Hydrodynamics beyond hydrodynamics

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Hydrodynamics is an established field with a venerable history and many applications.

However, today I will not talk about applications.

Rather, I would like to highlight some foundational questions that came to light in the last few years.

What is hydrodynamics?

Set of equations that tell you how stuff flows. “Stuff” can be water, air, a cold atomic cloud, hot primodial matter in early Universe, electron fluid in a solid, etc.

As a student, you open a book with “hydrodynamics” or “fluid dynamics” in the title. You often see derivations, approximations, and applications all mixed together.

If, as a student, you are also learning about vector calculus and partial differential eq-s at the same time, it can be hard to see the big picture.

But the big picture of hydrodynamics
is in fact quite simple

Conserved quantities

Fundamentally, hydrodynamics is a classical theory of things that can not disappear, i.e. are conserved.

$$\frac{\partial}{\partial t}\rho_a = -\nabla \cdot \mathbf{j}_a$$

density of some conserved quantity a

flux of the same conserved quantity a

a = energy, momentum, number of particles,…

Constitutive relations

One eq-n $\partial_t \rho = -\nabla \cdot \mathbf{j}$ for both ρ and $\mathbf{j}$. Hydro assumption: assume $\mathbf{j} = \mathbf{j}[\rho]$, then have eq-n for ρ only, can solve!

More generally, take some useful quantities γ, (temperature, velocity,…), express $\rho_a = \rho_a(\gamma),\ \mathbf{j}_a = \mathbf{j}_a(\gamma)$, then get eq-s for γ_a only, can solve!

Example: a = energy, $\gamma = T$ = temperature, then:

$\rho_\epsilon = \epsilon(T)$, constitutive relation $\mathbf{j}_\epsilon = -\kappa \boldsymbol{\nabla} T$,

$T=T_0+\delta T$, get diffusion equation for δT :

$$\boxed{\frac{\partial}{\partial t}\delta T = -D\nabla^2 \delta T}$$

$D = \kappa/\epsilon'(T_0)$

Summary of hydrodynamics

Conservarion laws: $\frac{\partial}{\partial t}\rho_a = -\nabla \cdot \mathbf{j}_a$

Constitutive relations (γ = temperature, fluid velocity,…) :

$$\rho_a = \rho_a(\gamma, \nabla\gamma, \nabla^2\gamma, \ldots),$$

$$\mathbf{j}_a = \mathbf{j}_a(\gamma, \nabla\gamma, \nabla^2\gamma, \ldots)$$

The procedure is the same, whether the fundamental constituents are classical or quantum, relativistic or not, normal fluid or superfluid, magnetic fields present or not, fluid is chiral or not.

Questions to ask about every theory

Every theory in physics is only approximately “correct”, limited by its domain of applicability.

Whenever we write down any equations that attempt to describe physical phenomena, we have to answer:

1. Do the equations make physical sense?
2. Can we improve the equations to capture more physics?
3. What kind of physics is beyond our equations?

What will follow are three stories, one for each question.

First story:

Do hydrodynamic equations even make sense?

Relativistic things

Say, you are a student in subatomic physics or astrophysics, and you want to learn about relativistic Navier-Stokes eq-s: quark-gluon plasma, neutron star mergers

Open "Fluid Mechanics" by Landau and Lifshitz:

some hydrodynamic equations

Open "Gravitation and Cosmology" by Weinberg*:

some hydrodynamic equations

And… these equations look very different!

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*Formulation of hydrodynamics due to Eckart (1940)

The equations look different, so what?

Let’s shut up and calculate. As a simple example, solve for linear perturbations of the thermal equilibrium state.

Both Landau-Lifshitz’ and Eckart’s equations predict that:

a) thermal equilibrium does not exist

b) things propagate faster than light

Hiscock, Lindblom, 1984

Hiscock, Lindblom, 1987

What exactly is the problem?

Perturbations $e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}}$, solve hydro equations: $\omega=\omega(\mathbf{k})$

Gapless modes: $\omega(\mathbf{k}\rightarrow 0)=0$, b/c of conserved charges. These correspond to normal hydrodynamics (sound etc).

But the equations also predict gapped modes $\omega(\mathbf{k}\rightarrow 0)\neq 0$, moreover with $\mathrm{Im}(\omega)>0$. These are unphysical modes.

These “fake” modes are outside of the validity regime of the low-energy hydro approximation. These are UV modes.

But if you want to actually solve the hydro equations in practice, these unphysical modes ruin predictability: cutoff-scale physics messes up the infrared behavior.

How is the problem fixed?

So the classic-textbook hydrodynamics is *not* what you solve in practice e.g. to study the quark-gluon plasma.

Most popular fix is the Israel-Stewart theory: the hydro equations are coupled to extra UV degrees of freedom, which in turn kill the unphysical UV modes.

These extra degrees of freedom are the dynamical stresses and heat fluxes, in addition to the dynamical T, u^α, μ.

The extra degrees of freedom of the Israel-Stewart theory play the role of a UV regulator. Note that in the non-relativistic Navier-Stokes eq-s, no extra d.o.f. are needed.

Other regulators?

Can one find a regulator of hydrodynamics that does not involve introducing extra UV degrees of freedom?

E.g. in field theory, the Pauli-Villars regularization introduces extra UV degrees of freedom, but dimreg does not.

Claim:

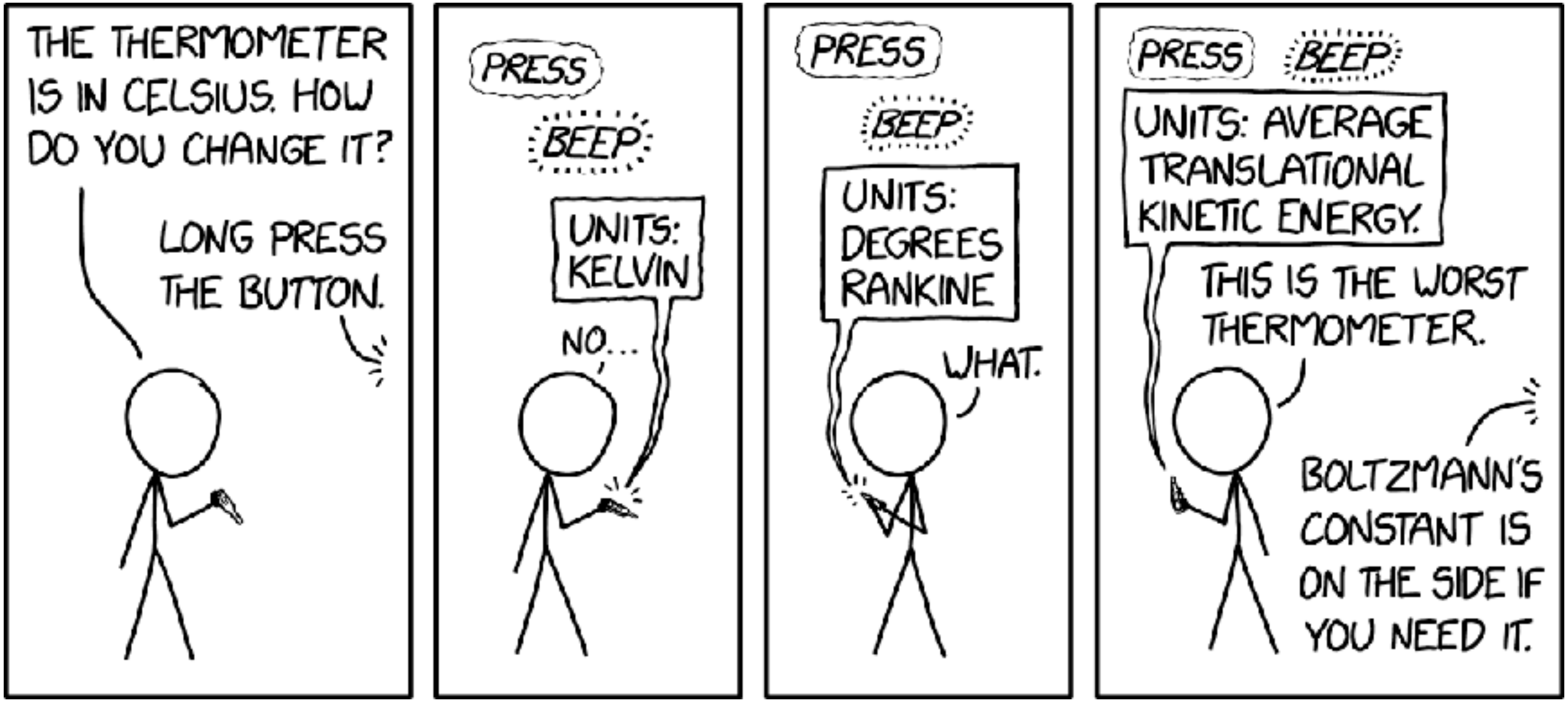
There *is* a sensible relativistic hydrodynamics whose only variables are T, u^α, μ, and no extra UV d.o.f. You need to choose a suitable out-of-equilibrium definition of T, u^α, μ.

Bemfica, Disconzi, Noronha, arXiv:1708.06255, arXiv:1907.12695

PK, arXiv:1907.08191, Hoult, PK, arXiv:2004.04102

Let’s first talk about physics

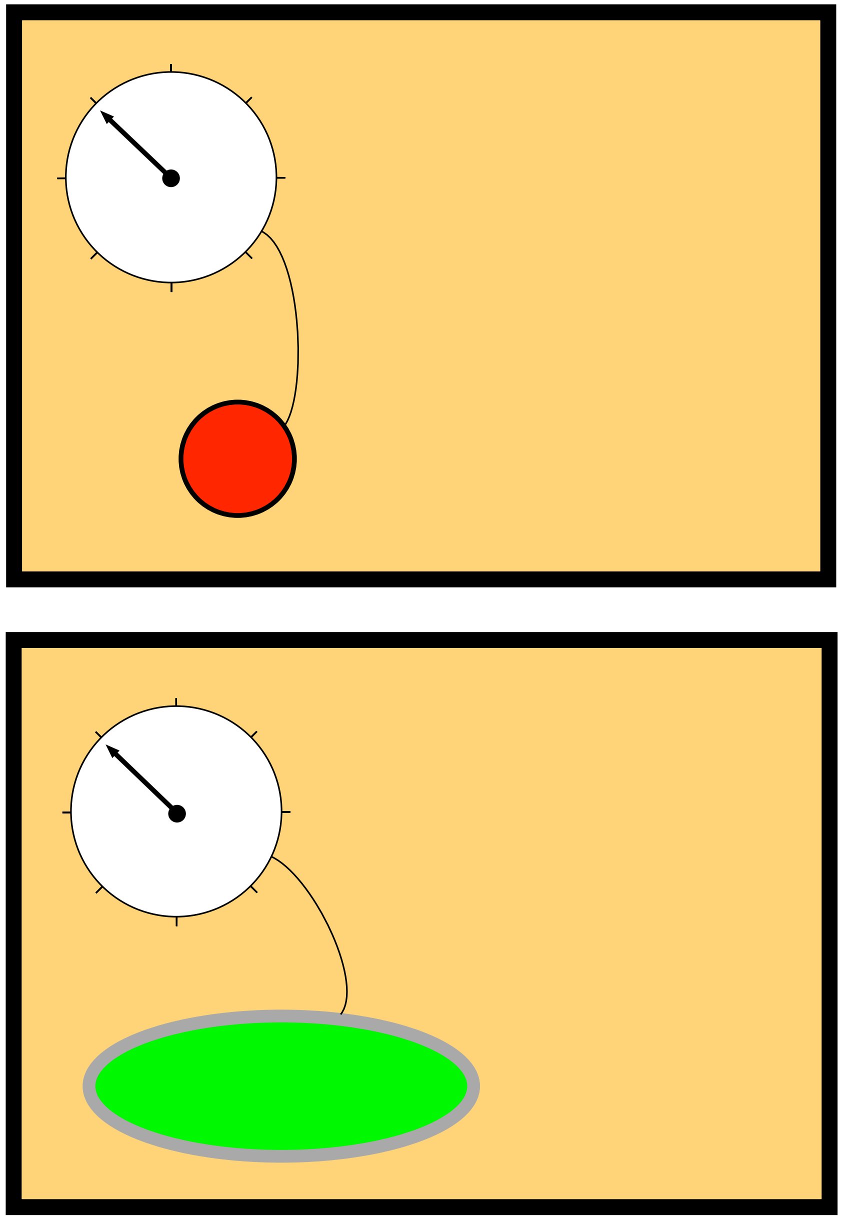
Example: Temperature



https://xkcd.com/2292/

Example: Temperature

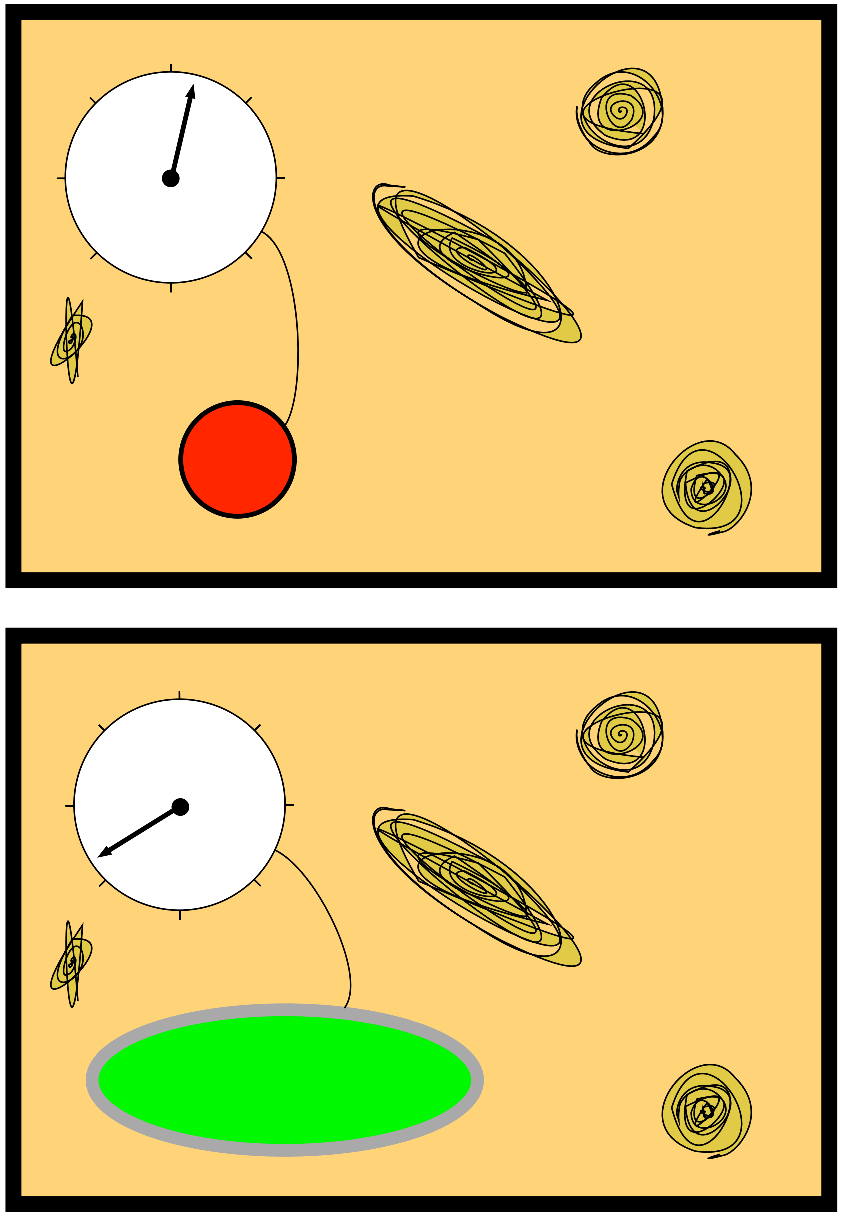
Temperature is something that is only unambiguously defined in equilibrium. By definition, T is the quantity that is measured by a thermometer.



In identical equilibrium states, two different (but properly calibrated) thermometers will show the same temperature.

Example: Temperature

Temperature is something that is only unambiguously defined in equilibrium. By definition, T is the quantity that is measured by a thermometer.



But in *identical* non-equilibrium states, the same two thermometers will show *different* temperatures!

Non-equilibrium conventions

So there is arbitrariness in what one means by “fluid temperature”: one’s choice of thermometer is a convention.

Same arbitrariness in what one means by “fluid velocity”: one’s choice of velocimeter is a convention.

Landau-Lifshitz's version of Navier-Stokes uses one convention, Eckart's version of Navier-Stokes uses another.

Note: there is no such thing as “the” Navier-Stokes eq-s until you specify *your arbitrarily chosen* convention.

Important:

Different conventions give rise to different, *mathematically inequivalent*, Navier-Stokes equations. These conventions have real consequences.

This is because the Navier-Stokes eq-s only give a crude approximation of a real fluid. The difference between the conventions is hidden in the crudeness of this approximation*.

Landau-Lifshitz and Eckart adopt different conventions for Navier-Stokes, but both are bad, and both lead to nonsensical predictions.

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*Using an analogy with quantum field theory, the choices of Landau-Lifshitz and Eckart are analogous to adopting UV regulators which violate unitarity.

What’s wrong with the classics?

Both Landau-Lifshitz and Eckart define T by:

Exact non-equilibrium energy density
=
$\epsilon(T)$ given by the equation of state

This means: as the local energy density changes, the thermometer adjusts its temperature *instantaneously*.

Such thermometers violate relativity, and lead to superluminal propagation in relativistic fluid dynamics.

Now let’s be slightly more formal

What we do in Effective Field Theory

1) Identify the low-energy variables

2) Write down all the terms allowed by the symmetry,

3) Do this up to a given dimension, e.g.:

$$S = \int d^4x \left(a(\partial_\mu \varphi)^2 + b\varphi^2 + c\varphi^4 \right)$$

4) Constrain the coefficients a, b, c so that the physics is sensible, e.g. $c < 0$ for stability of the vacuum

Do the same in hydro

1) Identify the low-energy variables: $T,\ u^\alpha,\ \mu$

2) Write down all possible terms in the constitutive relations consistent with the symmetry

3) Do this up to a given order (say, 1-st order) in the derivative expansion

4) Constrain the coefficients so that the physics is sensible, e.g. demand stability of equilibrium

Example: diffusion

Conservation law $\partial_t J^0 + \partial_i J^i = 0$, need $J^0 = J^0(\mu)$, $J^i = J^i(\mu)$

$$J^i = -\sigma \partial^i \mu + O(\partial^2) \qquad J^0 = \chi\mu + \nu\dot{\mu} + O(\partial^2)$$

If ignore the $\dot{\mu}$ term: diffusion eq-n

If keep the $\dot{\mu}$ term: both $\partial_t^2 \mu$ and $\partial_i^2 \mu$, hyperbolic eq-n

The term $\nu\dot{\mu}$ can be eliminated by a "field redefinition" of μ and truncating the derivative expansion. So it's not physical for the purpose of correlation functions. But it changes the structure of the PDE for μ(t,x).

Coef-t ν can be interpreted as specifying your μ-meter

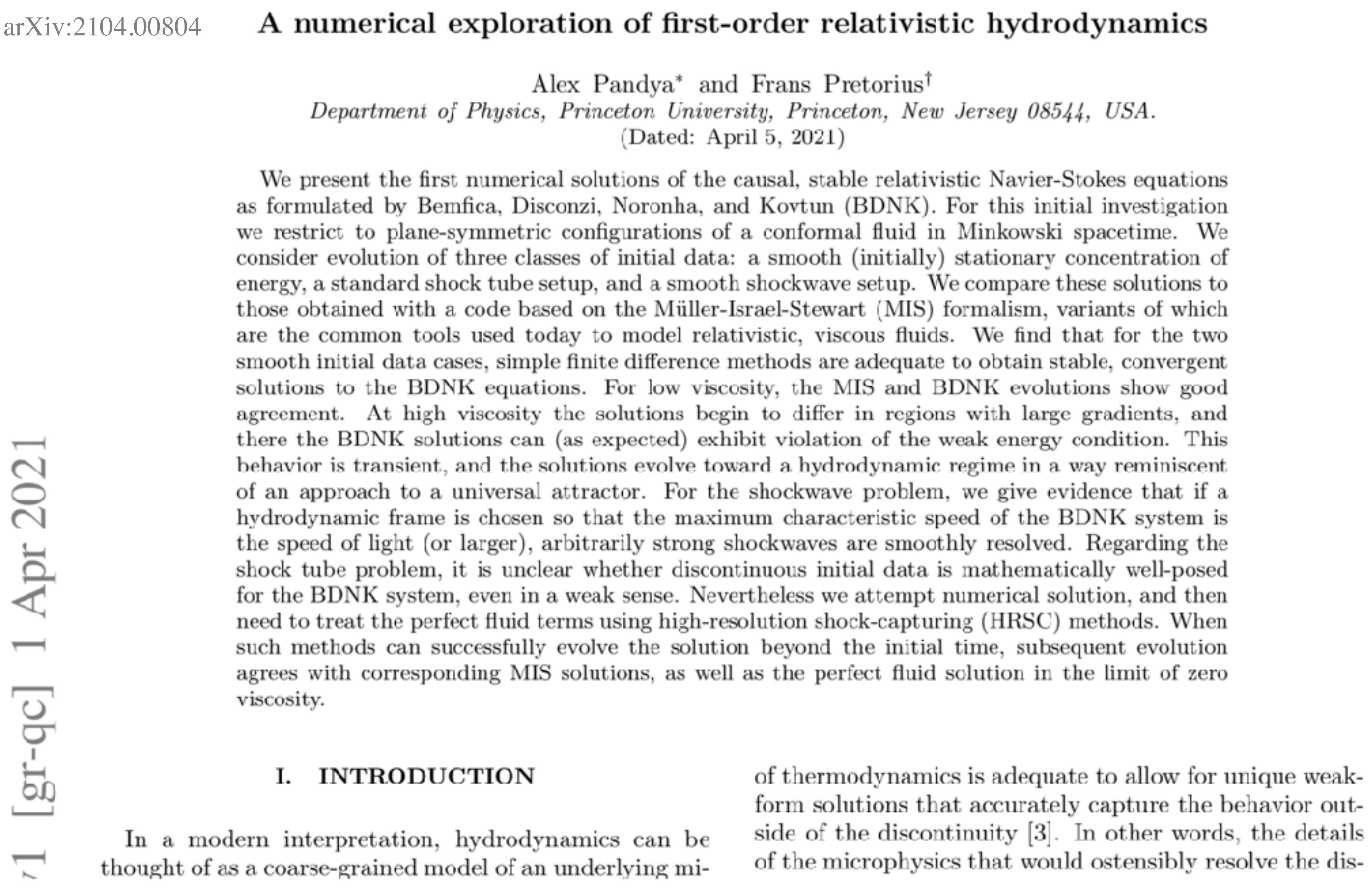
Claim:

The above example of diffusion can be generalized to full non-linear relativistic hydrodynamics, and it actually works. There is no need to introduce extra UV degrees of freedom.

Bemfica, Disconzi, Noronha, arXiv:1708.06255, arXiv:1907.12695

PK, arXiv:1907.08191, Hoult, PK, arXiv:2004.04102

First story



A numerical exploration of first-order relativistic hydrodynamics

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Department of Physics, Princeton University, Princeton, New Jersey 08544, USA.

(Dated: April 5, 2021)

We present the first numerical solutions of the causal, stable relativistic Navier-Stokes equations as formulated by Bemfica, Disconzi, Noronha, and Kovtun (BDNK). For this initial investigation we restrict to plane-symmetric configurations of a conformal fluid in Minkowski spacetime. We consider evolution of three classes of initial data: a smooth (initially) stationary concentration of energy, a standard shock tube setup, and a smooth shockwave setup. We compare these solutions to those obtained with a code based on the Müller-Israel-Stewart (MIS) formalism, variants of which are the common tools used today to model relativistic, viscous fluids. We find that for the two smooth initial data cases, simple finite difference methods are adequate to obtain stable, convergent solutions to the BDNK equations. For low viscosity, the MIS and BDNK evolutions show good agreement. At high viscosity the solutions begin to differ in regions with large gradients, and there the BDNK solutions can (as expected) exhibit violation of the weak energy condition. This behavior is transient, and the solutions evolve toward a hydrodynamic regime in a way reminiscent of an approach to a universal attractor. For the shockwave problem, we give evidence that if a hydrodynamic frame is chosen so that the maximum characteristic speed of the BDNK system is the speed of light (or larger), arbitrarily strong shockwaves are smoothly resolved. Regarding the shock tube problem, it is unclear whether discontinuous initial data is mathematically well-posed for the BDNK system, even in a weak sense. Nevertheless we attempt numerical solution, and then need to treat the perfect fluid terms using high-resolution shock-capturing (HRSC) methods. When such methods can successfully evolve the solution beyond the initial time, subsequent evolution agrees with corresponding MIS solutions, as well as the perfect fluid solution in the limit of zero viscosity.

I. INTRODUCTION

In a modern interpretation, hydrodynamics can be thought of as a coarse-grained model of an underlying mi- of thermodynamics is adequate to allow for unique weak-form solutions that accurately capture the behavior outside of the discontinuity [3]. In other words, the details of the microphysics that would ostensibly resolve the dis-

Question:

Sensible relativistic hydrodynamics seems to require that its eq-s contain non-hydrodynamic parameters. Does hydrodynamics actually have predictive power for non-linear evolution?*

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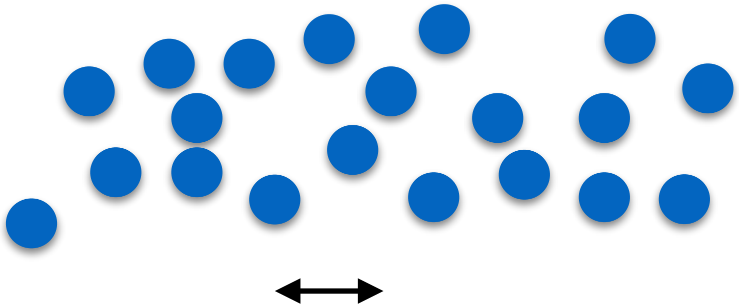
*Can the low-energy predictions be made independent of the UV regulator, à la RG?

Second story:

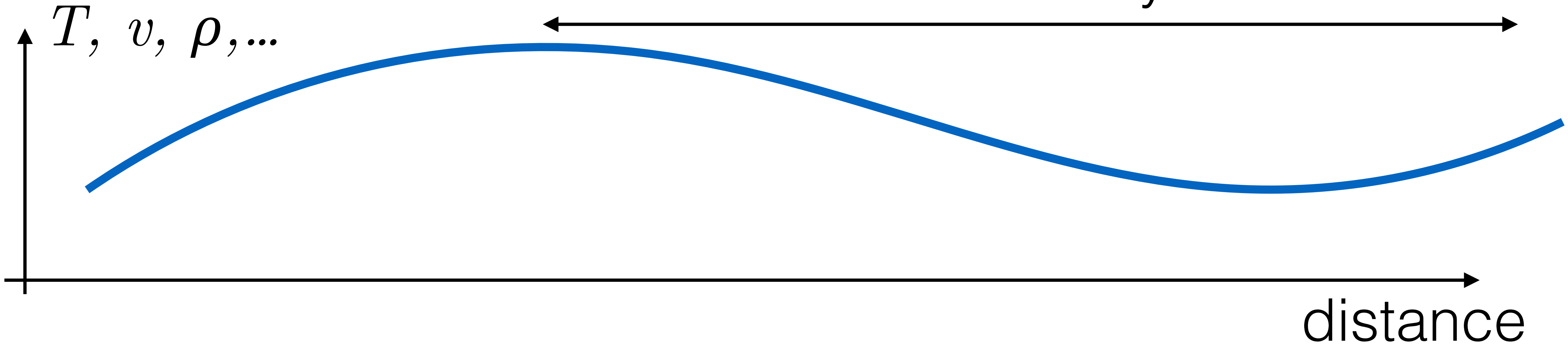
Are there limits to improving the hydrodynamic equations?

A first look at the limitations of hydrodynamics



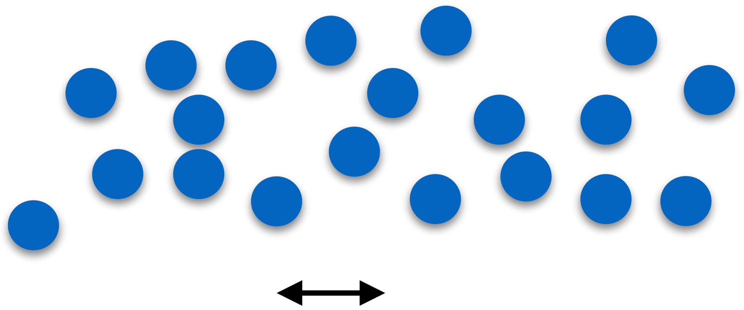
ℓ = typical microscopic distance scale

L = distance scale of hydro solutions



Hydrodynamics probably OK for $L \gg \ell$, small derivatives

A first look at the limitations of hydrodynamics



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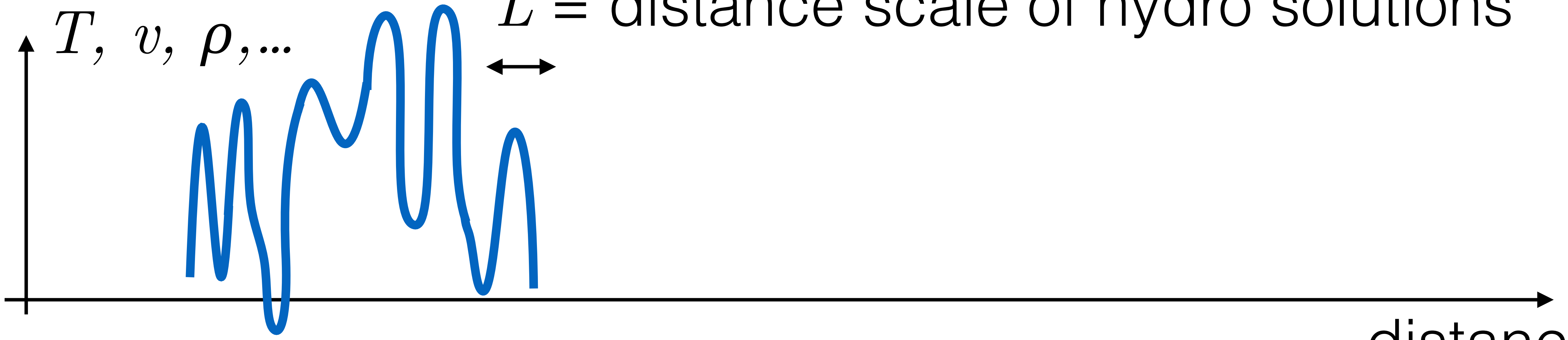


Hydrodynamics probably <u>not OK</u> for $L \sim \ell$, large derivatives

Hydrodynamics as an expansion in derivatives

Conservarion laws: $\frac{\partial}{\partial t}\rho_a = -\nabla \cdot \mathbf{j}_a$

Constitutive relations (γ = temperature, fluid velocity,…) :

$$\rho_a = \rho^{(0)}(\gamma) + \rho^{(1)}(\nabla\gamma) + \rho^{(2)}(\nabla^2\gamma, (\nabla\gamma)^2) + \dots$$

$$\mathbf{j}_a = \mathbf{j}^{(0)}(\gamma) + \mathbf{j}^{(1)}(\nabla\gamma) + \mathbf{j}^{(2)}(\nabla^2\gamma, (\nabla\gamma)^2) + \dots$$

Can we keep improving forever? Let's say we generate an infinite series in the gradients.

Q: Does this series converge, or does it diverge?

If converges: Hydrodynamics can be systematically improved to include more transport phenomena. 😀

If diverges: Hydrodynamics is not supposed to work… then why does it? Is hydrodynamics a fluke? 😕

What exactly is the expansion?

For a plane wave, $e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}$, derivatives are $\frac{\partial}{\partial \mathbf{x}} = i\mathbf{k}$, so the hydrodynamic expansion is the expansion in powers of $\mathbf{k}$.

Example: sound wave $\omega_{\text{sound}}(k) = \pm v_s k - i\Gamma k^2 + ...$

Do the series $\omega(k) = ak + bk^2 + ck^3 + dk^4 + ...$
converge or diverge in hydrodynamics?

Not clear how to answer in general, let's look at examples.

Analytic examples

There are fluids which can be studied analytically using holography. These fluids are similar to the quark-gluon plasma produced in nuclear collisions.

In the simplest solvable examples* we find that $\omega_{\mathrm{sound}}(k)$ is an analytic function of k, convergent for $|k|<k_c$, with

$$\boxed{k_c = \sqrt{2}(2\pi)\frac{k_{\mathrm{B}}T}{\hbar c}}$$

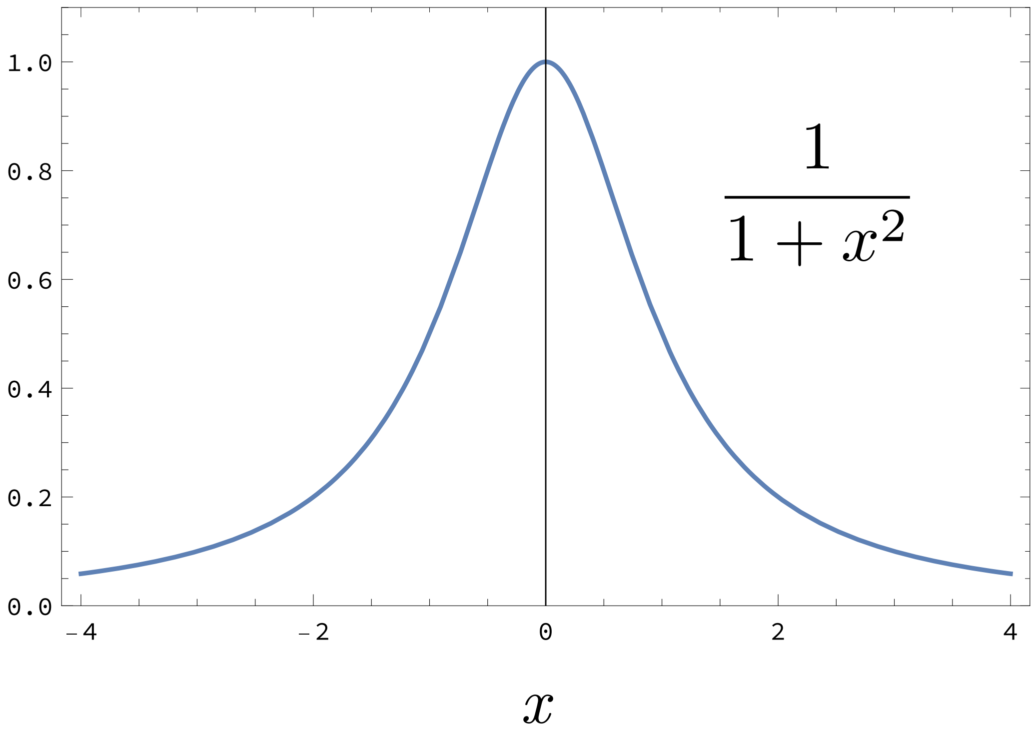
Grozdanov, PK, Starinets, Tadić, arXiv:1904.01018

The convergence is important. Gives one hope that hydrodynamics is improvable.

* $\mathcal{N}$=4 supersymmetric Yang-Mills theory in 3+1 dimensions and its cousins.

So… why is there a critical value $|k|=k_c$ in $\omega(k)$?

Example from basic math: function f(x) = 1/(1+x²)



Function is perfectly smooth for all $-\infty<|x|<\infty$. But the small-x Taylor expansion only converges for $|x|<1$. To understand why, take x complex.

Similarly, $\omega(k)$ is a smooth function at real k. To understand why it only converges for $|k|<k_c$, we must take k complex.

Complex ω and k

In classical physics, dispersion relations $\omega=\omega(k)$ come about by solving $F(\omega,k^2)=0$, where F is determined by the equations (hydrodynamics, Maxwell's eq-s in matter, etc).

Example: diffusion equation $\partial_t \, \delta T + D\nabla^2 \delta T = 0$ gives

$$F(\omega,k^2) = -i\omega - Dk^2 = 0.$$

$\delta T \sim e^{i\,\mathbf{k}\cdot\mathbf{x} - i\omega t}$

Example: sound waves in a viscous fluid give

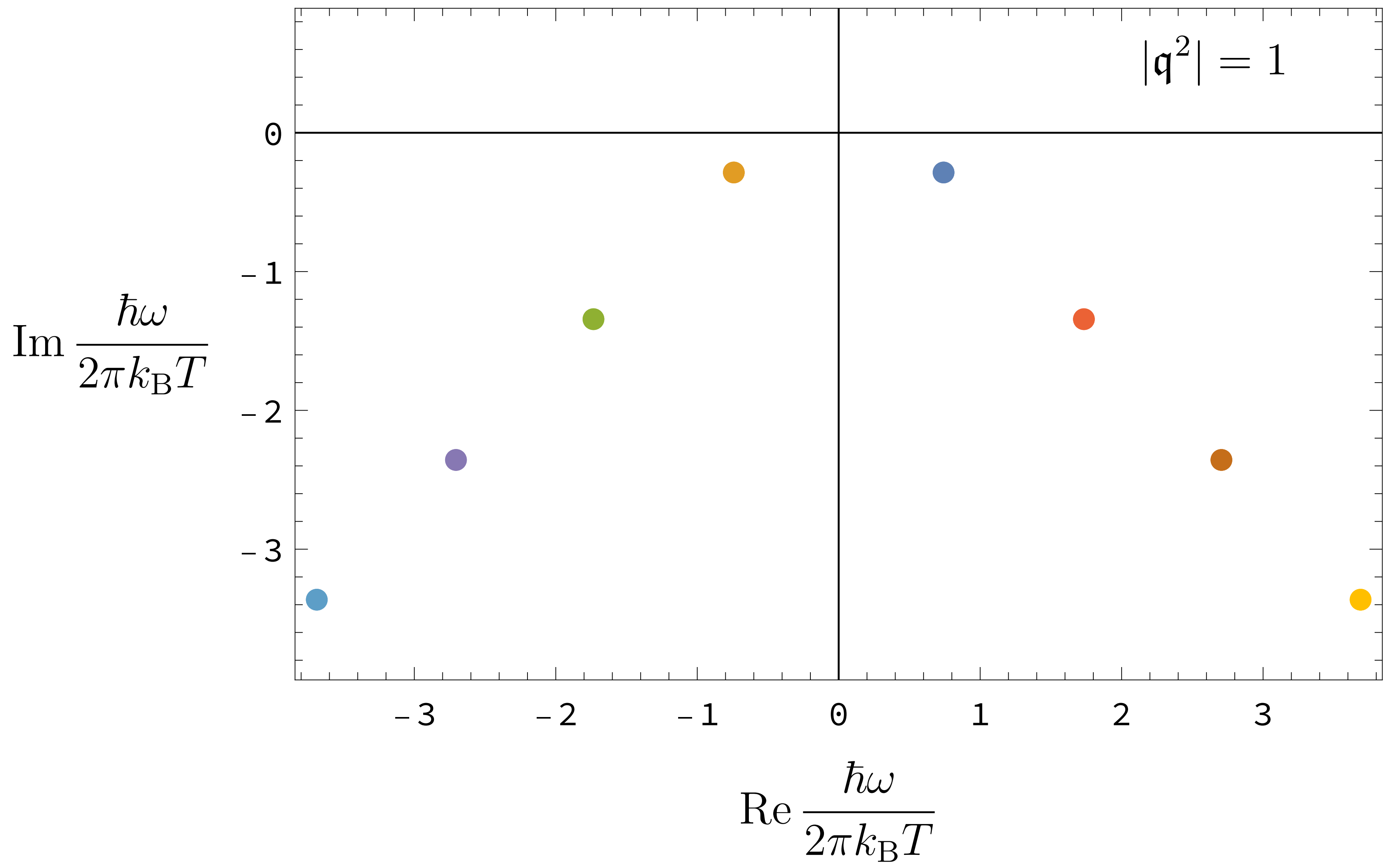
$$F(\omega,k^2) = \omega^2 - v_s^2\, k^2 + 2i\Gamma\omega k^2 = 0.$$

Take ω real, then $F(\omega,k^2) = 0$ gives $k(\omega)$, in general complex.
Imaginary part of k ⇒ damping length/penetration depth

Take k real, then $F(\omega,k^2) = 0$ gives $\omega(k)$, in general complex.
Imaginary part of ω ⇒ relaxation time

But what if *both* ω and k are complex?

Oscillation modes of a fluid, real k



Solutions to F(ω,k²)=0* in the plane of complex ω

$$\mathfrak{q} \equiv \frac{\hbar c\, k}{2\pi k_B T}$$

PK, Starinets, arXiv:hep-th/0506184

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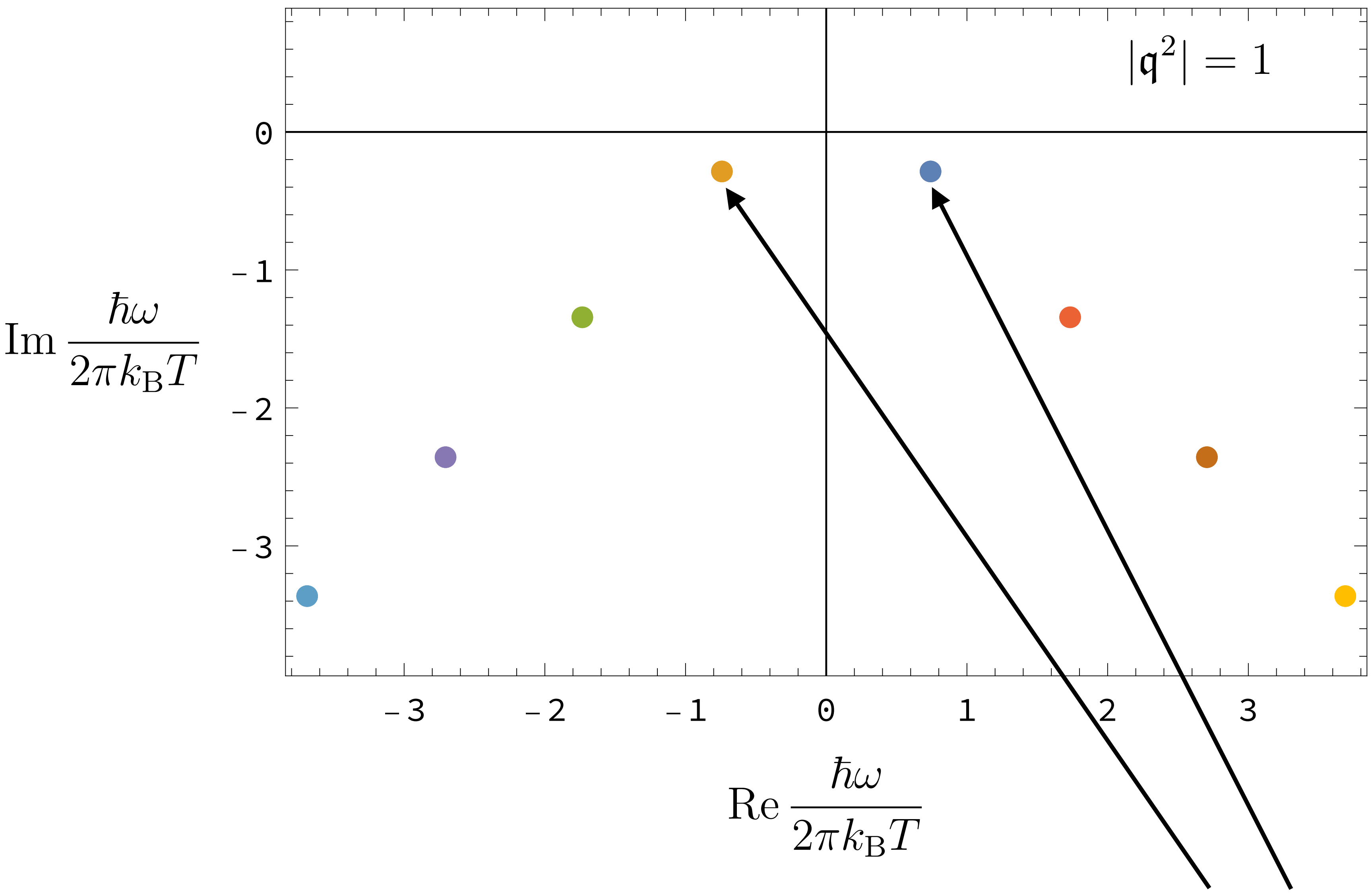
* Poles of the exact retarded Green's function of the energy density.

Oscillation modes of a fluid, real k



Solutions to F(ω,k²)=0* in the plane of complex ω

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PK, Starinets, arXiv:hep-th/0506184

Sound waves, $\omega(k) = \pm v_s k - i\Gamma k^2 + ...$

Macroscopic, classical, approach the origin as $k \to 0$.

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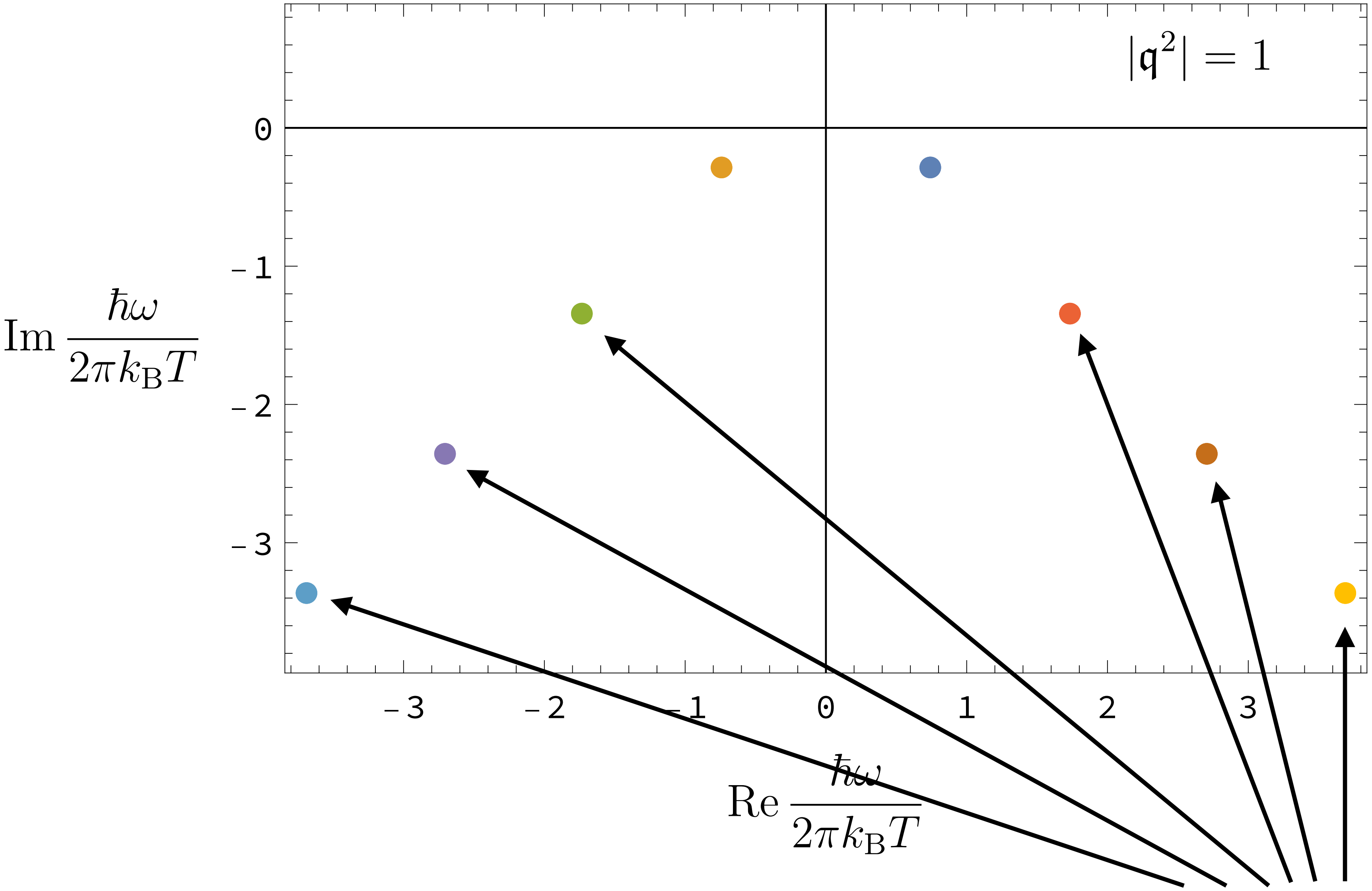
* Poles of the exact retarded Green's function of the energy density.

Oscillation modes of a fluid, real k



Solutions to F(ω,k²)=0* in the plane of complex ω

$$\mathfrak{q} \equiv \frac{\hbar c\, k}{2\pi k_B T}$$

PK, Starinets, arXiv:hep-th/0506184

microscopic oscillation modes, no classical interpretation, stay away from the origin as $k \to 0$.

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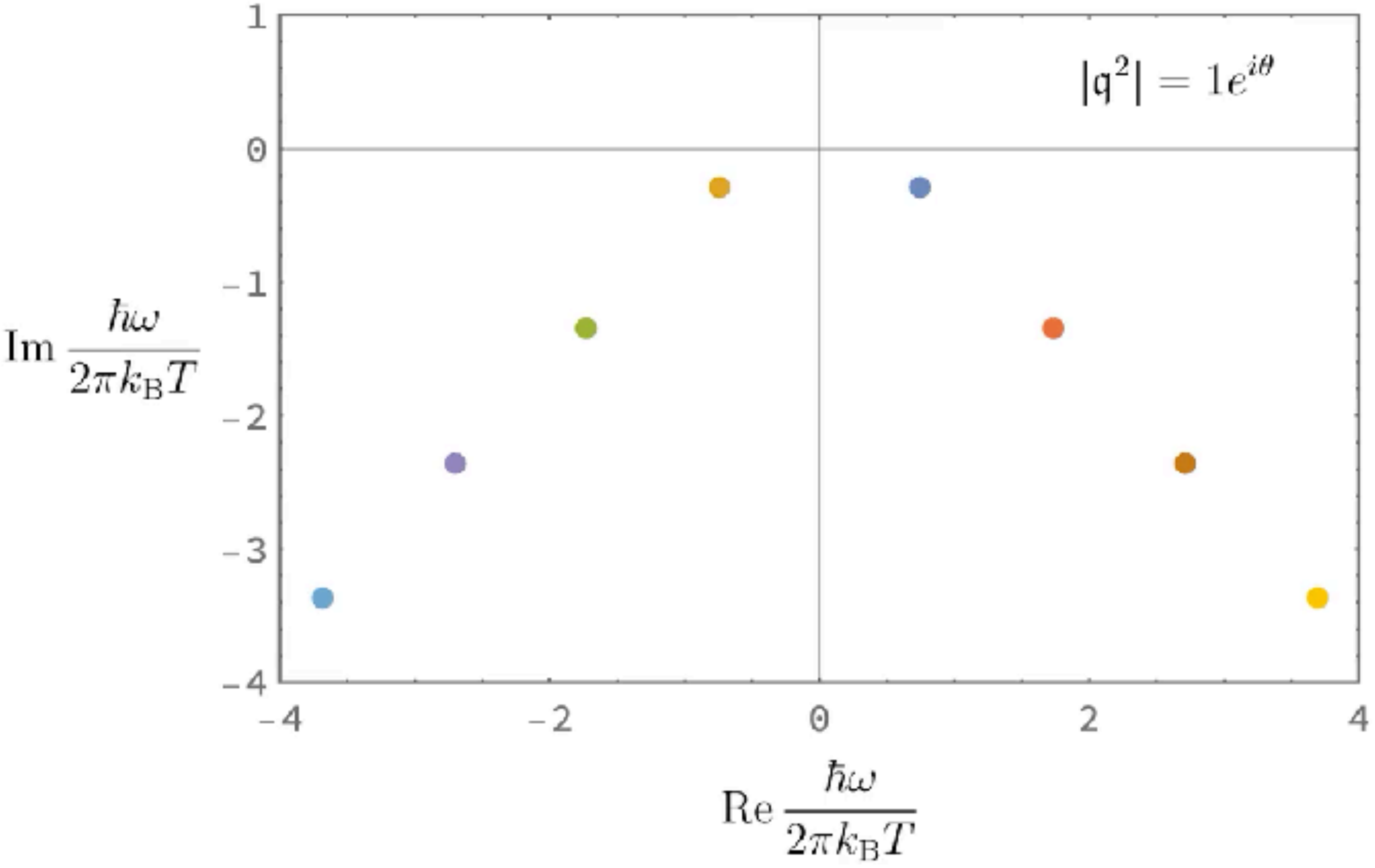
* Poles of the exact retarded Green's function of the energy density.

Oscillation modes of a fluid, complex k

Now take k to be complex, $|\mathfrak{q}^2| = 1e^{i\theta}$, and vary θ from $0\to 2\pi$.

$$\mathfrak{q} \equiv \frac{\hbar c\, k}{2\pi k_{\mathrm{B}} T}$$

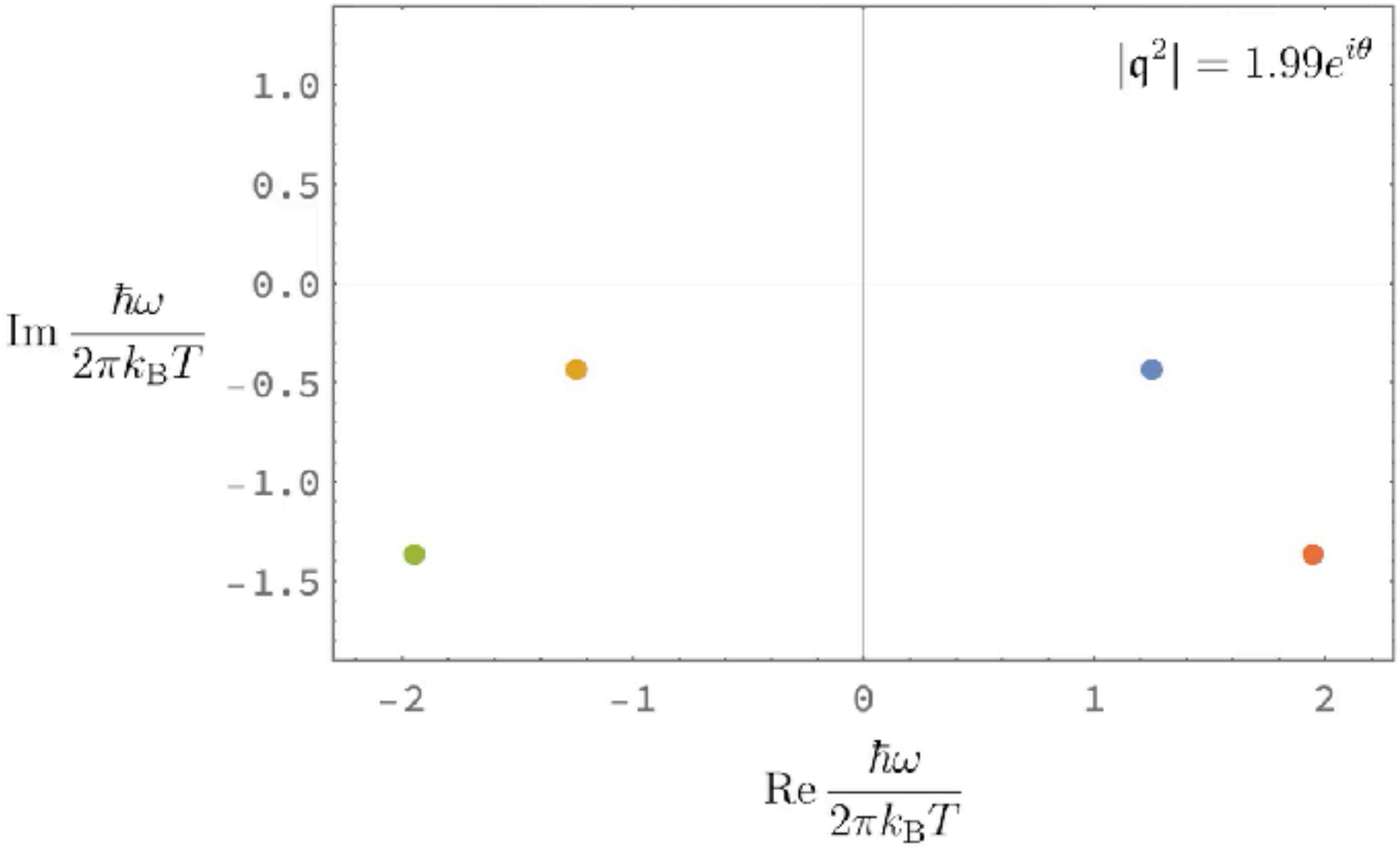


Sound modes (blue and gold) swap places, but remain sound modes when θ becomes 2π.

Oscillation modes of a fluid, complex k

Now take k to be complex, $|\mathfrak{q}^2| = 1.99\, e^{i\theta}$, and vary θ from $0 \to 2\pi$.

$$\mathfrak{q} \equiv \frac{\hbar c\, k}{2\pi k_{\mathrm{B}} T}$$

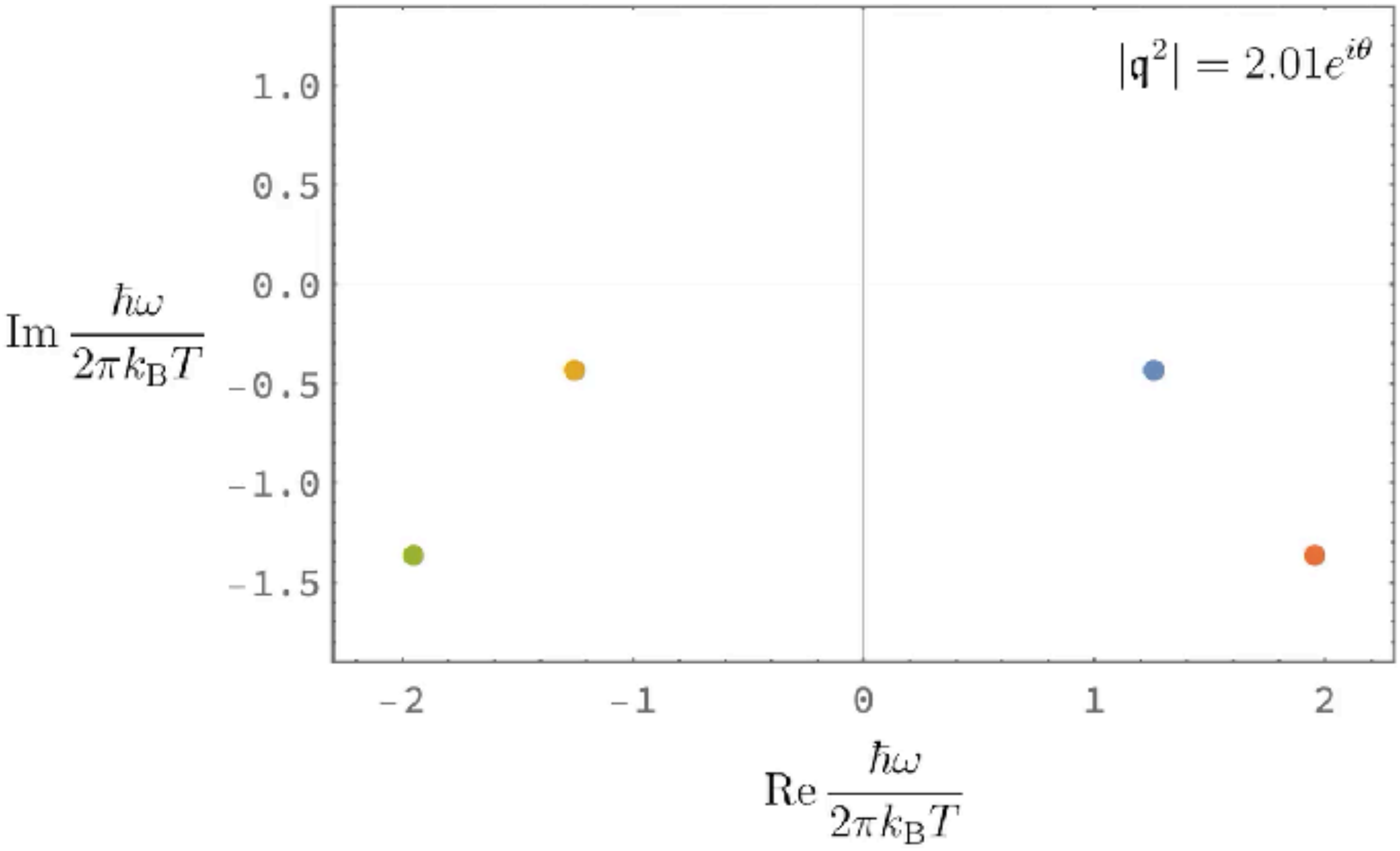


Sound modes (blue and gold) swap places, but remain sound modes when θ becomes 2π.

Oscillation modes of a fluid, complex k

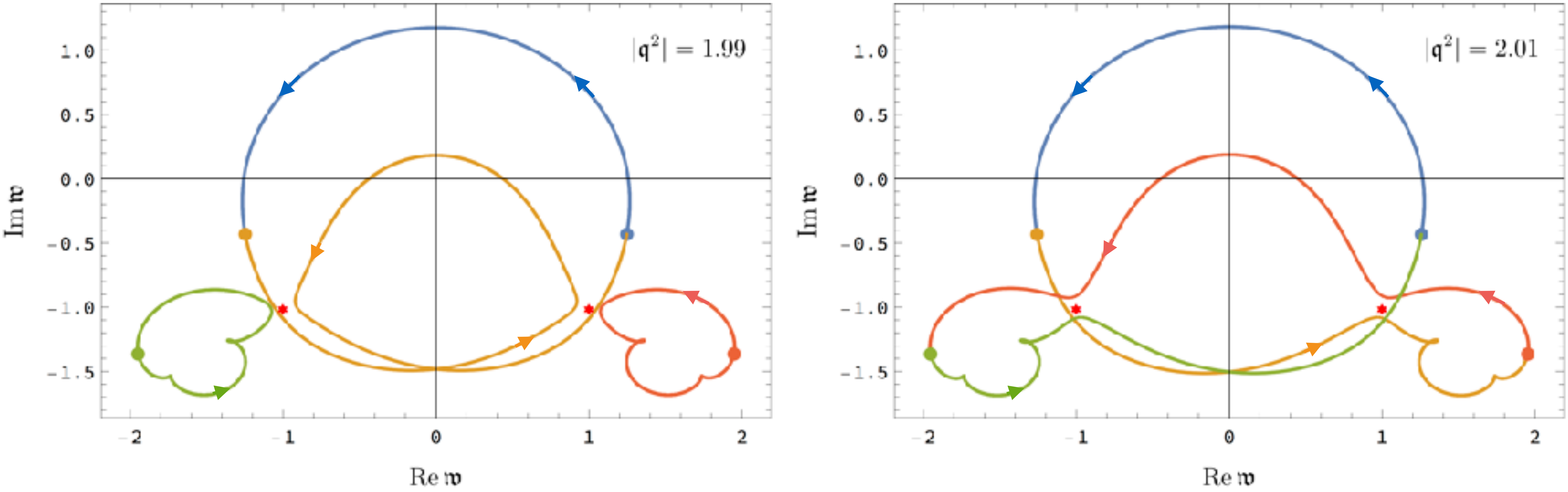
Now take k to be complex, $|\mathfrak{q}^2| = 2.01\, e^{i\theta}$, and vary θ from $0 \rightarrow 2\pi$.

$$\mathfrak{q} \equiv \frac{\hbar c\, k}{2\pi k_{\mathrm{B}} T}$$



Sound mode (gold) becomes one of the non-classical modes when θ becomes 2π!

Trajectories of the modes at $|\mathbf{q}^2|=1.99$ and $|\mathbf{q}^2|=2.01$



At $|\mathfrak{q}^2|=2$, modes collide, and the topology of the trajectories changes.

This is *level-crossing in macroscopic dissipative systems*, when a classical (hydrodynamic) excitation becomes a non-classical excitation.

This determines the convergence of the hydro expansion, and gives the critical wavelength for sound $\lambda_c=2\pi/k_c$,

$$\boxed{\lambda_c = \frac{1}{\sqrt{2}}\frac{\hbar c}{k_{\mathrm{B}} T}}$$

Grozdanov, PK, Starinets, Tadić, arXiv:1904.01018

Bottomline:

In many solvable examples, classical macroscopic excitations and non-classical microscopic excitations are merely different branches of the same multi-valued complex function.

Comment: Complex momentum is also related to quantum chaos through out-of-time-ordered correlation functions

Maldacena, Shenker, Stanford, arXiv:1503.01409

Grozdanov, Schalm, Scopelliti, arXiv:1710.00921

Blake, Lee, Liu, arXiv:1801.00010

Blake, Liu, arXiv:2102.11294

… many more papers

Final story:

Why everything I said so far is wrong

Let us look at very simple solutions of classical hydrodynamics: how the near-equilibrium physical system reacts to infinitesimal external sources.

Response functions

Classical hydro gives predictions for retarded functions of conserved densities and fluxes in equilibrium. E.g.:

$$G^{t,t}_{\text{ret}}(\omega, \mathbf{k}) = \frac{D\chi\mathbf{k}^2}{i\omega - D\mathbf{k}^2}$$

D = diffusion constant,
χ=∂n/∂μ susceptibility

response of the current to external gauge field
response of the stress tensor to external metric

$$G^{xy,xy}_{\text{ret}}(\omega, \mathbf{k}{=}0) = p - i\omega\eta + O(\omega^2)$$

p = pressure,
η = shear viscosity

PK, arXiv:1205.5040

These do not depend on which version of hydro one is using, and are universal predictions of classical hydro for low-energy observables. Allow to compute η, D, etc.

Are these predictions of classical hydro actually correct?

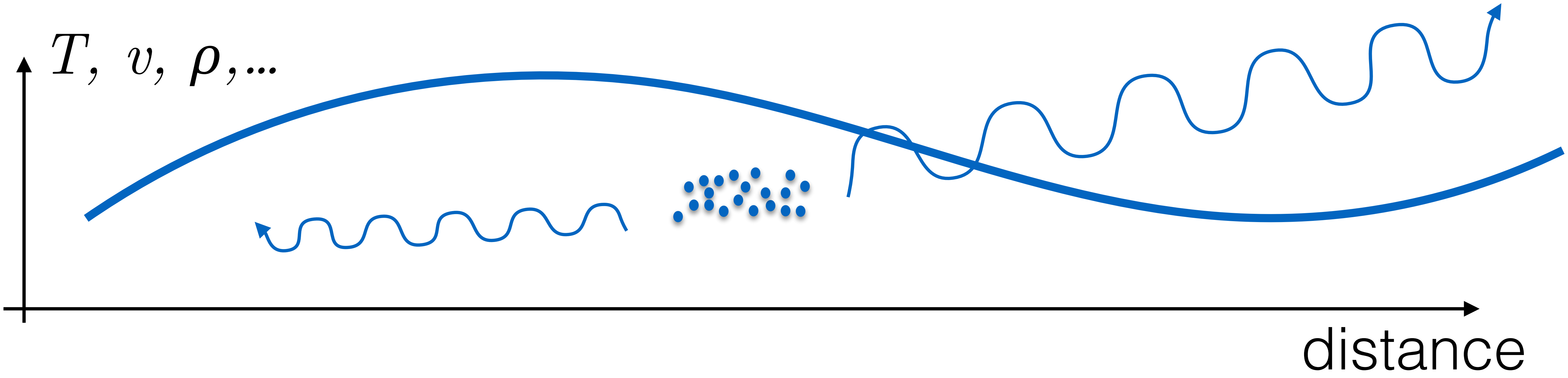
After all, classical hydro (i.e. hydro=PDEs) is but a toy model of fluids, and nature may or may not care for our toy models.

A second look at the limitations of hydrodynamics

Macroscopic stuff is made out of microscopic stuff.

In a quantum vacuum, virtual particles are constantly produced and absorbed due to quantum fluctuations. Similarly, in a macro-state, virtual sound waves are constantly produced and absorbed due to statistical fluctuations.

These sound waves will back-react on the macroscopic physics because hydro is non-linear and waves interact.



Claim

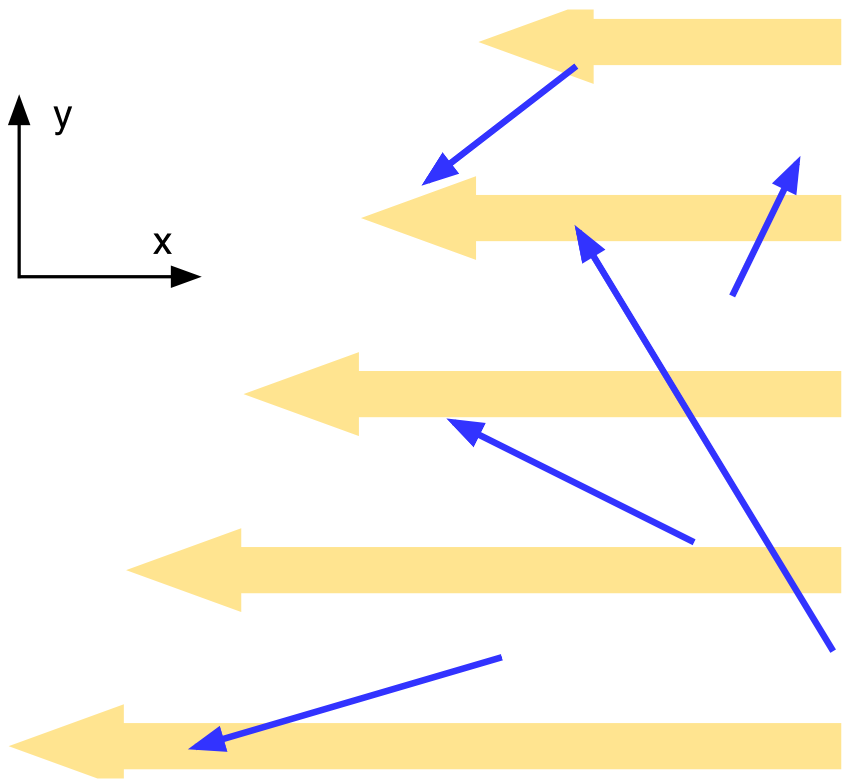
Even if you are interested in *macroscopic long-time, long-distance response* near static equilibrium, classical hydro eq-s which ignore the back-reaction can lead to predictions which are qualitatively wrong.

Alder, Wainwright, Phys. Rev. A 1, 18 (1970)

The rest of my talk will be comments on this claim.

Let’s first talk about physics

Example: viscosity



Momentum transfer between layers of fluid,

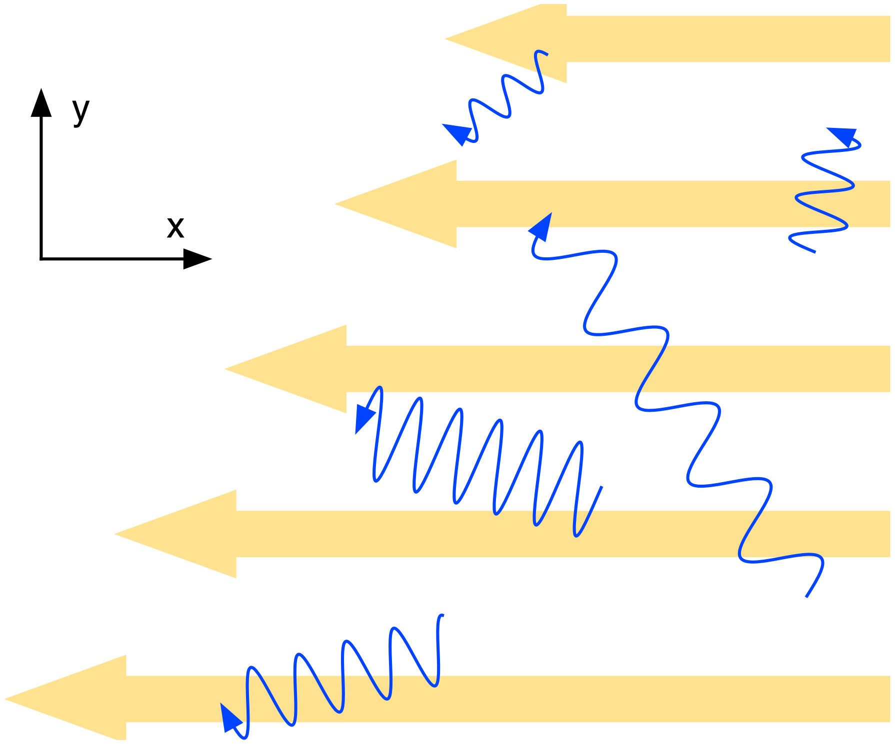
$T_{xy} = \eta\, \partial_y v_x + O(\partial^2)$

Related to correlations of stress:

$$\langle T^{xy} T^{xy} \rangle_{\mathrm{ret.}} = p - i\omega\eta + O(\omega^2)$$

In a gas: $\eta = \rho\, v_{th}\, \ell_{mfp}$

Example: viscosity



Momentum can also be transferred by collective excitations.

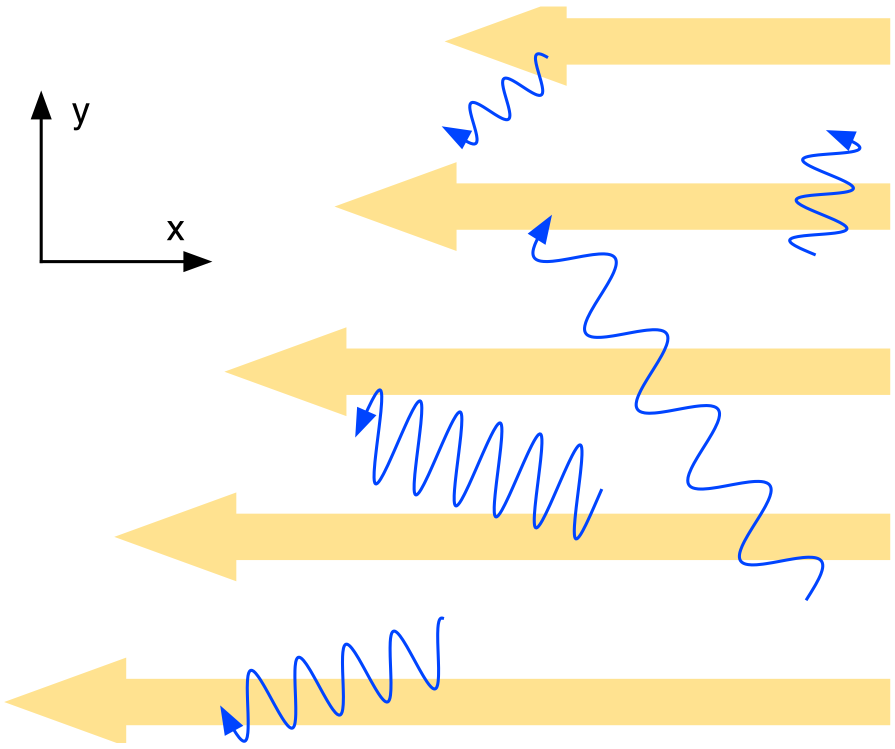
Gas of sound waves: $\ell_{\mathrm{mfp}} \sim \dfrac{1}{\frac{\eta}{\epsilon+p}\mathbf{k}^2}$

Contribution to viscosity:

$$\int^{\Lambda} d^D k \frac{T}{\frac{\eta}{\epsilon+p}\mathbf{k}^2} \rightarrow \frac{\Lambda T^2}{\eta/s} \quad \text{in D=3}$$

$$\rightarrow \text{IR divergent in D=2}$$

Example: viscosity



- This is the physics of thermal fluctuations. It is invisible if hydrodynamics is viewed just as a collection of partial differential equations.
- If you think η/s can be arbitrarily small, think again.
- Classical hydrodynamics may be irrelevant to physics in 2+1 dimensions.

Now let’s be slightly more formal

Stochastic hydrodynamics

Add noises: $T^{\mu\nu} = T^{\mu\nu}_{\text{class}}[\beta,\alpha] + \tau^{\mu\nu}$, $J^\lambda = J^\lambda_{\text{class}}[\beta,\alpha] + j^\lambda$

$\langle\ldots\rangle$ = average over the noise with some weight, i.e.

$$\langle\ldots\rangle = \int D\tau\, Dj\; e^{-S_{\text{noise}}[\tau,j]}$$

$S_{\text{noise}}[\tau,j]$ is Gaussian (quadratic in τ,j) for long-wavelength (small-k) fluctuations, by the central limit theorem.

Width of the Gaussian is determined by demanding that the equilibrium 2-point correlation functions satisfy the fluctuation-dissipation theorem

$$G_{\text{symm}}(\omega,\mathbf{k}) = \frac{2T}{\omega}\,\text{Im}\,G_{\text{ret}}(\omega,\mathbf{k})$$

Stochastic hydrodynamics

$$T^{ij} = \cdots + (\epsilon+p)v^i v^j + \cdots + \tau^{ij}$$

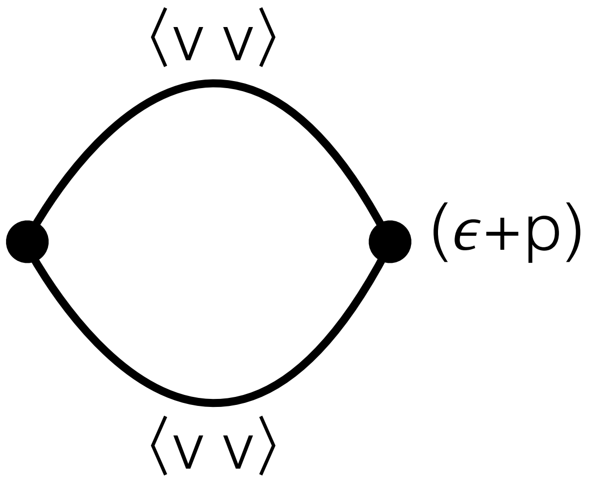
linear

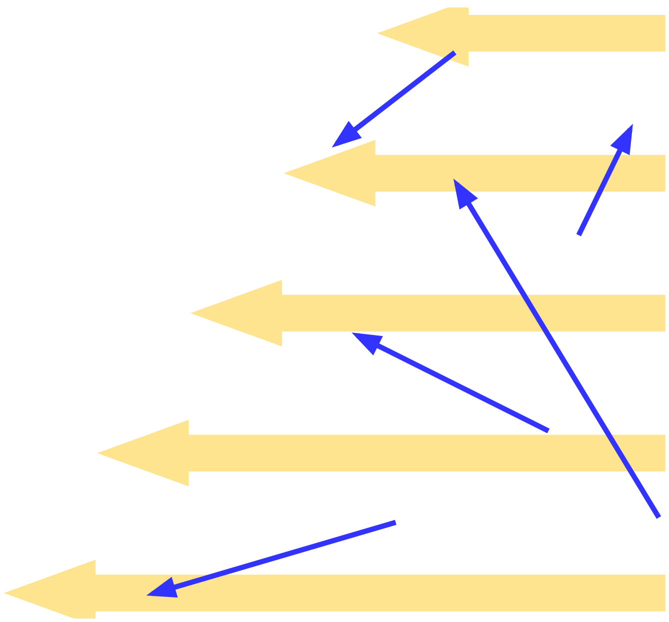
leading non-linear,
no derivatives term

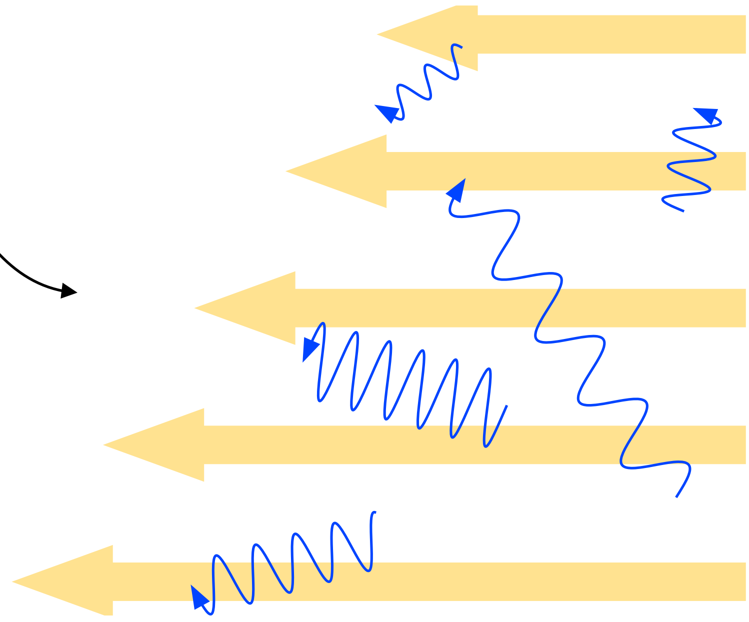
higher order

noise

$\langle T_{xy}\, T_{xy}\rangle = 2T\eta + (\epsilon+p)$







Stress correlations in 3+1 dimensions

$$\langle T_{xy}T_{xy}\rangle^R = \underset{\substack{\uparrow \\ \text{0-th order classical}}}{p} + \underset{\substack{\uparrow \\ \text{correction to p}}}{O(\Lambda^3 T)} - i\omega\left(\underset{\substack{\uparrow \\ \text{1-st order classical}}}{\eta} + \underset{\substack{\uparrow \\ \text{correction to }\eta}}{\frac{17T^2\Lambda}{120\pi^2\eta/s}}\right) + O\left(\underset{\substack{\uparrow \\ \text{cutoff-independent}}}{\frac{\omega^{3/2}}{(\eta/s)^{3/2}}}\right) + O(\underset{\substack{\uparrow \\ \text{2-nd order classical}}}{\omega^2})$$

PK, Yaffe, hep-th/0303010

PK, Moore, Romatschke, 1104.1586

- This is "one-loop" fluctuation correction to $\langle T^{xy} T^{xy}\rangle_{\text{ret}}$. Actual physical viscosity includes all such corrections.
- As expected, small η/s implies large corrections to η/s. Fluctuations are *mandatory* for small-viscosity physics.
- Fluctuations are more important than 2-nd order hydro. IR contributions determined by thermodynamics and η/s.

Stress correlations in 3+1 dimensions

$$\langle T_{xy}T_{xy}\rangle^R = p + O(\Lambda^3 T) - i\omega\left(\eta + \frac{17T^2\Lambda}{120\pi^2\eta/s}\right) + O\left(\frac{\omega^{3/2}}{(\eta/s)^{3/2}}\right) + O(\omega^2)$$

0-th order classical

correction to p

1-st order classical

correction to η

cutoff-independent

2-nd order classical

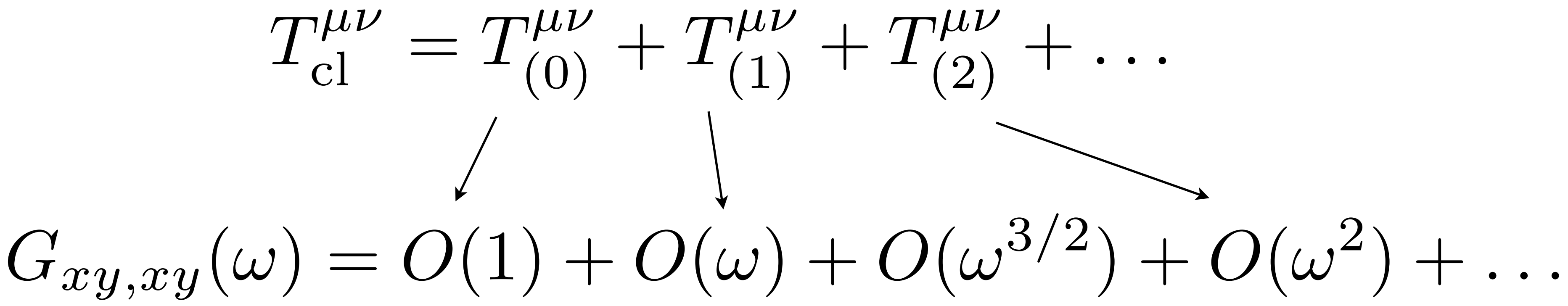
PK, Yaffe, hep-th/0303010

PK, Moore, Romatschke, 1104.1586

- In real time, $\omega^{3/2}$ gives $1/t^{3/2}$ decay of correlations, more important than first-order hydro (long-time tails).
- In 2+1 dimensions, $\omega^{3/2} \to \omega \ln(\omega)$. Kubo formula for viscosity does not exist.
- Large-N limit does not commute with the hydrodynamic limit, hence $\omega^{3/2}$ not visible in classical holography.

Analogy with quantum gravity

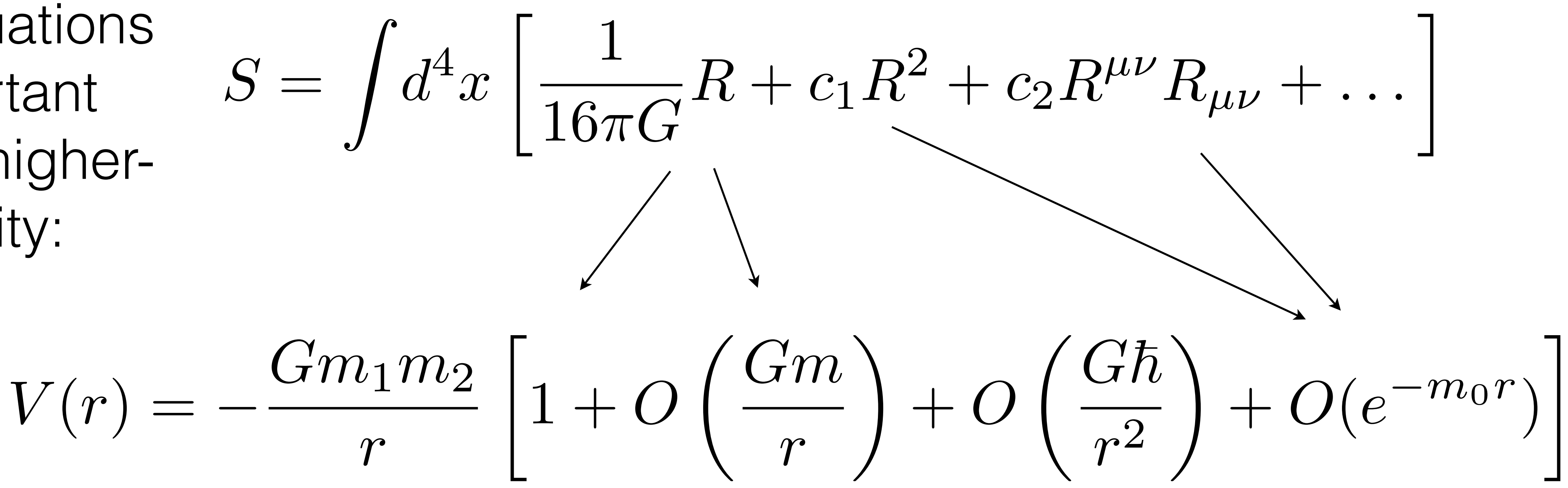
Thermal fluctuations are more important than classical higher-derivative hydro:



$$T^{\mu\nu}_{\mathrm{cl}} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + T^{\mu\nu}_{(2)} + \dots$$

$$G_{xy,xy}(\omega) = O(1) + O(\omega) + O(\omega^{3/2}) + O(\omega^2) + \dots$$

Quantum fluctuations are more important than classical higher-derivative gravity:



$$S = \int d^4x \left[\frac{1}{16\pi G} R + c_1 R^2 + c_2 R^{\mu\nu} R_{\mu\nu} + \dots \right]$$

$$V(r) = -\frac{G m_1 m_2}{r} \left[1 + O\left(\frac{Gm}{r}\right) + O\left(\frac{G\hbar}{r^2}\right) + O(e^{-m_0 r}) \right]$$

$$m_0 \sim (c_i G)^{-1/2}$$

Classical: Stelle 1978
Quantum: Bjerrum-Bohr, Donoghue, Holstein, 2002

Bottomline

Using classical hydrodynamics (i.e. hydro=PDEs) to evaluate the response to ω→0, k→0 sources is unreliable,

Short-time statistical fluctuations give rise to infrared *late-time* correlations,

The deviations from classical hydrodynamics are more pronounced at small viscosity,

Stochastic hydrodynamics predicts that these infrared correlations are universal, determined only by thermodynamics and transport coef-s (viscosity, conductivity etc).

But let’s dig deeper

Beyond the naive stochastic hydro

S_{noise} is only Gaussian as $k \to 0$. What if we want the derivative expansion, beyond the leading $k \to 0$ limit?

Width of the Gaussian was fixed by FDT for 2-point functions. But FDTs exist for all n-point functions. How do you impose those?

Heinz, Wang, hep-th/9809016

Classical hydro can only compute retarded functions. Non-retarded functions are invisible to classical hydro.

But higher-point non-retarded func-s can *not* be related to retarded func-s through FDTs, hence contain new low-energy info. How do you access that info?

Effective field theory

We are used to computing correlation f-s in field theory:

- Identify the low-energy degrees of freedom Ψ
- Identify the symmetries G
- Write a local action S[Ψ,A] invariant under G
- Get correlation functions by varying

$$e^{iW[A]} = \int D\Psi \; e^{iS[\Psi,A]}$$

with respect to A.

Can we do the same for near-equilibrium states? In such a EFT, classical hydro will be the Euler-Lagrange e.o.m. of the action, stochastic hydro will emerge as a leading-order truncation.

Building blocks of hydro EFT

Want locality, gauge- and difffeo-invariance, etc.

Want the Euler-Lagrange equations of motion to *be* the conservation laws $\partial_\mu T^{\mu\nu}=0$, $\partial_\lambda J^\lambda=0$. Different from the standard mechanics where EL e.o.m. *imply* conservation laws.

Want to access real-time correlation functions, hence formulate EFT using Schwinger-Keldysh fields (2 copies).

For equilibrium, want to impose Kubo-Martin-Schwinger constraints as a symmetry.

Martin, Siggia, Rose, Phys. Rev. A 8, 423 (1973)

Crossley, Glorioso, Liu, arXiv:1511.03646

Haehl, Loganayagam, Rangamani, arXiv:1511.07809

Jensen, Pinzani-Fokeeva,Yarom, arXiv:1701.07436

Glorioso, Liu, arXiv:1805.09331

Example: conserved current

- A_μ=external gauge field, φ=dynamical field, $B_\mu \equiv A_\mu + \partial_\mu \varphi$.

$$e^{iW[A]} = \int D\varphi\, e^{iS[B]}$$

This W[A] is gauge-invariant, and e.o.m. for φ is $\partial_\mu J^\mu=0$.

- Schwinger-Keldysh: need two copies $B_\mu^{1,2} \equiv A_\mu^{1,2} + \partial_\mu \varphi^{1,2}$. Let $\varphi^r = (\varphi^1 + \varphi^2)/2$, $\varphi^a = (\varphi^1 - \varphi^2)$, same for $A_\mu^{r,a}$. Want

$$W[A_\mu^r, A_\mu^a = 0] = 0$$

hence S = O(a) + O(a^2) + O(a^3) + …

- Shift symmetry: $\varphi^r \to \varphi^r + \lambda(\mathbf{x})$ determines SSB or not

- KMS: $B^r \to \Theta B^r$, $B^a \to \Theta B^a + (i/T_0)\Theta \partial_t B^r$, Θ = T or PT

Crossley, Glorioso, Liu, arXiv:1511.03646

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Diffusion in stochastic hydro (1)

Current $J^t = n(\mu)$, $J^k = -\sigma(\mu)\partial^k\mu + j^k$ satisfies $\partial_\lambda J^\lambda = 0$

Noise is Gaussian: $\langle j^i(x) j^k(x')\rangle = C^{ik}\delta(x - x')$

Enforce eom: $\delta[\partial_\lambda J^\lambda] = \int D\varphi^a \; e^{i\int \varphi^a\,\partial_\lambda J^\lambda}$

Noise average $\langle \ldots \rangle = \int Dj \; e^{-\frac{1}{2}\int j^i C^{-1}_{ik} j^k} \ldots$ generates $C_{ik}\partial_i\varphi^a\partial_k\varphi^a$

Get effective action

$$S[\mu, \varphi^a] = \int dt\, d^d x \left[i\varphi^a \left(\partial_t n(\mu) - \partial_k(\sigma(\mu)\partial_k\mu)\right) + \frac{1}{2} C_{ik}\partial_i\varphi^a\partial_k\varphi^a \right]$$

O(a), diffusion equation

O(a^2), “kinetic” term

Diffusion in stochastic hydro (2)

Effective action for diffusion:

$$S[\mu,\varphi^a] = \int dt\, d^dx \left[i\varphi^a \left(\partial_t n(\mu) - \partial_k(\sigma(\mu)\partial_k\mu)\right) + \tfrac{1}{2} C_{ik}\partial_i\varphi^a\partial_k\varphi^a \right]$$

Say, $C_{ik} = C(\mu)\,\delta_{ik} + \theta(\mu)\left[\delta_{ik}(\partial_j\mu)^2 - \partial_i\mu\partial_k\mu\right] + \ldots$. FDT for 2-point functions relates $C(\mu) = 2T\sigma(\mu)$ to $\sigma(\mu)$ of classical hydro.

$\theta(\mu)$ contributes to 4-point non-retarded function, which is not related to the retarded function by FDT.

Classical hydro only knows about retarded functions, hence classical hydro knows nothing about $\theta(\mu)$.

$\theta(\mu)$ is a stochastic transport coefficient. It can contribute to observable *retarded* functions through loops.

Diffusion in EFT

Identify $\mu \equiv B_{r\,t} = A_{r\,t} + \partial_t \varphi_r$, effective action for diffusion:

$$S_1 = \int dt\, d^d x \left[n(\mu) B_{a\,t} + i T_0 \sigma(\mu) B_{a\,i} \left(B_a^i + (i/T_0)\partial_t B_r^i \right) \right]$$

Crossley, Glorioso, Liu
arXiv:1511.03646

Beyond simple diffusion (recall $B_\mu = A_\mu + \partial_\mu \varphi$)

$$S_2 = \int dt\, d^d x\ \theta_1(\mu) B_{a\,i} B_{a\,j} \left[\partial_t B_r^i\, \partial_t B_r^j - \delta^{ij}\, \partial_t B_{r\,k}\, \partial_t B_r^k \right]$$
$$+ \int dt\, d^d x\ \theta_2(\mu)\, B_{a\,i} B_a^i \left(B_{a\,j} + (i/T_0)\partial_t B_{r\,j} \right) \left(B_a^j + (i/T_0)\partial_t B_r^j \right)$$

Jain, PK, arXiv:2009.01356

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transforms into a total derivative under KMS

transform into each other under KMS

Crossley, Glorioso, Liu
arXiv:1511.03646

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Crossley, Glorioso, Liu
arXiv:1511.03646

Beyond simple diffusion:

changes under KMS into stuff orthogonal to […]

KMS invariant

$$S_2 = \int dt\, d^d x\; \theta_1(\mu) \boxed{B_{a\,i} B_{a\,j}} \boxed{\left[\partial_t B_r^i\, \partial_t B_r^j - \delta^{ij}\, \partial_t B_{r\,k}\, \partial_t B_r^k \right]}$$
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Beyond simple diffusion:

transform into each other under KMS

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Crossley, Glorioso, Liu
arXiv:1511.03646

Beyond simple diffusion:

O(a^2), Gaussian noise, but not related to diffusion coefficient by FDT

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O(a^2)+O(a^3)+O(a^4), non-Gaussian noise

$$+ \int dt\, d^dx\, \theta_2(\mu)\, B_{a\,i} B_a^i \left(B_{a\,j} + (i/T_0)\partial_t B_{r\,j} \right) \left(B_a^j + (i/T_0)\partial_t B_r^j \right)$$

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Crossley, Glorioso, Liu
arXiv:1511.03646

Leading-order in derivatives

Derivative-suppressed compared to S_1

Beyond simple diffusion:

$$S_2 = \int dt\, d^d x\; \theta_1(\mu) B_{a\,i} B_{a\,j} \left[\partial_t B_r^i\, \partial_t B_r^j - \delta^{ij}\, \partial_t B_{r\,k}\, \partial_t B_r^k \right]$$
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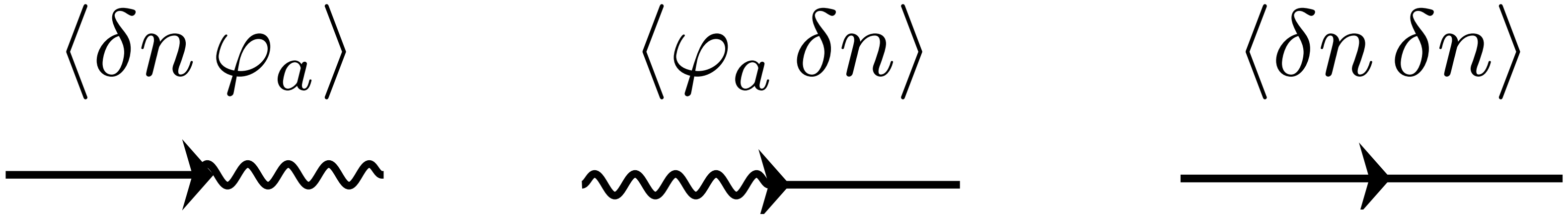
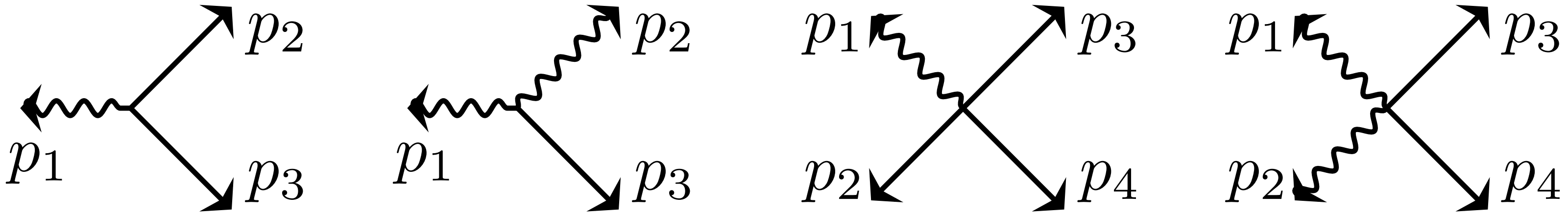
Jain, PK, arXiv:2009.01356

Propagators and vertices

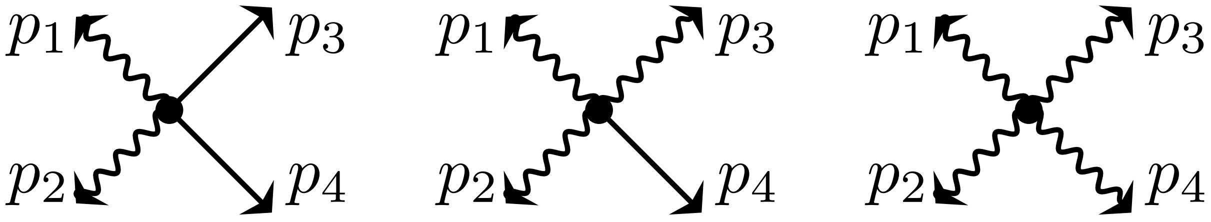
Propagators
(recall $\langle \varphi_a \varphi_a \rangle = 0$)



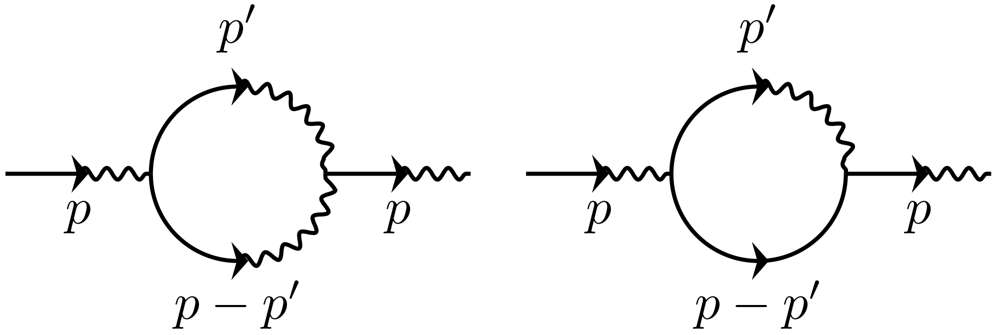
Vertices from S$_1$:



Vertices from S$_2$:



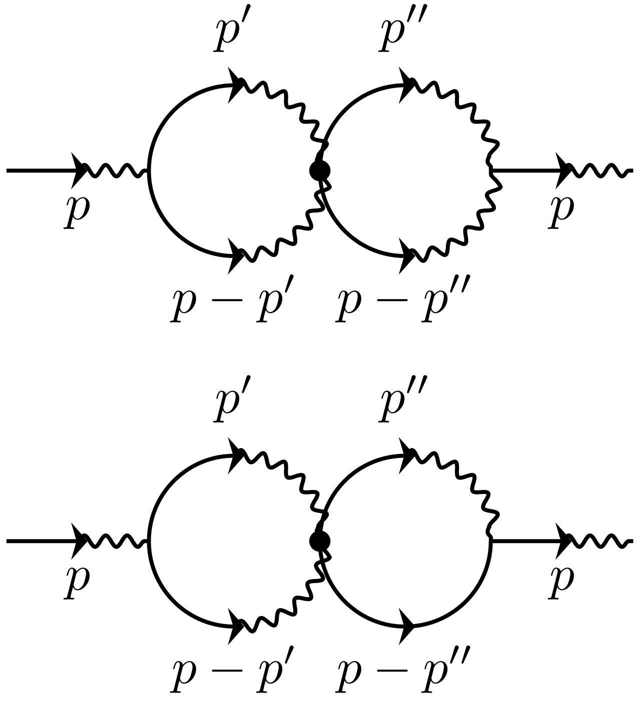
Density-density response function



$$\omega + iDk^2 \rightarrow \omega + iDk^2 + \#\omega k^2 \sqrt{k^2 - \frac{2i\omega}{D}}$$

Chen-Lin, Delacrétaz, Hartnoll, arXiv:1811.12540

Correction determined by thermo-dynamics and diffusion coefficient



$$\omega + iDk^2 \rightarrow \omega + iDk^2 + \#\omega k^2 \left(k^2 - \frac{2i\omega}{D}\right) k^4$$

Jain, PK, arXiv:2009.01356

Correction determined by stochastic transport coef-s invisible to classical hydrodynamics. 1-loop correction to 3-point function non-alalytic in ω,k.

Summary of EFT

Two types of fields: “classical” (φ_r) and “stochastic” (φ_a).

$S \supset \varphi_a F(\varphi_r)$ These are the non-linearities of classical hydro, fixed by the classical constitutive reltaions

$S \supset (\varphi_a)^2 G(\varphi_r)$ Some of these are fixed by FDT in terms of the classical transport coef-s (viscosity etc)

$S \supset (\varphi_a)^2 G(\varphi_r)$ Some of these are invisible to classical hydrodynamics

$S \supset (\varphi_a)^{n>2} H(\varphi_r)$ Invisible to classical hydrodynamics

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new interactions, “stochastic transport coefs” Jain, PK, arXiv:2009.01356

Bottomline:

Stochastic transport coef-s are invisible to classical hydro, but contribute to measurable correlation f-s through loops.

Stochastic transport coef-s give rise to *infrared* effects, derivative-suppressed in 3+1 compared to viscosity, etc. These are *different* effects from the usual long-time tails.

Stochastic transport coef-s can only be matched to the UV of the theory through non-retarded functions, unlike η,D,etc.

Thus knowing the macroscopic classical effective theory (to all orders in the derivative expansion) is not enough to correctly describe macroscopic *infrared* correlations. Different from “standard” QFT.

Encouragement for students

As a student, one often tends to think that all interesting questions in physics have already been answered many years ago by the great.

Hydrodynamics is an ancient field by modern standards, and people are still trying to figure out some very basic questions.

If, as a student, you work in a field that is less than 200 years old, do not lose hope: there surely are many fundamental things left to discover.

Thanks for listening!