



Reflected entropy for an evaporating black hole

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Based on arXiv:2006.10846 with Tianyi Li and Jinwei Chu

Introduction

- Quantum gravity is the key to understand the origin of our universe
- A simpler object involving quantum gravity is black hole. They have a temperature that leads to Hawking radiation.
- Black holes also have entropy, given by the Area of the horizons.
- The question is whether black holes behave like ordinary quantum systems. People believe they do (string theory, AdS/CFT) but do not know how.

- Importantly, there is a paradox if they do: consider a black hole formed by a pure state, after evaporation it becomes a thermal state (according to Hawking)-> information is lost
- You may argue that strange things can happen at the end of the evaporation. But the paradox already shows up near the middle age of BH.
- To understand this, we first introduce 2 different notions of entropy: fine-grained entropy and coarse-grained entropy.

Fine-grained < coarse-grained [\[Review: arXiv:2006.06872\]](#)

- 1st : Fine-grained entropy is simply the von Neumann entropy. It is Shannon's entropy with distribution replaced by density matrix. It is **invariant** under unitary time evolution.
- 2nd: Coarse-grained entropy is defined as follows. We only measure simple observables A_i . And consider all possible density matrices which give the same result as our system.

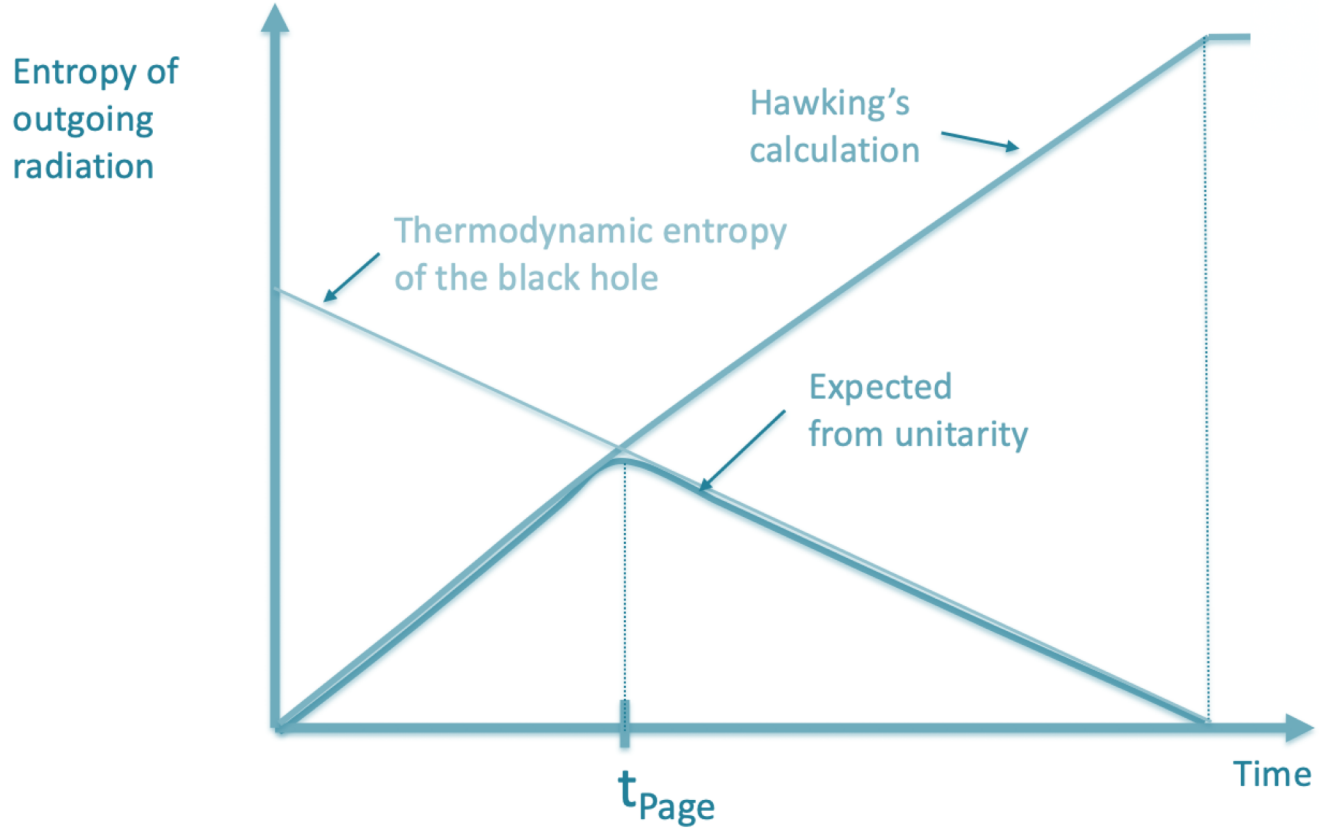
$$\text{Tr}[\tilde{\rho}A_i] = \text{Tr}[\rho A_i]$$

We then choose the maximal von Neumann entropy over all possible density matrices $S(\tilde{\rho})$. It **increases** under unitary time evolution. -> entropy in thermodynamics.

Information paradox

- Bekenstein-Hawking entropy is coarse-grained entropy.
- The thermal aspect of Hawking radiation comes from separating entangled outgoing Hawking quanta and interior Hawking quanta. Each side is a mixed state.
- As the entropy of radiation gets bigger and bigger, we run into trouble because, the entangled partners in black hole should have the same entropy, which exceeds the horizon entropy.
- In fact, the constantly increasing result was made by Hawking. Page suggested that the outgoing radiation entropy should follow **Page curve**

Page curve



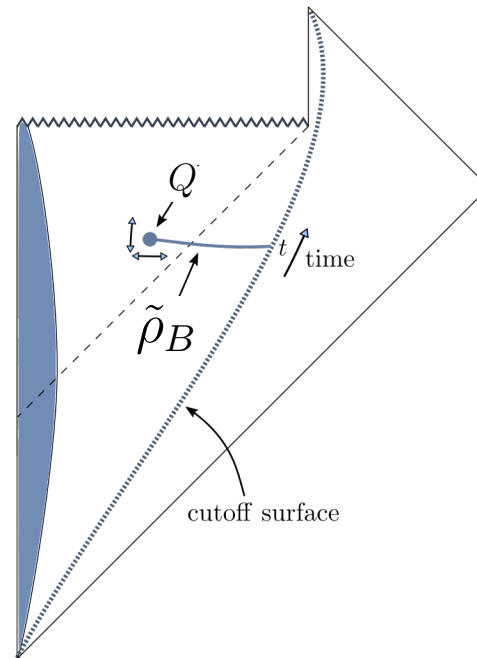
[Fig from arXiv:2006.06872]

How to reproduce Page curve?

QES formula for BH [Penington, Almheiri-Engelhardt-Marolf-Maxfield]

- The fine-grained entropy of black hole surrounded by quantum fields is given in terms of semiclassical entropy by

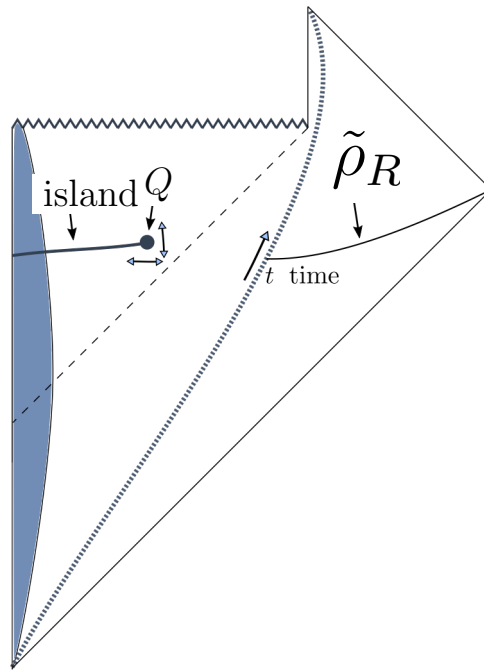
$$S_B = \text{ext}_Q \left\{ \frac{\text{Area}(Q)}{4G_N} + S(\tilde{\rho}_B) \right\} ,$$



Island formula for radiation [Almheiri-Mahajan-Maldacena-Zhao]

- Similarly, the fine-grained entropy of radiation is given in terms of semiclassical entropy by

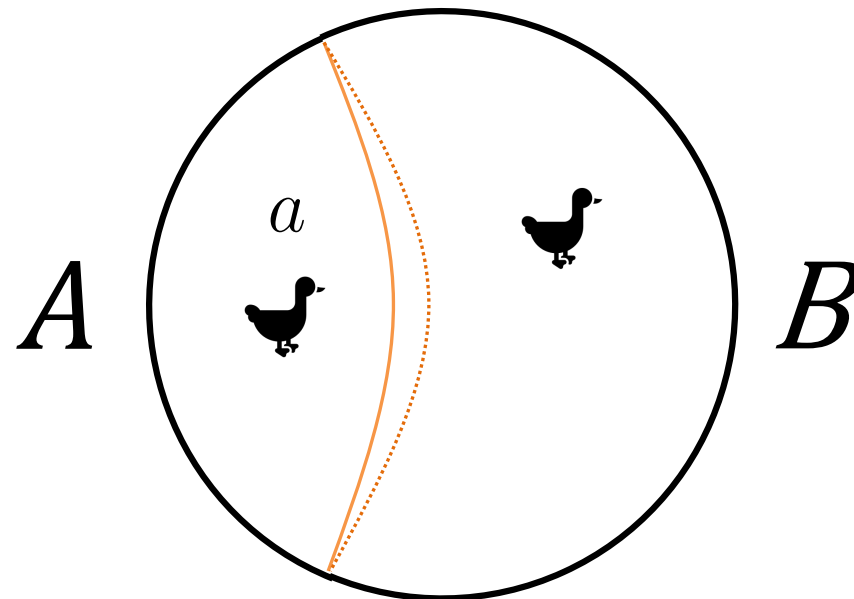
$$S(\rho_R) = \text{ext}_I \left\{ \frac{\text{Area}(\partial I = Q)}{4G_N} + S(\tilde{\rho}_{R \cup I}) \right\}$$



Quantum extremal surface [Engelhardt-Wall,RT,HRT]

- QES origins from holographic entanglement entropy in AdS/CFT with bulk matter

$$S(A) = \text{ext}_Q \left\{ \frac{\text{Area}(Q)}{4G_N} + S^{\text{bulk}}(a) \right\}$$



Motivations

- So far we only consider BH + radiation is pure, but what if BH + radiation is a mixed state?
- Are there other quantities which can have island formula?
- Can we read more information about the island?
- Can we compute the correlation between Hawking radiation A and B?

Von Neumann entropy
vs
Reflected entropy

(See also [arXiv:2006.10754](https://arxiv.org/abs/2006.10754) by V.Chandrasekaran,M.Miyaji,P.Rath)

Outline

- Reflected entropy and the holographic dual
- Quantum extremal cross section
- Gravitational reflected entropy
- Eternal black hole + CFT model

Canonical purification [Dutta-Faulkner]

- Consider a mixed state on a bipartite Hilbert space

$$\rho_{AB}$$

- Flipping Bras to Kets for the basis

$$|i\rangle \langle j| \longrightarrow |i\rangle \otimes |j\rangle$$

- A canonical purification

$$|\sqrt{\rho_{AB}}\rangle \in (\mathcal{H}_A \otimes \mathcal{H}_A^*) \otimes (\mathcal{H}_B \otimes \mathcal{H}_B^*)$$

$$\text{Tr}_{\mathcal{H}_A^* \otimes \mathcal{H}_B^*} |\sqrt{\rho_{AB}}\rangle \langle \sqrt{\rho_{AB}}| = \rho_{AB}$$

$$\rho_{AB} = \frac{1}{2} (|\uparrow\uparrow\rangle\langle\uparrow\uparrow|_{AB} + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|_{AB})$$



$$|\sqrt{\rho_{AB}}\rangle = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\uparrow\uparrow\rangle_{AA'BB'} + |\downarrow\downarrow\downarrow\downarrow\rangle_{AA'BB'})$$

- Reflected entropy

$$S_R(A : B) \equiv S(AA^*)_{\sqrt{\rho_{AB}}}$$

Reflected entropy

- Properties

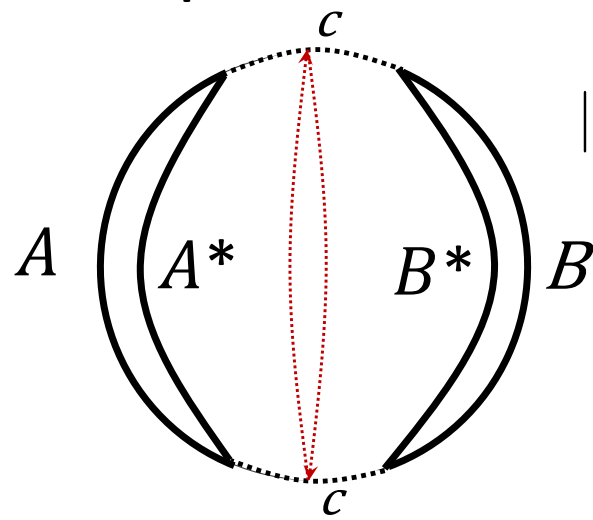
pure state : $S_R(A : B) = 2S(A)$,

factorized state : $S_R(A : B) = 0$,

bounded from below : $S_R(A : B) \geq I(A : B)$,

bounded from above : $S_R(A : B) \leq 2\min\{S(A), S(B)\}$

- Graph description



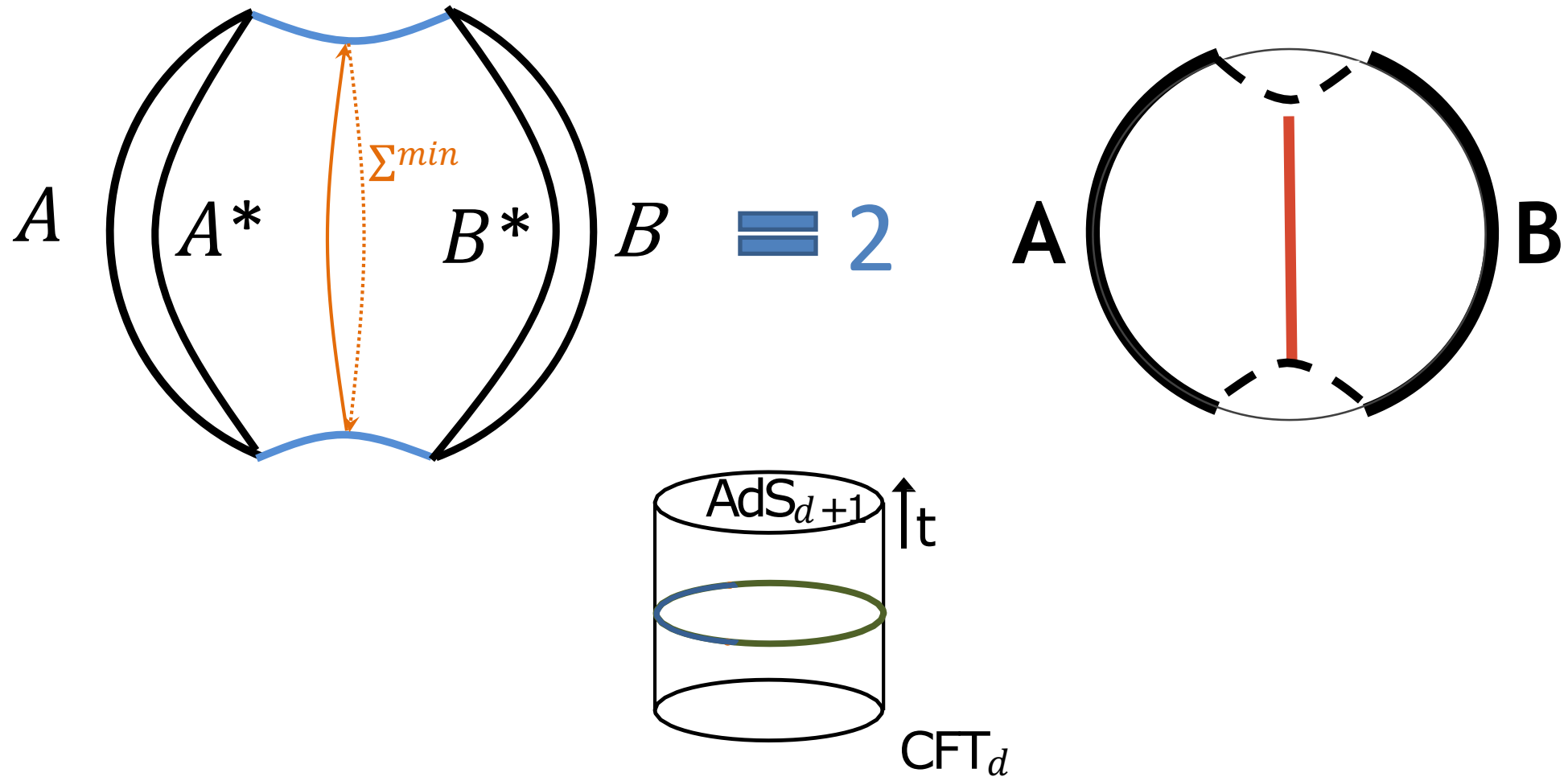
$$\psi_{ABc} \in \mathcal{H}_{ABc}$$

$$|\sqrt{\rho_{AB}}\rangle = |\sqrt{\text{Tr}_c |\psi\rangle\langle\psi|}\rangle \in (\mathcal{H}_A \otimes \mathcal{H}_{A^*}) \otimes (\mathcal{H}_B \otimes \mathcal{H}_{B^*})$$

$$\begin{aligned} S_R(A : B) &= S(AA^* : BB^*)_{\sqrt{\rho_{AB}}} \\ &= \text{Entanglement Entropy of Red Curve} \end{aligned}$$

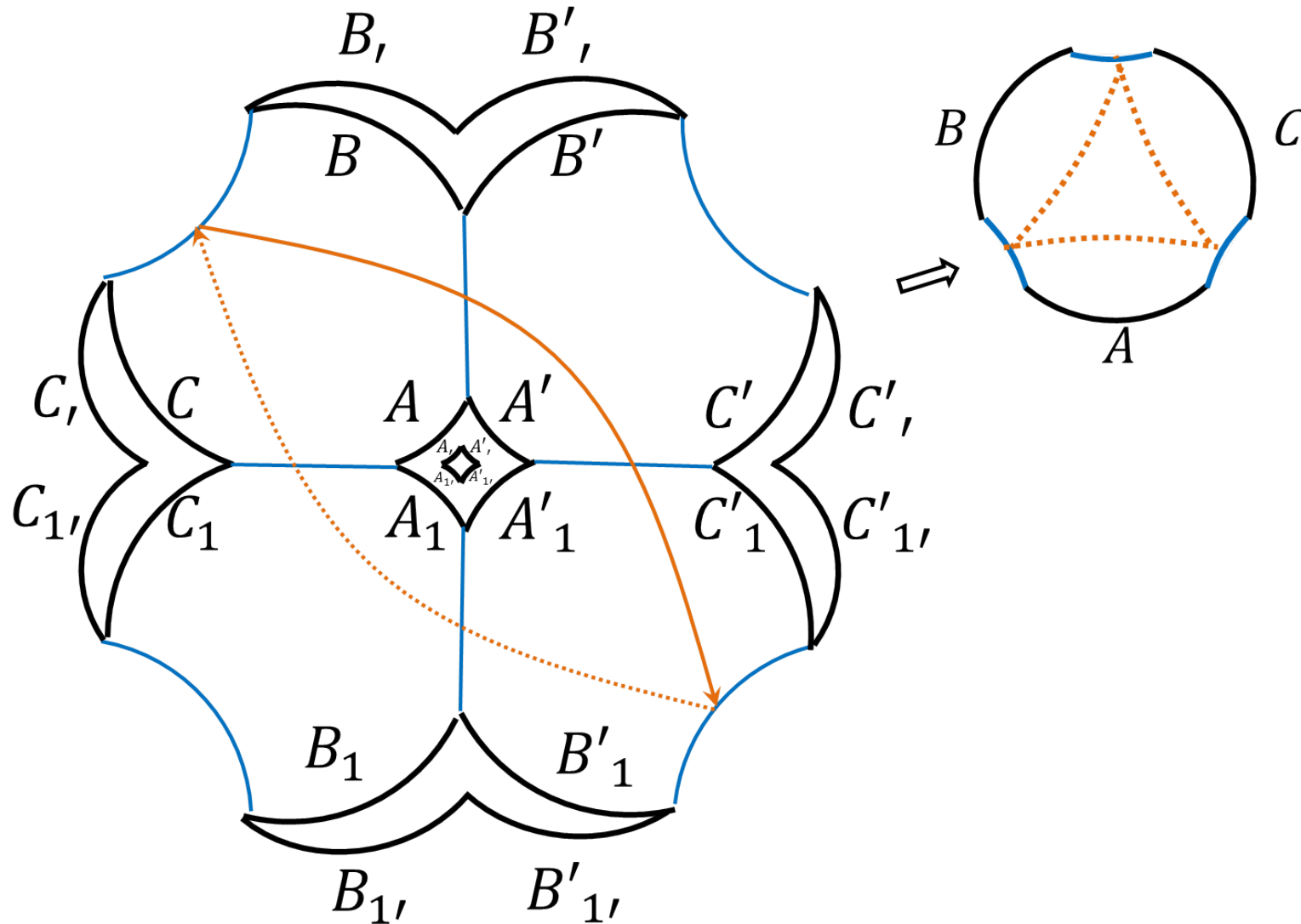
Holographic reflected entropy

[Dutta-Faulkner]




Multipartite reflected entropy $\Delta_W(A : B : C)$

[Chu-Qi-YZ,2019]



Replica trick

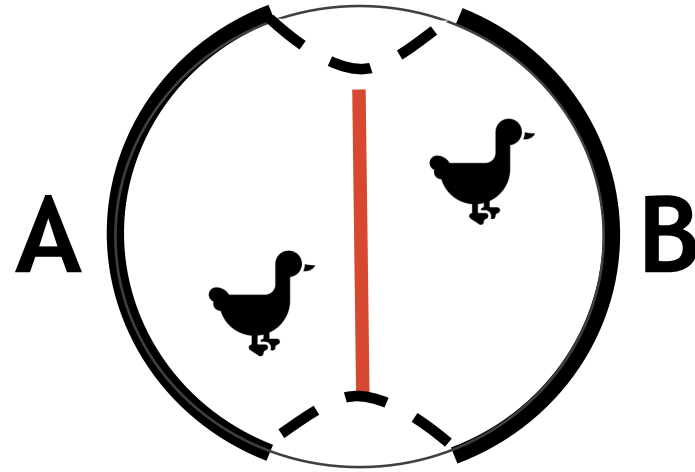
- Replica trick in canonical purifications

$$\begin{aligned} \psi_1 &= |\sqrt{\text{Tr}_c |\psi_{ABCabc}\rangle\langle\psi_{ABCabc}|} \rangle & \psi_1^{(m)} &= |(\text{Tr}_c \rho_0)^{\frac{m}{2}} \rangle, \\ \psi_2 &= |\sqrt{\text{Tr}_{bb'} |\psi_1\rangle\langle\psi_1|} \rangle & \psi_2^{(m)} &= |(\text{Tr}_{bb'} \rho_1^{(m)})^{\frac{m}{2}} \rangle, \\ \psi_3 &= |\sqrt{\text{Tr}_{aa'a''a'''} |\psi_2\rangle\langle\psi_2|} \rangle & \psi_3^{(m)} &= |(\text{Tr}_{aa'a''a'''} \rho_2^{(m)})^{\frac{m}{2}} \rangle \end{aligned}$$


- Replica trick in Renyi index

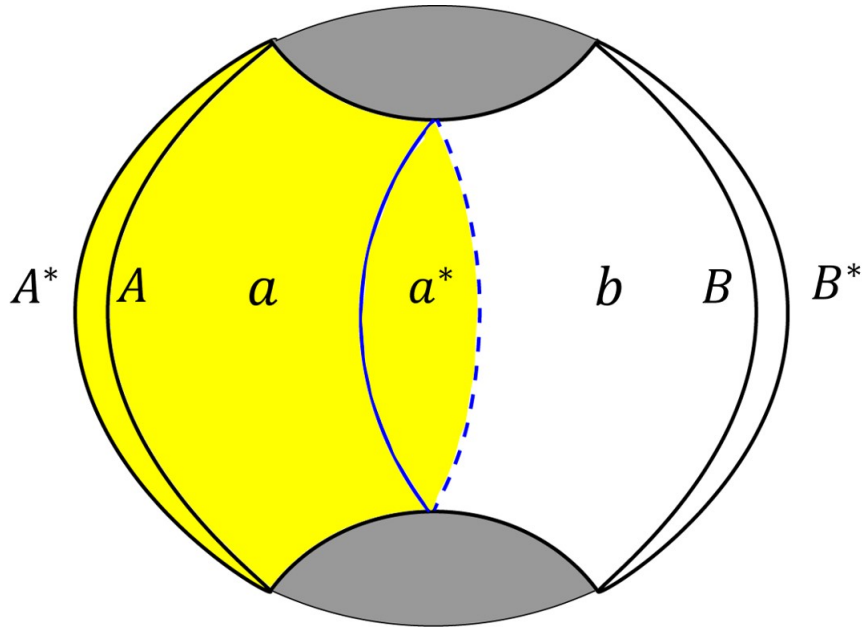
$$\Delta_R(A : B : C) = \lim_{\substack{\mathbf{n} \rightarrow 1 \\ m \rightarrow 1}} S_{\mathbf{n}}, \quad S_{\mathbf{n}} = \frac{1}{1 - \mathbf{n}} \ln \frac{\text{Tr}_R (\text{Tr}_L \rho_3^{(m)})^{\mathbf{n}}}{(\text{Tr} \rho_3^{(m)})^{\mathbf{n}}}$$

Quantum corrected reflected entropy



$$S_R(A : B) = \frac{2\langle \mathcal{A}[\partial a \cap \partial b] \rangle_{\tilde{\rho}_{ab}}}{4G_N} + S_R^{\text{bulk}}(a : b) + \mathcal{O}(G_N)$$

FLM on double replicas



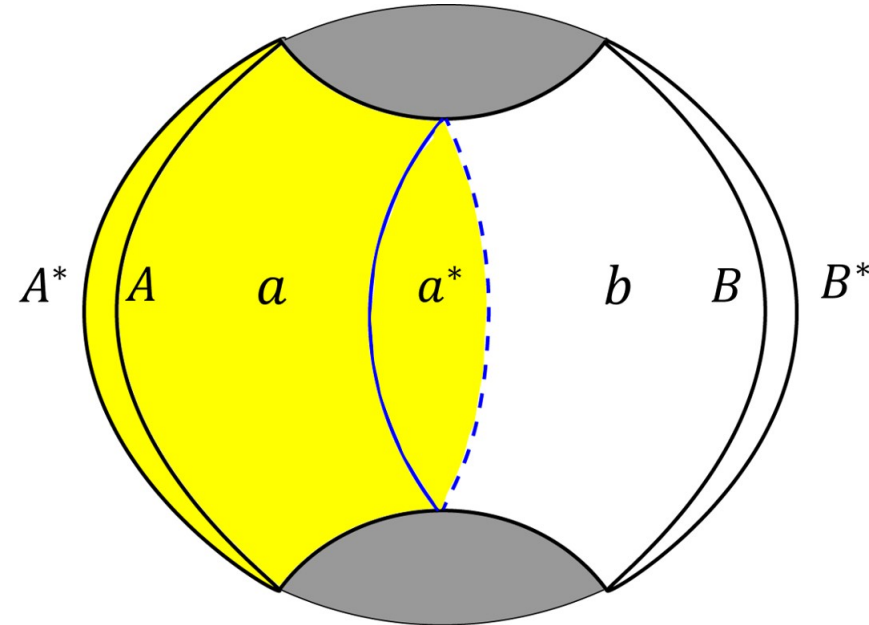
$$S(AA^*) = \frac{1}{4G_N} \langle \mathcal{A}[m(AA^*)] \rangle + S^{\text{bulk}}(aa^*) + \mathcal{O}(G_N)$$

$$\langle \mathcal{A}[m(AA^*)] \rangle = 2 \langle \mathcal{A}[\partial a \cap \partial b] \rangle$$

$$S_R^{\text{bulk}}(a : b) = S^{\text{bulk}}(aa^*)$$

Quantum extremal cross section

[Li-Chu-YZ,2020]



$$S(AA^*) = \text{ext}_Q \left\{ \frac{\text{Area}(Q)}{4G_N} + S^{\text{bulk}}(aa^*) \right\}$$

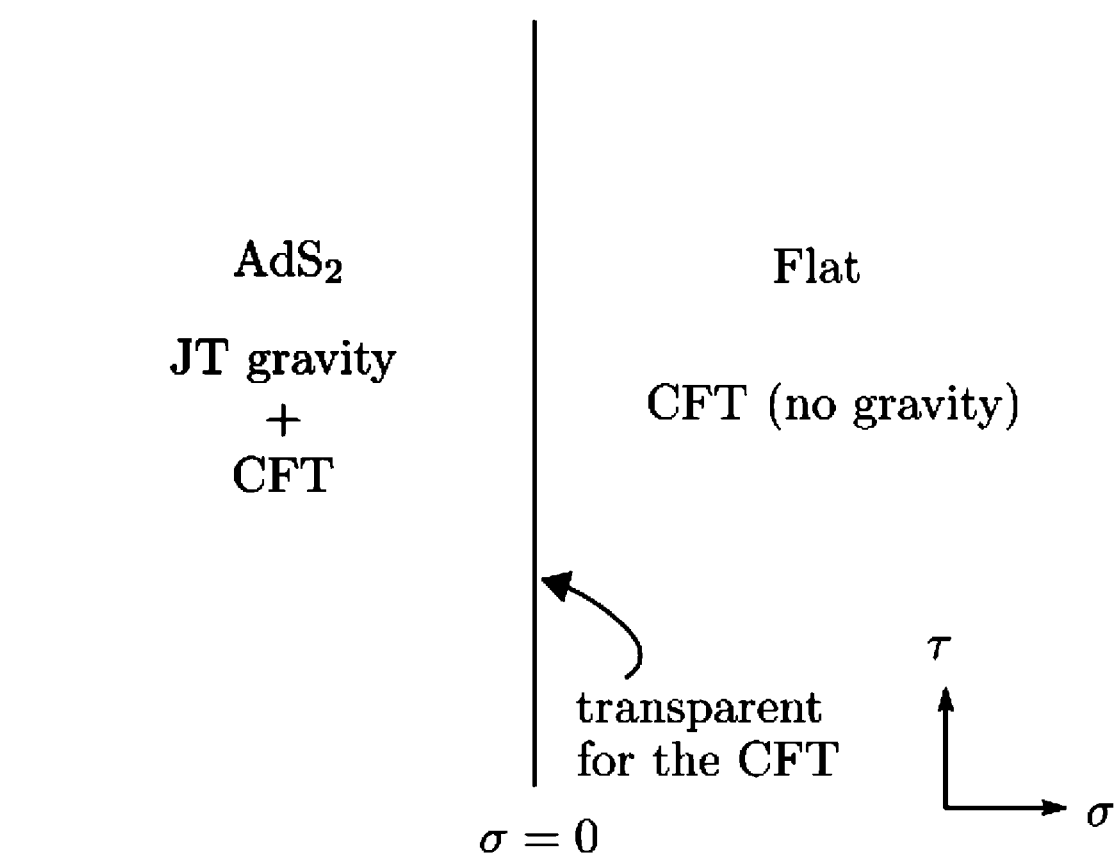
$$S_R(A : B) = \text{ext}_{Q'} \left\{ \frac{2\text{Area}(Q' = \partial a \cap \partial b)}{4G_N} + S_R^{\text{bulk}}(a : b) \right\}$$

Eternal black hole + CFT

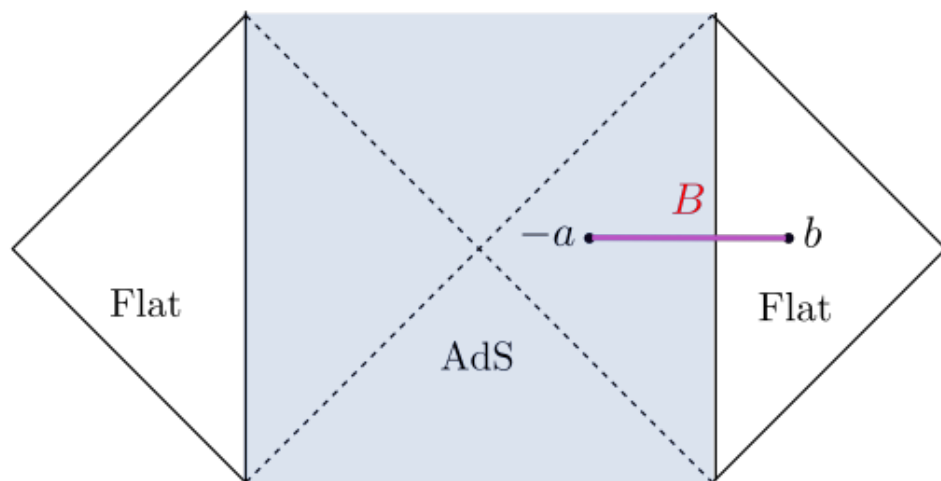
[Almheiri-Mahajan-Maldacena]

- AdS black holes do not evaporate.
- Information paradox can be realized in AdS spacetime joined to a Minkowski region, such that black hole can radiates into the attached Minkowski region
- Consider 2d Jackiw-Teiteboim gravity in AdS plus a CFT_2 (also in Minkowski), with a transparent boundary condition
- Explicit computations can be done in this model

$$I_{\text{total}} = -\frac{S_0}{4\pi} \left[\int_{\Sigma} R + \int_{\partial\Sigma} 2K \right] - \int_{\Sigma} (R + 2) \frac{\phi}{4\pi} - \frac{\phi_b}{4\pi} \int_{\partial\Sigma} 2K + S_{\text{CFT}}$$



QES for a single interval



$$ds_{\text{in}}^2 = \frac{4\pi^2}{\beta^2} \frac{dyd\bar{y}}{\sinh^2 \frac{\pi}{\beta}(y + \bar{y})}, \quad ds_{\text{out}}^2 = \frac{1}{\epsilon^2} dyd\bar{y}$$

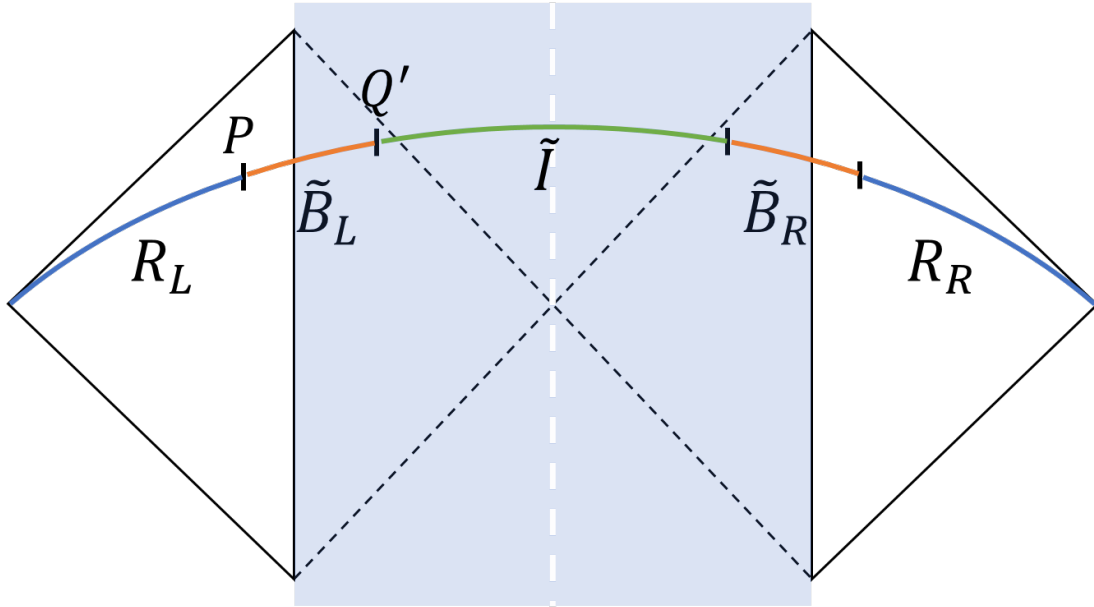
$$y = \sigma + i\tau, \quad \bar{y} = \sigma - i\tau, \quad \tau = \tau + \beta.$$

$$S_{\text{gen}} = S_0 + \phi(-a) + S_{\text{CFT}}([-a, b])$$

- $-a$ is determined following QES condition

$$\partial_a S_{\text{gen}} = 0 \quad \rightarrow \quad \sinh \left(\frac{2\pi a}{\beta} \right) = \frac{12\pi\phi_r}{\beta c} \frac{\sinh \left(\frac{\pi}{\beta}(b + a) \right)}{\sinh \left(\frac{\pi}{\beta}(a - b) \right)}$$

Gravitational reflected entropy (B-R)



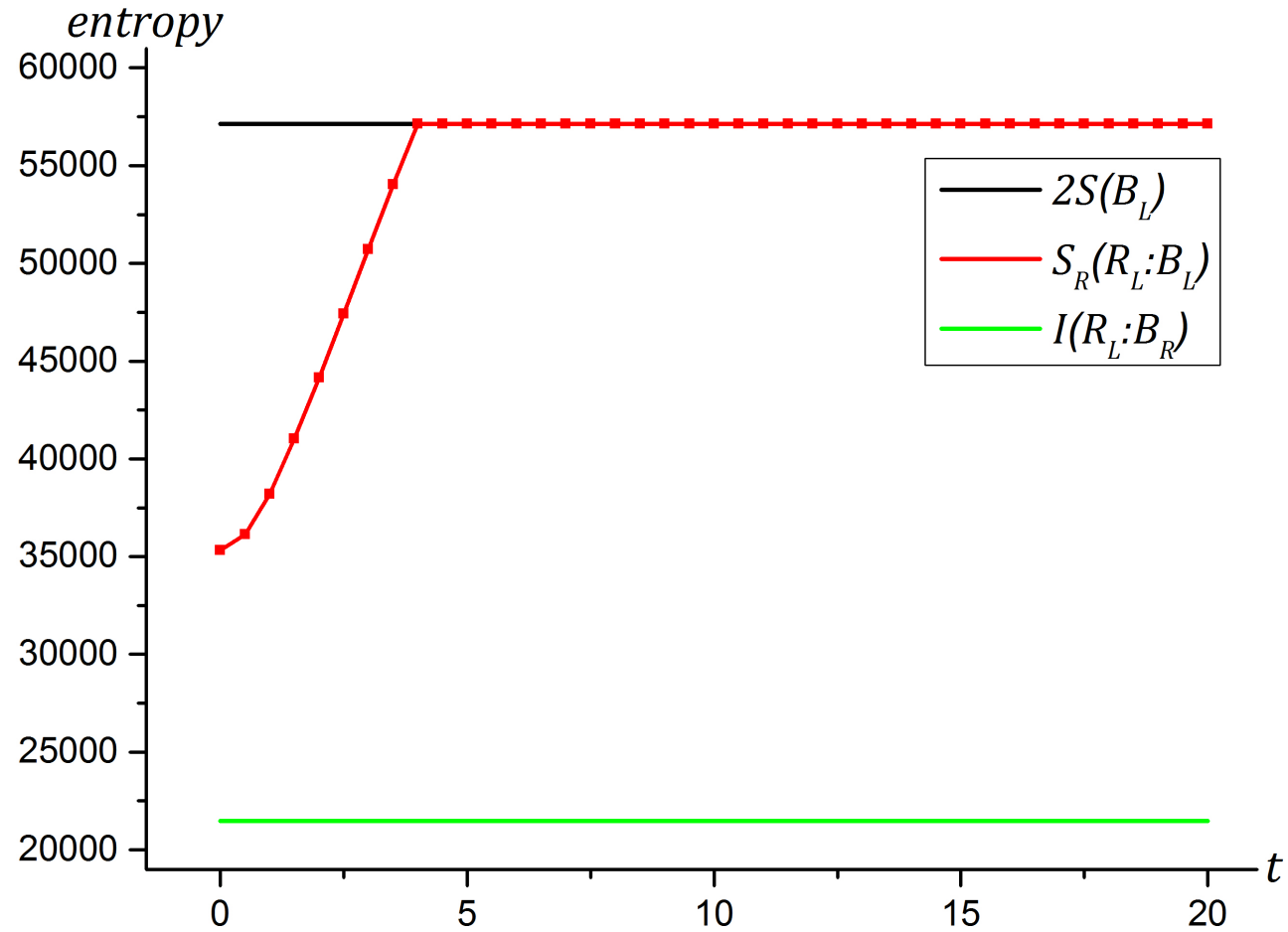
$$S(B_L) = \min \text{ext}_Q \left\{ \frac{A(Q)}{4G_N} + S(\tilde{\rho}_{B_L}) \right\}$$

$$S(R_L) = \min \text{ext}_Q \left\{ \frac{A(Q = \partial I_L)}{4G_N} + S(\tilde{\rho}_{R_L \cup I_L}) \right\}$$

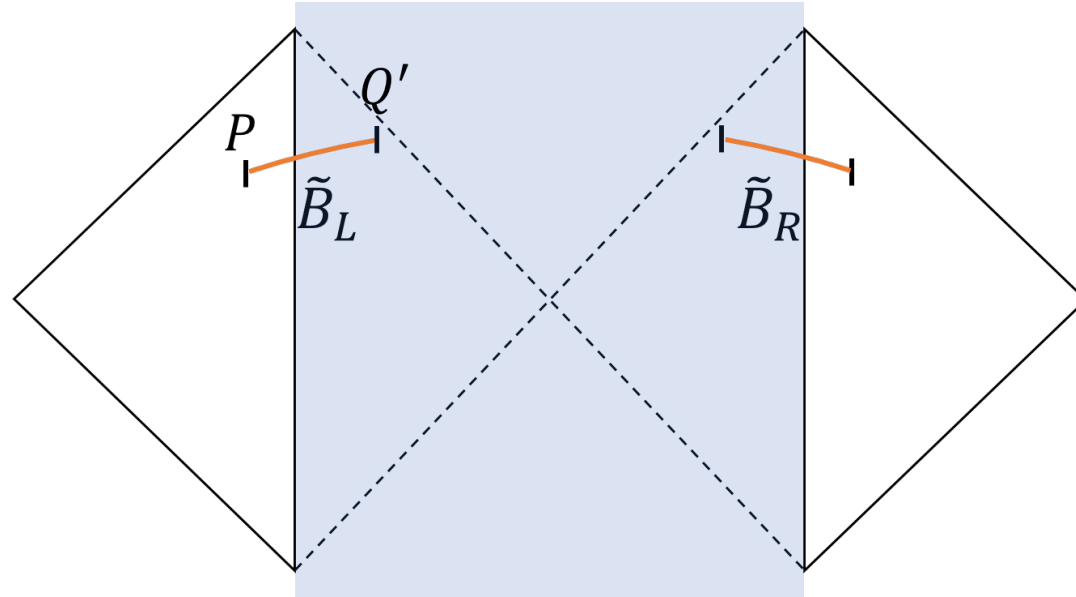
$$S_R(R_L : B_L) = S(R) = \min \text{ext}_I \left\{ \frac{A(\partial I)}{4G_N} + S(\tilde{\rho}_{R \cup I}) \right\}$$

$$S_R(R_L : B_L) = \min \text{ext}_{Q'} \left\{ \frac{2A(Q' = \partial \tilde{I}_L \cap \partial \tilde{B}_L)}{4G_N} + S_R(\tilde{\rho}_{R_L \cup \tilde{I}_L} : \tilde{\rho}_{\tilde{B}_L}) \right\}$$

$b = 0.01, \phi_r = 100, S_0 = c = 20000$ and $\epsilon_{UV} = 0.01$



B-B reflected entropy

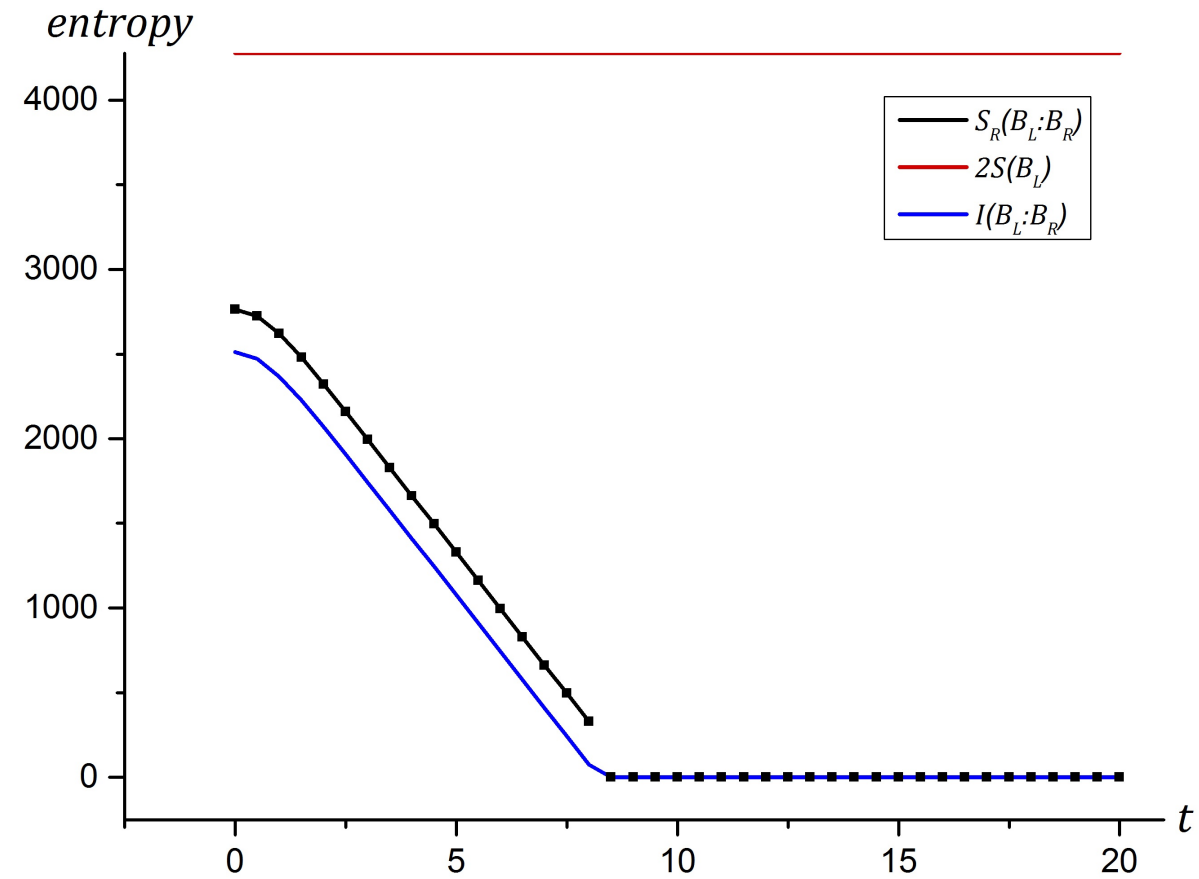


$$S_R(B_L : B_R) = \min \text{ext}_{Q'} \left\{ \frac{2A(Q' = \partial\tilde{B}_L \cap \partial\tilde{B}_R)}{4G_N} + S_R(\tilde{\rho}_{\tilde{B}_L} : \tilde{\rho}_{\tilde{B}_R}) \right\}$$

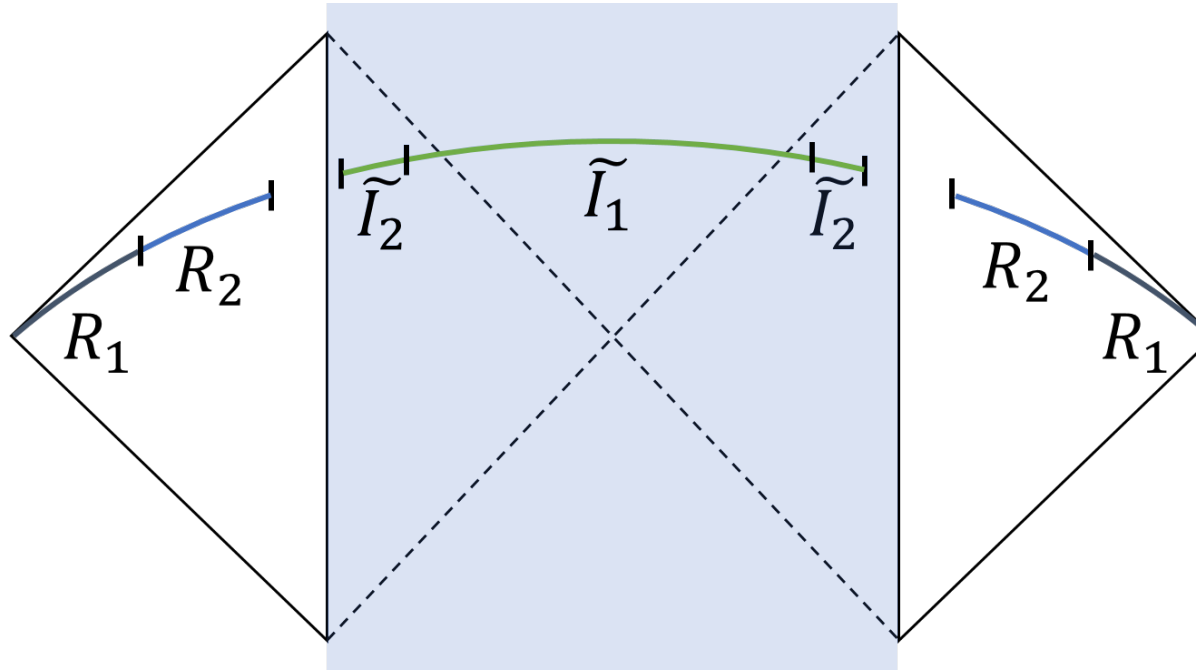
$$S_R(B_L : B_R) = 2S_0 + 2\phi_r + \frac{c}{3}(b - \ln \cosh t + \ln 2)$$

$$S_R(\tilde{\rho}_{\tilde{B}_L} : \tilde{\rho}_{\tilde{B}_R}) \sim c(-0.15\eta \ln \eta + 0.67\eta)$$

$$b = 1, \phi_r = 100, S_0 = c = 1000$$

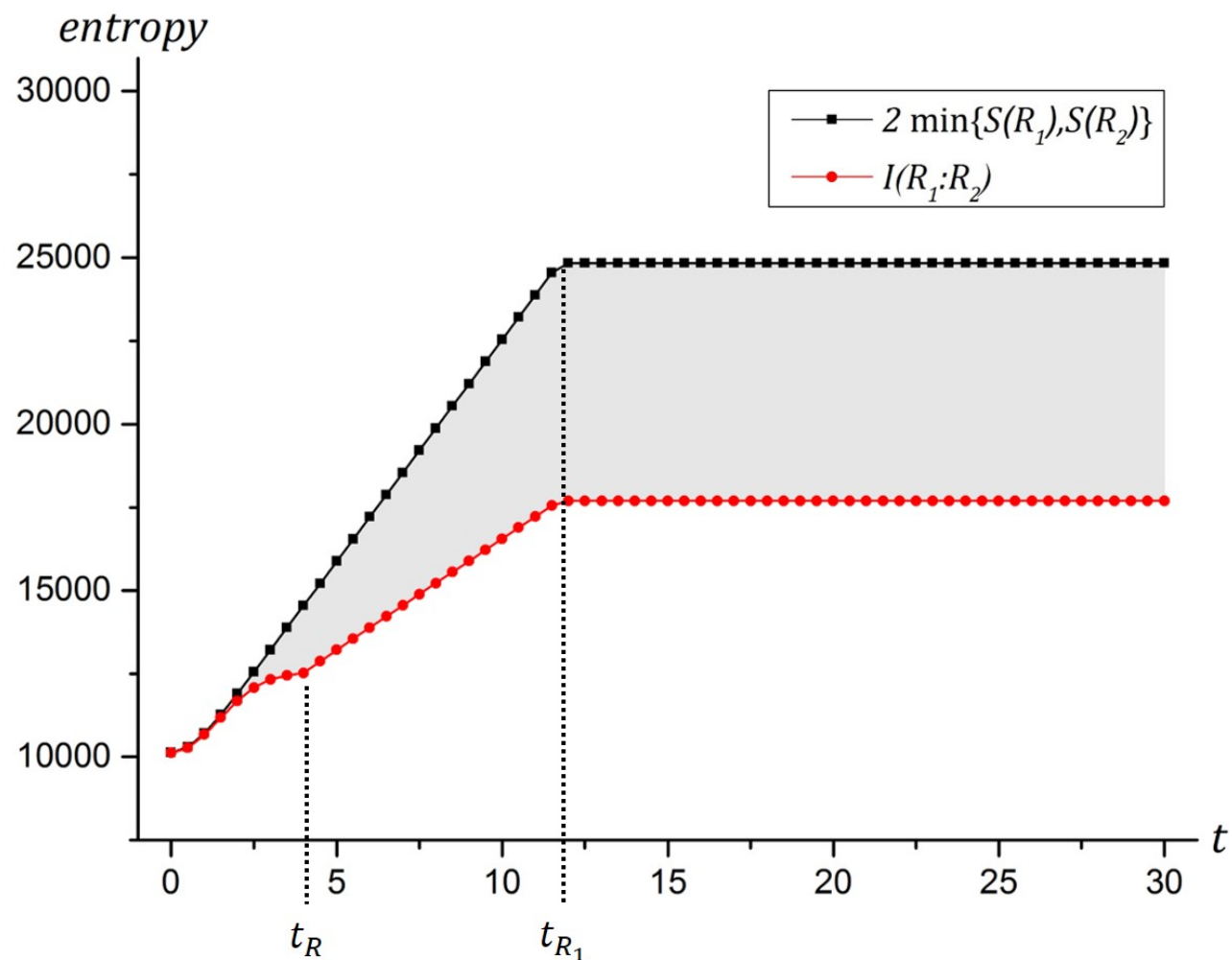


R-R reflected entropy



$$S_R(R_1 : R_2) = \min \text{ext}_{Q'} \left\{ \frac{2A(Q' = \partial\tilde{I}_1 \cap \partial\tilde{I}_2)}{4G_N} + S_R(\tilde{\rho}_{R_1 \cup \tilde{I}_1} : \tilde{\rho}_{R_2 \cup \tilde{I}_2}) \right\}$$

$b_1 = 0.01, b_2 = 5, \phi_r = 10, S_0 = c = 2000$ and $\epsilon_{UV} = 0.001$



Summary

- Reflected entropy curve is the analogy of Page curve for a globally mixed state
- Reflected entropy can be computed for R-R, B-B and B-R
- Reflected entropy has island cross-section as its area term
- Future direction: multipartite generalization (in progress)

Thank You!