively simplier decomposing summation procedures for $l = \lceil\frac{n}{2}\rceil - 2, \ldots, s$, which has to be done in this reverse order due to the property of $\omega_{\mathbf{k},i}^{(n)}$. Let us see how the recursion goes for $l =$

$\lceil \frac{n}{2} \rceil - 2, \ldots, s+1$

$$
\begin{aligned}
&-\sum_{k_{2l+1}=0}^{a_{2l+1}-1}\sum_{k_{2l+2}=0}^{a_{2l+2}-1}\sum_{m=0}^{a_{2l+2}-1} \mathbf{e}\left[(m-p_{l+1})\omega^{(n)}_{\mathbf{k},2l+2}\right] \\
=&-\sum_{k_{2l+1}=0}^{a_{2l+1}-1}\sum_{m=0}^{a_{2l+2}-1}\sum_{k_{2l+2}=0}^{a_{2l+2}-1} \mathbf{e}\left[(m-p_{l+1})\omega^{(n)}_{\mathbf{k},2l+2}\right] \\
=&-\sum_{k_{2l+1}=0}^{a_{2l+1}-1}\sum_{m=0}^{a_{2l+2}-1} a_{2l+2}\delta_{m,p_{l+1} \bmod a_{2l+2}} \mathbf{e}\left[\frac{-1}{a_{2l+2}}(m-p_{l+1})\omega^{(n)}_{\mathbf{k},2l+1}\right] \\
=&-a_{2l+2}\sum_{k_{2l+1}=0}^{a_{2l+1}-1} \mathbf{e}\left[q_l\omega^{(n)}_{\mathbf{k},2l+1}\right] \qquad q_l := \left\lfloor \frac{p_{l+1}}{a_{2l+2}} \right\rfloor \\
=&-a_{2l+2}a_{2l+1}\delta_{0,q_l \bmod a_{2l+1}} \mathbf{e}\left[\frac{-q_l}{a_{a_{2l+1}}}\omega^{(n)}_{\mathbf{k},2l}\right] \qquad p_l := \frac{q_l}{a_{a_{2l+1}}}
\end{aligned}
$$

for $l = s$,

$$
\begin{aligned}
&-\sum_{k_{2s+1}=0}^{a_{2s+1}-2}\sum_{k_{2s+2}=0}^{a_{2s+2}-1}\sum_{m=0}^{a_{2s+2}-1} \mathbf{e}\left[(m-p_{s+1})\omega^{(n)}_{\mathbf{k},2s+2}\right] \\
=&-a_{2s+2}\sum_{k_{2s+1}=0}^{a_{2s+1}-2} \mathbf{e}\left[q_s\omega^{(n)}_{\mathbf{k},2s+1}\right] \\
=&-a_{2s+2}\left(a_{2s+1}\delta_{0,q_s \bmod a_{2s+1}} - 1\right) \qquad \text{notice that } \omega^{(n)}_{\mathbf{k},2s} = 0,
\end{aligned}
$$

where $p \in \mathbb{Z}$ is from the previous step, i.e., the $(2l+3)$-th component. And the recursion terminates not only when $l = s$ but also when $\delta_{0,q_l \bmod a_{2l+1}} = 0$,

$$
q_l = \begin{cases}
\left\lfloor \frac{1}{a_{2l+2}}\frac{1}{a_{2l+3}}\left\lfloor \cdots \left\lfloor \frac{a-b}{a_n} \right\rfloor \right\rfloor \right\rfloor & n \text{ is even} \\
\left\lfloor \frac{a-b}{a_n} \right\rfloor & 2l+2 = n \\
\left\lfloor \frac{1}{a_{2l+2}}\frac{1}{a_{2l+3}}\left\lfloor \cdots \left\lfloor \frac{b-a}{a_{n-1}a_n} \right\rfloor \right\rfloor \right\rfloor & n \text{ is odd, for the first term, i.e., } (b-a) \text{ term instead of } (b-a-1) \text{term} \\
b-a & 2l+1 = n \text{ for the first term, i.e., } (b-a) \text{ term instead of } (b-a-1) \text{term}
\end{cases}.
$$

Eventually, we can finish the calculation of Seifert form.

Set $l_0$ to be the biggest $l \in \{0, 1, 2, \ldots, \frac{n}{2} - \frac{3}{4} - \frac{(-1)^n}{4}\}$ such that $\delta_{0,q_l \bmod a_{2l+1}} = 0$. If there is no such $l$, make a convention that $l_0 = -1$.

Consider the summation $\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor}$ in $\langle \alpha_a, \alpha_b \rangle$.

The term with $s < l_0$ vanishes due to $\delta_{0,q_l \bmod a_{2l+1}} = 0$.

We consider the easier case that $n$ is even.

The term with $s = \frac{n}{2}$ is $\frac{(-1)^{\frac{n}{2}}}{a_1a_3\ldots a_{n-1}}$. The term with $s = l_0$ is

$$
-\left(\frac{(-1)^{l_0}}{a_1a_3\cdots a_{2l_0-1}}\cdot\frac{d_{2l_0}}{d_n}\right)\left(\prod_{l=l_0+1}^{\frac{n}{2}-1} a_{2l+1}\right)\left(\prod_{l=l_0}^{\frac{n}{2}-1}(-a_{2l+2})\right) = \frac{(-1)^{\frac{n}{2}-1}}{a_1a_3\ldots a_{2l_0+1}}.
$$

The term with $s = l_0 + 1, \dots, \frac{n}{2} - 1$ is

$$\begin{aligned}
&\sum_{s=l_0+1}^{\frac{n}{2}-1} \left( \frac{(-1)^s}{a_1 a_3 \cdots a_{2s-1}} \cdot \frac{d_{2s}}{d_n} \right) (a_{2s+1} - 1) \left( \prod_{l=s+1}^{\frac{n}{2}-1} a_{2l+1} \right) \left( \prod_{l=s}^{\frac{n}{2}-1} (-a_{2l+2}) \right) \\
=& \sum_{s=l_0+1}^{\frac{n}{2}-1} \frac{(-1)^{\frac{n}{2}}}{a_1 a_3 \dots a_{2s+1}} (a_{2s+1} - 1) \\
=& (-1)^{\frac{n}{2}} \left( \frac{1}{a_1 a_3 \dots a_{2l_0+1}} - \frac{1}{a_1 a_3 \dots a_{n-1}} \right)
\end{aligned}$$

Therefore, we find that

$$\langle \alpha_a, \alpha_b \rangle = \begin{cases} 1 & l_0 = -1 \\ 0 & \text{otherwise} \end{cases}.$$

We can argue that the determinant of Seifert form above is 1. In particular, it is lower triangular matrix if we treat $a$ as row index and $b$ as column index. Obviously, if $a = b$, then $\langle \alpha_a, \alpha_b \rangle = 1$. Namely, diagonal entries are 1.

We want to show that if $a - b < 0, \quad a, b \in \{1, \dots, \mu_n\}$ then $\langle \alpha_a, \alpha_b \rangle = 0$. Equivalently, we will show that if $a - b < 0, \quad \langle \alpha_a, \alpha_b \rangle = 1$, then $a - b < 1 - \mu_n$. It is easy to find that if $a - b < 0$, then $q_l < 0$ regardless of the value of $l$.

Since $l_0 = -1$, we have $a_1 \mid q_0$. Due to the fact that $q_0 < 0$, the biggest possible $q_0$ is $-a_1$, i.e., we get $\left\lfloor \frac{1}{a_2} \frac{1}{a_3} q_1 \right\rfloor = -a_1$. One can find that the biggest possible $q_1$ such that $a_3 \mid a_1$ is $-a_1 a_2 a_3 + a_2 a_3 - a_3$. The same procedures can be done all the way until we find that the biggest possible $a - b$ is $-\mu_n$, i.e., $a - b < 1 - \mu_n$.

Then we can solve the case that $n$ is odd.

The first term, i.e., $(b - a)$-term, of the term with $s = l_0$ is

$$\frac{\mathbf{i}}{2} \left( \frac{(-1)^{l_0}}{a_1 a_3 \cdots a_{2l_0-1}} \cdot \frac{d_{2l_0}}{d_n} \right) \left( \prod_{l=l_0+1}^{\frac{n-1}{2}-1} a_{2l+1} \right) \left( \prod_{l=l_0}^{\frac{n-1}{2}-1} (-a_{2l+2}) \right) a_n = \frac{\mathbf{i}}{2} \frac{(-1)^{\frac{n-1}{2}}}{a_1 a_3 \dots a_{2l_0+1}}.$$

The first term, i.e., $(b - a)$-term, of the term with $s = l_0 + 1, \dots, \frac{n-1}{2}$ is

$$\begin{aligned}
&- \frac{\mathbf{i}}{2} \sum_{s=l_0+1}^{\frac{n-1}{2}-1} \left( \frac{(-1)^s}{a_1 a_3 \cdots a_{2s-1}} \cdot \frac{d_{2s}}{d_n} \right) (a_{2s+1} - 1) \left( \prod_{l=s+1}^{\frac{n-1}{2}-1} a_{2l+1} \right) \left( \prod_{l=s}^{\frac{n-1}{2}-1} (-a_{2l+2}) \right) a_n \\
&- \frac{\mathbf{i}}{2} \frac{(-1)^{\frac{n-1}{2}}}{a_1 a_3 \cdots a_{n-2}} \cdot \frac{d_{n-1}}{d_n} a_n \\
=& - \frac{\mathbf{i}(-1)^{\frac{n-1}{2}}}{2} \left( \frac{1}{a_1 a_3 \dots a_{n-2}} + \sum_{s=l_0+1}^{\frac{n-1}{2}-1} \frac{1}{a_1 a_3 \dots a_{2s+1}} (a_{2s+1} - 1) \right) \\
=& - \frac{\mathbf{i}(-1)^{\frac{n-1}{2}}}{2} \frac{1}{a_1 a_3 \dots a_{2l_0+1}}
\end{aligned}$$

Similarly, we find that

$$\text{The first term, i.e., } (b-a)\text{-term, of } \langle \alpha_a, \alpha_b \rangle = \begin{cases} 1 & l_0 = -1 \\ 0 & \text{otherwise} \end{cases}.$$

Denote by $l_0'$ the counterpart of the second term, i.e., $(b-a-1)$-term. Therefore,

$$\langle \alpha_a, \alpha_b \rangle = \begin{cases} 1 & l_0 = -1 \text{ and } l_0' \text{ non-negative} \\ -1 & l_0' = -1 \text{ and } l_0 \text{ non-negative} \\ 0 & \text{otherwise} \end{cases}.$$

Similarly, we can also argue that the determinant of Seifert form above is 1 by showing it is upper triangular matrix if we treat $a$ as row index and $b$ as column index and its diagonal entries are 1. For simplicity, we consider the contribution from the first term, i.e., $(b-a)$-term. Almost the same argument for the case that $n$ is even tells us that the biggest possible $b-a$ such that the first term, i.e., $(b-a)$-term of $\langle \alpha_a, \alpha_b \rangle$ equals to 1 is $-\mu_n - 1$. We can apply the result to the second term, i.e., $(b-a-1)$-term of $\langle \alpha_a, \alpha_b \rangle$. That is to say, the biggest possible $b-a$ such that the second term, i.e., $(b-a-1)$-term of $\langle \alpha_a, \alpha_b \rangle$ equals to $-1$ is $-\mu_n$. Combining these facts, Seifer form $\langle \alpha_a, \alpha_b \rangle$ is upper triangular matrix with diagonal entries 1.

## 4.3 K-theoretic interpretation of chain type

**Proposition 4.15** The group $G_{f_n^T}$ is a cyclic group of order $d_n$ generated by the element

$$g_{\text{gen.}} := \left( \mathbf{e}\left[ (-1)^{n-1} \frac{1}{d_1} \right], \ldots, \mathbf{e}\left[ (-1)^{i-1} \frac{1}{d_{n+1-i}} \right], \ldots, \mathbf{e}\left[ \frac{1}{d_n} \right] \right).$$

**Definition 4.16** For each non-negative integer $n$, define sets $B'_{f_n^T}$, $B_{f_n^T}$ of monomials in $\mathbb{C}[x_1, \ldots, x_n]$ inductively as follows: Let $B'_{f_0^T} := \{1\}$ and

$$B'_{f_n^T} := \{ x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} | 0 \le k_i \le a_i - 1 (i = 1, \ldots, n-1), 0 \le k_n \le a_n - 2 \}, \quad n \ge 1.$$

For $n = 0, 1$, let $B_{f_0^T} := B'_{f_0^T} = \{1\}$ and $B_{f_1^T} := B'_{f_1^T} = \{ x_1^{k_1} | 0 \le k_1 \le a_1 - 2 \}$.

For $n \ge 2$, let

$$B_{f_n^T} := B'_{f_n^T} \cup \left\{ \phi^{(n-2)}(x_1, \ldots, x_{n-2}) x_n^{a_n - 1} | \phi^{(n-2)}(x_1, \ldots, x_{n-2}) \in B_{f_{n-2}^T} \right\}.$$

**Proposition 4.17** The set $B_{f_n^T}$ defines a $\mathbb{C}$-basis of the Jacobian algebra $\mathrm{Jac}(f_n^T)$. Namely, we have $\mathrm{Jac}(f_n^T) = \langle [\phi^{(n)}(\mathbf{x})] | \phi^{(n)}(\mathbf{x}) \in B_{f_n^T} \rangle_{\mathbb{C}}$.

Set $d_0 := 1$ and $d_i := a_1 \cdots a_i$ for $i = 1, \ldots, n$. Define a positive integer $\widetilde{\mu}_n$ by $\widetilde{\mu}_n := \sum_{i=0}^{n} (-1)^{n-i} d_i$, which satisfies $\widetilde{\mu}_n = d_n - d_{n-1} + \widetilde{\mu}_{n-2}$.

**Corollary 4.18** The Milnor number $\mu_{f_n^T} = \dim_{\mathbb{C}} \mathrm{Jac}(f_n^T)$ is given by $\widetilde{\mu}_n$.

Let $\Omega^p(\mathbb{C}^n)$ be the complex vector space of regular $p$-forms on $\mathbb{C}^n$. Consider the complex vector space

$$\Omega_{f_n^T} := \Omega^n(\mathbb{C}^n)/df_n^T \wedge \Omega^{n-1}(\mathbb{C}^n).$$

If $n = 0$, then set $\Omega_{f_n^T} := \mathbb{C}$, the complex vector space of constant functions on a point. Note that $\Omega_{f_n^T}$ is naturally a free $\mathrm{Jac}(f_n^T)$-module of rank one, namely, by choosing a nowhere vanishing $n$-form $d\mathbf{x} := dx_1 \wedge \cdots \wedge dx_n$ we have the following isomorphism

$$\mathrm{Jac}(f_n^T) \xrightarrow{\cong} \Omega_{f_n^T}, \quad [\phi(\mathbf{x})] \mapsto [\phi(\mathbf{x})d\mathbf{x}].$$

Our first step is to show the dimension of complexified relative K-ring $K^0_{G_{f_n^T}}(\mathbb{C}^n, V_1^T) \otimes \mathbb{C}$. Let us state the following lemma.

**Lemma 4.19** If $n$ is even, $\left[\Omega_{f_n^T}\right]^{G_{f_n^T}}$ is spanned by the single $G_{f_n^T}$-invariant class of $n$-form $\left[\prod_{i=1}^{\frac{n}{2}} x_{2i}^{a_{2i}-1} d\mathbf{x}\right]$, i.e.,

$$\left[\Omega_{f_n^T}\right]^{G_{f_n^T}} = \mathbb{C} \cdot \left[\prod_{i=1}^{\frac{n}{2}} x_{2i}^{a_{2i}-1} d\mathbf{x}\right].$$

**Proposition 4.20**

$$\dim_{\mathbb{C}} \left( K^0_{G_{f_n^T}}(\mathbb{C}^n, V_1^T) \otimes \mathbb{C} \right) = \sum_{i=0}^{n} (-1)^i \frac{d_n}{d_i}.$$

**Proof** Consider $g = g^a_{\text{gen.}} \in G_{f_n^T}$ of the form $(1, \ldots, 1, *, \ldots, *)$ where the first $i$ components are 1 and the rest components are not 1, i.e., $*$ represent a number in $U(1) \setminus \{1\}$, where $i = 0, 1, \ldots, n$. Then, we obtain that $g = g^a_{\text{gen.}} \in G_{f_n^T}$ is of such a form if and only if $a \in I_i := \begin{cases} \{a \in d_i\mathbb{Z}/d_n\mathbb{Z} | d_{i+1} \nmid a\} & i = 0, 1, \ldots, n-1 \\ \{0\} & i = n \end{cases}$. In this case, $\mathrm{Fix}_g(\mathbb{C}^n) \cong \mathbb{C}^i$ and $\mathrm{Fix}_g(V_1^T) \cong \{x \in \mathbb{C}^i | f_i^T(x) = 1\}$ which is of homotopy type $\mathbb{S}^{i-1} \vee \cdots \vee \mathbb{S}^{i-1}$.

Let us apply the decomposition that we stated in the introduction

$$\mathrm{Tr}: \ K^0_{G_{f_n^T}}(\mathbb{C}^n, V_1^T) \otimes \mathbb{C} \xrightarrow{\cong} \bigoplus_{g \in G_{f_n^T}} \left[ K^0(\mathrm{Fix}_g(\mathbb{C}^n), \mathrm{Fix}_g(V_1^T)) \otimes \mathbb{C} \right]^{G_{f_n^T}},$$

When $i$ is odd, we have the following exact sequence of reduced K-groups for $K^0(\mathrm{Fix}_g(\mathbb{C}^n), \mathrm{Fix}_g(V_1^T)) = \widetilde{K}\left(\mathbb{C}^i / \{x \in \mathbb{C}^i | f_i^T(x) = 1\}\right)$,

$$\widetilde{K}^{-1}\left(\mathbb{C}^i\right) \longrightarrow \widetilde{K}^{-1}\left(\{x \in \mathbb{C}^i | f_i^T(x) = 1\}\right) \xrightarrow{\cong} \widetilde{K}\left(\mathbb{C}^i / \{x \in \mathbb{C}^i | f_i^T(x) = 1\}\right) \longrightarrow \widetilde{K}\left(\mathbb{C}^i\right),$$

where the isomorphism is due to the fact that the reduced K-groups of the two sides are trivial.

Therefore, $K^0(\mathrm{Fix}_g(\mathbb{C}^n), \mathrm{Fix}_g(V_1^T)) \cong \widetilde{K}^{-1}\left(\{x \in \mathbb{C}^i | f_i^T(x) = 1\}\right) \cong \widetilde{K}^{-1}\left(\mathbb{S}^{i-1} \vee \cdots \vee \mathbb{S}^{i-1}\right) = 0$.

When $i$ is even, the standard Chern character map gives an isomorphism

$$K^0(\mathrm{Fix}_g(\mathbb{C}^n), \mathrm{Fix}_g(V_1^T)) \otimes \mathbb{C} \to H^{\mathrm{ev}}(\mathrm{Fix}_g(\mathbb{C}^n), \mathrm{Fix}_g(V_1^T); \mathbb{C}).$$

Again, due to the fact that $\mathrm{Fix}_g(V_1^T) \cong \{x \in \mathbb{C}^i | f_i^T(x) = 1\}$ is of homotopy type $\mathbb{S}^{i-1} \vee \cdots \vee \mathbb{S}^{i-1}$, we have the following equation

$$H^{\text{ev}}(\text{Fix}_g(\mathbb{C}^n), \text{Fix}_g(V_1^T); \mathbb{C}) = H^i(\mathbb{C}^i, \{x \in \mathbb{C}^i | f_i^T(x) = 1\}; \mathbb{C}) \cong \Omega_{f_i^T}.$$

Applying the lemma, we have

$$\dim_{\mathbb{C}} \left[ K^0(\text{Fix}_g(\mathbb{C}^n), \text{Fix}_g(V_1^T)) \otimes \mathbb{C} \right]^{G_{f_n^T}} = \dim_{\mathbb{C}} \left[ \Omega_{f_i^T} \right]^{G_{f_i^T}} = 1.$$

Finally, since $|I_i| = \frac{d_n}{d_i} - \frac{d_n}{d_{i+1}} (i = 0, 1, 2, \ldots, n-1), |I_n| = 1$, we obtain that

$$\dim_{\mathbb{C}} \left( K^0_{G_{f_n^T}}(\mathbb{C}^n, V_1^T) \otimes \mathbb{C} \right) = \sum_{i \text{ even}} |I_i| = \sum_{i=0}^{n} (-1)^i \frac{d_n}{d_i}.$$

Our next step is to construct a $\mathbb{Z}$-basis of torsion-free part of the relative K-ring $K^0_{G_{f_n^T}}(\mathbb{C}^n, V_1^T)$.

Since $\mathbb{C}^n$ is $G_{f_n^T}$-homotopy equivalent to the origin, $K^0_{G_{f_n^T}}(\mathbb{C}^n)$ coincides with the representation ring of $G_{f_n^T}$, that is,

$$K^0_{G_{f_n^T}}(\mathbb{C}^n) = \mathbb{Z}[L_1, \ldots, L_n]/(L_1^{a_1} - 1, L_1 L_2^{a_2} - 1, \ldots, L_{n-1} L_n^{a_n} - 1) = \mathbb{Z}[L_n]/(L_n^{d_n} - 1),$$

where $L_i = \mathbb{C}^n \times \mathbb{C}$ is the trivial bundle with $G_{f_n^T}$-action $g \cdot (x, \lambda) := (gx, g_i \lambda)$ and $1 = \mathbb{C}^n \times \mathbb{C} =: \underline{\mathbb{C}}$ is the trivial bundle with $G_{f_n^T}$-action $g \cdot (x, \lambda) := (gx, \lambda)$. Note that $T\mathbb{C}^n \cong \sum_{i=1}^n L_i$ in the category of $G_{f_n^T}$-equivariant bundles.

Let us construct the following two types of complexes of $G_{f_n^T}$-line bundle

$$E^\bullet_{2s-1}: \quad \left( L^{-1}_{2s-1} = L_n^{(-1)^{n-2s} \frac{d_n}{d_{2s-1}}} \right) \xrightarrow{x_{2s-1}} \underline{\mathbb{C}}, \qquad s = 1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil,$$

$$F^\bullet_{2s-1}: \quad \underline{\mathbb{C}} \xrightarrow{x_{2s-1}^{d_{2s-1}}} \underline{\mathbb{C}}, \qquad s = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor,$$

and $E^\bullet_{2s-1}, F^\bullet_{2s-1}$ represent two elements of the relative K-ring $K^0_{G_{f_n^T}}(\mathbb{C}^n, \{x_{2s-1} \neq 0\})$. Note that $E^\bullet_{2s-1} \otimes F^\bullet_{2s'-1}$ represents an element of the relative K-ring $K^0_{G_{f_n^T}}(\mathbb{C}^n, \{x_{2s-1} \neq 0 \text{ or } x_{2s'-1} \neq 0\})$ and that there is an inclusion

$$K^0_{G_{f_n^T}}\left(\mathbb{C}^n, \{x_1 \neq 0 \text{ or } x_3 \neq 0 \text{ or } \cdots \text{ or } x_{2\lceil \frac{n}{2} \rceil - 1} \neq 0\}\right) \subseteq K^0_{G_{f_n^T}}\left(\mathbb{C}^n, V_1^T\right).$$

We consider the following exact sequence derived from the couple $(\mathbb{C}, V_1^T)$,

$$0 \xrightarrow{i} K^{-1}_{G_{f_n^T}}(V_1^T) \xrightarrow{j} K^0_{G_{f_n^T}}(\mathbb{C}^n, V_1^T) \xrightarrow{k} K^0_{G_{f_n^T}}(\mathbb{C}^n) \xrightarrow{i} K^0_{G_{f_n^T}}(V_1^T),$$

where the sequence starts from 0 is due to the fact that $K^{-1}_{G_{f_n^T}}(\mathbb{C}^n) = K^{-1}_{G_{f_n^T}}(\{0\}) = 0$. Furthermore, it is splitted up into the short exact sequence,

$$0 \longrightarrow K^{-1}_{G_{f_n^T}}(V_1^T) \xrightarrow{j} K^0_{G_{f_n^T}}(\mathbb{C}^n, V_1^T) \xrightarrow{k} \text{Im}\, k \longrightarrow 0,$$

**Proposition 4.21** The image of $k$ is a principal ideal of $K^0_{G_{f_n^T}}(\mathbb{C}^n)$ generated by

$$\prod_{s=1}^{\lceil \frac{n}{2} \rceil}(L_{2s-1}^{-1}-1),$$

which is a $\mathbb{Z}$-module of rank $\frac{d_n}{d_0}-\frac{d_n}{d_1}$. There is a morphism $u : \mathrm{Im}\, k \to K^0_{G_{f_n^T}}(\mathbb{C}^n, V_1^T)$ such that $k \circ u = \mathrm{id}_{\mathrm{Im}\, k}$, i.e., the above exact sequence right splits.

**Proof** We will prove the first statement by showing $\mathrm{Ker}\, i$ is the principal ideal. Let us denote $L_n^{(-1)^n}$ by $\widetilde{L}_n$. Then $K^0_{G_{f_n^T}}(\mathbb{C}^n) = \mathbb{Z}[\widetilde{L}_n]/(\widetilde{L}_n^{d_n}-1)$ and $L_{2s-1}^{-1} = \widetilde{L}_n^{\frac{d_n}{d_{2s-1}}}$. For any polynomial $\sum_{i=0}^{d_n-1} c_i \widetilde{L}_n^i \in \mathbb{Z}[\widetilde{L}_n]/(\widetilde{L}_n^{d_n}-1)$. Apply Euclidean division to the polynomial with divisor $(L_1^{-1}-1)$, though the degree of a non-zero polynomial here is in $\mathbb{Z}_{d_n}$. We will stop the division as a normal polynomial, i.e., stop the process as long as the degree smaller than $\frac{d_n}{d_1}$. Otherwise the process can be done endlessly due to the relation $\widetilde{L}_n^{d_n} = 1$. We obtain

$$\sum_{i=0}^{d_n-1} c_i \widetilde{L}_n^i = (L_1^{-1}-1)q_1(\widetilde{L}_n) + \widetilde{r}_1(\widetilde{L}_n),$$

where $\deg \widetilde{r}_1 < \frac{d_n}{d_1} (\widetilde{r}_1 \in \mathrm{span}_{\mathbb{Z}}\{\widetilde{L}_n^i\}_{i=0}^{\frac{d_n}{d_1}-1})$ and $\deg q_1 < \frac{d_n}{d_0} - \frac{d_n}{d_1}$. For $q_1$, it is divided by $(L_3^{-1}-1)$. However, we will do polynomial long division for it eventhough its degree is smaller than $\frac{d_n}{d_3}$ until its degree is zero, due to the relation $\widetilde{L}_n^{d_n} = 1$. But this time, the process will terminate since $\deg q_1 < \frac{d_n}{d_0} - \frac{d_n}{d_1} < d_n - \frac{d_n}{d_3}$. Then

$$q_1(\widetilde{L}_n) = (L_3^{-1}-1)q_3(\widetilde{L}_n) + \widetilde{r}_3(\widetilde{L}_n),$$

where $q_3(\widetilde{L}_n) \in \mathrm{span}_{\mathbb{Z}}\{\widetilde{L}_n^i\}_{i=-\frac{d_n}{d_3}}^{\frac{d_n}{d_0}-\frac{d_n}{d_1}-\frac{d_n}{d_3}}$ and $\widetilde{r}_3 \in \mathrm{span}_{\mathbb{Z}}\{\widetilde{L}_n^i\}_{i=-\frac{d_n}{d_3}}^{-1}$. The remaining dividing process ($s = 3, 4, \ldots, \lceil \frac{n}{2} \rceil$) follows the pattern of dividing $(L_3^{-1}-1)$, in other words, each time we do polynomial long division with at most $\frac{d_n}{d_0} - \frac{d_n}{d_1}$ steps. Finally, we get

$$\sum_{i=0}^{d_n-1} c_i \widetilde{L}_n^i = \prod_{s=1}^{\lceil \frac{n}{2} \rceil}(L_{2s-1}^{-1}-1)q_{\lceil \frac{n}{2} \rceil}(\widetilde{L}_n) + \sum_{j=1}^{\lceil \frac{n}{2} \rceil} \widetilde{r}_{2j-1}(\widetilde{L}_n)\prod_{s=1}^{j-1}(L_{2s-1}^{-1}-1),$$

where $q_{\lceil \frac{n}{2} \rceil}(\widetilde{L}_n) \in \mathrm{span}_{\mathbb{Z}}\{\widetilde{L}_n^i\}_{i=-\sum_{s=2}^{\lceil \frac{n}{2} \rceil}\frac{d_n}{d_{2s-1}}}^{\frac{d_n}{d_0}-\frac{d_n}{d_1}-\sum_{s=2}^{\lceil \frac{n}{2} \rceil}\frac{d_n}{d_{2s-1}}}$ and $\widetilde{r}_{2j-1}(\widetilde{L}_n) \in \mathrm{span}_{\mathbb{Z}}\{\widetilde{L}_n^i\}_{i=-\sum_{s=2}^{j}\frac{d_n}{d_{2s-1}}}^{-1-\sum_{s=2}^{j-1}\frac{d_n}{d_{2s-1}}}$. By multiplying $\widetilde{L}_n^{\sum_{s=2}^{\lceil \frac{n}{2} \rceil}\frac{d_n}{d_{2s-1}}}$ to both sides of the equation, we have that any polynomial in $K^0_{G_{f_n^T}}(\mathbb{C}^n)$ can be decompose into the following form

$$\prod_{s=1}^{\lceil \frac{n}{2} \rceil}(L_{2s-1}^{-1}-1)q(\widetilde{L}_n) + \sum_{j=1}^{\lceil \frac{n}{2} \rceil} r_{2j-1}(\widetilde{L}_n)\prod_{s=1}^{j-1}(L_{2s-1}^{-1}-1),$$

where $q(\widetilde{L}_n) \in \mathrm{span}_{\mathbb{Z}}\{\widetilde{L}_n^i\}_{i=0}^{\frac{d_n}{d_0}-\frac{d_n}{d_1}}$ and $r_{2j-1}(\widetilde{L}_n) \in \mathrm{span}_{\mathbb{Z}}\{\widetilde{L}_n^i\}_{i=\sum_{s=j+1}^{\lceil \frac{n}{2} \rceil} \frac{d_n}{d_{2s-1}}}^{-1+\sum_{s=j}^{\lceil \frac{n}{2} \rceil} \frac{d_n}{d_{2s-1}}}$.

Since $\{x_1 \neq 0 \text{ or } x_3 \neq 0 \text{ or } \cdots \text{ or } x_{2\lceil \frac{n}{2} \rceil -1} \neq 0\} \supseteq V_1^T$, the tensor product $\bigotimes_{s=1}^{\lceil \frac{n}{2} \rceil} E_{2s-1}^{\bullet}$ is exact on $V_1^T$, which gives a $G_{f_n^T}$-isomorphism between the two $G_{f_n^T}$-vector bundles from $\prod_{s=1}^{\lceil \frac{n}{2} \rceil}(L_{2s-1}^{-1}-1)$. Therefore, for $i : K^0_{G_{f_n^T}}(\mathbb{C}^n) \to K^0_{G_{f_n^T}}(V_1^T)$, we derived that

$$i\left(\prod_{s=1}^{\lceil \frac{n}{2} \rceil}(L_{2s-1}^{-1}-1)\right) = 0.$$

It is left to show that

$$i\left(\sum_{j=1}^{\lceil \frac{n}{2} \rceil} r_{2j-1}(\widetilde{L}_n)\prod_{s=1}^{j-1}(L_{2s-1}^{-1}-1)\right) = 0$$

if and only if $r_{2j-1} = 0, \forall j = 1, 2, \ldots \lceil \frac{n}{2} \rceil$.

Assume that $j_0 \in \{1, 2, \ldots \lceil \frac{n}{2} \rceil\}$ is the smallest index such that $r_{2j-1}$ is non-zero polynomial. We consider a subspace $V_1^T|_{2j_0-2} := \{(x_1, \ldots, x_n) \in V_1^T | x_{2j_0} = x_{2j_0+1} = \ldots x_n = 0\}$ of $V_1^T$, where $\dim V_1^T|_{2j_0-2} = 2j_0 - 2$. One may find that

$$\sum_{j=j_0+1}^{\lceil \frac{n}{2} \rceil} r_{2j-1}(\widetilde{L}_n)\prod_{s=1}^{j-1}(L_{2s-1}^{-1}-1)$$

has a factor $\prod_{s=1}^{j_0}(L_{2s-1}^{-1}-1)$. Thus, on the subspace, the complex $\bigotimes_{s=1}^{j_0} E_{2s-1}^{\bullet}$ yields a $G_{f_n^T}$-isomorphism between the two $G_{f_n^T}$-vector bundles from $\sum_{j=j_0+1}^{\lceil \frac{n}{2} \rceil} r_{2j-1}(\widetilde{L}_n)\prod_{s=1}^{j-1}(L_{2s-1}^{-1}-1)$.

While for $r_{2j_0-1}(\widetilde{L}_n)\prod_{s=1}^{j_0-1}(L_{2s-1}^{-1}-1)$, on the subspace $V_1^T|_{2j_0-2}$, we cannot get any exact complex from $r_{2j_0-1}(\widetilde{L}_n)$. From $\prod_{s=1}^{j_0-1}(L_{2s-1}^{-1}-1)$, there is no way to get $(j_0-1)$ two-term complexes so that one of them is exact on $\{x_1 \neq 0 \text{ or } x_3 \neq 0 \text{ or } \cdots \text{ or } x_{2j_0-1} \neq 0\} \supseteq V_1^T|_{2j_0-2}$

**Conjecture 4.22** Denote by $G_s^{\bullet}$ the complex

$$F_1^{\bullet} \otimes \cdots \otimes F_{2s-1}^{\bullet} \otimes E_{2s+1}^{\bullet} \otimes \cdots \otimes E_{2\lceil \frac{n}{2} \rceil -1}^{\bullet}.$$

The set

$$\left\{ G_s^{\bullet} \otimes L_n^{i_s-1} \middle| s = 0, 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor, 1 \le i_s \le \frac{d_n}{d_{2s}} - \frac{d_n}{d_{2s+1}}, \text{ if } 2s+1 > n \text{ then } \frac{d_n}{d_{2s+1}} = 0 \right\}$$

represents a $\mathbb{Z}$-basis of the relative K-ring $K^0_{G_{f_n^T}}(\mathbb{C}^n, V_1^T)$. Furthermore, the set

$$\left\{ G_0^{\bullet} \otimes L_n^{i_0-1} \middle| 1 \le i_0 \le \frac{d_n}{1} - \frac{d_n}{d_1} \right\}$$

represents a $\mathbb{Z}$-basis of direct summand $\operatorname{Im} u$ of the relative K-ring $K^0_{G_{f_n^T}}(\mathbb{C}^n, V_1^T)$. The set

$$\left\{ G_s^\bullet \otimes L_n^{i_s - 1} \middle| s = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor, 1 \le i_s \le \frac{d_n}{d_{2s}} - \frac{d_n}{d_{2s+1}}, \text{ if } 2s+1 > n \text{ then } \frac{d_n}{d_{2s+1}} = 0 \right\}$$

represents a $\mathbb{Z}$-basis of direct summand $\operatorname{Im} j$ of the relative K-ring $K^0_{G_{f_n^T}}(\mathbb{C}^n, V_1^T)$.

Let us calculate their image under $\iota^* \widetilde{\mathrm{ch}}$. Consider $g = g^a_{\mathrm{gen.}} \in G_{f_n^T}$, where $a \in I_i, i \in \{0, 1, 2, \ldots, n\} \cap 2\mathbb{Z}$. The component where $i < 2s - 1$ of $\iota^* \operatorname{Tr}(G_s^\bullet \otimes L_n^{i_s - 1})$ is 0. While, its component where $i > 2s - 1$ is

$$\mathbf{e}\left[\frac{-a(i_s - 1)}{d_n}\right] \prod_{j = \frac{i}{2}}^{\lceil \frac{n}{2} \rceil - 1} \left( \mathbf{e}\left[\frac{(-1)^{n-2j-1} a}{d_{2j+1}}\right] - 1 \right) \cdot G^\bullet_{s,i},$$

where $G^\bullet_{s,i} := F_1^\bullet \otimes \cdots \otimes F^\bullet_{2s-1} \otimes E^\bullet_{2s+1} \otimes \cdots \otimes E^\bullet_{i-1} \in \left[K^0(\mathrm{Fix}_g(\mathbb{C}^n), \mathrm{Fix}_g(V_1^T))\right]^{G_{f_n^T}}$ $(i > 2s - 1)$ and $G^\bullet_{0,0} := \mathbb{C} \in \left[K^0(\{0\}, \emptyset)\right]^{G_{f_n^T}}$.

Then, by using the multiplicativity of Chern character map on different pairs (different subspaces), the non-trivial (i.e., $i > 2s - 1$) component of $\iota^* \widetilde{\mathrm{ch}}(G_s^\bullet \otimes L_n^{i_s - 1}) = \mathrm{ch}\, \iota^* \operatorname{Tr}(G_s^\bullet \otimes L_n^{i_s - 1})$ is

$$a_1 a_3 \cdots a_{2s-1} \cdot \mathbf{e}\left[\frac{-a(i_s - 1)}{d_n}\right] \prod_{j = \frac{i}{2}}^{\lceil \frac{n}{2} \rceil - 1} \left( \mathbf{e}\left[\frac{(-1)^{n-2j-1} a}{d_{2j+1}}\right] - 1 \right) \cdot e_a,$$

where $e_a$ is the generator of $\left[H^i(\mathrm{Fix}_g(\mathbb{C}^n), \mathrm{Fix}_g(V_1^T); \mathbb{Z})\right]^{G_{f_n^T}} \cong \mathbb{Z}$.

We claim that $\iota^* \widetilde{\mathrm{ch}}(G_s^\bullet \otimes L_n^{i_s - 1})$ are linearly independent, and thus $G_s^\bullet \otimes L_n^{i_s - 1}$ represents a $\mathbb{C}$-basis of $K^0_{G_{f_n^T}}(\mathbb{C}^n, V_1^T) \otimes \mathbb{C}$.

In fact, let us consider a matrix with row index $(i, a), a \in I_i, i \in \{0, 1, 2, \ldots, n\} \cap 2\mathbb{Z}$ with the first $|I_0|$ indices $(0, a)$($a$ increases) then $i$ increases, and with column index $(s, i_s)$ with the first $|I_0|$ indices $(0, i_s)$($i_s$ increases) then $s$ increases. Note that when $i < 2s - 1$, entries are zero, which implies this matrix is a upper triangular block matrix of which block matrices on the main diagonal ($i = 2s$) are square matrices. We find that the block matrices on the main diagonal are essentially Vandermonde matrices whose determinants are not zero.

The $\Gamma$-class of the orbifold tangent bundle is

$$\widehat{\Gamma}\left(\sum_{k=1}^{n} L_k\right) = \prod_{k=1}^{n} \Gamma\left(1 - \frac{(-1)^{n-k} a \mod d_k}{d_k}\right).$$

Then, the non-trivial component of the $\Gamma$-class modification of the Chern character map $\mathrm{ch}_\Gamma(G_s^\bullet \otimes L_n^{i_s - 1})$ is

$$\frac{1}{2\pi} \cdot (2\pi \mathbf{i})^{\frac{i}{2}} \cdot a_1 a_3 \cdots a_{2s-1} \cdot \mathbf{e}\left[\frac{-a(i_s - 1)}{d_n}\right] \prod_{j = \frac{i}{2}}^{\lceil \frac{n}{2} \rceil - 1} \left( \mathbf{e}\left[\frac{(-1)^{n-1} a}{d_{2j+1}}\right] - 1 \right) \cdot \prod_{k=i+1}^{n} \Gamma\left(1 - \frac{(-1)^{n-k} a \mod d_k}{d_k}\right) \cdot e_a,$$

One can prove Conjecture 4.4 using the above result and Proposition 4.14.

Let us construct the following complexes of $G_{f_n^T}$-line bundle

$$E_{2s-1}^{\bullet}: \quad \left(L_{2s-1}^{-1} = L_n^{(-1)^{n-2s}\frac{d_n}{d_{2s-1}}}\right) \xrightarrow{x_{2s-1}} \underline{\mathbb{C}}, \qquad s = 1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil,$$

and $E_{2s-1}^{\bullet}$ represent an element of the relative K-ring $K^0_{G_{f_n^T}}(\mathbb{C}^n, \{x_{2s-1} \neq 0\})$. Note that $E_{2s-1}^{\bullet} \otimes E_{2s'-1}^{\bullet}$ represents an element of the relative K-ring $K^0_{G_{f_n^T}}(\mathbb{C}^n, \{x_{2s-1} \neq 0 \text{ or } x_{2s'-1} \neq 0\})$ and that there is an inclusion

$$K^0_{G_{f_n^T}}\left(\mathbb{C}^n, \{x_1 \neq 0 \text{ or } x_3 \neq 0 \text{ or } \cdots \text{ or } x_{2\lceil \frac{n}{2} \rceil - 1} \neq 0\}\right) \subseteq K^0_{G_{f_n^T}}\left(\mathbb{C}^n, V_1^T\right).$$

We state the following conjecture.

**Conjecture 4.23** Denote by $F^{\bullet}$ the complex $E_1^{\bullet} \otimes \cdots \otimes E^{\bullet}_{2\lceil \frac{n}{2} \rceil - 1}$. The set $\left\{F^{\bullet} \otimes L_n^{j-1} \middle| 1 \leq j \leq \mu_n\right\}$ represents a $\mathbb{Z}$-basis of the torsion-free part of the relative K-ring $K^0_{G_{f_n^T}}(\mathbb{C}^n, V_1^T)$.

**Remark 4.24** One only need to prove Conjecture 4.22 or Conjecture 4.23, since the other can be proved using the following relation

$$F_{2s-1}^{\bullet} = E_{2s-1}^{\bullet} \otimes \bigoplus_{i=1}^{d_{2s-1}} L_{2s-1}^{-i}.$$

Let us calculate the image of the basis in Conjecture 4.23 under $\iota^* \widetilde{\mathrm{ch}}$. Consider $g = g^a_{\text{gen.}} \in G_{f_n^T}$, where $a \in I_i, i \in \{0, 1, 2, \ldots, n\} \cap 2\mathbb{Z}$. The component of $\iota^* \operatorname{Tr}(F^{\bullet} \otimes L_n^{j-1})$ is

$$\mathbf{e}\left[\frac{-a(j-1)}{d_n}\right] \prod_{j=\frac{i}{2}}^{\lceil \frac{n}{2} \rceil - 1} \left(\mathbf{e}\left[\frac{(-1)^{n-2j-1}a}{d_{2j+1}}\right] - 1\right) \cdot F_i^{\bullet},$$

where $F_i^{\bullet} := E_1^{\bullet} \otimes \cdots \otimes E_{i-1}^{\bullet} \in \left[K^0(\mathrm{Fix}_g(\mathbb{C}^n), \mathrm{Fix}_g(V_1^T))\right]^{G_{f_n^T}}$ and $F_0^{\bullet} := \mathbb{C} \in \left[K^0(\{0\}, \emptyset)\right]^{G_{f_n^T}}$.

Then, by using the multiplicativity of Chern character map on different pairs (different subspaces), the component of $\iota^* \widetilde{\mathrm{ch}}(F^{\bullet} \otimes L_n^{j-1}) = \mathrm{ch}\, \iota^* \operatorname{Tr}(F^{\bullet} \otimes L_n^{j-1})$ is

$$\mathbf{e}\left[\frac{-a(j-1)}{d_n}\right] \prod_{j=\frac{i}{2}}^{\lceil \frac{n}{2} \rceil - 1} \left(\mathbf{e}\left[\frac{(-1)^{n-2j-1}a}{d_{2j+1}}\right] - 1\right) \cdot e_a,$$

where $e_a$ is the generator of $\left[H^i(\mathrm{Fix}_g(\mathbb{C}^n), \mathrm{Fix}_g(V_1^T); \mathbb{Z})\right]^{G_{f_n^T}} \cong \mathbb{Z}$.

We claim that $\iota^* \widetilde{\mathrm{ch}}(F^{\bullet} \otimes L_n^{j-1})$ are linearly independent, and thus $F^{\bullet} \otimes L_n^{j-1}$ represents a $\mathbb{C}$-basis of $K^0_{G_{f_n^T}}(\mathbb{C}^n, V_1^T) \otimes \mathbb{C}$.

In fact, let us consider a matrix with row index $(i, a), a \in I_i, i \in \{0, 1, 2, \ldots, n\} \cap 2\mathbb{Z}$ with the first $|I_0|$ indices $(0, a)$ ($a$ increases) then $i$ increases, and with column index $j = 1, \ldots, \mu_n$. We find that the matrix can be decomposed into a Vandermonde matrix times a diagonal matrix with entries $\prod_{j=\frac{i}{2}}^{\lceil \frac{n}{2} \rceil - 1} \left(\mathbf{e}\left[\frac{(-1)^{n-2j-1}a}{d_{2j+1}}\right] - 1\right) \neq 0$. Thus the matrix is invertible.

The Γ-class of the orbifold tangent bundle is

$$\widehat{\Gamma}\left(\sum_{k=1}^{n} L_k\right) = \prod_{k=1}^{n} \Gamma\left(1 - \frac{(-1)^{n-k}a \mod d_k}{d_k}\right).$$

Then, the non-trivial component of the Γ-class modification of the Chern character map $\mathrm{ch}_{\Gamma}(G_s^{\bullet} \otimes L_n^{i_s-1})$ is

$$\frac{1}{2\pi} \cdot (2\pi\mathbf{i})^{\frac{i}{2}} \cdot \mathbf{e}\left[\frac{-a(j-1)}{d_n}\right] \prod_{j=\frac{i}{2}}^{\lceil \frac{n}{2} \rceil -1} \left(\mathbf{e}\left[\frac{(-1)^{n-1}a}{d_{2j+1}}\right] - 1\right) \cdot \prod_{k=i+1}^{n} \Gamma\left(1 - \frac{(-1)^{n-k}a \mod d_k}{d_k}\right) \cdot e_a.$$

Similarly, Conjecture 4.4 can be proved using the above result and Proposition 4.14.

# Bibliography

[1] Daisuke Aramaki and Atsushi Takahashi. Maximally-graded matrix factorizations for an invertible polynomial of chain type. *Adv. Math.*, 373:107320, 23, 2020.

[2] V. I. Arnol'd. *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1989. Translated from the Russian by K. Vogtmann and A. Weinstein.

[3] V. I. Arnol'd, S. M. Guseĭn-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Vol. II*, volume 83 of *Monographs in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1988. Monodromy and asymptotics of integrals, Translated from the Russian by Hugh Porteous, Translation revised by the authors and James Montaldi.

[4] M. Atiyah. *K-theory*. CRC Press, 2018.

[5] M. F. Atiyah, R. Bott, and A. Shapiro. Clifford modules. *Topology*, 3(suppl, suppl. 1):3–38, 1964.

[6] M. F. Atiyah and F. Hirzebruch. Vector bundles and homogeneous spaces. In *Proc. Sympos. Pure Math., Vol. III*, pages 7–38. American Mathematical Society, Providence, R.I., 1961.

[7] Michael Atiyah and Raoul Bott. On the periodicity theorem for complex vector bundles. *Acta Math.*, 112:229–247, 1964.

[8] Michael Atiyah and Graeme Segal. On equivariant Euler characteristics. *J. Geom. Phys.*, 6(4):671–677, 1989.

[9] Bojko Bakalov and Todor Milanov. W-constraints for simple singularities. *Compositio Math.*, 149:840–888, 2013.

[10] Per Berglund and Tristan Hübsch. A generalized construction of mirror manifolds. *Nuclear Phys. B*, 393(1-2):377–391, 1993.

[11] Alessandro Chiodo, Hiroshi Iritani, and Yongbin Ruan. Landau-Ginzburg/Calabi-Yau correspondence, global mirror symmetry and Orlov equivalence. *Publ. Math. Inst. Hautes Études Sci.*, 119:127–216, 2014.

[12] Alessandro Chiodo and Jan Nagel. The hybrid Landau-Ginzburg models of Calabi-Yau complete intersections. In *Topological recursion and its influence in analysis, geometry, and topology*, volume 100 of *Proc. Sympos. Pure Math.*, pages 103–117. Amer. Math. Soc., Providence, RI, 2018.

[13] Boris Dubrovin. Geometry of 2D topological field theories. In *Integrable systems and quantum groups (Montecatini Terme, 1993)*, volume 1620 of *Lecture Notes in Math.*, pages 120–348. Springer, Berlin, 1996.

[14] Boris Dubrovin. Painlevé transcendents in two-dimensional topological field theory. In *The Painlevé property*, CRM Ser. Math. Phys., pages 287–412. Springer, New York, 1999.

[15] Wolfgang Ebeling and Atsushi Takahashi. Strange duality of weighted homogeneous polynomials. *Compos. Math.*, 147(5):1413–1433, 2011.

[16] Huijun Fan, Tyler Jarvis, and Yongbin Ruan. The Witten equation, mirror symmetry, and quantum singularity theory. *Ann. of Math. (2)*, 178(1):1–106, 2013.

[17] Anatoly Fomenko and Dmitry Fuchs. *Homotopical topology*, volume 273 of *Graduate Texts in Mathematics*. Springer, Switzerland, 2016.

[18] Edward Frenkel, Alexander Givental, and Todor Milanov. Soliton equations, vertex operators, and simple singularities. *Funct. Anal. Other Math.*, 3(1):47–63, 2010.

[19] Masahiro Futaki and Kazushi Ueda. Homological mirror symmetry for singularities of type D. *Math. Z.*, 273(3-4):633–652, 2013.

[20] Alexander B. Givental. Gromov-Witten invariants and quantization of quadratic Hamiltonians. volume 1, pages 551–568, 645. 2001. Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary.

[21] Alexander B. Givental. Semisimple Frobenius structures at higher genus. *Internat. Math. Res. Notices*, (23):1265–1286, 2001.

[22] Alexander B. Givental and Todor E. Milanov. Simple singularities and integrable hierarchies. In *The breadth of symplectic and Poisson geometry*, volume 232 of *Progr. Math.*, pages 173–201. Birkhäuser Boston, Boston, MA, 2005.

[23] Robin Hartshorne. *Residues and duality*. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin-New York, 1966. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64, With an appendix by P. Deligne.

[24] Claus Hertling. *Frobenius manifolds and moduli spaces for singularities*, volume 151 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2002.

[25] Yuki Hirano and Genki Ouchi. Derived factorization categories of non-thom–sebastiani-type sums of potentials, 2018.

[26] E. L. Ince. *Ordinary Differential Equations*. Dover Publications, New York, 1944.

[27] Hiroshi Iritani. An integral structure in quantum cohomology and mirror symmetry for toric orbifolds. *Adv. Math.*, 222(3):1016–1079, 2009.

[28] Birger Iversen. *Cohomology of sheaves*. Springer–Verlag, Berlin Heidelberg, 1986.

[29] Victor Kac. *Vertex algebras for beginners*, volume 10 of *University Lecture Series*. AMS, Providence, RI, 1996.

[30] Maxim Kontsevich. Intersection theory on the moduli space of curves and the matrix Airy function. *Comm. Math. Phys.*, 147(1):1–23, 1992.

[31] Marc Krawitz. *FJRW rings and Landau-Ginzburg mirror symmetry*. ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)–University of Michigan.

[32] Maximilian Kreuzer and Harald Skarke. On the classification of quasihomogeneous functions. *Comm. Math. Phys.*, 150(1):137–147, 1992.

[33] Yuri I. Manin. *Frobenius manifolds, quantum cohomology, and moduli spaces*, volume 47 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1999.

[34] Todor Milanov. *Primitive forms and Frobenius structures on the Hurwitz spaces*, arXiv:1701.00393.

[35] Todor Milanov. *Lectures on Painleve property for semi-simple Frobenius manifolds*, arXiv:1702.06630.

[36] Todor Milanov. *The Period map for quantum cohomology of* $\mathbb{P}^2$. arXiv:1706.04323, Adv. in Math. Volume 351, 31 July 2019, Pages 804-869.

[37] Todor Milanov and Kyoji Saito. Primitive forms and vertex operators. book in preparation.

[38] John Milnor. *Singular points of complex hypersurfaces*. Annals of Mathematics Studies, No. 61. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968.

[39] Haruo Minami. A Künneth formula for equivariant $K$-theory. *Osaka Math. J.*, 6:143–146, 1969.

[40] Madhav V. Nori. Zariski's conjecture and related problems. *Ann. Sci. École Norm. Sup. (4)*, 16(2):305–344, 1983.

[41] Takumi Otani and Atsushi Takahashi. Gamma integral structure for an invertible polynomial of chain type. *Adv. Math.*, 409(part B):Paper No. 108681, 50, 2022.

[42] Helge Ruddat, Nicolò Sibilla, David Treumann, and Eric Zaslow. Skeleta of affine hypersurfaces. *Geom. Topol.*, 18(3):1343–1395, 2014.

[43] Kyoji Saito. On the periods of primitive integrals I. RIMS-preprint 412, Kyoto Univ, 1982.

[44] Morihiko Saito. On the structure of Brieskorn lattice. *Ann. Inst. Fourier (Grenoble)*, 39(1):27–72, 1989.

[45] I. Satake. On a generalization of the notion of manifold. *Proc. Nat. Acad. Sci. U.S.A.*, 42:359–363, 1956.

[46] Graeme Segal. Equivariant $K$-theory. *Inst. Hautes Études Sci. Publ. Math.*, (34):129–151, 1968.

[47] Ichiro Shimada. Lectures on Zariski Van-Kampen theorem. Unpublished lecture notes, URL: `http://www.math.sci.hiroshima-u.ac.jp/shimada/LectureNotes/LNZV.pdf`.

[48] Atsushi Takahashi. Weighted projective lines associated to regular systems of weights of dual type. In *New developments in algebraic geometry, integrable systems and mirror symmetry (RIMS, Kyoto, 2008)*, volume 59 of *Adv. Stud. Pure Math.*, pages 371–388. Math. Soc. Japan, Tokyo, 2010.

[49] Constantin Teleman. The structure of 2D semi-simple field theories. *Invent. Math.*, 188(3):525–588, 2012.

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