Action Principle for Self-dual Forms

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Allahabad, April 2021

Plan:

1. A brief introduction to the issues

2. A brief review of closed superstring field theory

3. Action for self-dual forms

4. Hamiltonian formulation

5. Some speculations

References:

A.S., arXiv:1511.08220, arXiv:1903.12196

In 4n+2 space-time dimensions we can consider theories of 2n-form fields C with self-dual field strength F:

$$\mathbf{F} = \mathbf{dC}, \quad *\mathbf{F} = \mathbf{F}$$

*: Hodge dual

There is no conventional manifestly Lorentz invariant action describing this theory.

Normally we impose the self-duality constraint by hand.

There are several proposals to get around this problem.

1. Give up manifest Lorentz invariance

Henneaux, Teitelboim; Schwarz; Belov, Moore; $\cdots$

2. Introduce infinite number of auxiliary fields

McClain, Yu, Wu; Wotzasek; Martin, Restuccia; Devecchi, Henneaux

Faddeev, Shatashvili; Bengtsson, Kleppe; Berkovits

3. Introduce a non-polynomial action containing 1$/(\partial_\mu\phi\partial^\mu\phi)$ multiplying a term in the action

Pasti, Sorokin, Tonin, Bandos, Lechner, Nurmagambetov, Dall'Agata, $\cdots$

4. Other approaches **Siegel; Floreanini, Jackiw; Castellani, Pesando; Witten**

In this talk we shall follow a different approach motivated by superstring field theory.

Why superstring field theory?

For a long time, construction of an action for superstring field theory had suffered from an obstruction in the RR sector.

If we could construct a field theory for type IIB string theory, then by taking its low energy limit we could get a field theory for type IIB supergravity

– but this should not be possible since type IIB supergravity has a chiral four form field which does not have an action.

However eventually an action for superstring field theory was formulated

– suggests that we should be able to construct an action for type IIB suergravity!

A brief review of the structure of classical type II superstring field theory:

$\mathcal{H}_{\text{total}} = \oplus_{m,n} \mathcal{H}_{m,n}$: A graded vector space, $m,n \in \mathbb{Z}/2$

$\langle A|B\rangle$: an inner product between $A \in \mathcal{H}_{m,n}$ with $B \in \mathcal{H}_{-2-m,-2-n}$

String field has two components, $|\psi\rangle$ and $|\phi\rangle$

$$|\psi\rangle \in \mathcal{H} \equiv \mathcal{H}_{-1,-1} \oplus \mathcal{H}_{-1,-1/2} \oplus \mathcal{H}_{-1/2,-1} \oplus \mathcal{H}_{-1/2,-1/2}$$

$$|\phi\rangle \in \widetilde{\mathcal{H}} \equiv \mathcal{H}_{-1,-1} \oplus \mathcal{H}_{-1,-3/2} \oplus \mathcal{H}_{-3/2,-1} \oplus \mathcal{H}_{-3/2,-3/2}$$

$$|\psi\rangle \in \mathcal{H} \equiv \mathcal{H}_{-1,-1} \oplus \mathcal{H}_{-1,-1/2} \oplus \mathcal{H}_{-1/2,-1} \oplus \mathcal{H}_{-1/2,-1/2}$$

$$|\phi\rangle \in \widetilde{\mathcal{H}} \equiv \mathcal{H}_{-1,-1} \oplus \mathcal{H}_{-1,-3/2} \oplus \mathcal{H}_{-3/2,-1} \oplus \mathcal{H}_{-3/2,-3/2}$$

Action

$$\mathbf{S} = \langle\phi|\mathcal{Q}|\psi\rangle - \frac{1}{2}\langle\phi|\mathcal{Q}\,\mathbf{G}|\phi\rangle + \sum_{n} \frac{1}{n!}\{\psi^{n}\}$$

$\mathcal{Q}$**: maps** $\mathcal{H} \mapsto \mathcal{H}$**,** $\widetilde{\mathcal{H}} \mapsto \widetilde{\mathcal{H}}$**,**

G: maps $\widetilde{\mathcal{H}} \mapsto \mathcal{H}$ **(identity on** $\mathcal{H}_{-1,-1}$**)**

$\{\mathbf{A}_1 \cdots \mathbf{A}_n\}$**: maps** $\mathcal{H}^{n} \mapsto \mathbb{R}$**.**

$$\mathcal{Q}\,\mathbf{G} = \mathbf{G}\,\mathcal{Q}$$

$$\mathbf{S} = \langle\phi|\mathcal{Q}|\psi\rangle - \frac{1}{2}\langle\phi|\mathcal{Q}\,\mathbf{G}|\phi\rangle + \sum_{\mathbf{n}} \frac{1}{\mathbf{n}!}\{\psi^{\mathbf{n}}\}$$

Derive equations of motion from S:

$$\mathcal{Q}(|\psi\rangle - \mathbf{G}|\phi\rangle) = \mathbf{0}$$

$$\mathcal{Q}|\phi\rangle + \sum_{\mathbf{N}} \frac{1}{\mathbf{N}!}[\psi^{\mathbf{N}}] = \mathbf{0}$$

For $\mathbf{A_1}, \cdots, \mathbf{A_N} \in \mathcal{H}$, $[\mathbf{A_1} \cdots \mathbf{A_N}] \in \widetilde{\mathcal{H}}$ is defined via:

$$\langle\mathbf{A}|[\mathbf{A_1}\cdots\mathbf{A_N}]\rangle = \{\mathbf{A}\,\mathbf{A_1}\cdots\mathbf{A_N}\}$$

Combine the two equations to get the interacting field equation:

$$\mathcal{Q}|\psi\rangle + \sum_{\mathbf{N}} \frac{1}{\mathbf{N}!}\,\mathbf{G}\,[\psi^{\mathbf{N}}] = \mathbf{0}$$

$$\mathcal{Q}|\psi\rangle + \sum_{\mathbf{N}} \frac{\mathbf{1}}{\mathbf{N}!}\,\mathbf{G}\,[\psi^{\mathbf{N}}] = \mathbf{0}$$

$$\mathcal{Q}|\phi\rangle + \sum_{\mathbf{N}} \frac{\mathbf{1}}{\mathbf{N}!}[\psi^{\mathbf{N}}] = \mathbf{0}$$

Once we solve the first equation for $|\psi\rangle$, the solution for $|\phi\rangle$ to the second equation is fixed up to addition of solutions to free field equations:

$$\mathcal{Q}|\phi\rangle = \mathbf{0}$$

Furthermore which solution we choose does not affect the equations of motion for $|\psi\rangle$.

Therefore even though we have doubled the number of degrees of freedom, $\cdots$

$\cdots$ one set of degrees of freedom describe free fields that completely decouple from the interacting sector, and are unobservable.

$$\mathcal{Q}(\mathbf{G}|\phi\rangle - |\psi\rangle) = \mathbf{0}$$

$$\mathcal{Q}|\phi\rangle + \sum_{\mathbf{N}} \frac{\mathbf{1}}{\mathbf{N}!}[\psi^{\mathbf{N}}] = \mathbf{0}$$

$$|\psi\rangle \in \mathcal{H} \equiv \mathcal{H}_{-1,-1} \oplus \mathcal{H}_{-1,-1/2} \oplus \mathcal{H}_{-1/2,-1} \oplus \mathcal{H}_{-1/2,-1/2}$$

$$|\phi\rangle \in \widetilde{\mathcal{H}} \equiv \mathcal{H}_{-1,-1} \oplus \mathcal{H}_{-1,-3/2} \oplus \mathcal{H}_{-3/2,-1} \oplus \mathcal{H}_{-3/2,-3/2}$$

On $\mathcal{H}_{-1,-1}$, G is identity

Therefore, in this sector we can solve the first equation by setting $\phi = \psi$

– doubling of the degrees of freedom not needed

NSNS sector fields, including the metric, reside in $\mathcal{H}_{-1,-1}$ and therefore doubling of the metric degrees of freedom is not needed.

Gauge transformations

The gauge transformation parameters are also elements of $\mathcal{H}, \widetilde{\mathcal{H}}$.

$$|\theta\rangle \in \mathcal{H}, \qquad |\widetilde{\theta}\rangle \in \widetilde{\mathcal{H}}$$

$$|\delta\psi\rangle = \mathcal{Q}|\theta\rangle + \sum_{\mathbf{n}} \mathbf{G}\,|[\theta\psi^{\mathbf{n}}]\rangle, \qquad |\delta\phi\rangle = \mathcal{Q}|\widetilde{\theta}\rangle + \sum_{\mathbf{n}} |[\theta\psi^{\mathbf{n}}]\rangle$$

$|\widetilde{\theta}\rangle$ **free field gauge transformation**

– does not affect the interacting field $|\psi\rangle$

Non-trivial gauge transformations like general coordinate transformation are in θ.

Peculiar property: Transformation law of ϕ involves ψ, not ϕ

– strange for general coordinate transformation

This construction holds also for type IIB string theory.

It contains a self-dual 5-form field strength in D=10.

Furthermore compactification of type IIB on K3 has self-dual 3-form field strengths in D=6.

IIB compactification on K3$\times$K3 has chiral scalar fields in D=2.

Therefore the low energy effective action of superstring field theory must contain Lorentz invariant action for these chiral fields.

This suggests that we should be able to construct an action for chiral fields in 4n+2 dimensions by doubling the degrees of freedom.

In principle it should be possible to begin with the string field theory action and take the low energy limit to get the action for type IIB supergravity.

But we shall use the insights from string field theory and directly write down an action for type IIB supergravity and other theories with chiral field.

We shall take the minimalist approach where we only double the sector containing the chiral field, treating all other fields as if they are in $\mathcal{H}_{-1,-1}$ where we can avoid doubling.

Proposed field content:

– a 2n-form potential P (like $|\phi\rangle$)

– a (2n+1)-form self-dual field strength Q (like $|\psi\rangle$)

Proposed action:

$$\mathbf{S} = \frac{1}{2}\int \mathbf{dP} \wedge *\mathbf{dP} - \int \mathbf{dP} \wedge \mathbf{Q} + \int \mathcal{L}_{\mathbf{I}}(\mathbf{Q}, \chi)\,,$$

χ: all other fields including the metric

Q satisfies $*\mathbf{Q} = \mathbf{Q}$,   *: Hodge dual with metric $\eta_{\mu\nu}$

$\mathcal{L}_I$: A general interacting Lagrangian density
(assumed to not contain derivatives of Q when we discuss Hamiltonian formalism)

$$\mathbf{S} = \frac{1}{2}\int \mathbf{dP}\wedge *\mathbf{dP} - \int \mathbf{dP}\wedge\mathbf{Q} + \int \mathcal{L}_{\mathbf{I}}(\mathbf{Q},\chi)\,,$$

Equations of motion:

$$\mathbf{d}(*\mathbf{dP}-\mathbf{Q}) = \mathbf{0}, \quad \mathbf{dP} - *\mathbf{dP} + \mathbf{R}(\mathbf{Q},\chi) = \mathbf{0}, \quad \frac{\delta}{\delta\chi}\int \mathcal{L}_{\mathbf{I}} = \mathbf{0}$$

R is an anti-selfdual (2n+1)-form defined through:

$$\delta\int \mathcal{L}_{\mathbf{I}} = -\frac{1}{2}\int \mathbf{R}\wedge\delta\mathbf{Q}$$

Adding the exterior derivative of the second equation to the first equation we get

$$\mathbf{d}(\mathbf{R}-\mathbf{Q}) = \mathbf{0}\,.$$

$$d(*dP - Q) = 0, \quad dP - *dP + R(Q,\chi) = 0, \quad d(Q - R) = 0$$

Once we find Q by solving the third equation, the anti-self-dual part of dP is fixed by the middle equation.

Writing the first equation as

$$d(dP + *dP - Q) = 0$$

we see that the self-dual part of dP is also fixed except for the freedom of adding a self-dual form A satisfying

$$dA = 0$$

This is a free field degree of freedom.

Furthermore the choice of A does not affect the interacting field equations $d(Q - R) = 0$.

Net content: An unobservable free field and an observable interacting field Q.

$$S = \frac{1}{2}\int dP \wedge *dP - \int dP \wedge Q + \int \mathcal{L}_I(Q, \chi)\,,$$

Q satisfies $*Q = Q$,   *: Hodge dual with metric $\eta_{\mu\nu}$

Eventually we need a field that is self-dual with respect to the dynamical metric contained in χ, not with respect to $\eta_{\mu\nu}$.

Also the field strength involves Chern-Simons terms involving fields χ.

Nevertheless, string field theory suggests that it should be possible to encode this into the framework described here.

Peculiar feature: Under general coordinate transformation, both δP and δQ must involve Q, not P.

A brief summary of the construction

In the usual formulation of a self-dual form coupled to the metric, we consider an action of the form:

$$S = S_1 + S_2(\chi)$$

$$S_1 \equiv -\frac{1}{2}\int (dC + Y) \wedge *_g (dC + Y) + \int dC \wedge Y\,,$$

Constraint

$$(dC + Y) = *_g (dC + Y)$$

Y: a (2n+1)-form constructed out of the other fields χ

S_2 is a function of the other fields χ, including the metric

$*_g$: Hodge dual with respect to the metric $g_{\mu\nu}$.

Question: How do we replace the self-duality constraint with $g_{\mu\nu}$ for dC+Y by self-duality constraint with respect to $\eta_{\mu\nu}$ for Q?

It turns out that it is possible, via a non-standard coupling of the metric to Q.

$$S_1 \equiv -\frac{1}{2}\int (dC + Y) \wedge *_g (dC + Y) + \int dC \wedge Y\,,$$

can be replaced by

$$\begin{aligned} S_1' &= \frac{1}{2}\int dP \wedge *dP - \int dP \wedge Q \\ &+ \int d^{4n+2}x\,\left[\frac{1}{16}Q^T\mathcal{M}\,Q + \frac{1}{2}Q^T\left\{\frac{1}{2}\mathcal{M}\,Y - (\zeta - \varepsilon)\,Y\right\} - \frac{1}{2}Y^T\zeta\,Y + \frac{1}{4}Y^T\mathcal{M}\,Y\right]. \end{aligned}$$

$\mathcal{M}$: a $\binom{4n+2}{2n+1} \times \binom{4n+2}{2n+1}$ matrix determined in terms of the metric.

ζ, ε: constant matrices

Identification of variables: $Q = dC + *dC$

$*dY$ acts as the source for the (2n+1)-form field strength.

This construction has also been generallized to the case when the self-duality constraint is non-linear in the field strength, generalizing the DBI type action to chiral fields.

$$\mathbf{S} = \frac{\mathbf{1}}{\mathbf{2}} \int \mathbf{dP} \wedge *\mathbf{dP} - \int \mathbf{dP} \wedge \mathbf{Q} + \int \mathcal{L}_{\mathbf{I}}(\mathbf{Q}, \chi)\,,$$

Note that the decoupling does not happen at the level of the action but only at the level of the equations of motion

– may seem a little unsatisfactory.

We shall now show that the same decoupling can be seen at the level of the Hamiltonian.

In the following we shall treat P and Q in the canonical formulation but the other fields χ in the Lagrangian formulation.

Result: Routhian (although we shall denote it by H)

Conjugate momenta:

$$\Pi_P^{i_1\cdots i_{2n}} = -(\partial_0 P_{i_1\cdots i_{2n}} + \partial_{[i_1} P_{i_2\cdots i_{2n}]0}) - \sum_{\substack{j_1,\cdots,j_{2n+1} \\ j_1<j_2<\cdots<j_{2n+1}}} \epsilon^{i_1\cdots i_{2n} j_1\cdots j_{2n+1}} Q_{j_1\cdots j_{2n+1}},$$

$$\Pi_P^{0 i_1\cdots i_{2n-1}} = 0, \qquad \Pi_Q^{i_1\cdots i_{2n+1}} = 0\,.$$

Note: The last two are constraint equations

$$H = \sum_{\substack{i_1,\cdots,i_{2n} \\ i_1<i_2\cdots<i_{2n}}} \int d^{4n+1}x\, \Pi_P^{i_1\cdots i_{2n}} \partial_0 P_{i_1\cdots i_{2n}} - L$$

up to addition of constraints.

1. Compute the Poisson brackets of the constraints with H to generate more constraints

2. Analyze Poisson brackets of the constraints with each other to sort out first and second class constraints.

Define new variables:

$$\Pi_{\pm}^{i_1\cdots i_{2n}} \equiv \frac{1}{2}\left(\Pi_{P}^{i_1\cdots i_{2n}} \pm \epsilon^{i_1\cdots i_{2n} j_1\cdots j_{2n+1}}\,\partial_{[j_1} P_{j_2\cdots j_{2n+1}]}\right)$$

Result: Second class constraints:

$$\Pi_{Q}^{i_1\cdots i_{2n+1}} = 0$$

$$Q_{j_1\cdots j_{2n+1}} + 2\,\epsilon^{i_1\cdots i_{2n} j_1\cdots j_{2n+1}}\,\Pi_{-}^{i_1\cdots i_{2n}} + \frac{\partial \mathcal{L}_I}{\partial Q_{j_1\cdots j_{2n+1}}} = 0$$

– can be used to eliminate $Q_{j_1\cdots j_{2n+1}}$ and $\Pi_{Q}^{i_1\cdots i_{2n+1}}$.

First class constraints generating gauge transformations:

$$\Pi_{P}^{0 i_1\cdots i_{2n-1}} = 0, \qquad \partial_{i_1}\Pi_{P}^{i_1\cdots i_{2n}} = 0$$

The second constraint is equivalent to:

$$\Rightarrow \partial_{i_1}\Pi_{\pm}^{i_1\cdots i_{2n}} = 0$$

$\Pi_{\pm}$ are gauge invariant.

The Routhian is given by $\mathbf{H} = \mathbf{H}_+ + \mathbf{H}_-$ where

$$\mathbf{H}_+ = -\int \mathbf{d}^{4n+1}\mathbf{x}\,\Pi_+^{i_1\cdots i_{2n}}\Pi_+^{i_1\cdots i_{2n}},$$

$$\mathbf{H}_- = \int \mathbf{d}^{4n+1}\mathbf{x}\,\Pi_-^{i_1\cdots i_{2n}}\Pi_-^{i_1\cdots i_{2n}} + \int \mathbf{d}^{4n+1}\mathbf{x}\,\mathbf{g}(\Pi_-,\chi)$$

$$\mathbf{g}(\Pi_-,\chi) = \left[-\frac{1}{2}\sum_{\substack{j_1\cdots j_{2n+1}\\ j_1<j_2<\cdots<j_{2n+1}}}\left(\frac{\partial\mathcal{L}_I}{\partial \mathbf{Q}_{j_1\cdots j_{2n+1}}}\right)^2 - \mathcal{L}_I(\mathbf{Q},\chi)\right]$$

evaluated at the solution to the constraint:

$$\mathbf{Q}_{j_1\cdots j_{2n+1}} + 2\,\epsilon^{i_1\cdots i_{2n}j_1\cdots j_{2n+1}}\,\Pi_-^{i_1\cdots i_{2n}} + \frac{\partial\mathcal{L}_I}{\partial \mathbf{Q}_{j_1\cdots j_{2n+1}}} = 0$$

Dirac brackets:

$$\{\Pi_+^{i_1\cdots i_{2n}}(t,\vec{x}),\Pi_+^{k_1\cdots k_{2n}}(t,\vec{y})\}_{DB}=\frac{1}{2}\epsilon^{i_1\cdots i_{2n}j_1k_1\cdots k_{2n}}\frac{\partial}{\partial x^{j_1}}\delta(\vec{x}-\vec{y})\,,$$

$$\{\Pi_-^{i_1\cdots i_{2n}}(t,\vec{x}),\Pi_-^{k_1\cdots k_{2n}}(t,\vec{y})\}_{DB}=-\frac{1}{2}\epsilon^{i_1\cdots i_{2n}j_1k_1\cdots k_{2n}}\frac{\partial}{\partial x^{j_1}}\delta(\vec{x}-\vec{y})\,,$$

$$\{\Pi_+^{i_1\cdots i_{2n+1}}(t,\vec{x}),\Pi_-^{j_1\cdots j_{2n+1}}(t,\vec{y})\}_{DB}=0\,.$$

Recall that $H=H_++H_-$ where

$$H_+=-\int d^{4n+1}x\,\Pi_+^{i_1\cdots i_{2n}}\Pi_+^{i_1\cdots i_{2n}}\,,$$

$$H_-=\int d^{4n+1}x\,\Pi_-^{i_1\cdots i_{2n}}\Pi_-^{i_1\cdots i_{2n}}+\int d^{4n+1}x\,g(\Pi_-,\chi)$$

– sum of a free Hamiltonian H_+ and an interacting Routhian H_-

– confirms the result based on the analysis of equations of motion.

Using this formalism one can address some apparent puzzles involving compactification of theories of self-dual forms.

e.g. if we start with a self-dual 3-form in D=6, upon compactification on T^2 we should get U(1) gauge theory in D=4

In standard compactification the dimensionally reduced action is multiplied by the area $R_1 R_2$ of T^2

– should act as the inverse coupling.

But the expected coupling from S-duality is R_1/R_2! Witten

In the new formulation, due to the non-standard coupling of the metric, the dimensionally reduced action no longer has $R_1 R_2$ as an overall multiplicative factor.

A detailed analysis shows that the dimensionally reduced action and the associated Routhian are indeed S-duality invariant.

Speculations

This formalism allows us to construct the action of a single M5-brane

What about multiple M5-branes?

– expected to have tensionless strings carrying charge under the three form field strength

The possibility of including these degrees of freedom exist in this formulation since $*dY$ acts as the source for the self-dual field.

The rest of the talk will be some speculations on how we might construct appropriate Y representing tensionless string source.

Go back to type IIB on K3.

The self-dual 3-forms arise from components of self-dual 5-form along a harmonic two form on K3.

D3-branes wrapped on the dual 2-cycles are charged under the resulting 3-form field strength in 6D.

The area modulus X of the 2-cycle is part of the same supermultiplet as the 3-form field.

When X approaches zero, the D3-brane wrapped on the 2-cycle has zero tension

– a tensionless string

– related to the interacting part of the theory of multiple M5-branes.

Question: Is there a field theory that can describe these strings, but does not introduce other fields?

Strings can be generated as solitons, e.g. instanton solutions in 6D Yang-Mills theory. $\cdots$, Kim, Kim, Park

However this also introduces Yang-Mills fields in the theory.

One example where this does not happen is tachyon condensation in open string theory.

Tachyons can produce solitons without generating perturbative excitations around the vacuum.

In type IIB on K3, we can consider a pair of D7-branes and a pair of anti-D7 branes wrapped on the 2-cycle of size X.

Tachyon condensation on this system can produce a D3-brane wrapped on the 2-cycle – the string that we want to generate.

Open string field theory on the D7-$\overline{\text{D7}}$ brane system, coupled to the abelian tensor multiplet, will contain the desired CFT

The relevant part of the open string field theory action:

$$X\, g_s^{-1}\, S_{open}$$

g_s: string coupling (eventually taken to 0)

S_{open}: Open string field theory action

*dY: related to ‘Ellwood invariant’ measuring D3-brane charge wrapped on 2-cycle of size X

Question: Is there a truncation to field theory with finite number of fields?

Can we write down a tachyon action of the form

$$\mathbf{S}_{\text{tachyon}} = \mathbf{X}\,\mathbf{S}(\mathbf{T}, \cdots)$$

with T a 2×2 complex matrix valued scalar, describing the physics of tachyon condensation?

We could then identify the source dY appearing earlier to the 4-form constructed from T that measures the D3-brane charge

$$\mathbf{dY} \propto \mathrm{Tr}(\mathbf{dT} \wedge \mathbf{dT}^\dagger \wedge \mathbf{dT} \wedge \mathbf{dT}^\dagger)$$

At present the only known perturbatively UV finite theory of this kind is p-adic open bosonic string field theory with action

$$\int \left[-\frac{1}{2} T p^{-\Box/2} T + \frac{1}{p+1} T^{p+1} \right], \qquad p : \text{a prime number},$$

with a real field T.

Freund, Olson; Freund, Witten; Brekke, Freund, Olson, Witten

– captures the physics of tachyon condensation in open bosonic string theory quite well

Ghoshal, A.S.

A generalization of this to superstring theory could give us a quantum field theory on multiple M5-branes