Scrambling *vs* chaos

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Chaos/butterfly effect: exponential sensitivity to initial perturbation

$$\left| \frac{\partial q_i(t)}{\partial q_j(0)} \right| \sim e^{\lambda_1 t}$$

λ_1 **(largest) Lyapunov exponent**

More generally, in a $2N$-dimensional phase space,

$$\left(\frac{\partial q_i(t)}{\partial q_j(0)} \right)_{ij} \xrightarrow{\text{SVD}} \lambda_1, \lambda_2, \ldots, \lambda_{2N}$$

Lyapunov Spectrum

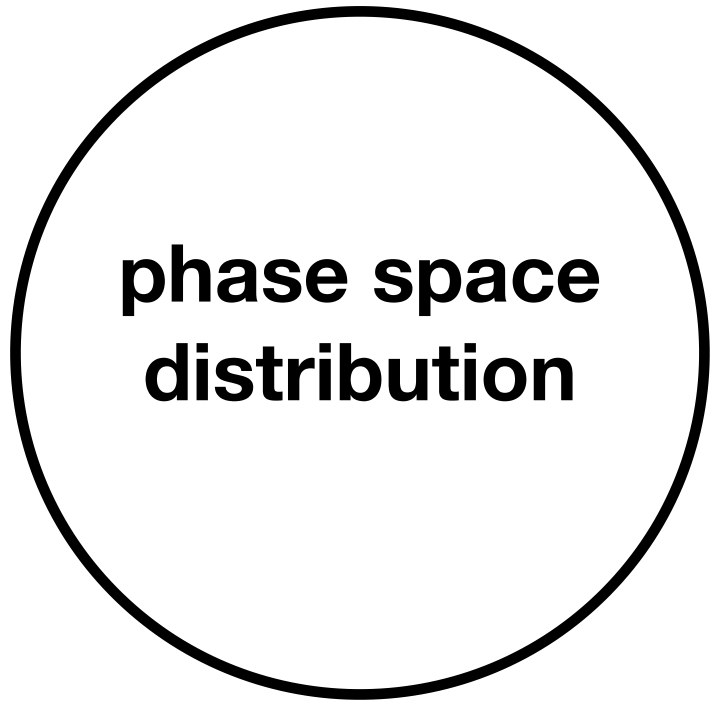
Chaos/butterfly effect

$$\left(\frac{\partial q_i(t)}{\partial q_j(0)}\right)_{ij} \xrightarrow{\text{SVD}} \lambda_1, \lambda_2, \ldots, \lambda_{2N}$$

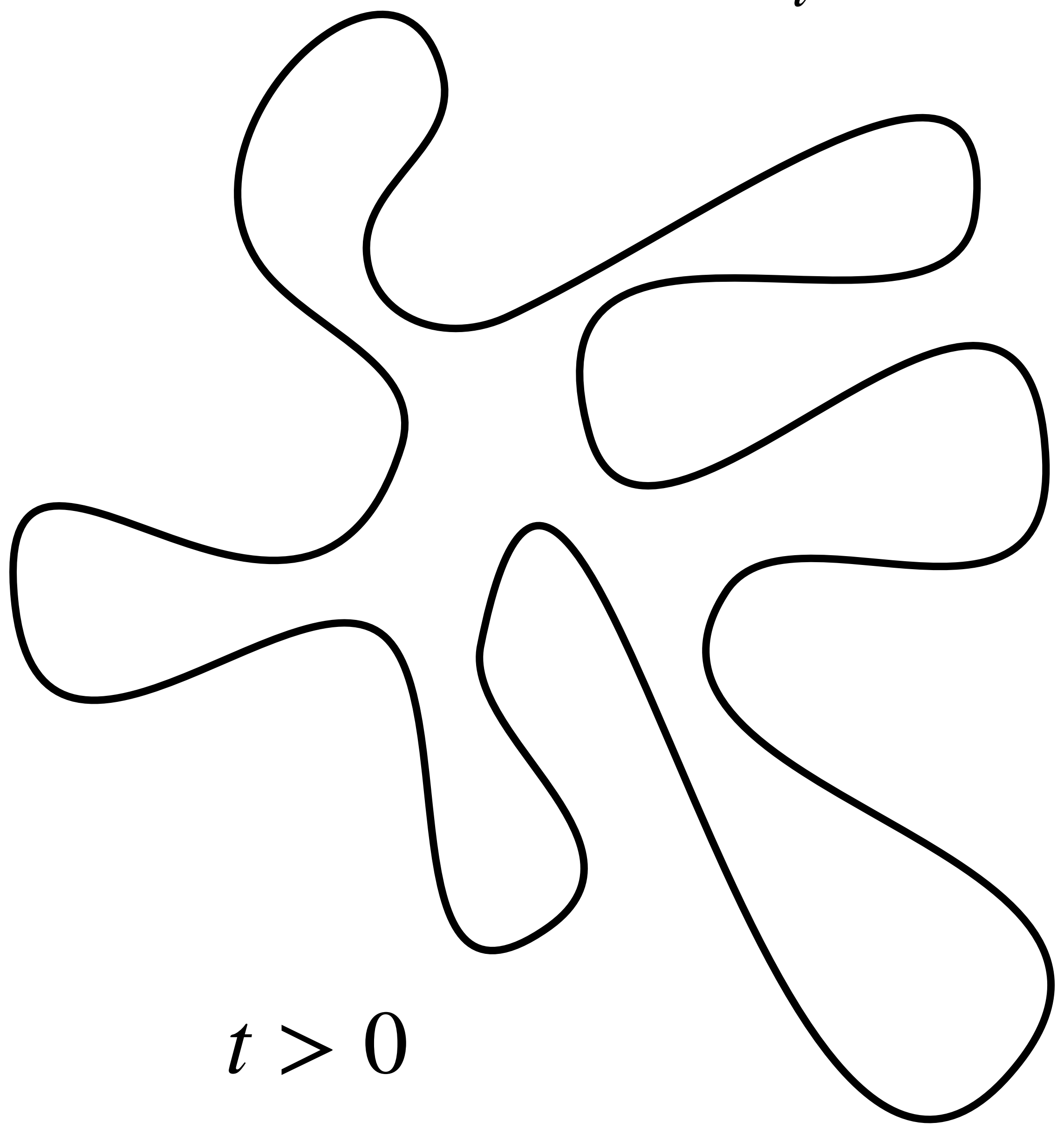
Liouville theorem $\Rightarrow \sum_i \lambda_i = 0$

Thermalization (Entropy growth)

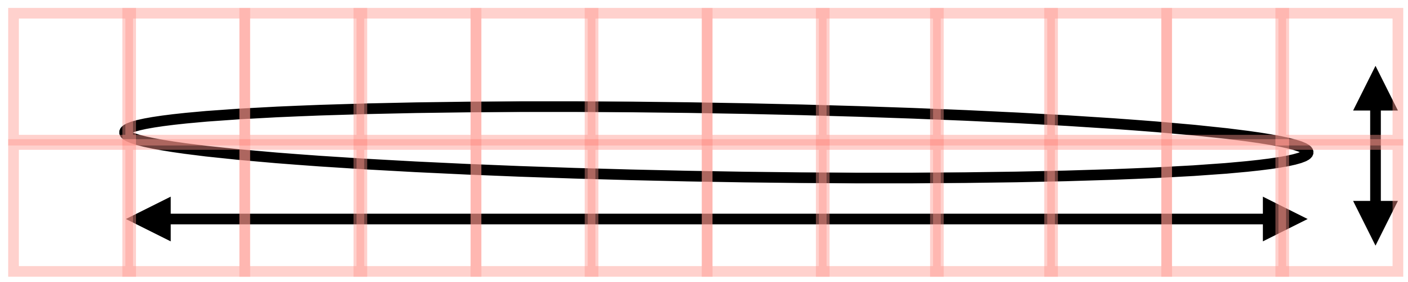
$$\frac{dS}{dt} = \sum_{\lambda_i > 0} \lambda_i =: \Lambda_+$$



$t = 0$



$t > 0$

$\sim$  $e^{-\Lambda_+ t}$

$e^{\Lambda_+ t}$

With fixed resolution,

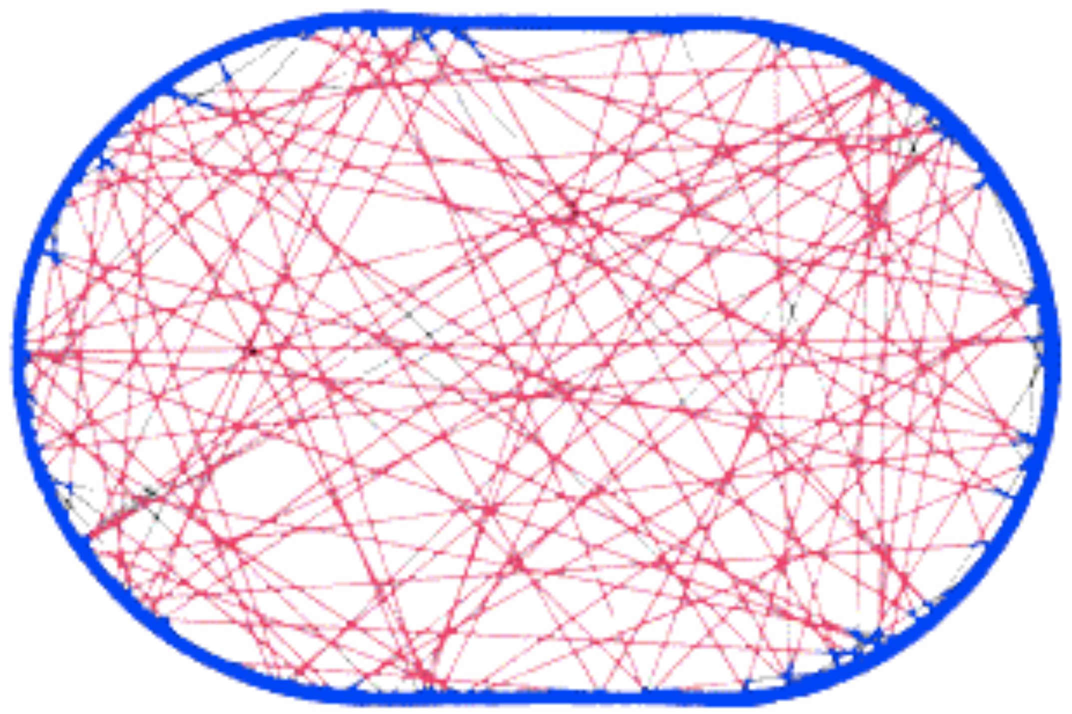
$V \sim e^{\Lambda_+ t} \Rightarrow S \sim \Lambda_+ t$

[Latora,Baranger, PRL 99]

Butterfly effect as (numerical) diagnostics of integrability

Non-integrable

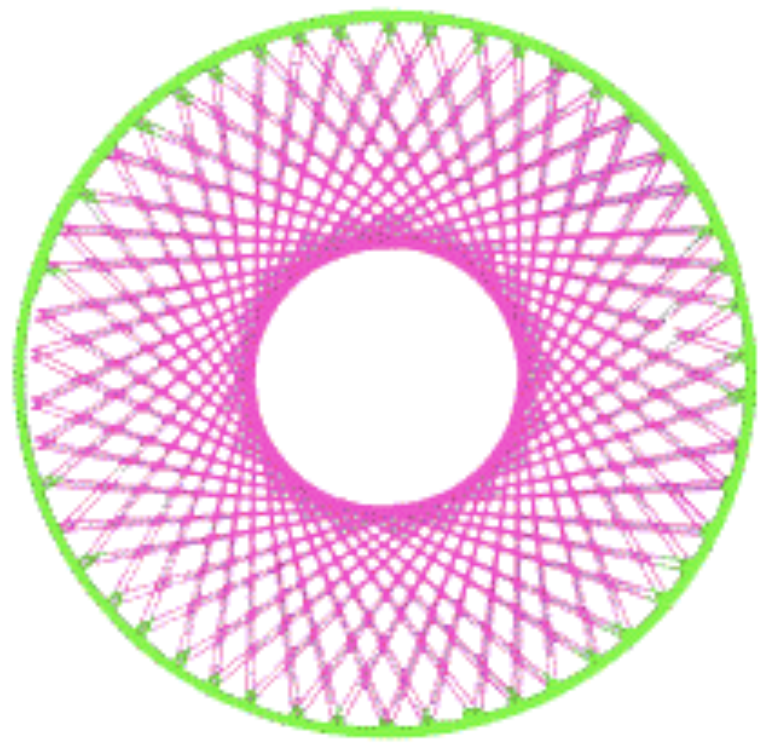
$$\left|\frac{\partial q_i(t)}{\partial q_j(0)}\right| \sim e^{\lambda_L t}\,, \lambda_L > 0$$



Integrable

$$\left|\frac{\partial q_i(t)}{\partial q_j(0)}\right| \sim t^{\alpha}\,, \lambda_L = 0$$

(Regular motion on invariant tori)



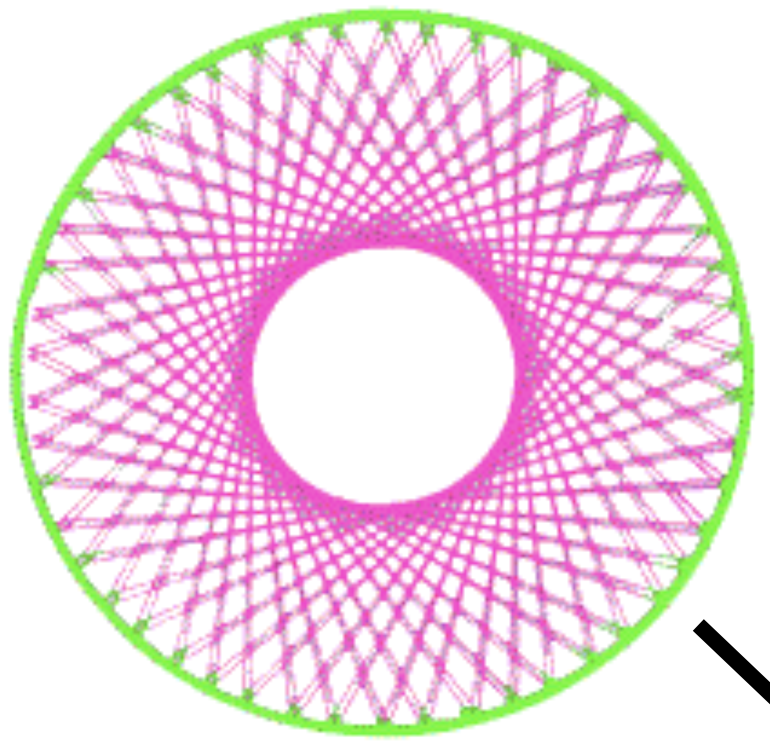
Note: a saddle point is not chaotic!
[exponential sensitivity only for $t \leq \mathcal{O}(1)$]

"Old-school" quantum chaos
Casati, Berry-Tabor, Bohigas-Giannoni-Schmit

Level statistics

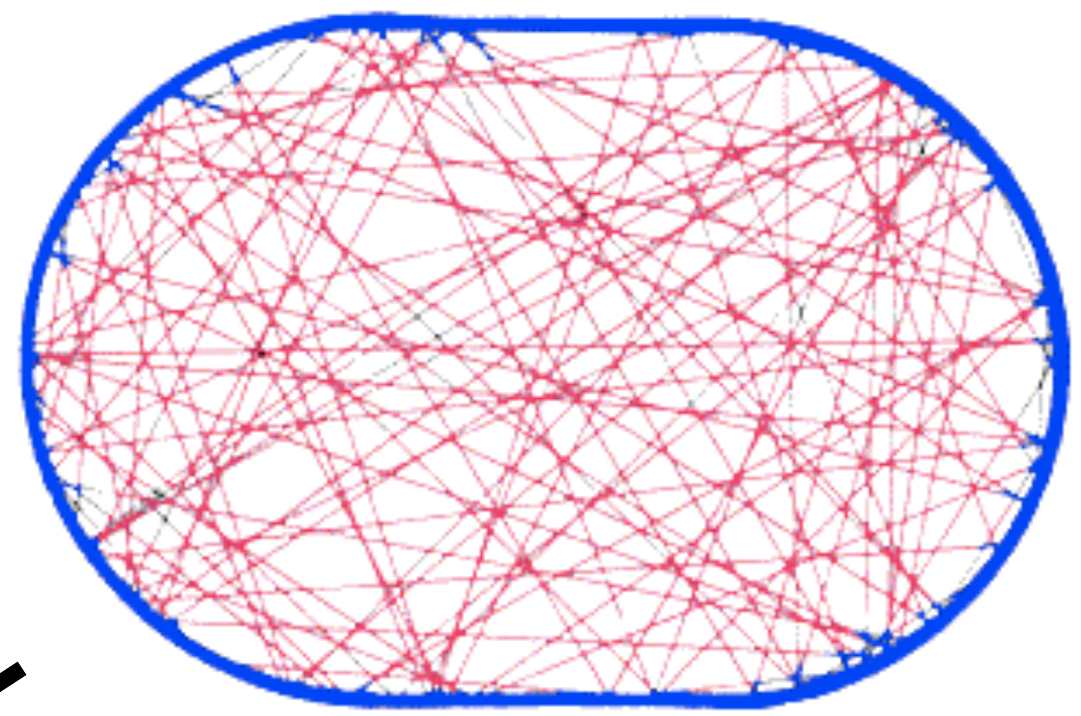
$$H|n\rangle = E_n|n\rangle$$

related: spectral form factor



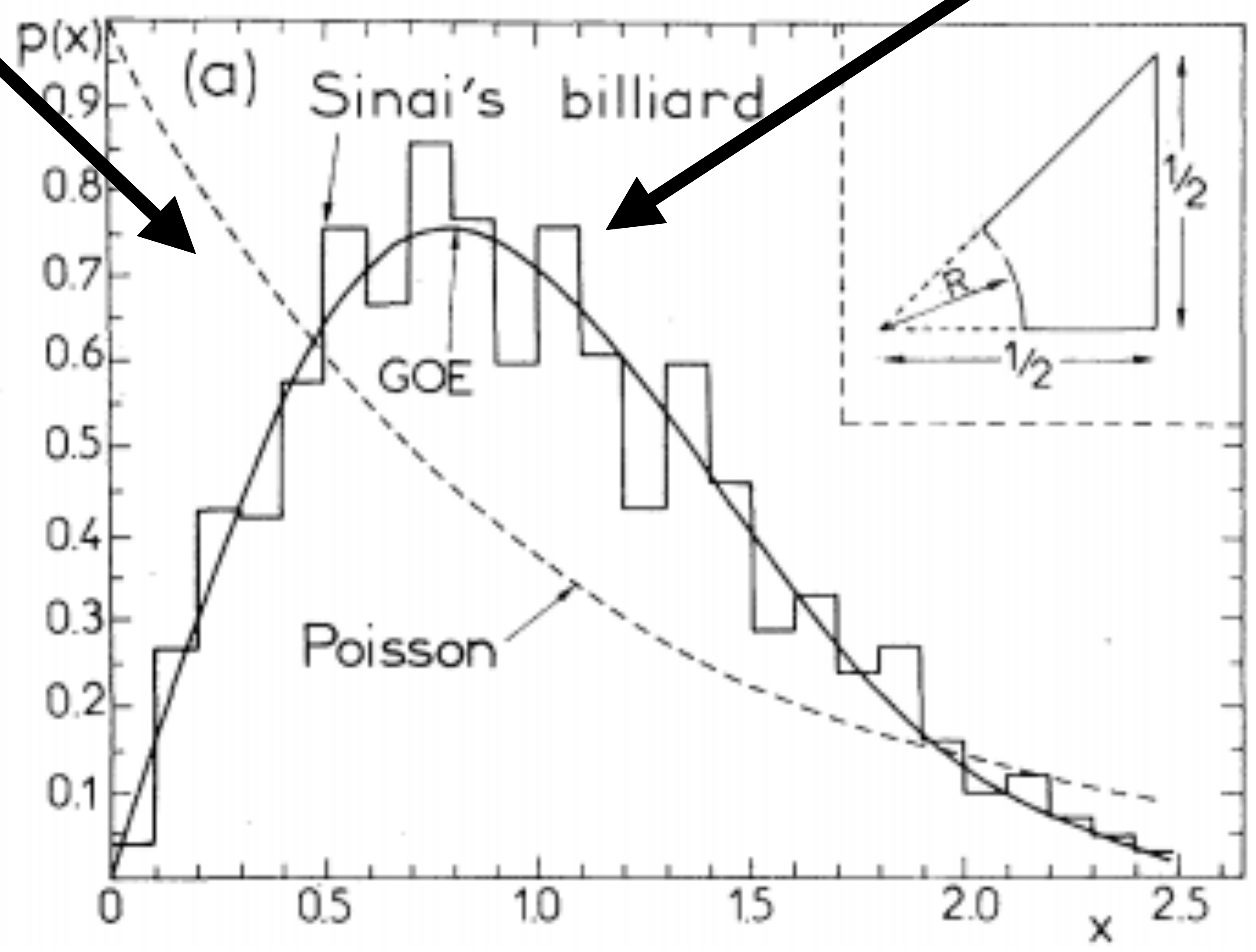
Integrable

Poisson statistics



Non-integrable

Random matrix theory



$E_n - E_{n-1}$

Powerful diagnostics

But: no λ_L (different time-scale)

Defining quantum λ_L (butterfly effect)

Larkin, Ovchinnikov
Maldacena, Shenker, Stanford

Out-of-time-order correlators

$$C(t) := \left\langle [V(t), W]^\dagger [V(t), W] \right\rangle$$

$\langle \dots \rangle$ Some expectation value

Semiclassical intuition:

$$\partial q(t)/\partial q(0) = \{q(t), p\}_{\text{P.B.}} \approx \frac{1}{i\hbar}[q(t), p]$$

In this talk, “scrambling” means
exponential growth of OTOCs

$$C(t) := \left\langle [V(t), W]^\dagger [V(t), W] \right\rangle \sim e^{\lambda_L t}$$

Does scrambling equal chaos?

(and does it have similar diagnostics power?)

I. In the classical limit (w/ Xu, Scaffidi)

II. In the large-N limit (w/ Kim, Altman)

III. Away from any limits?

Classical limit: Scrambling $\supsetneq$ chaos

$$C(t) = \left\langle [q(t), p][q(t), p]^{\dagger} \right\rangle = \int \left(\frac{\partial q(t)}{\partial q} \right)^2 \rho(q)$$

$\hbar = 1$

$$= \int \rho(q) e^{\lambda_L(q) t} \geq e^{\langle \lambda_L \rangle t} = e^{2\lambda_1 t}$$

[Galitski et. al.]

$$\lambda_L \geq 2\lambda_1$$

Chaos phase-space-averages over the log.

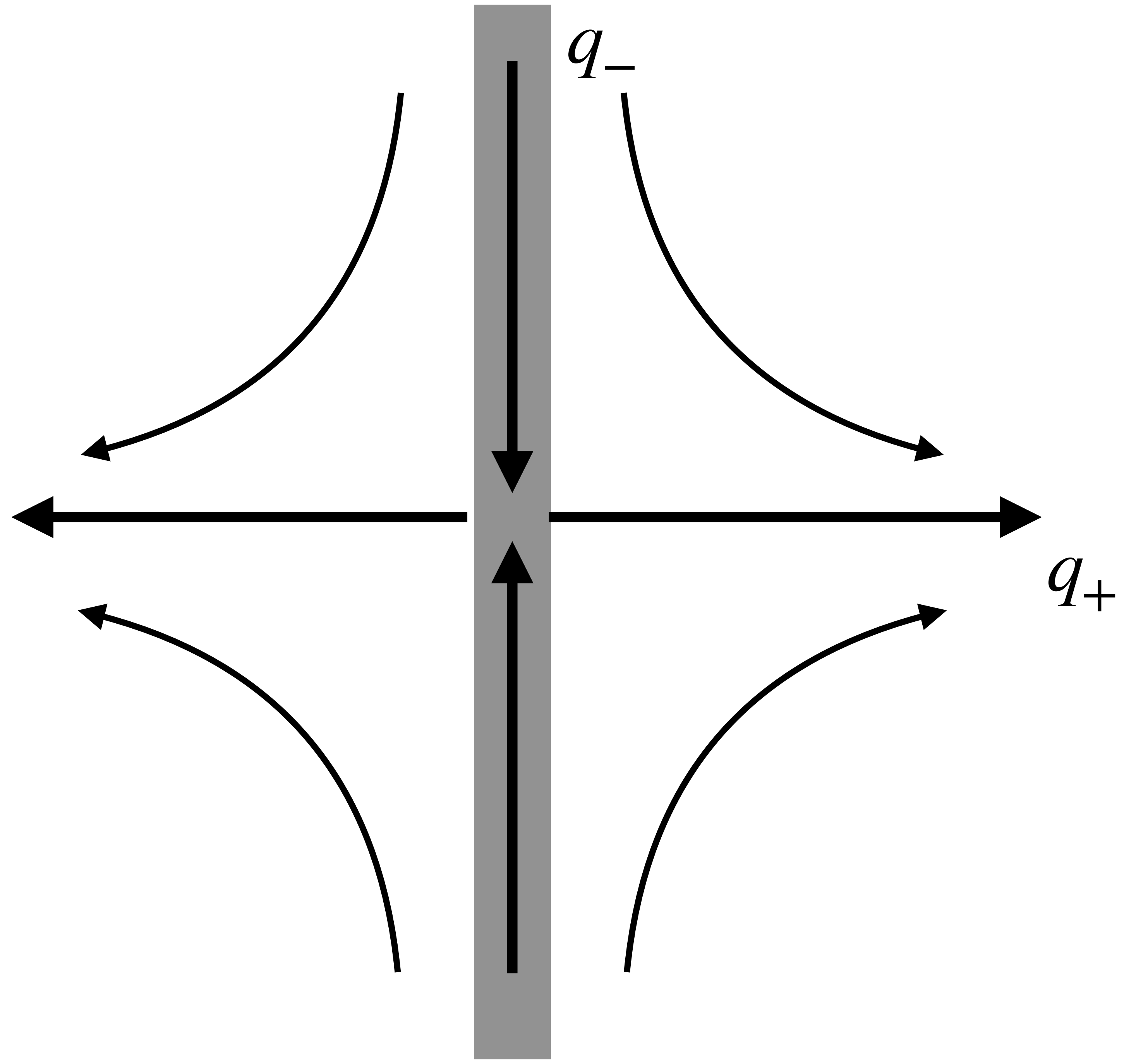
Scrambling averages before taking the log.

Quantitative detail?

Qualitative difference!

Scrambling can result from mechanisms other than chaos, e.g., saddle points.

Demo (2d phase space)



$$q_{\pm}(t) = e^{\pm \mu t} q_{\pm}$$ **(locally near a saddle)**

$$C(t) \geq \int_{q_+ \leq \epsilon} e^{2\mu t} dq_+ dq_- \sim \epsilon e^{2\mu t} = e^{\mu t}$$

$$\epsilon \sim e^{-\mu t}$$

So that the trajectory stay close at time t.

Hence, $\lambda_L \geq \mu$ without chaos!

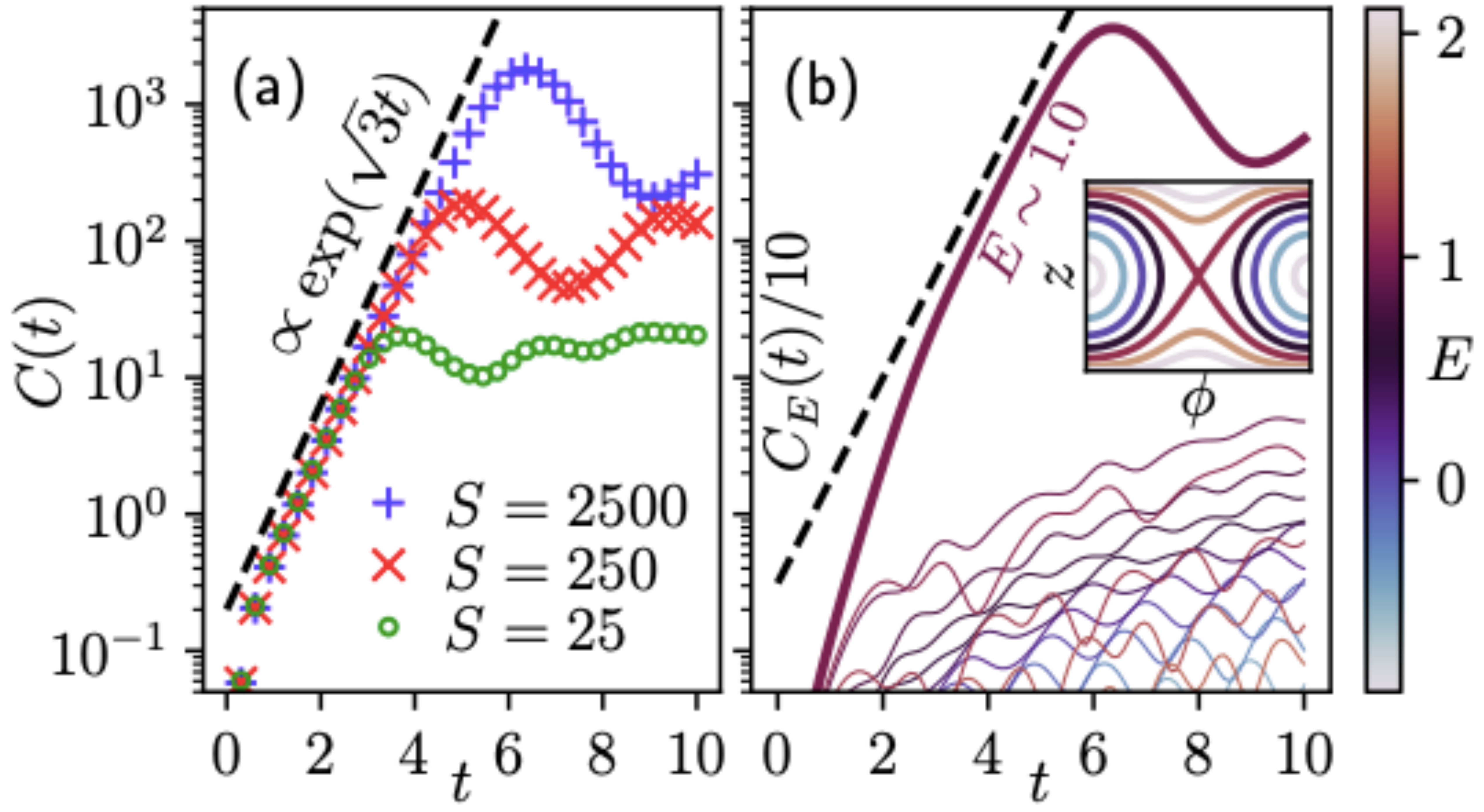
Finite T: [Hashimoto, Huh, Kim, Wanatabe]

Example: Lipkin-Meshkov-Glick model

$$H = x + 2z^2, \{x, y\} = z, \ldots$$

2d phase space, trivially integrable, but has a saddle with $\mu = \sqrt{3}$.

Saddle-dominated scrambling
$\lambda_L = \mu$

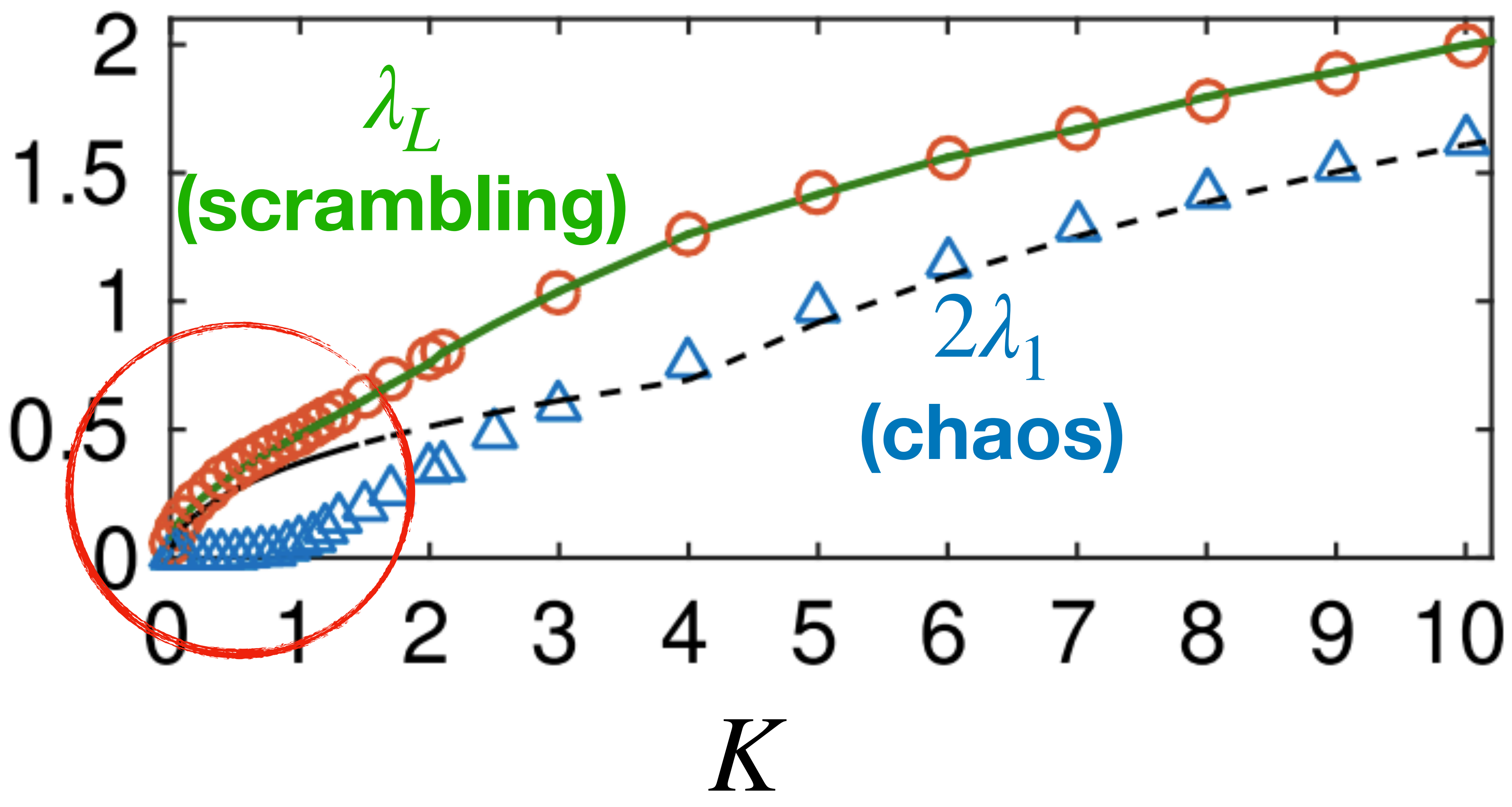


Infinite T OTOC **Microcanonical OTOCs**

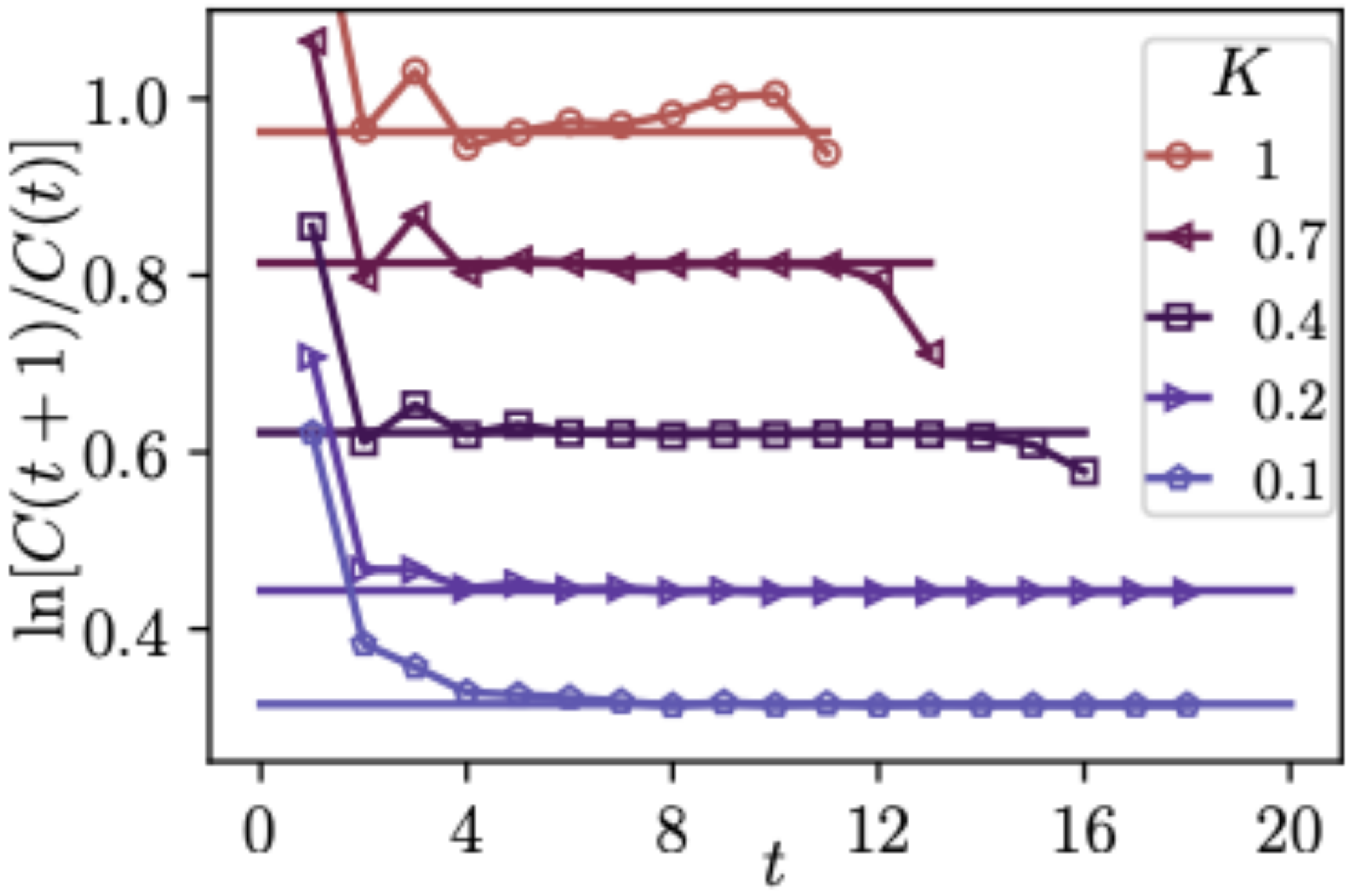
OTOCs are computed in quantized model, with $\hbar = 1/S$. The exponential growth saturates at Ehrenfest time $t_E \sim \ln S$.

Example: Kicked rotor

$$x, p \mapsto x + p, p + K \sin(x) \qquad (x = x + 2\pi)$$



$K \lesssim 1$: **scrambling without classical chaos**



$$\lambda_L = \mu$$

Saddle ($x, p = 0{,}0$)-dominated scrambling

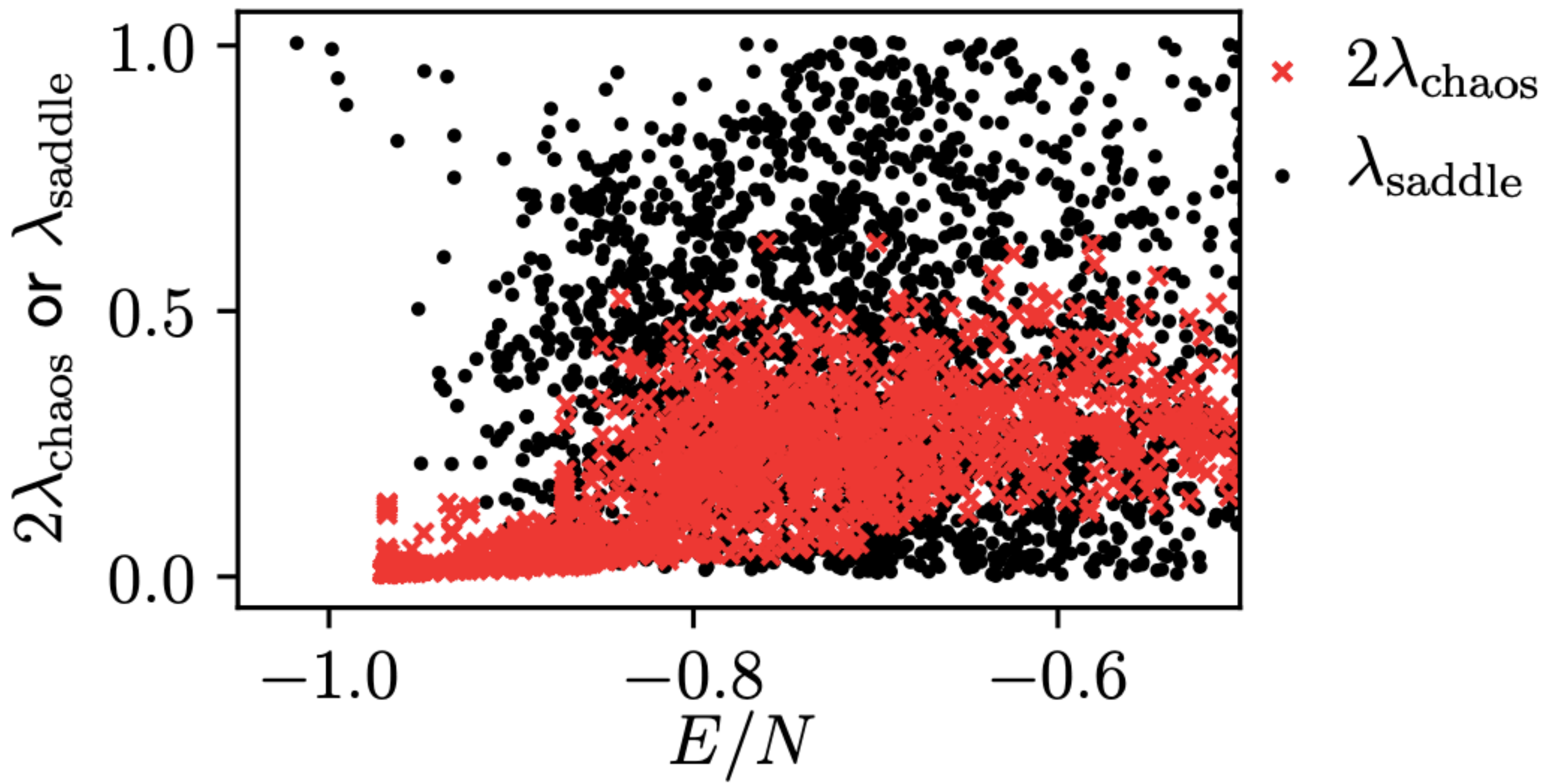
[Rozenbaum,Ganeshan,Galitski, PRL17]

Remark

Saddle-dominated scrambling can occur

- In higher dimension/many-body phase space;
- In presence of chaos.

It remains unclear how generally that happens.



Scrambling in large-N, low-T

Example: Sachdev-Ye-Kitaev

$$H = \sum_{ijkl=1}^{N} J_{ijkl}\gamma_i\gamma_j\gamma_k\gamma_l \qquad C(t) = \left\langle \{\gamma_i(t),\gamma_j\}\{\gamma_i(t),\gamma_j\} \right\rangle_T \sim e^{\lambda_L t}\,, t \lesssim \ln N$$

$T \ll J$: $\lambda_L = 2\pi T$, "fast scrambling"

What are some other behaviors? How do $\lambda_L(T)$ depend on the IR fixed point?

$\gamma_i\gamma_j + \gamma_j\gamma_i = \delta_{ij}\,, \overline{J_{ijkl}J_{i'j'k'l'}} = \frac{J^2}{N^3}\delta_{ijkl,i'j'k'l'}$

[Kitaev][Maldacena,Shenker,Stanford], …

Another example: mass-deformed SYK

$$H = \sum_{ij=1}^{N} \kappa_{ij}\, i\gamma_i\gamma_j + \sum_{ijkl=1}^{N} J_{ijkl}\gamma_i\gamma_j\gamma_k\gamma_l$$

Relevant perturbation, resulting in weakly-coupled IR fixed point ($\Delta_\gamma = 1/2 \neq \Delta_{\gamma,SYK} = 1/4$)

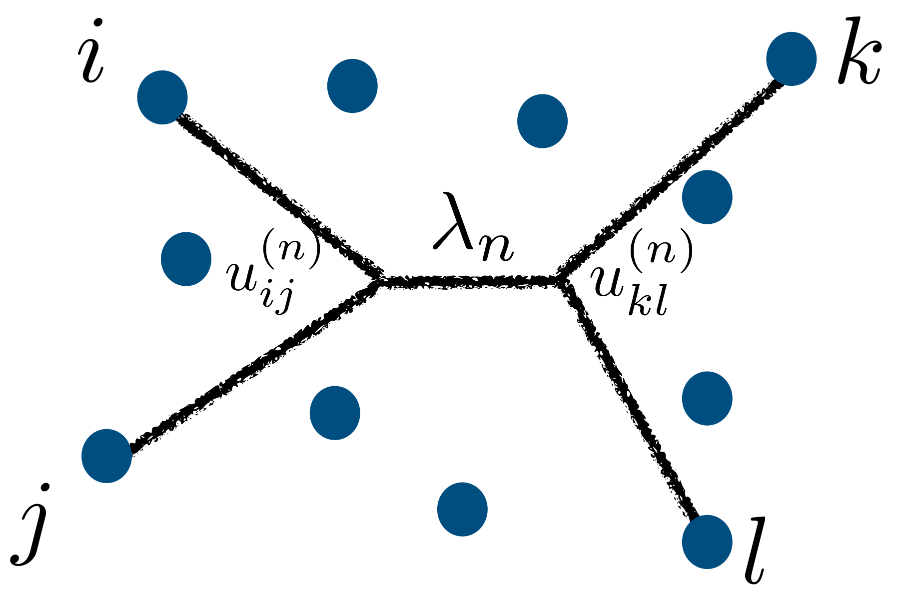
$$\lambda_L(T) = ?$$

García-García, Loureiro, Romero-Bermúdez, Tezuka (PRL 18)
$\lambda_L = 0, T < T_c$ [transition to no scrambling]

Banerjee, Altman (PRB 17, similar model)
$\lambda_L \propto T^2$ [$\ll T$, but non-vanishing]

Can we have a more general understanding by interpolating between IR fixed points?

Low-rank/Yukawa SYK

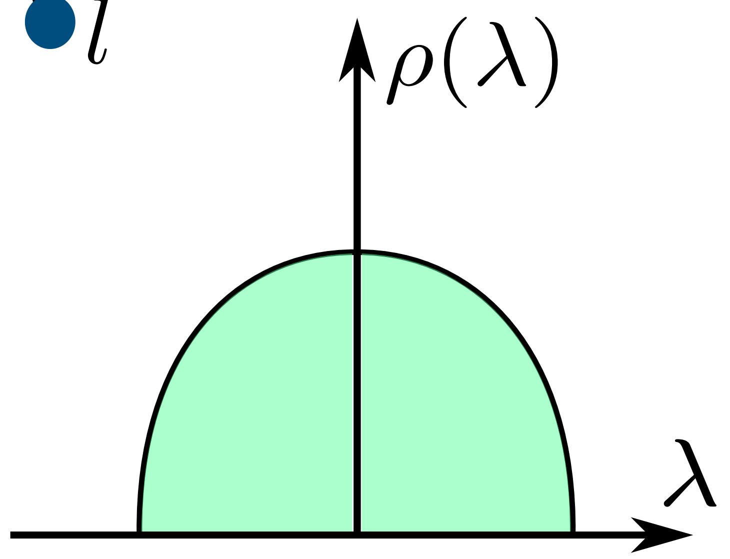


$$H = \sum_{ijkl=1}^{N} J_{ij,kl}\gamma_i\gamma_j\gamma_k\gamma_l \qquad J_{ij,kl} = \sum_{n=1}^{\gamma N} \lambda_n u_{ij,n} u_{kl,n}$$

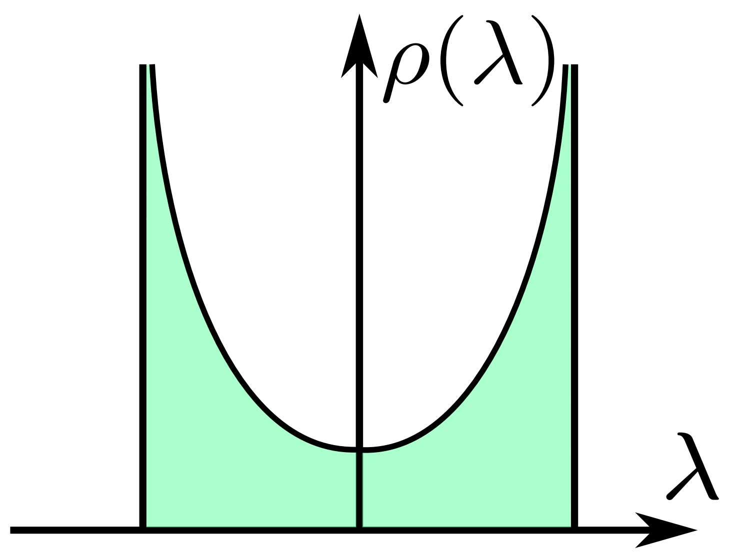
γ

# of mediating bosons per fermion

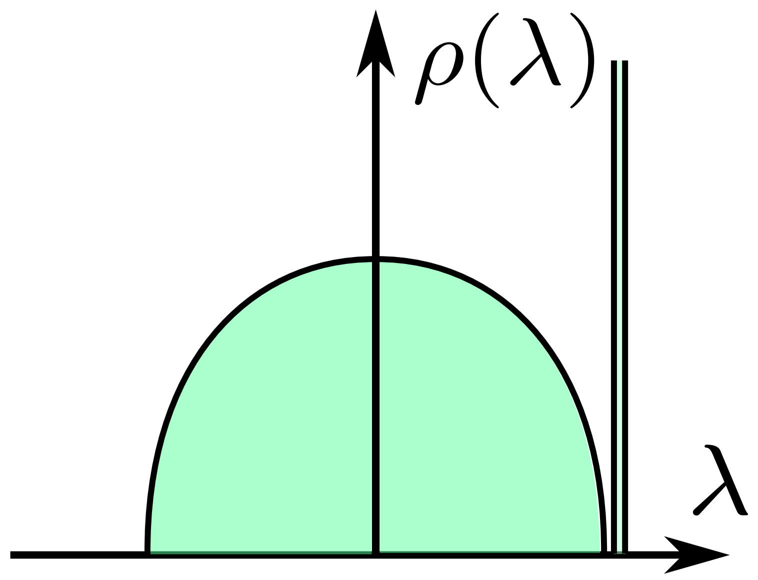
$$\rho(\lambda) = \sum_n \delta(\lambda - \lambda_n)$$



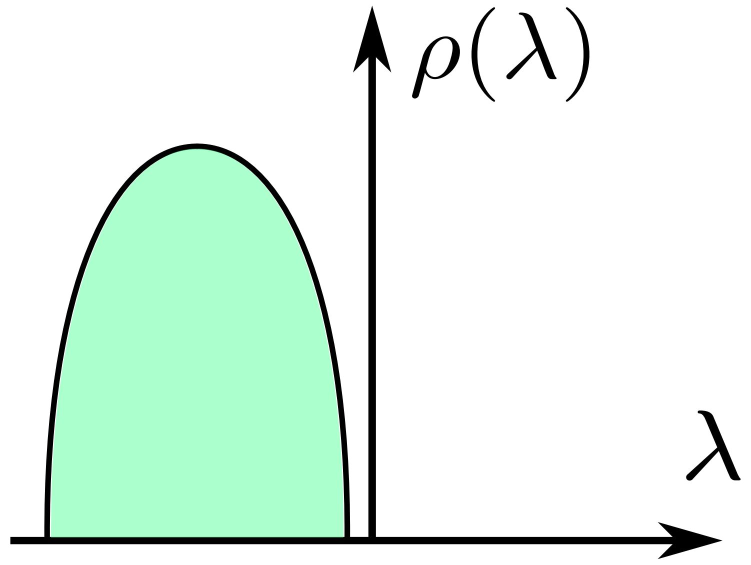
Fermi Liquid



Non Fermi liquid



Fast scrambler



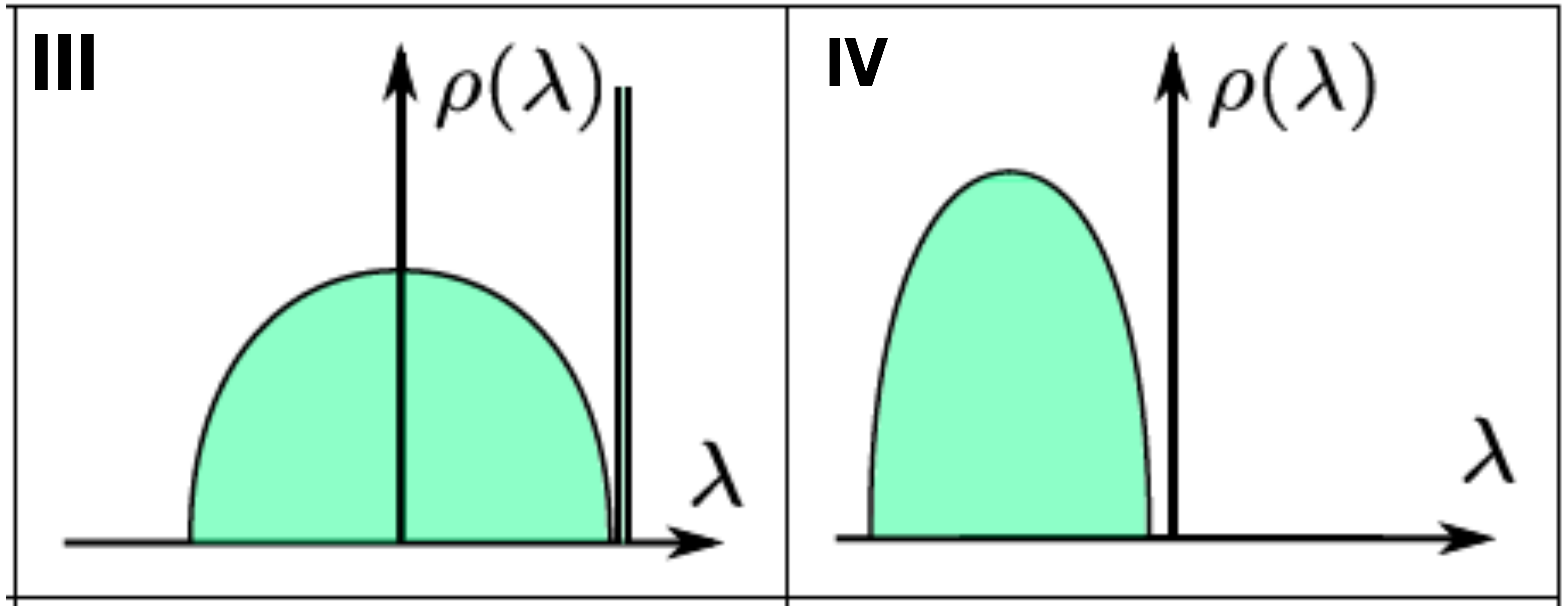
Fast scrambler

ex: graphene flake
[Franz, etc., PRL 18]

λ_L $\quad \ll T, \ \propto g \qquad \propto T \qquad 2\pi T$

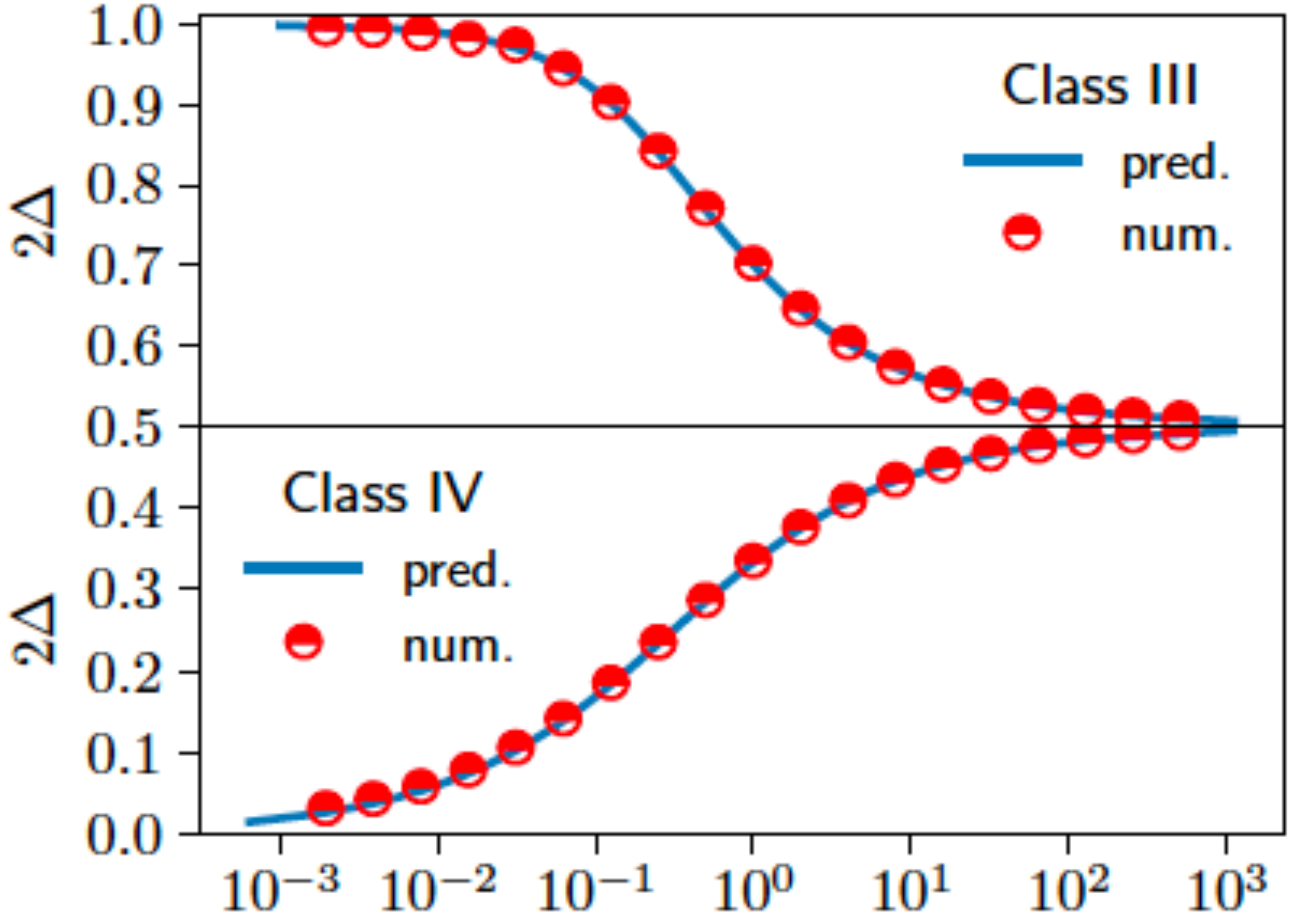
[Kim, XC, Altman PRB 2020, preprint 2006.02485, Kim, XC, 2004.05313] [Phys. Rev. Lett. 120, 241603 (2017)]

Fast scramblers (class III and IV)



$\rho \propto \delta(\lambda - \lambda_{\max}) + \ldots$

$\lambda_{max} < 0$



R/N (at the delta peak for class III)

qSYK-like with tunable scaling dimension

$G(\tau) \sim |\tau|^{-2\Delta}$, conformal invariance

Maximal Lyapunov exponent $\lambda_L = 2\pi T$ (like SYK)

Class IV: contains SUSY SYK

Class III: applications to superconductivity

Class III: Esterlis and Schmalian PRB 2019,
Yuxuan Wang PRL 2019

Class IV: Zhen Bi et. al. PRB 2017

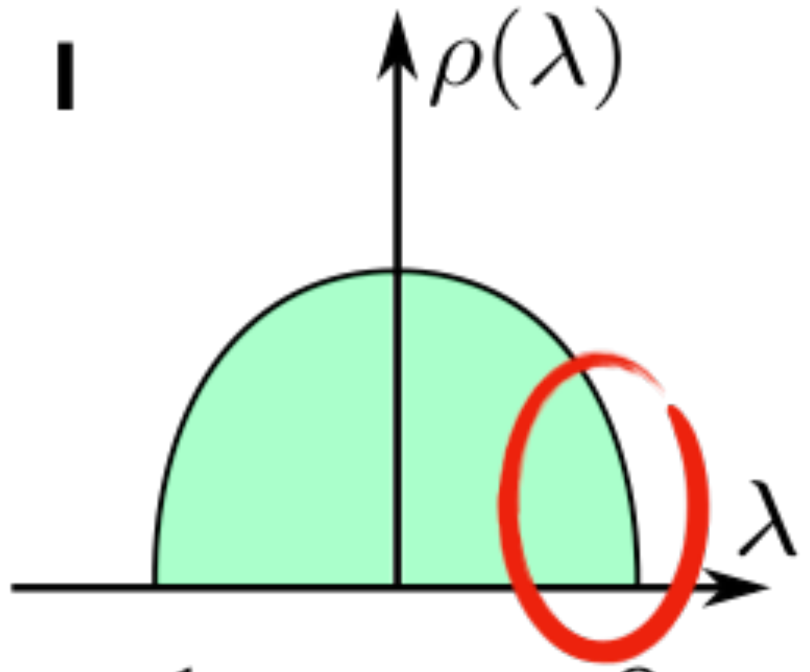
Fermi and non-Fermi liquids (class I & II)

$$\rho(\lambda) \sim (\lambda_{\max} - \lambda)^{\eta}$$

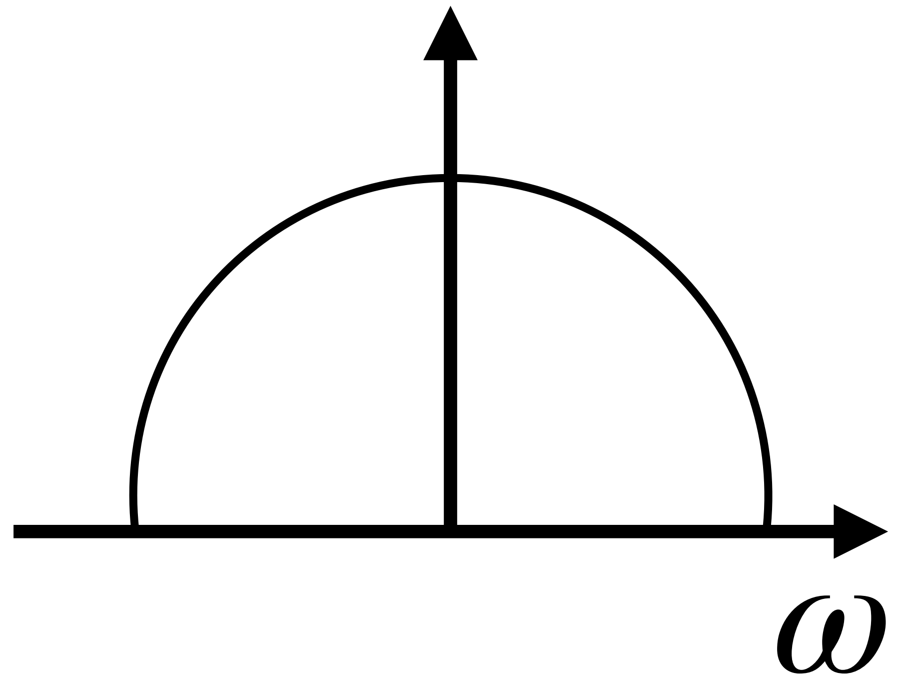
"free-fermion" leading scaling of Green function
+ sub-leading self-energy Σ ("quasiparticle decay")

Spectral density

I



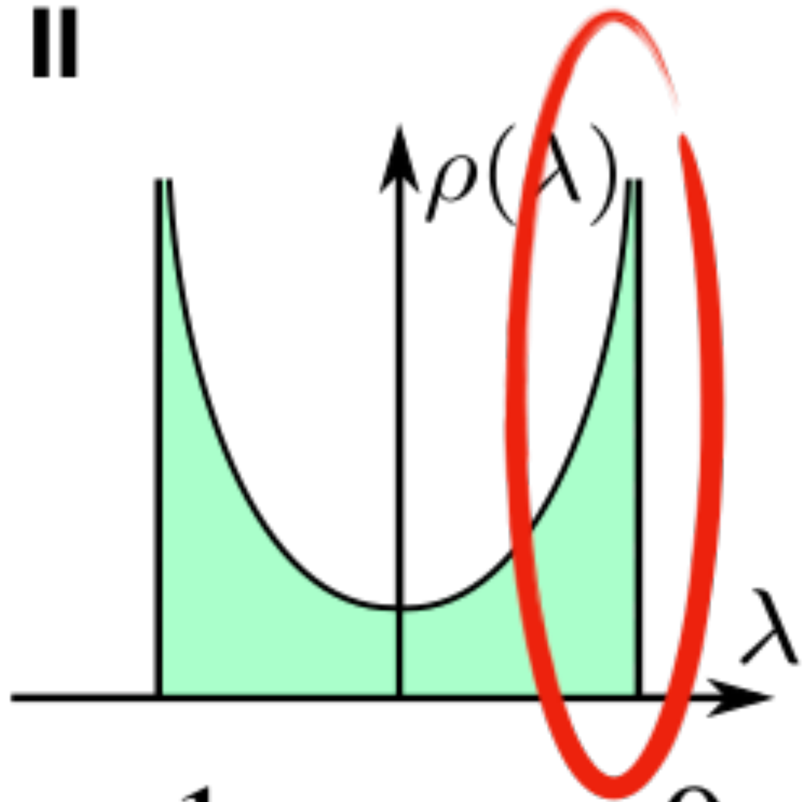
$1 > \eta > 0$



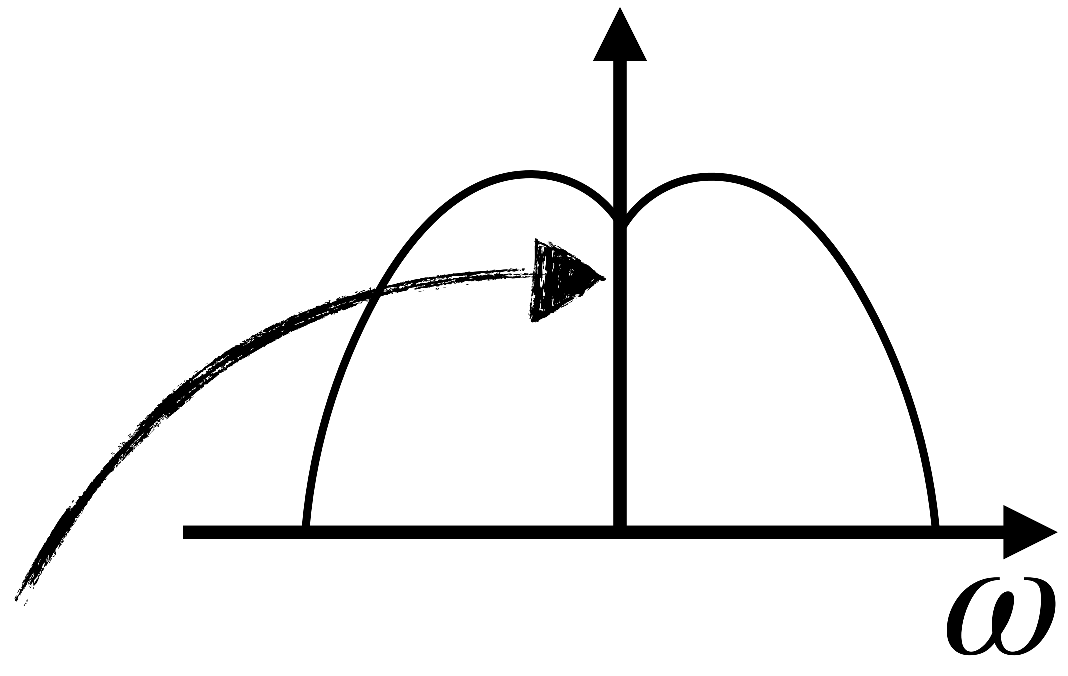
Fermi liquid $\eta > 0$

$$G(\omega) \sim \mathrm{sign}(\omega)\,, |\Sigma(\omega)| \sim |\omega|^{1+\eta} \ll |\omega|$$

II



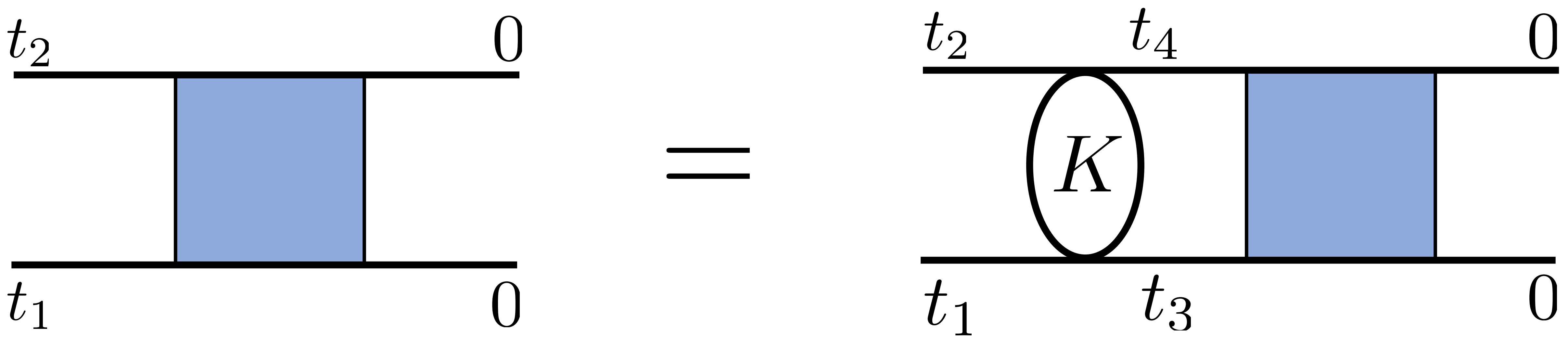
$-1 < \eta < 0$



Non- Fermi liquid $\eta < 0$

$$G(\omega) \sim \mathrm{sign}(\omega)\,, |\Sigma(\omega)| \sim |\omega|^{1+\eta} \gg |\omega|$$

λ_L(OTOC) is determined by ladder diagrams generated by stacking kernels

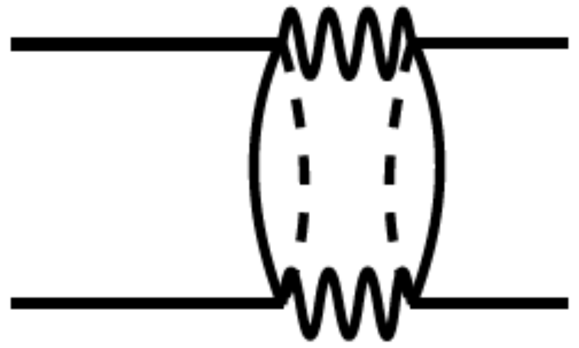


$$\int K(t_{1,\ldots,4})F(t_1,t_2) = F(t_3,t_4)\,,\; F(t_1,t_2) = f(t_2 - t_1)e^{\lambda_L(t_1+t_2)/2}$$

i.e., λ_L solved by requiring the largest eigenvalue be 1.

[Kitaev]

The ladder kernel in class I & II

$K_b =$  $K_f =$ 

$$[K_b(\lambda_L) + K_f(\lambda_L)]\, f = f$$

Class III, IV: Conformal solution gives $\lambda_L = 2\pi T$ after direct calculation (like SYK4)

Class I, II: Perturbation theory in the coupling $\gamma \sim R/N$ appearing in $\rho(\lambda) \approx \gamma(\lambda_{\max} - \lambda)^\eta$

$$\left[1 - \frac{\lambda_L}{F_0} + \gamma \frac{T^{\eta+1}}{F_0^{2\eta+2}} \tilde{K}\right] f = f \quad \Longrightarrow \quad \lambda_L = \gamma \frac{T^{\eta+1}}{F_0^{2\eta+1}} k(\lambda_L / 2\pi T)$$

Conformal (SYK_2)

Kinetic (*)

Interaction+decay

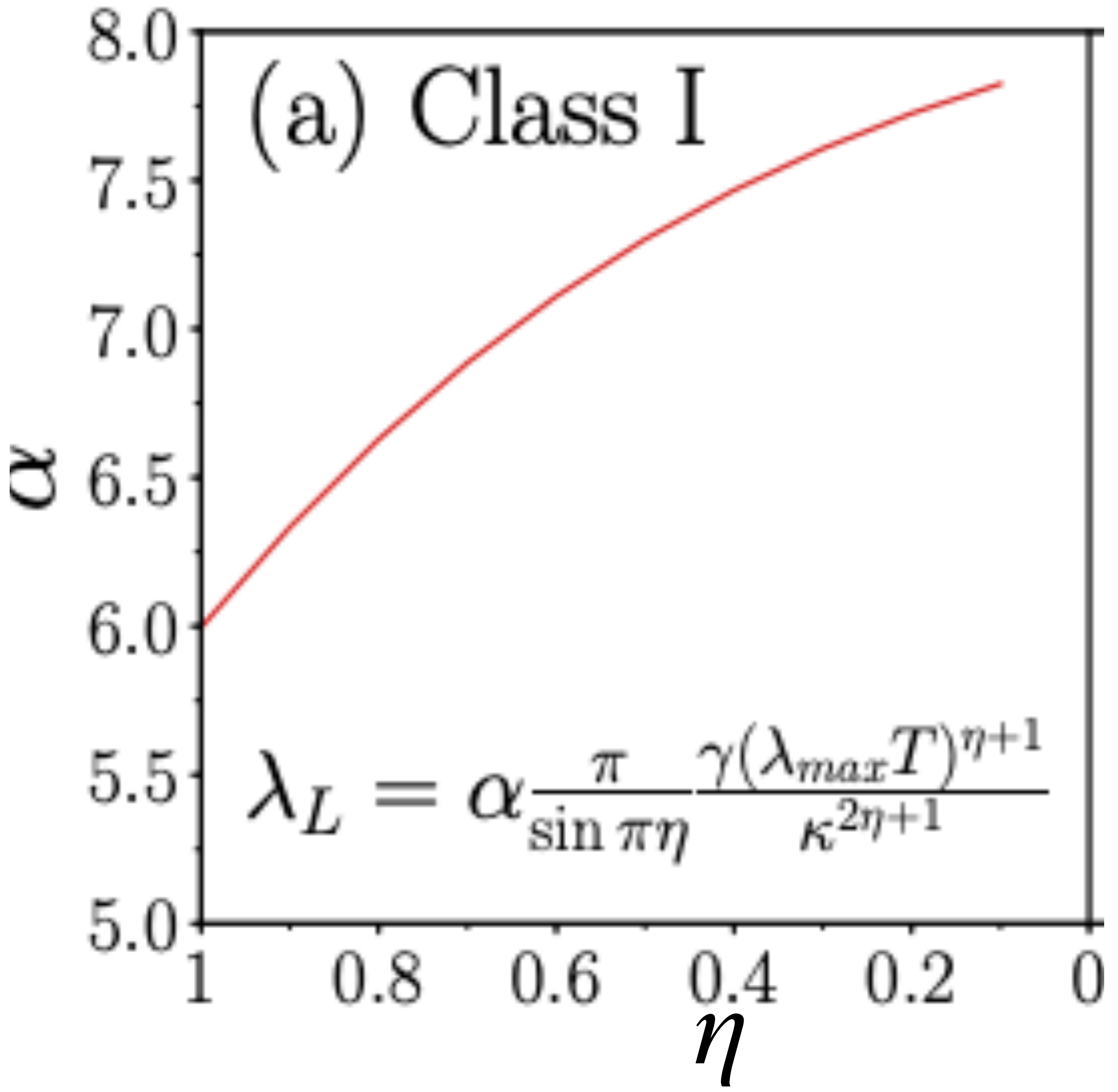
($\tilde{K}$: dimensionless kernel)

Most positive eigenvalue of $\tilde{K}$

F_0: **condensate generated SYK2 coupling**

(*): coming from the kinetic term in $G^{-1} = \partial_t - \Sigma$

$$\lambda_L = \gamma \frac{T^{\eta+1}}{F_0^{2\eta+1}} k(\lambda_L/2\pi T)$$

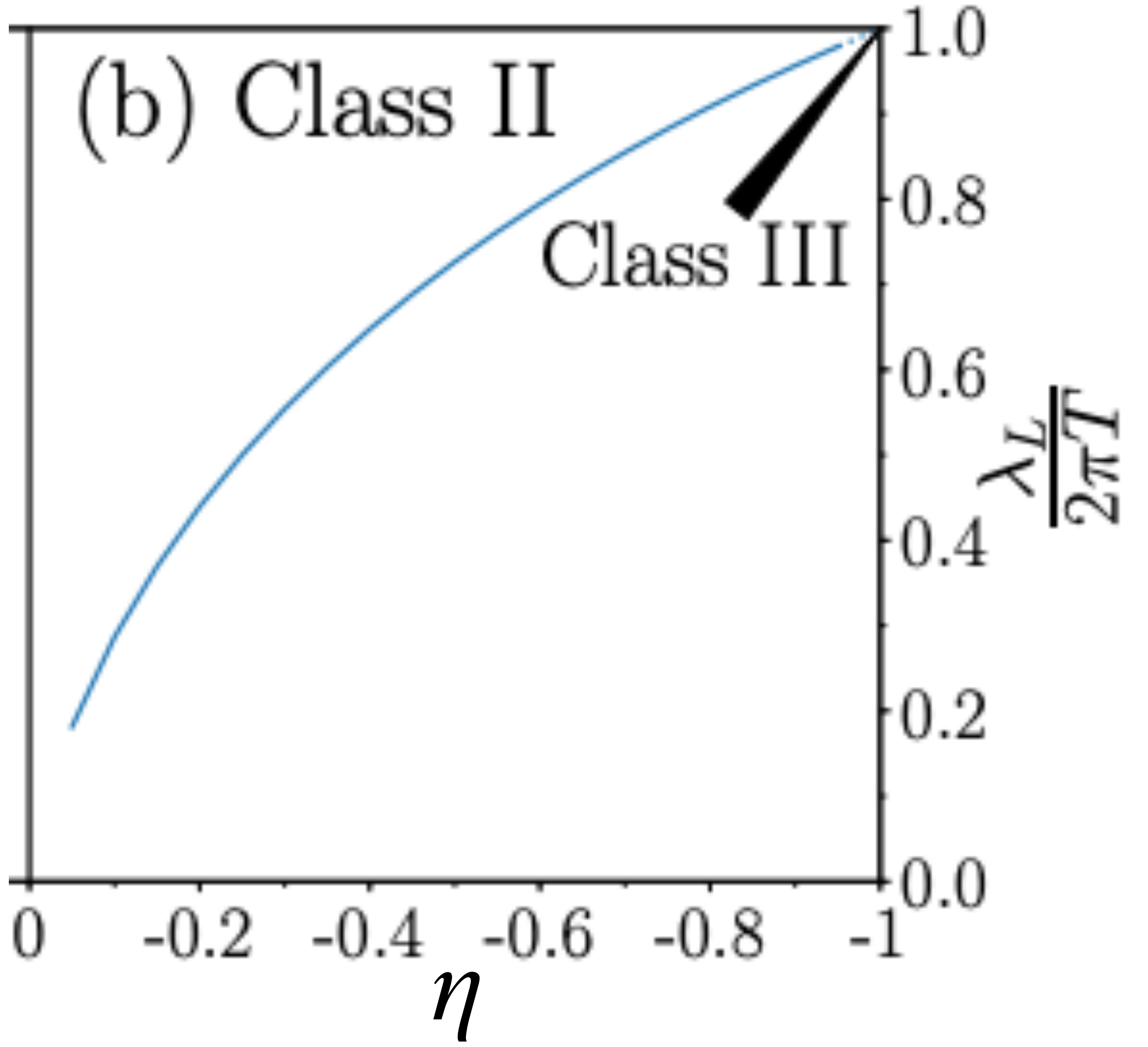


Fermi liquid (η > 0): $\lambda_L \propto \gamma T^{\eta+1} k(\lambda_L/2\pi T) \sim \gamma T^{\eta+1} k(0)$

Because RHS $\sim T^{\eta+1} \ll T$ we can take $k(0)$ to leading order in T

$\lambda_L \ll T, \lambda \propto \gamma$ (scrambling is significantly non-maximal, and perturbative in coupling constant)

Non-Fermi liquid (η < 0):

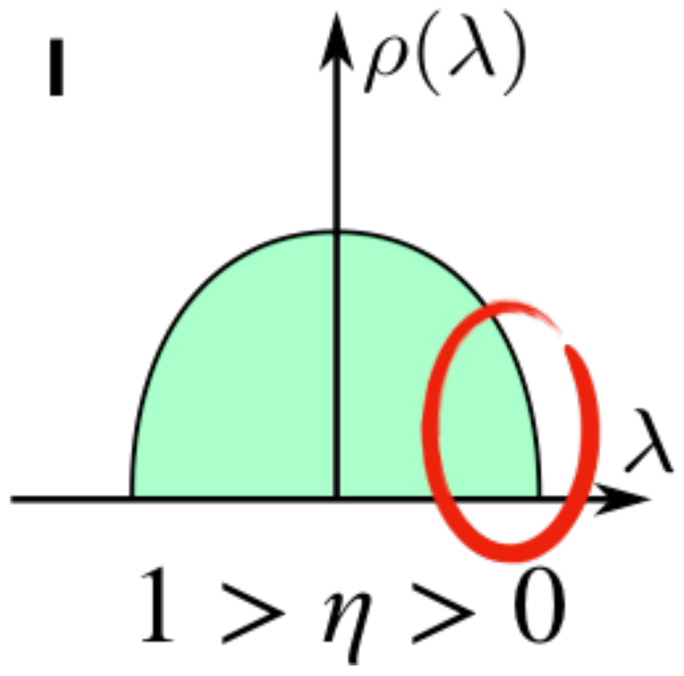


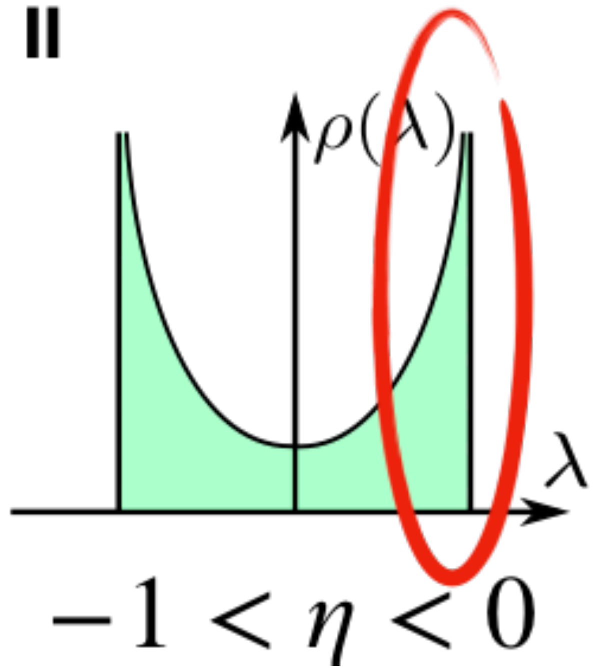
The LHS (kinetic term) is negligible compared to the interaction term

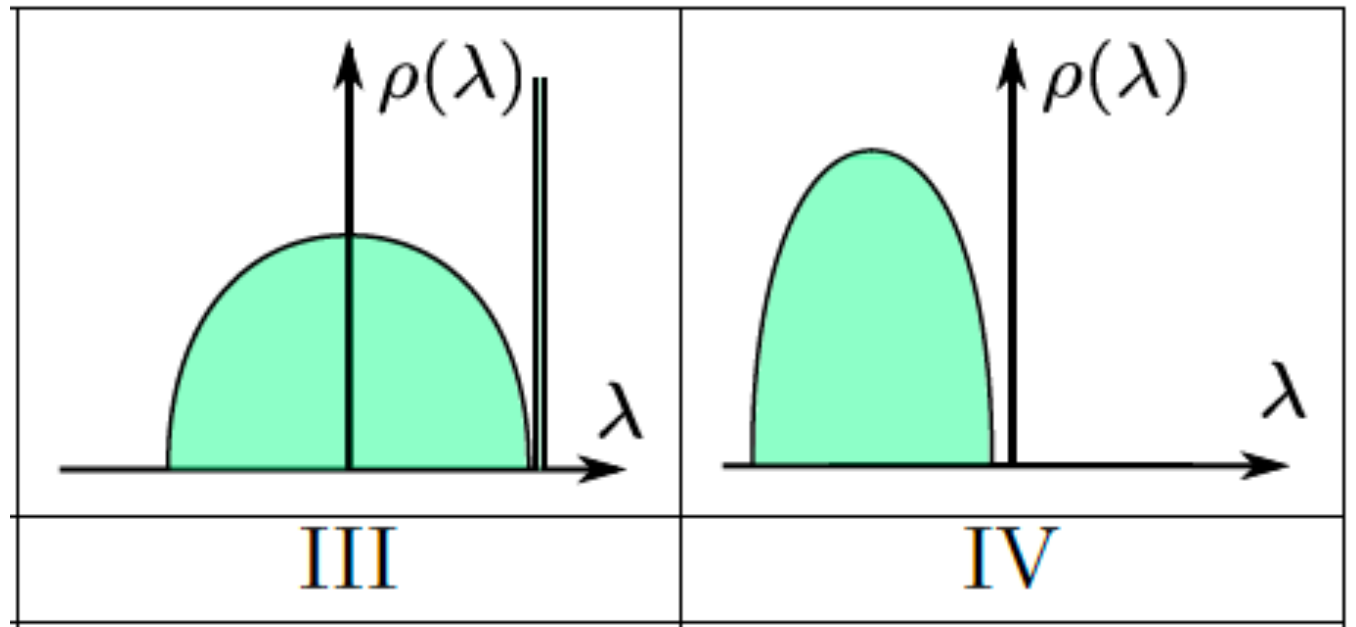
➡ $k(\lambda_L/2\pi T) = 0$ Determines λ_L

$\lambda_L = C_\eta T$ More universal. Independent of the coupling constant

Scrambling and quasiparticle decay in low rank SYK models





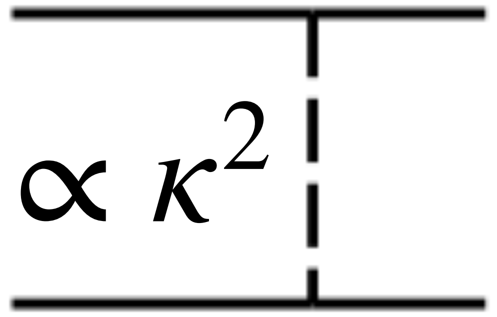
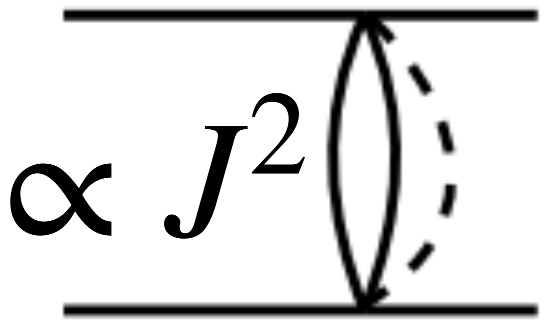


$$H = \sum_{n=1}^{R} \lambda_n \, u_{ij}^{(n)} u_{ij}^{(n)} \gamma_i \gamma_j \gamma_k \, \gamma_l$$

	quasiparticle decay $1/\tau$	λ_L
Fermi liquid	$\sim gT^{1+\eta},\ \eta > 0$	$\sim gT^{1+\eta}$
marginal FL	$\sim gT\ln(1/T)$	$\sim g\ln(1/g)T$
non-Fermi liquid	$\sim gT^{1+\eta},\ \eta < 0$	$= C_\eta T$
Fast scrambler	No quasiparticles	$2\pi T$

Biased opinion: OTOCs are good diagnostics of the IR fixed point's nature.

Back to mass-deformed SYK

$$K = \propto \kappa^2$$  $$+ \propto J^2$$ 

Similar to class I, with $1/\tau \sim T^2$ decay rate.

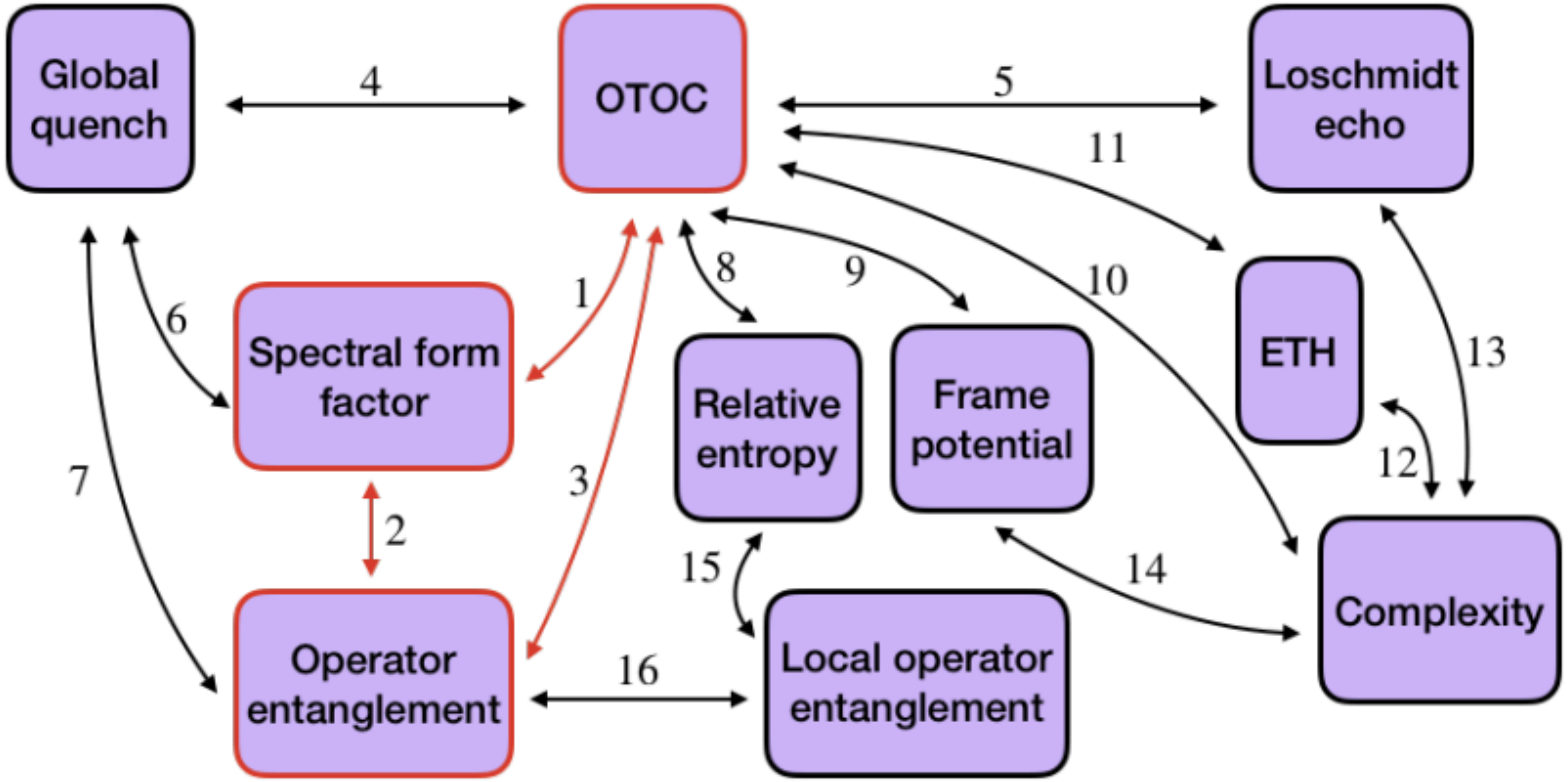
$$\Rightarrow \lambda_L \propto T^2$$

To ensure a positive prefactor requires further calculation, which will show

$$\lambda_L = \frac{3T^2J^2}{\kappa^3}\,, T, J \ll \kappa$$

What about away from large N or classical limit?

~~$\text{OTOC} \sim e^{\lambda_L t}$~~



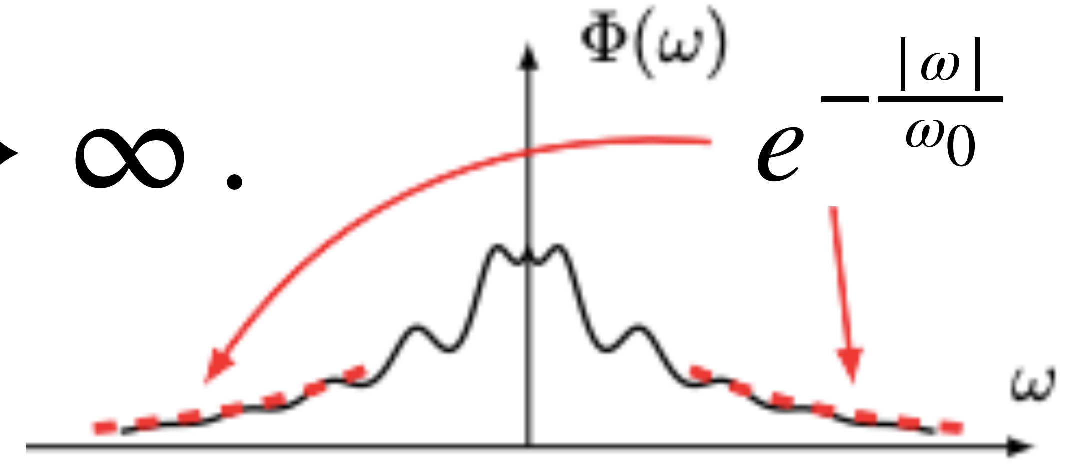
Web of chaos diagnostic [Kudler-Flama, Nie, Ryu]

[w/ Parker, Scaffidi, Advoshkin, Altman, PRX 19] [to appear 2020]

$$C(t) = \langle O(t)O(0) \rangle_{T=\infty} \qquad \Phi(\omega) = \int C(t)e^{i\omega t}dt$$

Hypothesis à la Bohigas-Giannoni-Schmit For non integrable systems and non-conserved operators O,

$$\Phi(\omega) \sim \exp(-|\omega|/\omega_0), \omega \to \infty .$$



For integrable systems, $\Phi(\omega)$ decay faster.

Theorem The above holds for chaotic Ising chain (ZZ+X+Z).

“Theorem” When λ_L is well-defined, it is bounded by (at $T = \infty$)

$$\lambda_L \leq \omega_0 \pi$$

But, at low temperature, ω_0 is too sensitive to UV details…

Technical note: there are sub-leading log corrections in 1d.

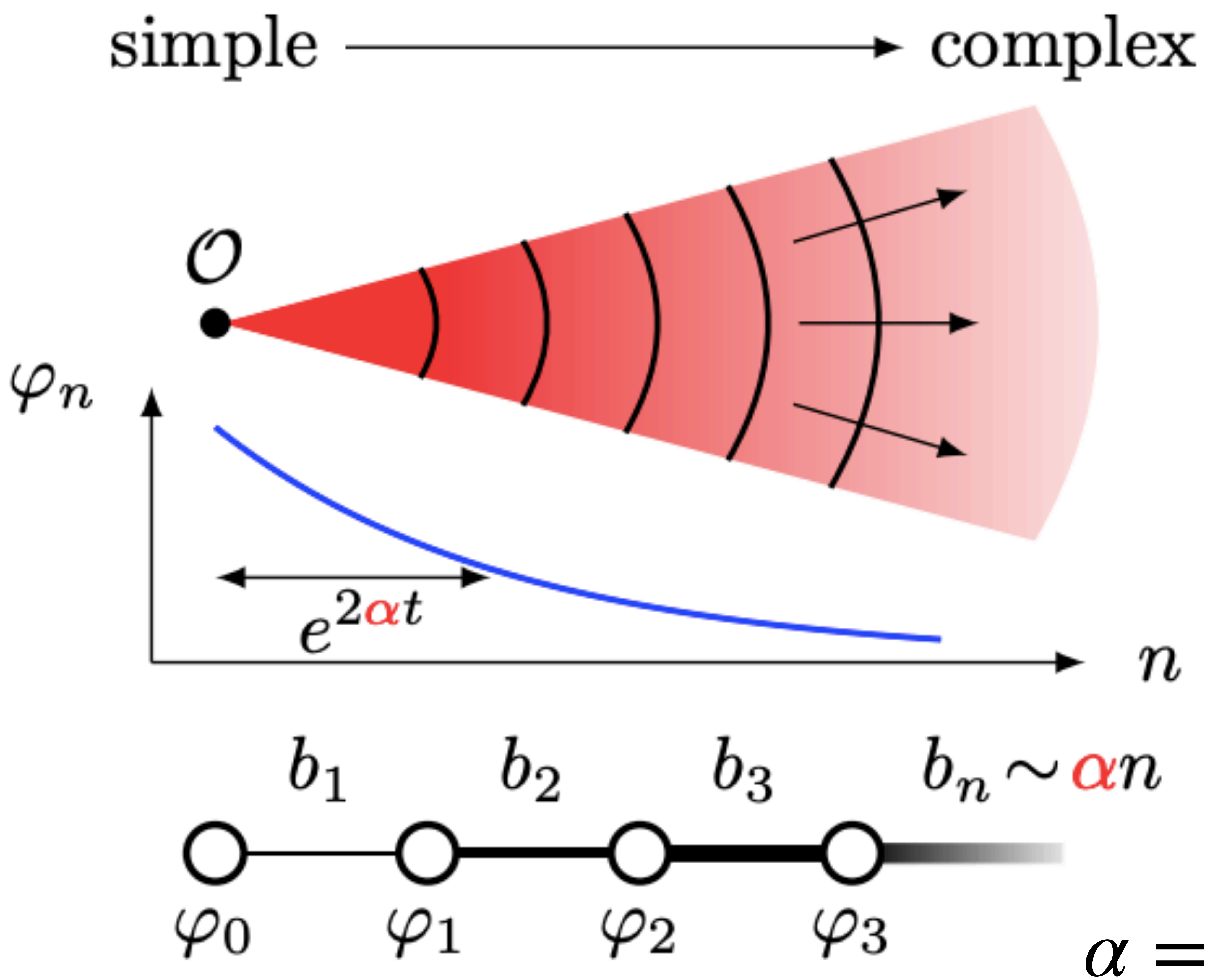
Does scrambling equal chaos?

I. In the classical limit: **No**

II. In the large-N, low-T limit: **works as intended**

III. Generic quantum: **I don’t know**

Thank you!



$$\alpha = \frac{\omega_0 \pi}{2}$$

$$G(z) = \cfrac{1}{z - \cfrac{b_1^2}{z - \cfrac{b_2^2}{z - \cfrac{b_3^2}{z - b_4^2 \ldots}}}}$$

$$[H, O_n] = b_n O_{n-1} + b_{n+1} Q_{n+1}$$

$\{O_n\}$: **Krylov basis**

$$O(t) = \sum_n i^n \varphi_n(t) O_n$$

$$(n)_t := \sum_n n \, |\varphi_n(t)|^2$$

"Krylov-complexity"

$$\text{OTOC} \leq C(n)_t$$

$$\boxed{\Rightarrow \lambda_L \leq \omega_0 \pi}$$