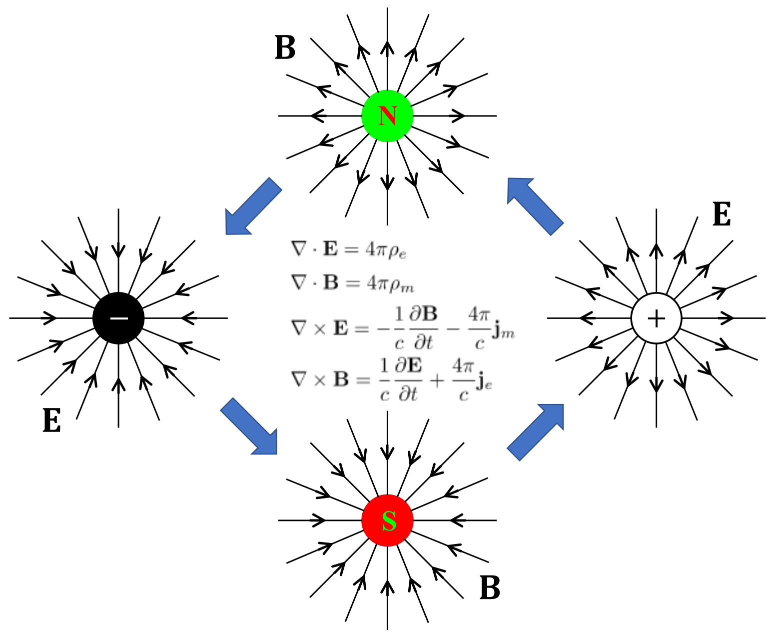
Anomaly of the Electromagnetic Duality of Maxwell Theory



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East Asian String Webinar

May 29, 2020

Collaboration

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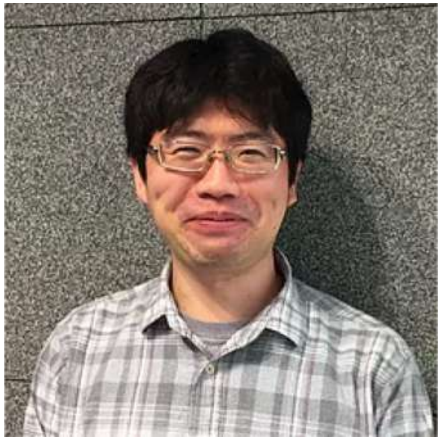

CTH-Tachikawa-Yonekura, **arXiv: 1905.08943**, **PRL 123, 161601 (2019)**
CTH-Tachikawa-Yonekura, **arXiv: 2003.11550**

Outline

- Maxwell theory × EM duality × Anomaly

- Anomalies: self-dual fields vs. chiral fermions

1. 2d
2. 4d
3. 6d

- Summary

Outline

- Maxwell theory $\times$ EM duality $\times$ Anomaly

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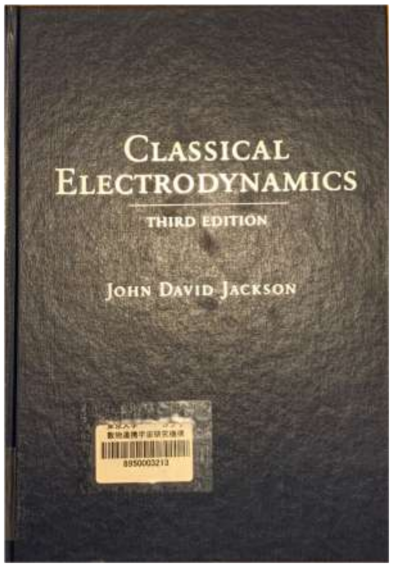
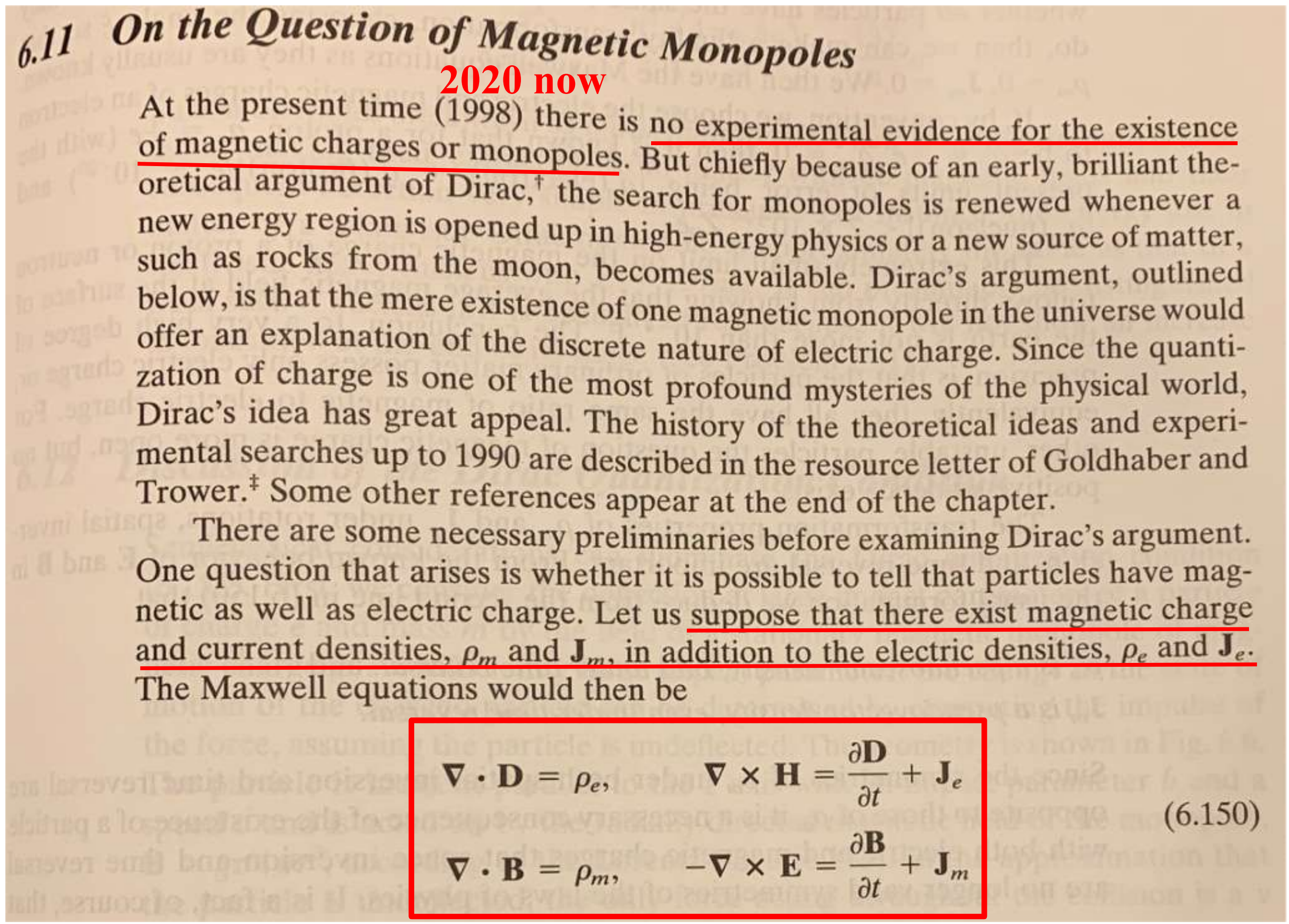
Maxwell theory and EM duality

From Jackson, we know





6.11 On the Question of Magnetic Monopoles

2020 now

At the present time (1998) there is no experimental evidence for the existence of magnetic charges or monopoles. But chiefly because of an early, brilliant theoretical argument of Dirac,† the search for monopoles is renewed whenever a new energy region is opened up in high-energy physics or a new source of matter, such as rocks from the moon, becomes available. Dirac's argument, outlined below, is that the mere existence of one magnetic monopole in the universe would offer an explanation of the discrete nature of electric charge. Since the quantization of charge is one of the most profound mysteries of the physical world, Dirac's idea has great appeal. The history of the theoretical ideas and experimental searches up to 1990 are described in the resource letter of Goldhaber and Trower.‡ Some other references appear at the end of the chapter.

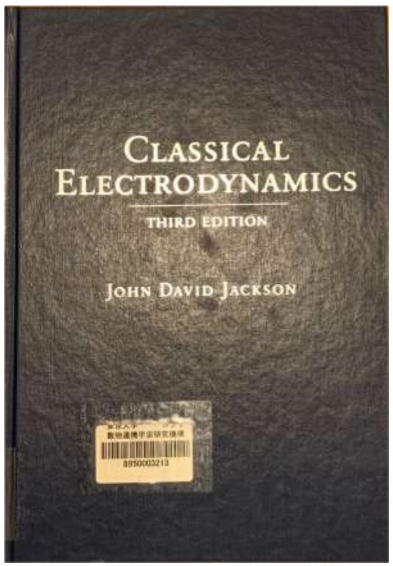
There are some necessary preliminaries before examining Dirac's argument. One question that arises is whether it is possible to tell that particles have magnetic as well as electric charge. Let us suppose that there exist magnetic charge and current densities, ρ_m and $\mathbf{J}_m$, in addition to the electric densities, ρ_e and $\mathbf{J}_e$. The Maxwell equations would then be

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_e, & \nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_e \\ \nabla \cdot \mathbf{B} &= \rho_m, & -\nabla \times \mathbf{E} &= \frac{\partial \mathbf{B}}{\partial t} + \mathbf{J}_m \end{aligned}} \tag{6.150}$$

Maxwell theory and EM duality

and

$$Z_0 = \sqrt{\mu_0/\epsilon_0} \text{ (in SI units) or 1 (in Gaussian units)}$$



The magnetic densities are assumed to satisfy the same form of the continuity equation as the electric densities. It appears from these equations that the existence of magnetic charge and current would have observable electromagnetic consequences. Consider, however, the following *duality transformation**:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}' \cos\xi + Z_0\mathbf{H}' \sin\xi, & Z_0\mathbf{D} &= Z_0\mathbf{D}' \cos\xi + \mathbf{B}' \sin\xi \\ Z_0\mathbf{H} &= -\mathbf{E}' \sin\xi + Z_0\mathbf{H}' \cos\xi, & \mathbf{B} &= -Z_0\mathbf{D}' \sin\xi + \mathbf{B}' \cos\xi \end{aligned} \tag{6.151}$$

For a real (pseudoscalar) angle ξ, such a transformation leaves quadratic forms such as $\mathbf{E} \times \mathbf{H}$, $(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$, and the components of the Maxwell stress tensor $T_{\alpha\beta}$ invariant. If the sources are transformed in the same way,

$$\begin{aligned} Z_0\rho_e &= Z_0\rho_e' \cos\xi + \rho_m' \sin\xi, & Z_0\mathbf{J}_e &= Z_0\mathbf{J}_e' \cos\xi + \mathbf{J}_m' \sin\xi \\ \rho_m &= -Z_0\rho_e' \sin\xi + \rho_m' \cos\xi, & \mathbf{J}_m &= -Z_0\mathbf{J}_e' \sin\xi + \mathbf{J}_m' \cos\xi \end{aligned} \tag{6.152}$$

then it is straightforward algebra to show that the generalized Maxwell equations (6.150) are invariant, that is, the equations for the primed fields $(\mathbf{E}', \mathbf{D}', \mathbf{B}', \mathbf{H}')$ are the same as (6.150) with the primed sources present.

The invariance of the equations of electrodynamics under duality transformations shows that it is a matter of convention to speak of a particle possessing an electric charge, but not magnetic charge. The only meaningful question is

Maxwell theory and EM duality

- Classical Maxwell theory: SO(2) duality symmetry

(It can be extended to a larger symm, e.g. SL(2, $\mathbb{R}$) [Gaillard-Zumino (81])

- Quantum mechanically, we know the electric and magnetic charges must obey the **Dirac** quantization condition

$$qm = 2\pi n\hbar, \quad n \in \mathbb{Z}$$

or more generally, the **Dirac-Zwanziger-Schwinger** quantization condition

$$q_1 m_2 - q_2 m_1 = 2\pi n\hbar, \quad n \in \mathbb{Z}$$

dyons

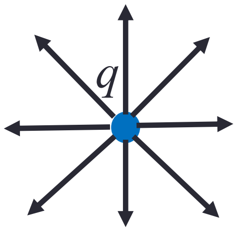
Maxwell theory and EM duality

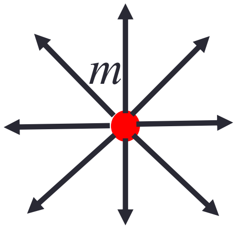
A heuristic derivation (**not** Dirac's original argument in 1931) :

[Jackson, Ch. 6.12]

angular momentum of the EM field:

$$\vec{L}_{\text{em}} = \frac{1}{c^2}\int_{\mathbb{R}^3} \vec{x} \times (\vec{E} \times \vec{H}) d^3x$$





$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} \qquad \vec{H} = \frac{m}{4\pi\mu_0}\frac{\vec{r'}}{r'^3}$$

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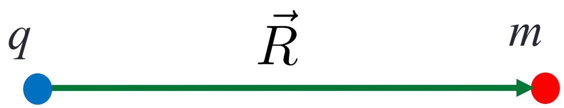

$$\vec{L}_{\text{em}} = \frac{qm}{4\pi}\frac{\vec{R}}{R}$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} \qquad\qquad \vec{H} = \frac{m}{4\pi\mu_0}\frac{\vec{r'}}{r'^3}$$

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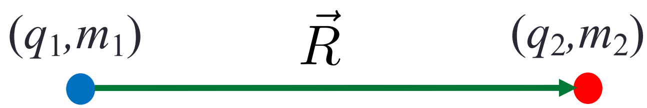

$$\vec{L}_{\text{em}} = \left(\frac{q_1 m_2}{4\pi} - \frac{m_1 q_2}{4\pi}\right)\frac{\vec{R}}{R}$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} \qquad\qquad \vec{H} = \frac{m}{4\pi\mu_0}\frac{\vec{r'}}{r'^3}$$

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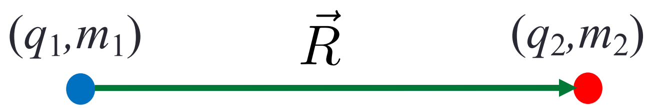

$$\vec{L}_{\text{em}} = \left(\frac{q_1 m_2}{4\pi} - \frac{m_1 q_2}{4\pi}\right)\frac{\vec{R}}{R}$$

But QM tells us $\quad (L_{\text{em}})_{\hat{R}} = \frac{1}{2} n\hbar, \quad n \in \mathbb{Z}$ ✓

Maxwell theory and EM duality

The quantum EM duality must **preserve** the charge quan. cond.

$$q_1 m_2 - q_2 m_1 = \det \begin{pmatrix} q_1 & q_2 \\ m_1 & m_2 \end{pmatrix} = 2\pi n \hbar, \quad n \in \mathbb{Z}$$

and thus is represented by an SL(2, $\mathbb{Z}$) group, namely

$$\begin{pmatrix} q \\ m \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q \\ m \end{pmatrix}$$

$$\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

i.e. $a, b, c, d \in \mathbb{Z}$

$ad - bc = 1$

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and thus is represented by an $\mathrm{SL}(2, \mathbb{Z})$ group, namely

generators

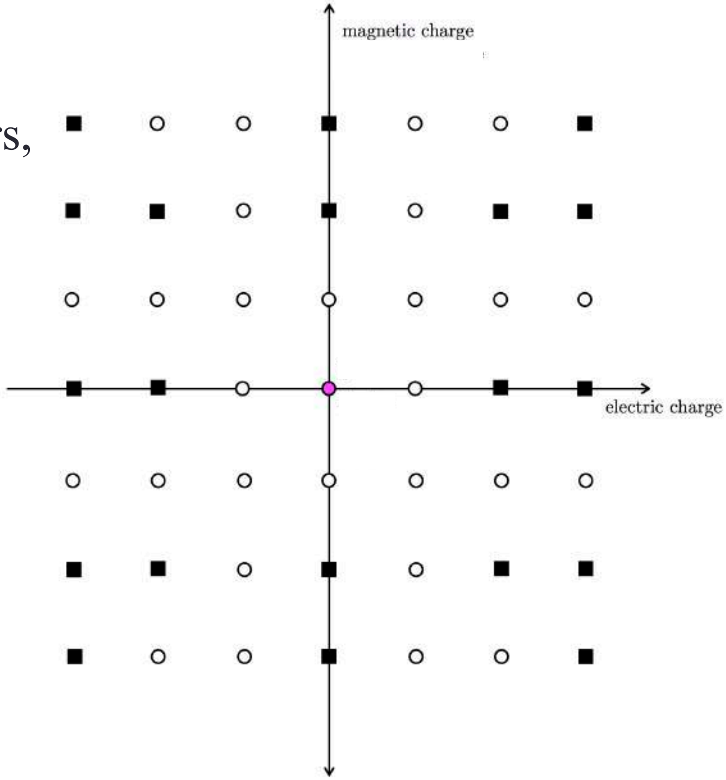
$$S : \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ standard EM duality (S-duality)

$$T : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$ Witten effect (2π shift of top. $\theta-$term)

More on Maxwell theory

In ordinary Maxwell theory, there is no relation btw charges (q, m) and spin/statistics. We could have both neutral boson and fermion as the "origin" of the charge lattice

Open circles: primitive vectors, representing *single-particle states* (SL(2, $\mathbb{Z}$) invariant)



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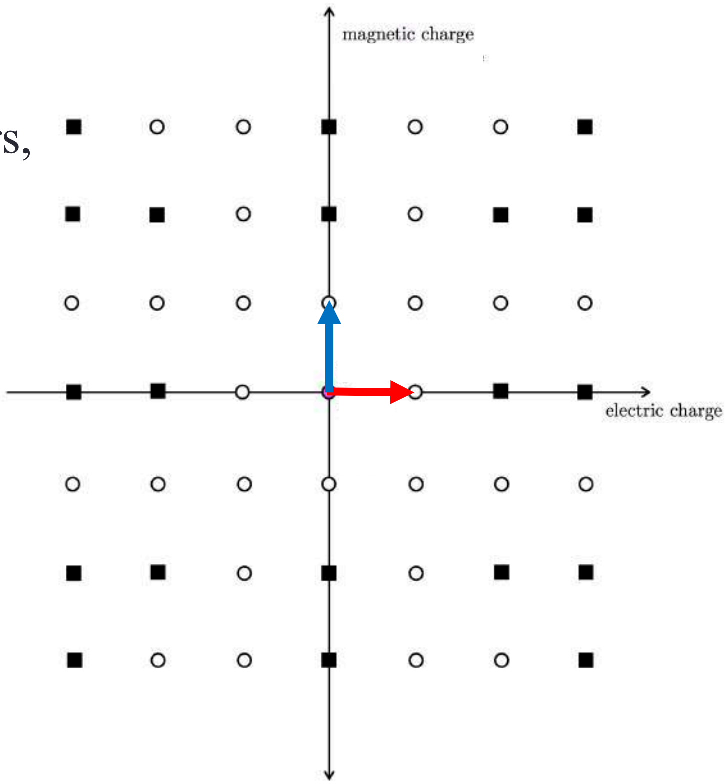
(q=1, m=0) and (q=0, m=1) span the charge lattice



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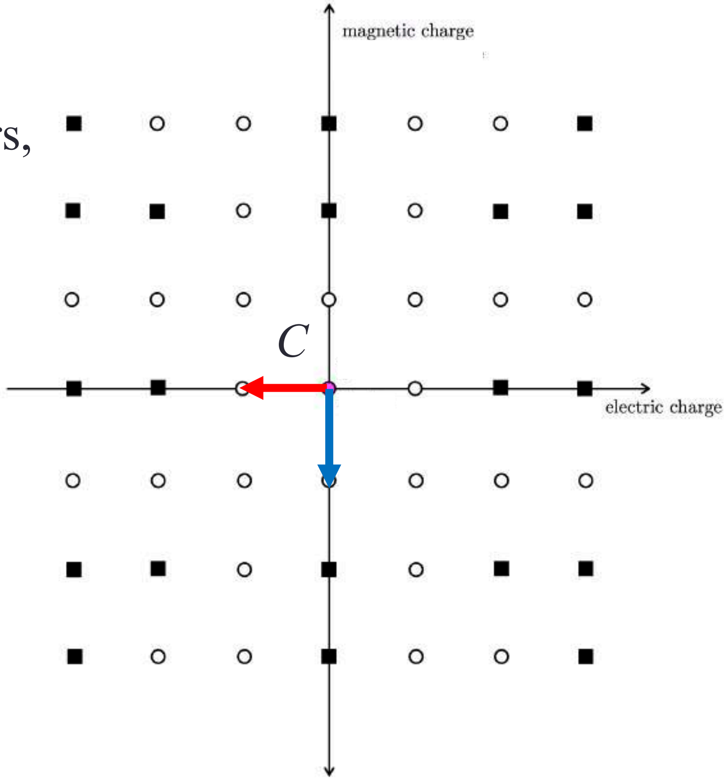
Open circles: primitive vectors, representing *single-particle states* (SL(2, $\mathbb{Z}$) invariant)

(q=-1, m=0) and (q=0, m=-1) span the charge lattice



$$C : \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

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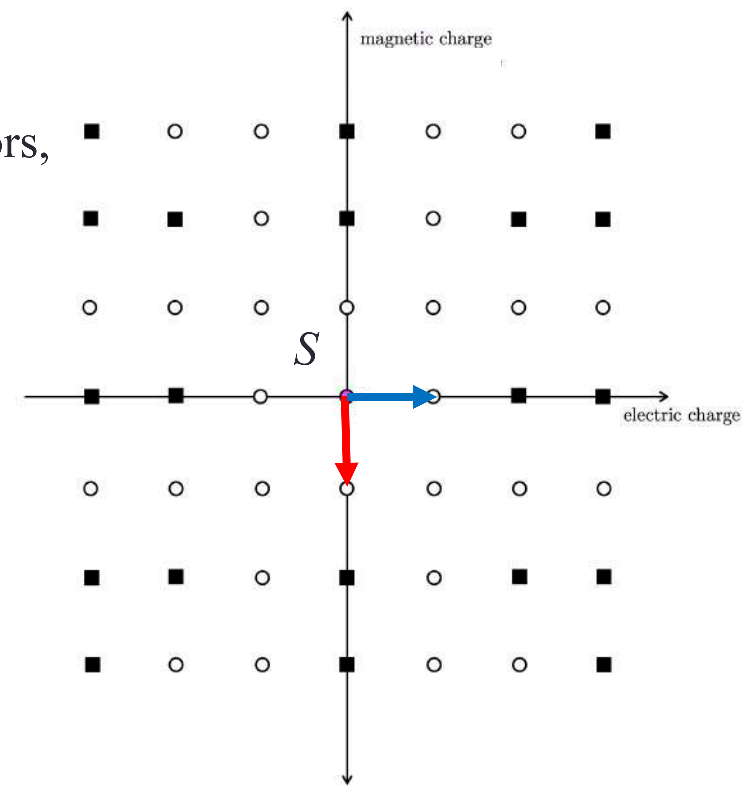
(q=0, m=-1) and (q=1, m=0) span the charge lattice



$$C : \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S : \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

More on Maxwell theory

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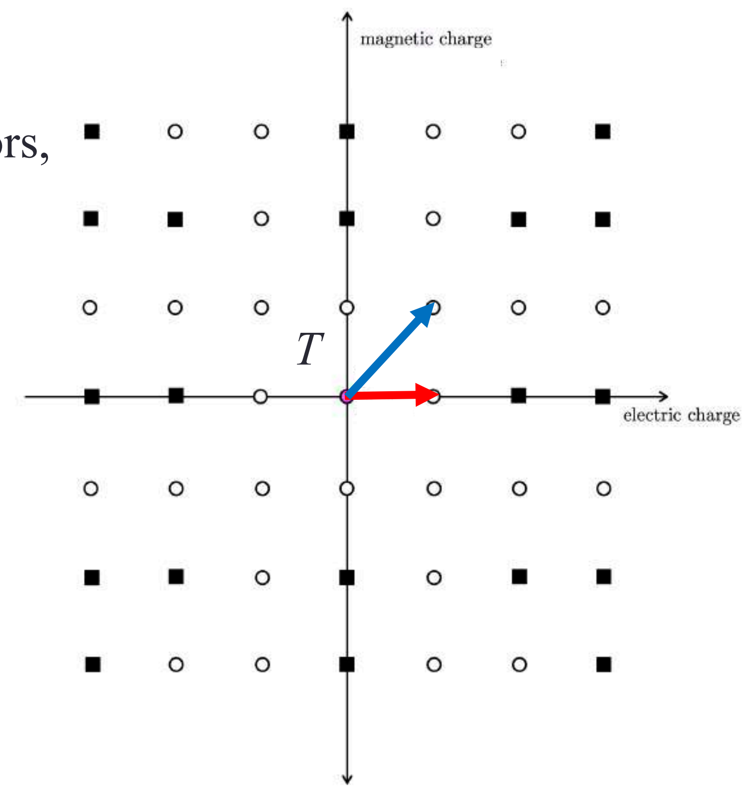

$$C : \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

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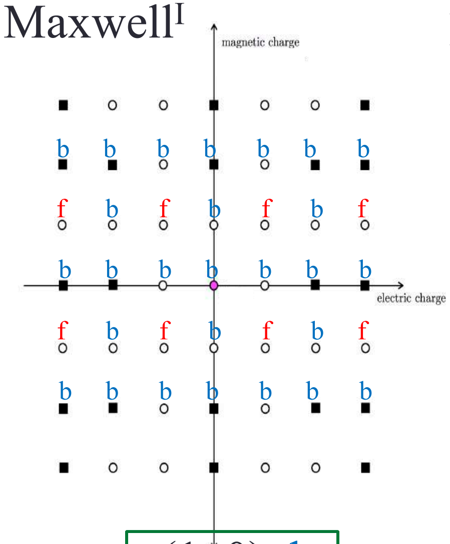
$$T : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

More on Maxwell theory

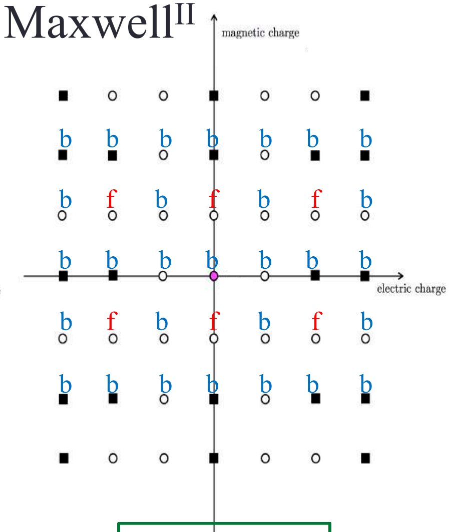
Now let's image a world where neutral particles are bosons. Then the charge lattice can be associated w/ various *"charge-spin relations"*, resulting extra **four** versions of Maxwell theory.

Maxwell$^{\mathrm{I}}$



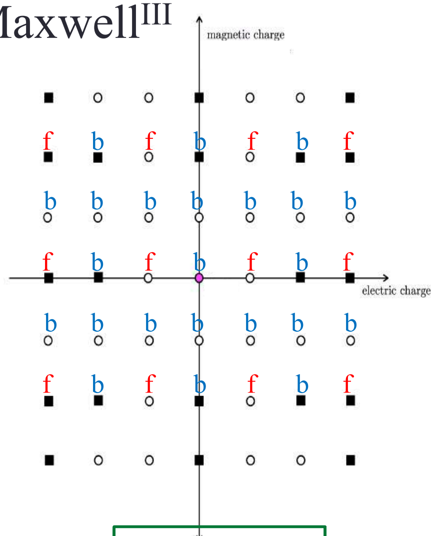
(1, 0): b
(0, 1): b
(1, 1): f

Maxwell$^{\mathrm{II}}$



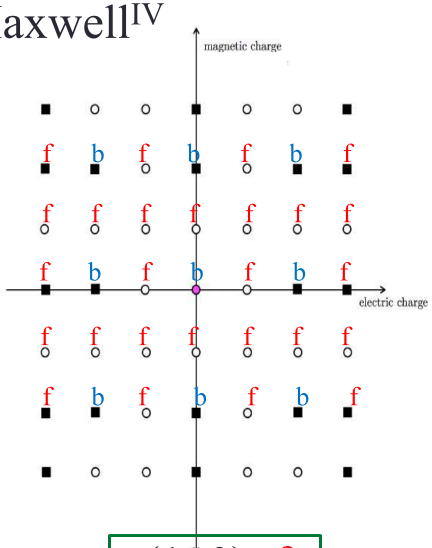
(1, 0): b
(0, 1): f
(1, 1): b

Maxwell$^{\mathrm{III}}$



(1, 0): f
(0, 1): b
(1, 1): b

Maxwell$^{\mathrm{IV}}$

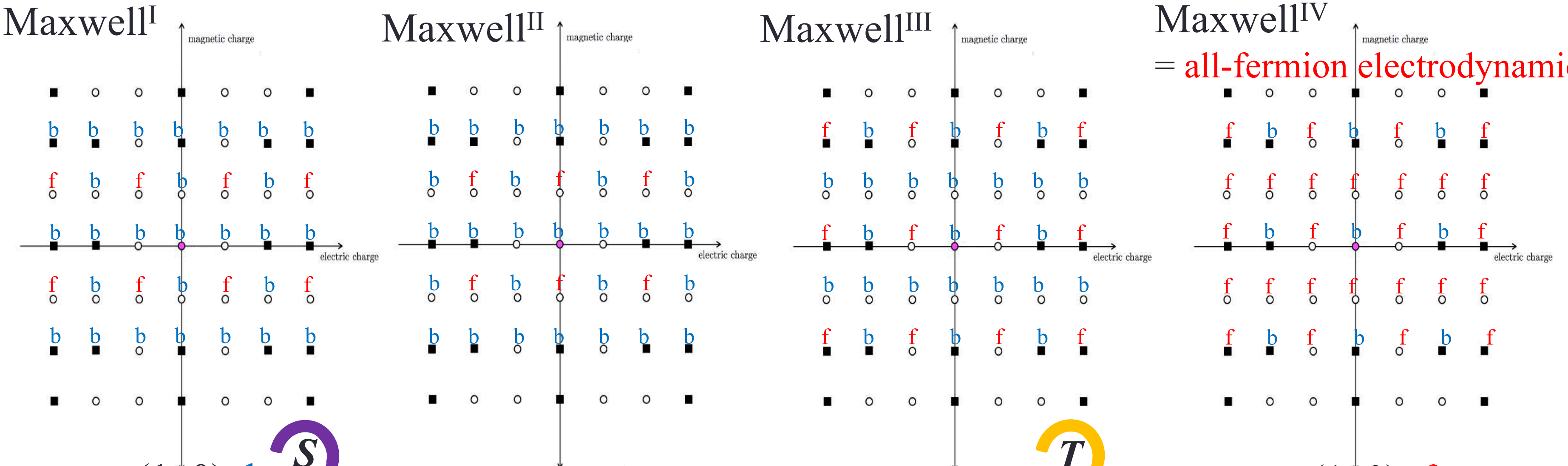


(1, 0): f
(0, 1): f
(1, 1): f

More on Maxwell theory

Now let's image a world where neutral particles are bosons. Then the charge lattice can be associated w/ various *"charge-spin relations"*, resulting extra **four** versions of Maxwell theory.

Maxwell$^{\mathrm{I}}$ | Maxwell$^{\mathrm{II}}$ | Maxwell$^{\mathrm{III}}$ | Maxwell$^{\mathrm{IV}}$ = all-fermion electrodynamics



Maxwell$^{\mathrm{I}}$		Maxwell$^{\mathrm{II}}$		Maxwell$^{\mathrm{III}}$	Maxwell$^{\mathrm{IV}}$
(1, 0): b	$\overset{T}{\longleftrightarrow}$	(1, 0): b	$\overset{S}{\longleftrightarrow}$	(1, 0): f	(1, 0): f
(0, 1): b		(0, 1): f		(0, 1): b	(0, 1): f
(1, 1): f		(1, 1): b		(1, 1): b	(1, 1): f

$\underbrace{\text{Maxwell}^{\mathrm{I}},\ \text{Maxwell}^{\mathrm{II}},\ \text{Maxwell}^{\mathrm{III}}}_{\text{permuted by SL(2, }\mathbb{Z})}$ $\underbrace{\text{Maxwell}^{\mathrm{IV}}}_{\text{SL(2, }\mathbb{Z})\text{ invariant}}$

All-fermion electrodynamics

- It seems natural to have these modified Maxwell theories
- However, Maxwell$^{\mathrm{IV}}$= all-ferm ED is anomalous (while the other three are not), in the sense that it cannot exist in purely 4d if microscopic DOF are only bosons [Wang-Potter-Senthil (13); Kravec-McGreevy-Swingle (14); Thorngren (14); Wang-Wen-Witten (18)]

Such a theory (in the absence of neutral fermions)

- does **not** have a bosonic regulator (e.g. 4d U(1) lattice gauge theory)
- does **not** have a well-defined part. func. on some spacetime (e.g.$\mathbb{CP}^2$)
- is the IR theory of some anomalous UV theory (e.g. ferm of isospin $4r+3/2$ w/ a refined SU(2) anomaly)
- must live on the boundary of a 5d bulk (w/ part. func. $(-1)^{\int_{M_5} w_2 w_3}$)

Symmetries of Maxwell theory

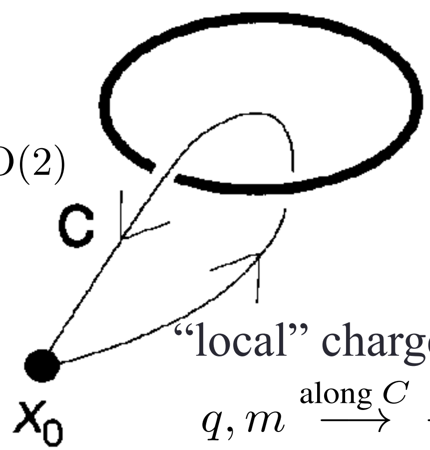
Symmetries of Maxwell theory might also be anomalous, and we'd like to know which symm is anomaly-free and thus can be gauged.

- One example is to consider Maxwell theory w/ extra **dynamical** gauge fields, e.g. *Alice electrodynamics* [Schwarz (82)]

a "Alice string/loop" with the "Cheshire charge" [Wilczek et al (90)]

Alice EM

$\mathrm{U}(1) \rtimes \mathbb{Z}_2 = \mathrm{O}(2)$



"local" charge conjugation

$$q, m \xrightarrow{\text{along } C} -q, -m$$

Fig from [Bucher-Lo-Preskill (92)]

Symmetries of Maxwell theory

Symmetries of Maxwell theory might also be anomalous, and we'd like to know which symm is anomaly-free and thus can be gauged.

- Another example is to consider the *Janus configuration* [Bak et al. (92); Giaotto-Witten (08)] where the spacetime has a duality twist

$$\mathbf{E}(x+L,y,z)=\mathbf{B}(x,y,z)$$
$$\mathbf{B}(x+L,y,z)=-\mathbf{E}(x,y,z)$$

[Ganor et al (08 ,10, 12, 14)]

Janus from wiki: God of beginnings, gates, transitions, time, duality, doorways, passages, and ending

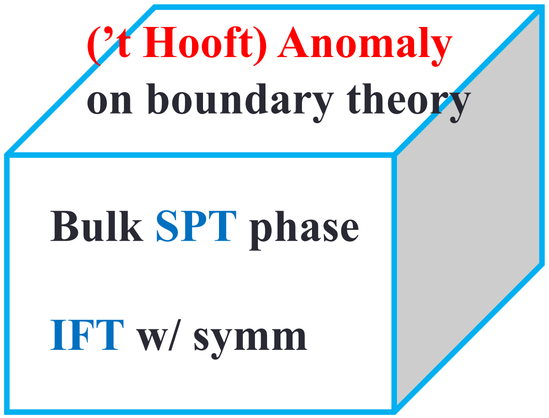
Anomaly of symmetries

- More precisely, we want to know the *'t Hooft anomaly* of a given symm, which obstructs the gauging of the symm.

- Such an anomaly manifests in a *controlled* manner and can be understood by anomaly inflow argument.

➢ *Modern view*: An n-dim anomalous theory is (most naturally) realized as a **boundary mode** of a (n+1)-dim symmetry protected topological (SPT) phase or invertible field theory (IFT) in (n+1)d

Hilbert space (on any closed manifold) is 1-dim



Anomaly of symmetries

☆☆☆ Fact :

$\boxed{\textbf{anomaly} \text{ in } n \text{ dim} \Leftrightarrow \text{part. func. of } (n{+}1)\text{d bulk } \textbf{IFT} \text{ on closed manifolds}}$

Outline

- Maxwell theory × EM duality × Anomaly

- Anomalies: self-dual fields vs. chiral fermions

1. (1+1)d
2. (3+1)d
3. (5+1)d

- Summary

Before discussing 4d Maxwell theory and its corresp 5d bulk theory, let’s look at a simpler but related example:

2d chiral boson and 3d Chern-Simons theory

Outline

- Maxwell theory × EM duality × Anomaly

- Anomalies: self-dual fields vs. chiral fermions

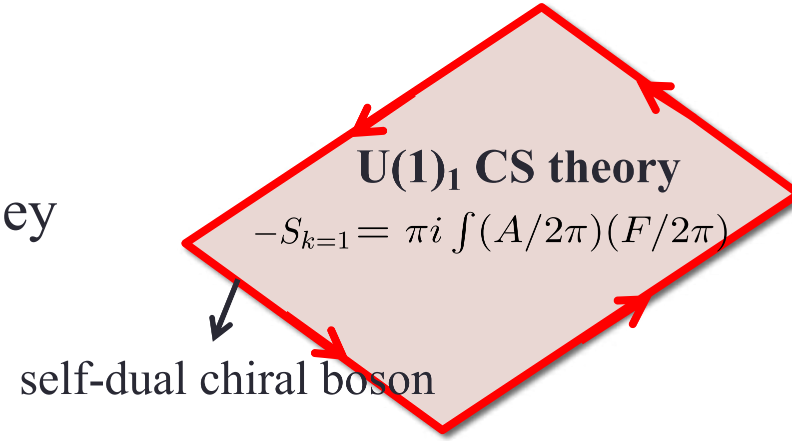
1. 2d
2. 4d
3. 6d

- Conclusion

Anomaly of 2d chiral boson in terms of 3d U(1) Chern-Simons theory

Boundary theory of 3d U(1)$_1$ CS theoey

= 2d (self-dual) chiral boson



The (grav) anomaly of the chiral boson is characterized by part. func. of the U(1)$_1$ CS a closed spin 3-manifold M_3:

$$Z_{\mathrm{U(1)CS}}(M_3) = \left[\int [DA]_{\text{top.trivial}} e^{\pi i \int (A/2\pi)(F/2\pi)} \right] \times \left[\sum_{A:\text{flat}} e^{\pi i \int (A/2\pi)(F/2\pi)} \right]$$

one-loop determinant　　　　classical saddle points

Anomaly of 2d chiral boson in terms of 3d U(1) Chern-Simons theory

$$Z_{\mathrm{U(1)CS}}(M_3) = \left[\int [DA]_{\text{top.trivial}} e^{\pi i \int (A/2\pi)(F/2\pi)}\right] \times \left[\sum_{A:\text{flat}} e^{\pi i \int (A/2\pi)(F/2\pi)}\right]$$

The phase of Z is given by [Hsieh-Tachikawa-Yonekura (19, 20)]

[Witten (89); Monnier-Moore (18)]

contributed by only top. nontrivial flat A

$$\frac{1}{2\pi} \operatorname{Arg} Z_{\mathrm{U(1)CS}}(M_3) = -\frac{1}{8}\underbrace{\eta_{\text{signature}}}_{\substack{\text{eta invariant of the signature op } (\star d + d\star) \\ = \frac{1}{2}\left(\sum \operatorname{sign}(\lambda)\right)_{\text{reg}}}} + \underbrace{\operatorname{Arf}(q)}_{\substack{\text{Arf invariant of quadratic refinement of the torsion} \\ \text{pairing on } H^2(M_3, \mathbb{Z})}}$$

Anomaly of 2d chiral boson in terms of 3d U(1) Chern-Simons theory

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[Atiyah-Patodi-Singer (75); Brumfiel-Morgan (73)]

$$= -\tfrac{1}{8}\left(\int_{X_4;\ \partial X_4 = M_3} \underset{\substack{\| \\ p_1/3}}{L_1} - \sigma(X_4)\right) + \left(-\tfrac{1}{8}\sigma(X_4)\right)$$

$$\left(p_1 = -\tfrac{1}{8\pi^2}\mathrm{tr}R^2\right)$$

Anomaly of 2d chiral boson in terms of 3d U(1) Chern-Simons theory

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$$= -\tfrac{1}{8}\left(\int_{X_4;\ \partial X_4 = M_3} \underset{\substack{\| \\ p_1/3}}{L_1} - \sigma(X_4)\right) + \left(-\tfrac{1}{8}\sigma(X_4)\right)$$

$$\left(p_1 = -\tfrac{1}{8\pi^2}\mathrm{tr} R^2\right)$$

$$= \int_{X_4;\ \partial X_4 = M_3} \underset{\substack{\| \\ -p_1/24}}{\hat{A}_1}$$

Anomaly of 2d chiral boson in terms of 3d U(1) Chern-Simons theory

$$Z_{\mathrm{U(1)CS}}(M_3) = \left[\int [DA]_{\text{top.trivial}} e^{\pi i \int (A/2\pi)(F/2\pi)}\right] \times \left[\sum_{A:\text{flat}} e^{\pi i \int (A/2\pi)(F/2\pi)}\right]$$

The phase of Z is given by [Hsieh-Tachikawa-Yonekura (19, 20)]

$$\frac{1}{2\pi}\operatorname{Arg} Z_{\mathrm{U(1)CS}}(M_3) = -\tfrac{1}{8}\eta_{\text{signature}} + \operatorname{Arf}(q)$$

$$= -\tfrac{1}{8}\left(\int_{X_4;\ \partial X_4 = M_3} L_1 - \sigma(X_4)\right) + \left(-\tfrac{1}{8}\sigma(X_4)\right)$$

$$= \int_{X_4;\ \partial X_4 = M_3} \hat{A}_1 \ = \eta_{\text{Dirac}} = \frac{1}{2\pi}\operatorname{Arg} Z_{\text{ferm}}(M_3)$$

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- This means 2d chiral boson (0-form gauge field) and 2d chiral fermion have the same anomaly (perturb grav anomaly); it is as expected since the two theories are actually *identical* in 2d (traditional sense of "**bosonization**").

Anomaly of 2d chiral boson in terms of 3d U(1) Chern-Simons theory

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- Nevertheless, the analysis here can be generalized to higher dimensions, where one can still relate the anomalies of p-form gauge fields to those of fermions, even though they are different theories!

Anomaly of 2d chiral boson in terms of 3d U(1) Chern-Simons theory

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- Nevertheless, the analysis here can be generalized to higher dimensions, where one can still relate the anomalies of p-form gauge fields to those of fermions, even though they are different theories!
  (A formal treatment of generic p-form gauge theories is by using *differential cohomology* [Cheeger-Simons (85); Hopkins-Singer (02); Córdova et al. (19); Hsieh-Tachikawa-Yonekura (20); etc.])

Outline

- Maxwell theory × EM duality × Anomaly

- Anomalies: self-dual fields vs. chiral fermions

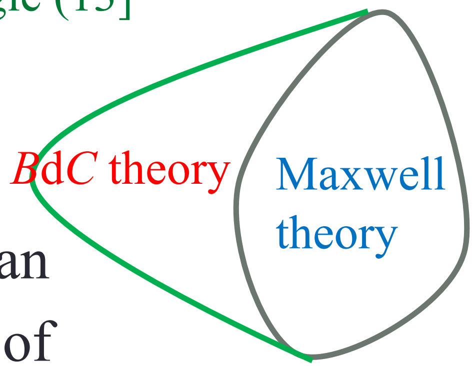
1. 2d
2. 4d
3. 6d

- Conclusion

Anomaly of Maxwell theory in terms of 5d top *BdC* theory

The 5d bulk theory corresp to Maxwell theory is a TQFT w/ two 2-form gauge fields: [Verlinde (95); Kravec-McGreevy-Swingle (13]

$$\pi i \int [(B/2\pi)d(C/2\pi) - (C/2\pi)d(B/2\pi)]$$



- At the level of differential form, this 5d theory has an SL(2, $\mathbb{Z}$) symm on (B, C), corresp to duality symm of the Maxwell theory.

- However, to make the action sensible when top nontrivial B, C are considered, we require the action to be a quadratic refinement of (diff-cohomology) paring of B, C, i.e. $2\pi i q(B, C)$, which might **break** the SL(2, $\mathbb{Z}$) symm in general.

Anomaly of Maxwell theory in terms of 5d top BdC theory

- The choice of quadratic refinement $q(B, C)$ is not unique, and in general we have

$$q_{(X,Y)}(B,C) = \int \frac{B}{2\pi}\frac{dC}{2\pi} + \int \frac{dB}{2\pi}\frac{X}{2\pi} + \int \frac{Y}{2\pi}\frac{dC}{2\pi}$$

- For $X=Y=0$, $q_{(0,0)}(B, C)$ is SL(2, $\mathbb{Z}$) invariant only on **spin**-manifolds. Its boundary theory is the ordinary Maxwell theory. [Witten (98); Gomi (04)]

- $q_{(X,Y)}(B, C)$ can be made SL(2, $\mathbb{Z}$) invariant on any 5d manifold if we take $X/2\pi = Y/2\pi = w_2$ (2nd Stiefel-Whitney class of M_5). In this case the corresp boundary theory is all-fermion ED.

Anomaly of Maxwell theory in terms of 5d top *BdC* theory

- The phase of the part func is [Hsieh-Tachikawa-Yonekura (19, 20)]

$$\frac{1}{2\pi}\,\mathrm{Arg}\,Z_{\mathrm{BdC}}(M_5) = -\frac{1}{4}\eta_{\mathrm{signature}} + \mathrm{Arf}(q)$$

- Before coupling this system to any (duality) symm background, let's see what the phase can tell us. In this situation,

$\beta : H^2(M_5, \mathbb{R}/\mathbb{Z}) \to H^3(M_5, \mathbb{Z})$

$$\eta_{\mathrm{signature}} = 0 \quad \text{and} \quad \mathrm{Arf}(q_{(X,Y)}) = \int (X/2\pi)\beta(Y/2\pi)$$

$X, Y\,(\in H^2(M_5, U(1)))$ are flat

- Taking $X/2\pi = Y/2\pi = w_2$ (i.e. all-fermion ED), we get

$$Z_{\mathrm{BdC}}(M_5) = \underbrace{|Z_{\mathrm{BdC}}(M_5)|}_{=1}\,e^{\underbrace{\pi i \int w_2 w_3}_{=\pm 1 \text{ (a } Z_2 \text{ grav anomaly)}}}$$

Anomaly of Maxwell theory in terms of 5d top BdC theory

- Now we consider the case when a nontrivial SL(2, $\mathbb{Z}$) background is present. Note that there are multiple choices of the symm structure, depending on the value of charge-conjugation square C^2 ($=S^4$)

Anomaly of Maxwell theory in terms of 5d top *BdC* theory

- Now we consider the case when a nontrivial SL(2, $\mathbb{Z}$) background is present. Note that there are multiple choices of the symm structure, depending on the value of charge-conjugation square C^2 ($=S^4$)

- Let's focus on the case $C^2 = (-1)^F$, the ferm number parity. The corresp symm structure is

$$\text{spin-}\,\mathrm{Mp}(2,\mathbb{Z}) := (\text{spin} \times \mathrm{Mp}(2,\mathbb{Z}))/\mathbb{Z}_2$$

where the *metaplectic group* Mp(2, $\mathbb{Z}$) is the double cover of SL(2, $\mathbb{Z}$)

$$Mp(2,\mathbb{Z}) := \langle S, T \mid S^2 = (T^{-1}S)^3,\ S^8 = 1\rangle$$

Anomaly of Maxwell theory in terms of 5d top *BdC* theory

- Canonical examples are 5d lens spaces $S^5/\mathbb{Z}_k$, k = 2, 3, 4, 6, where going around the generator of $\pi_1(S^5/\mathbb{Z}_k) = \mathbb{Z}_k$ comes with the duality action by an element of order k in SL(2, $\mathbb{Z}$)

Anomaly of Maxwell theory in terms of 5d top *BdC* theory

- Canonical examples are 5d lens spaces $S^5/\mathbb{Z}_k$, k = 2, 3, 4, 6, where going around the generator of $\pi_1(S^5/\mathbb{Z}_k) = \mathbb{Z}_k$ comes with the duality action by an element of order k in SL(2, $\mathbb{Z}$)

- Observation: on (some) $S^5/\mathbb{Z}_k$'s, we have the following fact

$$\frac{1}{2\pi}\,\mathrm{Arg}\, Z_{\mathrm{BdC}}(M_5) = -\frac{1}{4}\eta_{\mathrm{signature}} + \mathrm{Arf}(q) = 56\eta_{\mathrm{fermion}} \quad \mathrm{mod}\ 1$$

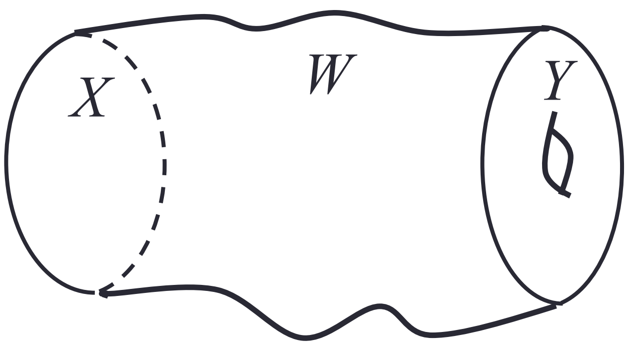
	$S^5/\mathbb{Z}_2$	$S^5/\mathbb{Z}_3$	$S^5/\mathbb{Z}_4$	$S^5/\mathbb{Z}_6$
$\eta_{\mathrm{signature}}$	0	$-\frac{1}{9}$	$-\frac{1}{2}$	$-\frac{14}{9}$
$H^3(M_5,(\mathbb{Z}^2)_\rho)$	$(\mathbb{Z}_2)^2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_1$
$\mathrm{Arf}(q)$	$+\frac{1}{2}$	$-\frac{1}{4}$	$+\frac{1}{8}$	0
$\frac{1}{2\pi}\,\mathrm{Arg}\, Z$	$+\frac{1}{2}$	$-\frac{2}{9}$	$+\frac{1}{4}$	$+\frac{7}{18}$
η_{fermion}	$-\frac{1}{16}$	$-\frac{1}{9}$	$-\frac{5}{32}$	$-\frac{35}{144}$

Anomaly of Maxwell theory in terms of 5d top *BdC* theory

- Well, it might just be a coincidence

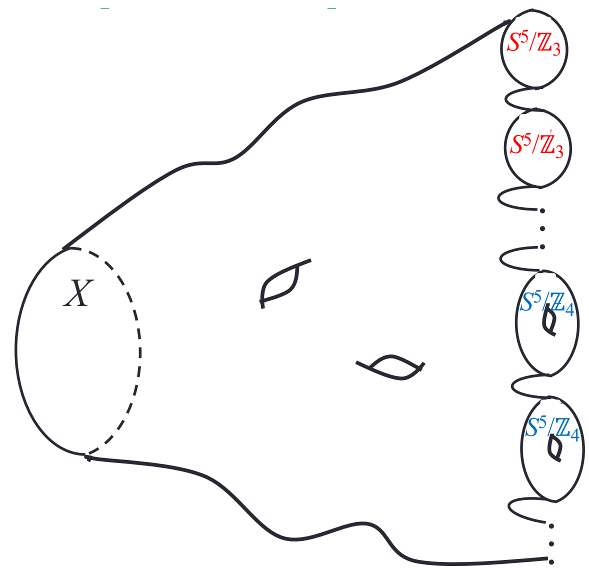
But we would really like to know whether such an identity holds for any 5-manifold w/ a spin-Mp(2, $\mathbb{Z}$) structure

➢The answer is YES, once we know these $S^5/\mathbb{Z}_k$*'s* are, under (co)*bordism*, generators of any 5d spin-Mp(2, $\mathbb{Z}$) manifold!



bordism: $[X] = [Y]$ if $\partial W = X \sqcup Y$

Part. func. Z of an IFT, e.g. Z_{BdC} and $\mathrm{e}(-2\pi i\eta_{\mathrm{ferm}})$, is a (co)bordism invariant



$$Z(X) = Z(S^5/\mathbb{Z}_3)^m Z(S^5/\mathbb{Z}_4)^n \cdots$$

Anomaly of Maxwell theory in terms of 5d top *BdC* theory

- Well, it might just be a coincidence

But we would really like to know whether such an identity holds for any 5-manifold w/ a spin-Mp(2, ℤ) structure

➢The answer is YES, once we know these $S^5/\mathbb{Z}_k$*'s* are, under (co)*bordism*, generators of any 5d spin-Mp(2, ℤ) manifold!

Math fact: all spin-Mp(2, ℤ) 5-mflds are classified by an abelian group

$$\Omega_5^{\text{spin-Mp}(2,\mathbb{Z})} = \mathbb{Z}_9 \oplus \mathbb{Z}_{32} \oplus \mathbb{Z}_2$$

generators: $S^5/\mathbb{Z}_3$ $S^5/\mathbb{Z}_4$ $[(S^5/\mathbb{Z}_4)' + 9(S^5/\mathbb{Z}_4)]$

Anomaly of Maxwell theory in terms of 5d top *BdC* theory

Therefore,

$$\boxed{\frac{1}{2\pi}\,\mathrm{Arg}\,Z_{\mathrm{BdC}}(M_5) = 56 \times \frac{1}{2\pi}\,\mathrm{Arg}\,Z_{\mathrm{fermion}}(M_5) \mod 1}$$

is true on any 5-manifold w/ a spin-Mp(2, $\mathbb{Z}$) structure

Namely, the anomaly of EM duality of the Maxwell theory is 56 times that of a 4d chiral fermion

It is still abstract (and somehow mysterious), however. Where does this number 56 come from?

- ➢ We provide an answer using the property of some 6d SCFT, known as the *E-string theory* [Ganor-Hanany (96); Seiberg-Witten (96)]

Outline

- Maxwell theory × EM duality × Anomaly

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Anomaly of Maxwell theory in terms of 6d SCFT

The E-string theory has two branches of vacua, the ***tensor branch*** and the ***Higgs branch***

- On the ***Higgs branch*** the E_8 symm is Higgsed to E_7, which acts on 28 fermions via its 56 dim fundamental rep
- When one moves to the ***tensor branch***, the E_8 symm is restored and a self-dual tensor field appears

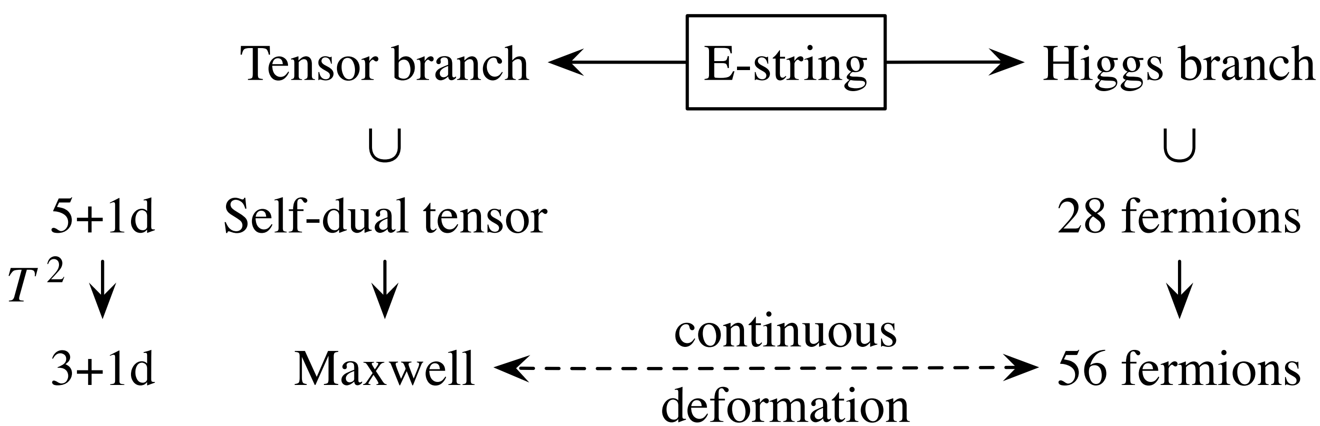
$$\begin{array}{cccc} & \text{Tensor branch} & \longleftarrow \boxed{\text{E-string}} \longrightarrow & \text{Higgs branch} \\ & \cup & & \cup \\ \text{5+1d} & \text{Self-dual tensor} & & \text{28 fermions} \end{array}$$

$$\boxed{\begin{aligned} \frac{1}{2\pi}\,\mathrm{Arg}\,Z_{\mathrm{CdC}}(M_7) &= -\frac{1}{8}\eta_{\mathrm{signature}} + \mathrm{Arf}(q) = 28\eta_{\mathrm{Dirac}} \\ &= 28 \times \frac{1}{2\pi}\,\mathrm{Arg}\,Z_{\mathrm{ferm}}(M_7) \end{aligned}}$$

[Hsieh-Tachikawa-Yonekura (20)]

Anomaly of Maxwell theory in terms of 6d SCFT

By compactifying this system on T^2, one finds that one Maxwell field is continuously connected to 56 chiral fermions, showing that they should have the same anomaly. The EM duality is formulated as the SL(2, $\mathbb{Z}$) acting on this T^2



- So far I have discussed the anomaly of SL(2, $\mathbb{Z}$) duality of one version of Maxwell theory (i.e. all-fermion electrodynamics).

- It is interesting to find out anomalies of any subgroups of SL(2, $\mathbb{Z}$) in other versions of Maxwell theory.

Here I listed the result w/o details:

symm str		None	charge conj. $\mathbb{Z}_2^C$	S-duality $\mathbb{Z}_4^S$	ST-symm. $\mathbb{Z}_3^{ST}$	full EM duality $SL(2,\mathbb{Z})$
spin$\times G$	Maxwell^o	0	0	0	$\mathbb{Z}_9$	$\mathbb{Z}_9$
so (G=none) & spin-G^f	Maxwell^I	0	$\mathbb{Z}_2$	$\mathbb{Z}_4$	-	-
	Maxwell^{II}	0	$\mathbb{Z}_2$	-	-	-
	Maxwell^{III}	0	$\mathbb{Z}_2$	-	-	-
	Maxwell^{IV}	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_9$	$\mathbb{Z}_{36}$
spin$\times G$	Weyl, $G \times \mathbb{Z}_2^f$	0	0	$\mathbb{Z}_4$	$\mathbb{Z}_9$	$\mathbb{Z}_{36}$
spin-G^f	Weyl, G^f	0	$\mathbb{Z}_{16}$	$\mathbb{Z}_{32}$	$\mathbb{Z}_9$	$\mathbb{Z}_{288}$

"-": no symm "0": no anomaly "$\mathbb{Z}_k$": mod-k anomaly

Anomalies of 4d Weyl ferm under the same symm was determined in [Hsieh (18)], and we have the following result in general (on either spin$\times G$ or spin-G^f mflds)

$$\boxed{\frac{1}{2\pi}\,\mathrm{Arg}\, Z_{\mathrm{BdC}}(M_5) = 56 \times \frac{1}{2\pi}\,\mathrm{Arg}\, Z_{\mathrm{fermion}}(M_5) \mod 1}$$

Outline

- Maxwell theory × EM duality × Anomaly

- Anomalies: self-dual fields vs. chiral fermions

1. (1+1)d
2. (3+1)d
3. (5+1)d

- Summary

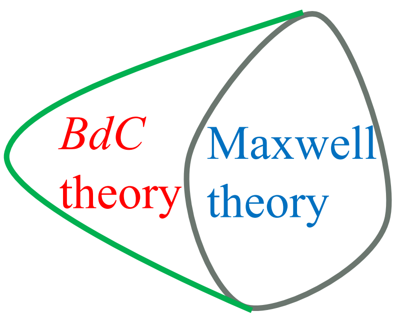
Summary

We consider various versions of 4d Maxwell theory and their duality symmetries, and compute the corresp 't Hooft anomalies

- In particular, we found

$\boxed{\textit{Anomaly of duality symm of Maxwell} = \mathbf{56}\ \textit{times that of a chiral fermion}}$

- The interpretation is twofold: one is by the **5d bulk SPT** (top. *BdC* theory) phase characterizing the anomaly, and the other is by the properties of a **6d SCFT** (E-string theory)



- Our result reproduces, as a special case, the known anomaly of the **all-fermion electrodynamics** discovered in the last few years

Thank you for your attention!