The day before yesterday

Basics of 6d N=(2,0) theory. S-duality of 4d N=4.

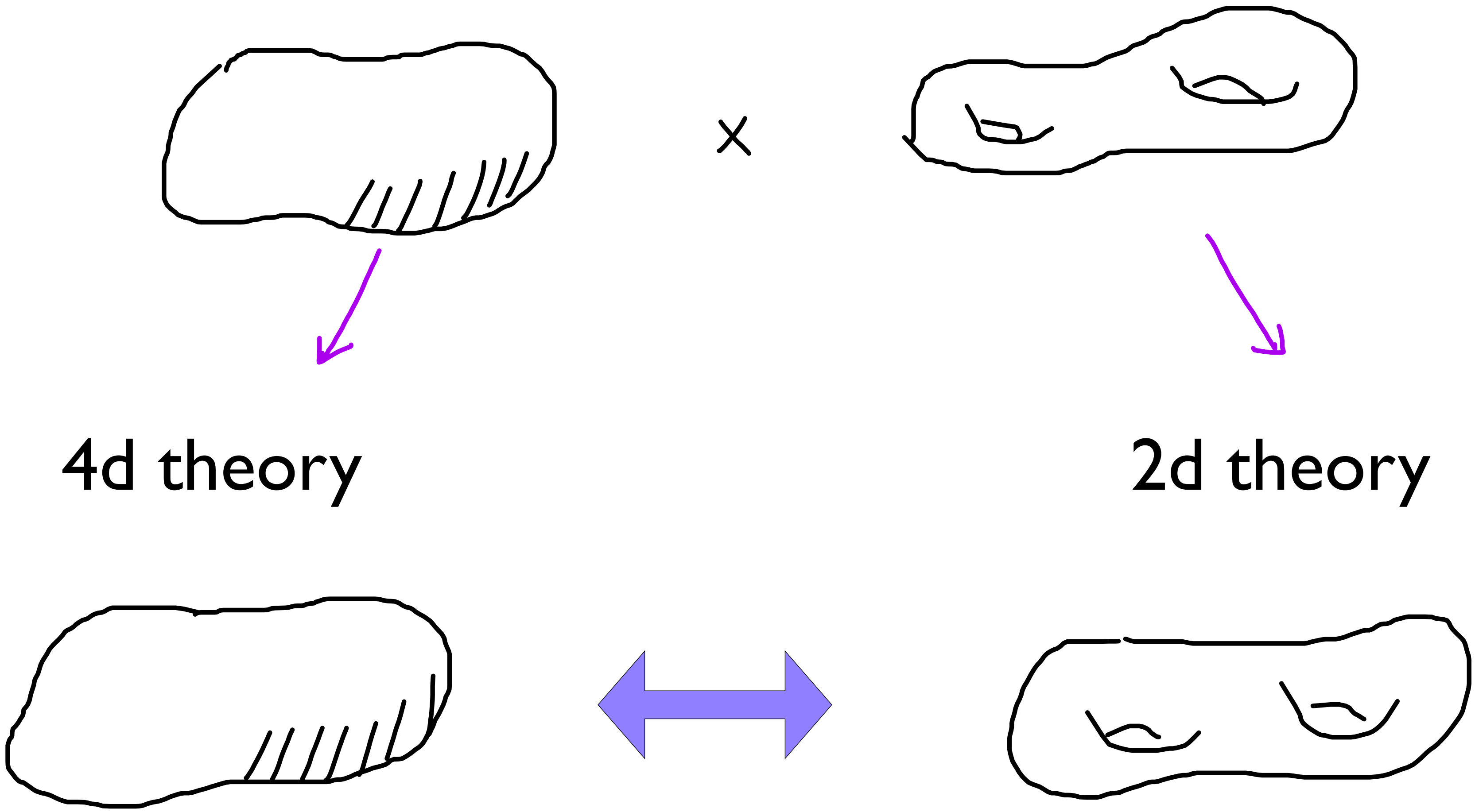
Yesterday

4d N=2 as 6d N=(2,0) compactified on C

Today

Relation with 2d CFT

6d N=(2,0) theory



M on K3

$E_8 \times E_8$ Heterotic on T^3

0 1 2 3 4 5 6 | $\underbrace{7\ 8\ 9\ 10}_{K3}$

0 1 2 3 4 5 6 | $\underbrace{7\ 8\ 9}_{T3}$

1 M5
wrapped on K3

unwound
heterotic string, that has a complicated worldsheet spectrum!

n M5
wrapped on K3

???

Today I study a "simpler" compactification:

K M5s on $R^4_{\varepsilon_1,\varepsilon_2}$

$R^4_{\varepsilon_1,\varepsilon_2}$ sounds non-compact, but $\varepsilon_{1,2}$
introduce an effective centripetal potential

The result is the 2d Toda theory of type A_{K-1},
with $b^2=\varepsilon_1/\varepsilon_2$.

2d Toda theory of type A_1 is the Liouville theory.

I need to talk about

- A bit more about 6d N=(2,0) theory
- What's the 2d Toda theory
- Why you care about this funny compactification on $R^4_{\varepsilon_1,\varepsilon_2}$

I'm afraid I don't have enough time.

Hopefully you'll be well-motivated and well-prepared by then, and can start reading papers on arXiv.

A bit more about 6d N=(2,0) theory

6d N=(2,0) theory is chiral.

Chiral fermions can have anomalies.
Self-dual tensor fields have anomalies, too.

For one free tensor multiplet,

$$I_8 = \qquad + \qquad$$

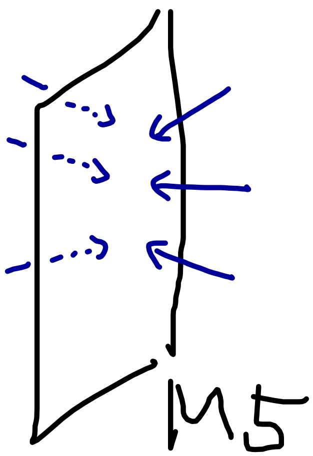
The part. func. depends on the choice of the gauge of SO(1,5) Lorentz rotation and $SO(5)_R$ symmetry.

OK for field theory purposes.

The part. func. depends on the choice of the gauge of SO(1,5) Lorentz rotation and $SO(5)_R$ symmetry.

Not OK for the full M-theory.

We integrate over the 11d metric!



$$I_8 = \underbrace{\square}_{\text{canceled by } \int_{11d} C_3 \wedge \mathrm{tr} R^4} + \underbrace{\square}_{\text{canceled by } \int_{11d} C_3 \wedge dC_3 \wedge dC_3}$$

Called as the anomaly inflow.

Goes back to [Callan-Harvey, '85].

$$I_8\,[\text{an M5}] = \underbrace{\boxed{}}_{\int C_3 \wedge \mathrm{tr} R^4} + \underbrace{\boxed{}}_{\int C_3 \wedge dC_3 \wedge dC_3}$$

What happens for K M5s? We don't know the action.
We can't calculate I_8 directly.
But we know how much anomaly inflow there is.

$$I_8\,[\text{K M5}] = K\,\underbrace{\boxed{}}_{\int C_3 \wedge \mathrm{tr} R^4} + K^3\,\underbrace{\boxed{}}_{\int C_3 \wedge dC_3 \wedge dC_3}$$

because M5 sources $G_4 = dC_3 \overset{*}{\longleftrightarrow} G_7 = dC_6$.

[Harvey-Minasian-Moore '98]

Anomaly ~ (left-moving dof) − (right-moving dof)

#dof ~ K^3

$$I_8\,[K\ M5] = K\,{\color{green}\square} + K^3\,{\color{red}\square}$$

$$= K\left({\color{green}\square} + {\color{red}\square}\right) + (K^3 - K)\,{\color{red}\square}$$

So

$$I_8\,[A_{K-1}] = (K-1)\left({\color{green}\square} + {\color{red}\square}\right) + (K^3 - K)\,{\color{red}\square}\ .$$

$$I_8\,[D_K] = 2K\left({\color{green}\square} + {\color{red}\square}\right) + (2K-2)(2K-1)\,K\,{\color{red}\square}\ .$$

[P.Yi '01]

found by carefully studying inflows to 2K M5s and M-orientifolds.

$$I_8\,[G] = \mathrm{rank}\,G\,\left(\square + \square\right) + h^\vee_G \dim G\ \square$$

conjectured by [Intriligator '00]

	SU(K)	SO(2K)	E_6	E_7	E_8
rank G	$K-1$	K	6	7	8
$h^\vee_G$	K	$2K-2$	12	18	30
dim G	K^2-1	$K(2K-1)$	78	133	248

$$I_8[G] = \mathrm{rank}\,G\,(\square + \square) + h^\vee_G \dim G\,\square$$

The same combination was known from late 80s:

$$c = \mathrm{rank}\,G + h^\vee_G \dim G \left(b + \frac{1}{b}\right)^2$$

The central charge of the 2d Toda theory of type G.
[Hollowood-Mansfield '90]

(similarity noted by [Bonneli-Tanzini '09])

Is there any relation?

6d N=(2,0) theoy on type G on $R^4_{\epsilon_1,\epsilon_2}$

$$\downarrow$$

2d Toda theory of type G with $b^2 = \epsilon_1/\epsilon_2$

KK-reduction of the 6d anomaly reproduces the 2d anomaly "c". [Alday-Benini-YT '09]

What's a 2d Toda theory?

[Bouwknegt-Schoutens '92]
is the standard ref. on W-alg.

Consider a 2d free boson w/ background charge:

$$S = \int d^2x \sqrt{g} \left(\partial_\mu \phi \partial_\mu \phi + \sqrt{2} Q \phi R \right)$$

The central charge is

$$\boxed{c = 1 + 6Q^2}$$

The operator $e^{-\sqrt{2} b \phi}$ has $\boxed{\text{dim} = 2b(Q-b)}$.

An operator is marginal when dim $= 2$.

So let's take $Q = b + 1/b$ and add $e^{-\sqrt{2} b \phi}$ to S.

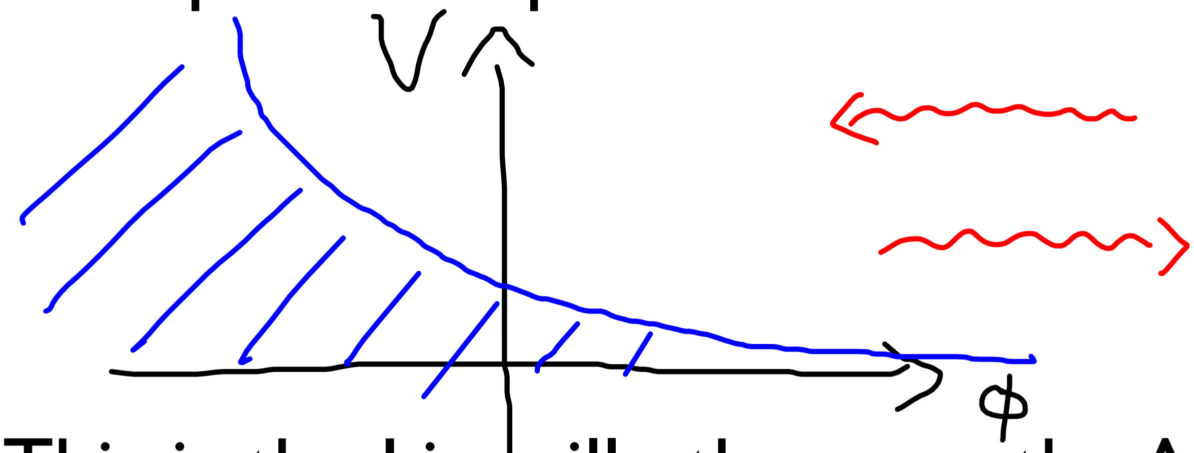
Believed to define an interacting CFT.

The action is

$$S = \int d^2x \sqrt{g} \left(\partial_\mu \phi \partial_\mu \phi + \sqrt{2}\, Q \phi R + \mu e^{-\sqrt{2} b \phi} \right)$$

central charge c = $1 + 6Q^2 = 1 + 6(b + 1/b)^2$.

It describes a wave in the Φ space reflecting off an exponential potential wall



This is the Liouville theory = the A_1 Toda theory.

Which part of this is A_1 ?

Use two bosons Φ_1 , Φ_2

$$S = \int d^2x \sqrt{g} \left(\partial_\mu \phi_i \partial_\mu \phi_i + Q_i \phi_i R \right)$$

The potential $e^{b(\phi_1 - \phi_2)}$ is marginal when

$\dim = b(Q_1 - b) + (-b)(Q_2 + b)$

$$(Q_1, Q_2) = \left(b + \frac{1}{b}\right)(1, -1) .$$

$$c = \sum_i (1 + 3Q_i^2) = 1 + \left[1 + 6(b + 1/b)^2\right]$$

$\Phi_1 + \Phi_2$ is free; $\Phi_1 - \Phi_2$ is interacting.

Φ_1 is reflected off to be Φ_2 .

Think of diag(Φ_1 , Φ_2) . $\Phi_1 \leftrightarrow \Phi_2$ is the Weyl reflection.

Think of them as the diagonal of 2×2 matrix.

Use K bosons $\Phi_1, \Phi_2, \ldots, \Phi_K$

$$S = \int d^2x\sqrt{g}\left(\partial_\mu\phi_i\partial_\mu\phi_i + Q_i\phi_i R\right).$$

The potentials $\boxed{e^{b(\phi_i - \phi_{i+1})}}$ is marginal when

$$\left\{\begin{array}{l} e^{b(\phi_1-\phi_2)} \\ e^{b(\phi_2-\phi_3)} \\ \vdots \\ e^{b(\phi_{K-1}-\phi_K)} \end{array}\right.$$

$$\vec{Q} = (b + 1/b)(K-1, K-3, \cdots, 1-K).$$

$$2 \quad 2 \quad \cdots \quad 2$$

$\Sigma\,\Phi_i$ is free; the others are interacting.

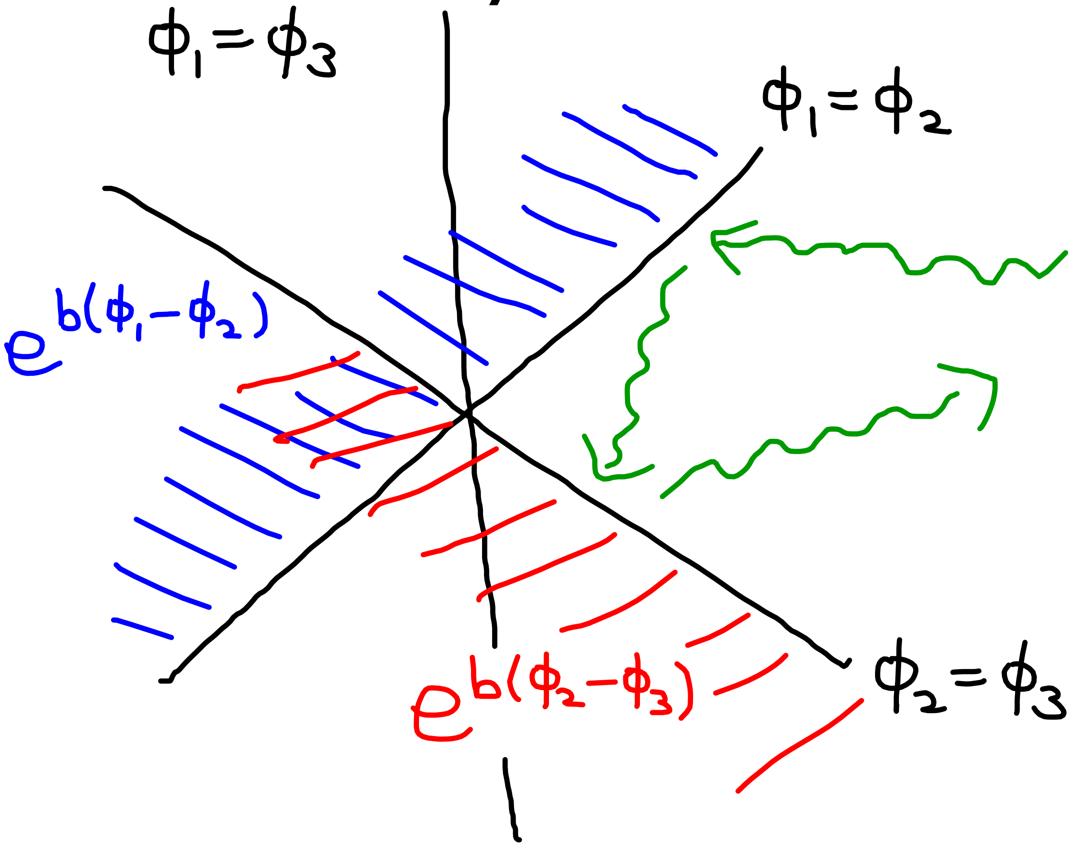
Potentials realize the Weyl reflections $\Phi_i \leftrightarrow \Phi_{i+1}$

$$c = K + 3\left[(K-1)^2 + (K-3)^2 + \cdots + (1-K)^2\right](b + 1/b)^2$$

$$= 1 + \underbrace{(K-1) + (K^3-K)(b+1/b)^2}_{\text{interacting part}}.$$

free part

A$_2$ Toda theory describes a wave bouncing off the wall of the Weyl chamber: of $SU(3)$.



Use r bosons $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_r)$

$$S = \int d^2x \sqrt{g} \left(\partial_\mu \phi_i \partial_\mu \phi_i + Q_i \phi_i R \right)$$

Introduce the potentials $e^{b\vec{\alpha}\cdot\vec{\phi}}$ for the simple roots.

For them to be marginal, the background charge

$$\vec{Q} = 2\left(b + \frac{1}{b}\right)\vec{\rho}$$

needs to satisfy

$$\vec{\alpha}\cdot\vec{\rho} = 1 \quad \text{for all simple roots.}$$

Potentials realize the Weyl reflections.

$$\begin{aligned} c &= r + 12\left(b + \frac{1}{b}\right)^2 |\vec{\rho}|^2 \\ &= \text{rank}\, G + h^\vee_G \dim G \left(b + 1/b\right)^2 . \end{aligned}$$

Toda theories are not just CFT.

For type A_{K-1}, we had $\Phi_1, \ldots, \Phi_k$.

Define $T(z)=W_2(z)$, $W_3(z), \ldots, W_K(z)$ via

$$:(\partial_z-\partial_z\phi_1)(\partial_z-\partial_z\phi_2)\cdots(\partial_z-\partial_z\phi_K):$$
$$=\partial_z^K+T(z)\partial_z^{K-2}+W_3(z)\partial_z^{K-3}+\cdots+W_K(z).$$

They are conserved currents,
and generates the $W(A_{K-1})$-algebra.
You saw something similar yesterday.

positions of M5s

K M5 on C was described by

$$(\lambda - \phi_{(1)}(z))(\lambda - \phi_{(2)}(z)) \cdots (\lambda - \phi_{(K)}(z))$$

$$= \lambda^K + u_2(z)\lambda^{K-2} + u_3(z)\lambda^{K-3} + \cdots u_K(z).$$

A_{K-1} Toda theory has

$$:(\partial_z - \partial_z\phi_1)(\partial_z - \partial_z\phi_2)\cdots(\partial_z - \partial_z\phi_K):$$

$$= \partial_z^K + W_2(z)\partial_z^{K-2} + W_3(z)\partial_z^{K-3} + \cdots W_K(z).$$

We postulate $\quad u_i(z) \longleftrightarrow \langle W_i(z) \rangle dz^i$

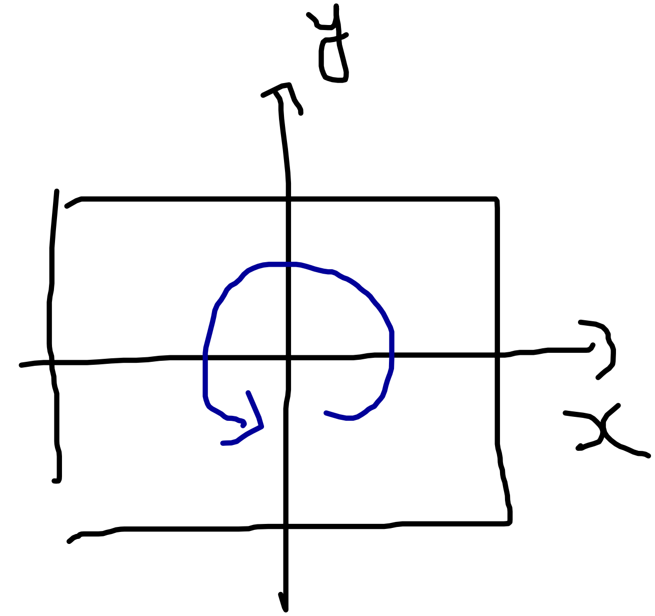
when compactifed on $R^4_{\varepsilon_1,\varepsilon_2}$.

This underlies all of the correspondence
(which people call in the way I can't).

What is this compactification on $R^4_{\varepsilon_1, \varepsilon_2}$, anyway?

Suppose you're asked the volume of R^2 .

$$\int dx\,dy = \infty \quad \cdots$$



You have a rotational symmetry.

Think of x & y canonically conjugate.

Rotation is generated by x^2+y^2.

Regularize by it:

$$\int dx\,dy \;\exp(-\varepsilon(x^2+y^2))$$

$$= \pi/\varepsilon .$$

We get something meaningful.

and diverges when ε is taken away.

In general, for a (path) integral,

$$Z = \int [D\phi] \, \exp(-S)$$

If the integration region has

- the structure of the phase space
- symmetries generated by H_i

You can regularize it using the Hamiltonians:

$$Z_{\varepsilon_i} = \int [D\phi] \, \exp\left(-S + \sum \varepsilon_i H_i\right)$$

Compactification on $R^4_{\varepsilon_1,\varepsilon_2}$ is one instance of this.

$$\begin{cases} \varepsilon_1 : \text{rotation of } x^1 - x^2 \text{ plane} \\ \varepsilon_2 : \text{rotation of } x^3 - x^4 \text{ plane} \end{cases}$$

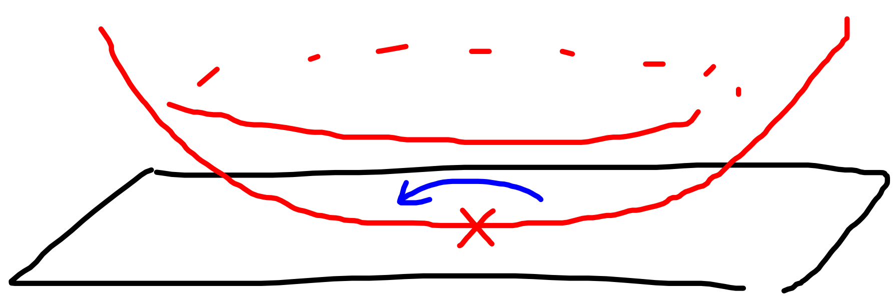
Rotational symmetries of R^4 induce symmetries on the space of configurations of fields.

$$\phi(x_i) \longrightarrow \phi(x_i)' \equiv \phi(R^j{}_i x_j).$$

You add the "Hamiltonians" for that symmetry to the action.

$$Z_{\varepsilon_1,\varepsilon_2} = \int [D\phi] \exp(-S + \varepsilon_1 H_1 + \varepsilon_2 H_2).$$

Note that H_i breaks Lorentz Invariance.



Why do we care?

For an N=2 gauge theory, the partition function on R^4

$$Z_{\varepsilon_1,\varepsilon_2} = \int [DAD\psi D\phi] \exp\left(-S - \varepsilon_1 H_1 - \varepsilon_2 H_2\right)$$

is finite, and behaves in the limit $\varepsilon_1, \varepsilon_2 \to 0$ as

$$\log Z_{\varepsilon_1,\varepsilon_2} = \frac{1}{\varepsilon_1 \varepsilon_2} F(a_1 \cdots a_r) + \text{finite terms}$$

where $F(\underbrace{a_1, a_2, \ldots, a_r}_{\text{vev of the vector-multiplet scalar.}})$ is the low-energy prepotential.

[Nekrasov '03]

And it's computable.

[Moore-Nekerasov-Shatashvili '97]

(Knowing prepotential = knowing the SW curve)

For pure N=2 gauge theory, Nekrasov's Z

$$Z_{\varepsilon_1,\varepsilon_2} = \int [DAD\psi D\phi]\, \exp(-S - \varepsilon_1 H_1 - \varepsilon_2 H_2)$$

becomes

$$= \sum_k \int d^{4Nk}X \, \exp\left(-S[A[X;\vec{x}]] + \varepsilon_1 H_1[A[X;\vec{x}]] + \varepsilon_2 H_2[A[X;\vec{x}]]\right)$$

parametrize the instanton configurations solving $F^a_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma} F^a_{\rho\sigma}$

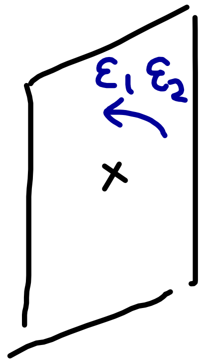
instanton number $k = \frac{1}{32\pi^2}\int \mathrm{tr}\, F\wedge F$

= the sum of

the regularized volumes of

the instanton moduli spaces of G.

6d N=(2,0) theory



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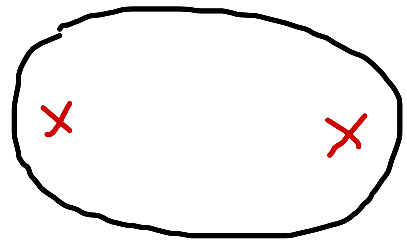


4d theory

pure $\mathcal{N}=2$ $SU(N)$ on $\mathbb{R}^4_{\varepsilon_1,\varepsilon_2}$

$$\sum_k \Lambda^{4Nk} \left(\text{reg. vol. of inst. moduli with } \#\,\text{inst} = k\right)$$

$\longleftrightarrow$

2d theory

A_{N-1} Toda on two-punc. sphere

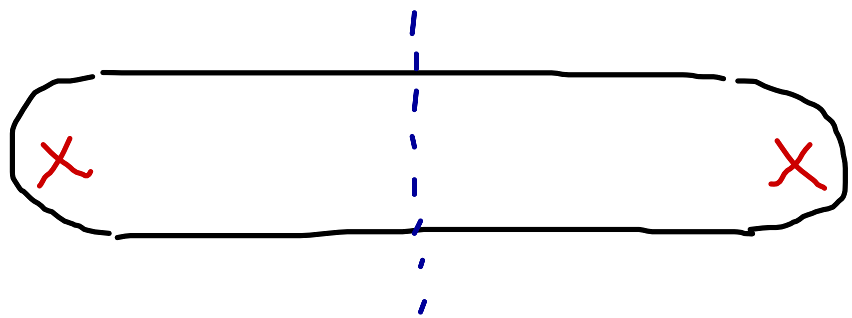
???

The M5 configuration was

$$\lambda^N + u_2(z)\,\lambda^{N-2} + u_3(z)\,\lambda^{N-3} + \cdots + u_N(z) = 0$$

where

$$u_2(z) = \underline{u_2}\left(\frac{dz}{z}\right)^2, \quad u_3(z) = \underline{u_3}\left(\frac{dz}{z}\right)^3, \quad \ldots, \quad u_N(z) = \left(\frac{\Lambda^N}{z} + \underline{u_N} + \Lambda^N z\right)\left(\frac{dz}{z}\right)^N.$$

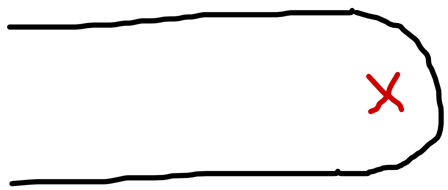


$$\langle \Psi | \Psi \rangle$$

This is an inner product of a state with itself.

What is the state?

For simplicity, consider A_1 theory (i.e. 2 M5 branes.)



$|\Psi\rangle$

$$u_2(z) \sim \underline{u_2}\,\frac{dz^2}{z^2} + \Lambda^2 \frac{dz^2}{z^3} \quad \text{at } z \sim 0$$

We identify $u_2(z)$ with $T(z)dz^2$:

$$T(z)dz^2 \sim L_0 \frac{dz^2}{z^2} + L_1 \frac{dz^2}{z^3} + L_2 \frac{dz^2}{z^4} + \cdots$$

$$\longrightarrow \quad L_0|\Psi\rangle \sim \underline{u_2}\,|\Psi\rangle$$
$$L_1|\Psi\rangle \sim \Lambda^2\,|\Psi\rangle.$$

creation ops $\longleftarrow$ $\longrightarrow$ annihilation ops

$$\cdots \; L_{-3} \; L_{-2} \; L_{-1} \; L_0 \; L_1 \; L_2 \; L_3 \; \cdots$$

This is a coherent state of the Virasoro algebra.

A bit more detail: a rep. of Virasoro is generated from

$$L_0|\Delta\rangle = \Delta|\Delta\rangle\,,\quad L_1|\Delta\rangle = 0\,,\quad L_2|\Delta\rangle = 0\,,\quad \cdots$$

Then

$$\begin{tikzcd} & & (L_{-1})^2|\Delta\rangle & \\ |\Delta\rangle \arrow[r, bend left, "L_{-1}"] \arrow[rr, bend right, "L_{-2}"'] & L_{-1}|\Delta\rangle \arrow[ur, bend left, "L_{-1}"] & L_{-2}|\Delta\rangle & \cdots \end{tikzcd}$$

Take $|\psi\rangle = |\Delta\rangle + \square\, L_{-1}|\Delta\rangle + \bigcirc\, L_{-1}^2|\Delta\rangle + \bigcirc\, L_{-2}|\Delta\rangle + \cdots$

Demand

$$L_1|\psi\rangle = q|\psi\rangle\,.$$

Then

$$\square\, L_1 L_{-1}|\Delta\rangle = q|\Delta\rangle \longrightarrow \square = \frac{q}{2\Delta}$$

$$\therefore\ \langle\psi|\psi\rangle = 1 + \frac{q^2}{2\Delta} + \cdots\,.$$

On the instanton side, the 1-inst. moduli space of SU(2) is

$$\mathbb{R}^4 \times \mathbb{R}^4/\mathbb{Z}_2 \quad \begin{pmatrix} a & \\ & -a \end{pmatrix}$$

$$\varepsilon_1, \varepsilon_2 \qquad \left(\frac{\varepsilon_1+\varepsilon_2}{2}+a\right), \left(\frac{\varepsilon_1+\varepsilon_2}{2}-a\right)$$

$$\therefore \text{reg. vol.} = \frac{1}{\varepsilon_1\varepsilon_2} \cdot \frac{1}{2} \frac{1}{\frac{\varepsilon_1+\varepsilon_2}{2}+a} \cdot \frac{1}{\frac{\varepsilon_1+\varepsilon_2}{2}-a}$$

$$\therefore \sum_k \Lambda^{4k} (\text{reg. vol}) = 1 + \frac{\Lambda^4}{(\varepsilon_1\varepsilon_2)^2} \frac{1}{2} \frac{1}{\frac{(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2} - \frac{a^2}{\varepsilon_1\varepsilon_2}} + \cdots$$

$$q = \Lambda^2/\varepsilon_1\varepsilon_2 \qquad \updownarrow \qquad \Delta = \frac{(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2} - \frac{a^2}{\varepsilon_1\varepsilon_2}$$

$$\langle \psi | \psi \rangle = 1 + \frac{q^2}{2\Delta} + \cdots$$

Of course the 1st terms agree by choosing the dictionary carefully. The point is that the subleading corrections also work.

Virasoro: easily calculable.

$$\langle\psi|\psi\rangle = 1 + \frac{q^2}{2\Delta} + \frac{q^4(c+8\Delta)}{4\Delta\left[(1+2\Delta)c - 10\Delta + 16\Delta^2\right]} + \cdots$$

Instantons : use the formula of Moore-Nekrasov-Shatashvili

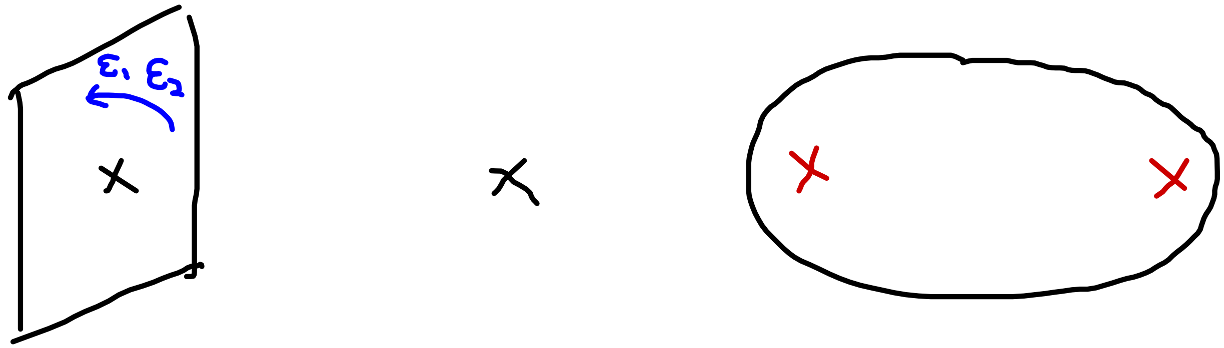
$$\boxed{\begin{aligned} q &= \Lambda^2/\varepsilon_1\varepsilon_2 \\ c &= 1 + 6(\varepsilon_1+\varepsilon_2)^2/\varepsilon_1\varepsilon_2 \\ \Delta &= \frac{(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2} - \frac{a^2}{\varepsilon_1\varepsilon_2}. \end{aligned}}$$

$$\sum_k \Lambda^{4k}(\text{reg. vol.})$$

$$= 1 + \frac{\Lambda^4}{2\varepsilon_1\varepsilon_2}\frac{1}{(\varepsilon_1+\varepsilon_2)^2 - a^2}$$

$$+ \frac{\Lambda^8\left[(\varepsilon_1+\varepsilon_2)^2 + \frac{\varepsilon_1\varepsilon_2}{8} - a^2\right]}{8(\varepsilon_1\varepsilon_2)^2\left(\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)^2 - a^2\right)\left(\left(\varepsilon_1+\frac{\varepsilon_2}{2}\right)^2 - a^2\right)\left(\left(\frac{\varepsilon_1}{2}+\varepsilon_2\right) - a^2\right)} + \cdots$$

6d N=(2,0) theory



4d theory

$\mathcal{N}=2$ pure $SU(N)$
on $\mathbb{R}^4_{\varepsilon_1,\varepsilon_2}$

$\sum_k$ (reg. vol. inst. mod.)
#inst $=k$

2d theory

A_{N-1} Toda theory
on two-punc. sphere

$\langle \psi | \psi \rangle$

a special coherent state.

The sum of

the regularized volumes of

the instanton moduli spaces of G.

should be equal to

The inner product with itself of

the coherent state in the Verma module of

the W-algebra associated to G.

[Gaiotto, '09]

Aug.

Calculating them in both sides, order by order, is easy.

Algorithms are both known; you just have to
implement them in Mathematica/Maple,
or if you're a faculty, let the student do it.

I don't have the luxury yet.

Anyway. Both sides produce horrible expressions,
but they should agree.

For SU(2), the equality was proved. [Fateev-Litvinov, '09]

For SU(N), even the order-by-order check ~~was~~ has not been done.

Please do if you're interested.

It would be a good exercise for you to learn

- details of the analysis of instantons in 4d
- details of the structure of the W-algebras in 2d