1+1d Adjoint QCD and non-invertible topological lines

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based on WIP with

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Introduction and summary

- 1+1d Adj. QCD was studied extensively in '90s:

  [Klebanov, Dalley '93], [Gross, Klebanov, Matytsin, Smilga '95], [Gross, Klebanov, Hashimoto '98]...

  [Kutasov '93][Boorstein, Kutasov '94],[Kutasov, Schwimmer '95],

- When massless, claimed to be in **deconfined** phase, although fermion cannot screen a probe in fundamental representation.

- [Cherman, Jacobson, Tanizaki, Unsal '19] revisited the problem.

- They analyzed symmetry (incl. one-form) and its anomaly. Concluded it is in confined (or partially deconfined) phase when $N \geq 3$.

- Symmetry is not enough. **Non-invertible topological line** accounts for deconfinement.

- First (non-topological) **gauge theory** example of non-invertible top. op.

1+1d massless Adjoint QCD

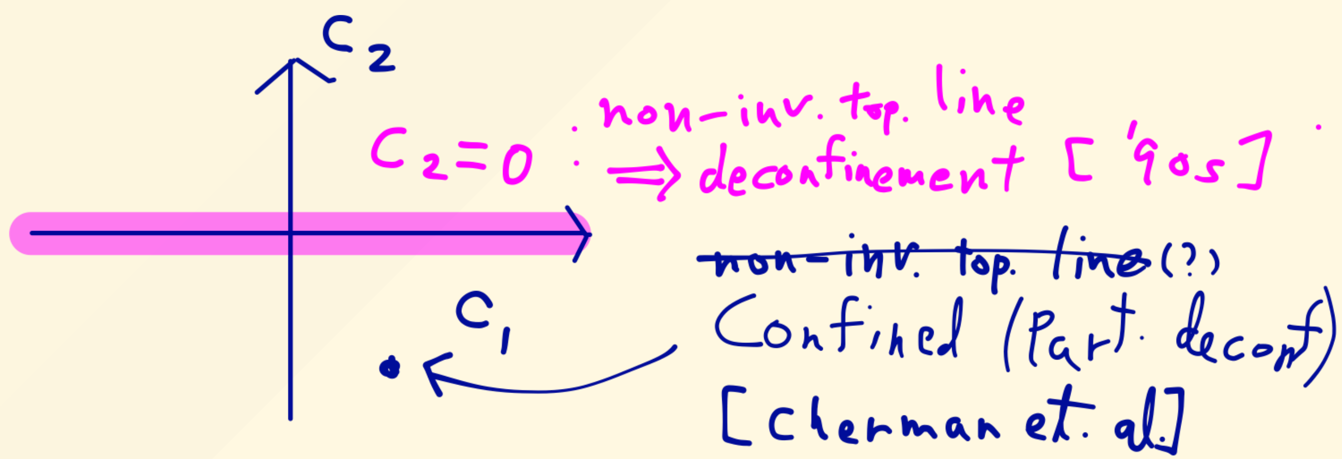
- 1+1d gauge theory with $G = \mathrm{SU}(N)$ with massless Majorana fermions $(\psi_L^{i\bar{j}}, \psi_R^{i\bar{j}})$ ($\sum_i \psi_{L,R}^{i\bar{i}} = 0$)
- $\mathcal{L} = \mathrm{Tr}\left(-\frac{1}{4g^2}F^2 + \mathrm{i}\psi_L \partial \psi_L + \mathrm{i}\psi_R \bar{\partial}\psi_R + j_L A_z + j_R A_{\bar{z}}\right)$
- $j_{L,R}^{i\bar{j}} = \sum_k \psi_{L,R}^{i\bar{k}} \psi_{L,R}^{k,\bar{j}}$
- Symmetry: $\mathbb{Z}_2^C \times \mathbb{Z}_2^\chi \times \mathbb{Z}_2^F$ ($\times \mathbb{Z}_N^{(1)}$: one-form (a.k.a. center) symmetry)

Quartic couplings

- Two independent classically marginal couplings preserving all the symmetry
- $\mathcal{L}_q = c_1\mathcal{O}_1 + c_2\mathcal{O}_2$
  $\mathcal{O}_1 = \mathrm{Tr}(\psi_+\psi_+\psi_-\psi_-) = \mathrm{Tr} j_L j_R,$
  $\mathcal{O}_2 = \left((\mathrm{Tr}(\psi_+\psi_-))^2 - \frac{2}{N}\mathrm{Tr}(\psi_+\psi_+\psi_-\psi_-)\right)$
- In the $N^2 - 1$ free fermion theory, $\mathcal{O}_2$ is a sum of $SU(N)_N$ primaries
- **Fusion rule** : $\langle \mathcal{O}_2(0)\, j_L(z_1) j_L(z_2)\cdots j_R(w_1) j_R(w_2)\cdots\rangle_{\text{free } \psi} = 0$
- $\langle \mathcal{O}_2\rangle_{\text{adj QCD}} = \int \mathcal{D}A\, e^{-\mathcal{S}_{YM+\text{cntr}}[A]} \left\langle \mathcal{O}_2\, e^{\int j^\mu A_\mu} \right\rangle_{\text{free } \psi} = 0$
- **No Feynman diagram** that can generate $\mathcal{O}_2$ with $j_L A_z + j_R A_{\bar{z}}$ and $\mathcal{O}_1$ coupling in adj QCD!

Protection by non-invertible line

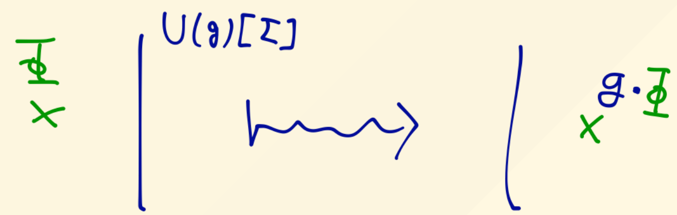
- What protects $\mathcal{O}_2$ from radiative generation?
- There is **no symmetry** that $\mathcal{O}_2$ violates.
- We claim that **non-invertible top. lines** protects it.
- The same set of lines also explains **deconfinement**.
- Parameter space:



- We expect that $\mathcal{O}_2$ def. breaks all the non-invertible lines and thus leads us to the picture of [Cherman, Jacobson, Tanizaki, Unsal '19] but have not succeeded to proof.

Symmetry and top. op.s

- Symmetry $G \implies$ Topological codim.-1 op $U(g)[\Sigma]$ for $g \in G$
  For $e^{\mathrm{i}\alpha} \in \mathrm{U}(1)$, $U(e^{\mathrm{i}\alpha})[\Sigma] = e^{\mathrm{i}\alpha \int_\Sigma J_\mu \mathrm{d}S^\mu}$

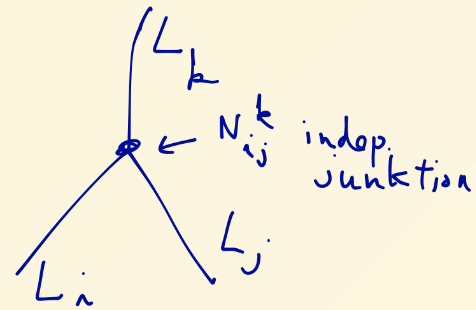


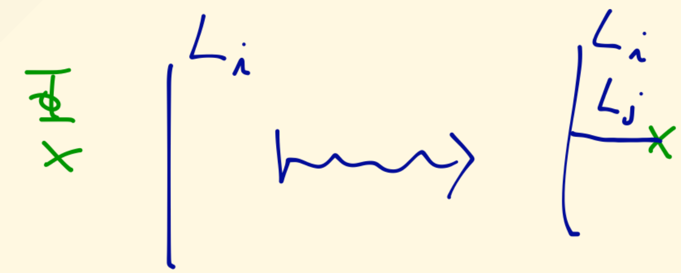
- $U(g)[\Sigma]$ is **invertible**: $U(g)[\Sigma]U(g^{-1})[\Sigma] = \mathbf{1}$
- "Higher-form" symmetry $\iff$ invertible top. op. with **higher codimension**.
  [Gaiotto,Kapustin,Seiberg,Willett '14]
- Not all topological operators have its inverse: **non-invertible** top. op.s.

Non-invertible topological lines

- Top. lines have **fusion rule**:

$$L_i \,\Big|\ \Big|\, L_j = \sum_k N_{ij}^{k} \,\Big|\, L_k$$





- Data of lines and topological junctions = **Fusion category**
- Should be regarded as **generalization of symmetry**, as they shares key features with symmetry (+anomaly): gauging, **RG flow invariance**.

  [Brunner, Carqueville, Plencner '14],[Bhardwaj, Tachikawa, '17],[Chang, Lin, Shao, Wang, Yin, '18]

- E.g. Tricritical ($c = \frac{7}{10}$) Ising + $\sigma'_{\frac{7}{16},\frac{7}{16}}$ relevant perturbation preserves W line with fusion $W^2 = 1 + W \implies$ asymmetric 2 vacua (First noticed by integrability)

  [Chang, Lin, Shao, Wang, Yin, '18]

- Massless Adj. QCD is another example, without (known) integrability.

Outline

- Charge q massless Schwinger model
- Non-abelian bosonization
- Non-invertible lines and confinement in 1+1d massless adj. QCD

Charge q massless Schwinger model

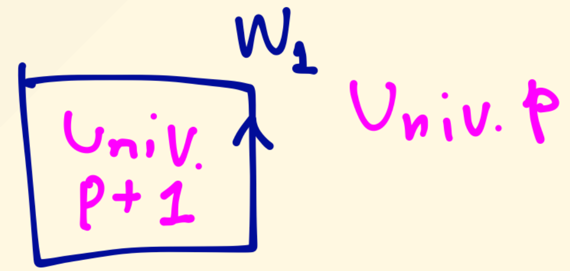
- 1+1d $\mathrm{U}(1)$ gauge theory with charge q massless Dirac fermion Ψ_q ($q > 1$), $\Psi_q \to e^{\mathrm{i}q\alpha}\Psi_q$
- (Ordinary) Symmetry: $\mathbb{Z}_2^C \times \mathbb{Z}_q^\chi$
- $\mathbb{Z}_q \subset \mathrm{U}(1)$ acts trivially on Ψ_q: $\mathbb{Z}_q^{(1)}$ one-form (a.k.a. center) symmetry
- Wilson line $W_p = e^{2\pi \mathrm{i} p \oint A}$: worldline of heavy probe with charge p
- W_q is screened by Ψ_q and deconfined.
- How about W_p when $p \neq 0 \mod q$?

One-form symmetry in 1+1d and "Universe"

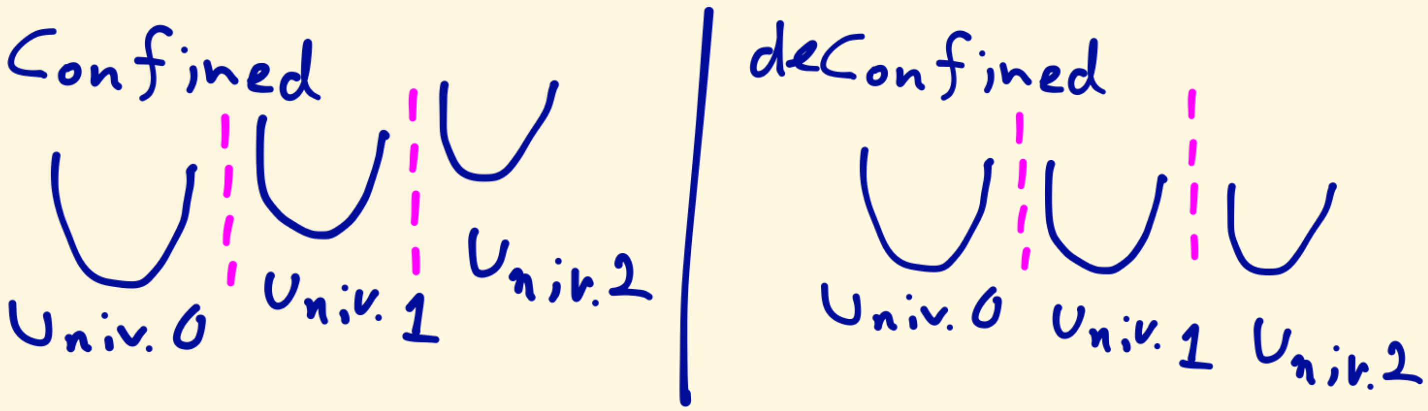
- The electric field $\frac{1}{e^2}F_{01}$ (classically in $\mathbb{Z}$) fluctuates because of Ψ_q, but jumps only by q.
- $U_k = e^{\frac{2\pi i k}{qe^2}F_{01}}$ is $\mathbb{Z}_q \subset \mathrm{U}(1)$ valued **topological** local (codim-2) operator
- Interpreted as the **symmetry operator** for $\mathbb{Z}_q^{(1)}$
- Clustering energy eigenstates (on $\mathbb{R}$) diagonalizes U_k: $U_k\left|p\right\rangle = e^{\frac{2\pi i k p}{q}}\left|p\right\rangle$
- Even on S^1, $\left|p_1\right\rangle$ and $\left|p_2\right\rangle$ does not mix if $p_1 \neq p_2 \mod q$
- No domain wall between $\left|p_1\right\rangle$ and $\left|p_2\right\rangle$ with finite tension
- Separated sectors even on **compact** space: **"universes"** labelled by eigenvalue p of U_1.

"Universe" and (de)confinement

- Wilson line (worldline of infinitely heave partible) separates "universes":
  $$U_k W_p = e^{\frac{2\pi i k p}{q}} W_p U_k$$
- Wilson loop contains another "universe" in it:



- $E_p \neq E_{p+1} \implies$ area law, confinement
  $E_p = E_{p+1} \implies$ perimeter law, deconfinement



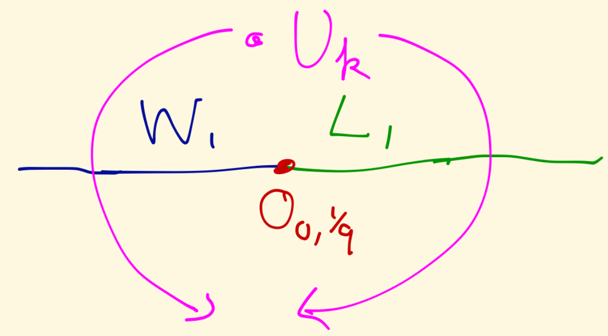
Abelian bosonization

- A way to study the charge q Schwinger model is the bosonization: Dirac fermion $\Psi \Longleftrightarrow \phi$, where ϕ: **periodic scalar** (set to be 2π)
- $\mathcal{O}_{n,w} = e^{in\phi + iw\tilde{\phi}}$
  $\Delta = n^2 + \frac{1}{4}w^2$, $S = nw$, $Q = wq$, $\Psi_q = \mathcal{O}_{\frac{1}{2},1}$
- The dual description is: $\frac{1}{8\pi}(\partial\phi)^2 - \frac{1}{4e^2}F^2 + \frac{q}{2\pi}\phi F$
- $\mathbb{Z}_q^\chi : \phi \to \phi + \frac{2\pi k}{q}$ for $k \in \mathbb{Z}_q^\chi$.
- Naively, IR limit seems equivalent to $e \to \infty$. If true, theory is BF theory (G/G TQFT with $G = U(1)_q$) describing q vacua $\Longrightarrow$ **deconfined**.
- **UV reason?**

$\mathbb{Z}_q^\chi \times \mathbb{Z}_q^{(1)}$ anomaly and deconfinement

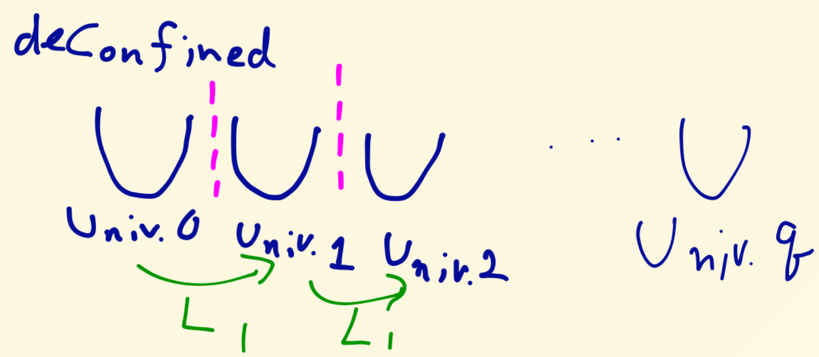
- $\mathcal{O}_{0,\frac{1}{q}} = e^{i\frac{1}{q}\tilde{\phi}}$: **defect operator** at the edge of $\mathbb{Z}_q^\chi$ line L_1 in free boson, $Q = 1$.
  $$\mathcal{O}_{n,w}(z)\mathcal{O}(0)_{0,\frac{1}{q}} = e^{\frac{2\pi i n}{q}}\mathcal{O}_{n,w}(e^{2\pi i}z)\mathcal{O}_{0,\frac{1}{q}}(0)$$

- After gauging $\mathrm{U}(1)$, $\mathcal{O}_{0,\frac{1}{q}}$ connects W_1 and L_1:



- $U_k W_p = e^{\frac{2\pi i k p}{q}} W_p U_k \implies U_k L_p = e^{\frac{2\pi i k p}{q}} L_p U_k$: $\mathbb{Z}_q^\chi \times \mathbb{Z}_q^{(1)}$ **anomaly**

- L_1 is **topological**:
  $L_1|\psi\rangle$ and $|\psi\rangle$ have degenerate energy:



- $\mathrm{SU}(N)$ adj QCD has a smiliar story but requires to consider **non-invertible** top. lines when $N \geq 3$.

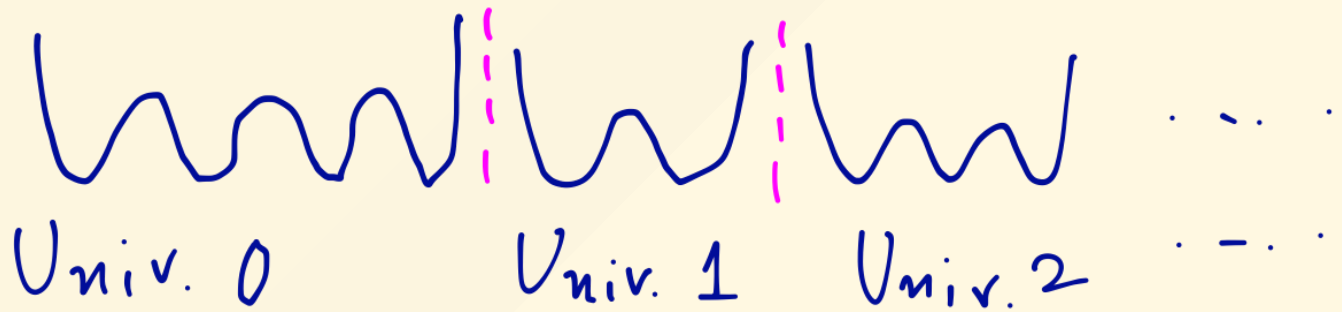
Nonabelian bosonization

- We would like to repeat a similar analysis for massless adjoint QCD with $\mathrm{SU}(N)$ gauge group.
- Dualize N^2-1 Maj. fermions while keeping the $\mathrm{SU}(N)$ symmetry manifest.
  $\Longrightarrow$ **Nonabelian bosonization** [Witten '84]
- $n\,\psi$ (Maj.) $/(-1)^F \Longleftrightarrow \mathrm{Spin}(n)_1$ WZW model [Ji, Shao, Wen '19]
- $\mathrm{PSU}(N) \subset \mathrm{Spin}(N^2-1)$, $\widehat{\mathfrak{su}}(N)_N \subset \widehat{\mathfrak{spin}}(N^2-1)_1$:
  conformal embedding ($c(\widehat{\mathfrak{su}}(N)_N) = c(\widehat{\mathfrak{spin}}(N^2-1)_1)$)
- (Adj. QCD with $g_{\mathrm{YM}} \to \infty$)$/(-1)^F \Longleftrightarrow \mathrm{Spin}(N^2-1)_1/\mathrm{SU}(N)_N$ coset TQFT
- "Gauge back" $(-1)^F$ by gauging $\mathbb{Z}_2^{\mathrm{spinor}}$ with Arf twist.
  [Alvarez-Gaume, Bost, Moore, Nelson, Vafa '87],... [Thorngren '18],[Karch, Tong, Turner '19]
- Adj. QCD with $g_{\mathrm{YM}} \to \infty \Longleftrightarrow \mathrm{Spin}(N^2-1)_1/\mathrm{SU}(N)_N/_{\mathrm{Arf}}\mathbb{Z}_2^{\mathrm{spinor}}$
  Precise version of bosonization prediction by [Kutasov '93],[Boorstein, Kutasov '94], [Kutasov, Schwimmer '95]

$\mathrm{Spin}(N^2-1)_1/\mathrm{SU}(N)_N/_{\mathrm{Arf}}\mathbb{Z}_2^{\mathrm{spinor}}$

- Coset counting $\Longrightarrow$ 2^{N-1} **vacua**. :[Kutasov '93]

  Most of them are not because of SSB
- All the N universes (due to $\mathbb{Z}_N^{(1)}$) are degenerate = **deconfined**.



- Naively IR limit = $g \to \infty$, as g is super-renormalizable. However it is not very clear whether the flow generate other terms in the strongly coupled regime.
- **UV reason** of deconfinement and exponentially many vacua? : **Topological lines**

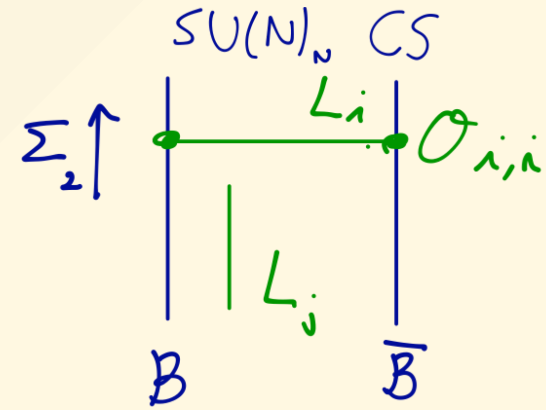
Topological lines in adj QCD

- Topological lines in adj QCD = $\mathfrak{su}(N)$ preserving (commutes with j) top. lines in free fermions:

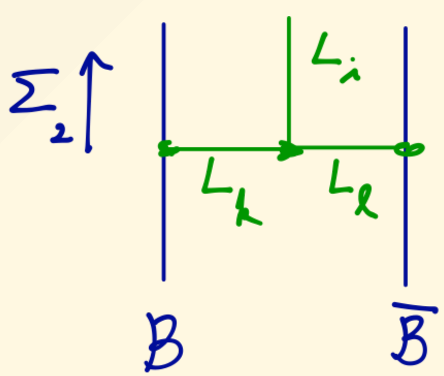
$$\langle L, \mathcal{O}, \cdots \rangle_{\text{adj QCD}} = \int \mathcal{D}A\, e^{-\mathcal{S}_{YM+\text{cntr}}[A]} \left\langle L, \mathcal{O}, \cdots\ e^{\int j^\mu A_\mu} \right\rangle_{\text{free }\psi}$$

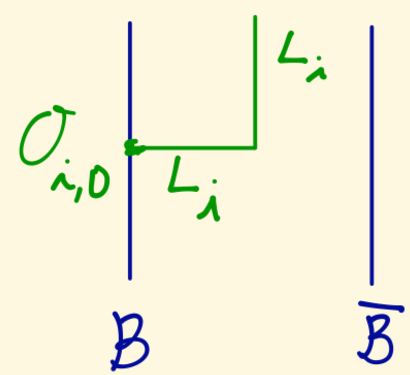
- No classification of top. lines in general 1+1d free theory.
  [Fuchs, Gabrdiel, Runkel, Schweigert '07] for S^1 theory
- $N^2 - 1$ Majorana fermions $\supset \widehat{\mathfrak{spin}}(N^2-1)_1 \supset \widehat{\mathfrak{su}}(N)_N$
- $\mathfrak{su}(N)_N$ **non-diagonal (spin-)RCFT**
- General theory on top. lines in RCFT
  [Fuchs,Runkel,Schweigert '02]...
- Much easier in diagonal RCFT ($SU(N)$ WZW models) $\Longrightarrow$ **Verlinde lines**

Verlinde lines in diagonal RCFT



- Diagonal RCFT = CS theory on a interval.
- $\mathcal{O}_{i,i}$: Line L_i bridging boundaries
- Chiral alg. pres. topological line in RCFT = topological line in CS along Σ_2: **Verlinde line** ($O(2^N)$ of them) (L_i is the topological Wilson line of the auxiliary gauge field in 3d bulk. Not to be confused with the Wilson line W_{i} of the physical gauge field in adj QCD.)
- $L_i \otimes L_j = \bigoplus_k N_{i,j}^k L_k$
- **Defect operator** at the edge of L_i: $\bigoplus_{k,l} N_{i,k}^l V_k \otimes \overline{V}_l$:



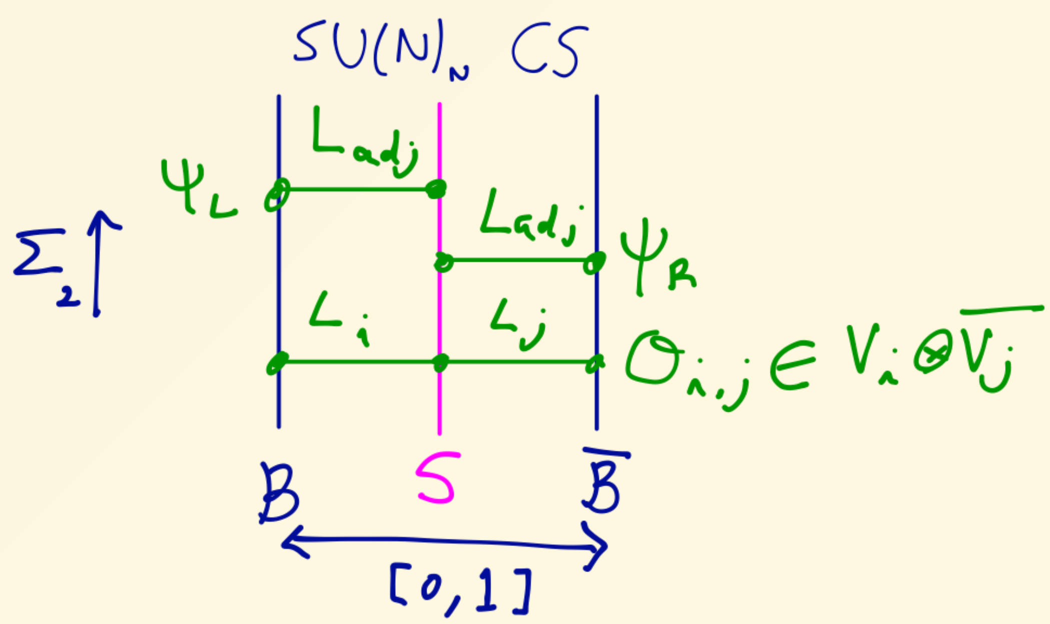


Fermions as $\mathfrak{su}(N)_N$ RCFT

- Topological lines in adj QCD = **$\mathfrak{su}(N)$ preserving top. lines** in fermions
- $\mathfrak{su}(N)_N$ **non-diagonal (spin-)RCFT**

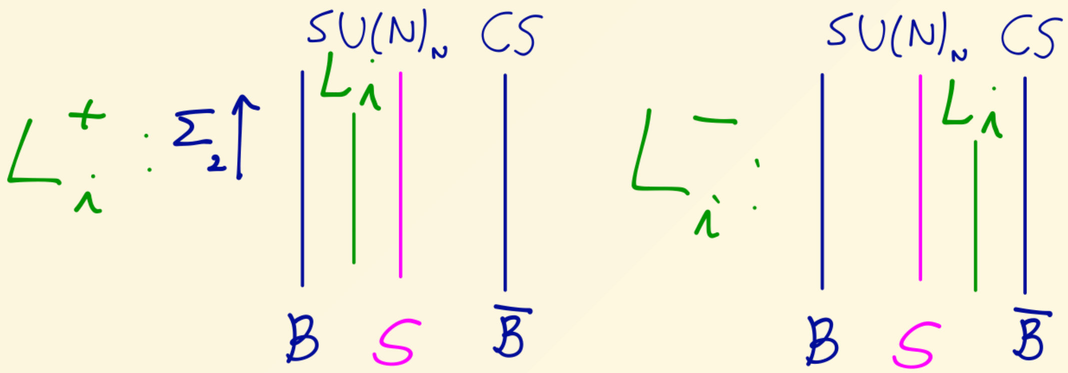
- Non-diagonal RCFT = CS theory on a interval with surface op. insertion:

  [Kapustin Saulina '10], [Fuchs, Schweigert, Valentino '12], [Carqueville, Runkel, Schaumann '17]

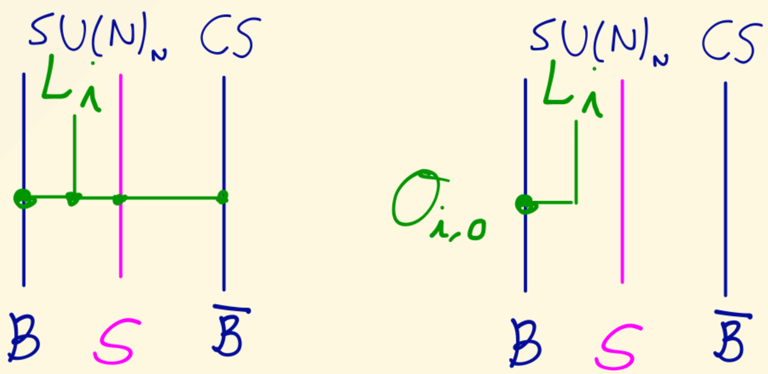


$SU(N)_N$ preserving topological lines in $\psi^{i\bar{j}}$

- Subset of topological lines : $L_i^{\pm}$ defined by



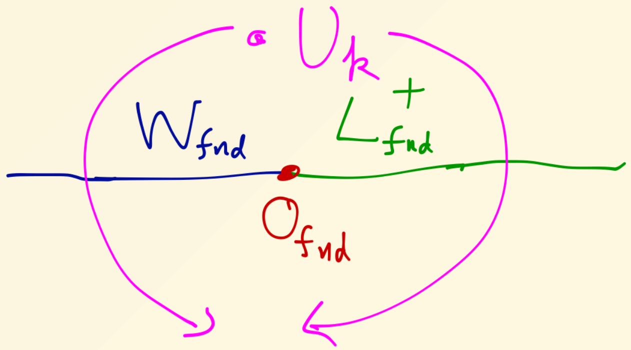
- Defect operator at the edge of L_i^+: $\bigoplus_{k,l,m} N_{l,i}^k Z_{k,m} V_l \otimes \overline{V}_m$



- In particular, $\mathcal{O}_i \in V_i \otimes \overline{V}_0$ always exists.

Topological line - $\mathbb{Z}_N^{(1)}$ mixed anomaly

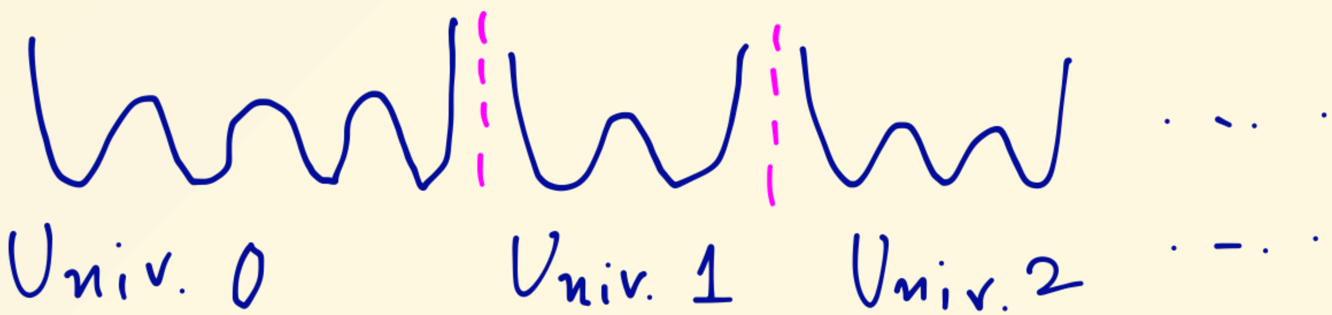
- L^+_{fnd} in free $\psi^{i\bar{j}}$ theory have the defect op. $\mathcal{O}_{\mathrm{fnd},0}$, which is in fnd of $\mathrm{SU}(N)$.
- When gauging $\mathrm{SU}(N)$, $\mathcal{O}_{\mathrm{fnd},0}$ becomes a **line changing operator** between W_{fnd} and L^+_{fnd}:



- $\mathbb{Z}_N^{(1)}$ one-form sym. acts on Wilson line: W_{fnd} by $U_k W_{\mathrm{fnd}} = e^{\frac{2\pi i k}{N}} W_{\mathrm{fnd}} U_k$
- $U_k L^+_{\mathrm{fnd}} = e^{\frac{2\pi i k}{N}} L^+_{\mathrm{fnd}} U_k$: **"(non-invertible) top. line - $\mathbb{Z}_N^{(1)}$ mixed anomaly"**
- $|0\rangle$ and $L_{\mathrm{fnd}}|0\rangle$ are degenerate and in different universes: **Deconfinement**

IR TQFT?

- The IR TQFT fixed point should admit the whole set of $\mathrm{SU}(N)$ preserving top. lines in $\psi^{i\bar{j}}$ theory.
- A candidate is $\mathrm{Spin}(N^2-1)/SU(N)/_{\mathrm{Arf}}\mathbb{Z}_2$ (= CS theory on S^1 with S insertion). **Other candidates**?
- The full structure of the lines (fusion category) is complicated.
- Classifying TQFTs that admit given a set of lines is not easy. ($\cong$ classifying modular invariants of the chiral algebra ($\mathfrak{su}(N)_N$))
- Analysing small N ($\sim 3, 4$, or 5).



Summary and prospect

- 1+1d massless adj. QCD has many ($\mathcal{O}(2^N)$) **topological line operators**, most of which are non-invertible.
- Topological line is an interface between different "universes" due to **"top. line - $\mathbb{Z}_N^{(1)}$ mixed anomaly"** $\implies$ **deconfinement**
- We expect non-invertible lines will be broken by the **double trace quartic** $\mathcal{O}_2$ $\implies$ confinement (of probe in fundamental rep)
- **Higher dimensions?**
  - Math? ("Fusion n-category?") [Douglas, Reutter '18]
  - **Concrete examples** of non-invertible topological operators in higher dimensional non-topological QFT? Free theory? gauging?

Sym. = inv'ble Codim 1 top. surface $\xrightarrow{\text{gen.}}$ p-form Sym. p-grp

Non-inv'ble Codim 1 top. surface (Fusion Cat.) $\longrightarrow$ All top. ops. (??)