Yesterday

Basics of 6d N=(2,0) theory. S-duality of 4d N=4.

Today

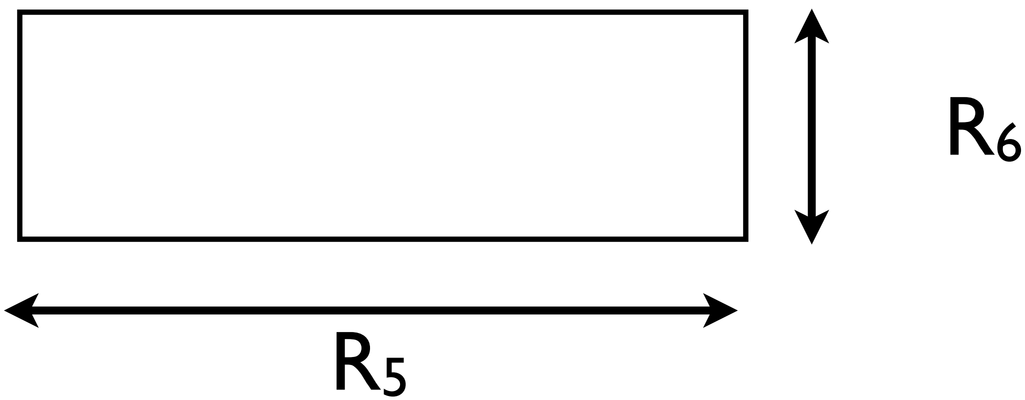
4d N=2 as 6d N=(2,0) compactified on C

Tomorrow

Relation with 2d CFT

Yesterday's talk's summary

- 6d N=(2,0) theory comes in types G= A,D,E
- Put on a torus of edges of lengths R_5 and R_6 ,



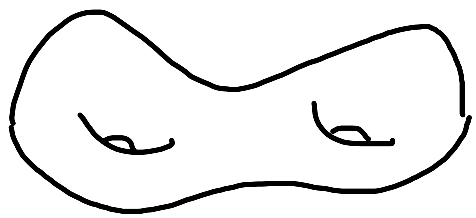
You get 4d N=4 theory with gauge group G,

at coupling $\frac{1}{g^2} = \frac{R_5}{R_6}$ S- duality !

How dow we get 4d N=2 theory?

We need to break SUSY, but not too much.

Instead of a flat torus, use a general Riemann surface

C : 

If you do this naively, it breaks all SUSY,
because there's no covariantly constant spinor!

$$D_\mu \varepsilon = (\partial_\mu + \Gamma_\mu{}^{\bullet}{}_{\bullet})\, \varepsilon$$

along C

The way out:

compensate the spacetime curvature
with the R-charge curvature.

Spinor is in $\mathbf{4}$ of SO(1,5) $\otimes$ $\mathbf{4}$ of SO(5)$_R$

with the reality condition.

$$\psi_{\alpha a} = J_{\alpha\beta} J_{ab} \psi^{*\beta b}$$

$\dim_{\mathbb{R}} = 16$. We'd like to preserve half of them.

We're splitting SO(1,5) to $\overset{\text{spacetime}}{SO(1,3)} \times \overset{\text{Riemann surface}}{SO(2)}$

Let's also split $SO(5)_R$ to $SO(3)_R \times SO(2)_R$

Spinor was in $\mathbf{4}$ of SO(1,5) $\otimes$ $\mathbf{4}$ of $SO(5)_R$
with the reality condition.

$$SO(1,3) \times SO(2) \times SO(3)_R \times SO(2)_R$$

We get

$$(\mathcal{D}_+ \oplus \overline{\mathcal{D}}_-) \otimes (\mathcal{D}_+ \oplus \mathcal{D}_-)$$

$$\longrightarrow \quad \mathcal{D}_+ \otimes \mathcal{D}_+$$

$$\oplus \quad \mathcal{D}_+ \otimes \mathcal{D}_-$$

Reality condition removes half of them.

$$SO(1,3) \times SO(2) \times SO(3)_R \times SO(2)_R$$

The spinors are in $\mathcal{Q}_+ \otimes \mathcal{Q}_{\oplus}$

$\oplus$ $\mathcal{Q}_+ \otimes \mathcal{Q}_{\ominus}$

We're turning on SO(2) curvature. Destroys all SUSY.

$$D_\mu \varepsilon = (\partial_\mu + \Gamma_\mu)\,\varepsilon\,.$$

Let's set $SO(2)_R$ curvature = SO(2) curvature.

$$D_\mu \varepsilon = (\partial_\mu + \Gamma_\mu \pm A_\mu)\,\varepsilon \qquad \text{with } \Gamma_\mu = A_\mu$$

The spinors in $\mathcal{Q}_+ \otimes \mathcal{Q}_-$ can be constant!

$SO(3,1)$ $\quad$ $SO(3)_R \simeq SU(2)_R$

4d $\mathcal{N}=2$!

Five scalars were vectors of $SO(5)_R$

$$\underbrace{\phi_1 \quad \phi_2}_{SO(2)_R} \quad \underbrace{\phi_3 \quad \phi_4 \quad \phi_5}_{SO(3)_R}$$

Two of them couple to $SO(2)_R$, now set to SO(2)

They now effectively form a (co)tangent vector of the surface.

$$D_\mu(\phi_1 + i\phi_2) = (\partial_\mu + A_\mu)(\phi_1 + i\phi_2)$$
$$= (\partial_\mu + \Gamma_\mu)\ (\phi_1 + i\phi_2).$$

Let $\Phi(z,\bar{z}) = (\phi_1 + i\phi_2)\, dz$.

$z = x^1 + ix^2$: local coordinate on $\mathbb{C}$.

We can now define 4d N=2 supercharges.
When are they preserved?

$\delta_\varepsilon\psi = 0$ leads to the conditions

$$\begin{cases} d\Phi(z,\bar{z}) = \bar{\partial}\Phi \wedge d\bar{z} = 0 \rightsquigarrow \Phi = \Phi(z). \\ \partial\phi_3 = \partial\phi_4 = \partial\phi_5 = 0 \end{cases}$$

We're forced to set $\phi_{3,4,5} = \text{const.}$

but $\Phi(z)$ can be nontrivial !

We forget $\phi_{3,4,5}$ for the rest of the talk.

So far we talked about 1 M5-brane on C.

How about N M5-branes on C? We have one-forms

$$\phi^{(1)}(z), \quad \phi^{(2)}(z), \quad \cdots, \quad \phi^{(N)}(z).$$

But we can't distinguish an M5 from another.

Let λ be an auxiliary one-form. Then

$$(\lambda-\phi^{(1)}(z))(\lambda-\phi^{(2)}(z))\cdots(\lambda-\phi^{(N)}(z))$$

$$= \lambda^N + u_1(z)\lambda^{N-1} + u_2(z)\lambda^{N-2} + \cdots + u_N(z)$$

contains the invariant info. $u_k(z)$ are k differentials:

$$u_k(z) = a(z)\, dz^k \qquad \longrightarrow b(w) = a(z)\left(\frac{dz}{dw}\right)^k.$$

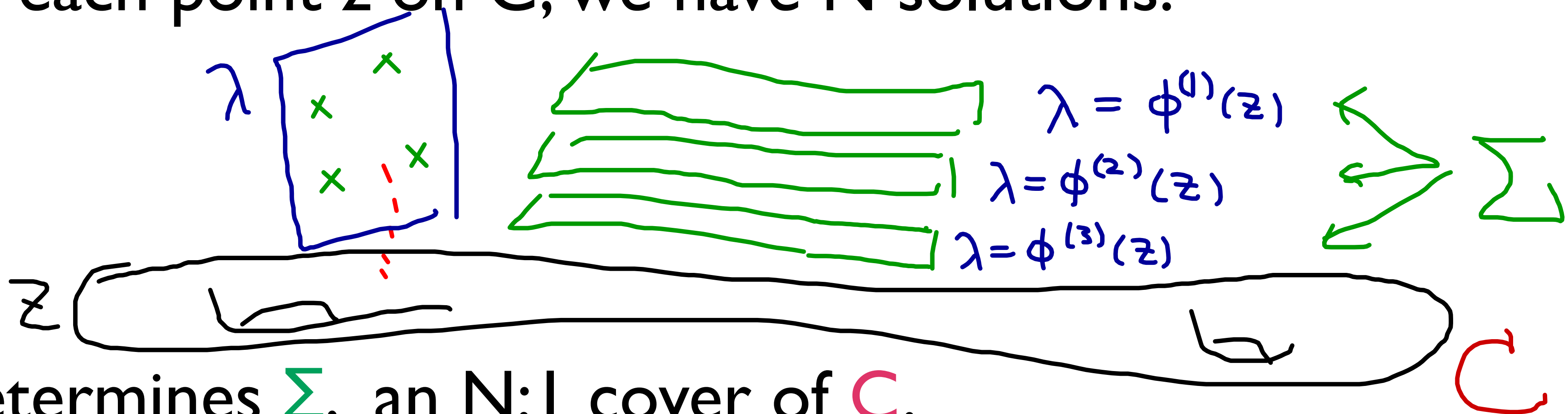
$$= b(w)\, dw^k$$

The equation

$$0 = \lambda^N + u_1(z)\lambda^{N-1} + u_2(z)\lambda^{N-2} + \cdots + u_N(z)$$

characterize N M5-branes wrapped on C.

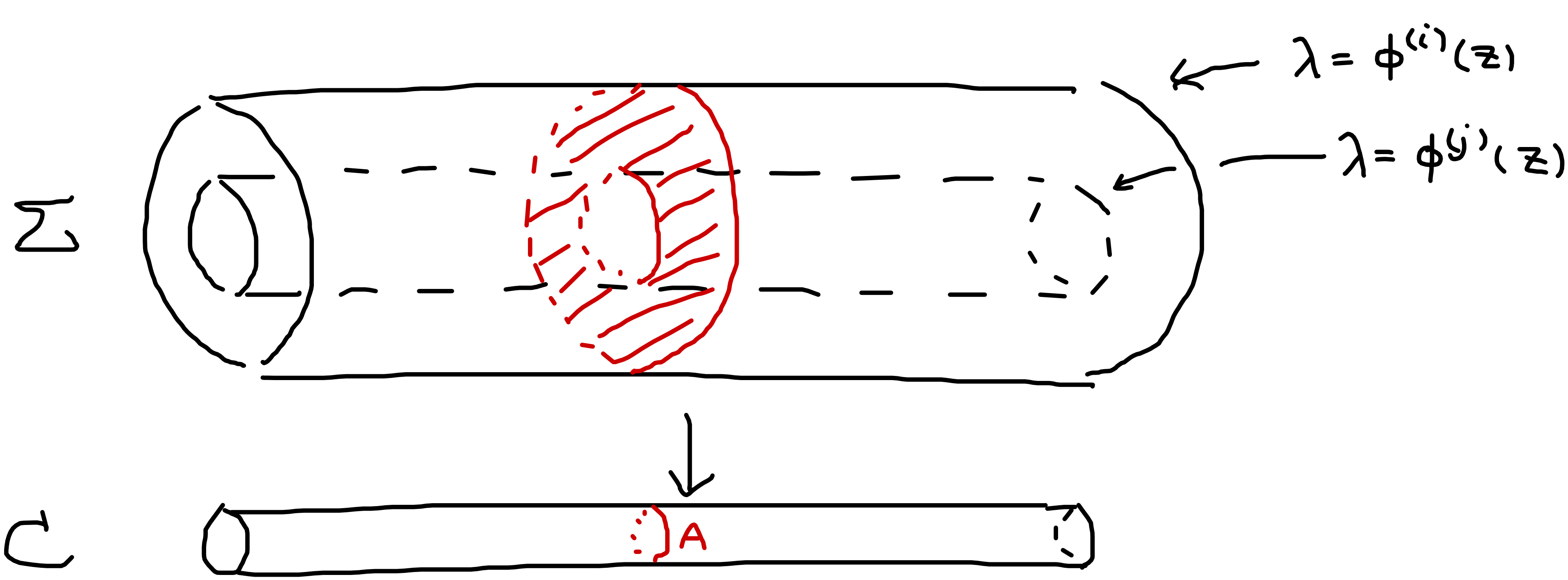
At each point z on C, we have N solutions:



Determines Σ, an N:1 cover of C.

What are the supersymmetric states?

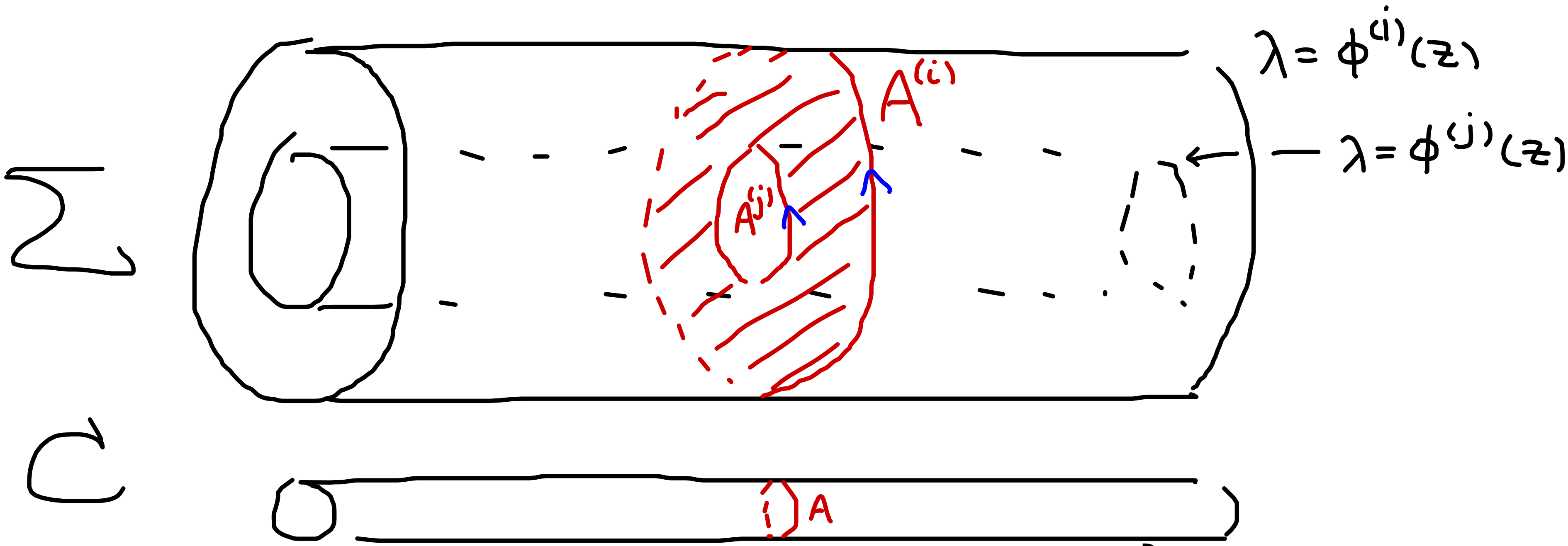
A string can extend between 2 M5-branes.



$$\text{mass} = \int_A |\phi^{(i)} - \phi^{(j)}| \geq \left| \int_A \phi^{(i)} - \phi^{(j)} \right| .$$

This is known/believed to give an N=2 vector multiplet.

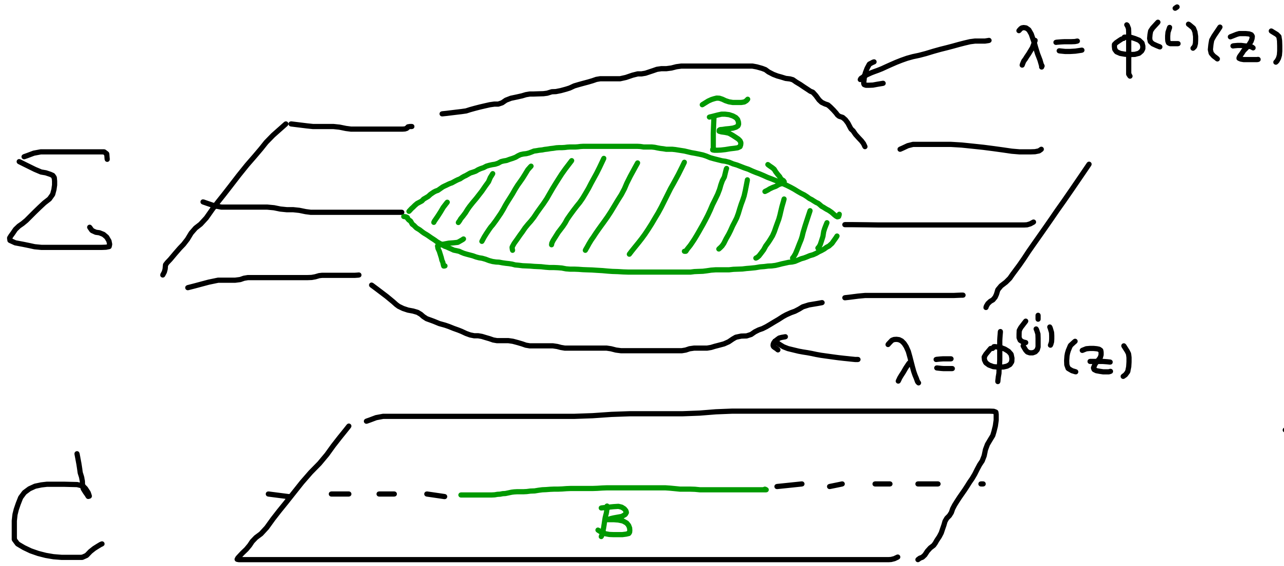
Instead of thinking of an integral over C,



This can be thought of an integral of λ over Σ

$$\int_A \left(\phi^{(i)} - \phi^{(j)} \right) = \int_{A^i - A^j} \lambda$$

Another possibility is



$$\text{mass} = \int_B |\phi^{(i)} - \phi^{(j)}| = \int_{\widetilde{B}} \lambda .$$

This is known to give an N=2 hypermultiplet.

There would be more possibilities,
but not well understood.

Summarizing, starting from 6d N=(2,0) theory of type A_{N-1}, we get an 4d N=2 theory characterized by

$$\Sigma : \ 0 = \lambda^N + u_1(z)\lambda^{N-1} + u_2(z)\lambda^{N-2} + \cdots + u_N(z)$$

where BPS particles have masses

$$\int_A \lambda \qquad , \ A: \text{a closed contour of } \lambda.$$

(Gaiotto-Moore-Neitzke '09 Sec. 3 is a nice review on the topics covered so far.)

Klemm-Lerche-Mayr-Vafa-Warner '96

But what's this N=2 theory???

Conversely, Seiberg and Witten observed that

given an N=2 theory with gauge group G
and matter fields in the rep. R,

there will be a pair of

a Riemann surface Σ
and a one-form λ on it

such that masses of BPS particles are given by

$$\int_A \lambda \quad , \quad A: \text{closed path on } \Sigma$$

But how can we find (Σ, λ) given (G,R) ?

Nobody has been able to answer this question in full generality so far.

It's YOU who will solve this important problem.

That said, there are a few methods developed:

- Guess and check consistency (SW, 1994~)
- Geometric engineering (Vafa et al. 1997~)
- Plumbing M5-branes (Gaiotto et al. 2009~)
- Instanton integral (Nekrasov et al. 2003~)

I'd be happy to talk about each of them in detail, but the time constraint doesn't allow me.

Let's consider N=2 pure SU(N) theory.

$$\begin{tikzcd} & A_\mu \arrow[dl, bend right, "Q_1"'] \arrow[dr, bend left, "Q_2"] & \\ \lambda^1_\alpha \arrow[dr, bend right, "Q_2"'] & & \lambda^2_\beta \arrow[dl, bend left, "Q_1"] \\ & \phi & \end{tikzcd}$$

ϕ : complex scalar in adj.

The potential is

$$V = \mathrm{tr}[\phi, \phi^\dagger]^2$$

So there's a family of vacua.

$$\phi = \mathrm{diag}(a_1, a_2, \cdots, a_N)$$

for $G = SU(N)$

One-loop running of coupling is easy to calculate:

$$\frac{\partial}{\partial \log \Lambda_{cut}} \frac{8\pi^2}{g^2}(\Lambda_{cut}) = 2N$$

The dynamical scale is

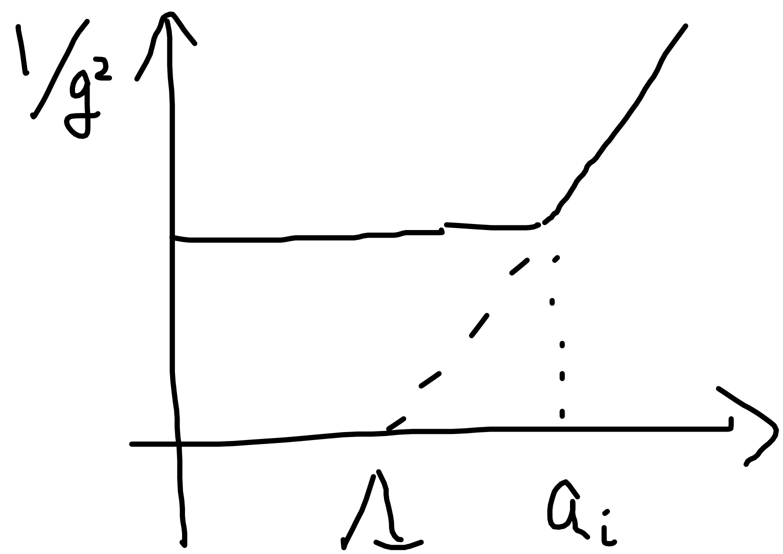
$$\Lambda^{2N} = \Lambda_{cutoff}^{2N} \exp\left(\frac{8\pi^2}{g^2}(\Lambda) + i\theta \right)$$

Consider the weakly-coupled regime

$$\Lambda \ll a_i$$

We have W-bosons with mass

$$m = |a_i - a_j|$$



We also have monopoles:

Take an SU(2) 't Hooft-Polyakov monopole,
and embed into the (i,j)-th block

$$\begin{array}{c c} i \rightarrow & \\ j \rightarrow & \\ & \end{array}\begin{array}{c} \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ i & j \end{array} \end{array}$$

with the mass

$$\frac{4\pi}{g^2(\text{"}a\text{"})}|a_i-a_j| \sim \left(\frac{2N}{2\pi}\log\frac{\text{"}a\text{"}}{\Lambda}\right)|a_i-a_j|$$

Compare them to the mass of W-bosons:

$$|a_i-a_j|$$

The ratio encodes the running of the gauge coupling.

The configuration of N M5-branes is this:

$$0 = \lambda^N + u_2(z)\,\lambda^{N-2} + \cdots + u_N(z) \quad \text{where}$$

$$u_2(z) = \underline{u_2}\left(\frac{dz}{z}\right)^2,\ u_3(z) = \underline{u_3}\left(\frac{dz}{z}\right)^3 \cdots\ u_{N-1}(z) = \underline{u_{N-1}}\left(\frac{dz}{z}\right)^{N-1},\ u_N(z) = \left(\Lambda^N z + \underline{u_N} + \frac{\Lambda^N}{z}\right)\left(\frac{dz}{z}\right)^N$$

Writing $\lambda = \frac{x\,dz}{z}$, we have

$$-\left(\frac{\Lambda^N}{z} + \Lambda^N z\right) = x^N + \underline{u_2}\,x^{N-2} + \underline{u_3}\,x^{N-3} + \cdots + \underline{u_N}$$

which is a form of the Seiberg-Witten curve due to [Martinec-Warner '96] for pure $SU(N)$ theory.

But that won't work with this younger audience.

quoting of a once well-known fact

The curve is

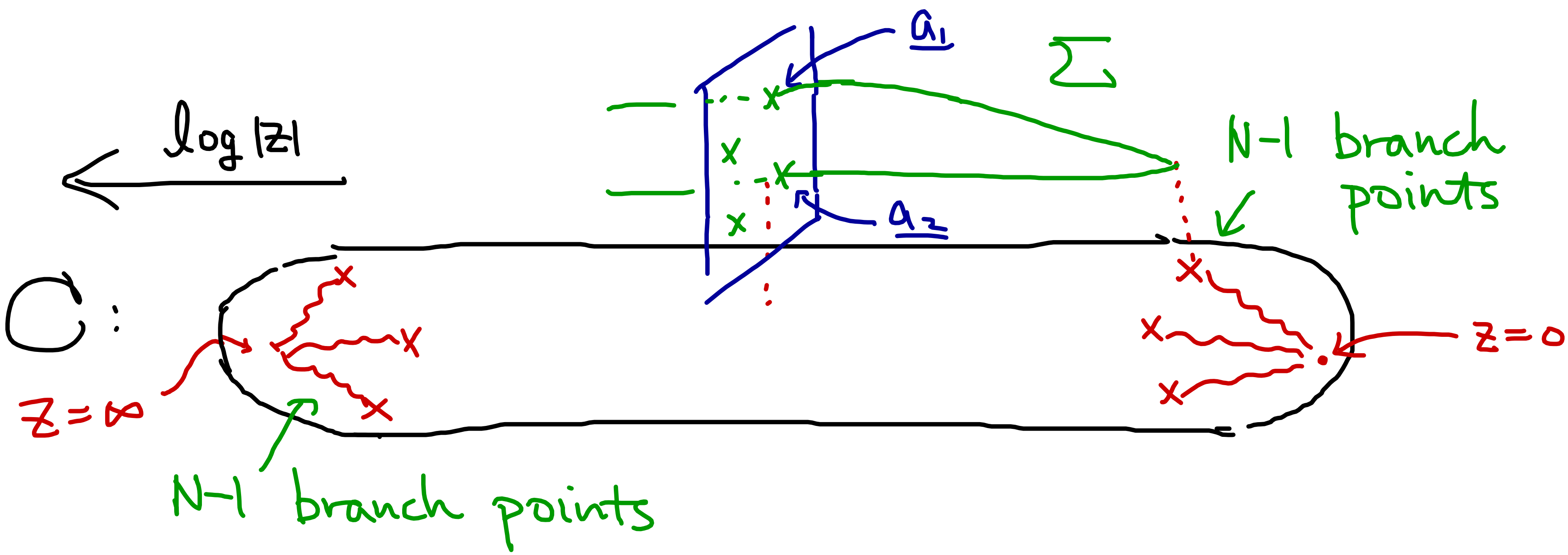
$$-\Lambda^N\left(z+\frac{1}{z}\right) = x^N + \underline{u_2}\, x^{N-2} + \underline{u_3}\, x^{N-3} + \cdots + \underline{u_N}$$

Let's study the situation $\quad \Lambda \ll |u_k|^{1/k}$

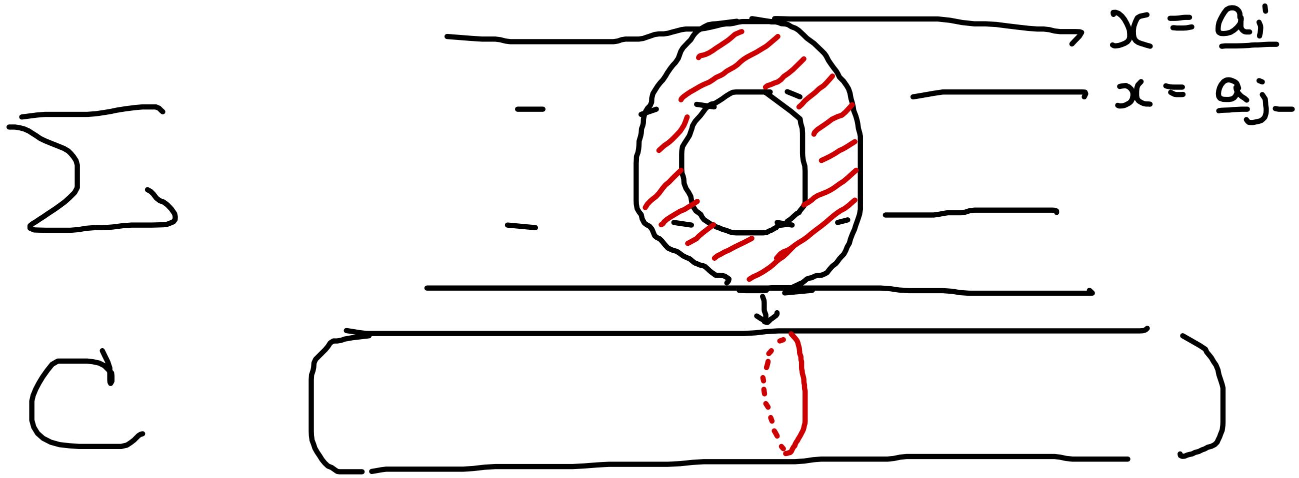
Factorize the u's as follows

$$-\Lambda^N\left(z+\frac{1}{z}\right) = (x-\underline{a_1})(x-\underline{a_2})\cdots(x-\underline{a_N})$$

Then the curve looks like



First, you see W-bosons:

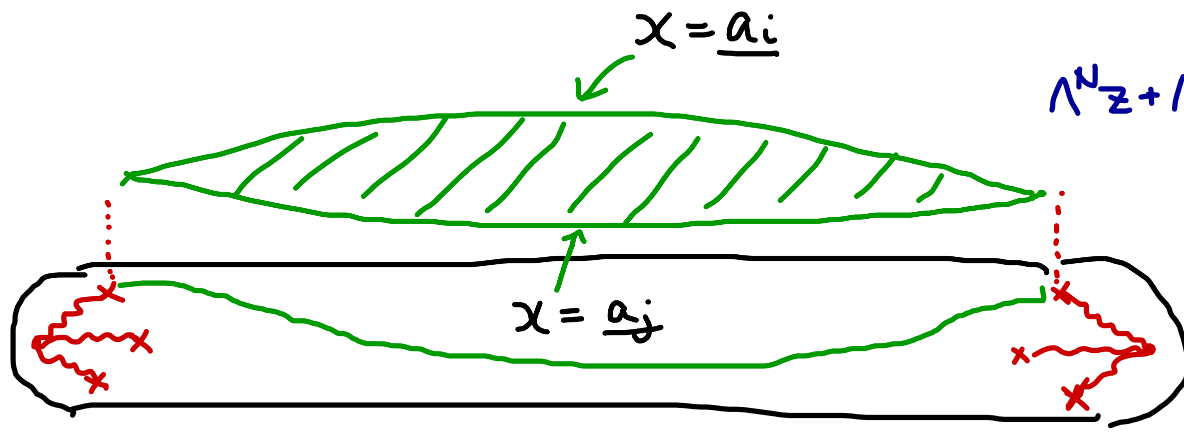


They exist for each pair of (i,j).

The mass is approximately

$$\int \lambda = \int x \frac{dz}{z} = \int (\underline{a}_i - \underline{a}_j) \frac{dz}{z} = 2\pi i (\underline{a}_i - \underline{a}_j)$$

Second, you see monopoles:



$\Lambda^N z + \Lambda^N/z = (x-\underline{a_1})(x-\underline{a_2}) \cdots (x-\underline{a_N})$

$z \sim \dfrac{\Lambda^N}{\text{“}a^N\text{”}}$

The mass is roughly given by

$$\int \lambda = \int x \frac{dz}{z} \sim \left(2 \log \frac{\Lambda^N}{\text{“}a^N\text{”}}\right)(\underline{a_i} - \underline{a_j})$$

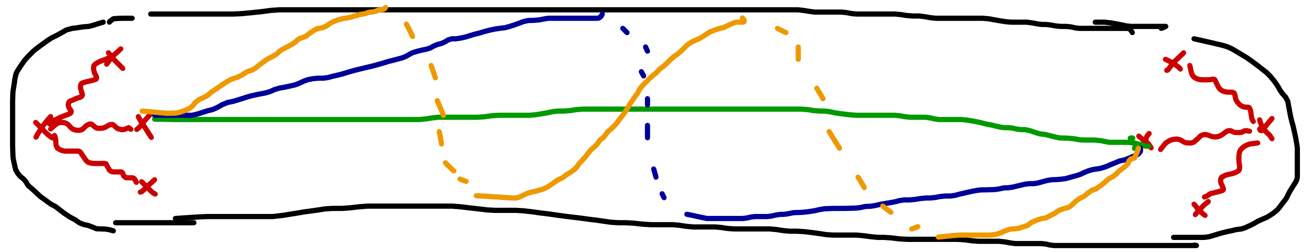
so, we get the expected log running, with correct β.

To see this, compare it with W-boson : $2\pi i(\underline{a_i} - \underline{a_j})$

$$\rightsquigarrow \frac{4\pi}{g^2} \sim \frac{2N}{2\pi} \log \frac{\text{“}a\text{”}}{\Lambda}.$$

Explicit form of “a” can be found from the curve.

Log has branches, reflected by the existence of dyons.

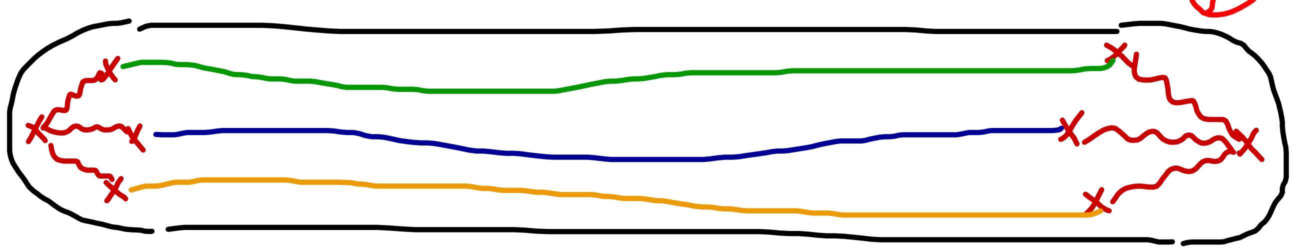


whose mass is

※ Resolvent of
$x^N + u_2 x^{N-2} + \cdots + u_N + \varepsilon = 0$
is an degree $(N-1)$ poly in ε.

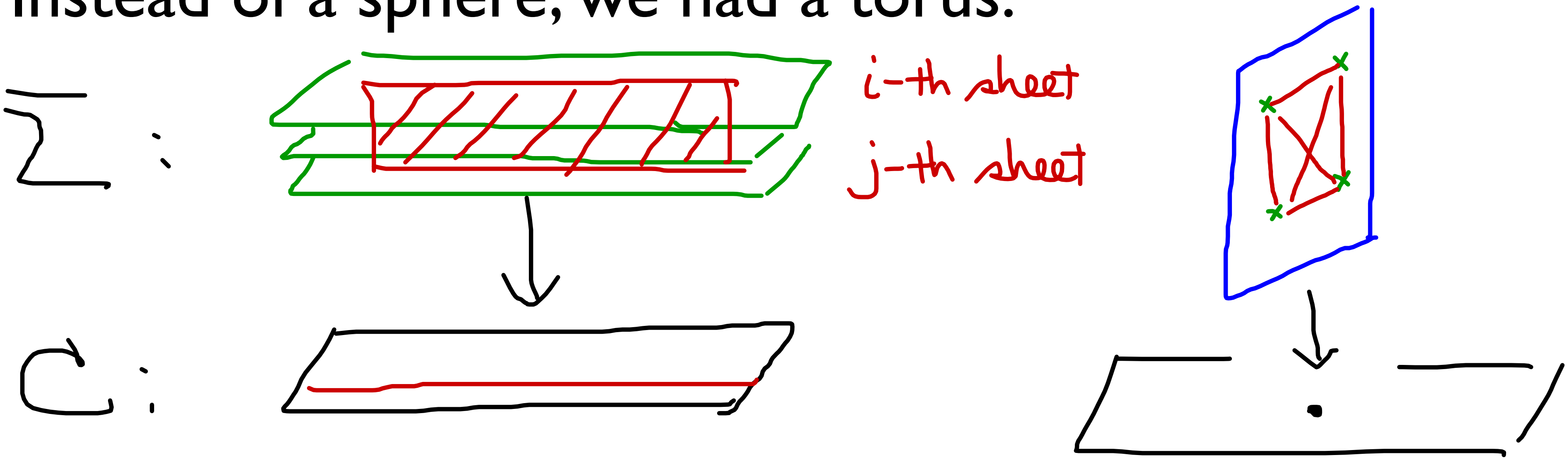
$$\left(2\log\frac{\Lambda^N}{\text{“}a^N\text{”}} + 2\pi i m\right)(\underline{a}_i - \underline{a}_j)$$

Note that there are only (N−1) tower of dyons,



not for every pair of (i,j). This agrees with an old semi-classical analysis of the monopole moduli space.

Compare this with N=4 SU(N) SYM.
Instead of a sphere, we had a torus.



So, we have monopoles for each pair of (i,j)

This is supported by semi-classical analysis on the monopole moduli space.

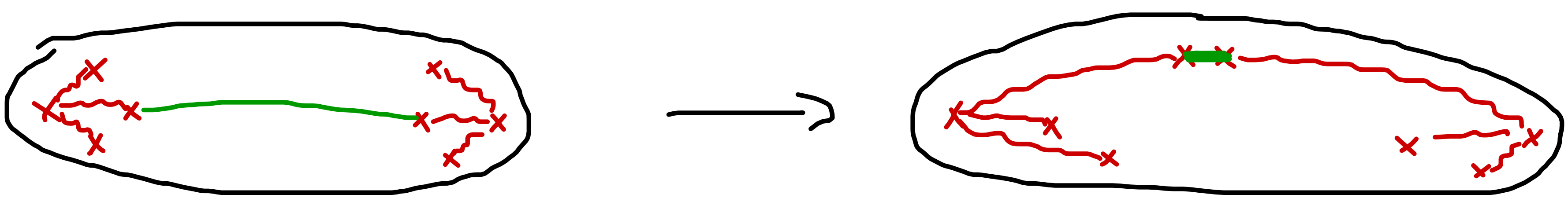
Anyway, the configuration

$$\Lambda^N\left(\frac{1}{z}+z\right) = x^N + u_2 x^{N-2} + \cdots + u_N$$

reproduces the spectrum of pure SU(N) when weakly coupled, $\Lambda \ll |u_k|^{1/k}$

Holomorphy of N=2 low-energy Lagrangian guarantees it should then be OK for all values of u_k.

Two branch points can collide, for example,



producing almost massless monopoles, etc.

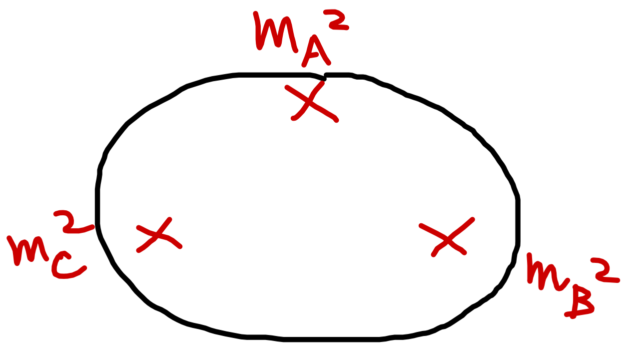
Let's consider just 2 M5-branes and a 3-punctured sphere

$$x^2 - u_2(z) = 0$$

$$u_2(z) \sim m_A^2 \frac{dz}{z} \quad \text{at } z \sim 0$$

$$\sim m_B^2 \frac{dz}{z-1} \quad \text{at } z \sim 1$$

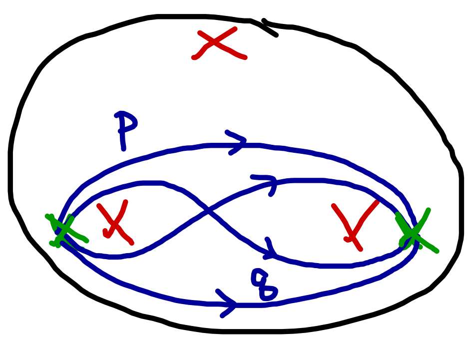
$$\sim m_C^2 \frac{d(1/z)}{1/z} \quad \text{at } z \sim \infty$$



You see four paths, producing four hypermultiplets with masses

$$P \rightarrow \frac{1}{2}(m_A + m_B + m_C) \qquad \frac{1}{2}(m_A - m_B + m_C)$$

$$\frac{1}{2}(m_A + m_B - m_C) \qquad \frac{1}{2}(m_A - m_B - m_C) \leftarrow q$$

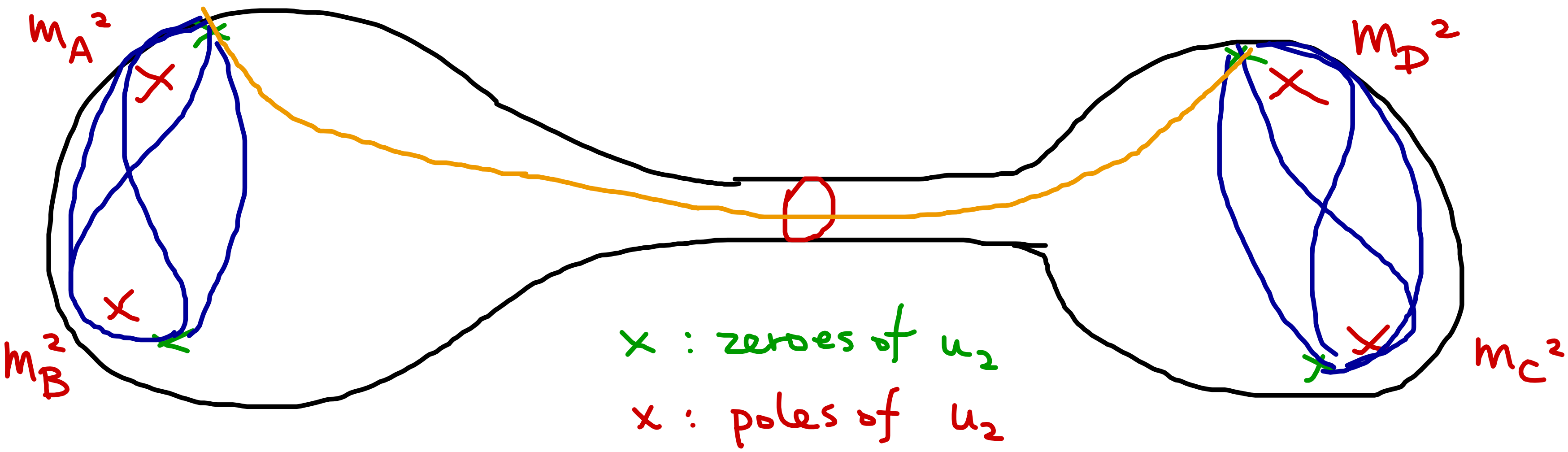


$\times$: poles of u_2

$\times$: zeros of u_2

[Gaiotto 0904.2275]

Take two copies, and connect them

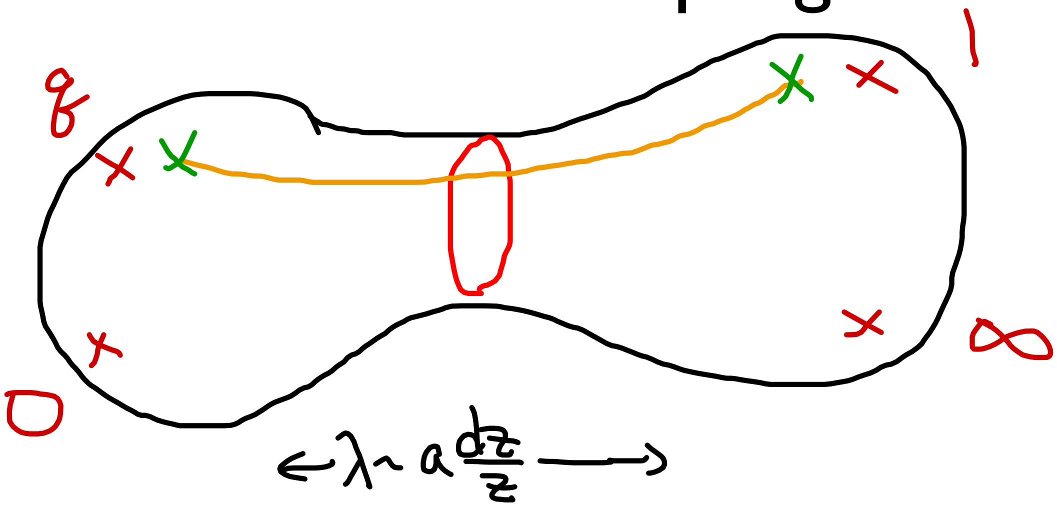


We find:

- an SU(2) vector boson
- two doublets with mass $\frac{1}{2}(m_A \pm m_B)$ from the left
- two doublets with mass $\frac{1}{2}(m_C \pm m_D)$ from the right
- monopoles connecting the left and the right

It's SU(2) with four flavors.

Note that the coupling is tunable.

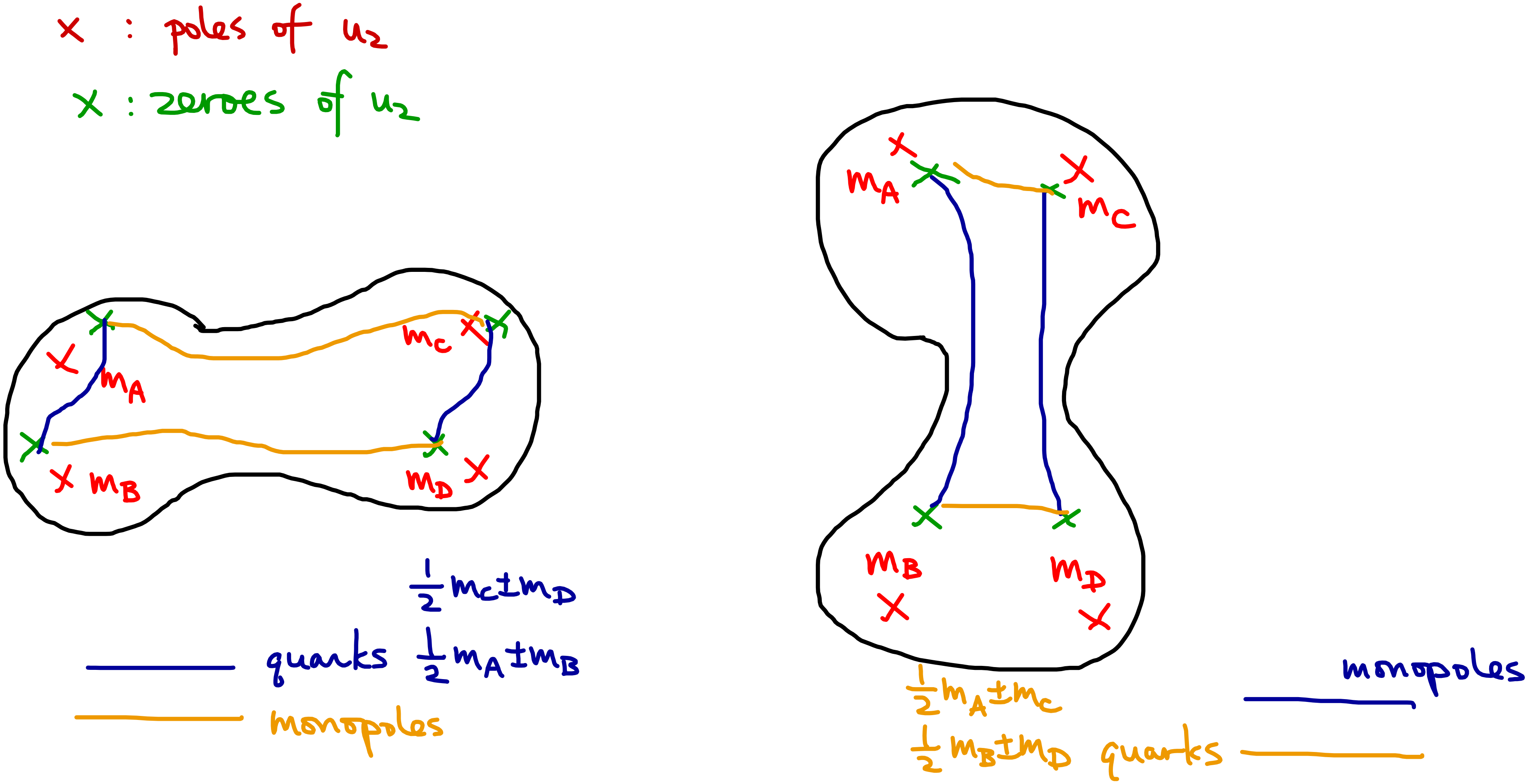


W-boson $\sim a$

monopole $\sim a \log q$

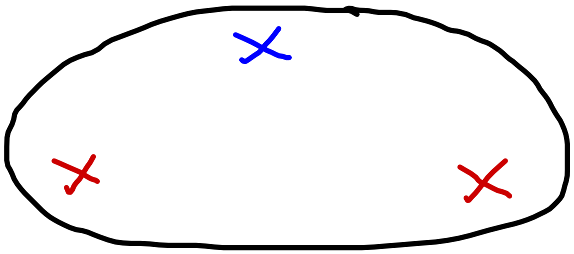
Agrees with the fact that β=0. for SU(2) + 4 flavors

What happens when it becomes very strong?



You get SU(2) with four flavors again,
but the mass is shuffled; new quarks were monopoles.
Reproduces the S-duality originally found in [SW '94]

Similarly, for N M5-branes,

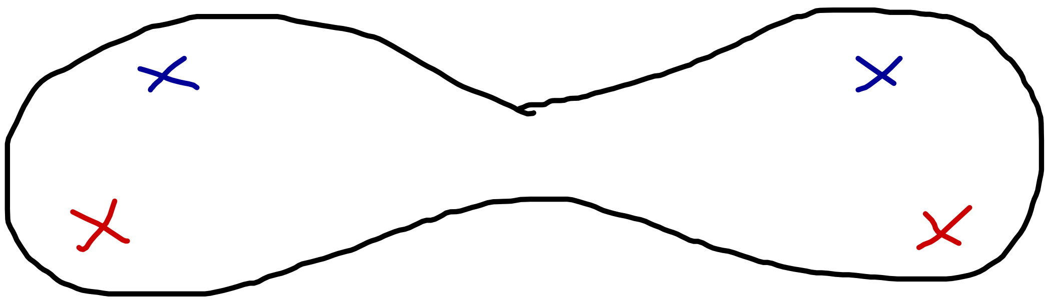


$\times$ $\lambda \sim \mathrm{diag}(m, m, \cdots, (1-N)m)\frac{dz}{z}$

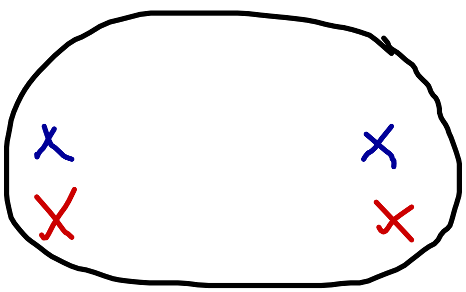
$\times$ $\lambda \sim \mathrm{diag}(m_1, m_2, \cdots, m_N)\frac{dz}{z}$

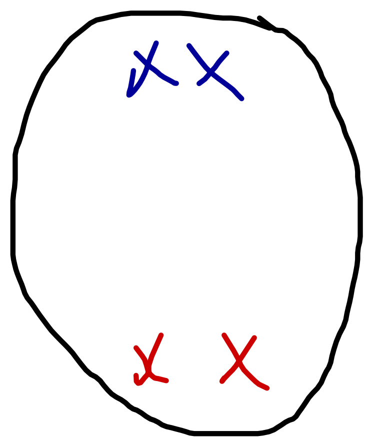
the same for N=2

represents N x N hypermultiplets. Then



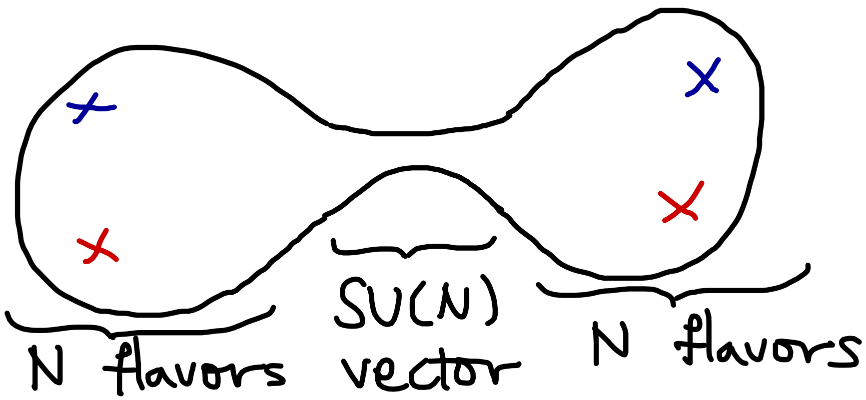
is SU(N) with 2N flavors, with β=0.

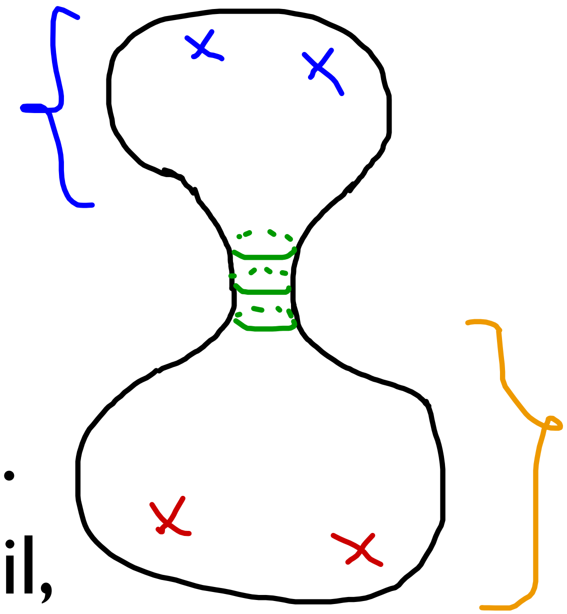




The coupling is tunable.

What happens when it’s become very strong?





In contrast to SU(2), it’s not the same.

I don’t have time to describe it in detail,
but the conclusion is that SU(N) with 2N flavors is dual to

an SU(2) vector boson coupled to
- a doublet hypermultiplet
- a CFT called R_N [Chacaltana-Distler ’10]

Moral: you started from a strange thing (M5).
So you got a strange thing in 4d.