Chiral Boundary States for Fermions

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Based on work with Philip Boyle Smith
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What’s to Come

1. Review of the Mod 2 Anomaly
2. Chiral Boundary States
3. Boundary RG Flows
4. The $\mathbf{Z}_8$ Classification of d=2+1 SPTs
5. An Open Problem: ‘t Hooft Lines

The Mod 2 Anomaly

The Mod 2 Anomaly

A single, quantum mechanical Majorana fermion, λ, is inconsistent:

$$Z_{\mathrm{Majorana}} = \sqrt{2}$$

But there is no problem with two Majorana fermions, or a Dirac fermion:

$$\{\lambda_i, \lambda_j\} = \delta_{ij}$$

and $$Z_{\mathrm{Dirac}} = Z^2_{\mathrm{Majorana}} = \mathrm{Tr}(\mathbb{1}) = 2$$

The Mod 2 Anomaly in Boundary Conditions

Consider a *massive* Majorana fermion χ in d=1+1 dimensions in the presence of a boundary. We have two choices of boundary conditions:

$$\chi_L = \pm \chi_R$$

One such choice has a localised zero mode; the other does not

$$\chi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \exp\left(\mp m x\right) \chi(0)$$

The choice with the zero mode has an anomalous boundary theory.

Kitaev 2000; Dijkgraaf and Witten 2018

The Mod 2 Anomaly in Boundary Conditions

Consider a *massive* Dirac fermion, $\psi = \chi_1 + i\chi_2$, with the following choices of boundary conditions:

$$\psi_L = \psi_R$$

Here there are either two zero modes or none.

$$\psi_L = \psi_R^\dagger$$

c.f. Andreev reflection

Here there is a single Majorana zero mode.

Note: The anomaly is independent of the mass.

The Mod 2 Anomaly in Boundary Conditions

Ways to cancel the anomaly:

- Put the same boundary condition on each end of an interval
    - But you can’t put different choices on different ends.
- Add an extra boundary Majorana mode in by hand.
- Anomaly inflow through the Arf SPT phase.

c.f. a beautiful D-brane story: Witten ‘18, Kaidi, Parra-Martinez and Tachikawa, ‘19

Multiple Fermions

Chiral Boundary Conditions

Consider 2*N massless* Majorana fermions or, equivalently, N Dirac fermions in d=1+1 dimensions.

Question:
- What symmetries can be preserved by boundary conditions?

Answer:
- Any that don’t suffer a ‘t Hooft anomaly.

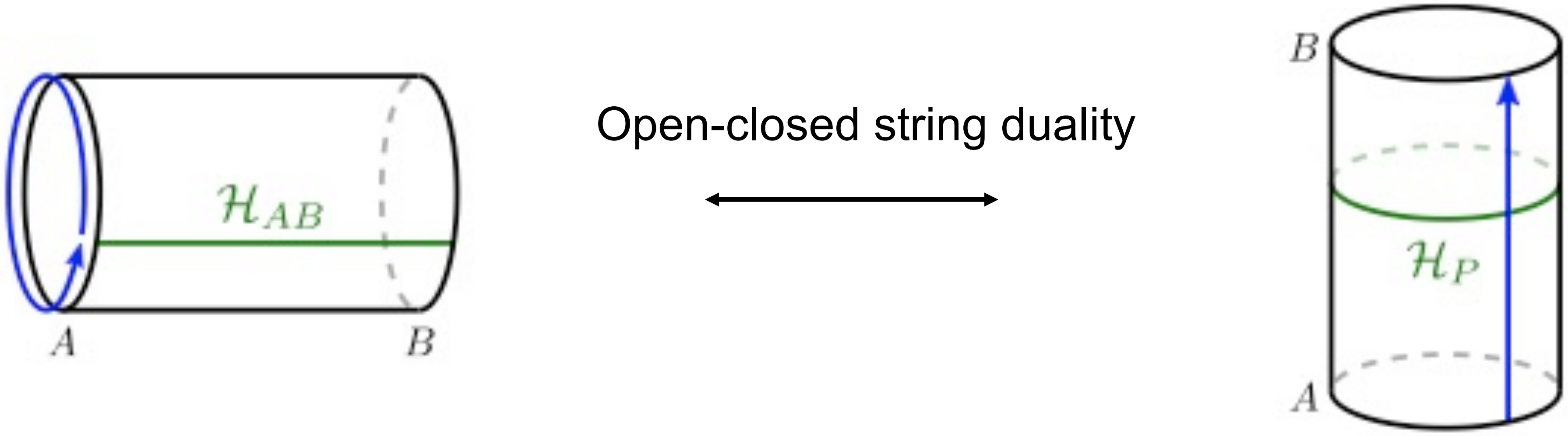
e.g. a *U(1)* symmetry that acts on *N* Dirac fermions with charges obeying

$$\text{left-movers} \longrightarrow \sum_{i=1}^{N} Q_i^2 = \sum_{i=1}^{N} \bar{Q}_i^2 \longleftarrow \text{right-movers}$$

Chiral Boundary States

In general, these boundary conditions cannot be implemented directly on the fermion fields.

We could introduce new boundary degrees of freedom, but at low energies the relevant physics is captured by *boundary state*.



Open-closed string duality

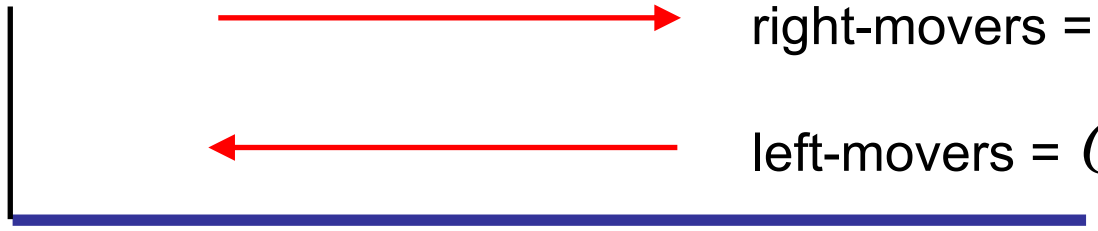
Boundary conditions captured by states $|A\rangle$ and $|B\rangle$

Cardy '89

Chiral Boundary Conditions

We consider boundary conditions that preserve a *U(1)*^{N} symmetry.

Charges:



right-movers = $\bar{Q}_{\alpha i}$

left-movers = $Q_{\alpha i}$

$\alpha = 1, \ldots, N$

$i = 1, \ldots, N$

No ‘t Hooft anomalies ⟺ $\displaystyle\sum_{i=1}^{N} Q_{\alpha i} Q_{\beta i} = \sum_{i=1}^{N} \bar{Q}_{\alpha i} \bar{Q}_{\beta i}$

Chiral Boundary Conditions

It turns out that the boundary state is specified by a rational, orthogonal matrix

$$\mathcal{R}_{ij} = (\bar{Q}^{-1})_{i\alpha} Q_{\alpha j}$$

e.g. $\psi_L = \psi_R$ for each fermion gives $\mathcal{R} = \mathbb{1}$

$\psi_L = \psi_R^\dagger$ for each fermion gives $\mathcal{R} = -\mathbb{1}$

The Charge Lattice

It turns out that the boundary state is specified by a rational, orthogonal matrix

$$\mathcal{R}_{ij} = (\bar{Q}^{-1})_{i\alpha} Q_{\alpha j}$$

All our results are expressed in terms of the following *charge* lattice

$$\Lambda[\mathcal{R}] = \left\{ \ \lambda \in \mathbb{Z}^N \ : \ \mathcal{R}\lambda \in \mathbb{Z}^N \ \right\} = \mathbb{Z}^N \cap \mathcal{R}^{-1}\mathbb{Z}^N$$

A Quick Look at the Boundary State

$$\boxed{|\theta;\mathcal{R}\rangle = g_{\mathcal{R}} \sum_{\lambda\in\Lambda[\mathcal{R}]} e^{i\gamma_{\mathcal{R}}(\lambda)}\, e^{i\theta\cdot\lambda} \| \lambda, \bar{\lambda} = -\mathcal{R}\lambda\rangle\!\rangle}$$

with the Ishibashi states given by

$$\|\lambda,\bar{\lambda}\rangle\!\rangle = \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathcal{R}_{ij}\bar{J}_{i,-n}J_{j,-n}\right)|\lambda,\bar{\lambda}\rangle$$

ground states labelled by left and right-moving charges

See, e.g. Recknagel and Schomerus, ‘98;
Cho, Shiozaki, Ryu and Ludwig ’16;
Han, Tiwari, Hsieh and Ryu ’17

The Boundary Central Charge

The normalization $g_{\mathcal{R}} = \langle 0,0\,|\,\theta;\mathcal{R}\rangle$ is important. This is the boundary central charge, first introduced by Affleck and Ludwig in 1991. It captures the boundary contribution to the free energy.

$$\boxed{g_{\mathcal{R}} = \sqrt{\mathrm{Vol}(\Lambda[\mathcal{R}])}}$$

i.e. the volume of the primitive unit cell of the lattice*

Boyle Smith and Tong ’19

e.g.
- $\mathcal{R} = \pm\mathbb{1} \implies g_{\mathcal{R}} = 1$
- *N*=*2* fermions with charges $a^2 + b^2 = c^2 + 0 \implies g_{\mathcal{R}} = \sqrt{c}$

(*This same formula arose as the tension of a D-brane in Bachas, Brunner and Roggenkamp ‘12.)

To Which $\mathbf{Z}_2$ SPT Phase Does Each State Belong?

Consider imposing a boundary state $\mathcal{R}$ on one end of an interval, and $\mathcal{R}'$ on the other.

The number of ground states of the system is given by

$$G[\mathcal{R},\mathcal{R}'] = \frac{\sqrt{\mathrm{Vol}(\Lambda[\mathcal{R}])\,\mathrm{Vol}(\Lambda[\mathcal{R}'])}}{\mathrm{Vol}(\Lambda[\mathcal{R},\mathcal{R}'])}\,\sqrt{\mathrm{det}'(\mathbb{1}-\mathcal{R}^T\mathcal{R}')}$$

It can be shown that,

$$G[\mathcal{R},\mathcal{R}'] \in \begin{cases} \mathbb{Z} & \text{if same class} \\ \sqrt{2}\mathbb{Z} & \text{if different class} \end{cases}$$

Boyle Smith and Tong ’19

Flows Between Boundary States

Boundary RG Flows

The d=0+1 boundary behaves, in many ways, like any other QFT. Boundary operators are classified as:

- Irrelevant
- Marginal ⇨ moves among different boundary conditions.
- Relevant ⇨ induces an RG flow to a new boundary condition.

The g-theorem: the boundary central charge decreases under RG.

Affleck and Ludwig '91, Friedan and Konechny '03, Casini, Landea and Torroba '16

Relevant Operators

We can use the state-operator map to determine the boundary operators. They are labelled by:

$$\rho \in \Lambda[\mathcal{R}]^{\star}$$

This determines their dimension and charges

$$L_0 = \frac{1}{2}\rho^2 \quad \text{and} \quad \mathcal{Q}_\alpha = Q_{\alpha i}\rho_i$$

If you turn on a relevant operator, where do you flow?

Boundary RG Flows

We turn on a relevant boundary operator, labelled by

$$\rho \in \Lambda[\mathcal{R}]^\star$$

This breaks the symmetry:

$$U(1)^N \rightarrow U(1)^{N-1}$$

Assumption:

At the end of the RG flow, we restore a (typically different) $U(1)^N$ symmetry

Boundary RG Flows

We turn on a relevant boundary operator, labelled by

$$\rho \in \Lambda[\mathcal{R}]^\star$$

Assuming that we land among our general class of boundary states, there is a unique candidate:

$$(\mathcal{R}_{IR})_{ik} = (\mathcal{R}_{UV})_{ij} \left(\delta_{jk} - \frac{2}{\rho^2} \rho_j \rho_k \right)$$

One might naively think that the IR central charge is

$$g_{\mathrm{naive}} = \sqrt{\mathrm{Vol}(\Lambda[\mathcal{R}_{IR}])}$$

…but this is too quick.

Boundary RG Flows

There are a number of subtleties that we must address:

1. The RG flow can move us from one SPT phase to the other! This is only consistent if the IR boundary state is accompanied by an extra Majorana mode. This increases the central charge by $\sqrt{2}$.

2. We could start with an extra Majorana mode and use it to dress a *fermionic* operator to initiate an RG flow.

3. The RG flow may preserve a discrete symmetry

$$U(1)^N \rightarrow U(1)^{N-1} \times \mathbb{Z}_n$$

Typically the IR boundary state does not. We must then sum over the images. This again increases the central charge.

Boundary RG Flows

Once these subtleties are taken into account, we find the following simple result:
We perturb by an operator $\mathcal{O}$. Then the infra-red central charge is

$$\boxed{g_{IR} = g_{UV} \sqrt{\dim \mathcal{O}}}$$

Note: this immediately satisfies the g-theorem.

The $\mathbf{Z}_8$ Classification of d=2+1 SPT Phases

The View from the Edge

We could also view our d=1+1 fermions as the edge of a d=2+1 SPT phase.

There is a **Z**$_8$ classification which means that 8 Majorana fermions in d=1+1 can be gapped preserving, chiral fermion parity $(-1)^L \times (-1)^R$.

Fidkowski and Kitaev ‘09, Ryu and Zhang ‘12, Qi ‘12

This, in turn, means that it should be possible to construct a boundary condition for 8 Majorana fermions preserving $(-1)^L \times (-1)^R$.

The simplest such boundary state was constructed by Maldacena and Ludwig in 1995, and, in addition, preserves an SO(8) symmetry:

$$\mathcal{R}_{ij} = \delta_{ij} - \frac{1}{2}$$

(and recognized as such by Cho, Shiozaki, Ryu and Ludwig ‘16)

The **Z**$_8$ Classification

Which of our boundary states preserve $(-1)^L \times (-1)^R$?

Can show:

$$(-1)^L|\lambda,\bar{\lambda}\rangle = (-1)^{\lambda^2}|\lambda,\bar{\lambda}\rangle$$

$$(-1)^R|\lambda,\bar{\lambda}\rangle = (-1)^{\bar{\lambda}^2}|\lambda,\bar{\lambda}\rangle$$

i.e. preservation if the lattice $\Lambda[\mathcal{R}]$ is even

Claim: This can only happen if the lattice has dimension $N = 4k$

Claim: The stable boundary state with this property has $g_{\mathcal{R}} = \sqrt{\frac{N}{2}}$

An Open Question

‘t Hooft Lines in Chiral Gauge Theories

- ‘t Hooft lines in d=3+1 are defined by boundary conditions for fields
- Decompose fermions in angular momentum modes to reduce to d=1+1
- The lowest angular momentum modes of chiral fermions require chiral boundary conditions

Callan ’83, Polchinski ‘84

Affleck and Sagi ‘93; Maldacena and Ludwig ‘95

‘t Hooft Lines in Chiral Gauge Theories

e.g. U(1) gauge theory with Weyl fermions with charges 1, 5, -7, -8, 9

⇨ chiral boundary conditions preserving $SU(2)_{\rm rot} \times U(1)$ with

Left movers: $\mathbf{1}_1,\ \mathbf{5}_5,\ \mathbf{9}_9$

Right movers: $\mathbf{7}_7,\ \mathbf{8}_8$

Boundary state unknown!

Thank you for your attention