Nonperturbative Mellin Amplitudes

EPFL

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based on: • arXiv: 1912.11100 with Joao Silva and Alexander Zhiboedov
• in progress with Dean Carmi, " "

East Asian Strings Webinar

26/06/2020

Motivation

$\boxed{\text{Conformal Bootstrap}}$ $\longleftrightarrow$ $\boxed{\text{S-matrix Bootstrap}}$

$\begin{cases} \text{QFT} \\ \text{QG} \end{cases}$ on AdS_{d+1}

$\begin{cases} \text{QFT} \\ \text{QG} \end{cases}$ on $\mathbb{M}^{d+1}$

Mellin Amplitudes $\xrightarrow[\text{Limit}]{\text{Flat Space}}$ Scattering Amplitudes

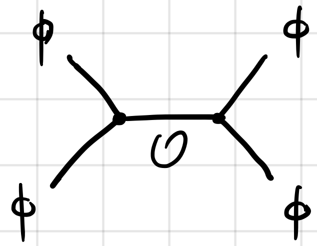
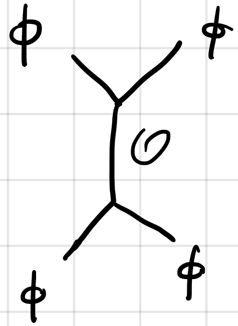
$\Updownarrow$

Boundary correlators

$$\langle \phi(x_1) \dots \phi(x_4) \rangle_{CFT_d}$$

Motivation

- Conformal Bootstrap in Mellin space ?

$$\langle \phi(x_1) \ldots \phi(x_4) \rangle = \sum_{\mathcal{O}} C^2_{\mathcal{O}\phi\phi}$$  $$= \sum_{\mathcal{O}} C^2_{\mathcal{O}\phi\phi}$$ 

→ More efficient Bootstrap?

Mellin-Polyakov Bootstrap [Gopakumar, Sinha, Sen, Kaviraj, Dey, Ghosh, ...]

Holographic Correlators [Rastelli, Zhou, Alday, Bissi, Perlmutter, Pufu, Chester, Ray; Gonçalves, Pereira, Roumpedakis, Caron-Huot, Drummond, Paul, Santagata, ...]

...

- When does the Mellin representation of the 4pt-function exist ?
- How to impose crossing, OPE and unitarity in Mellin space ?

Outline

* Motivation
* Mellin Amplitudes Basics
* Existence and Analyticity
* Polyakov conditions
* Regge boundedness
* Dispersion relations
* Sum rules
* Open Questions

Mellin Amplitudes basics

['09 Mack]

Scaling dimension of ϕ

$$\boxed{\sum_{\substack{j=1\\ j\neq i}}^{4} \gamma_{ij} = \Delta}$$

scalar primary $\dim[\phi]=\Delta$

$$\langle \phi(x_1)\cdots\phi(x_4)\rangle_{CFT} = \int [d\gamma]\, \boxed{M(\gamma_{ij})} \prod_{1\le i<j\le 4} \frac{\Gamma(\gamma_{ij})}{(x_i-x_j)^{2\gamma_{ij}}}$$

Mellin Amplitude

$$= \frac{1}{(x_{13}^2\, x_{24}^2)^{\Delta}} \int_{-i\infty}^{i\infty} \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2} \underbrace{\boxed{M(\gamma_{12},\gamma_{14})}\, \Gamma^2(\gamma_{12})\, \Gamma^2(\gamma_{14})\, \Gamma^2(\Delta-\gamma_{12}-\gamma_{14})}_{\hat{M}(\gamma_{12},\gamma_{14})}\, u^{-\gamma_{12}}\, v^{-\gamma_{14}}$$

cross ratios

$$u = \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2}$$

$$v = \frac{x_{14}^2\, x_{23}^2}{x_{13}^2\, x_{24}^2}$$

Crossing : $\gamma_{12} + \gamma_{13} + \gamma_{14} = \Delta$

$$M(\gamma_{12}, \gamma_{14}) = M(\gamma_{14}, \gamma_{12}) = M(\gamma_{12}, \gamma_{13})$$

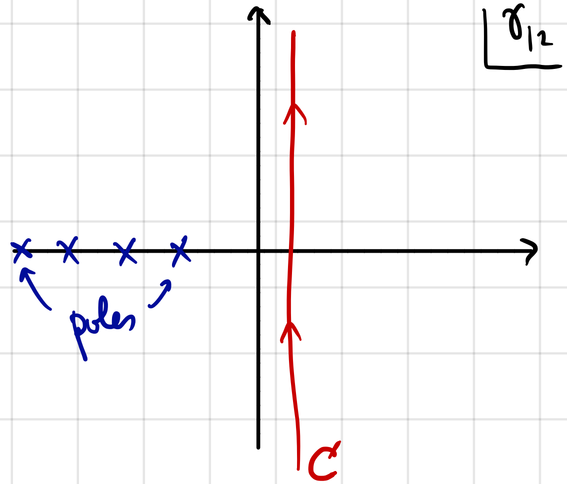
Crossing :

$$\gamma_{12} + \gamma_{13} + \gamma_{14} = \Delta$$

$$M(\gamma_{12}, \gamma_{14}) = M(\gamma_{14}, \gamma_{12}) = M(\gamma_{12}, \gamma_{13})$$

Operator Product Expansion (OPE)

$$\phi(x_1)\,\phi(x_2) = \sum_{\mathcal{O}} c_{\phi\phi\mathcal{O}}\,(x_{12}^2)^{\Delta_\phi - \frac{\Delta_{\mathcal{O}}}{2}} \left[\mathcal{O}(x_2) + \text{descendants}\right]$$



Compare with Mellin representation :

$$\langle \phi(x_1)\,\phi(x_2) \cdots \rangle = \int_C d\gamma_{12}\, (x_{12}^2)^{-\gamma_{12}} \cdots M(\gamma_{12}, \ldots)$$

$\Rightarrow$ Poles

Mack polynomial

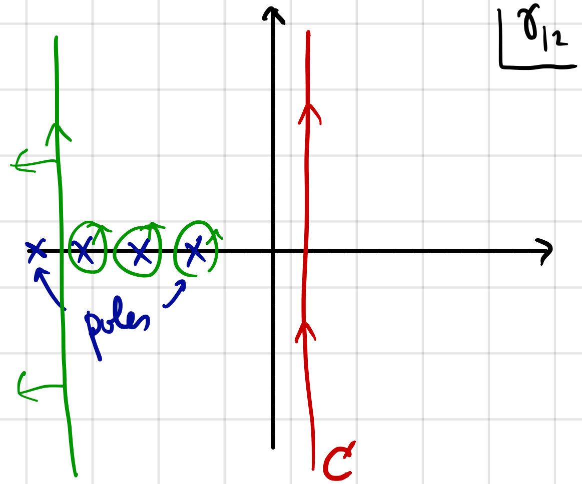
$$M(\gamma_{12}, \gamma_{14}) \approx \frac{c_{\phi\phi\mathcal{O}}^2\, Q(\gamma_{14})}{\gamma_{12} - \Delta + \frac{1}{2}\Delta_{\mathcal{O}}}$$

Crossing : $\gamma_{12} + \gamma_{13} + \gamma_{14} = \Delta$

$$M(\gamma_{12}, \gamma_{14}) = M(\gamma_{14}, \gamma_{12}) = M(\gamma_{12}, \gamma_{13})$$

Operator Product Expansion (OPE)

$$\phi(x_1)\,\phi(x_2) = \sum_{\mathcal{O}} C_{\phi\phi\mathcal{O}} \left(x_{12}^2\right)^{\Delta_\phi - \frac{\Delta_{\mathcal{O}}}{2}} \left[\mathcal{O}(x_2) + \text{descendants}\right]$$



Compare with Mellin representation :

$$\langle \phi(x_1)\,\phi(x_2) \cdots \rangle = \int_{\mathcal{C}} d\gamma_{12} \left(x_{12}^2\right)^{-\gamma_{12}} \cdots \; M(\gamma_{12}, \ldots)$$

$\Rightarrow$ **Poles**

Mack polynomial

$$M(\gamma_{12}, \gamma_{14}) \approx \frac{C_{\phi\phi\mathcal{O}}^2 \; Q_{J,m}(\gamma_{14})}{\gamma_{12} - \Delta + \frac{\tau}{2} + m}$$

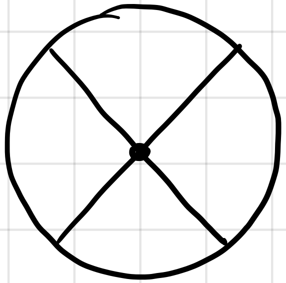
$$m = 0, \underbrace{1, 2, 3, \ldots}_{\text{descendants}}$$

twist $\tau = \Delta_{\mathcal{O}} - J_{\mathcal{O}}$ spin

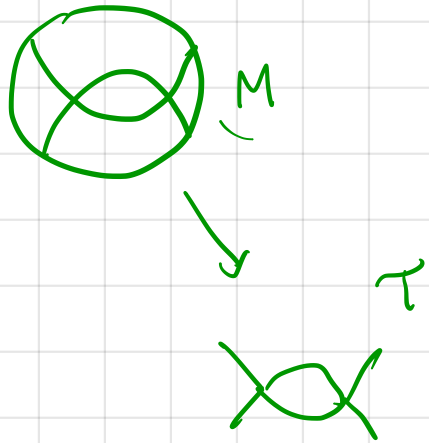
→ **Analogy** : Mellin Amplitudes ⟷ Scattering Amplitudes

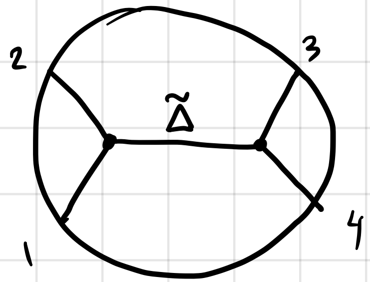
$\gamma_{ij} \longleftrightarrow p_i \cdot p_j$

Simplicity of Witten diagrams



$\rightarrow \quad M(\gamma_{12},\gamma_{14}) = \text{const.}$





$$\rightarrow \quad M(\gamma_{12},\gamma_{14}) = \sum_{m=0}^{\infty} \frac{a_m}{\gamma_{12} - \Delta + \frac{\tilde{\Delta}}{2} + m}$$

Δ fixed   if massless in M

"$\Delta \to \infty$" if massive in M

$M^2 R^2 = \Delta(\Delta - d)$

Flat space Limit Formulas

$$\text{Mellin Amp.} \xrightarrow[R_{AdS}\to\infty]{\gamma_{ij}\to\infty,\ \gamma_{ij}/\gamma_{12}\ \text{fixed}} \text{Scattering Amp.}$$

$\gamma_{ij}/\sqrt{\lambda} \to \alpha' p_i \cdot p_j$

$$\frac{p_i \cdot p_j}{p_1 \cdot p_2} = \frac{\gamma_{ij}}{\gamma_{12}}$$

Existence and Analyticity

$$F(u,v) \equiv \left(x_{13}^2\, x_{24}^2\right)^{\Delta} \langle \phi(x_1) \dots \phi(x_4) \rangle = \int_{-i\infty}^{i\infty} \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2}\, \hat{M}(\gamma_{12}, \gamma_{14})\, u^{-\gamma_{12}}\, v^{-\gamma_{14}}$$

$$\Rightarrow \qquad \cancel{\hat{M}(\gamma_{12}, \gamma_{14}) = \int_0^\infty \frac{du\, dv}{u\, v}\, u^{\gamma_{12}}\, v^{\gamma_{14}}\, \underbrace{F(u,v)}_{\substack{u\to 0 \\ u^{-\Delta}}}}$$

but inverse Mellin <u>does not converge</u>!

Existence and Analyticity

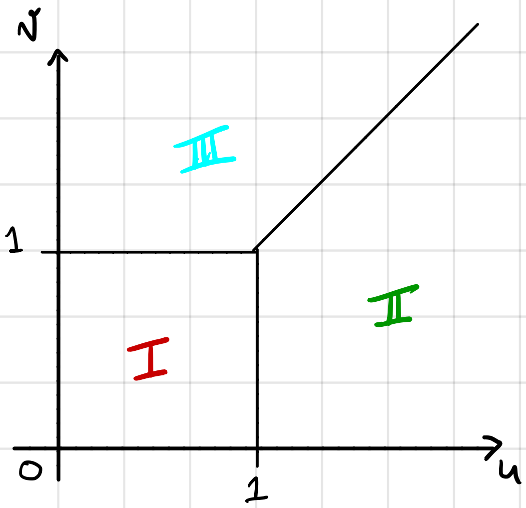
$$F(u,v) \equiv \left(x_{13}^2\, x_{24}^2\right)^{\Delta} \langle \phi(x_1) \cdots \phi(x_4) \rangle = \int_{-i\infty}^{i\infty} \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2}\, \hat{M}(\gamma_{12}, \gamma_{14})\, u^{-\gamma_{12}}\, v^{-\gamma_{14}}$$

$$\Rightarrow \quad \hat{M}(\gamma_{12}, \gamma_{14}) = \int_0^{\infty} \frac{du\, dv}{u\, v}\, u^{\gamma_{12}}\, v^{\gamma_{14}}\, F(u,v)$$

$\sim u^{-\Delta}$

but inverse Mellin does not converge!

Idea: split the integral



$$\boxed{K(\gamma_{12}, \gamma_{14}) \equiv \int_0^{1} \frac{du\, dv}{u\, v}\, u^{\gamma_{12}}\, v^{\gamma_{14}}\, F(u,v)}$$

Analytic for $\mathrm{Re}\,\gamma_{12} > \Delta$ and $\mathrm{Re}\,\gamma_{14} > \Delta$.

$$F(u,v) = \int_{\mathrm{Re}(\gamma_{12},\gamma_{14}) > \Delta} \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2} K(\gamma_{12},\gamma_{14})\, u^{-\gamma_{12}} v^{-\gamma_{14}} \qquad \mathrm{I}$$

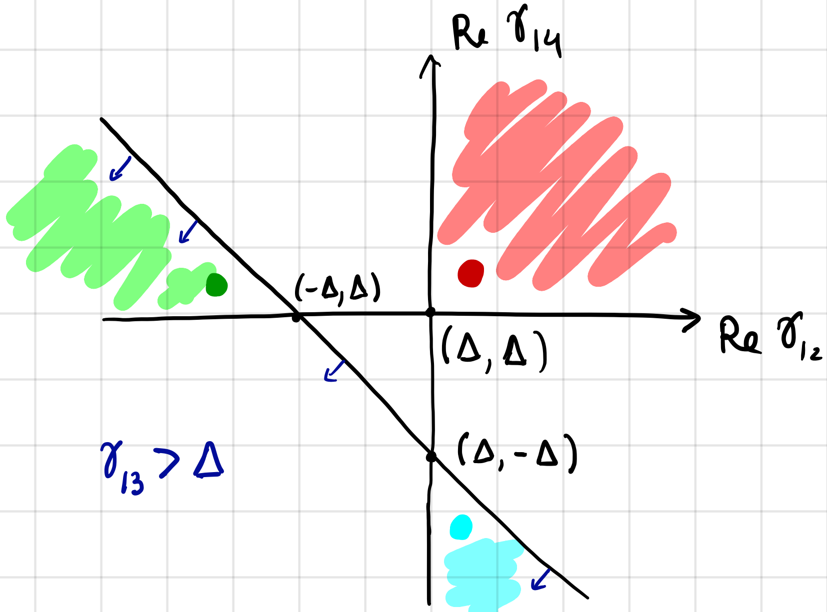
$\propto \Theta(1-u)\Theta(1-v)$

$$+ \int_{\mathrm{Re}(\gamma_{13},\gamma_{14}) > \Delta} \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2} K(\gamma_{13},\gamma_{14})\, u^{-\gamma_{12}} v^{-\gamma_{14}} \qquad \mathrm{II}$$

$$+ \int_{\mathrm{Re}(\gamma_{12},\gamma_{13}) > \Delta} \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2} K(\gamma_{12},\gamma_{13})\, u^{-\gamma_{12}} v^{-\gamma_{14}} \qquad \mathrm{III}$$

where we used crossing.

$$\gamma_{12} + \gamma_{13} + \gamma_{14} = \Delta$$



If we can bring the 3 integrals to the same contour then

$$\boxed{\hat{M}(\gamma_{12}, \gamma_{14}) \equiv K(\gamma_{12}, \gamma_{14}) + K(\gamma_{12}, \gamma_{13}) + K(\gamma_{13}, \gamma_{14})}$$

If we can bring the 3 integrals to the same contour then

$$\boxed{\hat{M}(\gamma_{12}, \gamma_{14}) \equiv K(\gamma_{12}, \gamma_{14}) + K(\gamma_{12}, \gamma_{13}) + K(\gamma_{13}, \gamma_{14})}$$

Example : GFF

$$F(u, v) = 1 + u^{-\Delta} + v^{-\Delta}$$

$$K(\gamma_{12}, \gamma_{14}) = \frac{1}{\gamma_{12}\gamma_{14}} + \frac{1}{\gamma_{12}(\gamma_{14}-\Delta)} + \frac{1}{(\gamma_{12}-\Delta)\gamma_{14}}$$

$$\hat{M} = 0$$

→ Disconnected part of the correlator does not contribute to M

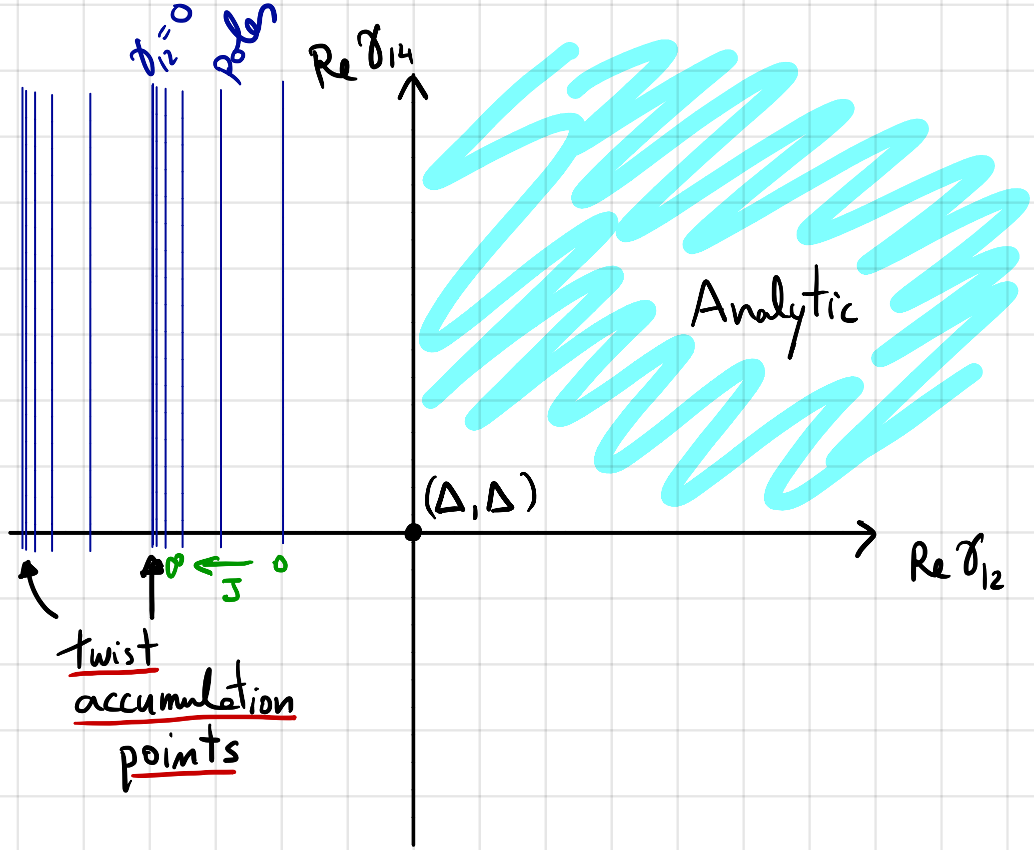
Contour deformation requires analytic continuation of K.

$$\boxed{K(\gamma_{12},\gamma_{14}) \equiv \int_0^1 \frac{du\,dv}{u\,v}\, u^{\gamma_{12}}\, v^{\gamma_{14}}\, F(u,v)}$$

$$F(u,v) = \sum_{\tau\geq 0} u^{\tau/2-\Delta} f_\tau(v)$$

$$\Downarrow$$

$$\boxed{\text{poles at } \gamma_{12} = \Delta - \frac{\tau}{2}}$$



τ = twist of $\mathcal{O} \subset \phi\times\phi$

$\exists$ double twist operators

$$\tau = 2\Delta + 2n + \gamma(n,J)$$

$$\Big\downarrow J\to\infty$$

$$0$$

['12 Komargodski, Zhiboedov]
['12 Fitzpatrick, Kaplan, Poland, Simmons-Duffin]

Twist spectrum discrete or continuous?

['18 Kusuki] ['18 Collier et al.] 2D ↵

OPE :

$$F(u,v) = \sum_{\tau,\ell} \underbrace{a_{\tau,\ell}}_{\geq 0} u^{\tau/2-\Delta} (z^\ell + \bar{z}^\ell)$$

$u = z\bar{z}$
$v = (1-z)(1-\bar{z})$

$$K(\gamma_{12},\gamma_{14}) = \underbrace{\int_0^1 \frac{du\,dv}{uv} u^{\gamma_{12}} v^{\gamma_{14}} \Big[F_{sub}(u,v)}_{\substack{\text{Analytic for} \\ \mathrm{Re}\,\gamma_{14} > \Delta \\ \mathrm{Re}\,\gamma_{12} > \Delta - \frac{\tau_{max}}{2}}} + \underbrace{\sum_{\tau<\tau_{max}} \sum_{\ell} a_{\tau,\ell} u^{\tau/2-\Delta} (z^\ell + \bar{z}^\ell) \Big]}_{\text{Integrate term by term}}$$

Integrate term by term

$\Downarrow$

poles at $\gamma_{12} = \Delta - \tau/2$

analytic for $\mathrm{Re}\,\gamma_{14} > \Delta$

OPE :

$$F(u,v) = \sum_{\tau,\ell} \underbrace{a_{\tau,\ell}}_{\geq 0}\, u^{\tau/2-\Delta} \left(z^\ell + \bar z^\ell\right)$$

$u = z\bar z$
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Integrate term by term

$\Downarrow$

poles at $\gamma_{12} = \Delta - \tau/2$

analytic for $\mathrm{Re}\,\gamma_{14} > \Delta$

Double-Light cone limit bound

$$F_{sub} \lesssim u^{-\Delta + \frac{\tau_{max}}{2}}\, v^{-\Delta}$$

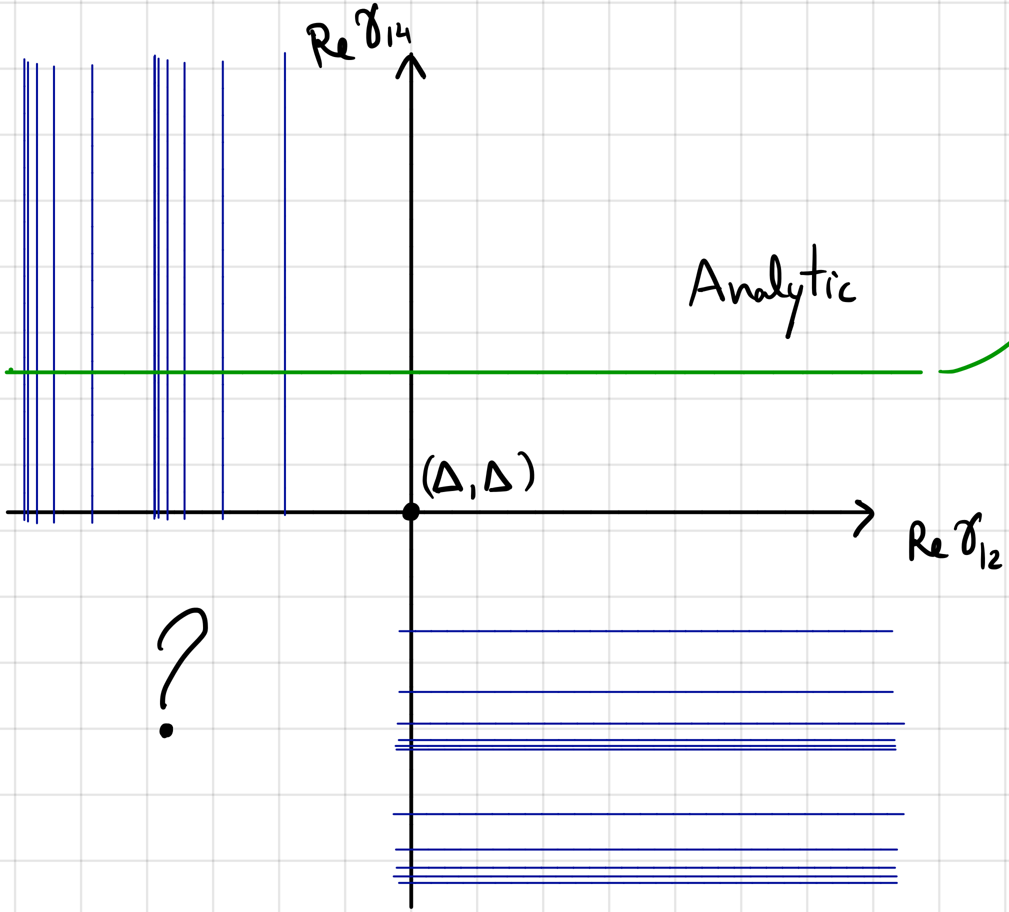
$$\boxed{u \sim v \to 0}$$

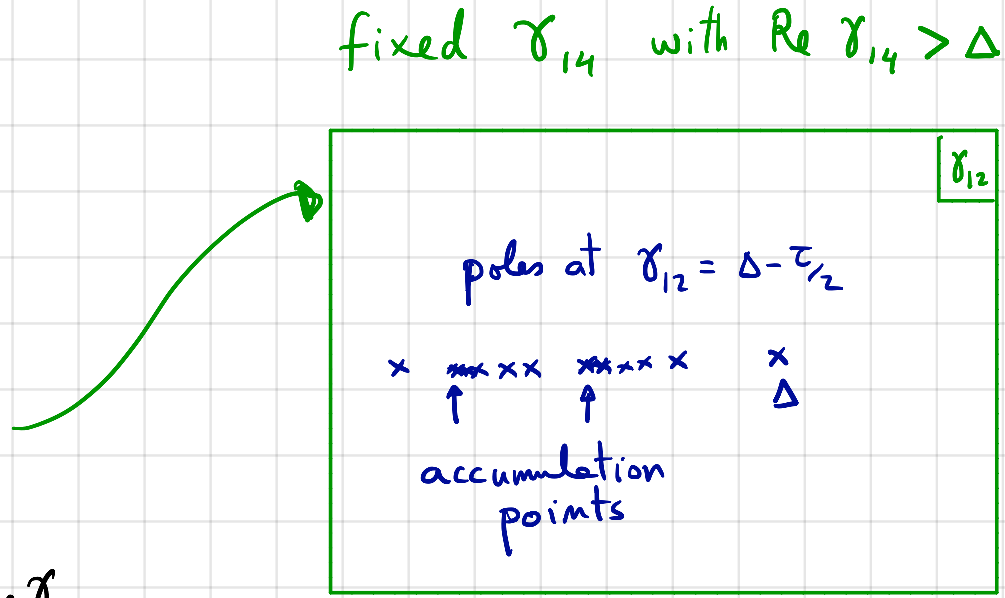
This follows from :

$$F_{sub} = \sum_{\tau \geq \tau_{max}} \sum_{\ell} a_{\tau,\ell} (z\bar z)^{\tau/2-\Delta} \left(z^\ell + \bar z^\ell\right) \Rightarrow (z_1 \bar z)^{\Delta - \frac{\tau_{max}}{2}} F_{sub}(z_1,\bar z) < (z_2 \bar z)^{\Delta - \frac{\tau_{max}}{2}} F_{sub}(z_2,\bar z)$$

$0 < z_1 < z_2 < 1 \quad , \quad 0 < \bar z < 1$

Using the OPE, we showed that



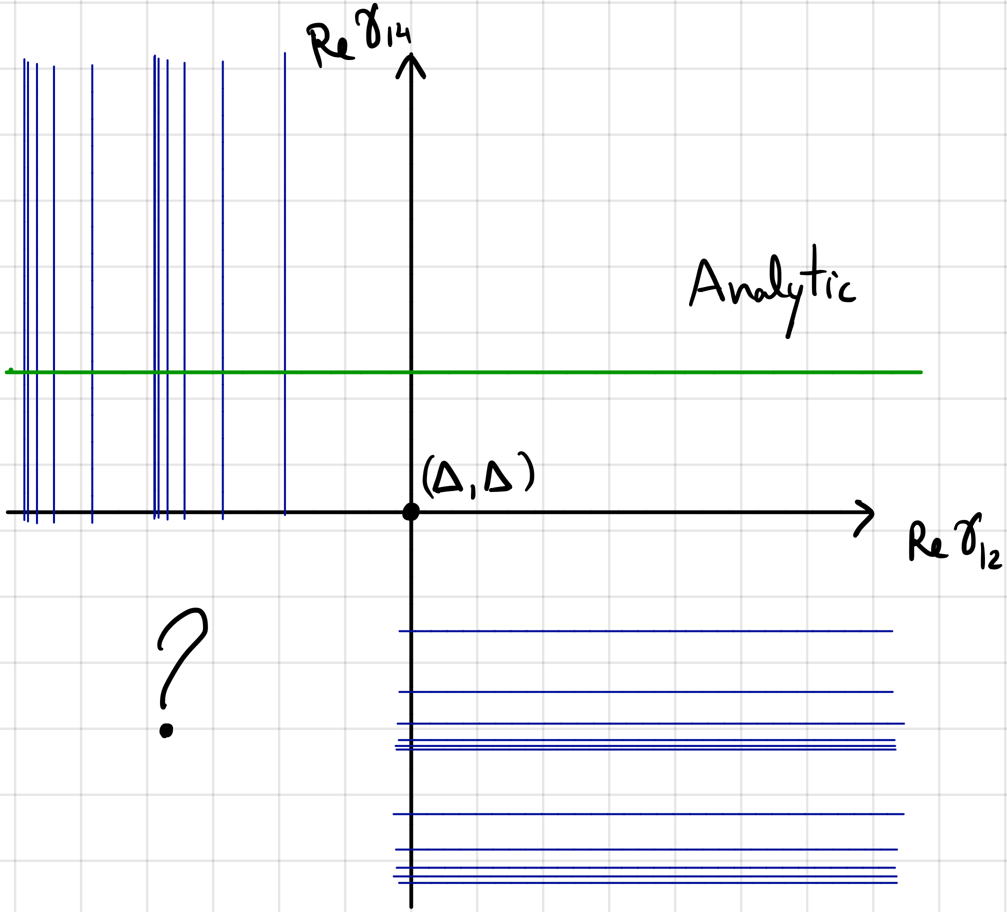


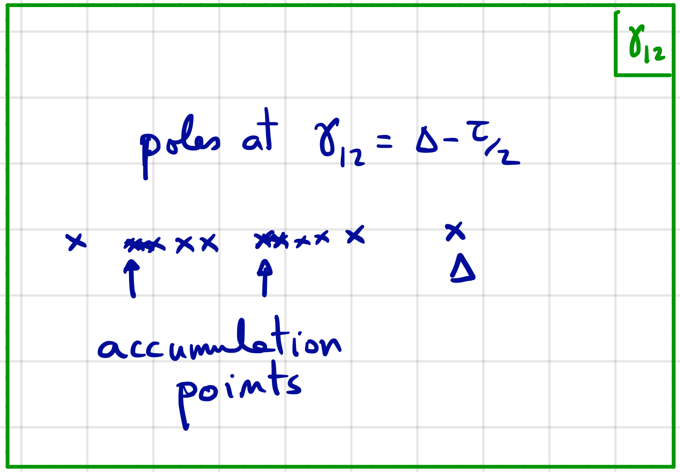
Maximal Mellin Analyticity Conjecture : The OPE poles are the only singularities

[If no accumulation points $\Longrightarrow$ No more singularities]
Tube or Bochner's theorem

Using the OPE, we showed that

fixed γ_{14} with $\mathrm{Re}\,\gamma_{14} > \Delta$

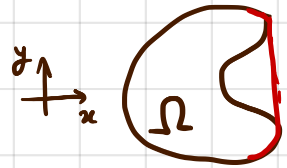




Tube Theorem

$F(x,y)$ analytic for $\mathrm{Re}(x,y) \in \Omega$

$\Rightarrow F(x,y)$ analytic for $\mathrm{Re}(x,y) \in \Omega_0$



$\Omega_0 = \text{convex hull}(\Omega)$

Maximal Mellin Analyticity Conjecture: The OPE poles are the only singularities

[If no accumulation points $\Longrightarrow$ No more singularities]

Tube or Bochner's theorem

If Δ = minimal twist, then we proved MMA in the blue region that contains the crossing symmetric point $\gamma_{12} = \gamma_{14} = \gamma_{13} = \frac{\Delta}{3}$.

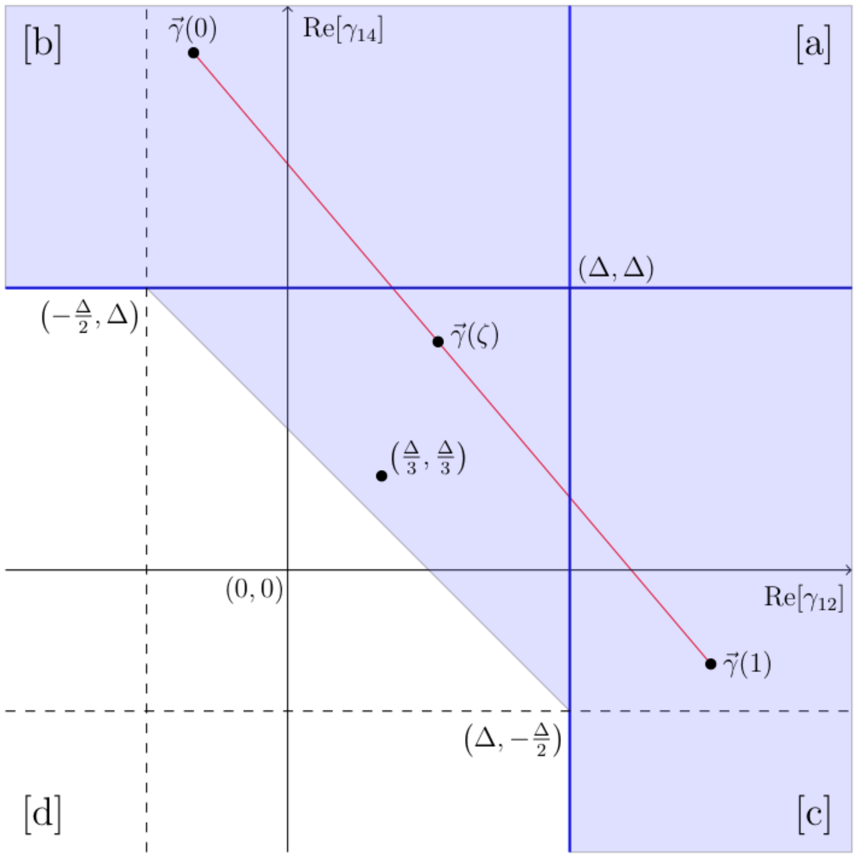


Figure 19: Construction that leads to equations (317) and (318). We represent the case $\Delta = \tau_{lightest}$ and therefore $\tau_* = 2\Delta$. We can establish analyticity in the shaded domain without crossing the accumulation point of accumulation points of triple-twist operators (marked with dashed lines).

Straight Contour

$$F(u,v) = \int_{\mathrm{Re}(\gamma_{12}) = \mathrm{Re}(\gamma_{14}) = \frac{\Delta}{3}} \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2} \hat{M}(\gamma_{12}, \gamma_{14})\, u^{-\gamma_{12}}\, v^{-\gamma_{14}}$$

$$+ \sum_{0 \leq \tau < \frac{4\Delta}{3}} c_\tau^2 \left[u^{\frac{\tau}{2} - \Delta} g_{\tau, \ell(\tau)}(v) + v^{\frac{\tau}{2} - \Delta} g_{\tau, \ell(\tau)}(u) + v^{-\tau/2} g_{\tau, \ell(\tau)}\left(\frac{u}{v}\right) \right]$$

$\hookrightarrow$ collinear block

Straight Contour

$$F(u,v) = \int_{\mathrm{Re}(\gamma_{12}) = \mathrm{Re}(\gamma_{14}) = \frac{\Delta}{3}} \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2} \hat{M}(\gamma_{12}, \gamma_{14})\, u^{-\gamma_{12}}\, v^{-\gamma_{14}}$$

$$+ \sum_{0 \leq \tau < \frac{4\Delta}{3}} c_\tau^2 \left[u^{\frac{\tau}{2}-\Delta}\, g_{\tau,\ell(\tau)}(v) + v^{\frac{\tau}{2}-\Delta}\, g_{\tau,\ell(\tau)}(u) + v^{-\tau/2}\, g_{\tau,\ell(\tau)}\!\left(\frac{u}{v}\right) \right]$$

↳ collinear block

If $\boxed{\Delta < \frac{3}{4}\tau_{gap}}$ then $F - (\text{subtractions}) = F_{\text{connected}}$.

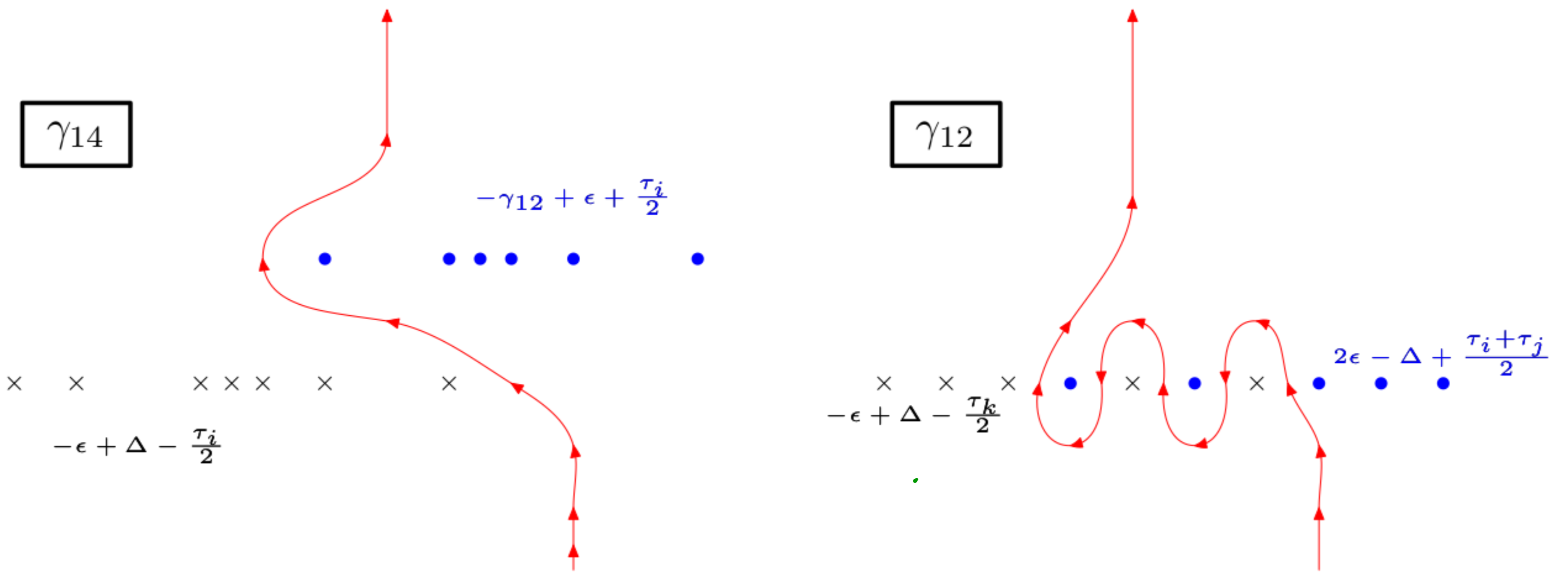
Example : 3D Ising $\Delta = 0.518$ $\tau_{gap} = 1$

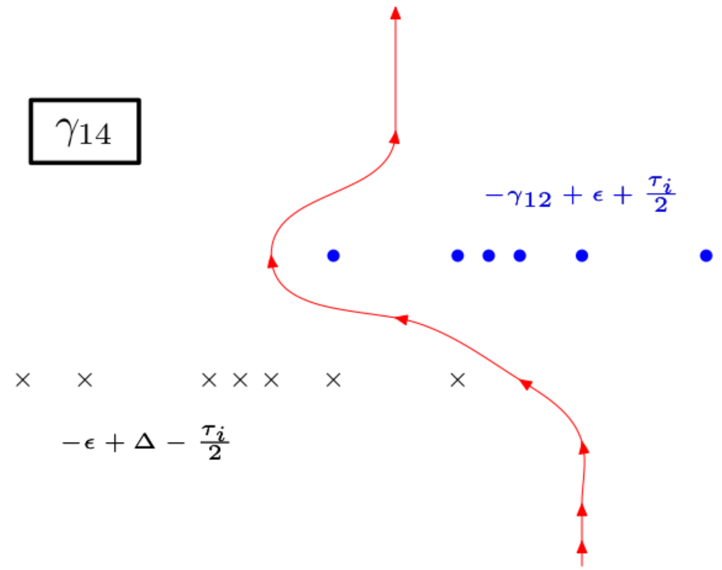
$$\boxed{(x_{13}^2\, x_{24}^2)^\Delta \langle \sigma(x_1) \cdots \sigma(x_4) \rangle_{\text{conn}} = \int_{\mathrm{Re}(\gamma_{12}) = \mathrm{Re}(\gamma_{14}) = \frac{\Delta}{3}} \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2} \hat{M}(\gamma_{12}, \gamma_{14})\, u^{-\gamma_{12}}\, v^{-\gamma_{14}}}$$

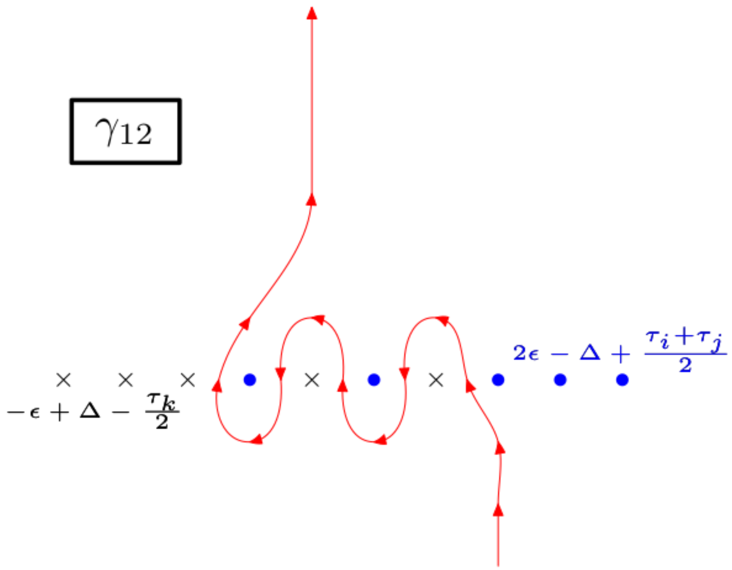
Deformed Contour

$$F(u,v) = \int_C \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2} \hat{M}(\gamma_{12}, \gamma_{14})\, u^{-\gamma_{12}}\, v^{-\gamma_{14}}$$

$$\left.\begin{aligned} &+ 1 + u^{-\Delta} + v^{-\Delta} \\ &+ C^2_{\phi\phi\phi}\left[u^{-\Delta/2} g_{\Delta,0}(u) + v^{-\Delta/2} g_{\Delta,0}(v) + v^{-\Delta/2} g_{\Delta,0}\left(\frac{u}{v}\right)\right] \end{aligned}\right\} \text{"pimches"}$$

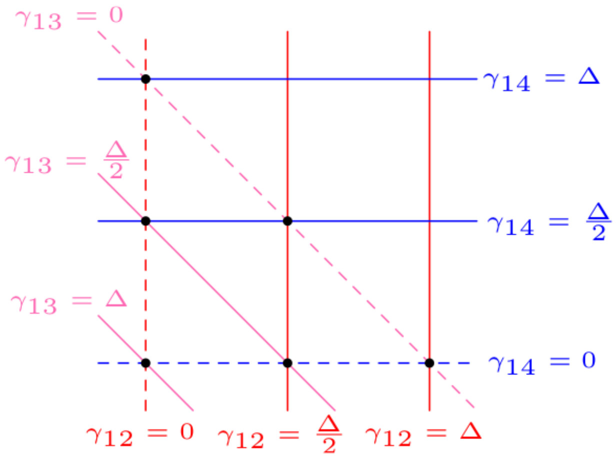


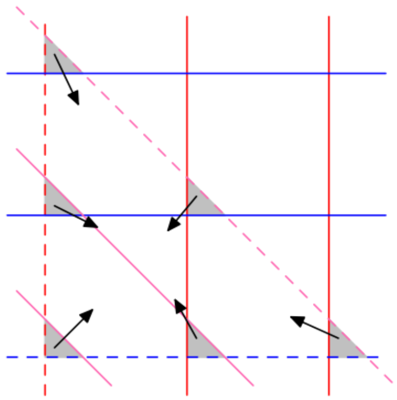




Deformed contour pinched if $\quad \tau_i + \tau_j + \tau_k = 4\Delta$

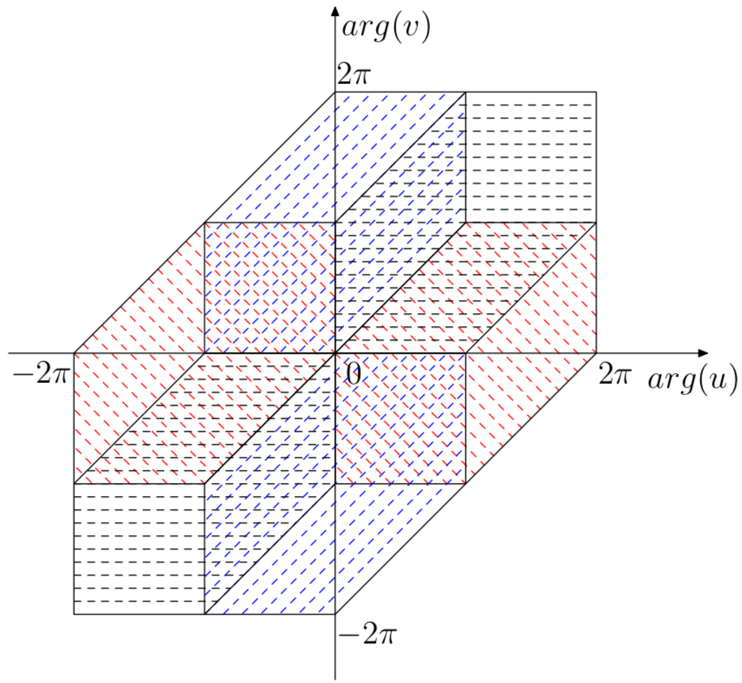
Generic CFT $\longrightarrow$ 6 pinches.





Sectorial Domain of Analyticity

$F\left(|u|\, e^{i \arg u},\ |v|\, e^{i \arg v}\right)$ analytic $\forall\ |u|>0,\ |v|>0,\ (\arg u, \arg v) \in \Theta_{CFT}$

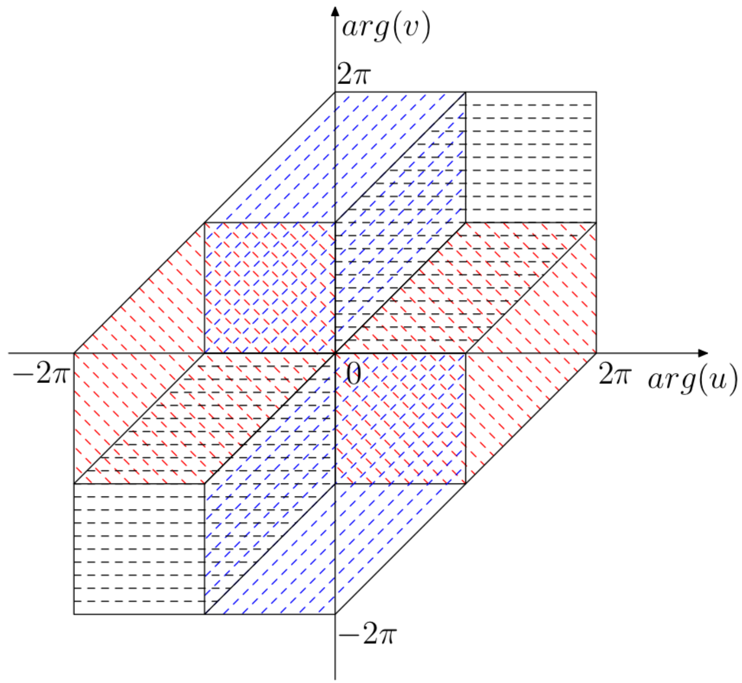


OPE in 3 channels $\Rightarrow$ Θ_{CFT} $=$

Sectorial Domain of Analyticity

$$F\left(|u|\,e^{i\arg u},\ |v|\,e^{i\arg v}\right) \quad \text{analytic} \quad \forall\ |u|>0,\ |v|>0,\ (\arg u, \arg v) \in \Theta_{CFT}$$

$$\text{OPE} \implies \Theta_{CFT} =$$



$$F(u,v) = \int_{-i\infty}^{i\infty} \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2}\, |u|^{-\gamma_{12}}\, |v|^{-\gamma_{14}}\, \hat{M}(\gamma_{12}, \gamma_{14})\, \exp\left(\arg u\, \mathrm{Im}\,\gamma_{12} + \arg v\, \mathrm{Im}\,\gamma_{14}\right)$$

$$\overset{*}{\implies} \quad \hat{M}(\gamma_{12}, \gamma_{14}) \sim \exp\left[-\operatorname{Sup}_{\Theta_{CFT}}\left(\arg u\, \mathrm{Im}\,\gamma_{12} + \arg v\, \mathrm{Im}\,\gamma_{14}\right)\right]$$

$$\parallel$$

$$\Gamma^2(\gamma_{12})\,\Gamma^2(\gamma_{14})\,\Gamma^2(\Delta - \gamma_{12} - \gamma_{14})\,M(\gamma_{12}, \gamma_{14}) \implies$$ No exponential behavior for M

Sectorial Domain of Analyticity

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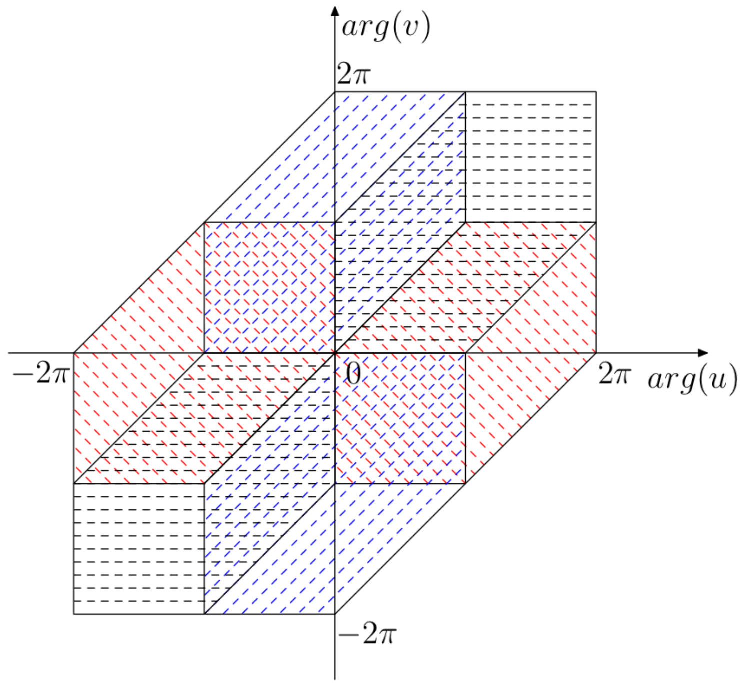

OPE $\Rightarrow$ Θ_{CFT} $=$

* Also requires polynomial boundness of F inside sectorial domain.

$$F(u,v) = \int \frac{d\gamma_{12}\, d\gamma_{14}}{(2\pi i)^2} |u|^{-\gamma_{12}} |v|^{-\gamma_{14}} \hat{M}(\gamma_{12}, \gamma_{14}) \exp\left(\arg u \operatorname{Im}\gamma_{12} + \arg v \operatorname{Im}\gamma_{14}\right)$$

$$\overset{*}{\Rightarrow} \quad \hat{M}(\gamma_{12}, \gamma_{14}) \sim \exp\left[-\operatorname{Sup}_{\Theta_{CFT}}\left(\arg u \operatorname{Im}\gamma_{12} + \arg v \operatorname{Im}\gamma_{14}\right)\right]$$

$||$

$\Gamma^2(\gamma_{12})\, \Gamma^2(\gamma_{14})\, \Gamma^2(\Delta - \gamma_{12} - \gamma_{14})\, M(\gamma_{12}, \gamma_{14}) \Rightarrow$ No exponential behavior for M

Polyakov Conditions

Polyakov Conditions

Naively $\quad M(\gamma_{12},\gamma_{14}) \xrightarrow[\gamma_{12}\to -n]{} 0 \quad$ and $\quad \dfrac{M(\gamma_{12},\gamma_{14})}{\gamma_{12}+n} \xrightarrow[\gamma_{12}\to -n]{} 0 \qquad n=0,1,2,\dots$

because

$$M(\gamma_{12},\gamma_{14}) = \frac{\hat{M}(\gamma_{12},\gamma_{14})}{\Gamma^2(\gamma_{12})\,\Gamma^2(\gamma_{14})\,\Gamma^2(\Delta-\gamma_{12}-\gamma_{14})} .$$

However, $\quad \gamma_{12}=-n \quad$ are accumulation points of poles !

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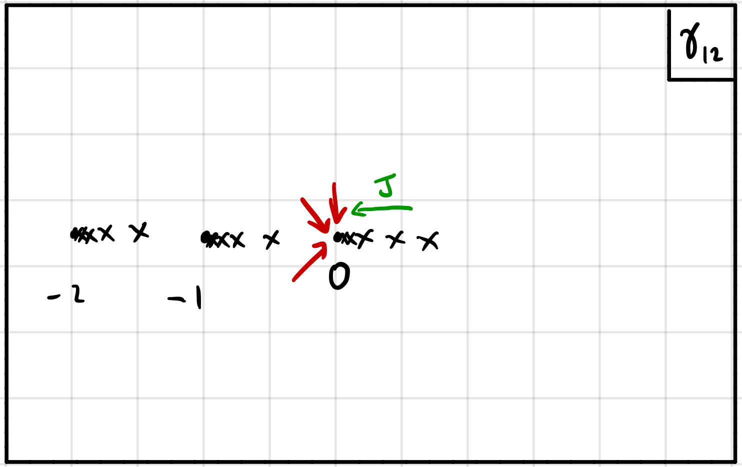
However, $\quad \gamma_{12} = -n \quad$ are accumulation points of poles !



double-twist Regge trajectories

$$\tau = 2\Delta + 2n + \gamma(n,J)$$

$$\Rightarrow \text{poles at } \gamma_{12} = -n - \tfrac{1}{2}\gamma(n,J)$$

$$\gamma(n,J) \sim \frac{f(n)}{J^{\tau_{gap}}} , \qquad J\to\infty$$

Analogous to $s\to\infty$ with fixed t for Veneziano amplitude.

$$A(s,t) \sim s^{\alpha(t)}\,\beta(t) \qquad s\to\infty\, e^{i\theta} ,\ \theta\neq 0,\pi$$

Near the accumulation point $\boxed{\gamma_{12}=0}$

$$\gamma_{12}\,\hat{M}(\gamma_{12},\gamma_{14}) \sim \sum_{\substack{J=0\\ J\text{ even}}}^{\infty} \frac{1}{\gamma_{12}+\gamma(J)/2}\ \mathop{\mathrm{Res}}_{\gamma_{12}=-\gamma(J)/2} \hat{M}(\gamma_{12},\gamma_{14})\,\gamma_{12}$$

$$\left[\frac{\gamma(J)}{2} \sim -\frac{a}{J^{\tau_{gap}}}\right] \sim \int_1^{\infty} dJ\ \frac{1}{\gamma_{12}-\frac{a}{J^{\tau_{gap}}}}\ \underbrace{\left[C_J^2\right]^{GFF} Q_J(\gamma_{14})\,\gamma(J)}_{J^{2\gamma_{14}-\tau_{gap}-1}}$$

$$f(\gamma_{14}) + \left[\gamma_{14} \longleftrightarrow \Delta-\gamma_{14}\right]$$

Near the accumulation point $\gamma_{12}=0$

$$\gamma_{12}\hat{M}(\gamma_{12},\gamma_{14}) \sim \sum_{\substack{J=0\\ J\text{ even}}}^{\infty} \frac{1}{\gamma_{12}+\gamma(J)/2} \operatorname*{Res}_{\gamma_{12}=-\gamma(J)/2} \hat{M}(\gamma_{12},\gamma_{14})\,\gamma_{12}$$

$$\left[\frac{\gamma(J)}{2} \sim -\frac{a}{J^{\tau_{gap}}}\right] \sim \int_1^{\infty} dJ \, \frac{1}{\gamma_{12}-\frac{a}{J^{\tau_{gap}}}} \underbrace{\left[C_J^2\right]^{GFF} Q_J(\gamma_{14})\,\gamma(J)}_{J^{2\gamma_{14}-\tau_{gap}-1} f(\gamma_{14}) + [\gamma_{14}\leftrightarrow\Delta-\gamma_{14}]}$$

converges for $\boxed{\Delta-\tau_{gap}/2 < \gamma_{14} < \tau_{gap}/2}$

$$\gamma_{12}\hat{M}(\gamma_{12},\gamma_{14}) \sim (-\gamma_{12})^{-\frac{2\gamma_{14}}{\tau_{gap}}} g(\gamma_{14}) + (\gamma_{14}\leftrightarrow\Delta-\gamma_{14})$$

"Regge behavior"

Polyakov condition : $\boxed{M(\gamma_{12}=0,\gamma_{14}) = 0 \quad , \quad \Delta-\frac{\tau_{gap}}{2} < \gamma_{14} < \frac{\tau_{gap}}{2}}$

[Preliminary]

$$\boxed{\frac{M(\gamma_{12},\gamma_{14})}{\gamma_{12}+n} \xrightarrow[\gamma_{12}\to -n]{} 0}$$

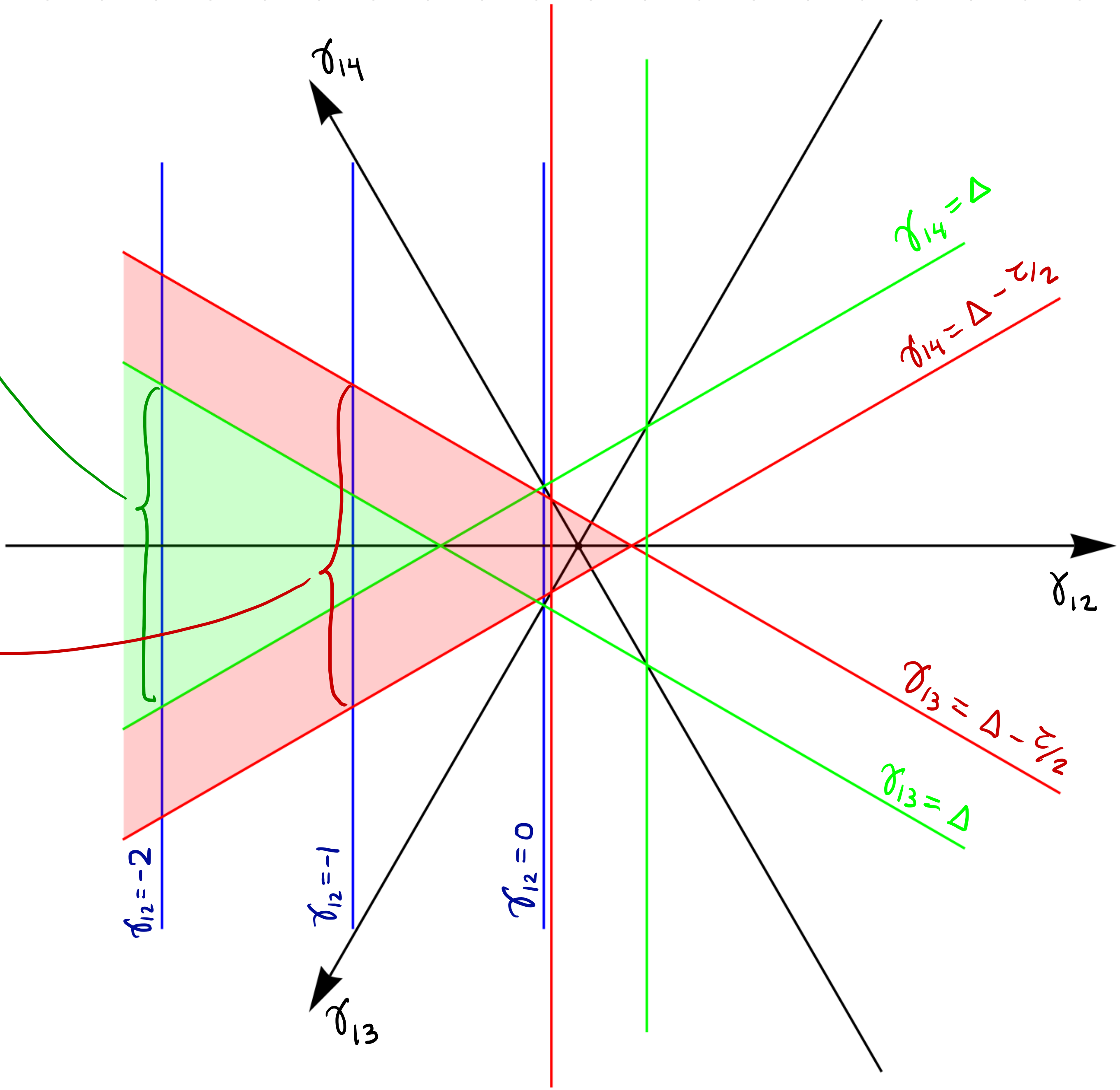
$$\boxed{M(\gamma_{12},\gamma_{14}) \xrightarrow[\gamma_{12}\to -n]{} 0}$$

For $\gamma_{12} \to -n$:

$$M(\gamma_{12},\gamma_{14}) \sim (\gamma_{12}+n)^{1+\frac{2(\gamma_{14}-\Delta)}{\tau_{gap}}}$$

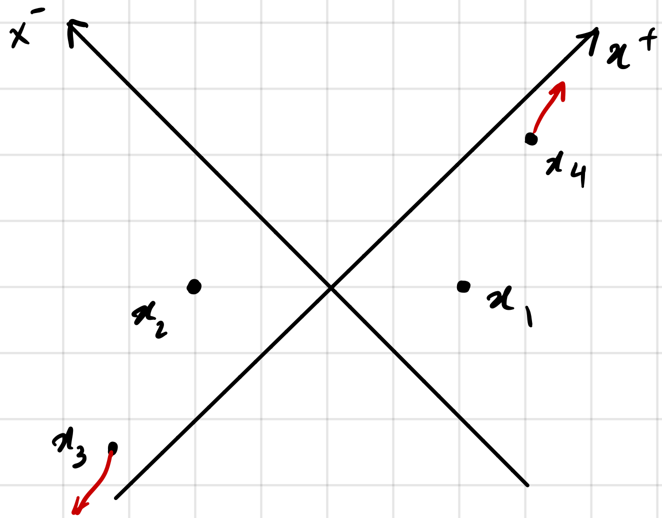
$$+ (\gamma_{12}+n)^{1+\frac{2(\gamma_{13}-\Delta)}{\tau_{gap}}}$$

$$+ O(\gamma_{12}+n)^2$$



Regge limit

$$\begin{cases} x_1^{\pm} = \pm 1 \\ x_2^{\pm} = \mp 1 \\ x_3^{\pm} = \mp e^{\rho \pm t} \\ x_4^{\pm} = \pm e^{\rho \pm t} \end{cases}$$



$t \to \infty \iff \gamma_{14} \to \infty$

$$\begin{aligned} u &= 16\, e^{-2t} \xrightarrow[t\to\infty]{} 0 \\ v &\simeq 1 - 8\, e^{-t} \cosh\rho \xrightarrow{} 1 \end{aligned}$$

Unitarity + OPE $\Rightarrow$ $\displaystyle \lim_{t\to\infty} \left| \frac{F(u,v)}{F_{\text{disc}}(u,v)} \right| \leq 1$

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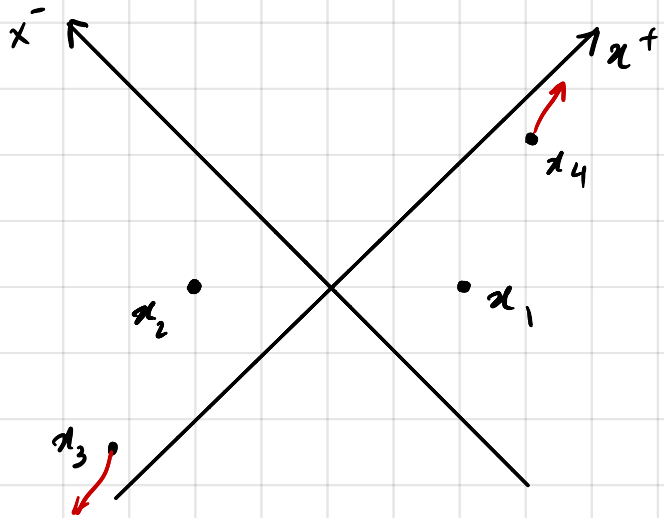

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Unitarity + OPE $\Rightarrow$ $\displaystyle \lim_{t\to\infty} \left| \frac{F(u,v)}{F_{disc}(u,v)} \right| \leq 1$

From time ordering

$$F(u,v) = \int \frac{d\gamma_{12}}{2\pi i}\, u^{-\gamma_{12}}\, \Gamma^2(\gamma_{12}) \int \frac{d\gamma_{14}}{2\pi i} \underbrace{\Gamma^2(\gamma_{14})\, \Gamma^2(\Delta - \gamma_{12} - \gamma_{14})\, (v e^{2\pi i})^{-\gamma_{14}}}_{\substack{2 \\ (\gamma_{14})^{2(\Delta - \gamma_{12} - 1)}\, e^{-\gamma_{14} \log v}}} \underbrace{M(\gamma_{12}, \gamma_{14})}_{\substack{2 \\ \gamma_{14}^{j_0}\, m(\gamma_{12})}}$$

Regge boundedness

$$\lim_{\gamma_{14}\to i\infty} M(\gamma_{14}, \gamma_{12}) \leq c\,|\gamma_{14}| \quad , \quad \mathrm{Re}\,\gamma_{12} > \Delta - \frac{\tau_{gap}}{2}$$

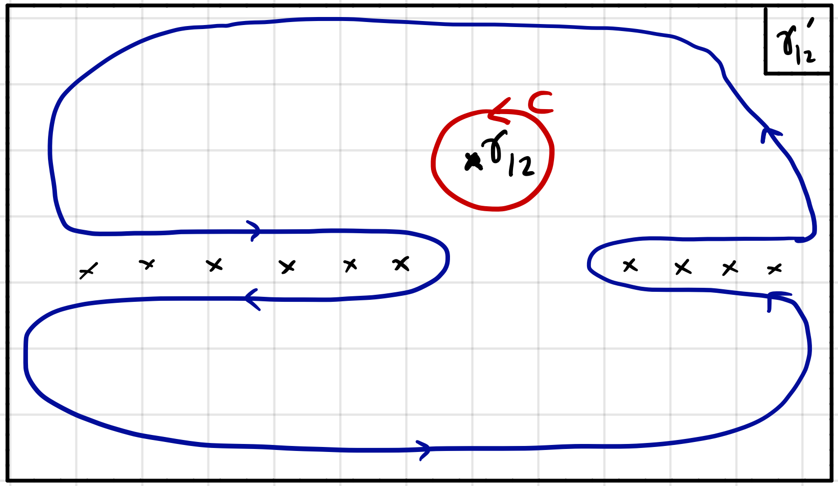
Conjecture:

$$\boxed{\lim_{|\gamma_{14}|\to \infty} M(\gamma_{14}, \gamma_{12}) \leq c\,|\gamma_{14}| \quad , \quad \mathrm{Re}\,\gamma_{12} > \Delta - \frac{\tau_{gap}}{2}}$$

Validity at the crossing symmetric point $\Rightarrow$ $\Delta < \frac{3}{4}\tau_{gap}$

$$\gamma_{12} = \gamma_{13} = \gamma_{14} = \frac{\Delta}{3}$$

Dispersion Relations



$$\frac{M(\gamma_{12},\gamma_{14})}{\gamma_{12}\,\gamma_{13}} = \oint_C \frac{d\gamma_{12}'}{2\pi i}\,\frac{1}{\gamma_{12}'-\gamma_{12}}\,\frac{M(\gamma_{12}',\gamma_{14})}{\gamma_{12}'(\Delta-\gamma_{14}-\gamma_{12}')} =$$

$$\boxed{\Delta-\frac{\tau_{gap}}{2} < \gamma_{14} < \frac{\tau_{gap}}{2}}$$

fixed

$$= \sum_\tau \frac{\underset{\gamma_{12}=\Delta-\tau/2}{\mathrm{Res}}\, M(\gamma_{12},\gamma_{14})}{(\Delta-\tau/2)\,(\frac{\tau}{2}-\gamma_{14})}\left[\frac{1}{\gamma_{12}-\Delta+\tau/2}+\frac{1}{\gamma_{13}-\Delta+\tau/2}\right]$$

with $\gamma_{13}=\Delta-\gamma_{12}-\gamma_{14}$

Dispersion Relations

$$\frac{M(\gamma_{12},\gamma_{14})}{\gamma_{12}\,\gamma_{13}} = \oint \frac{d\gamma_{12}'}{2\pi i}\,\frac{1}{\gamma_{12}'-\gamma_{12}}\,\frac{M(\gamma_{12}',\gamma_{14})}{\gamma_{12}'\,(\Delta-\gamma_{14}-\gamma_{12}')} =$$

$$\boxed{\Delta-\frac{\tau_{gap}}{2}<\gamma_{14}<\frac{\tau_{gap}}{2}}$$

$$= \sum_{\tau} \frac{\operatorname*{Res}_{\gamma_{12}=\Delta-\tau/2} M(\gamma_{12},\gamma_{14})}{(\Delta-\tau/2)\left(\frac{\tau}{2}-\gamma_{14}\right)} \left[\frac{1}{\gamma_{12}-\Delta+\tau/2}+\frac{1}{\gamma_{13}-\Delta+\tau/2}\right]$$

with $\gamma_{13}=\Delta-\gamma_{12}-\gamma_{14}$

$$\boxed{M(\gamma_{12},\gamma_{14}) = \sum_{\tau,J} C_{\tau,J}^{2}\, Q_J(\gamma_{14})\,\frac{\gamma_{12}\,\gamma_{13}}{(\Delta-\tau/2)\left(\frac{\tau}{2}-\gamma_{14}\right)}\left[\frac{1}{\gamma_{12}-\Delta+\tau/2}+\frac{1}{\gamma_{13}-\Delta+\tau/2}\right]}$$

Crossing: $\gamma_{12}\longleftrightarrow\gamma_{13}$ manifest ✓

$\gamma_{12}\longleftrightarrow\gamma_{14}$ $\Rightarrow$ sum rules $\boxed{M(\gamma_{12},\gamma_{14})-M(\gamma_{14},\gamma_{12})=0}$

Sum rules

For example,
$$\frac{\partial}{\partial y} M\Big(\underbrace{\frac{\Delta}{3}+y}_{\gamma_{12}},\ \underbrace{\frac{\Delta}{3}-y}_{\gamma_{14}}\Big)\Big|_{y=0} = 0$$

$$\Delta < \frac{3}{4}\tau_{gap}$$

$$\Rightarrow \quad \boxed{\sum_{\tau,J} C^2_{\tau,J}\,\alpha_{\tau,J} = 0}$$

Mack polynomials

$$\alpha_{\tau,J} = -\frac{(\tau-\Delta)\,Q_J\left(\frac{\Delta}{3}\right)}{\left(\tau-\frac{4\Delta}{3}\right)^2\left(\tau-\frac{2\Delta}{3}\right)^2} + \frac{\Delta}{3}\,\frac{Q'_J\left(\frac{\Delta}{3}\right)}{\left(\tau-\frac{4\Delta}{3}\right)\left(\tau-\frac{2\Delta}{3}\right)(\tau-2\Delta)}$$

Sum rules

For example, $\quad \left.\frac{\partial}{\partial y} M\left(\frac{\Delta}{3}+y, \frac{\Delta}{3}-y\right)\right|_{y=0} = 0$

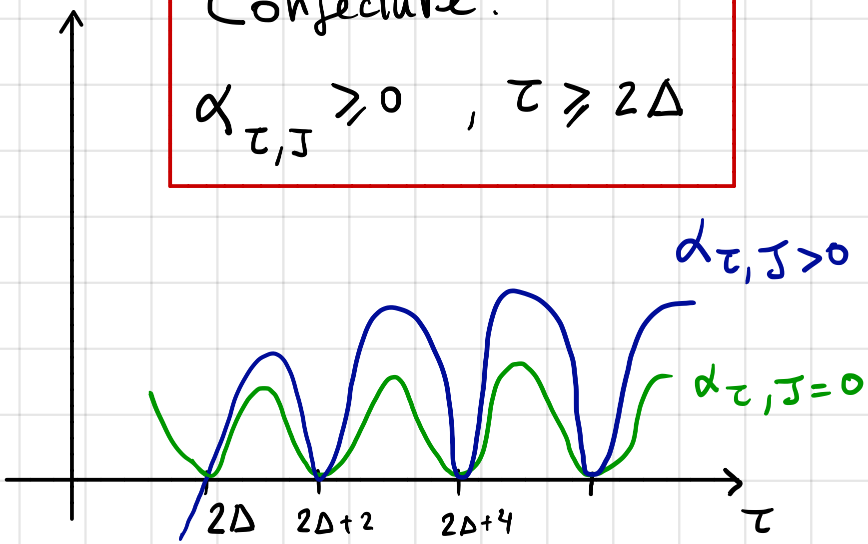
$$\Rightarrow \quad \boxed{\sum_{\tau, J} C^2_{\tau, J}\, \alpha_{\tau, J} = 0}$$

Mack polynomials $\times \dfrac{1}{\Gamma^2(\Delta - \tau/2)}$

$$\alpha_{\tau, J} = -\frac{(\tau - \Delta)\, Q_J\left(\frac{\Delta}{3}\right)}{\left(\tau - \frac{4\Delta}{3}\right)^2 \left(\tau - \frac{2\Delta}{3}\right)^2} + \frac{\Delta}{3} \frac{Q'_J\left(\frac{\Delta}{3}\right)}{\left(\tau - \frac{4\Delta}{3}\right)\left(\tau - \frac{2\Delta}{3}\right)(\tau - 2\Delta)}$$

Conjecture:

$\alpha_{\tau, J} \geq 0 \quad , \tau \geq 2\Delta$



$\Rightarrow$

Extremal Functional

for max $\tau_{gap} = ?$

Solution : GFF $\quad \tau_{gap} = 2\Delta$

Example : 3D Ising

$$\underbrace{\sum_{\text{Leading Regge traj.}} C^2_{\tau,J}\,\alpha_{\tau,J}}_{<0} + \underbrace{\sum_{\text{Rest}} C^2_{\tau,J}\,\alpha_{\tau,J}}_{>0} = 0$$

['16 DSD]

$$<0 = -0.028968\ (T_{\mu\nu}) - 0.012122\ (J=4) - 0.029107\ (6\leq J\leq 30) - 0.0222\ (J>30) = -0.0924$$

$$>0 = +0.084569\ (\epsilon) + 0.0018\ \left([\sigma,\sigma]_{n=1}^{0\leq J\leq 30}\right) + 0.0016\ \left([\epsilon,\epsilon]_{n=0}^{4\leq J\leq 30}\right) + 0.0014\ \left([\epsilon,\epsilon]_{n=0}^{J>30}\right) + \cdots = 0.0894 + \ldots$$

They cancel up to 3% error !

Example : Large N CFTs

single-trace : $C^2_{\phi\phi\mathcal{O}_{st}} \sim \frac{1}{N^2}$

double-trace : $C^2_{\phi\phi\mathcal{O}_{dt}} \sim N^0$ $\quad \tau\left[\mathcal{O}_{dt} = :\phi\, \partial^{2n} \partial^J \phi :\right] = 2\Delta + 2n + \frac{1}{N^2}\gamma(n,J)$

Sum rule at order $\frac{1}{N^2}$:

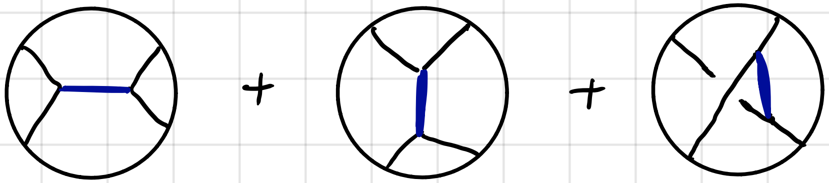
$$\boxed{\underbrace{\sum_{\mathcal{O}_{st}} C^2_{\phi\phi\mathcal{O}_{st}}\, \alpha_{\tau,J}}_{\text{single-trace exchanges}} + \underbrace{\sum_{J=2}^{\infty} C^{2\,\text{GFF}}_{\phi\phi[\phi\phi]^J_{n=0}} \frac{\gamma(0,J)}{N^2} \left.\frac{\partial \alpha_{\tau,J}}{\partial \tau}\right|_{\tau=2\Delta}}_{\text{Leading double-Trace trajectory}} + \underbrace{\left(\text{Rest}_{UV} \geq 0\right)}_{\text{Heavy operators}} = O\left(\frac{1}{N^4}\right)}$$

$$\Rightarrow \quad H\left[\left\{\tau, J, C^2\right\}_{st}, \left\{\gamma(0,J)\right\}_{dt}\right] \leq 0$$

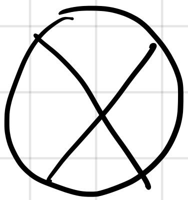
$\uparrow$ Calculable from EFT in AdS

EFTs in AdS

- $\lambda \varphi^2 \chi$ $\qquad \lambda \sim \frac{1}{N}$ $\qquad \Delta < \frac{3}{4}\Delta_\chi$

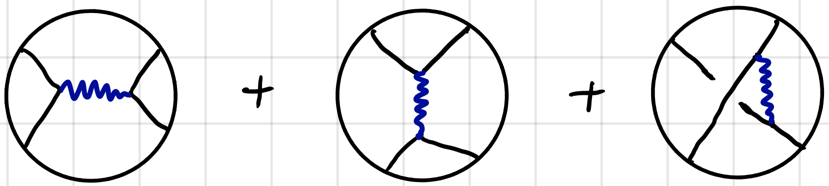


$\rightarrow \quad H = 0$



$\lambda(\partial\varphi)^4 \qquad H \propto -\lambda$

- Minimally coupled scalar $\qquad G_N \sim \frac{1}{N^2}$



$\rightarrow \quad H < 0$

$$\frac{d-2}{2} < \Delta < \frac{3}{4}(d-2)$$

Open Questions

- Prove the Maximal Mellin Analyticity conjecture
- Prove Regge bound
- Construct a basis of extremal functionals
- More efficient numerical bootstrap?
- Extend beyond $\boxed{\Delta < \frac{3}{4}\tau_{gap}}$
- Extend to spinning and non-identical operators
- Higher point functions
- BCFT
- Connection to S-matrix via Flat Space limit
- Connection to Polyakov-Mellin bootstrap

Thank you!