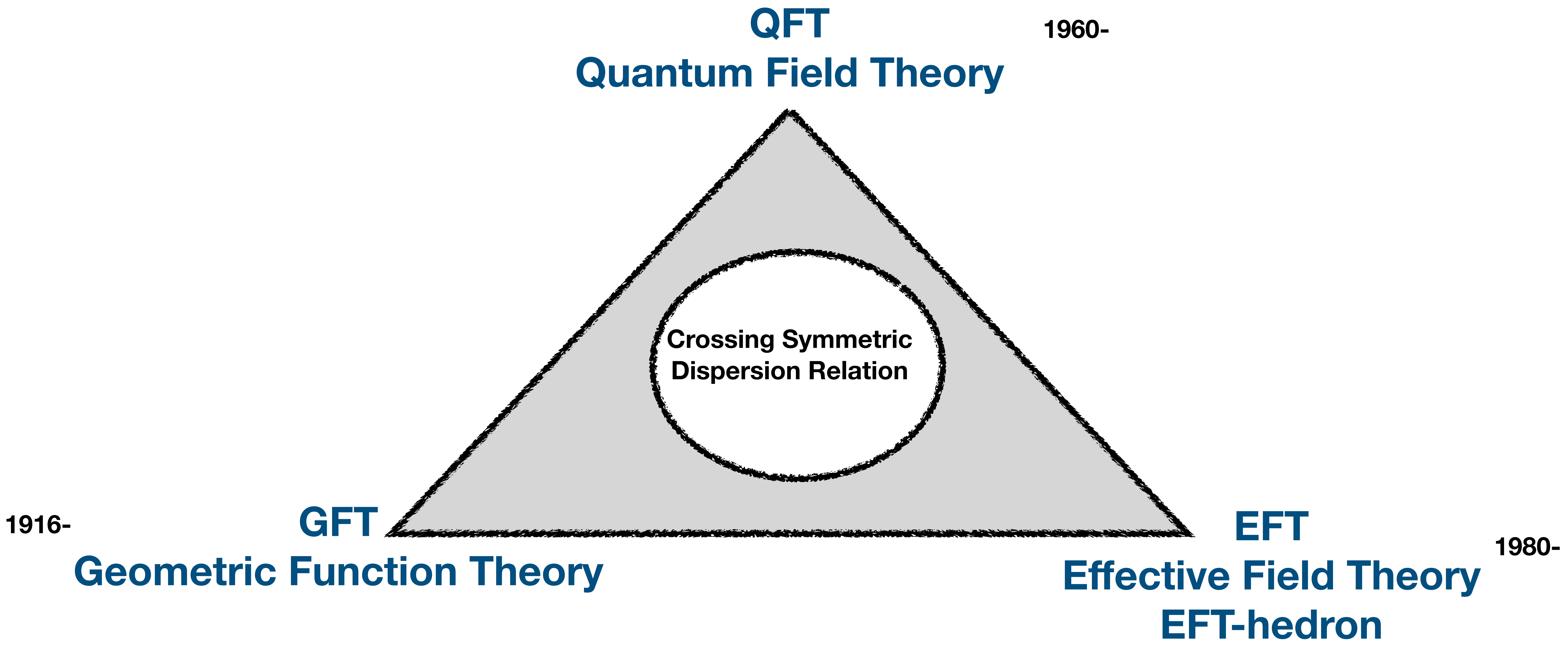
QFT, EFT and GFT

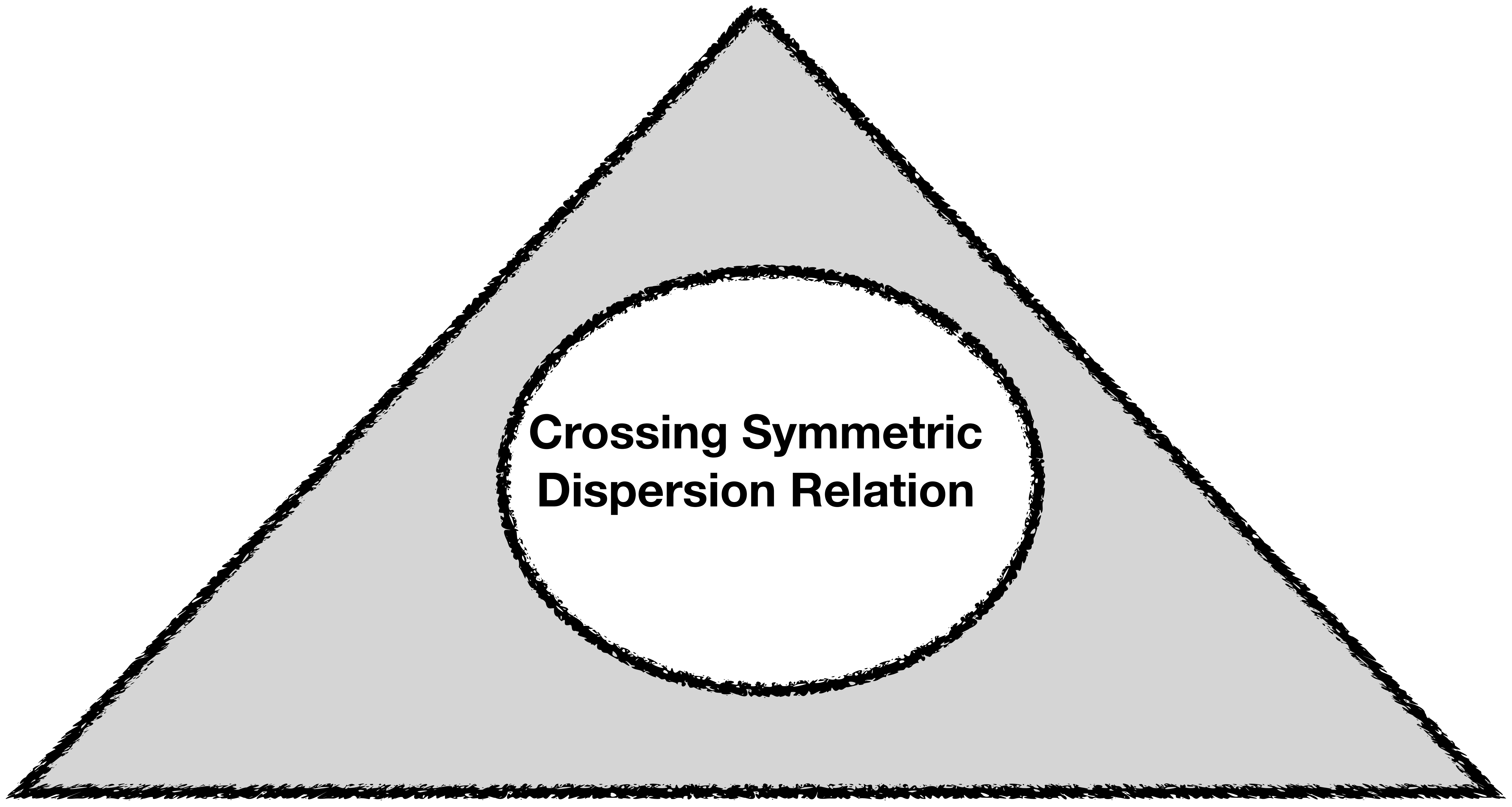
Aninda Sinha
Centre for High Energy Physics,
Indian Institute of Science,
Bangalore, India.

East Asian Strings Webinar, 2021

Big picture


QFT
Quantum Field Theory
1960-
Crossing Symmetric
Dispersion Relation
1916-
GFT
Geometric Function Theory
EFT
Effective Field Theory
EFT-hedron
1980-

Big picture: amplified

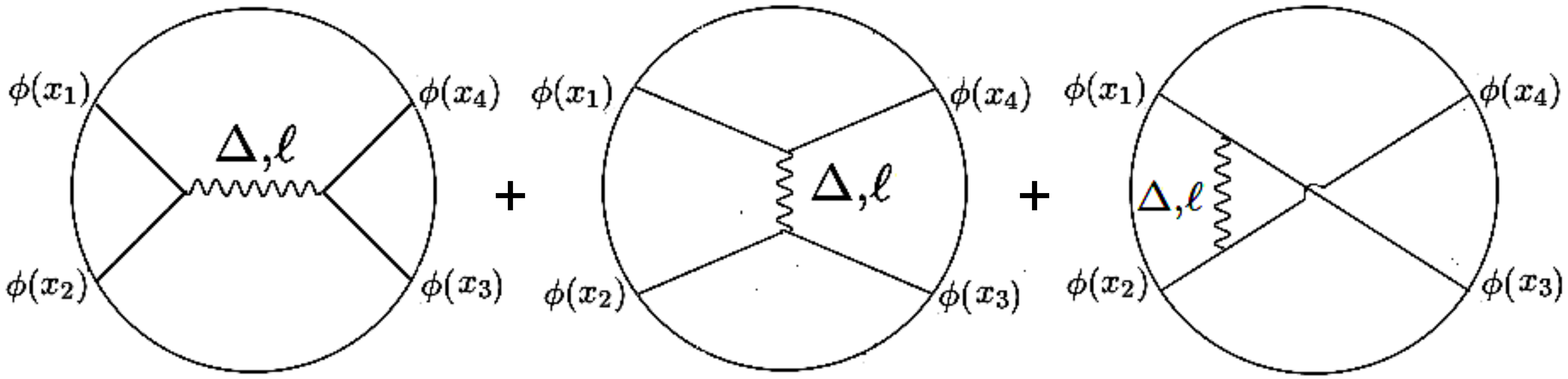


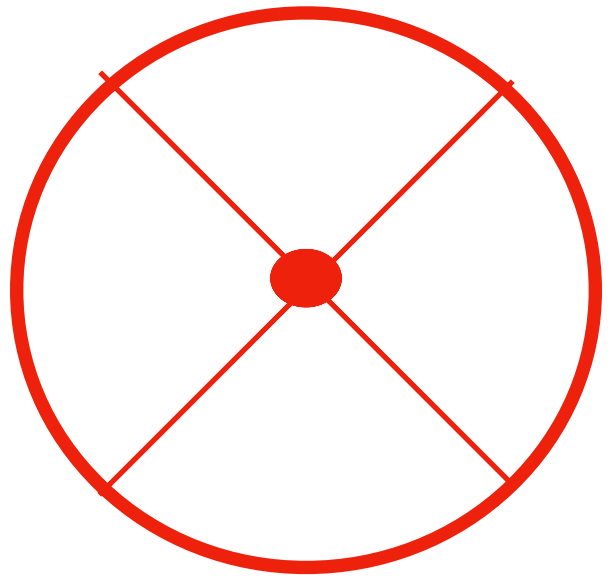
QFT+EFT motivation

- Consider 2-2 scattering of identical massive scalars.
- In perturbative QFT we use Feynman diagrams. These manifest crossing symmetry.
- A non-perturbative representation of the amplitude follows from fixed-t dispersion relation. This loses crossing symmetry, which needs to be imposed as a constraint.

- Where are the Feynman diagrams then from the dispersion relation point of view?
- There is an analogous question one can ask in conformal field theory—this was our motivation to look into this question: Polyakov in 1974 proposed a crossing symmetric bootstrap. This looked like……

$$\mathcal{M}(s,t) \overset{??}{=} \sum_{\Delta,\ell} c_{\Delta,\ell}$$



+  ??

Gopakumar, Kaviraj, Sen, AS PRL '16; Gopakumar, AS '18,…………

- How do we constrain Effective Field Theories? What role does the inaccessible high energies (unknown) play?
- What are the correct mathematical ingredients? Are there known mathematical theorems which can help us?

- So imagine you were an EFT specialist and had access to 2-2 scattering.
- Schematically you have low energy information of the form
$$\mathcal{M}(s_1, s_2, s_3) = \sum \mathcal{W}_{pq} x^p y^q\,, \quad x = s_1 s_2 + s_1 s_3 + s_2 s_3\,, \quad y = s_1 s_2 s_3$$
subject to $s_1 + s_2 + s_3 = 0$

- Typically you have information about the first few Wilson coefficients at low energies.
- Can these take on arbitrary values? Knowing these are there anything useful we can say about the higher order ones which we don't know anything about?

Arkani-hamed, Huang, Huang; Caron-huot, Mazac, Rastelli, Simmons-Duffin, van-Duong; Tolley, Wang, Zhou;

Bern, Kosmopoulous, Zhiboedov……..

$\mathcal{W}_{p,q}$	q=0	q=1	q=2	q=3	q=4	q=5
p=0	-5.22252	-0.0209238	0.000401094	-0.0000116118	3.9934×10^{-7}	-1.5104×10^{-8}
p=1	0.0663542	-0.0023309	0.0000983248	-4.4442×10^{-6}	2.0832×10^{-7}	-
p=2	0.00344623	-0.00027954	0.00001862	-1.1521×10^{-6}	-	-
p=3	0.000267396	-0.0000348355	3.1948×10^{-6}	-2.5174×10^{-7}	-	-
p=4	0.0000245812	-4.4442×10^{-6}	5.2081×10^{-7}	-	-	-
p=5	2.4827×10^{-6}	-5.7605×10^{-7}	-	-	-	-

Table 2: $\mathcal{W}_{p,q}$ for 1-loop ϕ^4 amplitude

$$\mathcal{M}^{(cl)}(s_1,s_2) = -\frac{\Gamma(1-s_1)\Gamma(1-s_2)\Gamma(s_1+s_2+1)}{s_1 s_2 (s_1+s_2)\Gamma(s_1+1)\Gamma(-s_1-s_2+1)\Gamma(s_2+1)} + \frac{1}{s_1 s_2 (s_1+s_2)} \tag{B.1}$$

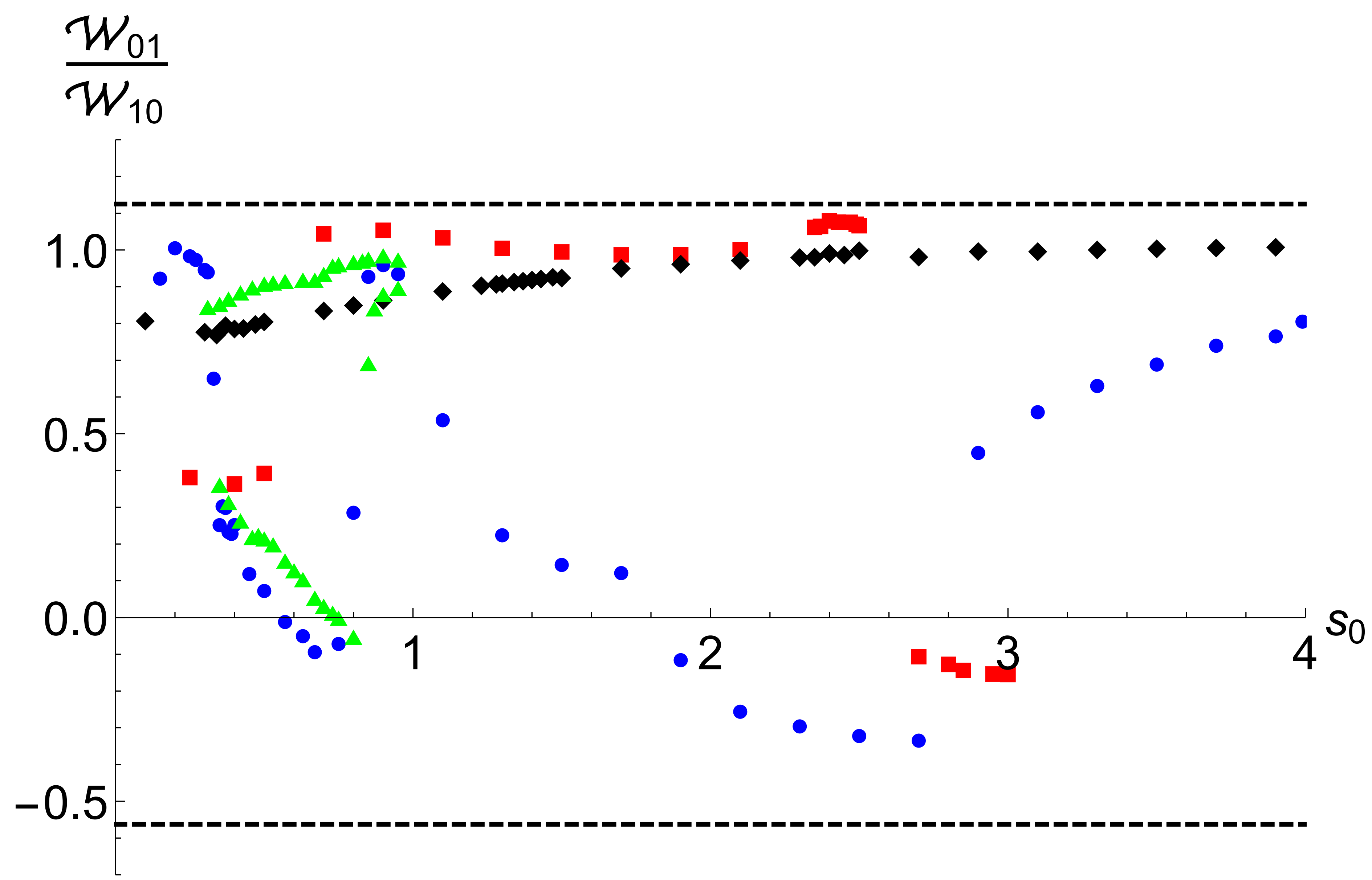
2103.12108

$\mathcal{W}_{p,q}$	q=0	q=1	q=2	q=3	q=4	q=5
p=0	2.40411	-2.88988	2.98387	-2.99786	2.99973	-2.99997
p=1	2.07386	-4.98578	7.99419	-10.9987	13.9998	-17.
p=2	2.0167	-6.99881	14.9984	-25.9995	39.9999	-57.
p=3	2.00402	-9.00023	23.9996	-49.9998	89.9999	-147.
p=4	2.00099	-11.0002	34.9999	-84.9999	175.	-322.
p=5	2.00025	-13.0001	48.	-133.	308.	-630.

Table 1: $\mathcal{W}_{p,q}$ for tree level type II superstring theory amplitude

$\mathcal{W}_{p,q}$	q=0	q=1	q=2	q=3	q=4	q=5
p=0	-1.90562	5.02671	-0.249527	0.0118008	-0.000555517	0.0000262344
p=1	5.72161	0.395863	-0.0520982	0.00402939	-0.000264317	-
p=2	0.642298	0.0217519	-0.00787377	0.000904172	-	-
p=3	0.0796397	-0.000836409	-0.000995454	0.000166504	-	-
p=4	0.0101505	-0.000579411	-0.000103708	-	-	-
p=5	0.0013093	-0.000136893	-	-	-	-

Table 3: $\mathcal{W}_{p,q}$ for pion scattering from S-matrix bootstrap with $s_0 = 0.35$



- Based on 2012.04877 (PRL, 21) with Ahmadullah Zahed and 2101.09017 (PRL, 21) with Rajesh Gopakumar and Ahmadullah Zahed, 2103.12108 with Parthiv Haldar and Ahmadullah Zahed and work in progress with Prashanth Raman.

PHYSICAL REVIEW D VOLUME 6, NUMBER 10 15 NOVEMBER 1972

Rigorous Parametric Dispersion Representation with Three-Channel Symmetry*

G. Auberson and N. N. Khuri
Rockefeller University, New York, New York 10021
(Received 30 June 1972)

Starting with an analyticity domain in the two Mandelstam variables which is contained in the domain obtained by Martin, we derive a parametric dispersion representation for scattering amplitudes in the equal-mass case. For pion-pion scattering this representation is a rigorous consequence of the axioms of local field theory; it displays in a symmetric and explicit way the contributions of all three channels, and it has only "physical" absorptive parts. This representation is useful for deriving sum rules involving only absorptive parts and relating all three channels. Some of these sum rules are given in this paper, the most important of which form a set of independent physical relations that lead to necessary and sufficient conditions ensuring full crossing symmetry.

A relatively unknown paper from 1972!

Note added in proof. One should note that Eq. (4.13) is tremendously simplified in the fully symmetric case, $\pi^0\pi^0 \to \pi^0\pi^0$. In that case one obtains

$$F_0(\bar{s}, \bar{t}, \bar{u}) = \alpha_0 + \frac{1}{\pi}\int_{8/3}^{\infty} \frac{d\bar{s}'}{\bar{s}'} A(\bar{s}'; \bar{t}_+(\bar{s}'; \bar{s}, \bar{t}, \bar{u}))\, H(\bar{s}'; \bar{s}, \bar{t}, \bar{u}),$$

where $H(\bar{s}'; \bar{s}, \bar{t}, \bar{u}) = [\bar{s}(\bar{s}' - \bar{s})^{-1} + \bar{t}(\bar{s}' - \bar{t})^{-1} + \bar{u}(\bar{s}' - \bar{u})^{-1}]$, and $\bar{t}_+(\bar{s}'; \bar{s}, \bar{t}, \bar{u}) = t_+(\bar{s}'; \bar{a})$ with $\bar{a} = \bar{s}\,\bar{t}\,\bar{u}(\bar{s}\,\bar{t} + \bar{t}\bar{u} + \bar{s}\,\bar{u})^{-1}$ and $\bar{t}_+(\bar{s}', \bar{a})$ given in Eq. (5.20). This representation holds for *any* point (s, t) for which $\tau \equiv \bar{t}_+(\bar{s}'; \bar{s}, \bar{t}, \bar{u}) + \frac{4}{3}$ lies in the Martin-Lehmann ellipses $E(s')$ for $A(s', \tau)$ given in Eq. (A2). The similarity of this representation to the Cini-Fubini approximation[4] is striking. This representation follows most directly from Eq. (5.2) by transforming from the (z, a) variables to the s, t, u variables.

The most useful/ encouraging formula was in a NOTE ADDED!

ACKNOWLEDGMENTS

We are indebted to F. J. Dyson and G. Wanders for useful comments.

See also Mahoux, Roy, Wanders 1974.

Crossing symmetric dispersion relation

$$\mathcal{M}_0(s_1,s_2)=\alpha_0+\frac{1}{\pi}\int_{\frac{2\mu}{3}}^{\infty}\frac{ds_1'}{s_1'}\mathcal{A}\left(s_1';s_2^{(+)}\left(s_1',a\right)\right)\times H\left(s_1';s_1,s_2,s_3\right)$$

$$H\left(s_1';s_1,s_2,s_3\right)=\left[\frac{s_1}{\left(s_1'-s_1\right)}+\frac{s_2}{\left(s_1'-s_2\right)}+\frac{s_3}{\left(s_1'-s_3\right)}\right]$$

$$s_2^{(+)}\left(s_1',a\right)=-\frac{s_1'}{2}\left[1-\left(\frac{s_1'+3a}{s_1'-a}\right)^{1/2}\right],$$

$$s_1+s_2+s_3=0 \quad \mu=4m^2$$

$$a=\frac{s_1s_2s_3}{s_1s_2+s_1s_3+s_2s_3}$$

$$\alpha_0\equiv\mathcal{M}(s_1=0,s_2=0)$$

NB: $s_1=s-\frac{4}{3}m^2, s_2=t-\frac{4}{3}m^2, s_3=u-\frac{4}{3}m^2$

Expand

In terms of poles in s_1, s_2 & $s_3 = -s_1 - s_2$

$$-s_1 s_2 (s_1+s_2)\mathcal{M}(s_1,s_2) = \frac{\Gamma\left(1-s_1\right)\Gamma\left(1-s_2\right)\Gamma\left(s_1+s_2+1\right)}{\Gamma\left(s_1+1\right)\Gamma\left(-s_1-s_2+1\right)\Gamma\left(s_2+1\right)}$$

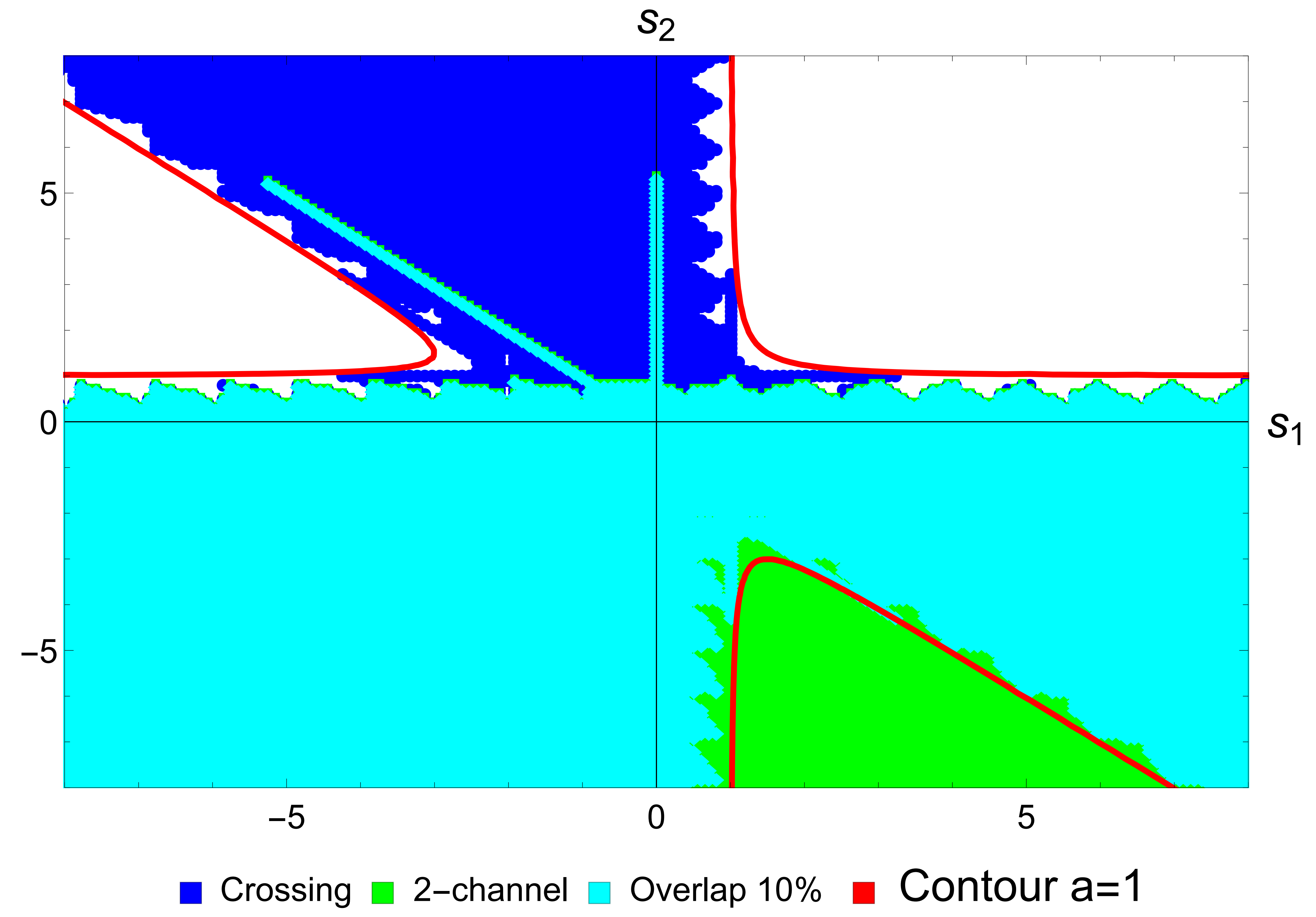
Answer

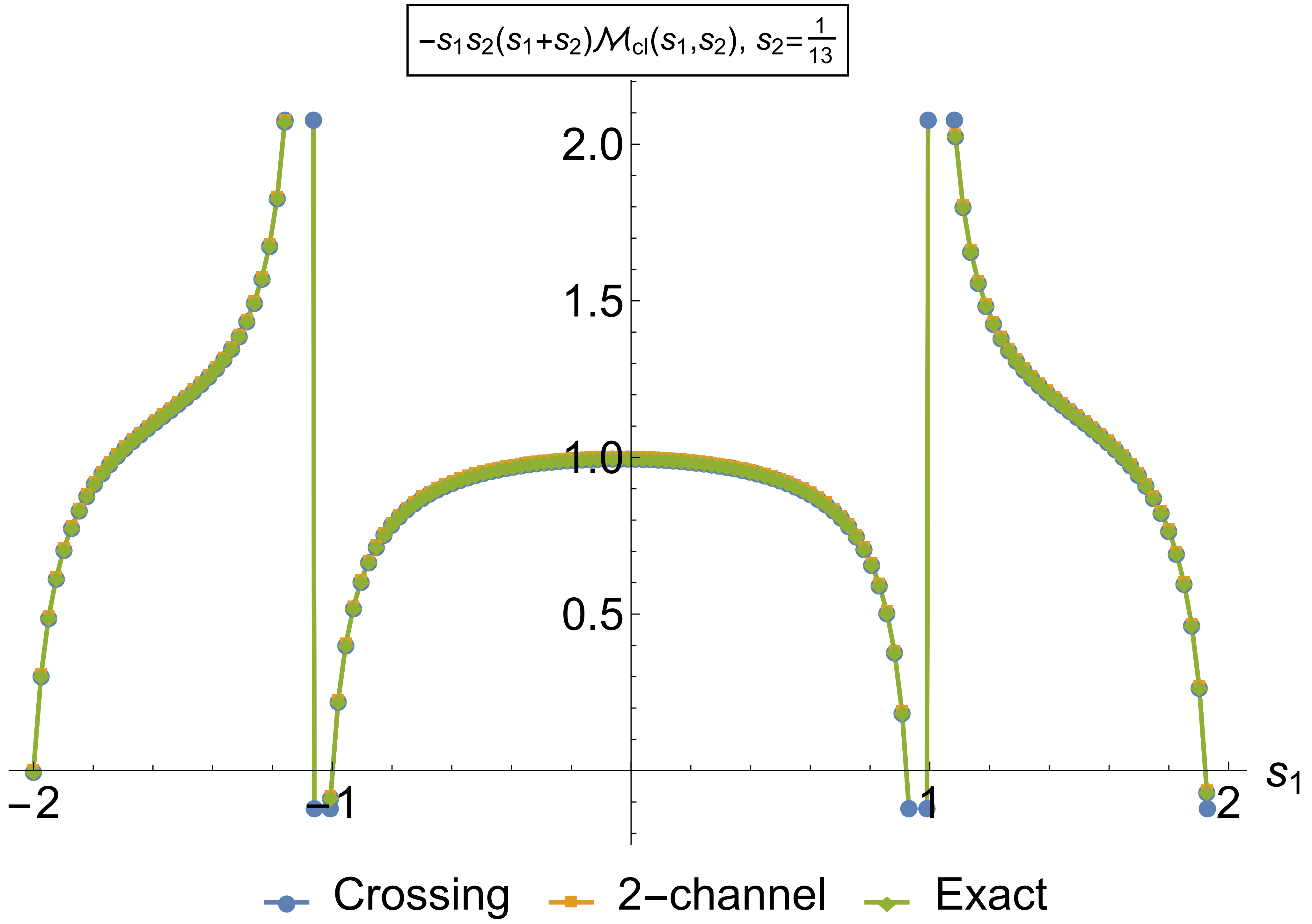
$$-s_1 s_2 (s_1+s_2)\mathcal{M}(s_1,s_2)^{(crossing)} = 1 + \sum_{k=0}^{\infty}\left[\frac{(-1)^k}{k!(k+1)!}\left(\frac{1}{k-s_1+1}+\frac{1}{k-s_2+1}+\frac{1}{k-s_3+1}-\frac{3}{k+1}\right)\right.$$
$$\left.\times\frac{\Gamma\left(\frac{1}{2}\left(-k\sqrt{\frac{4a}{-a+k+1}+1}+k-\sqrt{\frac{4a}{-a+k+1}+1}+3\right)\right)\Gamma\left(\frac{1}{2}\left(k\sqrt{\frac{4a}{-a+k+1}+1}+k+\sqrt{\frac{4a}{-a+k+1}+1}+3\right)\right)}{\Gamma\left(\frac{1}{2}\left(k\left(\sqrt{\frac{4a}{-a+k+1}+1}-1\right)+\sqrt{\frac{4a}{-a+k+1}+1}+1\right)\right)\Gamma\left(\frac{1}{2}\left(-\sqrt{\frac{4a}{-a+k+1}+1}-k\left(\sqrt{\frac{4a}{-a+k+1}+1}+1\right)+1\right)\right)}\right],$$

$$\sim \frac{\Gamma(\frac{3+k}{2}-\lambda)\Gamma(\frac{3+k}{2}+\lambda)}{\Gamma(\frac{1-k}{2}-\lambda)\Gamma(\frac{1-k}{2}+\lambda)}$$

$$a = \frac{s_1 s_2 s_3}{s_1 s_2 + s_1 s_3 + s_2 s_3} \equiv \frac{y}{x}$$

Numerical checks





Similar expansion exists for the 2d Ising Mellin Amplitude. Also similar expansion exists for the open string amplitude with only s,t symmetry (w P. Raman).

s	t	Exact	$k_{\max}$=100	$k_{\max}$=400
$\frac{46}{13}$	$\frac{1}{10}$	1.32322	1.32361	1.32325
$\frac{83}{10}$	$-\frac{8}{5}$	-0.000619309	-0.00061931	-0.000619309
$\frac{41}{5}+\frac{21i}{10}$	$-\frac{8}{5}-\frac{43i}{10}$	$0.200577 - 0.0884721i$	$0.200577 - 0.088472i$	$0.200577 - 0.0884721i$
$\frac{3}{5}+\frac{31i}{10}$	$\frac{1}{5}+\frac{i}{2}$	$-0.242057+2.28081i$	$-0.247887+2.28194i$	$-0.242315+2.28022i$
$\frac{31i}{10}$	$\frac{i}{5}$	$0.769274 + 0.638919i$	$0.769435 + 0.63905i$	$0.769279 + 0.638931i$

- Good match for complex values

- Notice something nontrivial. If we consider just a fixed k.
- Expand around a=0.
- You will get negative powers of x.
- But LHS has no such powers. This means that once we sum over k, these negative powers will cancel. Keep this in mind.
- Lesson: To have a crossing symmetric expansion, it seems we have to introduce "spurious", "non-local" singularities to the basis elements. LOCALITY CONSTRAINTS/NULL CONSTRAINTS

A heuristic derivation of the crossing symmetric dispersion

- Auberson, Khuri 1972; AS, Zahed '20.

The key steps

- We can parametrize the solution using a new variable z

$$s_k = a - a\frac{(z - z_k)^3}{z^3 - 1}$$

- Here z_k are the cube roots of unity. Satisfying

  $z_1 + z_2 + z_3 = 0$

The key idea

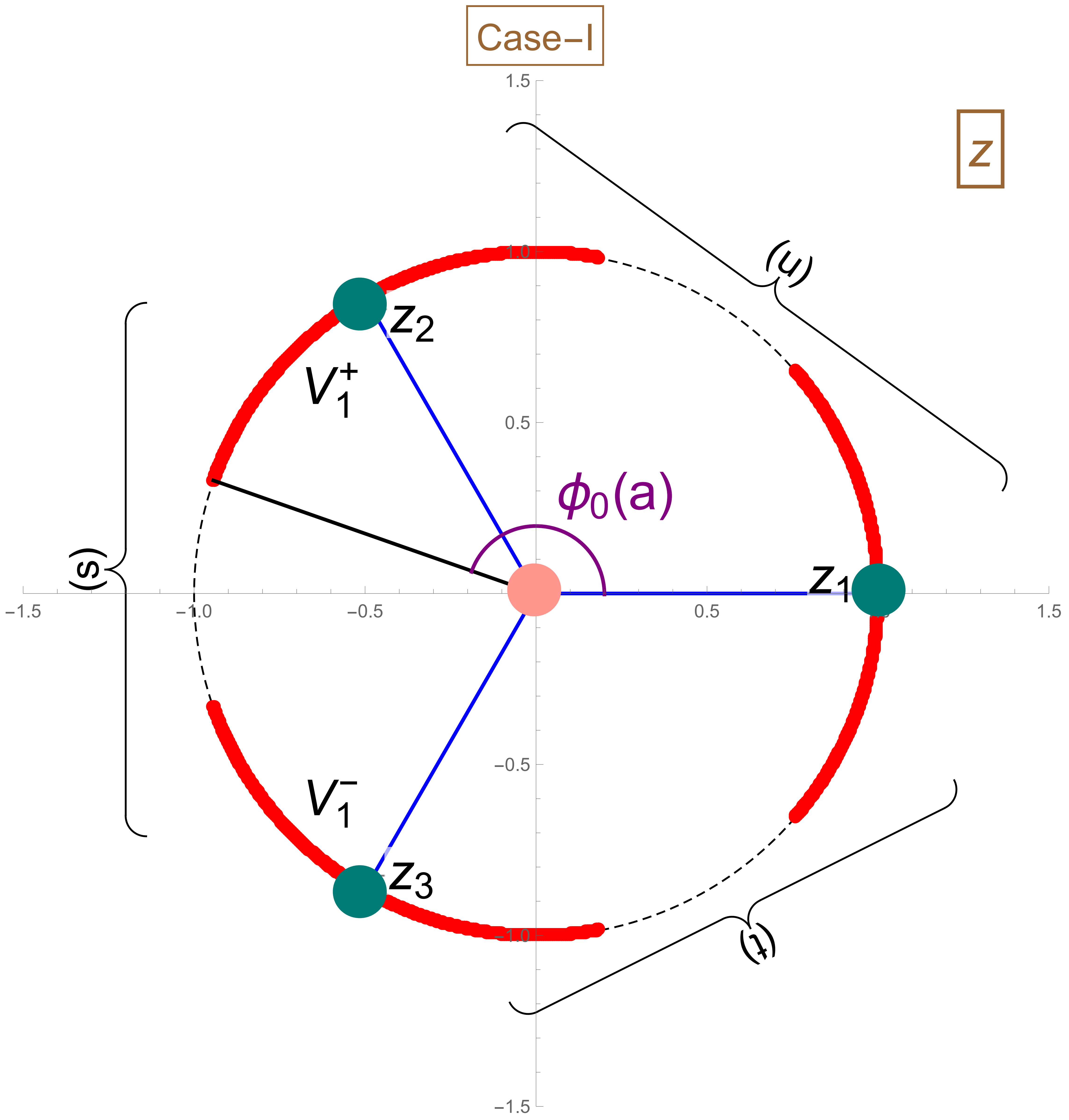
- Idea now is to write a dispersion relation in z for fixed a.
- Conveniently one can show

$$-y \equiv s_1 s_2 s_3 = \frac{27a^3z^3}{(z^3-1)^2}$$

$$-x \equiv s_1 s_2 + s_2 s_3 + s_1 s_3 = \frac{27a^2z^3}{(z^3-1)^2}$$

Make a mental note of the forms of these. These are what are called Koebe functions in the context of univalent functions.

- So that an expansion in powers of x, y is an expansion in powers of a, z^3.



$$s_k = a - a\frac{(z - z_k)^3}{z^3 - 1}$$

$$z_1 + z_2 + z_3 = 0$$

$$a = \frac{s_1 s_2 s_3}{s_1 s_2 + s_1 s_3 + s_2 s_3} = \frac{s_1 s_2 (s_1 + s_2)}{s_1 s_2 + s_1^2 + s_2^2}$$

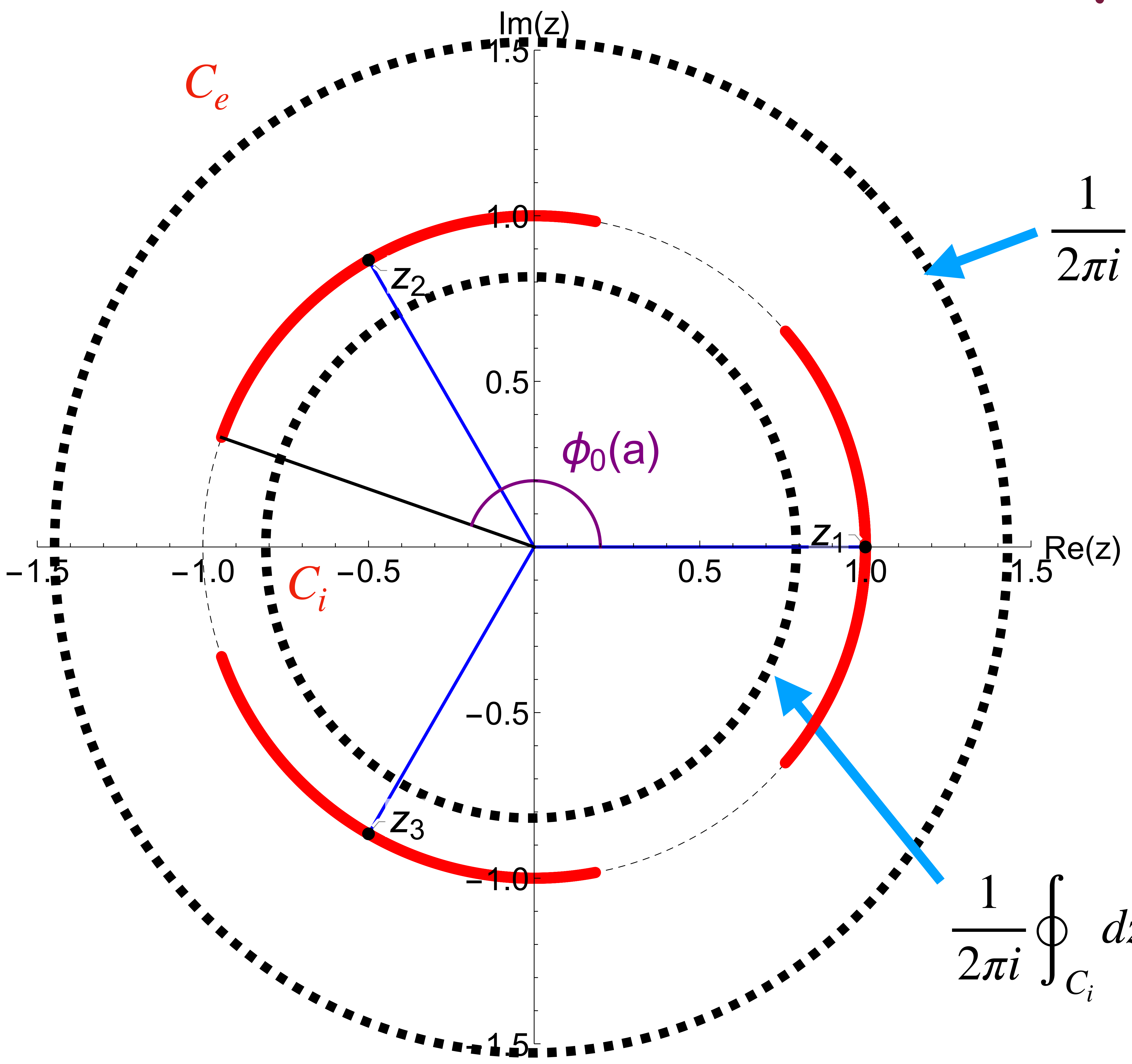
Low energy

High energy

$$-\frac{8}{9}m^2 < a \le 0$$

$s_1 \to a, |s_2|, |s_3| \to \infty$ when $z \to z_1$ etc

The key idea



$$\frac{1}{2\pi i}\oint_{C_e} dz' \frac{z'^3 - 1}{z'^3(z'-z)}\mathscr{M}(z',a) = \mathscr{M}(s_1 = 0, s_2 = 0)$$

$$\frac{1}{2\pi i}\oint_{C_i} dz' \frac{z'^3 - 1}{z'^3(z'-z)}\mathscr{M}(z',a) = \frac{z^3-1}{z^3}\mathscr{M}(z,a) + \frac{\mathscr{M}(s_1 = 0, s_2 = 0)}{z^3}$$

The key steps

- Imposing $s_1 + s_2 + s_3 = 0$ the equation

$$a = \frac{s_1 s_2 s_3}{s_1 s_2 + s_1 s_3 + s_2 s_3}$$

- Becomes a quadratic equation giving s_2 in terms of s_1. Call these $s_2^{\pm}(s_1, a)$

$$s_2^{(\pm)}(s_1, a) = -\frac{s_1}{2}\left[1 \mp \left(\frac{s_1 + 3a}{s_1 - a}\right)^{1/2}\right]$$

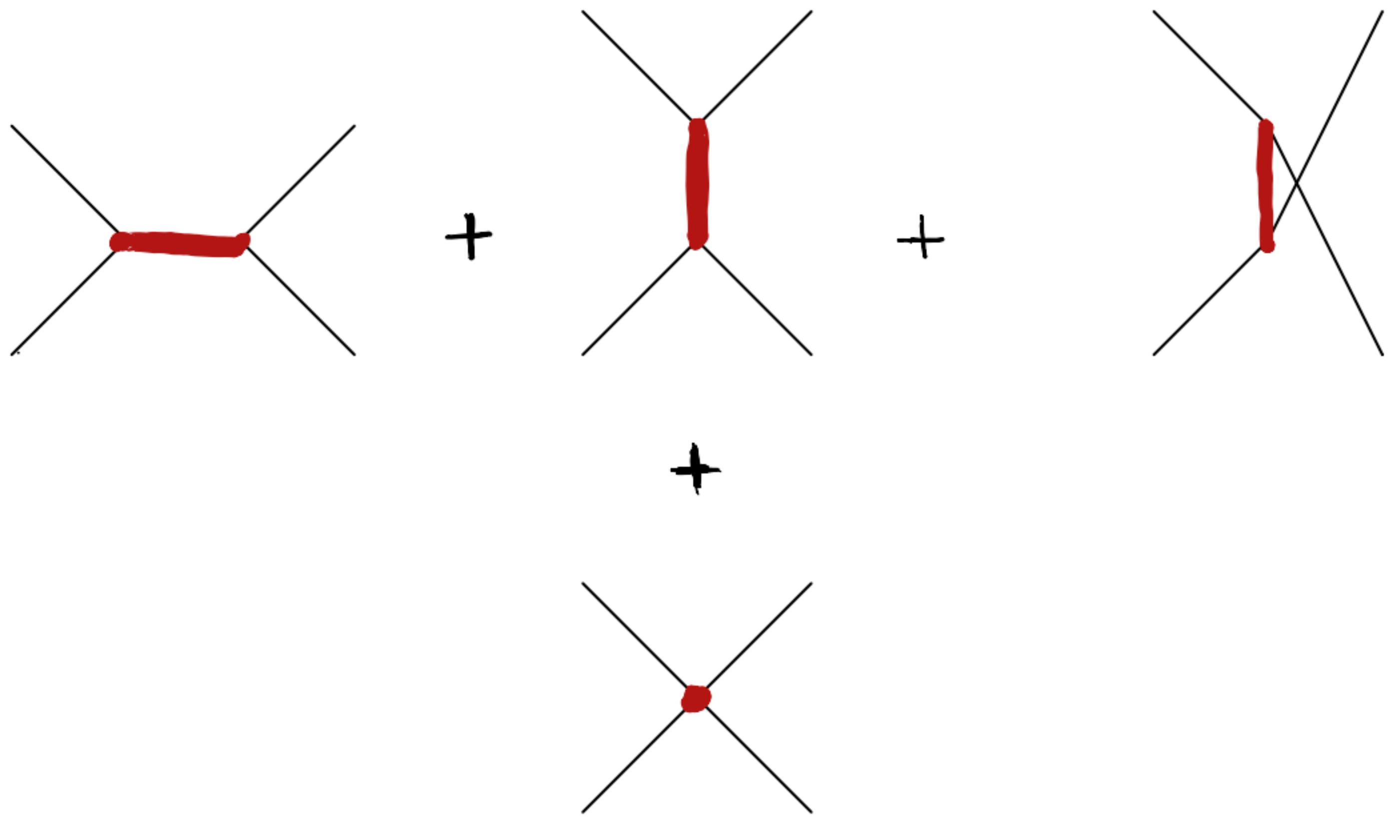
Structure of "Dyson" and "Feynman" blocks

$$\mathcal{M}(s_1,s_2) = \mathcal{M}(0{,}0) + \frac{1}{\pi}\sum_{\ell}(2\ell+2\alpha)\int_{\frac{2\mu}{3}}^{\infty}\frac{d\sigma}{\sigma}H(\sigma;s_i)a_\ell(\sigma)C_\ell^{(\alpha)}(\sqrt{\xi(\sigma,a)})$$

Dyson block expansion —"Regge bounded"

$$\mathcal{M}(s_1,s_2) = \mathcal{M}(0{,}0) + \frac{1}{\pi}\sum_{\ell}(2\ell+2\alpha)\int_{\frac{2\mu}{3}}^{\infty}\frac{d\sigma}{\sigma(\sigma-\frac{2\mu}{3})^\ell}a_\ell(\sigma)\left[\frac{Q_\ell(s_1,s_2)}{\sigma-s_1}+\frac{Q_\ell(s_2,s_3)}{\sigma-s_2}+\frac{Q_\ell(s_3,s_1)}{\sigma-s_3}+poly_\ell\right]$$

Feynman block expansion —not "Regge bounded"

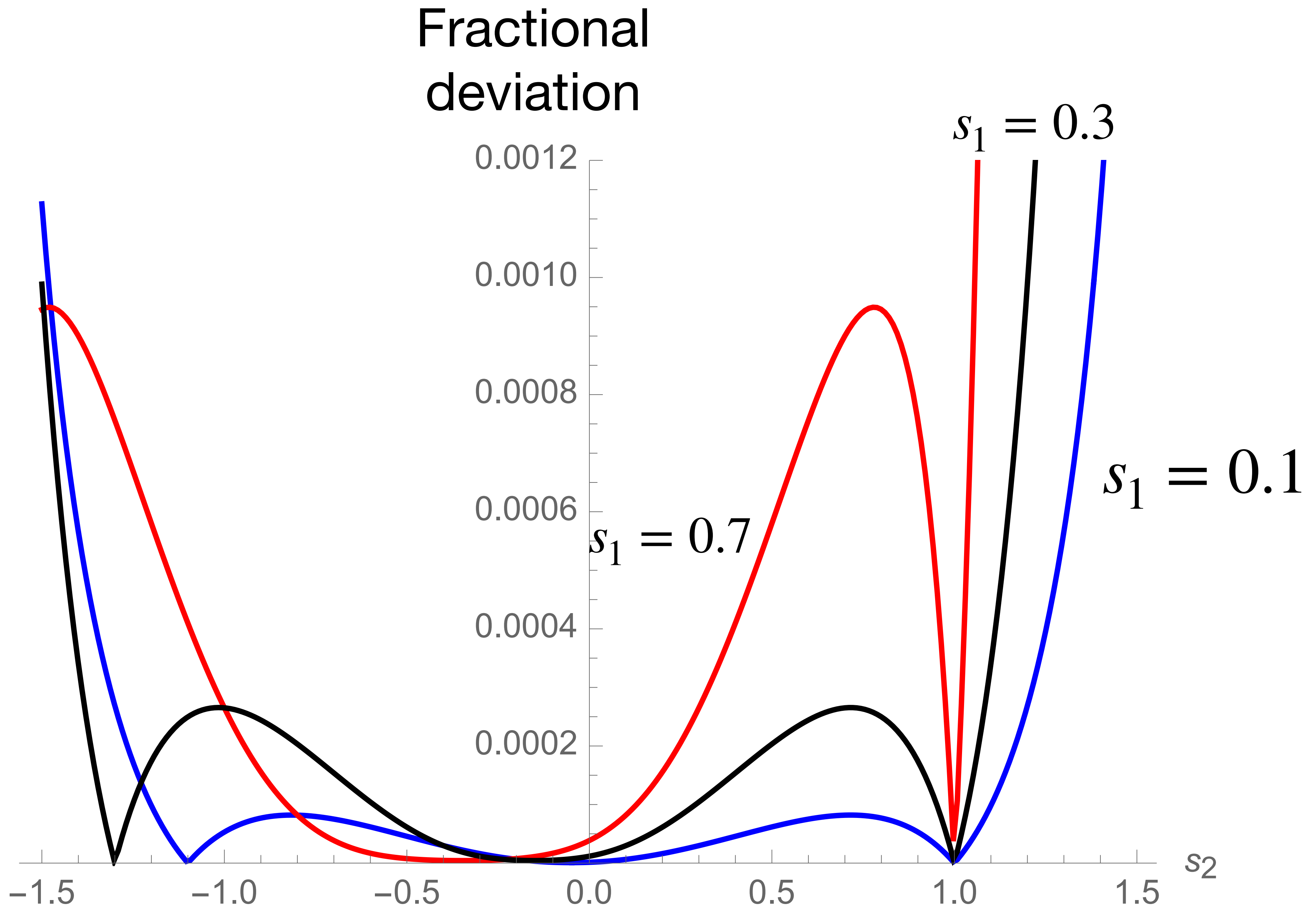


$eg. \quad poly_2 = c_1x + c_2y$

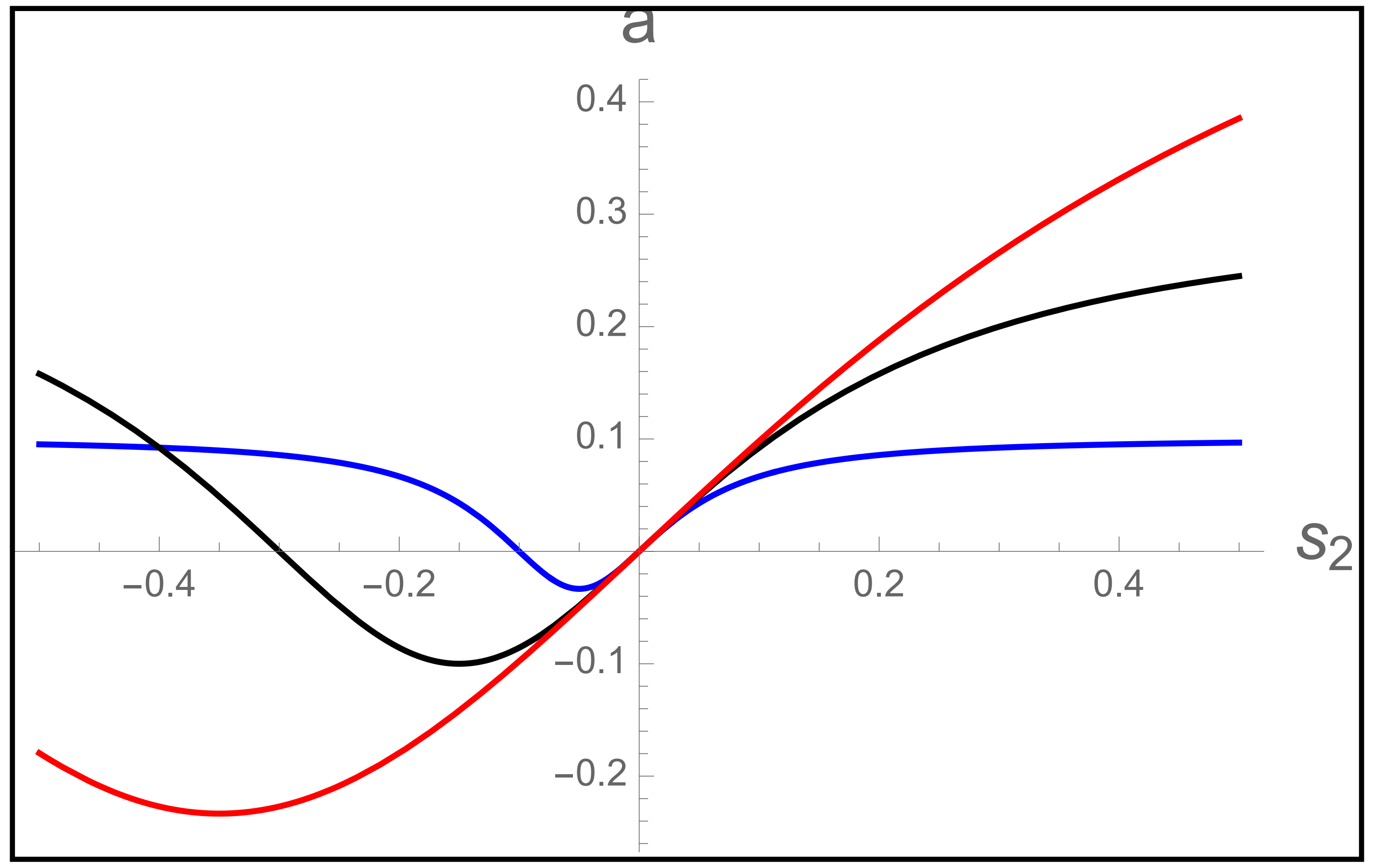
* $Q_\ell(s_1,s_2) = s_1(s_1-\frac{2\mu}{3})^\ell C_\ell^{(\alpha)}(\cos\theta)$

"Dyson" block expansion

Expansion of the massless pole subtracted dilation amplitude in terms of crossing symmetric partial waves (locality constraints implicit).

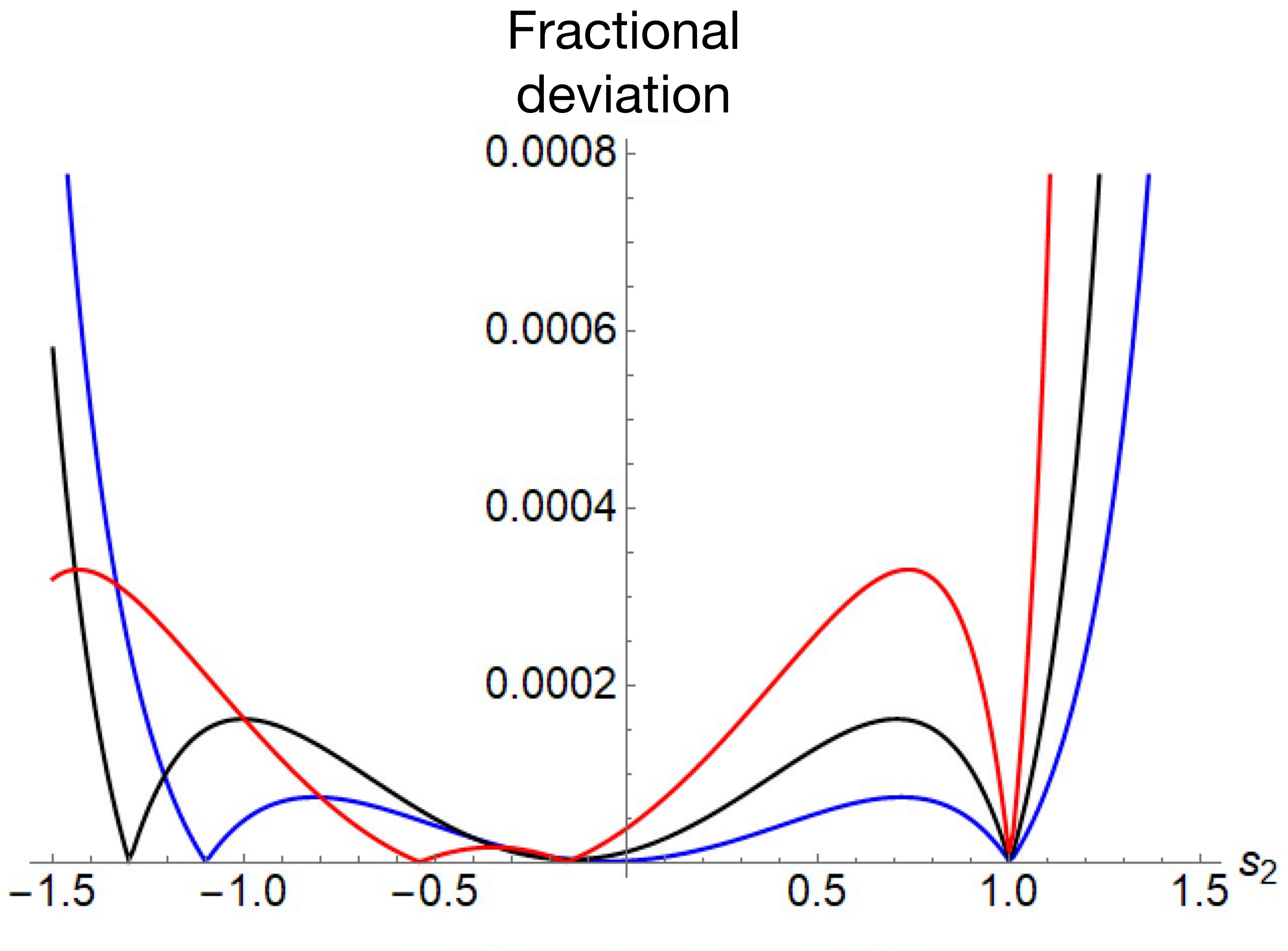


$\ell_{max} = 6, \quad k_{max} = 6$



"Feynman" block expansion

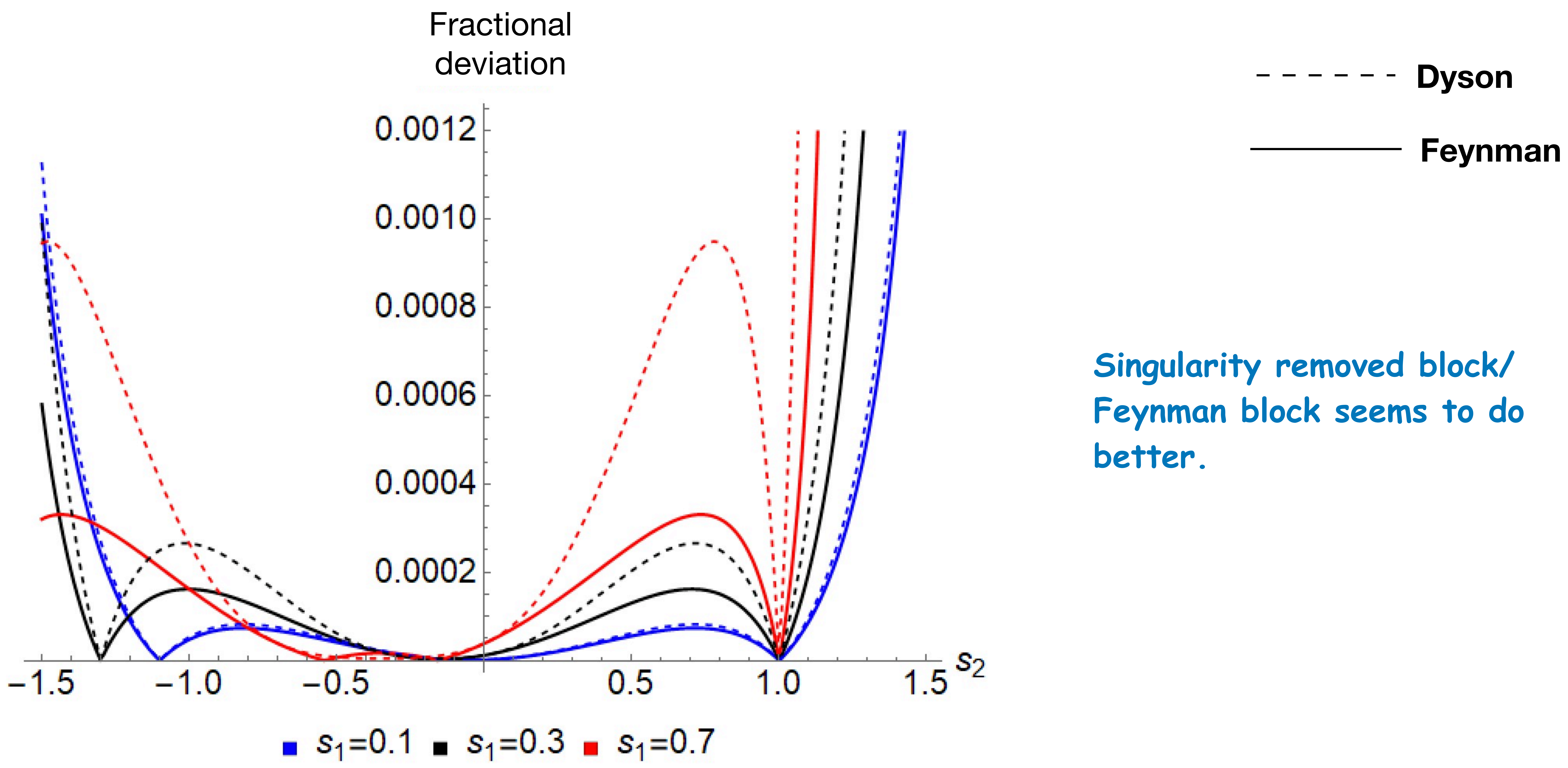
Expansion of the massless pole subtracted dilation amplitude in terms of crossing symmetric partial waves (locality constraints imposed—operationally throw away negative powers of x, partialwave*kernel wise).



$$\ell_{max} = 6, \quad k_{max} = 6$$

Importance of contact terms

s_1	s_2	Exact	Feynman	Without contact terms	Dyson
$\frac{1}{13}$	$\frac{1}{10}$	2.45013	2.45012	2.44898	2.45012
$\frac{7i}{27}$	$1+\frac{i}{10}$	0.886092 + 10.2404 i	0.886074 + 10.2397 i	0.834015 + 10.2278 i	0.886395 + 10.24 i
$\frac{3}{10}+\frac{31i}{10}$	$\frac{1}{11}+\frac{3i}{13}$	0.327184 + 0.214067 i	0.332949 + 0.217049 i	0.790025 + 0.0663346 i	0.32976 + 0.219688 i
$\frac{1}{13}+\frac{33i}{10}$	$\frac{5i}{13}$	0.205184 + 0.179331 i	0.211081 + 0.187269 i	0.775999 + 0.0550316 i	0.201598 + 0.183881 i
$\frac{31i}{10}$	$\frac{i}{5}$	0.312277 + 0.112769 i	0.315664 + 0.116603 i	0.802506 + 0.0667358 i	0.313287 + 0.116089 i



Singularity removed block/ Feynman block seems to do better.

Proposed nomenclature

	QFT	**CFT**
	Partial wave with singularity—Dyson block	Conformal partial wave with singularity—Polyakov block
	Partial wave with singularities removed—Feynman block	CPW with singularities removed—Witten block
	QFT	**CFT**

GFT

- 2103.12108 with P. Haldar and A. Zahed; but mainly work in progress with P. Raman

- Is dispersion (closing of contour, dropping of arcs) enough to give bounds?
- No. What gives two-sided bounds is crossing symmetry and locality. On the math side we have additional analytic properties like univalence, typically realness etc on either a disc, punctured disc or annulus.

- 2103.12108 with P. Haldar and A. Zahed; but mainly work in progress with P. Raman

Power of univalence: a simple example

$$g(z_1) = g(z_2) \implies z_1 = z_2$$

Definition

$$f(z) = z + az^2, \quad |z| < 1$$

Example in a disk

$$f(z_1) = f(z_2) \implies (z_1 - z_2)(1 + az_1 + az_2) = 0$$

$$|1 + az_1 + az_2| \geq |1 - |a||z_1 + z_2||$$

Reverse triangle

$$|a||z_1 + z_2| \leq |a|(|z_1| + |z_2|) \leq 2|a|$$

Triangle

$$|a| < \frac{1}{2}$$

BOUND!!

A bit of math history

Univalent $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad |z| < 1$$

Bieberbach 1916 conjectured $|a_n| \leq n$

Real version proved by Rogosinski,… 1931

Most general version proved by Louis de Branges in 1985

Interesting subclasses, more relevant to give stronger bounds

- 2103.12108 with P. Haldar and A. Zahed; but mainly work in progress with P. Raman

“Convex sum of typically real univalent functions is typically real”

$Im\, f(z)\, Im\, z > 0$

Kernel

$$H(a, s_1, \tilde{z}) = \frac{27a^2(3a - 2s_1)}{s_1^3} \frac{\tilde{z}}{1 + \gamma\tilde{z} + \tilde{z}^2}$$

Mobius transformation of Koebe function— univalent with real coefficients. Univalent functions with real coefficients are also typically Real**

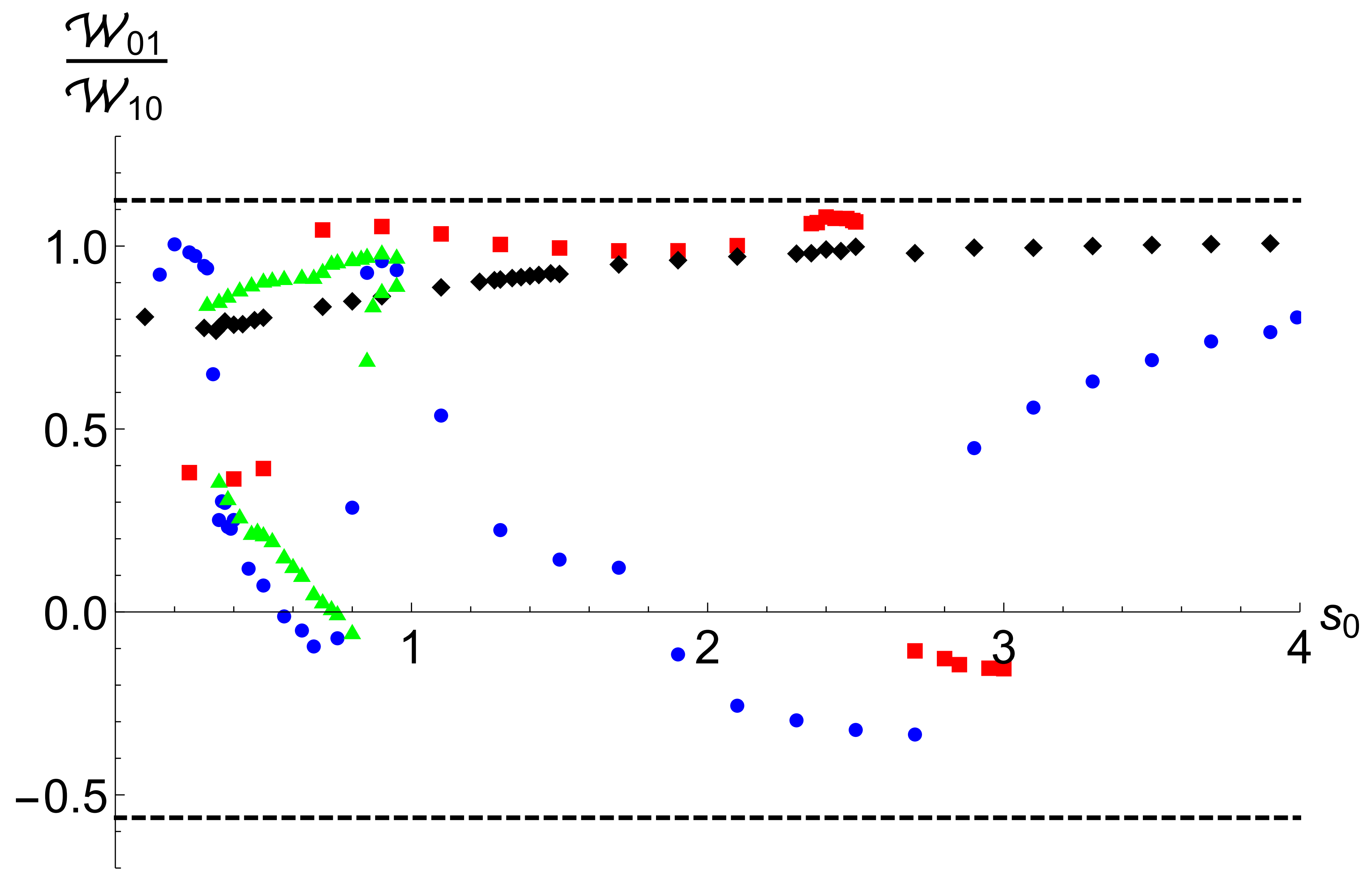
When absorptive part>0, we have a convex sum

$$\mathcal{M}(s_1, s_2) \sim \int \text{absorptive part} \times \text{Kernel}$$

**** Convex sum of univalent functions are not necessarily univalent BUT convex sum of typically real functions are typically real!** **With P. Raman, to appear**

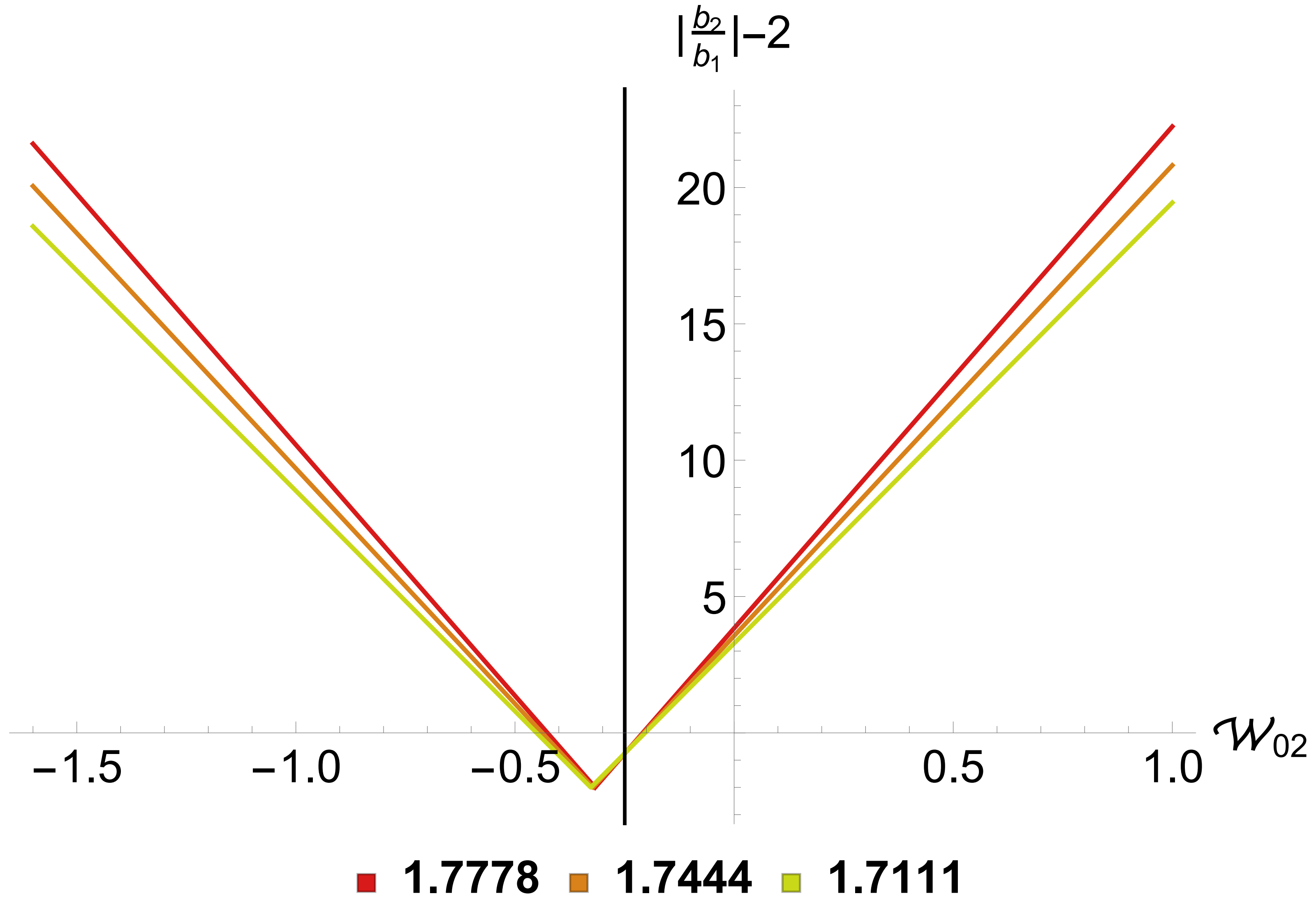
Geometric function theory—Bieberbach, Rogosinski, Wigner, Nehari, Schwarz, Komatu, Goodman, papers dating over a 100 years or so.

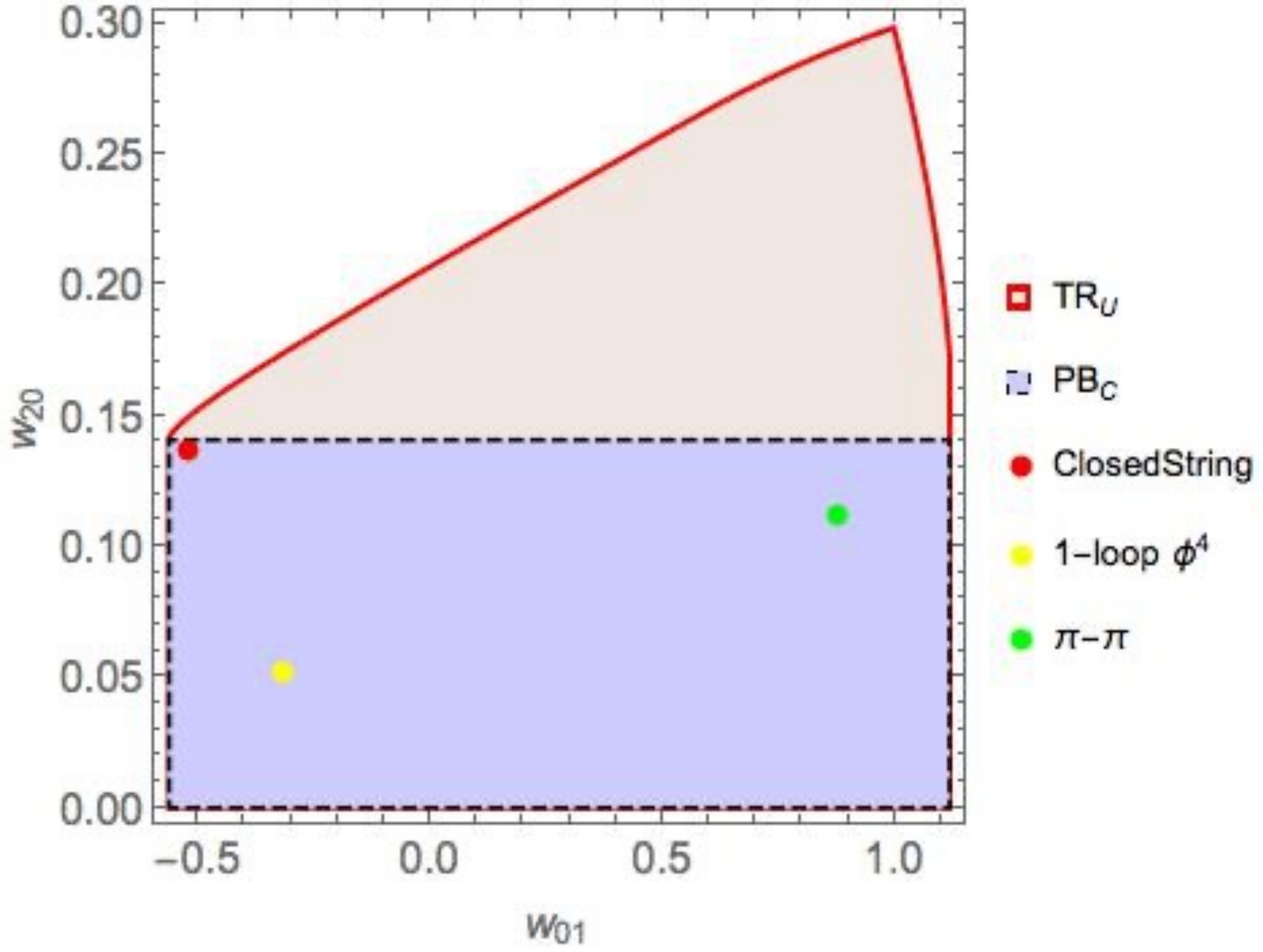
$$-\frac{9}{16} < \frac{\mathscr{W}_{0,1}}{\mathscr{W}_{1,0}} < \frac{9}{8}$$

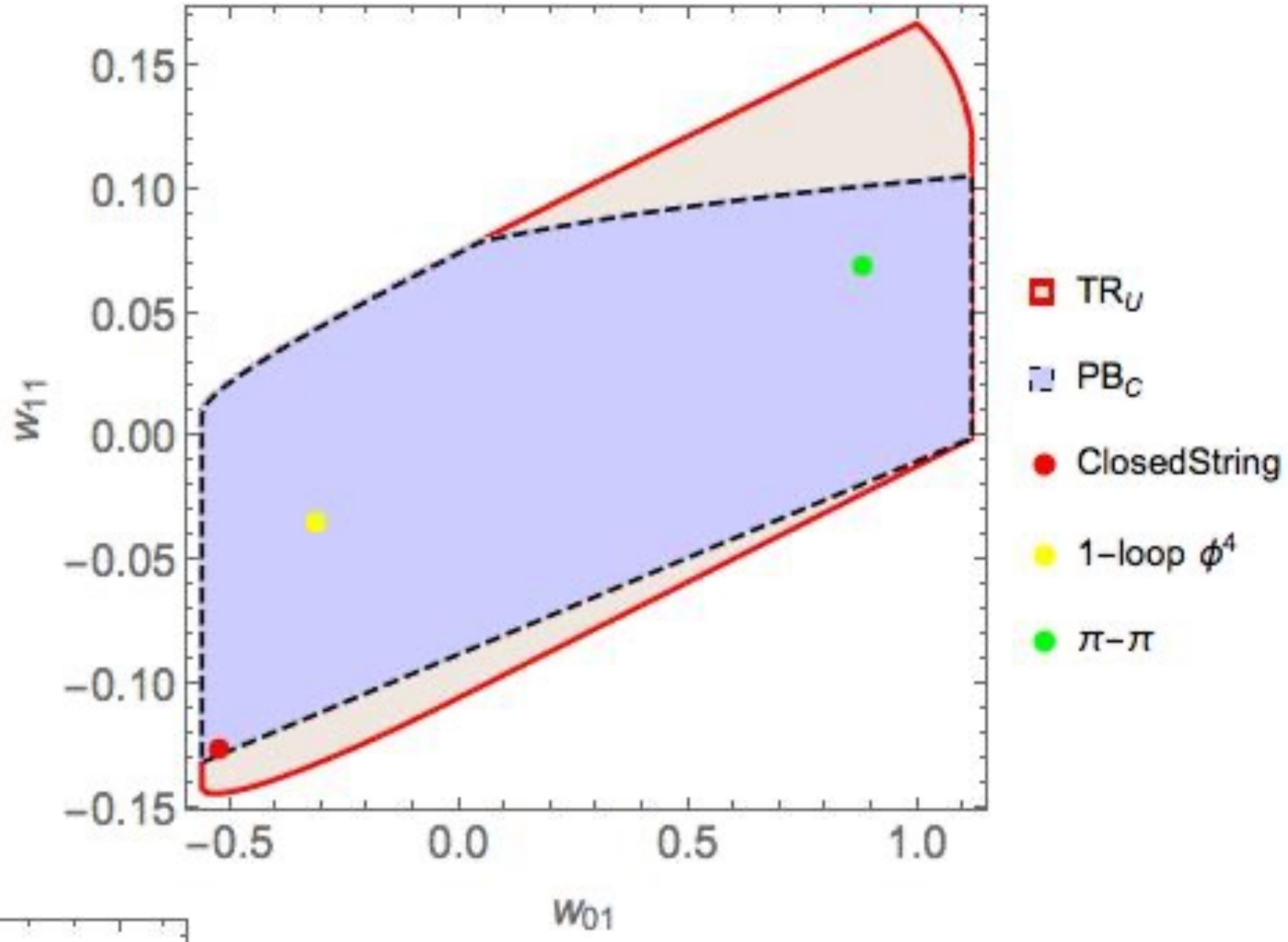


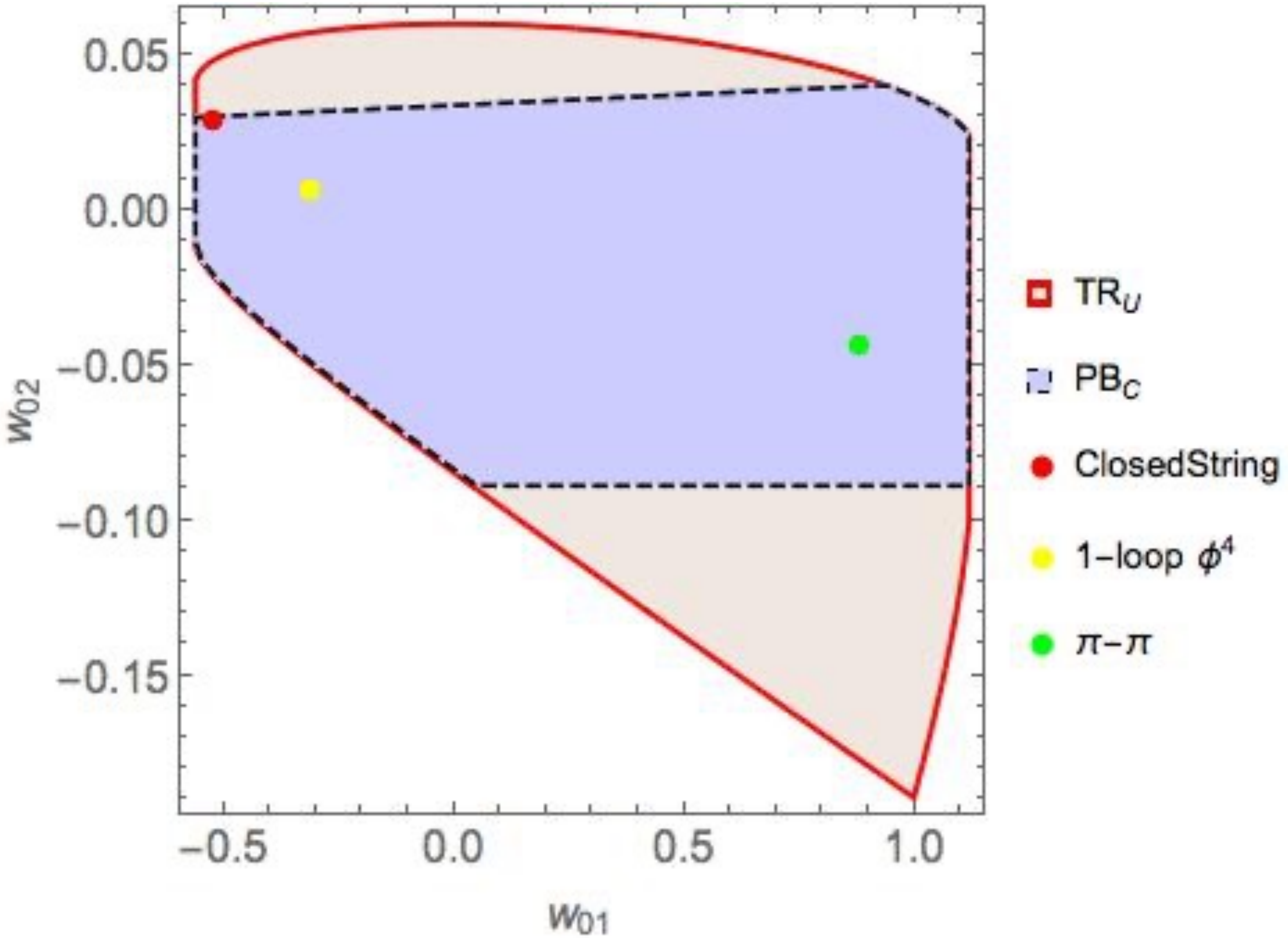
2103.12108; 2012.04877

$$-2 \leq 2 - \frac{27a^2(a(a\mathscr{W}_{02} + \mathscr{W}_{11}) + \mathscr{W}_{20})}{a\mathscr{W}_{01} + \mathscr{W}_{10}} \leq 2$$

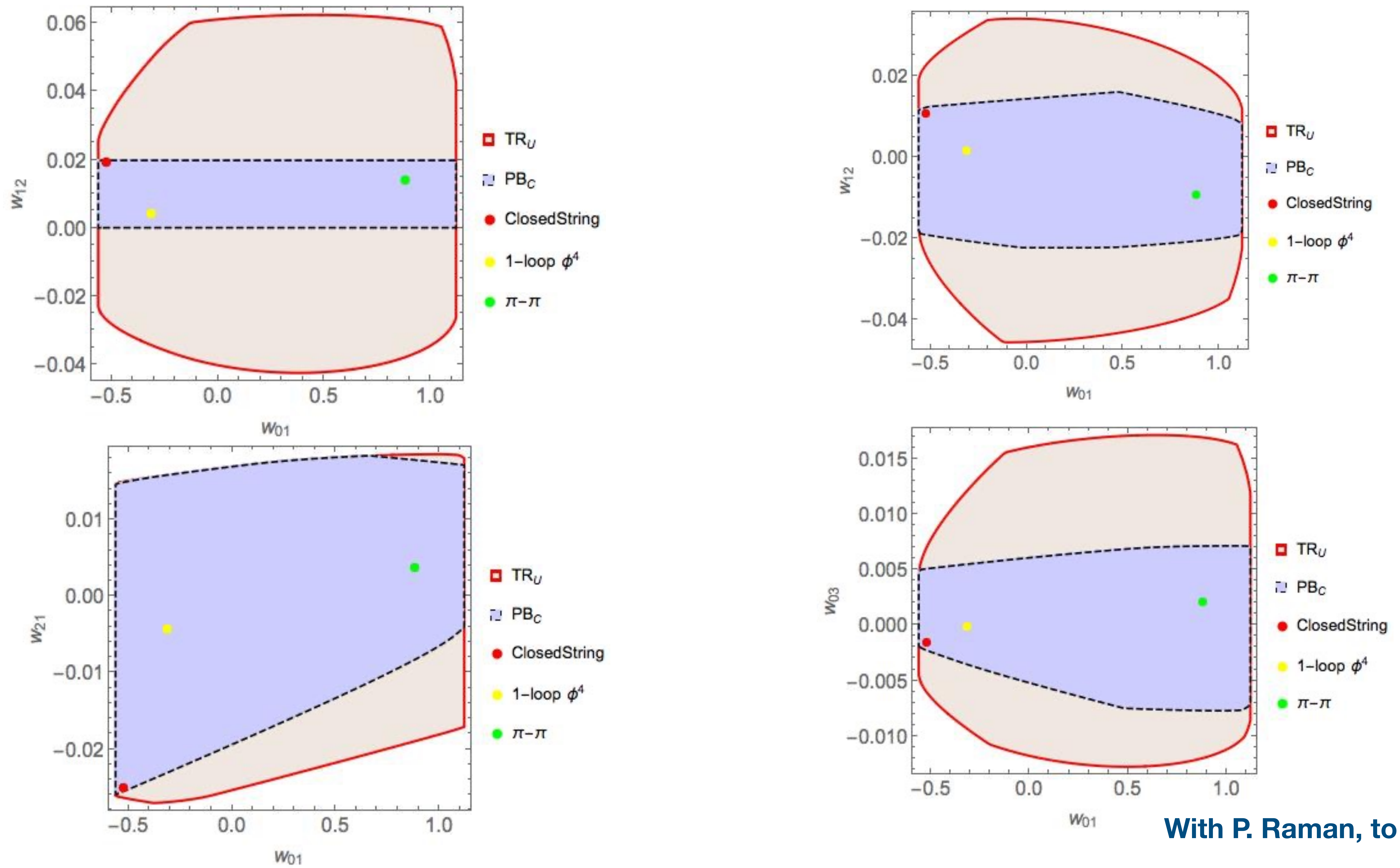








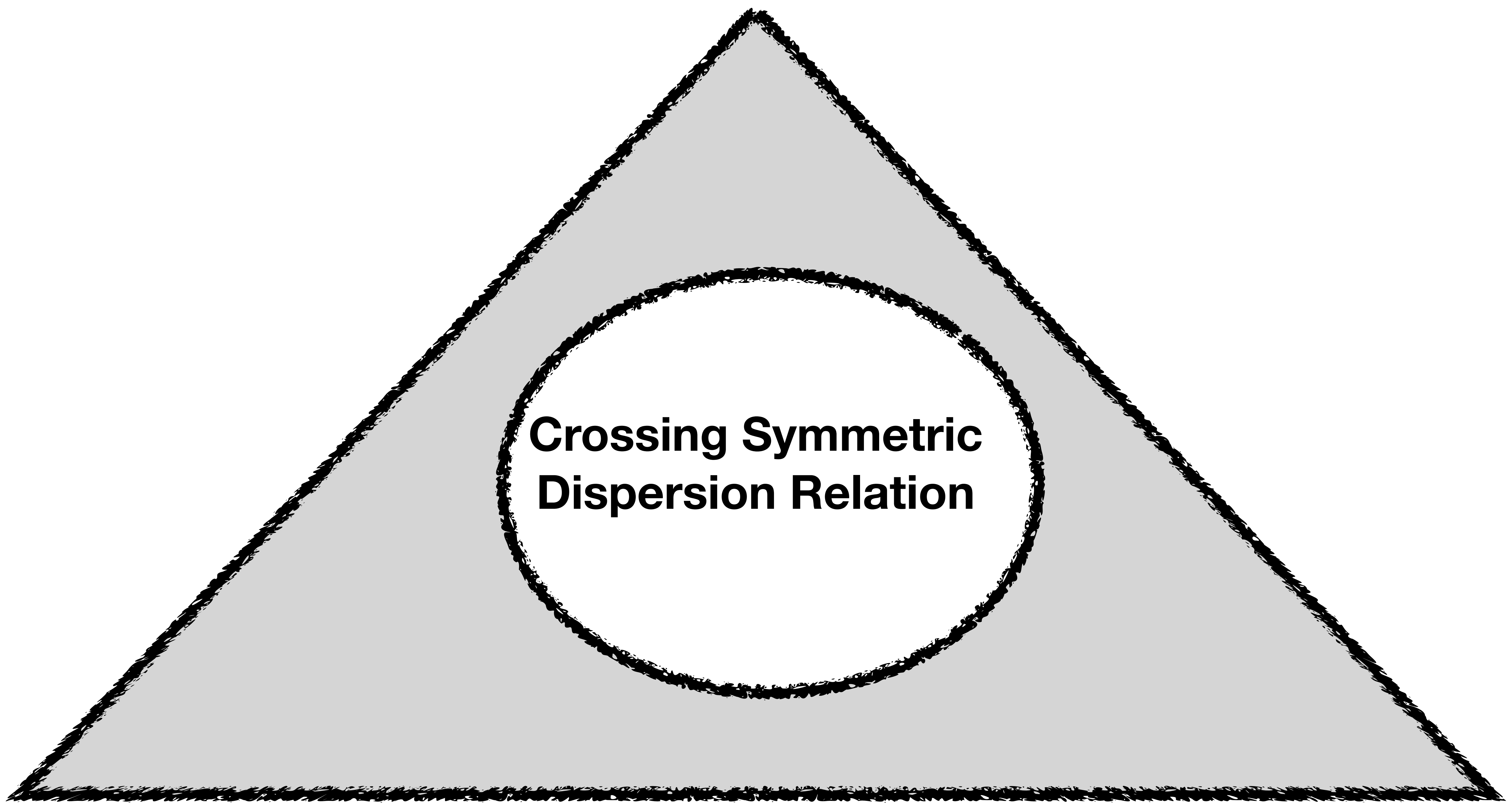
With P. Raman, to appear



With P. Raman, to appear

Future is bright

QFT 1960-
Quantum Field Theory



1916- **GFT**
Geometric Function Theory

EFT 1980-
Effective Field Theory
EFT-hedron