Lecture II-1: Symmetry breaking

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In this semester we will continue the discussion of quantum field theory, but now mostly dynamics rather than perturbation theory and purely formal things. The topics we will discuss are much more difficult to deal with rigorously than the formal theory we studied in the fall semester. We will try to do it when possible, but it will not always be possible. In general, we will try to form an intuitive picture of what is going on, and illustrate it by considering concrete examples.

1.0. Theories and realizations. [This section is explanatory and was written by the preparers of these notes in order to create a false sense of security i.e. an unsubstantiated feeling that we understand what we are talking about in the rest of the lecture. Unfortunately, this is not the case, at least if "understand" means what it usually means among mathematicians.]

For the purposes of the present and forthcoming lectures, it will be important for us to distinguish theories and their realizations. So let us explain what we mean by a theory and what we mean by its realization. This explaination is not a mathematical definition (in fact, it is hard to give a definition which is both rigorous and useful), but we hope that it will make clear what we are talking about.

Recall that a classical physical system Σ is usually described by defining the space of states X of Σ (a symplectic manifold, maybe infinite-dimensional) and a 1-parameter group g^t of time translations which preserves symplectic structure. Then g^t produces a Hamiltonian flow, which is defined (ignoring topological problems) by a Hamiltonian function H. This function is called the Hamiltonian, or energy function of the system. One should remember that H is defined only up to adding a (locally) constant function.

In most examples, H is bounded from below. In such a case, H is always normalized in such a way that the infinum of H on (each connected component of) X is zero.

In this situation, by **a theory** we mean a pair (X, H), where X is a symplectic manifold, and $H : X \to \mathbb{R}$ the energy function, defined up to a (locally) constant function. By **a vacuum state of this theory** we mean a lowest energy equilibrium state $x \in X$ of the system Σ, i.e. a state where the energy functional H attains a global minimum.

The same theory can have different vacuum states. For example, if we have a particle on the line with potential energy $U(x) = \frac{g}{4!}(x^2 - a^2)^2$, then the space X is the phase plane $\mathbb{R}^2$ with coordinates (x, p), the Hamiltonian is $\frac{p^2}{2} + U(x)$, and there are two vacuum states $(a, 0)$ and $(-a, 0)$.

Often a classical system Σ can be described by a Lagrangian $\mathcal{L}$, defined on some space of fields S on the spacetime V. In this case its space of states is the space $X \subset S$ of extremals of $\mathcal{L}$. As we have seen before, the space X carries a natural closed 2-form ω, which is nondegenerate, and thus defines a symplectic structure on X. Also, the group of time translations acts on X and preserves ω. Therefore, the flow on X generated by this 1-parameter group is Hamiltonian, and is defined by a Hamiltonian function H.

However, one should remember that the description of a theory by a Lagrangian is not intrinsic, since different Lagrangians defined on different spaces of fields S may define the same theory. Indeed, consider, for example, the Lagrangian $\mathcal{L}_1 = \int x'(t)^2 dt/2$ defined on $S_1 = C^\infty(\mathbb{R}, M)$, where M is a Riemannian manifold. This Lagrangian defines the geodesic flow. The space of states in this case is T^*M, and the Hamiltonian function is $p^2/2$. On the other hand, we can write the Lagrangian $\mathcal{L}_2 = \int(-x'(t)p(t) + p^2(t)/2)dt$ defined on the space $S_2 = C^\infty(\mathbb{R}, T^*M)$. It is easy to see that these two Lagrangians define the same theory.

We will consider relativistically invariant theories (X, H), i.e. theories with an action of the

Poincare group **P** on X, which preserves the symplectic structure (in the case when the system is defined by a Lagrangian, the group **P** acts on S and preserves the Lagrangian, and therefore, it acts on X preserving the Poisson structure). The group of time translations is a subgroup of **P**, so **P** also preserves the energy function H. In this case, when we talk about vacuum states of the theory (X, H), we mean a vacuum state invariant under **P**.

Recall that to define a symplectic manifold X is the same thing as to define the Poisson algebra $A = C^\infty(X)$ (X can be reconstructed as the spectrum of A). Therefore, we may say that a theory is a pair (A, H), where A is a Poisson algebra and $H \in A$. The Poisson algebra A is called the algebra of observables of the system.

Now let us consider quantum systems. The definition of a theory in this case is similar to the classical case. Namely, we will define **a quantum theory** to be a pair (A, H), where A is a *-algebra (not necessarily commutative), and H is a selfadjoint element of A, defined up to adding a real number. The algebra A is called the algebra of quantum observables (operators). The element H, as before, is called the Hamiltonian. Everything here is dependent on a real positive parameter $\hbar$ (the Planck constant).

By **a realization (or solution)** of a quantum theory (A, H) we will mean an irreducible *-representation of the algebra A in some Hilbert space $\mathcal{H}$, such that the spectrum of the operator H is bounded from below (representations are considered up to an isomorphism which preserves H). We will always normalize H so that the lowest point of its spectrum is zero. The space $\mathcal{H}$ is called the quantum space of states of the system (in this realization). Of course, as in the classical case, the same theory can have different realizations, as the same algebra can have different representations.

As we have mentioned, we will be interested in relativiatically invariant theories, i.e. theories with an action of the Poincare group on A, so that the subgroup of time translations acts by $a \to e^{itH/\hbar} a e^{-itH/\hbar}$. When we talk about realizations of such a theory, we will assume that **P** acts in $\mathcal{H}$ by unitary operators, with the group of time translations acting by $e^{itH/\hbar}$.

By a vacuum state we mean a vector $\Omega \in \mathcal{H}$ such that $Hv = 0$. In a relativistically invariant situation, a vacuum state is the same thing as a **P**-invariant vector.

Remark. In general, as we will see, an irreducible realization of a theory can have many vacuum states. Therefore, the notions of a realization and of a vacuum state are not equivalent. However, if the algebra of observables is commutative (i.e. in the classical theory), each irreducible representation of this algebra is 1-dimensional, and there is no real difference between the notions of a realization and a vacuum state. Therefore, the word "realization" is not usually used when one refers to the classical theory.

Suppose that we have a classical theory (A_0, H_0) which has been quantized, and the corresponding $\hbar$-dependent family of quantum theories is (A, H) (here by an $\hbar$-dependent family we mean a family depending on the dimensionless parameter $\hbar/S_0$, where S_0 is a characteristic scale of action). This means that we have a quantization map – some linear map $A_0 \to A$, given by $a \to \hat{a}$, such that $\hat{H}_0 = H$, and $[\hat{a}, \hat{b}] = i\hbar\widehat{\{a, b\}} + o(\hbar)$, $\hbar \to 0$. In this case, we will say that a state $v \in \mathcal{H}$ of norm 1 is localized near a classical solution $x \in X = \mathrm{Spec}A_0$ if for any $a \in A_0$ $\langle v, \hat{a}v\rangle \to a(x)$, $\hbar \to 0$.

Let us now explain the connection between the classical and the quantum notions of a vacuum state. Suppose we have a quantum vacuum state Ω of norm 1 which is localized near a classical state x. In this case x is a classical vacuum state. Indeed, $H_0(x) = \lim_{\hbar\to 0}\langle\Omega, H\Omega\rangle = 0$, and for any $F \in A_0$ $\{F, H_0\}(x) = \lim_{\hbar\to 0} \frac{1}{i\hbar}\langle\Omega, [\hat{F}, H]\Omega\rangle = 0$, so x is a stationary point of H_0.

Remark. Note any quantum vacuum state is localized near a classical vacuum state. Sometimes a quantum vacuum state is "spread" with some density over the set of classical vacuum states. We will see examples of this in today's lecture.

Given a quantum theory (A, H), it is convenient to represent its realizations by correlation func-

tions. Namely, given a realization $\mathcal{H}$ of this system, and a vacuum state $\Omega \in \mathcal{H}$, we can define correlation functions $\langle \Omega, L_1 ... L_n \Omega \rangle$, where $L_i \in A$. Since the action of A in $\mathcal{H}$ is irreducible, the realization $\mathcal{H}$ can be completely reconstructed from these correlation functions.

Remark. The irreducibility condition is not always satisfied in physically interesting examples. But here for simplicity we will assume that it is satisfied.

Sometimes a quantum system can be defined by a Lagrangian. Of course, as in the classical case, this is not always possible, and if possible, not in a unique way. However, such a presentation is very convenient for understanding the behaviour of the system. So let us explain (on examples) how to pass from a Lagrangian to the Hamiltonian and the operator algebra.

We will start with the case of quantum mechanics, when the spacetime is just the time line. Consider a Lagrangian for one boson:

$$\mathcal{L} = \int [\frac{(\phi')^2}{2} - U(\phi)]dt.$$

Then, by definition, the operator algebra is generated by operators ϕ_0, ϕ_0', H, with the canonical commutation relations

$$[\phi_0, \phi_0'] = i\hbar,\ [H, \phi_0] = -i\hbar\phi_0',\ [H, \phi_0'] = i\hbar U'(\phi_0).$$

Define the local operators $\phi(t) = e^{iHt/\hbar}\phi_0 e^{-iHt/\hbar}$, $\phi'(t) = e^{iHt/\hbar}\phi_0' e^{-iHt/\hbar}$. It follows from the above definition that $d\phi/dt = \phi'$, and ϕ satisfies the Newton's differential equation

$$\phi''(t) = -U'(\phi(t)).$$

The operators of the form $F(\phi(t), \phi'(t))$, where F is a polynomial, are called local operators at t (we order products in such a way that ϕ' stands on the right from ϕ).

The operator algebra is spanned (topologically) by operators $\phi(t_1)...\phi(t_n)$. Thus, a realization of the theory is defined by prescribing expectation values of these operators – the correlation functions.

In any realization of the theory, the Hamiltonian is given by the following explicit formula:

$$H = \frac{1}{2}(\phi')^2 + U(\phi) + C.$$

Indeed, the difference of the left and the right hand sides of this equation commutes with ϕ and ϕ', so by irreducibility it acts by a scalar.

Now consider quantum field theory. We first consider the theory in a spacetime $V = L \times \mathbb{R}$, where L is a lattice (finite or infinite). Let ∇_L be the discrete gradient operator on the lattice. Consider a Lagrangian

$$\mathcal{L} = \sum_{x \in L} \int dt[\frac{1}{2}(\phi_t^2(x,t) - (\nabla_L \phi(x,t))^2) - U(\phi(x,t))]$$

In this case the operator algebra is generated by operators $\phi(x,0), \phi_t(x,0)$, $x \in L$, and H, satisfying the commutation relations

$$[\phi(x,0), \phi_t(x,0)] = i\hbar,\ [H, \phi(x,0)] = -i\hbar\phi_t(x,0), \tag{1}$$

$$[H, \phi_t(x,0)] = i\hbar[-\Delta_L\phi(x,0) + U'(\phi(x,0))], \tag{2}$$

(where Δ_L is the lattice Laplacian) and such that $\phi(x,0), \phi'(x,0)$ commute with $\phi(y,0), \phi'(y,0)$ if $x \neq y$ (causality). The local operators $\phi(x,t), \phi'(x,t)$ are defined as above. (Observe that since H does not commute with ϕ, for $t_1 \neq t_2$ the operators $\phi(x_1,t_1)$, $\phi(x_2,t_2)$, in general, do not commute).

As before, the operator algebra is spanned by the operators $\phi(x_1,t_1)...\phi(x_n,t_n)$. Thus, a realization of the theory is determined by expectation values of these operators – the correlation functions.

As in the case of quantum mechanics, in any realization we can compute the Hamiltonian explicitly. Namely, the Hamiltonian is of the form $H = \sum_{x\in L} H_x$, where $H_x = \frac{1}{2}(\phi_t^2(x,0) + (\nabla_L\phi(x,0))^2) + U(\phi(x,0)) + C_x$.

Remark. Of course, if the lattice L is infinite, the definition of H can be problematic, since the sum over L may be divergent. However, since commutators of H with other operators are well defined, one may hope that the constants C_x can in fact be adjusted in such a way that the sum converges. This is indeed true in many situations.

Now let us consider field theory in continuous spacetime. In this case the operator algebra and the Hamiltonian are defined similarly to the case of discrete space, which was considered above. Namely, the operator algebra will be generated by the operators $\phi(x,0)$, $\phi_t(x,0)$, and also H_b, $b \in \mathfrak{p}$, where $\mathfrak{p}$ is the Lie algebra of the Poincare group $\mathbf{P}$, with the commutation relations between ϕ, ϕ_t and H_b similar to the above.

However, we will face an additional problem – now expressions like $\phi^2(x,t)$ may not be well defined, because of ultraviolet divergences. This problem can be cured by the ultraviolet renormalization theory, which we discussed last semester, if the Lagrangian we started with was renormalizable. In this case, the algebra of local operators is not quite an algebra, but an OPE algebra (an algebra with operator product expansion). It is almost never possible to compute the structure constants of this algebra exactly (rational conformal field theory in 2 dimensions is the main exception), but it is possible to compute them in perturbation expansion to any finite order. In general, for continuous space we have additional analytic difficulties (compared to the case of discrete space), but they will not be very important in the present lecture, so we will not discuss them here. We will just need the rough general picture, which has been outlined in this introduction.

Finally, let us say what we will mean by a **vacuum** for a quantum theory (A,H). We will mean by a vacuum for (A,H) one of two, roughly equivalent, things:

1. A linear functional $\langle,\rangle : A \to \mathbb{C}$ on the operator algebra (the expectation value), which satisfies some field theory axioms (e.g. axioms for Wightman functions);

2. A realization $\mathcal{H}$ of (A,H) together with a vacuum state Ω, normalized to unity.

The passage from 2 to 1 is trivial, and the passage from 1 to 2 is a part of the general formalism of field theory (see Kazhdan's lecture 1).

In general, a vacuum is not the same thing as a realization, since the same realization can have different vacua. For example, in the theory of Dirac operator on a manifold the space of vacua is the space of harmonic spinors. However, in a Wightman field theory in infinite volume, one can show that any realization has exactly one vacuum state.

1.1. What is symmetry breaking, and why it does not happen in quantum mechanics.

Suppose we have some classical physical theory (A,H), which has a symmetry group G. Let us ask the following question: does this theory have a G-invariant vacuum state?

If we have a quantum theory (A,H), which has a symmetry group G, then the correct analogue of this equestion is: does this theory have a G-invariant realization?

If the answer is no, one says that the symmetry is broken. If the answer is yes, one says that the symmetry is preserved.

In classical mechanics and classical field theory symmetry breaking can easily happen. For example, consider a classical particle of mass 1 on the line whose potential energy is $U(x) = g(x^2-a^2)^2/4!$, where $a > 0, g > 0$. The space of states of this particle is the plane with coordinates x,p, and its Hamiltonian is $\frac{p^2}{2} + U(x)$. There is an action of the group $G = \mathbb{Z}/2\mathbb{Z}$ on the space of states, by

$(x,p) \to (-x,-p)$, which preserves the equations of motion, and there are two lowest energy states: $s_+ = (a,0), s_- = (-a,0)$, which are permuted by G. But there is no lowest energy state which is G-invariant.

In quantum mechanics, symmetry breaking does not occur. This is a simple, but nontrivial and very important result. Let us show why this is true in the case of quantum mechanics of bosons, with a real Lagrangian. In this case, the operator algebra is generated by operators x_i, p_i, $i = 1, ..., n$, satisfying the Heisenberg algebra relations, and has a Hamiltonian $H = \frac{p^2}{2} + U(x)$. There is a realization of the theory in $\mathcal{H} = L^2(\mathbb{R}^n)$, with x_i acting by multiplication by the coordinate functions, and $p_i = -i\hbar\frac{\partial}{dx_i}$. It is well known this representation is irreducible. Any symmetry group G of the potential U also acts in $\mathcal{H}$. Thus, symmetry breaking does not occur.

Remark. In fact, according to the Stone-von-Neumann theorem, $\mathcal{H}$ is the unique realization of the theory.

In the case of bosons with real Lagrangian we can in fact make a stronger statement. Namely, not only is the realization unique, but the vacuum is also unique (and therefore invariant under any symmetry group of U). For simplicity we will show it in the case of only one boson on the line, but the argument we will give generalizes to any number of bosons in a space of any dimension.

We will consider a single boson on the line, in a field with potential $U(x)$ as above.

Theorem 1.1 *Let $H = -\frac{1}{2}\frac{d^2}{dx^2} + U(x)$ be any Schrödinger operator, such that the potential $U(x)$ tends to $+\infty$ at infinity (so that H has discrete spectrum). Let E_0 be the smallest eigenvalue of H. Then there exists a unique, up to a factor, function $\psi \in L^2(\mathbb{R})$ (called the vacuum state wave function) such that $H\psi = E_0\psi$.*

Proof For any $f \in L^2(\mathbb{R})$ we have

$$(f, Hf) = \int_{-\infty}^{\infty}\Big(\frac{1}{2}|f'(x)|^2 + U(x)|f(x)|^2\Big)dx \tag{1.1}$$

Thus, ψ is defined by the condition that it is a global minimum point for the energy functional $E(f) := (f, Hf)$ on the sphere $||f|| = 1$. The proof of uniqueness of ψ rests on the following Lemma.

Lemma. If ψ is a real global minimum point of (1.1) than ψ has constant sign.

Proof of the Lemma. Let $E(\psi) = E_0$. Suppose that ψ changes sign at the point x_0. Since ψ satisfies the Euler-Lagrange (=Schrödinger) equation $H\psi = E_0\psi$, we have $\psi'(x_0) \neq 0$. Consider the function $|\psi|$. It is clear that $E(|\psi|) = E(\psi) = E_0$, but $|\psi|$ is not smooth, so it does not satisfy the Euler-Lagrange equation $Hf = E_0 f$, and thus cannot be the global minimum of E. So, the smallest value of E on the sphere is less than E_0 – a contadiction.

Now it is easy to prove the theorem. If the space of solutions of $H\psi = E_0\psi$ is more than 1-dimensional, then there exist two linearly independent, orthogonal real solutions ψ_1, ψ_2. On the other hand, both of them have to be of constant sign, so $(\psi_1, \psi_2) \neq 0$ – a contradiction. □

It is useful to consider how symmetry breaking, which is absent in the quantum theory, arises in the quasiclassical limit. For this purpose, we should introduce the Planck's constant $\hbar$, and consider the $\hbar$-dependent Hamiltonian

$$H = -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + U(x), \tag{1.2}$$

where $U(x) = g(x^2 - a^2)^2/4!$. Let E_0, E_1 be the lowest eigenvalues of H in the space of even and odd functions, respectively, and ψ_0, ψ_1 the corresponding eigenvectors. We assume that ψ_0, ψ_1 have unit norm, and are normalized in such a way that $\psi_0(0) > 0$, $\psi_1'(0) > 0$. It is easy to show in the same

way as above that ψ_1 does not change sign in the regions $x > 0$, $x < 0$. Define $\psi_+ = \frac{1}{\sqrt{2}}(\psi_0 + \psi_1)$, $\psi_- = \frac{1}{\sqrt{2}}(\psi_0 - \psi_1)$.

Then it is possible to prove the following.

Theorem 1.2 *(i) As $\hbar \to 0$, $E_0, E_1 \sim a\hbar\sqrt{g/4!}$, and $E_1 - E_0 \sim Ce^{-S_0/\hbar}$, where S_0 is a positive constant.*

(ii) In the sense of distributions,

$$\lim_{\hbar\to 0} |\psi_\pm|^2 = \delta(x \mp a). \tag{1.3}$$

This theorem shows that for very small $\hbar$, although there is only one lowest energy state ψ_0 with energy E_0, there is another stationary state ψ_1 with energy E_1 almost indistinguishable from the lowest one, and in the 2-dimensional space spanned by ψ_0, ψ_1, there are two orthogonal states ψ_+, ψ_- localized near the classical equilibrium states $a, -a$. The states ψ_+, ψ_- are not stationary (i.e. are not eigenvectors of H), but their failure to be stationary is indistinguishable to any finite order in $\hbar$ (in fact, the angle between $H\psi_\pm$ and $\psi_\pm$ is dominated by $\text{const}e^{-S_0/\hbar}$. In particular, symmetry restoration in quantum theory is not seen at the perturbation theory level.

Thus, we have seen how symmetry is lost in the quasiclassical limit. One can consider this effect from a slightly different prospective, by looking how symmetry appears in the process of quantization.

Recall that we have two classical lowest-energy states $a, -a$, near which the operator H looks approximately like a harmonic oscillator. Therefore, we can look for eigenfunctions of H perturbatively, in the form

$$\tilde{\psi}_\pm(x) = f_\pm(\frac{x \mp a}{\sqrt{\hbar}}),\ f_\pm(z) = (\pi\hbar)^{-1/4} g^{1/8} e^{-\sqrt{g}z^2/2}(1 + u_1(z)\hbar^{1/2} + u_2(z)\hbar + ...) \tag{1.4}$$

(the 0-th term of this expansion is the lowest eigenfunction of the harmonic oscillator). It is easy to see that the real "eigenfunction" of the form (1.4), normalized to have unit norm, is unique for each sign.

Now, one can see the following.

(i) The formal series $f_\pm(z)$ do not converge. However, they represent asymptotic expansions of actual functions $\psi_\pm(z\sqrt{\hbar} \pm a)$ (where $\psi_\pm$ are as above), which are smooth from the right in $\hbar$ on $[0, \infty)$, but not analytic.

(ii) The formal series $\tilde{\psi}_\pm(x)$ are eigenfunctions of H with the same eigenvalue. On the other hand, the actual functions $\psi_\pm(x)$ are not eigenvectors of H, although their failure to be ones is exponentally small. The actual lowest eigenvector of H is unique up to a factor, and equals $\psi_0 = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-)$. In particular, it is G-invariant, unlike ψ_+, ψ_-.

(iii) Let $W_+^n(t_1, ..., t_n)$, $W_-^n(t_1, ..., t_n)$ be the correlation functions computed perturbatively (by Feynman calculus) using the formal vacua $\tilde{\psi}_+$, $\tilde{\psi}_-$. Then W_+^n, W_-^n do not serve as small $\hbar$ asymptotic expansions of the correlation functions of any realization of our quantum theory. However, the averages $\frac{1}{2}(W_+^n + W_-^n)$ do serve as such asymptotic expansions.

This shows how symmetry appears in quantization, when one goes from the perturbative to the nonperturbative setting.

1.2. Still no symmetry breaking in quantum field theory in finite volume.

Before going over to quantum field theory, we will give one more argument, at the physical level of rigor, which shows why there is no symmetry breaking in quantum mechanics (this argument is more of an explanation than a proof). It is based on the path integral approach. It does not use

representation theory of the Heisenberg algebra, nor the positivity of the vacuum wave function, and has the advantage that it also works for quantum field theory on a spacetime with a "space" part of finite volume.

As before, we will consider the $\mathbb{Z}/2\mathbb{Z}$-symmetric quartic potential $U(x)$. Suppose that symmetry breaking were the case. Then the two perturbative vacua $\tilde{\psi}_+, \tilde{\psi}_-$ do indeed exist nonperturbatively, i..e serve as asymptotic expansions of actual lowest eigenstates ψ_+, ψ_- of the Hamiltonian, in two different realizations of the theory, $\mathcal{H}_+, \mathcal{H}_-$. Consider the space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. The vectors $\psi_+, \psi_- \in \mathcal{H}$ are "localized" near $a, -a$, and are orthogonal to each other. Thus, the inner product $(\psi_+, e^{-Ht/\hbar}\psi_-)$ must vanish. Let us now compute the same inner product using path integrals.

Recall Feynman-Kac formula:

$$(\delta_{x_1}, e^{-Ht/\hbar}\delta_{x_2}) = \int_{\phi:[0,t]\to\mathbb{R},\phi(0)=x_1,\phi(t)=x_2} e^{-S(\phi)/\hbar} D\phi, \tag{1.5}$$

where

$$S(\phi) = \int_0^t [\frac{1}{2}(\phi')^2 + U(\phi)]ds. \tag{1.6}$$

The states ψ_+, ψ_- are "localized" near $a, -a$ (i.e. are close to delta-functions at $a, -a$ after a suitable normalization). So, if we believe the Feynman-Kac formula in this situation, we can substitute $\delta(x-a)$, $\delta(x+a)$ instead of them, and apply the Feynman-Kac formula. Using the small $\hbar$ stationary phase estimate on the right hand side of (1.5), we will get

$$(\psi_+, e^{-tH/\hbar}\psi_-) \sim Ce^{-S_*(t)/\hbar}, \tag{1.7}$$

where $S_*(t)$ is the least possible action of a path $\phi : [0, t] \to \mathbb{R}$ such that $\phi(0) = -a, \phi(t) = a$.

The least action $S_*(t)$ is attained at a classical trajectory $\phi = \phi_*(\tau)$, which is a solution of the Euler-Lagrange differential equation, i.e. the Newton's equation $\phi'' = \frac{g}{6}\phi(\phi^2 - a^2)$ with boundary conditions $\phi(0) = -a$, $\phi(t) = a$. (Such a solution exists and is unique).

Remark. The function $x = \phi_*(\tau)$ describes the motion of a ball in the potential field with potential $-U(x)$ (a camel's back), from one hump to the other. The initial velocity of the ball is such that the time needed to go from the top of one hump to the top of the other is t. The reason that the potential $U(x)$ is replaced here by $-U(x)$ is that we are doing a Euclidean path integral, which means that we performed a Wick rotation $t \to it$. This rotation transforms the Newton's equation for the potential U into the Newton's equation for the potential $-U$.

Formula (1.7) contradicts the fact that $(\psi_+, e^{-Ht/\hbar}\psi_-)$ vanishes. So our assumption that there are two vacua was false.

Remark. Formula (1.7) actually gives the correct estimate of the inner product $(\psi_+, e^{-tH}\psi_-)$. In particular, $S_0 = S_*(\infty)$. This estimate can be confirmed by rigorous methods.

In this argument, we have never used the fact that ψ_+, ψ_- are functions on the real line. All we used is that they are "localized" near classical equilibrium states $a, -a$, i.e. that for any local observable A the expectation value of A on $\psi_\pm$ is close to the value of the corresponding classical observable at the point $(\pm a, 0)$ in the phase space. Thus, our argument is independent of the realization of the space of states as $L^2(\mathbb{R})$. This makes it easy to generalize this argument to the case of field theory.

Consider a spacetime $M \times \mathbb{R}$, where M is the "space". We will assume that we have already performed a Wick rotation, so that the metric on the spacetime is a Riemannian product metric. We will assume that the volume of M is finite and equals V.

Consider the field theory on $M \times R$ with one scalar Bose field ϕ, described by the Euclidean Lagrangian

$$\mathcal{L} = \int (\frac{1}{2}(\nabla\phi)^2 + U(\phi))dx, \tag{1.8}$$

where U is the quartic potential as above. As before, we have two equilibrium states $a, -a$.

Now consider this theory quantum mechanically. Then we can see that there is still no symmetry breaking. The simplest way to see it is to consider first a discrete space M. In this case, the operator algebra is a finite tensor product of Heisenberg algebras corresponding to points of M, so it is a Heisenberg algebra itself, and the representation-theoretic argument the we used in the case of quantum mechanics shows that symmetry breaking does not occur.

However, it is more instructive to use the path integral argument. As before, assume that symmetry breaking occurs. Then we have two realizations $\mathcal{H}_+$, $\mathcal{H}_-$, and two orthogonal quantum vacua $\Omega_\pm \in \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ whic are "localized" near the equilibrium points $a, -a$ for small $\hbar$, in the sense that $\langle \Omega_\pm, \phi(x_1)...\phi(x_n)\Omega_\pm \rangle \to (\pm a)^n$, $\hbar \to 0$.

As in the quantum mechanics case, the inner product $(\Omega_+, e^{-tH/\hbar}\Omega_-)$ vanishes. On the other hand, computing it using the Feynman-Kac formula, we will get

$$(\Omega_+, e^{-tH/\hbar}\Omega_-) \sim e^{-S_*(t)V/\hbar}, \tag{1.9}$$

where $S_*(t)$ is as above. The reason is that the least action is attained on the space-independent classical solution $\phi_*^M(\mu, \tau) = \phi_*(\tau)$, $\mu \in M$, $\tau \in \mathbb{R}$.

Since (1.9) is nonzero (here it is essential that the volume V is finite), we get a contradiction.

As in quantum mechanics, we can define two sets of correlation functions W_+^n, W_-^n, evaluated by using perturbation theory near the lowest energy points $a, -a$ (of course, we can only define them in the renormalizable case, i.e. in 4 dimensions and below; also, one should remember that renormalization is not uniquely determined). Our reasonings show that W_+^n, W_-^n are not small $\hbar$ expansions of the correlation functions of a realization of our quantum theory. On the other hand, the functions $W_0^n = \frac{1}{2}(W_+^n + W_-^n)$ have a chance to be asymptotic expansions of the actual (nonperturbative) correlation functions.

For $\dim(M) = 1, 2$ the existence of the quantum theory $\mathcal{H}$ has been established in constructive field theory. In this case, it is possible to show that there exists a G-invariant vacuum Ω_0, such that the correlation functions of the theory with respect to Ω_0 indeed have the asymptotic expansion given by W_0^n.

1.3. Symmetry breaking in quantum field theory in infinite volume.

When the volume V of the space M becomes infinite, the arguments of the previous section fail. The representation-theoretic argument fails, because the canonical representation of the operator algebra, whioch we used in the case of finite volume, is now an infinite tensor product of spaces corresponding to points; so we have to make sense of it, and there may be no G-invariant way of doing so. The path integral argument also fails. Indeed, the right hand side of (1.9) vanishes, so both computations of $(\Omega_+, e^{-tH/\hbar}\Omega_-)$ give the same answer, and we can derive no contradiction. Moreover, if we assume that we have a representation $\mathcal{H}$ of the operator algebra with two vacua Ω_+, Ω_-, then using the formula

$$\langle \Omega_+, \phi(x_1^*)....\phi(x_n^*)\Omega_- \rangle = \int_{\phi: X \to \mathbb{R}, \phi \to \pm a, t \to \pm\infty} \phi(x_1)...\phi(x_n)e^{-S(\phi)/\hbar}D\phi, \tag{1.10}$$

where $x^* = (\mu, it)$ for $x = (\mu, t)$, we infer that the inner product (1.10) vanishes for any $x_1, ..., x_n$. This shows that the space $\mathcal{H}$ splits in an orthogonal direct sum: $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where the spaces $\mathcal{H}_\pm$ are the Hilbert spaces of separate realizations generated by the vacua Ω_+, Ω_-. This means, symmetry breaking could occur, like in the classical theory: quantum effects are not strong enough to restore symmetry.

This is what in fact happens in quantum field theory. More precisely, the situation is the following.

(i) If the symmetry group G is a finite group, symmetry can be broken in infinite volume starting with spacetime dimension 2. For example, in the example we considered symmetry breaking does occur.

(ii) If the symmetry group G is a connected Lie group, symmetry breaking can occur starting with spacetime dimension 3. The fact that it does not occur in dimension 2 is a remarkable and a very important fact, which we will discuss in the next section. This fact was proved by S.Coleman in 1973 (see S.Coleman's paper, CMP, vol. 31, page 259).

Thus, symmetry breaking is an "infrared" effect, associated with the behavior of the theory at large distances.

1.4. Infinite volume asymptotics of correlation functions.

Let us see how symmetry is broken in the infinite volume limit of finite volume quantum field theories. We will assume that M is a torus $T_r = (\mathbb{R}/r\mathbb{Z})^{d-1}$, and $r \to \infty$. In the limit, we hope to recover a Poincare invariant field theory corresponding to our Lagrangian. However, as we know, there are two such theories: $\mathcal{H}_+$ and $\mathcal{H}_-$. So which of them do we recover?

Let us formulate the question more precisely. Consider the correlation functions W_r^n, corresponding to the field theory with space being the torus T_r. Let also W_+^n, W_-^n be the correlation functions of the theories $\mathcal{H}_+, \mathcal{H}_-$. The question is, what is the asymptotics of W_r^n as $r \to \infty$, in terms of W_+^n, W_-^n?

The answer is: there exists an r-dependent normalization constant $C(r)$ such that

$$\lim_{r\to\infty} C(r)W_r^n = \frac{1}{2}(W_+^n + W_-^n). \tag{1.11}$$

Now consider a field theory defined by a Lagrangian $\mathcal{L} = \int(\frac{1}{2}(\nabla\phi)^2) + U(\phi))d^n x$, where $U(\phi)$ is a general potential, with a finite symmetry group G. Suppose that $a_1, ..., a_m$ are the global minimum points of U, transitively acted on by G, and $U(a_i) = 0$. Consider first the quantum field theory with space being the torus of volume $V = r^{d-1}$. We have seen that there is one realization $\mathcal{H}$ of this theory, and it has certain correlation functions W_r^n. On the other hand, in infinite volume we will have m different realizations $\mathcal{H}_1, ..., \mathcal{H}_m$, with correlation functions $W_1^n, ..., W_m^n$.

Let us consider the asymptotics of W_r^n as $r \to \infty$. The answer is the following: for a suitable V-dependent normalization constant $C(r)$,

$$\lim_{r\to\infty} C(r)W_r^n = \frac{1}{m}\sum W_i^n, \tag{1.12}$$

This shows that the limit of normalized finite volume correlation functions may, in general, fail to satisfy the cluster decomposition axiom, (see Kazhdan's lectures).

This story can be slightly generalized. Namely, we can make M a ball of radius r, and impose some boundary conditions B on fields at the boundary of the spacetime. This means, we will define correlation functions by the formula

$$W_r^n(B)(x_1, ..., x_n) = \int_{\phi\text{satisfying boundary conditions}} \phi(x_1)...\phi(x_n)e^{-S(\phi)}D\phi.$$

Then the infinite volume asymptotics of $W_r^n(B)$ looks like

$$\lim_{r\to\infty} C(r)W_r^n(B) = \sum p_i W_i^n, \tag{1.12}$$

The collection of numbers p_i represents the "density" with which the quantum vacuum in finite volume is spread over the set of classical minima (=quantum vacua in infinite volume). This density depends

on the boundary conditions. The word "density" should not be taken literally, however, because the numbers p_i are in general complex numbers.

If we take the simplest boundary condition $\phi = a_i$, then in the limit we will get the correlation functions W_i^n (i.e. $p_i = 1, p_j = 0$, $i \neq j$), so we get purely the i-th vacuum. However, if we impose some other boundary condition, we will, in general, get a mixture of vacua.

For example, in the case of $\mathbb{Z}/2\mathbb{Z}$-invariant quartic potential, we can impose boundary conditions $\phi = a$, $\phi = -a$, or $\phi = 0$ (do not worry that the action of all fields in the third case is infinite; since we are considering a normalized path integral, i.e. divided by the partition function, this infinity will cancel). Let $W_r^n(s)$ are the corresponding correlation functions $s = a, -a, 0$. Then in the first case $p_+ = 1, p_- = 0$ (as it it hard to get from the boundary anywhere except a with a small action), in the second case $p_+ = 0, p_- = 1$ (for a similar reason), and in the third case $p_+ = p_- = 1/2$.

1.5. Continuous symmetry breaking.

Now we will consider symmetry breaking in quantum field theory, when the symmetry group is a connected Lie group. The typical example is a complex valued Bose field ϕ, and the Lagrangian

$$\mathcal{L}(\phi) = \int d^d x \left(\frac{1}{2} |\nabla \phi|^2 + (|\phi|^2 - a^2)^2 \right). \tag{1.13}$$

This Lagrangian has a $U(1)$-symmetry, acting by $\phi \to e^{i\theta}\phi$. In the classical theory, we have lowest energy states $\phi = ae^{i\theta}$, $\theta \in [0, 2\pi)$, so we have symmetry breaking. We will try to find out whether symmetry breaking exists also in the quantum theory.

Above we showed that symmetry breaking does not occur in the 1-dimensional case (quantum mechanics). We did it for the case of a real boson ϕ, but for the complex boson the argument (with path integral) works even better. For instance, for Lagrangian (1.13) the set of classical minima of energy is the circle $|\phi| = a$, which is connected. Therefore, to go from one classical minimum, $ae^{i\theta_1}$, to another, $ae^{i\theta_2}$, one does not need to go over the "hump" of the potential, and can go along the circle of minima, so one can do it with even less action than before. Therefore, the path integral computation described above would show that in the complex case there is even more linking between the states ψ_+, ψ_- than in the real case. This effect might make us think that perhaps in field theory, continuous symmetry breaking does not happen as easily as discrete symmetry breaking. And indeed, it turns out that continuous symmetry breaking cannot happen in two dimensions, and can happen only in dimensions ≥ 3.

Unfortunately, the path integral method is too crude to show that continuous symmetry breaking does not occur in 2 dimensions. Indeed, it is easy to see that all paths in the integral which computes $(\Omega_+, e^{-tH}\Omega_-)$ have infinite action in infinite volume, so we can derive no contradiction. So let us demonstrate why symmetry is preserved in 2 dimensions and broken above 2 dimensions by considering the simplest example.

The simplest example is the theory of a free massless real scalar Bose field in d dimensions, defined by the Lagrangian

$$\mathcal{L}_0(\phi) = \int d^d x (\frac{1}{2} (\nabla \phi)^2). \tag{1.14}$$

This theory has a translation symmetry $\phi \to \phi + c$. Let us show that this symmetry is broken for $d > 2$, and preserved for $d = 2$.

For $d > 2$, symmetry breaking is obvious. Indeed, in this case we have a Wightman field theory generated by an elementary field ϕ, satisfying Wightman axioms (see Kazhdan's lectures). In this theory, the 1-point function of the operator ϕ is zero, while the 1-point function of the operator $\phi + c$ is c. Therefore, the transformation $\phi \to \phi + c$ does not preserve the 1-point function. This shows, that

in the case $d > 2$ we in fact have not a single realization, but a family of realizations $\mathcal{H}_c$ parametrized by points c of the line. The theory $\mathcal{H}_c$ is defined by the condition that $\langle \Omega_c, \phi\Omega_c \rangle = c$, where Ω_c is the vacuum of $\mathcal{H}_c$. The vacuum Ω_c is "localized" near the classical equilibrium state c. The map $\phi \to \phi + c$ transforms $\mathcal{H}_b$ to $\mathcal{H}_{b+c}$.

For $d = 2$, the situation is not the same. The problem is that the operator ϕ is not defined in 2 dimensions, although its derivatives are. Indeed, in 2 dimensions, we have

$$\langle \partial\phi(x)\partial\phi(y) \rangle = -\partial_x \partial_y \ln|x - y|, \tag{1.15}$$

so if the operator ϕ was defined in some way, we would have

$$\langle \phi(x)\phi(y) \rangle = -\ln|x - y| + C, \tag{1.16}$$

which contradicts positivity (the function on the RHS of (1.16) is not positive).

Let us say this more precisely. What we have in 2 dimensions is a quantum field theory generated by a Wightman map ϕ from Schwarz functions to operators, which is defined not on the whole space of Schwarz functions $S(V)$ but only on the space $S_0(V)$ of Schwarz functions on V with integral zero. Then derivatives of ϕ can then be defined on all Schwarz functions by

$$\partial_i \phi(f) = -\phi(\partial_i f)$$

(the right hand side makes sense since $\int \partial_i f = 0$).

However, in the theory defined by such ϕ the question of symmetry breaking does not arise, since there is no symmetry to begin with: on the space $S_0(V)$, the maps ϕ and $\phi + c$ are the same. So, in order to raise the question about symmetry breaking, we should consider an extension of our operator algebra, which will have a nontrivial action of symmetry. The most reasonable way of doing so is the following.

Instead of considering the theory of an $\mathbb{R}$-valued massless scalar ϕ, we will consider the theory of a circle-valued field ϕ with the same Lagrangian. We take the circle to be $S_\lambda = \mathbb{R}/2\pi\lambda\mathbb{Z}$. The Lagrangian is the same as before. In this case, the local functional ϕ is not defined, but instead we have local functionals $e^{ik\phi/\lambda}$, $k \in \mathbb{Z}$. The local functionals in this theory are Laurent polynomials in $e^{i\phi/\lambda}$ whose coefficients are differential polynomials in the derivatives of the field ϕ, but not in ϕ itself. The translational symmetry in this theory is the $U(1)$-symmetry: for a complex number z with $|z| = 1$, $z \circ e^{ik\phi/\lambda} = z^k e^{ik\phi/\lambda}$, and z acts trivially on the derivatives of ϕ. Classically, this theory has a family of vacua (equilibrium states), given by $\phi = c$, $c \in S_\lambda$.

Let us show that in this system symmetry is not broken quantum-mechanically. It is enough to show that the 1-point function $\langle \Omega, O(0)\Omega \rangle$ vanishes for any local operator O of the form $O = P(\phi)e^{ik\phi/\lambda}$ for $k \neq 0$. Let us show this in the case $P = 1$ (in general, the proof is analogous). Proof: From the OPE in the free theory (cf. Witten's lecture 3 from the fall term) we get

$$\langle \Omega, O(x)O^*(0)\Omega \rangle = |x|^{-k^2/\lambda^2},$$

so this 2-point function vanishes at infinity. But by clustering, the limit of this function at infinity is $|\langle \Omega, O(0)\Omega \rangle|^2$. So, $\langle \Omega, O(0)\Omega \rangle = 0$.

Another way to see that there is no symmetry breaking is as follows. The Hilbert space of the quantized theory is of the form $\mathcal{H} = F \otimes F^* \otimes l_2(\lambda\mathbb{Z}) = \oplus_{k\in\mathbb{Z}}(F \otimes F^*)_k$, where F is the Fock space. The operators corresponding to derivatives of ϕ respect this decomposition, while the operator $e^{ik\phi/\lambda}$ maps the space $(F \otimes F_*)_{k'}$ to $(F \otimes F^*)_{k+k'}$. The vacuum vector Ω belongs to the zero component $(F \otimes F^*)_0$. This implies that all correlation functions of operators in this theory are invariant under the action of

$U(1)$. In particular, $\langle e^{ik\phi/\lambda}\rangle = \delta_{0k}$, which shows that the vacuum Ω is not "localized" near any classical vacuum but is "spread" uniformly over the space S_λ of classical vacua. Thus, symmetry under $U(1)$ is not broken.

Remark 1. It is instructive to see why our argument that there is symmetry breaking for $d > 2$ fails in finite volume, where, as we know, there should be no symmetry breaking. The problem is that in finite volume the field ϕ is not defined, although its derivatives are. Indeed, since the spacetime is the product of a time line with a compact space, at large distances in looks simply like a line, the 2-point function of ϕ (which is the Green's function of the spacetime) at large $|t|$ looks like the 1-dimensional Green's function, i.e. $-|t| + C + o(1)$. This function is not bounded from below, so it violates positivity. Moreover, as follows from considering the case $d = 2$, the situation is the same if all spatial directions but one are compactified.

Remark 2. Above we have considered the theory of a free massless scalar ϕ in dimension $d > 2$ and found that its space of quantum vacua is the space of values of ϕ (i.e. the target space $\mathbb{R}$). More generally, if one considers the sigma-model with spacetime $\mathbb{R}^d$, $d \geq 2$, and target space M (a Riemannian manifold), the space of quantum vacua, as well as the space of classical vacua, will be M. (Here you should forget for a moment that this sigma-model for nonlinear M is not renormalizable, and so it is not clear what this statement means. We will clarify this point later.) In particular, M as a Riemannian manifold can be recovered from the quantum theory as moduli space of quantum vacua (see Remark 3 below).

On the other hand, we saw that for $d = 2$ the theory of a circle-valued field ϕ (which is the same as the 2-dimensional sigma-model with target space S_λ) has only one vacuum Ω. This is the case for 2-dimensional sigma-model in general: its moduli space of quantum vacua is generally very small, and does not coincide with the space of classical vacua (=the target space). In particular, the target space cannot be recovered intrinsically from the quantum theory. For example, the circle S_λ cannot be recovered intrinsically from the theory of maps into S_λ considered above. This, in fact, happens for a good reason – one can show that the theories attached to the circles S_λ and $S_{1/\lambda}$ are equivalent (for example, their partition functions coincide – see Gawedzki's lecture 1, formula (9)). This is the starting point for the theory of mirror symmetry.

Remark 3. Consider a field theory with the Lagrangian $\mathcal{L} = \int d^d x(\frac{1}{2}(\nabla\phi)^2 + U(\phi))$, where ϕ takes values in some Riemannian manifold M, and U is a potential function on M ($U \geq 0$). Let $M(0)$ is the set of zeros of U. Assume that $M(0)$ is nonempty and smooth, and that d^2U is nondegenerate on $T_xM/T_xM(0)$ for $x \in M(0)$. Suppose that there is a Lie group G which acts by isometries on M, fixes U, and acts transitively on M_0. In this case, one can show (at the physical level of rigor) that the "infrared behavior" of the theory described by $\mathcal{L}$ is the same as the "infrared behavior" of the sigma-model with target space being the space $M(0)$ of classical vacua. The precise meaning of this statement is explained in Section 1.7. (We can ignore nonrenormalizability problems by defining the theories by a cutoff path integral, where integration is taken over fields defined on a lattice with step Λ^{-1}, or over fields having only Fourier modes with $|k| < \Lambda$, with respect to some coordinate system; in this setting, the cutoff Λ is not sent to infinity). This fact can be explained heuristically: if a function ϕ has only low Fourier modes, it cannot oscillate rapidly, so in order to have a small action and thus give a noticeable contribution to the path integral, it has to stay closely to the minimum locus $M(0)$, i.e. has to be close to a map into $M(0)$.

Thus, continuous symmetry breaking, being an infrared effect, will occur in the theory described by $\mathcal{L}$ iff it occurs in the corresponding sigma-model. So, as follows from remark 2, symmetry breaking tends not to occur in dimension 2, but tends to occur in dimension > 2.

These are, however, mostly heuristic arguments. In the next section we will treat the issue of continuous symmetry breaking in a more systematic way, using Goldstone's theorem.

1.6. Goldstone's theorem.

Recall the standard formalism of Noether's theorem and currents in classical field theory. Suppose we have a Lagrangian $\mathcal{L} = \mathcal{L}(\phi)$ in a d-dimensional spacetime V. Denote the space of solutions of the corresponding Euler-Lagrange equations by X.

Let G^s be a 1-parameter symmetry group of this Lagrangian. Let $D_\phi := \frac{d}{ds}|_{s=0} G^s\phi$. We assume that D_ϕ is a local functional of ϕ.

Let $\eta \in \Omega^1(X, \Omega^{n-1}(V))$ be the canonical 1-form on the space of solutions that was discussed in Bernstein's lectures and in Witten's problem sets (the canonical 2-form on X was defined as $\int_C d_X\eta$, where C is an $n-1$-dimensional cycle). For instance, the formula for η for the free theory of a massless scalar is

$$\eta(\delta\phi)(x) = \delta\phi(x) * d\phi(x). \tag{1.17}$$

Let $J \in \Omega^0(X, \Omega^{d-1}(V))$ be defined by the formula $J = \eta(D_\phi)$. Since D_ϕ is local, so is J. Thus, J is a local functional on X with values in $\Omega^{d-1}(V)$. For instance, if $\mathcal{L}$ is the Lagrangian of the theory of a free massless scalar, and $G^s\phi = \phi + s$, then $D_\phi = 1$, so $J(x) = *d\phi(x)$. The local functional J is called the *current* corresponding to the symmetry G^s. The main property of the current is that it is *conserved*, i.e. $d_V J = 0$. This follows form the fact that $d_V\eta = 0$.

In finite volume it is useful to define the charge functional $Q = \int_C J(x)$, where C is some spacelike cycle (e.g. $t = \text{const}$). Since the current is conserved, this quantity is independent of the choice of the cycle, as long as it represents the fundamental homology class of "space". If $\Omega = \int_C d_X\eta$ is a nondegenerate 2-form on X, then Q is a Hamiltonian which defines the symmetry group G^s, in the sense that $\frac{d}{ds}|_{s=0}(G^s)^*F = \{F, Q\}$, where $\{,\}$ is the Poisson bracket on S, and F any local functional on X.

In infinite volume, the functional Q is not necessarily defined, since the integral does not converge. In this case, it is convenient to set $C = C_0 = \{t = 0\}$ (here we have chosen a time coordinate on the spacetime), and define the "cutoff charge functional"

$$Q_f = \int_{C_0} f(x)J(x), \tag{1.18}$$

where $f : C_0 \to \mathbb{R}$ is a Schwarz function. The limit $\lim_{f\to 1} Q_f$ in this case does not exist, but for any local functional F

$$\lim_{f\to 1}\{F, Q_f\} = \frac{d}{ds}|_{s=0}(G^s)^*F. \tag{1.19}$$

Remark. By $f \to 1$ we mean that f converges to 1 uniformly on any compact set, and all derivatives of f go to zero uniformly on the whole space.

In quantum theory, the story is the same, except that (local) functionals are replaced with local operators, and Poisson bracket with commutator times i. That is, to any one-parameter symmetry G^s there corresponds a $d-1$-form-valued local operator $J(x)$, which is conserved, and the charge operator $Q = \int_C J(x)$ (in finite volume) has the property

$$[F, Q] = -i\frac{d}{ds}|_{s=0}(G^s)^*F. \tag{1.20}$$

The case of infinite volume is dealt with in the same way as in the classical theory, by considering cutoff operators Q_f.

Now we will discuss Goldstone's theorem. Suppose that the symmetry G^s in the theory defined by $\mathcal{L}$ is broken not only classically but also quantum mechanically. In this case, if we have a solution

of the theory $\mathcal{H}$ (a Hilbert space with an action of the operator algebra), then there exists a scalar local operator ϕ whose 1-point function is not invariant under symmetry.

Remark. Strictly speaking, we only know that some n-point function is not invariant; but in all known situations with symmetry breaking there is also a non-invariant 1-point function.

Let Q_f be the cutoff charge operator for the symmetry G^s. We have

$$\langle\Omega|[Q_f, \phi(0)]|\Omega\rangle \neq 0, \tag{1.21}$$

for f sufficiently close to 1. Thus, $\langle\Omega|[J(x)\phi(0)]|\Omega\rangle \neq 0$ for some x.

Consider the 2-point functions

$$M_+(x) := \langle\Omega|J(x)\phi(0)|\Omega\rangle, M_-(x) := \langle\Omega|\phi(0)J(x)|\Omega\rangle \tag{1.22}$$

(these are $d-1$-forms on V). Since the symmetry is broken, $M_+ \neq M_-$, although by space-like separation, they coincide if x is spacelike.

Let $\mathcal{H} = \int_{p\in V_+} \mathcal{H}_p$ be the spectral decomposition of $\mathcal{H}$ with respect to the action of the translation group (here V_+ is the positive part of the full light cone). Using the decomposition of the inner product in a sum over intermediate states, ($\langle J(x)\Omega, \phi(0)\Omega\rangle = \sum_n \langle J(x)\Omega, n\rangle\langle n, \phi(0)\Omega\rangle$), we get

$$M_\pm(x) = \int_{V_+} K_\pm(x,p)dp, \tag{1.23}$$

where $K_+(x,p), K_-(x,p)$ are the contributions to M_+, M_- from intermediate states of 4-momentum p ($K_\pm$ are vector-valued distributions in p).

Because of Lorentz invariance, the distributions K_+, K_- look like

$$K_\pm = pe^{\pm ipx}\rho_\pm(-p^2), \tag{1.24}$$

where $\rho_\pm(s)$ are distributions on the half-line $s \geq 0$.

This yields

$$M_\pm(x) = i * d_V \int_0^\infty \rho_\pm(m^2)W_m(\pm x)dm^2, \tag{1.25}$$

where $W_m(x) = \int_{O_m^+} e^{ipx}dp$ is the Klein-Gordon propagator with mass m defined in Lecture 1 last term (O_m^+ is the upper sheet of the hyperboloid $p^2 = -m^2$).

Let $M(x) = M_+(x) - M_-(x)$. Since $W_m(x) = W_m(-x)$ when x is spacelike, for spacelike x (1.25) yields

$$M(x) = i * d_V \int_0^\infty (\rho_+(m^2) - \rho_-(m^2))W_m(x)dm^2. \tag{1.26}$$

However, as we have mentioned, by spacelike separation $M(x)$ vanishes for spacelike x. This implies that $\rho_+ = \rho_- = \rho$, and so (1.25) yields

$$M(x) = i * d_V \int_0^\infty \rho(m^2)(W_m(x) - W_m(-x))dm^2. \tag{1.27}$$

Differentiating both sides of (1.27), at a point x such that $x^2 > 0$, and using the conservation of the current and the Klein-Gordon equation $\nabla^2 W_m = -m^2 W_m$, we get

$$\int_0^\infty m^2\rho(m^2)(W_m(x) - W_m(-x))dm^2 = 0. \tag{1.28}$$

Taking the Fourier transform, we get

$$p^2\rho(-p^2) = 0. \tag{1.29}$$

Thus, $\rho(m^2) = c\delta(m^2)$, where c is a constant. The constant c cannot vanish, otherwise we will prove that $[J(x), \phi(0)]$ has a zero expectation value at the vacuum, which contradicts the assumption of symmetry breaking.

This argument shows that all contributions to the 2-point function $\langle \Omega, J(x)\phi(0)\Omega\rangle$ comes from intermediate states of zero mass. This implies that $L^2(O_0^+)$ is contained in the discrete spectrum of $\mathcal{H}$ (i.e. as an honest subrepresentation of the Poincare group). Such a subrepresentation is interpreted in quantum field theory as a massless particle of zero spin. Thus, we have proved the following statement, which goes under the name of Goldstone's theorem:

Theorem *In the Hilbert space of a realization of a field theory with continuous symmetry breaking, there is a massless scalar (i.e. a subrepresentation of the Poincare group isomorphic to $L^2(O_0^+)$) which is created by the current of the broken symmetry.*

Remark. "Created" means that $J(x)\Omega$ is not orthogonal to the subrepresentation.

Definition. The massless scalar which we found is called the Goldstone boson corresponding to the broken symmetry G^s.

Remark 1. If the symmetry with respect to G^s was not broken, then $\lim_{f\to 1} Q_f\Omega$ would be zero. On the other hand, when symmetry breaking occurs, Q_f creates Goldstone bosons from the vacuum. Thus, Goldstone bosons "measure" the failure of symmetry.

Remark 2. The Goldstone boson can be created not only by the current operator of the symmetry, but also by other local operators. In fact, as we saw in the proof of Goldstone's theorem, it will be created by any scalar local operator whose 1-point function is not invariant under the symmetry.

Remark 3. There is no claim in Goldstone's theorem that the Goldstone boson is free, i.e. that it can be created by a free field $\phi(x)$. In fact, as we will see, this is often not the case.

Remark 4. If continuous symmetry breaking occurs classically, the Goldstone boson can already be seen in perturbation theory. As an example consider Lagrangian (1.13). Consider the classical vacuum state $\phi = a$. This vacuum state is degenerate. Therefore, if we introduce real variables $\phi_1 = \mathrm{Re}\phi - a$, $\phi_2 = \mathrm{Im}\phi$, and rewrite the Lagrangian in terms of these variable, then because of the degeneracy of the minimum the field ϕ_2 will be classically massless. Therefore, if we compute the 2-point function of ϕ_2 it will have a pole at $k^2 = 0$ (modulo the perturbation parameters). Of course, in principle loop terms might shift this pole. In other words, the classically massless ϕ_2 may get nonzero mass quantum-mechanically. What Goldstone theorem tells us is that this will not happen if symmetry is broken.

Corollary from Goldstone theorem *Symmetry breaking does not happen in 2 dimensions.*

Proof. Otherwise, by Goldstone's theorem Goldstone bosons would have to exist. But in a 2-dimensional quantum field theory, there can be no massless particles created by a local operator. Indeed, the 2-point function of this operator in momentum space equals $w(k^2) = \int_0^\infty \frac{d\mu(m^2)}{k^2+m^2}$, where μ is the spectral measure. If there is a massless particle, this measure will have an atom at $m = 0$. But in this case the 2-point function $W(x)$ in position space will behave like $-C\ln x^2$ at infinity, i.e. would violate the positivity axiom. □

Now assume that we have a Lagrangian which has a Lie group G of symmetries. Assume that $\mathcal{H}$ is a realization of the quantum field theory defined by this Lagrangian, whose stabilizer is $H \subset G$. In this case one says that in the realization $\mathcal{H}$, the G-symmetry is spontaneously broken to H.

Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of G, H. Goldstone's theorem implies

Corollary *$\mathcal{H}$ contains in its discrete spectrum a subrepresentation isomorphic to $L^2(\mathcal{O}_0^+) \otimes (\mathfrak{g}/\mathfrak{h})$.*
Proof The proof is clear: if this is not so than there exists an element $Y \in \mathfrak{g}$, $Y \notin \mathfrak{h}$, such that $Q_f^Y \Omega \to 0$ (weakly) when $f \to 1$. This means that the symmetry with respect to Y is not broken – a contradiction.

The corollary means that Goldstone bosons corresponding to linearly independent broken infinitesimal symmetries are also linearly independent.

Example. Let $\phi : \mathbb{R}^d \to \mathbb{R}^N$ be a scalar field, and consider the Lagrangian

$$\mathcal{L}(\phi) = \int d^d x (\frac{1}{2}(\nabla\phi)^2 + \frac{g}{4!}(\phi^2 - a^2)^2). \tag{1.30}$$

This Lagrangian has an $SO(N)$-symmetry, and the space of its classical vacua is S^{N-1}. Therefore, classically the $SO(N)$-symmetry is broken to $SO(N-1)$. As we know, if $d > 2$, the same will happen quantum mechanically (for a weakly coupled theory), and so any realization (solution) of the theory has $N-1$ independent Goldstone bosons.

Let P be the classical minimum with coordinates $(0, 0, ..., 0, a)$. Consider the realization $\mathcal{H}_P$ of the theory, where the vacuum Ω is localized near P. Let $Y_i \in \mathfrak{so}_N$, $i = 1, ..., N-1$, be the infinitesimal rotations in the planes generated by basis vectors e_i, e_N of $\mathbb{R}^N$. The 1-point function of the operator ϕ_i is not invariant under Y_i, so ϕ_i creates the Goldstone boson corresponding to Y_i.

We can construct low energy (non-vacuum) states localized near other classical vacua than P. Indeed, let P' be another classical vacuum, and $Y \in \mathfrak{so}(N)$ is an element such that $e^Y P = P'$. Let J_Y be the current corresponding to Y, and Q_f^Y the corresponding cutoff charge. Then the state $e^{iQ_f^Y}\Omega$ is a low energy state localized near P'.

At long distances the theory will behave as a sigma-model into the space of classical vacua. This is an interesting statement for $N \geq 3$, because in this case the target (S^{N-1}) is not flat, so the sigma-model is not free. More precisely, at low energies (or long distances), the theory of bosons ϕ_i, $i = 1, ..., N-1$, will be free in the zero approximation, but in the first approximation it will not be free but will be described by the Lagrangian of the sigma-model into the sphere.

1.7. Infrared behavior of purely non-renormalizable field theories.

In this section we will clarify the meaning of the statement that for $d > 2$ a quantum field theory behaves in the infrared limit as a sigma-model into the space of classical vacua.

Suppose we have a purely nonrenormalizable field theory described by a Lagrangian $\mathcal{L}$. We will call a Lagrangian purely nonrenormalizable if all its couplings have negative dimension. An example of such a Lagrangian is the Lagrangian of a sigma-model for $d > 2$. Such Lagrangians are not good for perturbative renormalization in the UV limit, but create no problem in the IR limit, since all their interactions are IR irrelevant from the point of view of the Wilsonian renormalization group flow. Namely, if we introduce an UV momentum cutoff Λ (which is now not being sent to infinity), we can define correlation functions of $\mathcal{L}$ perturbatively: the correlation function is the sum of amplitudes of all Feynman diagrams, which are evaluated as usual, with integration carried out with cutoff $|q| < \Lambda$. Because $\mathcal{L}$ has no mass terms, there will be some IR divergences, but they can be dealt with in the same way as we dealt with UV divergences in Lectures 1-3 last semester. Moreover, since the theory is purely non-renormalizable, only finitely many graphs will be divergent for each number of external legs (like in a superrenormalizable theory in UV renormalization).

Now suppose that we want to compute the asymptotic expansion of the n-point function in momentum representation, around the point $k_i = 0$. We will have (for $\sum k_i = 0$):

$$G_n(k_1, ..., k_n) = k_1^{-2} ... k_n^{-2} G_n^0(k_1, ..., k_n), \tag{1.31}$$

where G_n^0 is a certain series, having a limit at $k_i = 0$ (In general, this will not be a power series; it may contain terms of the form $k^4 \ln k^2$).

The key property of this series, which follows from pure nonrenormalziability, is that modulo terms of any finite power, it is determined by finitely many Feynman graphs. Thus, we can obtain the IR asymptotics of the correlation functions to any order in k_i without having to sum the perturbation series.

Now suppose we have an actual quantum field theory, given by some renormalizable Lagrangian $\mathcal{L}'$. When we say that the theory defined by $\mathcal{L}'$ is described in the IR limit by a purely nonrenormalizable Lagrangian $\mathcal{L}$ (on the same fields), we mean that to a certain order in k_i (near $k_i = 0$), the functions G_n^0 given by (1.31) are the same for $\mathcal{L}$ as for $\mathcal{L}'$.

For instance, when at the end of the previous section we said that at low energies (or long distances), the theory of bosons ϕ_i, $i = 1, ..., N-1$, is free in the zero approximation, and described by the Lagrangian of the sigma-model into the sphere,

$$\mathcal{L}_\sigma = \int d^d x (\frac{1}{2} \sum (\nabla \phi_i)^2 + R(\sum \phi_i^2)(\sum (\nabla \phi_i)^2))$$

in the first approximation (where R is proportional to the curvature), we meant that the functions $G_n^0(k_1, ..., k_n)$ for ϕ_i are the same as in the free theory modulo $o(1)$, and the same as in the sigma-model modulo $o(k^2)$.

Computing higher terms of the k-expansion, one can construct a purely nonrenormalizable low energy effective theory, which will describe our theory in the infrared to any required accuracy.

Remark. The restriction of the function $G_0^n(k_1, ..., k_n)$ to the locus $k_i^2 = 0$ (but k_i is not necessarily zero) has a physical meaning: it is the scattering amplitude (S-matrix) of n Goldstone bosons. Thus, the statement is that scattering matrix of n Goldstone bosons in model (1.30) is like in the free theory for $N = 2$ (as the circle is flat), but has a quadratic correction due to curvature for $N > 2$.

Therefore, asymptotic expansion (1.31) can be computed to any order in k_i by computing amplitudes of finitely many diagrams.

Lecture II-2, part I: Infrared Behavior of Quantum Field Theories

Edward Witten[*1]

Given a quantum field theory X, we want to solve it, that is, to learn the most interesting things about it. A big piece of "solving" a theory is determining what it flows to in the infrared. Fairly often, the answer is: "nothing," that is, X flows to a trivial theory. This happens precisely when X has a mass gap, for then all (Euclidean) correlation functions decay exponentially. Showing that a given theory flows to a trivial theory may, however, be a rather deep result.

Very often, the infrared limit is not trivial but is a free theory of massless particles, together with an irrelevant interaction which goes to zero in the infrared. In fact, this happens in most of the simplest examples that we will meet. Note that an irrelevant interaction would, in the ultraviolet, be considered "unrenormalizable"; the perturbations that are ill-behaved in the ultraviolet are just the ones that vanish as one flows to the infrared limit.

When a theory is free in the infrared, the question then becomes: *which* massless particles is it a free theory of? They might not be related to the ones in the original Lagrangian. In fact, as we shall see in the second part of the lecture, the answer to this question may depend on the vacuum state we are in.

For the infrared limit to be trivial is a special case of the infrared limit being free; it is the case that there are no massless particles at all in the physical spectrum.

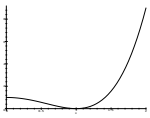
We begin with an example, and then discuss several general features of infrared limits.

1 Example from last time

Consider a theory which breaks $SO(3)$ to $SO(2) = U(1)$. We have three real scalars which can be combined to a 3-component object $\vec{\phi}$ which transforms in the 3-dimensional representation of $SO(3)$. The Lagrangian is

$$\mathcal{L} = \frac{1}{\lambda}\int d^n x \left[\frac{1}{2}(\partial_\mu \vec{\phi})^2 + V(|\vec{\phi}|)\right],$$

where, letting $\rho = |\vec{\phi}|$, $V(\rho)$ is a potential which has a nondegenerate minimum away from the origin.



The simplest such potential is $V(\rho) = \frac{1}{8}(\rho^2 - \rho_0^2)^2$, with a minimum at ρ_0. If we let $\Omega = \phi/\rho \in S^2$, we can rewrite the Lagrangian as

$$\mathcal{L} = \frac{1}{\lambda}\int d^n x \left[\frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}\rho^2(\partial_\mu \Omega)^2 + V(\rho)\right].$$

The term $\frac{1}{2}\rho^2(\partial\Omega)^2$ represents the round metric on S^2.

When λ is small, we can hope that the classical approximation will be good. Since ρ is a massive field, we can integrate it out of the theory by setting it equal to its minimum value ρ_0 and studying fluctuations

$$\rho = \rho_0 + w,$$

[1]*Notes by David R. Morrison

where w is now the quantum field which will appear in our path integrals. To first approximation, we would have an effective Lagrangian

$$\mathcal{L}_{\rm eff}(\Omega) = \frac{\rho_0^2}{2\lambda}\int d^n x\,(\partial\Omega)^2,$$

describing a nonlinear sigma model of maps to G/H, which in the present case is S^2. The massless fields are described by Ω.

Can this be an answer? In other words, could *any* quantum field theory flow to this sigma model in the infrared? In fact, it *is* a possible answer for spacetime dimension $n > 2$, because the nonlinear sigma-model is non-renormalizable above two dimensions, so the interaction we get is irrelevant. That is, if a_i are Riemann normal coordinates on near a point $P \in S^2$, we can expand schematically near P

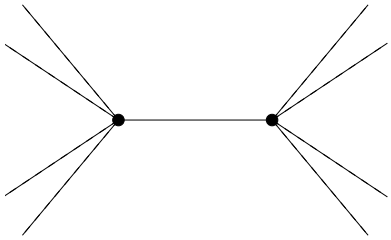
$$(\partial\Omega)^2 = (da)^2 + Ra^2(da)^2 + \dots$$

with R being the Riemann tensor of S^2. From this we see that the interaction is irrelevant above two dimensions. In fact, to give $(da)^2$ dimension n, we must assign dimension $(n-2)/2$ to a, whence the interaction has dimension $2n-2$, which exceeds n for $n > 2$. The fact that the sigma model is a possible answer for $n > 2$ but not for $n \leq 2$ is an aspect of the fact, already discussed last week, that spontaneous breaking of a continuous symmetry is possible for $n > 2$ but not for $n \leq 2$.

Is the sigma model the *correct* answer for the infrared behavior of our particular problem, at least for sufficiently small λ? We claim that it is. To study this point, let us treat the effects of w perturbatively. The interactions of w with the Goldstone boson come from the interaction term in the Lagrangian:

$$\frac{1}{2\lambda}\rho^2(\partial\Omega)^2 = \frac{1}{2\lambda}(\rho_0+w)^2(\partial\Omega)^2 = \frac{1}{2\lambda}(\rho_0^2 + 2\rho_0 w + w^2)(\partial\Omega)^2.$$

The operator $\partial\Omega$ is highly nonlinear, and can be thought of as emitting an arbitrary number of Goldstone bosons. We need to calculate Feynman diagrams involving w's, in order to find the effective Lagrangian. A typical diagram is



representing w with a solid line and a's with dotted lines. The w-propagator is $\frac{1}{k^2+m_w^2}$; since we are interested in small momenta k, we expand in powers of k. The leading term is

$$\text{const}\,(\partial\Omega)^2\frac{1}{m_w^2}(\partial\Omega)^2.$$

This has $SO(3)$ symmetry and is an irrelevant interaction. It is even more irrelevant than the terms, sketched above, that come by expanding $(\partial\Omega)^2$ in powers of a.

In fact, while it is instructive to study these diagrams, just to show that the sigma model is infrared-stable, we do not need the details of the diagrams. All we need to know is that the effective action has $SO(3)$ symmetry. So what possible terms could be generated in the effective action?

- A potential $V(\Omega)$ is not possible, because there is no $SO(3)$-invariant function on S^2. This is the basic reason that, given the $SO(3)$ symmetry, the sigma model is infrared stable. A potential function would change the picture completely. For instance, a generic potential would have an isolated, nondegenerate minimum, giving us a unique vacuum with an infrared-trivial massive theory, in contrast to a continuous family of vacua associated with spontaneously broken $SO(3)$.
- The only term with only two derivatives that respects all the symmetries of the problem is $(\partial\Omega)^2$ itself. So quantum corrections due to diagrams with w fields should definitely be expected to modify the coefficient of this term.
- Other possible terms like $((\partial\Omega^2)^2$ and $\partial\Omega\,\nabla^2(\partial\Omega)$ have more than two derivatives and are more and more irrelevant.

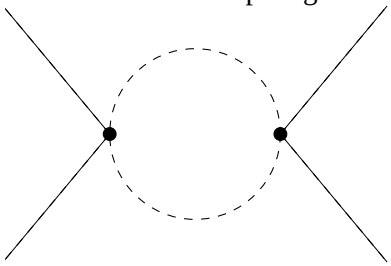
So the leading infrared behavior is determined by an effective action of the form $\frac{1}{f^2}(\partial\Omega)^2$ with

$$\frac{1}{f^2} = \frac{\rho_0^2}{2\lambda} + \text{ loop corrections.}$$

When this is expanded in Riemann normal coordinates about a given vacuum, that is a given point $P \in G/H$, one gets interactions (of which the first was sketched above) that involve the Riemann tensor of G/H and its covariant derivatives. If one uses these interactions to compute scattering amplitudes involving Goldstone bosons with small momenta of order k, the tree level amplitudes are all proportional to k^2 for k near zero, as the interaction terms all contain precisely two derivatives. Loop contributions are smaller for $k \to 0$, since the interactions are irrelevant in the infrared. To be more precise, loop amplitudes all either (i) renormalize the constant f in the $SO(3)$-invariant Lagrangian, or (ii) give corrections to the scattering amplitudes that vanish faster than k^2 for $k \to 0$.

Hence, the terms of order k^2 in the Goldstone boson scattering are all completely determined by the one constant f (or more generally by the choice of a G-invariant metric on the homogeneous space G/H). In the 1960's, it was discovered that the low energy scattering of pions beautifully fits such a description, with $G = SU(2) \times SU(2)$ and H a diagonal $SU(2)$. This is how it was discovered that the strong interactions have a spontaneously broken approximate chiral symmetry; the discovery played a very major role in the subsequent development of physics.

What happens if one wants to compute terms in the Goldstone boson scattering of higher order than k^2? It is clear that in order k^4, new constants will enter that can only be determined from microscopic calculations (or experiment), since there are G-invariant interactions with four derivatives (such as the $((\partial\Omega)^2)^2$ term found above from the explicit tree diagram considered). However, interestingly, in four spacetime dimensions, the lowest order correction to the k^2 amplitude for Goldstone bosons is not of order k^4 but of order $k^4 \ln k$. It comes from a loop diagram



with vertices drawn from the two-derivative part of the Lagrangian, and hence is uniquely detemined in terms of the same constant f that controls the k^2 terms in the scattering amplitudes. The analysis

of low-energy Goldstone boson interactions via the ideas I have explained is known as "current algebra." In particular, via current algebra relations, one can deduce from experiment what is the broken symmetry group G, and many of the parameters in the G-invariant effective Lagrangian.

One final comment about symmetry-breaking examples such as this one: if we begin with a G-invariant microscopic Lagrangian $\mathcal{L}_{\rm micro}$ which we perturb to

$$\mathcal{L}_{\rm micro} + \varepsilon(\delta\mathcal{L})$$

with the term $\delta\mathcal{L}$ not being G-invariant, then in the infrared we will get

$$\frac{1}{f^2}(\partial\Omega)^2 + \varepsilon V(\Omega) + O(\varepsilon^2),$$

with $V(\Omega)$ being a non-G-invariant operator – of which the most relevant part is of course a potential with no derivatives, as suggested in the notation V. $V(\Omega)$ is highly constrained by the fact that it must transform under G the same way that $\delta\mathcal{L}$ does. For example, in the case of strong interactions, a small $\delta\mathcal{L}$ term, breaking $SU(2)\times SU(2)$ to a diagonal $SU(2)$, is actually present; it selects a unique vacuum from what would otherwise be a continuous family, and gives small masses to the pions. In current algebra studies of pions, one really takes the momentum k to be of order the pion mass.

2 Which spins?

Now we consider in a general way infrared-free theories in 4 dimensions. (The considerations that follow generalize above 4 dimensions but become trivial below dimension 4). The general discussion seems to suggest that infrared-free theories might have massless particles of any spin. But in practice, in all interesting examples I am familiar with, one can argue *a priori* that any massless particles will have spins 0, 1/2 or 1.

Most theories of interest can be formulated not just on flat $\mathbf{R}^4$, but on a more general curved 4-manifold M^4 with a general metric g. In fact, any theory that is part of the description of nature has this property, since general relativity is part of nature and in nature, space-time is curved! In quantum field theory, the ability to work on a curved space-time implies the existence of a very special operator, called the stress tensor or energy-momentum tensor $T_{\mu\nu}(x)$. It measures the response to a change in the metric tensor g. We suppose that a theory is formulated with a general g by a Lagrangian $\mathcal{L}(\phi_i; g)$, which is invariant under diffeomorphisms acting both on the ϕ_i and on g. g is not one of the fields of the theory – it is arbitrary but is held fixed in studying the classical or quantum dynamics of the ϕ_i – and this diffeomorphism invariance means that the theory, if formulated in a spacetime (M, g), really depends on g only up to diffeomorphism. In this setup, the stress tensor is defined as

$$T_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}.$$

This implies obviously that T is a symmetric tensor

$$T^{\mu\nu} = T^{\nu\mu}.$$

T can also be shown to obey

$$D_\mu T^{\mu\nu} = 0$$

by virtue of diffeomorphism invariance. If our theory is actually *conformally invariant*, then T is traceless, that is $g^{\mu\nu}T_{\mu\nu} = 0$.

Having such a stress tensor leads to powerful statements even if one specializes to the case that M is flat Euclidean space. For instance, last fall, when we axiomatized quantum field theory, we required Poincaré invariance, with conserved charges $Q(K)$ for every Killing vector field K in spacetime. The existence of a conserved, symmetric stress tensor is a local statement that leads to Poincaré invariance globally. Given any Killing vector field K, one uses the Killing vector equation (which reads $D_\mu K_\nu + D_\nu K_\mu = 0$) plus symmetry and conservation of T to prove that the current

$$J^\nu(K) = K_\mu T^{\mu\nu}$$

is conserved. Then one obtains the conserved charge

$$Q(K) = \int_{\Sigma^{n-1}} * J(K)$$

where the integral is taken over an initial value hypersurface (such as time zero).

I have presented this as if one needs to have a Lagrangian so as to deduce the existence of a stress tensor $T = \delta\mathcal{L}/\delta g$. Though this is a powerful approach, one can also argue more abstractly. Consider any theory which can be formulated for any metric on M. To define an operator $T(y)$, we must give a definition, for any specified n points $x_1, \dots, x_n$ on M^4 distinct from each other and from y and operators $\mathcal{O}_1, \dots, \mathcal{O}_n$, the correlation function $\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) T(y) \rangle_g$ (here the subscript serves to emphasize that the correlation function depends on a metric g). We define this as the derivative of $\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)\rangle_g$ with respect to g:

$$\frac{\delta}{\delta g(y)} \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)\rangle_g = \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) T(y)\rangle_g.$$

The reader should verify that this definition agrees with the previous one in case a Lagrangian exists. T as defined in this way is obviously symmetric; it is conserved if the theory depends only on the diffeomorphism class of the metric g. Many of the properties of a local quantum field operator follow readily from this definition of T, and it is plausible to believe that they all do in general.

In any event,[2] one normally considers in practice theories that have a local, conserved, symmetric (and of course gauge-invariant) stress tensor. As I have essentially already noted, any theory that appears in nature has this property, since T appears directly in the Einstein equations!

Existence of T leads[3] to sharp restrictions on possible massless particles. The possible spins of a massless particle in a theory with a stress tensor are $0, 1/2$, and 1. A further and analogous restriction is the following. Let J^μ be a conserved current associated with a "global symmetry." Thus, J transforms as a vector under Poincaré, and the conserved charge

$$Q(J) = \int * J$$

is Poincaré invariant. Then Q annihilates any massless particle of spin 1.

The proofs are so similar that we consider the two cases together. In four dimensions, denote by $|p, j\rangle$, a massless one-particle state of momentum p and spin j (in general $j \in \mathbf{Z}/2$, and let $|p', j\rangle$ be a state of different momentum in the same Poincaré representation. Let $|p, j\rangle$ be an eigenstate of Q with eigenvalue q. Consider the matrix elements $\langle p', j|J^\nu|j, p\rangle$ and $\langle p', j|T^{\mu\nu}|j, p\rangle$. The latter cannot vanish at all, and the former cannot vanish unless $q = 0$. For in the limit that $p' \to p$, we have by Lorentz invariance

[2] Except in studying quantum gravity, which has a very different flavor from quantum field theory in a fixed spacetime, which is the subject of our lectures this spring.

[3] S. Weinberg and E. Witten, *Limits on massless particles*, Phys. Lett. B **96** (1980), 59–62.

1. $\langle p', j|J^\nu|j, p\rangle \sim p^\nu$, and

2. $\langle p', j|T^{\mu\nu}|j, p\rangle \sim p^\mu p^\nu$

The proportionality constant is q in the first case and 1 in the second, since (as Q and the momentum operators are obtained by integration of J or T) matrix elements of J or T with identical initial and final states measure the charge or momentum of the state.

On the other hand, one can prove using Lorentz invariance that for all $p' \neq p$, these matrix elements vanish in the first case for spin greater than $1/2$, and in the second case for spin greater than 1. The proof goes by noting first for for $p' \neq p$, the subgroup of the Lorentz group that leaves fixed both p and p' is a copy of $SO(2)$ (or $SO(n-2)$ if we are in n spacetime dimensions; the present considerations degenerate below four dimensions as $SO(n-2)$ is then trivial). One simply shows that $SO(2)$ invariance of $\langle p', j|J^\nu|j, p\rangle \sim Qp^\nu$ and $\langle p', j|T^{\mu\nu}|j, p\rangle$, assuming that these matrix elements are nonzero, implies that the spin is in absolute value $\leq 1/2$ or ≤ 1, in the two cases. A convenient way to perform this computation is to go to a Lorentz frame in which (writing the time coordinate first), $p = (1, 1, 0, 0)$ and $p' = (1, -1, 0, 0)$. The $SO(2)$ that leaves fixed both p and p' is the rotation of the last two coordinates. Under the generator of this $SO(2)$, the states $|j, p\rangle$ and $|j, p'\rangle$ have respectively eigenvalue j and $-j$. The minus sign for the $|j, p'\rangle$ state, which is crucial, arises because it describes a particle moving in the opposite direction from $|j, p\rangle$; they each have the same spin relative to their own directions of motion, but opposite spins if referred to a fixed axis. So $\langle p', j|J^\nu|j, p\rangle$ or $\langle p', j|T^{\mu\nu}|j, p\rangle$ can be nonzero only if, in the $SO(2)$ action on J or T, there is a term with spin or eigenvalue $-2j$. As the components of J transform under $SO(2)$ with spin ≤ 1 in absolute value, while for T one has components of spin ≤ 2, we get $|j| \leq 1/2$ or $|j| \leq 1$ in the two cases, as was claimed above.

We can actually be somewhat more precise about this result. We have so far used only representation theory, but in quantum field theory one has also a CPT theorem, which implies in four dimensions that every massless particle of spin j is accompanied by one of spin $-j$. So spins $\pm 1/2$ will go together, and likewise spins ± 1.

In general dimension n, similar reasoning gives the following result. The spin of a massless particle is classified by a representation of the "little group" $SO(n-2)$. If a stress tensor exists, the allowed representations for massless particles are the spinor representation(s), and exterior powers of the fundamental $n-2$-dimensional representation (including the trivial representation).[4] Global charges vanish except for massless particles transforming in the trivial or spinor representation. This n-dimensional formulation is related to the statement in four dimensions as follows: $j = \pm 1/2$ correspond to the two spinor representations, while $j = 0$ and $j = \pm 1$ come from the exterior powers of the fundamental representation.

3 Why are particles massless?

If the couplings in a theory are generic, massless particles must be massless for a reason. One possible reason is supersymmetry, but we won't discuss that now. Other possible reasons are as follows.

Spin 0 particles are massless when they are Goldstone bosons, that is, when there is a broken symmetry. Spin $1/2$ particles are massless when they are chiral fermions; their masslessness is due to an *unbroken* chiral symmetry. This means simply the following: the CPT theorem says that if one has n massless particles of spin $1/2$, one also has n such particles of spin $-1/2$. If there is an unbroken symmetry group G and the massless particles of spin $1/2$ transform in a representation R of G, then the

[4] For $n-2$ divisible by four, the middle exterior power can be decomposed into self-dual or anti-self-dual pieces which are each *real*; one can be present without the other.

massless particles of spin $-1/2$ transform, by the CPT theorem, in the representation $\overline{R}$ (the complex conjugate of R). If R and $\overline{R}$ are distinct. this spectrum cannot be perturbed in a G-invariant way to a give masses to the fermions.

The reasons just mentioned are really the only known reasons to have massless particles of spin 0 or 1/2 without supersymmetry and without adjusting some parameters to make particles massless. For spin 1 the situation is somewhat different. The Poincaré representation of a massless spin 1 particle, in four (or more) dimensions, simply cannot be perturbed to give a mass to the particle, unless there is a massless spin 0 particle that can combine with it in a Higgs mechanism as will be discussed in the second half of lecture. If our massless spin 0 particles are Goldstone bosons, then the broken symmetries that shift them ensure that they cannot participate and disappear from the massless spectrum in a Higgs mechanism. So as long as the massless spin 0 particles are Goldstone bosons, and *as long as the theory is infrared-free*, the existence of massless spin 1 particles is stable just from Poincaré symmetry.

This depends heavily on the theory being infrared-free since we applied group theory to one-particle states. If interactions are important even at low energies, the states with different numbers of particles can "mix," and we cannot draw a conclusion just by applying Poincaré invariance to the one-particle states.

What interactions can massless particles of these types have? In the case of spin 0 particles, which we assume to be Goldstone bosons, there are no relevant interactions. We have already seen at the beginning of this lecture that there are no relevant or marginal interactions of Goldstone bosons only. There are likewise no relevant or marginal interactions of fermions only above two dimensions – we explored such questions in the fall term – and with similar arguments and a little more care, one can show that there are no relevant or marginal couplings of Goldstone bosons to fermions.

Spin 1 particles are of a different nature, since they correspond to gauge fields, and gauge fields *can* have relevant interactions in the infrared. If G is the gauge group and A is the connection, the Lagrangian

$$\int \frac{1}{g^2} |F_A|^2 d^4x$$

is nonlinear. It contains couplings that in four dimensions are marginal classically. Whether the interactions are relevant or irrelevant quantum mechanically depends on the behavior of the β-function. An irrelevant nonlinearity in the infrared will allow the gauge theory to function as an effective description. On the other hand, if the nonlinearity is relevant in the infrared, then the gauge theory is *not* the answer. (In the intermediate case when $\beta = 0$ we would get a non-free theory in the infrared.)

Goldstone bosons are always invariant under gauge symmetries – a gauge group acting on them would violate the global symmetry that leads to having Goldstone bosons in the first place. So if we see massless spin 1 fields in the infrared without adjusting parameters to make it so, we should expect that either G must be abelian, or else there must be enough fermions in large enough representations of G so that $\beta > 0$. To explain their masslessness, the fermions are chiral (that is, the states of $j = 1/2$ and $j = -1/2$ transform differently) either under the gauge group itself, or under some unbroken global symmetry group that commutes with the gauge group.

Lecture II-2, Part II: Spontaneous breaking of gauge symmetry

Edward Witten

Notes by Pavel Etingof and David Kazhdan

In this lecture we will consider gauge symmetry breaking.

2.1. Gauge symmetry. Recall what gauge symmetry is. We have a spacetime $X = \mathbb{R}_{time} \times X_0$. We have a compact gauge group G. We have a field theory where a field configuration is a connection is some principal bundle over $\mathbb{R}^d$ and possibly some matter fields.

Recall the Hamiltonian approach to gauge theory. Let $\tilde{M}_0$ be the space of solutions to the classical equations of motion. On $\tilde{M}_0$ we have an action of the group $\hat{G}$ of gauge transformations. Let $M_0 \subset \tilde{M}_0$ be the space of all solutions where the G-bundle is trivialized in the time direction, and the connection is trivial in that direction. Such solutions as usual are completely determined by the pair $A(t_0)$, $\frac{dA}{dt}(t_0)$, where $t = t_0$ is a space cycle, and initial data for the matter fields. It is clear that any element of $\tilde{M}_0$ can be brought to M_0 by a gauge transformation, so M_0 still contains all solutions up to gauge transformations.

Suppose that $X_0 = \mathbb{R}^{d-1}$. In this case we may consider only trivial bundles, and connections which vanish at spatial infinity. In other words, if $\mathcal{A}$ is the space of connections A on the trivial G-bundle over $\mathbb{R}^{d-1}$ which vanish at ∞ then M_0 for pure gauge theory is $T^*\mathcal{A}$. If matter fields are present, then M_0 is a product of $T^*\mathcal{A}$ with some other space.

Define $\tilde{G}$ to be the group of elements of $Maps(\mathbb{R}^{d-1}, G)$ which have a limit at infinity, and $\tilde{G}_0$ to be the subgroup of $\tilde{G}$ consisting of functions which tend to 1 at ∞. We have $\tilde{G}/\tilde{G}_0 = G$. This quotient group is called the group of constant gauge transformations at ∞ and called G_∞.

The group $\tilde{G}$ acts symplectically on M_0. The physical phase space in gauge theory is the symplectic quotient $M = M_0//\tilde{G}_0$. Note that we only divide by $\tilde{G}_0$ and not by the whole group $\tilde{G}$, so that we inherit an action of the quotient G_∞ on M. It is this symmetry group whose breaking we will discuss.

2.2. Breaking of gauge symmetry and charges at infinity.

Definition. Suppose we have a (classical) gauge theory, and let $s \in M$ be its vacuum state. Let $H \subset G = G_\infty$ be the stabilizer of s. In this case we will say that at the vacuum state s the gauge symmetry is broken from G to H.

Thus, by symmetry breaking we mean essentially the same thing as for global symmetry: there is a symmetry of the Poisson algebra of functions on M which does not fix a particular vacuum state.

Important remark. The above expression "the same thing " should be taken with great care. There are some fundamental differences between the two situations, which will become clear below. They come from the fact that in the situation we are considering here, (unlike Lecture II1) the physical observables, being gauge invariant by definition, automatically commute with G and therefore do not, in general, separate points on M; i.e. not every function on M is "observable". In other words, the action of G on the "theory" (in the sense of Lecture II1) is trivial from the beginning.

Let us now compute the action of G (classically). First of all, we have a moment map $\mu : M_0 \to \tilde{\mathfrak{g}}_0^*$, where $\tilde{\mathfrak{g}}_0$ is the Lie algebra of $\tilde{G}_0$ – the algebra of functions from $\mathbb{R}^{n-1}$ to the Lie algebra $\mathfrak{g}$ of G which vanish at infinity. Thus, for any $\varepsilon \in \tilde{\mathfrak{g}}_0^*$ we have a Hamiltonian $Q(\varepsilon) \in C^\infty(M_0)$ defined by $Q(\varepsilon)(X) = \mu(X)(\varepsilon)$.

In fact, it is easy to compute $Q(\varepsilon)$ using Noether formalism. Namely,

$$Q(\varepsilon) = \int_{\mathbb{R}^{d-1}} Tr(\frac{\partial A}{dt} \nabla_A \varepsilon) d^{d-1}x + \text{ matter terms }, \tag{2.1}$$

On M, $Q(\varepsilon) = 0$ if ε vanishes at infinity. Thus, on M we have $\nabla_A^* \frac{dA}{dt} =$ matter terms . In particular, in pure gauge theory $\nabla_A^* \frac{dA}{dt} = 0$.

Taking this into account, we see that on M

$$Q(\varepsilon) = \int_{\mathbb{R}^{d-1}} Tr(\nabla_A(\varepsilon \frac{dA}{dt}))d^{d-1}x. \tag{2.2}$$

Using Stokes' formula, we can rewrite (2.2) as

$$Q(\varepsilon) = \lim_{r\to\infty} \int_{S^{n-2}(r)} *_{d-1} Tr(\varepsilon \frac{dA}{dt}) = \lim_{r\to\infty} \int_{S^{n-2}(r)} *_d Tr(\varepsilon F), \tag{2.3}$$

where F is the curvature of the spacetime connection corresponding to the given point of M_0, and $S^k(r)$ is the k-sphere of radius r. This formula defines the hamiltonians for the action of $G = G_\infty$ on M.

This formula shows that $Q(\varepsilon)$ vanishes for all gauge transformations (not necessarily vanishing at infinity) on a particular state if $F = o(r^{2-n})$, $r \to \infty$ on that state. However, if this is not the case, then $Q(\varepsilon)$ may be nonzero for a constant ε.

Example. Consider a $U(1)$ gauge theory with a charged complex scalar. The fields are a connection A on a hermitian line bundle and a section ϕ of this bundle. The Lagrangian is

$$\mathcal{L} = \frac{1}{4e^2} \int F^2 + \int |D_A\phi|^2 d^4x + \int \frac{\lambda}{8}(|\phi|^2 + v^2)^2 d^4x. \tag{2.3}$$

This is the most general renormalizable Lagrangian in these fields in 4 dimensions. Here e, λ, v are parameters and e^2, λ are positive while v^2 can be positive or negative. For simplicity we assume first that $v^2 \neq 0$.

This theory is not believed to exist in the UV, but we will regard it as an effective theory for some more fundamental theory.

Classically (and quantum mechanically for $e^2, \lambda << 1$) we have two cases.

1. $v^2 > 0$; the potential has a single minimum.

2. $v^2 < 0$; the potential has a circle of minima.

Let us consider how in these two cases the theory behaves in the infrared.

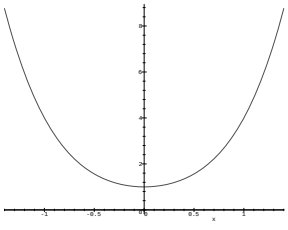


Figure 1. The potential for $v^2 > 0$.

Case 1. $v^2 > 0$. In this case the minimum of energy is attained when $\phi = 0$. First consider the case when the gauge coupling vanishes: $e^2 = 0$. In this case our theory is a direct product of a pure (free) abelian gauge theory and the ϕ^4 theory. Therefore, it has a unique vacuum, and the particles which occur at the lower part of the spectrum are a massless vector, or gauge boson (coming from gauge theory) and two massive real scalars (coming for ϕ^4 theory).

If we turn on small e^2 the situation should remain the same. Indeed, certainly nothing can happen to the massive scalars (the part of the Hilbert space with the nonzero charge, where these scalars are, has a mass gap, and massiveness is an open condition); moreover, their masses must be equal since

there is a $U(1)$ symmetry at infinity (the $Q(\varepsilon)$ for constant ε) which prohibits the masses to differ. The fact that $Q(\varepsilon) \neq 0$ is clear since this is so at $e^2 = 0$, when $Q(\varepsilon)$ represents the $U(1)$ global symmetry.

Furthermore, the massless vectors cannot become massive. Indeed, recall that a massless vector means an irreducible representation of $SO(3,1)$ with $p^2 = 0$ and spin 1, i.e. the space of sections of a 2-dimensional equivariant vector bundle over the light cone. This vector bundle cannot be deformed to an equivariant vector bundle over the hyperboloid, since the stabilizer group $SO(3)$ of a point on the hyperboloid does not have an irreducible 2-dimensional representation. Thus, the quantum theory for small coupling will have the same particles – two massive scalars (the real and imaginary part of ϕ) and a massless vector (the gauge boson).

Remark. The above argument on non-deformability of a massless vector fails in 3 and 2 dimensions. For example, in 3 dimensions, the massless vector is just the space of functions over the cone, which can be successfully deformed into the space of functions over a hyperboloid. This actually happens when in pure $U(1)$ gauge theory one introduces a Chern-Simons term $c\int A \wedge dA$. The theory remains free but becomes massive, yielding one massive scalar. In the theory we are considering (for 3 dimensions), this cannot happen dynamically since the Chern-Simons term is odd under change of orientation, but in other theories this could happen.

In fact, quantum mechanically the operator $Q(\varepsilon)$ (for a suitable normalization of ε) has integer eigenvalues, and thus defines (in quantum theory) a $\mathbb{Z}$-grading of the corresponding Hilbert space. In particular, since $Q(\varepsilon) \neq 0$, there are sectors of the Hilbert space which cannot be reached from the vacuum by applying local operators. This shows that we have a fundamental violation of Wightman axioms: the representation of the operator algebra in the physical Hilbert space is not irreducible. However, the theory still has one vacuum only: the minimal energy in the sectors with nonzero charge $Q(\varepsilon)$ is positive.

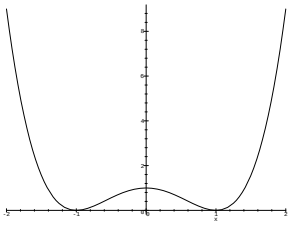


Figure 2. The potential for $v^2 < 0$.

Case 2. $v^2 < 0$. Let $v^2 = -b^2$. Then classically we have a minimum of energy on the circle $|\phi| = b$. This implies that any finite energy configuration has the property $\phi = be^{i\theta_0}$ at infinity, where θ_0 is a constant. Therefore, by a gauge transformation which has a finite limit at infinity, we can arrange that ϕ is real and positive: $\phi = b + w$ where w is a new real variable. Writing the Lagrangian in terms of the new variables, we will get something with the following quadratic part:

$$L_{quadratic} = \frac{1}{4e^2}\int F^2 + \int d^4x((dw)^2 + M^2w^2) + \int d^4x b^2A^2. \tag{2.4}$$

It is seen from (2.4) than now all fields are massive. Of course, Lagrangian (2.4) is not gauge invariant for A, since we have already "spent" the gauge symmetry on making ϕ real.

Thus, infrared limit of the corresponding quantum theory is trivial for small values of the couplings. In particular, there are no massless gauge bosons: they have been "eaten" by the ϕ-field. This situation is called Higgs phenomenon, or spontaneous breaking of gauge symmetry.

Note that in spite of the presence of a circle of zero energy states, our theory has only one vacuum. In other words, all points of the circle are regarded as the same state, on the grounds that they are gauge equivalent to each other and therefore define equivalent realizations (i.e. give the same expectation values of gauge invariant local operators) This is a fundamental difference between gauge and global symmetry breaking. In global symmetry breaking, the points of the circle represent different vacua (realizations) of the theory, since there exist non-symmetric operators which have different expectation values at different point of the circle.

Note also that the operator $Q(\varepsilon)$ doesn't act in the Hilbert space of states, since classically $Q(\varepsilon)$ generates a group which rotates the circle and permutes the zero energy states. In particular, in this case local operators act irreducibly in the Hilbert space, and there are no sectors which cannot be reached from the vacuum. This is the difference between case 2 and case 1: in case 1, as you remember, $Q(\varepsilon)$ acts in the Hilbert space nontrivially and defines a splitting into sectors.

Remark. If one tries to compute $Q(\varepsilon)$ in Case 2 (when the symmetry is broken) using formula (2.3), the answer will be zero since the integrand dies rapidly at infinity.

The particles which are found in the infrared in the situation of Case 2 are, according to (2.4), a massive vector (A) and a massive scalar (ϕ). There is only one scalar since ϕ is now real. Thus, at the level of representation theory the Higgs phenomenon arising in Case 2 boils down to a deformation of representations of the Poincare group: a massless vector plus a massless scalar is deformed to a massive vector. Recall for comparison that a massless vector separately cannot be deformed into a massive representation.

Finally, consider the special case $v^2 = 0$. In this case classically we have no symmetry breaking as for $v^2 > 0$, and the particles are a massless vector and two massless scalar. However, it is not expected to be the quantum answer, since this configuration is not stable under perturbations.

Remark. If the Lagrangian we start with is not IR free (say, it is the Lagrangian of an asymptotically free gauge theory) then the classical analysis we discussed above does not apply in quantum theory. In this case the infrared behavior of the theory is difficult to determine. In particular, it could happen that in the infrared the gauge group of the ultraviolet theory will be replaced with some completely different group, which is not even a subgroup in the original group.

2.3. Symmetry breaking and gauging. In conclusion, let us discuss the connection between global symmetry and gauge symmetry breaking. Suppose we have a Lagrangian L of a field theory (say in 4 dimensions) which has a global $U(1)$ symmetry. A typical example is when the theory contains some scalar fields ϕ_j which are sections of hermitian vector bundles, and $U(1)$ acts by multiplication of these sections by $e^{in_j\theta}$ This $U(1)$ symmetry can be gauged, by introducing a $U(1)$ gauge field A and new Lagrangian

$$L_{gauged} = \frac{1}{4e^2}\int F_A^2 + L_A, \tag{2.5}$$

where L_A is L in which all derivatives of ϕ_j are replaced by covariant derivatives.

The statement is that if L is infrared free then for small gauge couplings the symmetry breaking behavior of the theories defined by L and L_{gauged} is usually the same. Namely, if there is breaking of global symmetry for L then there is breaking of gauge symmetry for L_{gauged} and vice versa.

Indeed, let us consider both cases.

Case 1. No global symmetry breaking. In this case classically the minimum of energy is at $\phi_j = 0$, and thus there is a $U(1)$-invariant vacuum. In quantum theory, $U(1)$ acts in the Hilbert space, and there are no massless particles (Goldstone bosons) corresponding to $U(1)$. In this case, for small gauge coupling e, the matter part L_A of the Lagrangian almost decouples from the gauge part; so classically we get a massless gauge boson.

To consider the quantum mechanical situation, we assume that there are no massless particles in the ungauged theory. In this case, the above classical answer is also quantum mechanical for small couplings, by the non-defomability of representations from massless to massive. However, if massless particles (say Goldstone bosons corresponding to other symmetries which are broken) are present, this answer may not be true.

Case 2. Global symmetry breaking. In this case at the minimum of energy some of the ϕ_j is not zero. There is no invariant vacuum, and there is a Goldstone boson corresponding to this symmetry breaking. In this case, pick a vacuum state and a component $\phi_j^{(n)}$ which is not zero at this vacuum. In the gauged theory, we can perform a gauge transformation which will make this component real and positive. This shows that if there are no other massless particles (in particular, no other broken global symmetries), all fields in the theory will become massive. This happens classically, due to Higgs mechanism as in Case 2 above, and also quantum mechanically for small couplings. Thus, we have breaking of gauge symmetry.

Lecture II-3: Infrared behavior and the S-matrix of the 2-dimensional sigma-model with target space S^{N-1}

Edward Witten
Notes by Pavel Etingof and David Kazhdan

3.1. Infrared behaviour of 2-dimensional sigma-models with target space S^{N-1}.

Consider the theory of N scalar bosons, with the Lagrangian

$$\mathcal{L} = \int d^d x (\frac{1}{2}(\nabla\phi)^2 + \frac{g}{4!}(\phi^2 - a^2)^2), \tag{3.1}$$

where $\phi : \mathbb{R}^d \to \mathbb{R}^N$. In Lecture II-1 we saw that above $d = 2$ this theory has symmetry breaking, from $SO(N)$ to $SO(N-1)$. In particular, for any point y on the sphere $\phi^2 = a^2$ there exists a realization $\mathcal{H}_y$ of this theory, with $\langle\phi\rangle = y$. We also saw that the low energy effective theory for (2.1) is the sigma-model with target space S^{N-1}.

In 2 dimensions the first statement fails: there is no symmetry breaking, and there is only one realization, with an action of $SO(N)$. However, the second statement remains valid: the low energy effective theory is the sigma-model.

In $d > 2$, we know that the sigma-model is infrared free (in the zero approximation); more precisely, it converges in the infrared limit to a free theory of N-1 massless scalars (Goldstone bosons). This is not the case in 2 dimensions. Indeed, in 2 dimensions the sigma-model is renormalizable, so it has marginal interactions (i.e. interactions with a 0-dimensional coupling). On the other hand, it was shown in Gawedzki's lecture on sigma-models that the β-function of the 2-dimensional sigma model into a space of positive curvature is negative (the model is asymptotically free in the UV limit). Therefore, the marginal interactions are relevant in the infrared, and apriori we cannot conclude that the model is not infrared free, even in the zero approximation. In fact, what happens (as we will see today) is that instead of $N-1$ massless particles that we had in $d > 2$, we will have $N-1$ massive particles, so the infrared limit is trivial (the correlation functions are analytic at the origin in the momentum space, and decay exponentially at infinity in position space).

In the first half of the lecture, we will show that the 2-dimensional sigma-model is indeed infrared trivial for large N. Namely, we will show that all coefficients of the expansion of the correlation functions in a series in $1/N$ are analytic at the origin in momentum space, and the first term of this expansion gives the correlation functions for N free massive particles.

Remark 1. Since the β-function of the 2-dimensional sigma-model into the sphere is negative, it is believed that this theory actually exists. So we will now study the infrared behavior of a (probably) actual quantum field theory.

Remark 2. Since the theory defined by the quartic Lagrangian (3.1) exists rigorously in $d = 2$, it can be regarded as an ultraviolet cutoff of the sigma-model. The characteristic momentum scale Λ of this cutoff is the mass of the radial component of ϕ. This cutoff is called the linear sigma-model, as opposed to the nonlinear sigma-model, which is the model of maps to the sphere.

3.2. Computation of the infrared behavior in the $N \to \infty$ limit.

Now we will compute the infrared behavior of the sigma model in the limit $N \to \infty$ using the saddle point approximation in the path integral. We will operate with path integrals formally, without worrying whether they exist or not.

The path integral which defines the generating series for the Euclidean correlation functions of the

sigma-model is given by the formula

$$Z(J) = \int_{\phi:\phi^2=1} D\phi e^{-\frac{1}{2\lambda}\int d^2x|d\phi|^2 + Z_J \int d^2x J\phi}, \tag{3.2}$$

where $\phi, J : \mathbb{R}^2 \to \mathbb{R}^N$, and λ is a coupling constant. The factor Z_J is added because the operator ϕ has anomalous dimension (see Gross' lecture 3) and requires multiplicative renormalization. (As we know, Z_J is cutoff-dependent, and diverges as the cutoff goes to infinity.)

It is more convenient to integrate over maps to a linear space than over maps to a curved manifold. Therefore, it is useful to rewrite integral (3.2) in the form

$$Z(J) = \int_{\phi:\mathbb{R}^2\to\mathbb{R}^N} D\phi \prod_{x\in\mathbb{R}^2} \delta(\phi^2(x)-1) e^{-\frac{1}{2\lambda}\int d^2x|d\phi|^2 + Z_J\int d^2x J\phi}. \tag{3.3}$$

We will now use the formula

$$\delta(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ia\sigma} d\sigma. \tag{3.4}$$

Substituting (3.4) into (3.3), we get (up to a factor):

$$Z(J) = \int D\phi \int D\sigma e^{i\int d^2x(\frac{\sigma}{2}(\phi^2-1)) - \frac{1}{2\lambda}\int d^2x|d\phi|^2 + Z_J\int d^2x J\phi}. \tag{3.5}$$

Thus, the use of formula (3.4) led to introduction of an auxiliary scalar field σ.

Now we will change the order of integration and integrate with respect to ϕ. Our goal is to reduce the integral to the form in which N (the dimension of the target space) enters analytically. We are lucky that the sphere is a quadric, so the integral with respect to ϕ is Gaussian. So we can compute it explicitly, and get

$$Z(J) = \int D\sigma \det\Big(-\frac{\Delta}{\lambda} - i\sigma\Big)^{-N/2} e^{-i\int d^2x\frac{\sigma}{2} - \frac{1}{2}Z_J^2\int d^2x J\cdot(\Delta/\lambda + i\sigma)^{-1}J} \tag{3.6}$$

(the power $-N/2$ of the determinant appears because we are doing N independent Gaussian integrals in the components ϕ_i of ϕ). It is useful to raise the determinant into the exponential:

$$Z(J) = \int D\sigma e^{-\frac{N}{2}\mathrm{Tr}\ln(-\frac{\Delta}{\lambda} - i\sigma) - i\int d^2x\frac{\sigma}{2} - \frac{1}{2}Z_J^2\int d^2x J\cdot(\Delta/\lambda + i\sigma)^{-1}J}. \tag{3.7}$$

Now we want to bring integral (3.6) to the form where there is a factor of N in front of all terms in the exponential (except for the J-term, which serves to expand the answer in the formal series in J), and then evaluate the integral using the saddle-point approximation. For this purpose we have to make the change of variable $\rho = \sigma/N$ and rescale the coupling constant by $\tilde{\lambda} = \lambda N$. After these changes the integral, up to a factor, looks like

$$Z(J) = \int D\sigma e^{-\frac{N}{2}[\mathrm{Tr}\ln(-\Delta - i\tilde{\lambda}\rho) + i\int d^2x\frac{\rho}{\lambda} - \frac{\tilde{\lambda}}{2N}Z_J^2\int d^2x J\cdot(\Delta + i\tilde{\lambda}\rho)^{-1}J}. \tag{3.8}$$

In this form, we can already apply the saddle point approximation. (This is really saddle point and not stationary phase, as the function in the exponential is complex).

The most natural thing is to look for a Poincare invariant saddle point, since the integral we are considering is Poincare-invariant. Thus, we should look for a saddle point $\rho(x)$ which is a constant, $\rho(x) = \rho$. The value of ρ is found from the saddle point equation

$$\frac{d}{d\rho}[\mathrm{Tr}\ln(-\Delta - i\tilde{\lambda}\rho) + i\rho\int d^2x] = 0. \tag{3.9}$$

(at this point we are making an IR cutoff L, i.e. assume that the spacetime is a torus of size $L \times L$). Let us rewrite (3.9) using the momentum space reprsentation of the operator $-\Delta - i\tilde{\lambda}\rho$, and send L to infinity. Then we get

$$\tilde{\lambda} \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 - i\tilde{\lambda}\rho} = 1. \tag{3.10}$$

The integral on the left hand side is logarithmically UV divergent. So we should introduce an UV cutoff Λ in momentum space (i.e. consider the integral over the ball $|k| < \Lambda$). Then (3.10) becomes

$$\frac{\tilde{\lambda}}{4\pi} \ln \frac{\Lambda^2}{-i\tilde{\lambda}\rho} = 1. \tag{3.11}$$

This shows that

$$\rho = \frac{i\Lambda^2}{\tilde{\lambda}} e^{-\frac{4\pi}{\tilde{\lambda}}}. \tag{3.12}$$

In Gross' lecture 3 it was shown that in an asymptotically free theory, the effective coupling constant at scale Λ depends on the scale according to the formula

$$\tilde{\lambda}(\Lambda) = \frac{1}{A \ln \Lambda} + o(\frac{1}{\ln \Lambda}), \Lambda \to \infty, \tag{3.13}$$

where A is minus the coefficient of the 1-loop beta-function. In the sigma-model it turns out that $A = (2\pi)^{-1}\frac{N-2}{N}$, so it equals to $(2\pi)^{-1}$ modulo $1/N$ (see the end of Gawedzki lecture 3). It can be shown, that modulo $1/N$, the beta-function equals to its 1-loop approximation. This implies that there exists a limit $M^2 = \lim_{\Lambda\to\infty}(-i\tilde{\lambda}\rho)$, which is positive.

Remark. Thus, in the quantum theory we have a characteristic momentum scale defined by M, despite the fact that the classical theory is conformally invariant, and the Lagrangian has only a dimensionless coupling λ and defines no particular momentum (or length) scale. This phenomenon is called "dimensional transmutation". It gives a very vivid demonstration of the fact (which we already know), that there may be no canonical way of renormalizing a theory defined by a scale-invariant Lagrangian.

Recalling the form of the J-term in the integral $Z(J)$, we conclude that the J^2 term of the expansion of $Z(J)$ has order $1/N$. Thus, we should rescale the fields ϕ, $\phi \to N^{1/2}\phi$, to get a nonzero limit of correlation functions as $N \to \infty$. After this change it turns out that the 2-point functions of ϕ_i have a nonzero limit, while the connected $4, 6, ...$-point functions vanish modulo $1/N$. Thus, in the first order approximation in $1/N$, our theory is free. Moreover, the propagator of this first order theory is $\frac{1}{k^2+M^2}$, where M is defined as above. Thus, the limiting theory is the theory of N independent free massive scalars of mass M.

The connected 4-point function is of order $1/N$. Therefore, the effective coupling constant between the massive scalars is of order $1/N$.

The fact that the fields ϕ_i decouple in the large N limit and become massive scalar fields, has the following probabilistic explanation.

Consider the theory of N free massive bosons a_i, described by the Lagrangian

$$\mathcal{L} = \int (\prod Da_i) e^{-N\sum_i (|da_i|^2 + M^2 a_i^2)}. \tag{3.14}$$

Consider the expectation value $\langle a_i^2 \rangle$ of the operator a_i^2 in this theory for a fixed i. This value requires renormalization: it is given by the integral

$$\int \frac{d^2k}{(2\pi)^2} \frac{1}{N(k^2 + M^2)}, \tag{3.15}$$

which is UV divergent. So we should introduce an UV cutoff Λ. Then we have $\langle a_i^2\rangle = \frac{1}{4\pi N}\ln\Lambda^2/M^2$. Therefore, $\langle\sum_i a_i^2\rangle = \frac{1}{4\pi}\ln\Lambda^2/M^2$. On the other hand, by the law of large numbers, if d is the dispersion of a_i^2, then the dispersion of $\sum a_i^2$ is $d/\sqrt{N}$, so for large N the vector $(a_1, ..., a_N)$ stays close to the the sphere $\sum x_i^2 = \frac{1}{4\pi}\ln\Lambda^2/M^2$. Thus, the theory of N independent massive fields for large N behaves like the sigma-model for the sphere, whose radius varies logaritmically with the cutoff.

3.3. Computation of the S-matrix.

Now we will study the sigma-model into the sphere in a totally different way. Namely we will show that this model is integrable even for finite N, which means that its S-matrix can be computed explicitly.

Recall the Coleman-Mandula theorem (Bernstein's lecture 1): in a field theory of dimension $d > 2$ with a mass gap and a nondegenerate S-matrix, any even infinitesimal symmetry of the S-matrix is a linear combination of an element of the Poincare Lie algebra and a symmetry which commutes with the Poincare Lie algebra. In other words, if there exists a "forbidden" symmetry (not having such a decomposition), then the S-matrix, under some technical conditions, equals 1, i.e. the theory is free.

In two dimensions, the Coleman-Mandula theorem is false. That is, there exist 2-dimensional quantum field theories with $S \neq 1$ which have an infinitesimal symmetry forbidden by the Coleman-Mandula theorem. In fact, we will see that the 2-dimensional sigma-model has this property.

Thus, in 2 dimensions the information that there exists a "forbidden" symmetry is not enough to conclude that $S = 1$. However, this information still allows to compute the S-matrix. In this section we will show how to do it for the sigma-model into S^{N-1}.

We will work on a flat Minkowski spacetime with coordinates x_+, x_- and the metric dx_+dx_-. Let

$$\mathcal{L} = \frac{1}{2\lambda}\int d^2x(\partial_+\phi\partial_-\phi), \phi^2 = 1. \tag{3.16}$$

be the Lagrangian of the sigma-model with target S^{N-1} ($\phi \in \mathbb{R}^N$). Since this Lagrangian can be written naturally on a curved spacetime, it has a stress-energy tensor $T = \frac{\delta\mathcal{L}}{\delta g}$. This is a symmetric, rank 2 tensor, so it has 3 components in coordinates x_+, x_-: T_{++}, T_{--}, T_{+-}.

Let us consider some properties of the stress-energy tensor. Classically, we have

$$T_{++} = (\partial_+\phi)^2, T_{--} = (\partial_-\phi)^2. \tag{3.17}$$

Since the sigma-model is classically conformally invariant, we have $T_{+-} = 0$. However, we should expect that quantum mechanically conformal invariance is broken (in fact, we know this from the large N limit), so quantum mechanically, $T_{+-} \neq 0$; in fact, we will see that it is proportional to $\partial_+\phi\partial_-\phi$ (see below). Still, the current conservation equations

$$\partial_-T_{++} + \partial_+T_{+-} = 0, \partial_+T_{--} + \partial_-T_{+-} = 0, \tag{3.18}$$

(which classically reduce to $\partial_-T_{++} = 0, \partial_+T_{--} = 0$, because of vanishing of T_{+-}) are still satisfied quantum mechanically, if the operator T_{+-} is suitably renormalized (this follows from the fact that the Poincare symmetry exists in the quantum theory).

Let us give another demonstration of why the equation

$$\partial_-T_{++} + \partial_+T_{+-} = 0. \tag{3.19}$$

is satisfied under a suitable renormalization of T_{+-}. For this purpose recall that classically operators in our theory have a bigrading (d_+, d_-) (with respect to dilations of x_+, x_-). In quantum theory, since

conformal invariance is broken, only the diagonal grading $d_+ - d_-$ survives, and the bigrading becomes a (bi)filtration.

Let us classify $SO(N)$-invariant operators of various bidegrees (under (d_+, d_-) we will list operators of bidegree $d_+ - n, d_- - n$, $n \geq 0$).

(0,0): 1

(0,1): none

(1,0): none

(2,0): $(\partial_+\phi)^2 = T_{++}$

(0,2): $(\partial_-\phi)^2 = T_{--}$

(1,1): $\partial_+\phi\partial_-\phi, 1$

(2,1): $\partial_+^2\phi\partial_-\phi$

This table shows that (3.19) has to be satisfied, under a suitable renormalization of T_{ij}, for dimensional reasons (to be more specific, because any operator of bidegree (2.1) is ∂_+ of an operator of bidegree (1,1), and there is no operators of bidegree (1,0).

Remark. In fact, it can be shown that $T_{+-} = \beta\partial_+\phi\partial_-\phi$, where β is the beta-function. Thus, the beta-function measures the failure of conformal invariance.

We will now construct a symmetry forbidden by the Coleman-Mandula theorem using the (quantum) stress-energy tensor. The idea of constructing this symmetry is the following. Suppose that we found two local operators X_+, X_- such that $\partial_+X_- = -\partial_-X_+$, but $X_- \neq \partial_+Y$ for any Y. In this case the operator-valued form $J = X_+dx_+ - X_-dx_-$ is a nontrivial conserved current, and the charge operator $Q = \int_C J$, where C is a spacelike cycle, is an infinitesimal symmetry. So we should construct X_+, X_-.

Classically, $X_+ = T_{++}^2$ satisfies the above condition, as $\partial_-(T_{++})^2 = 0$. Quantum mechanically, however, this equation is not satisfied. So we will again use dimension counting to demonstrate the existence of a conservation law.

We want to show that there exist operators X_+, X_- of degrees (4,0) and (3,1), such that $\partial_-X_+ + \partial_+X_- = 0$, and $X_+ \neq \partial_+Y$. To do this, let us extend the above table of operators:

(3,0): $\partial_+\phi\partial_+^2\phi$

(3,1): $(\partial_-\phi\partial_+\phi)(\partial_+\phi)^2$, $\partial_-\phi\partial_+^3\phi$, $(\partial_+\phi)^2$.

(4,0): $((\partial_+\phi)^2)^2$, $(\partial_+^2\phi)^2$, $\partial_+^3\phi\partial_+\phi$

(4,1): $\partial_-\phi\partial_+^4\phi$, $(\partial_-\phi\partial_+\phi)(\partial_+\phi\partial_+^2\phi)$, $(\partial_-\phi\partial_+^2\phi)(\partial_+\phi)^2$, $\partial_+\phi\partial_+^2\phi$.

Let H_{d_+,d_-} be the space of operators of bidegree $d_+ - n, d_- - n$ for all $n \geq 0$. We have: $\dim H_{3,0} = 1$, $\dim H_{3,1} = 3$, $\dim H_{4,0} = 3$, $\dim H_{4,1} = 4$. We have two maps $\partial_+ : H_{3,1} \to H_{4,1}$, $\partial_- : H_{4,0} \to H_{4,1}$. For dimensional reasons, there is a 2-dimensional subspace Z in $H_{4,0}$ such that for $X \in Z$ $\partial_-X \in \mathrm{Im}\partial_+$. On the other hand, the image B of ∂_+ in $H_{4,0}$ is 1-dimensional. Therefore, the "cohomology" group $H = Z/B$ is not zero (it is 1-dimensional). This "cohomology" group represents the conservation law that we are looking for.

Remark. It is easy to define the complex for which Z are cocycles, B are coboundary, and H are cohomology.

Denote the charge operator corresponding to this conservation law by Q. It is easy to show that $[Q, P_\pm] = 0$, and $[K, Q] = 3Q$, where K is the infinitesimal boost operator of the Poincare group (the generator of $\mathfrak{so}(1,1)$), normalized in such a way that $[K, P_\pm] = \pm P_\pm$, where $P_\pm$ are infinittesimal translations of x_+, x_-.

Thus, Q is an infinitesimal symmetry violating the conclusion of the Coleman-Mandula theorem.

Remark. Of course, by interchanging + and − in the definition of Q, we can obtain another "forbidden" conserved charge Q', with $[K, Q'] = -3Q'$.

Now let us see how we can use Q. Let $H_m \subset \mathcal{H}$ be a subrepresentation of the Poincare group which represents a particle of mass m ($H = L^2(O_m^+)$). Let $|k\rangle$ denote the state of this particle in which its momentum is $k = (k_+, k_-)$, where $k_+k_- = m^2$. (Of course, $|k\rangle \notin H_m$, but for an L^2-function f on O_m^+, $\int f(k)|k\rangle dk \in H_m^+$.) We have $P_\pm|k\rangle = k_\pm|k\rangle$, and $Q|k\rangle = ck_+^3|k\rangle$, where c is a k-independent constant.

Let λ be the parameter on the hyperbola $k_+k_- = m^2$ defined by the equations $k_\pm = me^{\pm\lambda}$. As we know, a 1-particle state is represented by a wave function, which satisfies the Klein-Gordon equation:

$$\psi(x_+, x_-) = \int_{-\infty}^{\infty} \hat{\psi}(\lambda) e^{i(k_+x_+ + k_-x_-)} d\lambda, \tag{3.20}$$

The operator $e^{i\gamma P_\pm}$ of the Poincare group acts on the wave function by translating x_+, x_- by γ. Let us now compute how the "forbidden" symmetry $e^{i\delta Q}$ acts on the wave function:

$$e^{i\delta Q}\psi(x_+, x_-) = \int_{-\infty}^{\infty} \hat{\psi}(\lambda) e^{i(\delta c k_+^3 + k_+x_+ + k_-x_-)} d\lambda \tag{3.21}$$

Thus, the operator $e^{i\delta Q}$ transforms the weight function of a particle with momentum k by translating x_+ by the effective amount $\gamma_{\text{eff}} = \delta c k_+^2$.

Suppose now that we have a particle whose wave function is localized near a classical worldline $x = x_0 + vt$ ($t + x = x_+, t - x = x_-$). Then the symmetry $e^{i\delta Q}$ shifts the worldline in a way dependent on the velocity v. (This property is characteristic for all symmetries which don't commute with the Poincare group, and don't belong to it).

Now it is easy to explain the idea of proof of the Coleman-Mandula theorem, and why it fails in 2 dimensions.

In dimension $d > 2$, two lines generically don't intersect. So, in a theory with a "forbidden" symmetry, if we have two particles heading for a collision (i.e. their worldlines intersect), we can apply the "forbidden" symmetry and obtain two other worldlines which don't intersect, and are arbitrarily far from each other. Since the S-matrix is invariant under the symmetry, this argument shows that it equals 1.

In two dimensions, however, two lines generically do intersect, so we cannot conclude from the above argument that the S-matrix is 1. Still, since three lines generically do not intersect at the same point, the above argument shows that scattering reduces to a successive 2-particle collisions, in which two particles are produced.

The presence of symmetry Q also shows that in the 2-2 scattering, the momenta (and hence masses) of incoming particles are equal to those of outgoing particles. Indeed, let $1, 2$ label incoming particles, and $3, 4$ label outgoing particles. The charge Q acts on the state of the i-th particle by by multiplication by $C_i(k_i)_+^3$. So by charge conservation $C_1(k_1)_+^3 + C_2(k_2)_+^3 = C_3(k_3)_+^3 + C_4(k_4)_+^3$. On the other hand, by momentum conservation $(k_1)_+ + (k_2)_+ = (k_3)_+ + (k_4)_+$. So if k_1, k_2 are known, (k_3, k_4) can take only a discrete set of values. This implies that $(k_3, k_4) = (k_1, k_2)$ or $(k_3, k_4) = (k_2, k_1)$.

Now let us return to the sigma-model with target space S^{N-1}. In this case, as we know for large N (and believe for all $N \geq 3$), we have N massive particles of the same mass m, which form an N-dimensional representaion V of $SO(N)$ (i.e. the space of 1-particle states is $V \otimes L^2(O_m^+)$). The above arguments allow us to make the following conclusions about the scattering matrix.

(i) The scattering matrix is in this case an honest matrix, i.e. a function of the form $S(\lambda_1 - \lambda_2)$ with values in $\text{End}_{SO(N)}(V \otimes V)$, where λ_i are defined by $k_i = (me^{\lambda_i}, me^{-\lambda_i})$, and k_i are the momenta of the incoming particles. Thus,

$$S(\theta)_{ij}^{kl} = F(\theta)\delta_i^k\delta_j^l + G(\theta)\delta_i^l\delta_j^k + H(\theta)\delta_i^j\delta_k^l, \tag{3.22}$$

where F, G, H are complex-valued functions.

(ii) $S(\lambda)$ satisfies the quantum Yang-Baxter equation:

$$S_{12}(\lambda_1 - \lambda_2)S_{13}(\lambda_1 - \lambda_3)S_{23}(\lambda_2 - \lambda_3) = S_{23}(\lambda_2 - \lambda_3)S_{13}(\lambda_1 - \lambda_3)S_{12}(\lambda_1 - \lambda_2). \tag{3.23}$$

(iii) $S(\lambda)$ is unitary.

(iv) Crossing symmetry: S continues meromorphically to the complex plane, and $S_{ij}^{kl}(\lambda) = S_{li}^{jk}(\pi i - \lambda)$.

Property (i) we have already explained. Property (ii) follows from the fact that the 3-3 scattering matrix factors as a product of three 2-2 scattering matrices, and the order in which these three 2-2 collisions occur can be reversed by the "forbidden" symmetry. Property (iii) is satisfied for the S-matrix of any field theory. Property (iv), roughly, follows from the fact that the 2-2 S-matrix is the residue of the pole of the 4-point function, which extends to the complex values of momenta, so the S-matrix has to be invariant under the discrete part of the Poincare group, which permutes past ant future.

A.Zamolodchikov and Al.Zamolodchikov showed in 1979 (Annals of Physics, vol. 120) that the S-matrix is uniquely determined by these conditions for $N \geq 3$, up to a scalar θ-dependent factor, provided it is nontrivial (which we can get from the large N expansion). The functions F, G, H for this S-matrix is given by an easy explicit formula:

$$F(\theta) = 1, G(theta = -\frac{2\pi i}{N-2}\theta, H(\theta) = -\frac{2\pi i}{(N-2)(\pi i - \theta)}. \tag{3.24}$$

This solution of the quantum Yang-Baxter equation was one of the main examples from which the theory of quantum groups has originated. The quantum group corresponding to it is now called "the Yangian of $\mathfrak{so}(N)$" (see Drinfeld's talk "Quantum groups" at the International Congress of Mathematicians at Berkeley).

As we indicated, the S-matrix is defined by (3.24) up to multiplication by a scalar function $f(\theta)$, which satisfies the conditions $f(\theta)f(-theta) = 1$, and $f(\pi i - \theta) = f(\theta)$. Thus, the function f is $2\pi i$-periodic. Which periodic function corresponds to the actual S-matrix of the theory is a nontrivial question. A perturbative calculation indicates that this function should be chosen in such a way that the S-matrix has the smallest possible number of singularities. There are two such choices, and one of the is believed to be realized in the sigma-model. The other choice is believed to be realized similarly in the Gross-Neveu model.

Thus, we have shown the integrability of the sigma-model into S^{N-1}.

One can show in a similar way the the Gross-Neveu model introduced in Gross' lecture 4 is also integrable, in the sense that its S-matrix and 1-particle matrix elements of composite operators can be computed. The S-matrix turns out to be the same as above, up to the scalar factor.

But these two models are an exception. Already in the next lecture we will consider a slightly more complicated sigma-model – with target space $\mathbb{C}P^N$, for which the explicit form of the S-matrix is unknown.

Also define the space and time coordinates t, x by $t = \frac{1}{2}(x_+ + x_-)$, $x = \frac{1}{2}(x_+ - x_-)$.

At $t = 0$ we have

$$\psi(x, 0) = \int_{-\infty}^{\infty} \hat{\psi}(\lambda) e^{i(k_+ - k_-)x} d\lambda, \tag{3.21}$$

and the wave function is completely determined by its value at $x = 0$, via the Klein-Gordon equation, and the energy positivity condition.

Lecture II-4: The large N limit of the $\mathbb{CP}^{N-1}$ model

Edward Witten
Notes by Dan Freed

Remark *The lecture treated the σ-model into projective space, but these notes cover the generalization to a Grassmannian, as requested in Problem Set 3.*

In this lecture we discuss the large N behavior of the two dimensional σ-model into the Grassmannian $Gr(k,N)$ of k dimensional subspaces of $\mathbb{C}^N$. Here k is fixed as $N \to \infty$. We also consider the real Grassmannian. Since Grassmannians have positive Ricci curvature these field theories are asymptotically free, but in any case our task is to investigate the infrared behavior.

The Euclidean action of the σ-model is

$$S[\phi] = \frac{1}{g^2}\int_\Sigma d^2x\,|d\phi|^2 - i\theta\int_\Sigma \phi^*(\alpha), \tag{1}$$

where Σ is a Riemann surface, $\phi\colon \Sigma \to Gr(k,N)$ a map into the Grassmannian, $\alpha \in H^2(Gr(k,N),\mathbb{Z})$ a generator of the cohomology, and g,θ are parameters of the theory. We specify the metric on $Gr(k,N)$ shortly. The second term is a topological term;[5] for Σ closed the integral is integer-valued. Thus a shift $\theta \to \theta + 2\pi$ does not affect the model. The parameter θ is also a parameter of the quantum theory, but renormalization exchanges the dimensionless coupling constant g with a mass parameter μ.

The rescaled coupling constant $\tilde{g}$, defined by

$$g^2 = \tilde{g}^2/N, \tag{2}$$

is more natural in the large N limit, as we will see.

4.1 The Questions We ask specific questions about the behavior of the model.

1. Is there a mass gap?
2. What is the θ dependence of the partition function?

   Remark. For Σ small the partition function

   $$Z(\theta) = \int D\phi\ \ e^{-\frac{1}{g^2_{\text{eff}}}\int_\Sigma d^2x\,|d\phi|^2}\ e^{i\theta\int_\Sigma \phi^*(\alpha)}$$

   can be studied using perturbation theory. Here g_{eff} is the effective coupling, which varies with the distance scale in the theory set by the size of the surface Σ. (Classically the model is conformally invariant, so does not depend on the size of Σ, but quantum mechanically this is no longer true.) By asymptotic freedom this coupling is small at small distances, hence the assertion that perturbation theory applies in this regime. Now the classical solutions to (1) are harmonic maps $\Sigma \to Gr(k,N)$. Note that the only θ dependence is through the degree of ϕ, and so we split the integral as a sum over maps ϕ of varying degrees. In degree 0 we obtain

[5] The factor of i is present in the *Euclidean* action so that the action conjugates under orientation reversal. In this way its continuation to Minkowski space is real. See Freed's notes *Actions and Reality* for more details.

constant maps and in general for degree n some moduli space $\mathcal{M}_n$ of harmonic maps, which are the *instantons* of this model. Degree 1 instantons are (anti)holomorphic maps. The perturbation expansion around these solutions has the rough form

$$Z(\theta) \sim \frac{\mathrm{Vol}(Gr(k,N))}{\sqrt{\det(\cdot)}}(1+\cdots) + \sum_{\pm} e^{\pm i\theta} e^{-cN/g_{\mathrm{eff}}^2} \frac{\mathrm{Vol}\,\mathcal{M}_1}{\sqrt{\det(\cdot)}}(1+\cdots) + \text{higher instantons} \tag{3}$$

The constant c in the exponential is the action of a 1-instanton, which is independent of N. The factor of N comes from (2). If $g_{\mathrm{eff}} << 1$ we see that the θ dependent term of $Z(\theta)$ vanishes exponentially as $N \to \infty$. As the area of Σ increases the effective coupling g_{eff} also increases. Equation (3) is the answer for Σ compact and of small area, but we will find a vastly different result for $\Sigma = \mathbb{R}^2$.

3. *Symmetry Breaking.* Symmetry breaking in two dimensions is possible for a discrete symmetry, and in this model we have the parity symmetry

$$\begin{aligned} P\colon \Sigma &\longmapsto \overline{\Sigma} \qquad (\overline{\Sigma} \text{ is } \Sigma \text{ with the opposite orientation}) \\ \theta &\longmapsto -\theta \end{aligned} \tag{4}$$

   For $\Sigma = \mathbb{R}^2$ we implement the orientation reversal by an orientation-reversing isometry, i.e., a reflection. Then for $\theta = 0$ and $\theta = \pi$ the parity symmetry P acts on a fixed theory and we ask if it is broken in the quantum theory.

4. The group $PSU(N)$ of isometries of the Grassmannian acts in the classical theory. Since continuous symmetry groups are unbroken in two dimensions, this symmetry acts in the quantum theory as well. However, it is possible that a realization of the theory has a symmetry group which is a cover of $PSU(N)$, the latter being the group which acts on the operator algebra. Does that happen here?

4.2 An Equivalent Formulation To study the model we rewrite it, that is, we construct an action with the same classical and quantum physics. As a preliminary we recall some basic geometry of the Grassmannian. Over $Gr(k,N)$ lies a canonical sequence of vector bundles

$$0 \longrightarrow S \xrightarrow{s} Gr(k,N) \times \mathbb{C}^N \longrightarrow Q \longrightarrow 0, \tag{5}$$

where the fiber of S at a k-plane π is simply π viewed as a subspace of $\mathbb{C}^N$. Fix the standard metric on $\mathbb{C}^N$. It induces a metric on S and identifies $Q \cong S^\perp$. There is a canonical connection ∇ on S, obtained by projecting the natural connection on the trivial bundle $Gr(k,N) \times \mathbb{C}^N$. We easily compute

$$\nabla = d - s^* ds, \tag{6}$$

where s is the inclusion $S \xrightarrow{s} Gr(k,N) \times \mathbb{C}^N$. Then

$$\nabla s\colon T(Gr(k,N)) \longrightarrow \mathrm{Hom}(S, S^\perp) \tag{7}$$

is an isomorphism. We use it to induce a metric on $Gr(k,N)$, the metric needed to write down the σ-model action (1).

Now if $\phi\colon \Sigma \to Gr(k,N)$ we pullback (5) to obtain a sequence of bundles over Σ, and by (7) the lagrangian density of ϕ is

$$|d\phi|^2 = |(\phi^*\nabla)\phi^* s|^2. \tag{8}$$

Note $\phi^* s\colon \phi^* S \to \Sigma \times \mathbb{C}^N$ determines ϕ. The idea is to replace ϕ by such a bundle map, and so first to replace $\phi^* S$ by a fixed bundle. Note $\deg(\phi^* S) = \deg(\phi)$ so that the topology of $\phi^* S$ determines the cohomology class $\phi^*(\alpha)$, which appears in the second term of the action (1). Hence fix a vector bundle $E \to \Sigma$ of rank k and degree d. Also fix a hermitian metric on E. We introduce a new field

$$\hat{\phi}\colon E \longrightarrow \Sigma \times \mathbb{C}^N$$

which we constrain to be an isometric immersion:

$$\hat{\phi}^* \hat{\phi} = \mathrm{id}_E\,. \tag{9}$$

The image of $\hat{\phi}$ determines a map $\phi\colon \Sigma \to Gr(k, N)$ which is unchanged if we shift $\hat{\phi}$ by a unitary gauge transformation of E. To rewrite (8) in terms of $\hat{\phi}$ we need a connection on E, and as there is no natural choice we introduce a variable unitary connection A. Using $\hat{\phi}$ we identify E with a subbundle of the trivial bundle $\Sigma \times \mathbb{C}^N$, so can differentiate $\hat{\phi}$ using the usual derivative d. Writing A as a 1-form plus this trivial connection we find

$$\begin{aligned} |d_A \hat{\phi}|^2 &= |d\hat{\phi} + \hat{\phi} A|^2 \\ &= |d\hat{\phi}|^2 + 2\,\mathrm{Re}(d\hat{\phi}, \hat{\phi} A) + |A|^2, \end{aligned} \tag{10}$$

since $\hat{\phi}^* \hat{\phi} = \mathrm{id}_E$. This expression is quadratic in A, so if (10) is a classical lagrangian for A we can use the equations of motion to obtain

$$A_0 = -\hat{\phi}^* d\hat{\phi}. \tag{11}$$

Comparing with (6) we see that A_0 is the pullback of the canonical connection on S, and so by (8)

$$|d\phi|^2 = |d_{A_0} \hat{\phi}|^2.$$

In other words, the lagrangian (10) is equivalent to $|d\phi|^2$ for fields which satisfy the constraint (9). We impose the constraint via a lagrange multiplier field

$$\sigma\colon \Sigma \longrightarrow \mathrm{HermitianEnd}(E).$$

The θ term in the original action (1) can be computed using the (skew-Hermitian) curvature F_A via Chern-Weil theory. Altogether we obtain for our new action[6]

$$S[\hat{\phi}, A, \sigma] = \frac{1}{g^2} \int_\Sigma |d_A \hat{\phi}^2| - i \int_\Sigma \mathrm{Tr}\, \sigma(\hat{\phi}^* \hat{\phi} - \mathrm{id}_E) + \frac{\theta}{2\pi} \int_\Sigma \mathrm{Tr}\, F_A. \tag{12}$$

As we have explained, the classical equations of motion (and other classical constructs) computed from (12) are equivalent to those computed from the original action (1). The classical computation which led to (11) is valid quantum mechanically since the dependence of (12) on A is quadratic and the Hessian is the identity operator (see (10)). (We ignore the constant determinant factor which we obtain from the A integral.) Thus the quantum physics is the same as well.

[6]Some explanation about the lagrange multiplier term is in order. It is based on the general formula

$$\int e^{i(\sigma, x)} d\sigma = \delta(x),$$

where σ lies in the dual space to x, the measure $d\sigma$ is suitable normalized, and $\delta(x)$ is the δ-distribution supported at the origin. In the Minkowski space lagrangian the lagrange multiplier term is

$$\int_\Sigma \mathrm{Tr}\, \sigma(\hat{\phi}^* \hat{\phi} - \mathrm{id}_E),$$

and σ should be interpreted as a 2-form (or density); it lies in the dual space to the function $\hat{\phi}^* \hat{\phi} - \mathrm{id}_E$. Rotation to Euclidean space yields the second term of (12).

4.3 The Large N Effective Theory The argument here is almost identical to that for the large N σ-model into a sphere (see lecture II–3). So we will be brief.

First, the $\hat\phi$ integral is Gaussian, so the partition function is

$$\begin{aligned} Z &= \sum_E \int \frac{\mathcal{D}A}{\mathrm{vol}} D\sigma D\hat\phi\, e^{-S[\hat\phi,A,\sigma]} \\ &= \sum_E \int \frac{\mathcal{D}A}{\mathrm{vol}} D\sigma \exp - \left[\mathrm{Tr}\ln\left(\frac{d_A^* d_A}{g^2} - i\sigma\right) + i\int_\Sigma \mathrm{Tr}\,\sigma + \frac{\theta}{2\pi}\int_\Sigma \mathrm{Tr}\,F_A \right]. \end{aligned} \tag{13}$$

Here $\sum_E$ is the sum over bundles E. We take $\Sigma = \mathbb{R}^2$. Then the sum over E is irrelevant, as is the topological term. In a first approach to this problem, the topology of the bundle will not be important, since it is spread over an infinite volume. Once we get a basic understanding of what the quantum vacuum looks like, it will not be hard to go back and see how the topology enters. We evaluate the leading behavior in large N by evaluating at a stationary point of the exponential in the integrand, i.e., a classical solution of the effective action.

The only Poincaré invariant (that is, Euclidean invariant) possibility for the gauge field is $A = 0$. In that case we rewrite minus the integrand in the last line of (13) as

$$S_{\mathrm{eff}}[A,\sigma] = N\,\mathrm{Tr}\ln\left(\frac{d^*d}{g^2} - i\sigma\right) + i\int \mathrm{Tr}\,\sigma,$$

where the operator in the first term acts on sections of E^*. Now the only Poincaré invariant possibility is σ constant, and so we diagonalize σ and pass to k one dimensional problems for an eigenvalue. At this point we rescale

$$\begin{aligned} \sigma &= N\tilde\sigma \\ g^2 &= \tilde g^2/N, \end{aligned} \tag{14}$$

and so for each eigenvalue we have exactly the problem we had for the large N σ-model in to a sphere. Thus the solution $\tilde\sigma_0$ has all eigenvalues equal and is specified by

$$-i\tilde g^2\tilde\sigma_0 = \Lambda^2 e^{-4\pi/\tilde g^2} = M^2$$

for Λ an ultraviolet cutoff and now $\tilde g^2$ the running coupling constant. We define M^2 to be this dynamically generated mass squared.

So in the large N effective action $\tilde\sigma$ acquires a mass. Note that the A field has no transverse degrees of freedom—it is a gauge field in two dimensions—so does not enlarge the spectrum of the model (though, of course, it affects the Hamiltonian as we will see, and in fact diminishes the spectrum). Thus the large N limit has a mass gap. This is the answer to the first in our list of questions.

To answer the other questions we need to compute something more precise, namely the leading approximation to the large N effective action. This means that we do perturbation theory for the action (12) about the point $\hat\phi_0 = 0, A_0 = 0, \tilde\sigma_0 = iM^2/\tilde g^2$. So shifting $\tilde\sigma$ by $\tilde\sigma_0$ and rescaling $\hat\phi$ we have from (12)

$$S'[\hat\phi, A, \tilde\sigma] = \int_\Sigma |d_A\hat\phi|^2 \;+\; M^2\int_\Sigma |\hat\phi|^2 \;-\; i\tilde g^2\int_\Sigma \mathrm{Tr}\,\tilde\sigma\hat\phi^*\hat\phi \;+\; \frac{\theta}{2\pi}\int_\Sigma \mathrm{Tr}\,F_A. \tag{15}$$

Here we omit a constant term (which only shifts the partition function by a constant) and a linear term in $\tilde\sigma$ (since we are expanding around a solution of the effective action).

The effective action is computed in perturbation theory using one particle irreducible Feynman diagrams with external lines for A and σ and with internal $\hat\phi$ lines. So the inverse propagator for $\tilde\sigma$ is

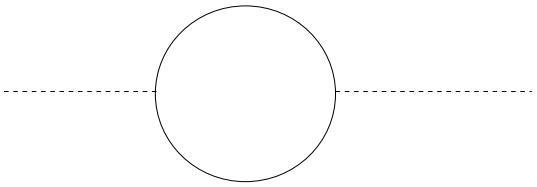


Figure 1: The inverse $\tilde{\sigma}$ propagator

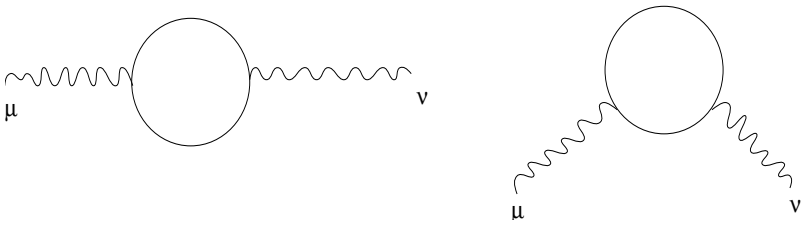


Figure 2: The inverse A propagator

computed from the diagram shown in Figure 1, where the solid line represents $\hat{\phi}$ and the dotted line represents σ. (Note that (15) has no quadratic term in $\tilde{\sigma}$, else Figure 1 would be a correction to such a term.) We evaluate the diagram in momentum space as

$$\begin{aligned} -\tilde{g}^4 Nk \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2+M^2} \frac{1}{(p-q)^2+M^2} &= -\frac{\tilde{g}^4 Nk}{4\pi^2} \int_0^1 d\alpha \int d^2q \, \frac{1}{[q^2+(M^2+\alpha(1-\alpha)p^2)]^2} \\ &= -\frac{\tilde{g}^4 Nk}{\pi} \int_0^1 \frac{d\alpha}{M^2+\alpha(1-\alpha)p^2} \\ &= -\frac{\tilde{g}^4 Nk}{\pi M^2} + O(p^2) \qquad \text{as } p \to 0. \end{aligned} \tag{16}$$

This corresponds to a term

$$C|\tilde{\sigma}|^2$$

in the effective action, with $C > 0$. (The minus sign comes since in the Euclidean theory diagrams compute negative contributions to the effective action.) This is the dominant term in the infrared, which means that $\tilde{\sigma}$ is massive and does not affect the long range behavior of the theory.

The inverse propagator for A is computed by the diagrams in Figure 2, which come from the first term in (15). Here the wavy line represents A. To compute these diagrams we need the Feynman rules for the vertices indicated in Figure 3, which correspond to the terms

$$2\,\mathrm{Re}(d\hat{\phi}, \hat{\phi}A)$$
$$|\hat{\phi}A|^2$$

in the action (15). The indices refer to a standard orthonormal basis for $\mathbb{R}^2$. The Feynman rule (in momentum space) for the second vertex is easy:

$$II = -\delta_{\mu\nu}.$$

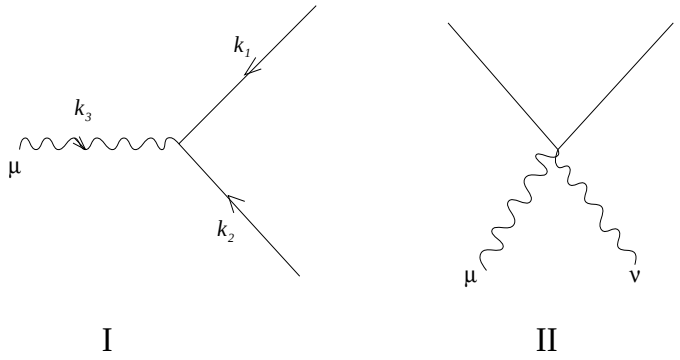


Figure 3: The interaction vertices for the A field

(There is a minus sign since the Euclidean functional integral involves e^{-S}.) For the vertex I we must remember that $\hat{\phi}$ is complex and that A is skew-Hermitian:

$$2\,\mathrm{Re}(\partial_\mu\hat{\phi}, \hat{\phi}A_\mu) = -\partial_\mu\hat{\phi}A_\mu\hat{\phi}^* + \hat{\phi}A_\mu\partial_\mu\hat{\phi}^*.$$

Thus the vertex is

$$I = -i(k_1 - k_2)_\mu.$$

Note that one of the solid lines in the vertex represents $\hat{\phi}$ and the other solid line represents $\hat{\phi}^*$. So the sum of the diagrams in Figure 2 is

$$\int \frac{d^2q}{(2\pi)^2} \frac{(p-2q)_\mu (p-2q)_\nu}{(q^2+M^2)((p-q)^2+M^2)} \;-\; 2\delta_{\mu\nu} \int \frac{d^2q}{(2\pi)^2} \frac{1}{(q^2+M^2)}. \tag{17}$$

The factor of 2 in the second term is from the two ways of attaching the A lines to the external vertices; the corresponding factor of 2 in the first term is canceled by the symmetry which exchanges the two internal vertices in the first diagram. (That is, there is a factor of $1/2$ from the expansion of the exponential, since we have two triple vertices.) Each term in (17) is divergent, but the divergences cancel in the difference, and after some computation similar to (16) the answer to leading order in p is

$$\frac{N}{12\pi M^2}(p_\mu p_\nu - p^2\delta_{\mu\nu}). \tag{18}$$

This corresponds to a term

$$\frac{N}{24\pi M^2}|F_A|^2 \tag{19}$$

in the effective action. (Again we must recall that diagrams contribute negatively to the effective action.) In fact, (18) and (19) correspond precisely if $k = 1$ (the σ-model into projective space), since then A is an abelian connection. In the nonabelian case there are cubic and quartic terms, but by gauge invariance their leading contribution must be as in (19). We remark that (19) is the lowest order gauge invariant term we can write, and so its appearance can be predicted from gauge invariance alone, but of course we must do a computation to determine the coefficient.

So, finally, the long distance behavior of the two dimensional σ-model into the Grassmannian $Gr(k, N)$, in the large N limit, is equivalent to the long distance behavior of a two dimensional gauge theory with

gauge group $U(k)$ and charged massive scalars. The action is

$$\frac{N}{2e^2}\int_\Sigma |F_A|^2 + \frac{\theta}{2\pi}\int_\Sigma \operatorname{Tr} F_A + \int_\Sigma |d_A\phi|^2 + M^2\int_\Sigma |\phi|^2. \tag{20}$$

The first term in (20) was just computed, where the coupling e^2 summarizes the numerical factor in (19). The second term is the θ term in (15). The last two terms are the first two terms in (15), except we now drop the '$\hat{}$' for convenience. So ϕ is a section of $(E^*)^{\oplus N}$.

4.4 Real Grassmannians From the beginning we can replace the complex Grassmannian with the real Grassmannian. In that case the bundle E is real, $\hat{\phi}\colon E \to \Sigma \times \mathbb{R}^N$, and there is no other change except in the representation of the θ term. For simplicity we consider the σ-model into the Grassmannian $Gr^0_{\mathbb{R}}(k,N)$ of *oriented* k-planes in $\mathbb{R}^N$. (It is a double cover of the Grassmannian of unoriented k-planes.) Now $H^2(Gr^0_{\mathbb{R}}(k,N);\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and the θ term is meant to detect this class. Thus the second term of (1) is replaced by

$$i\theta \deg_2 \alpha,$$

where $\deg_2$ is the mod 2 degree (0 or 1) and $\theta = 0$ or $\theta = \pi$. In the reformulation of the problem E is an oriented real k-plane bundle over Σ, and the topological term in (12) and subsequent formulas is

$$i\theta w_2(E)[\Sigma], \tag{21}$$

where w_2 is the second Stiefel-Whitney class. For $\Sigma = \mathbb{R}^2$ we can rewrite (21) in terms of a *Wilson line* operator. Namely, a field configuration with finite action (12) has A essentially flat at infinity. Since $\mathbb{R}^2$ is contractible we can lift the $SO(k)$ connection A to a Spin(k) connection $\tilde{A}$, and the holonomy on a large loop $C \subset \mathbb{R}^2$ is approximately ± 1 depending on the Stiefel-Whitney class of the induced bundle on S^2. (We let '-1' denote the nontrivial element of Spin(k) covering $1 \in SO(k)$.) Choose a representation R of Spin(k) and consider

$$W_R(C) = \frac{\operatorname{Tr}_R \operatorname{hol}_C(A)}{\dim R}. \tag{22}$$

In the limit where the loop C becomes large, this computes the exponential of (21), where $\theta = 0$ if R is a representation of $SO(k)$ and $\theta = \pi$ if R is a representation of Spin(k) but not of $SO(k)$.

The generalization to an arbitrary connected compact Lie group G is clear. A G bundle over Σ has a characteristic class in $H^2(\Sigma, \pi_1 G)$. It pairs with a homomorphism $e^{i\theta}\colon \pi_1 G \to \mathbb{T}$ to give a term in the exponentiated action. Here $\mathbb{T} = U(1)$ is the circle group. On the other hand a representation of the simply connected covering group induces a homomorphism $e^{i\theta}\colon \pi_1 G \to \mathbb{T}$, so we can use the Wilson operator (22) to write the θ term on $\mathbb{R}^2$.

4.5 Pure Gauge Theory We still must determine the quantum behavior of the theory with effective action (20). For this, we will first practice by analyzing the pure gauge theory in two dimensions, whose action is the sum of the first two terms of (20). We may as well consider an arbitrary connected compact Lie group G. The norm in the first term of (20) is defined via a bi-invariant metric on G. We quantize the theory on the circle S^1_V of circumference (=volume) V. In general, to quantize a gauge theory in n dimensions on an n-manifold Y, we consider connections on $\mathbb{R} \times Y$ in *temporal gauge* and take solutions to the equations of motion up to time-independent gauge transformations. (See Kazhdan's lectures on the quantization of gauge theories.) For pure gauge theory without the θ term the resulting space is the (co)tangent bundle of the space of connections on Y modulo gauge

transformations. For $Y = S^1_V$ we first fix a basepoint; then a connection up to gauge equivalence is specified by the holonomy, an element of G. A change of basepoint conjugates the holonomy, and so the quantum Hilbert space is

$$\mathcal{H}_{\text{gauge}}(\theta = 0) \;=\; L^2(G)^G,$$

the space of conjugacy invariant functions on G. A basis for this space is the set of characters of irreducible representations.

Next we compute the Hamiltonian. Let $A_t = A_t(x)dx$ be a connection on $\mathbb{R} \times S^1_V$ (with coordinates t, x) in temporal gauge, relative to some trivialization, and let $g_t \in G$ be the corresponding path of holonomies. In a gauge where A_t is constant, we have $g_t = e^{VA_t}$. Then the first term in the action (20) is

$$\frac{N}{2e^2} \int\int \left|\frac{dA}{dt}\right| \, dtdx = \frac{N}{2e^2 V} \int |\dot{g}|^2 \, dt. \tag{23}$$

This is the lagrangian for a classical particle of mass N/e^2V moving on the Riemannian manifold G; the corresponding quantum Hamiltonian is

$$H_{\text{gauge}} = \frac{e^2 V}{2N} \Delta_G, \tag{24}$$

where Δ_G is the laplacian on G. The eigenfunctions are the characters of the irreducible representations with eigenvalues proportional to the Casimir.

Now consider the θ term in (20). For $G = U(k)$ we have a natural closed imaginary 1-form $\alpha \in i\Omega^1_G$ which is the trace of the Maurer-Cartan form. Then in terms of the path g_t of holonomies the θ term in the action is

$$\frac{\theta}{2\pi} \int g^* \alpha. \tag{25}$$

This is a topological term—it is invariant under reparametrizations of the path g_t. More geometrically, $\theta\alpha$ is a flat connection on a topologically trivial hermitian line bundle L_θ over G, and up to equivalence it is given by an element in $H^1(G; \mathbb{T})$. Then (25) is parallel transport in this flat bundle.[7] More generally, we can "twist" our mechanical system by any hermitian line bundle L with connection. Physically this describes a particle moving in an electromagnetic field. The quantum Hilbert space is the the space of sections of L with Hamiltonian the laplacian for such sections. In our case we obtain the space

$$\mathcal{H}_{\text{gauge}}(\theta) = L^2(G, L_\theta)^G$$

of invariant sections with Hamiltonian (24). For arbitrary G an element $e^{i\theta} \in H^1(G; \mathbb{T})$ corresponds to a homomorphism $e^{i\theta} \colon \pi_1 G \to \mathbb{T}$. Recall that $\pi_1 G$ is a subgroup of the center of G. Then the eigenfunctions of the laplacian acting on $\mathcal{H}_{\text{gauge}}(\theta)$ are the characters of representations of the simply connected cover of G whose restriction to $\pi_1 G$ is $e^{i\theta}$; the eigenvalue is again proportional to the Casimir.

For the unitary group $G = U(k)$ we identify $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ as before. The smallest Casimir occurs for a representation $\det^{\tilde{\theta}/2\pi}$, where $\tilde{\theta} \in \mathbb{R}$ is a representative of θ of smallest absolute value. If $\theta \neq \pi$ there is a unique such $\tilde{\theta}$, but for $\theta = \pi$ there are two possibilities: $\tilde{\theta} = \pi$ and $\tilde{\theta} = -\pi$. For $G = SO(k)$ the simply connected cover is $\tilde{G} = \text{Spin}(k)$. For $\theta = 0$ there is a unique lowest representation, the trivial representation of $SO(k)$. For $\theta = \pi$ the lowest representation is the spin representation if k is odd, and the two half spin representations if k is even. In the large volume limit only the lowest eigenvalue

[7] In general, the classical field theory action on a manifold with boundary—here the interval—is not a number, but we do not stress that point here.

survives, so we have two vacuua for $\theta = \pi$ in the complex ($U(k)$) case and for k even in the real ($SO(k)$) case. Observe that the two vacuua correspond under the involution $g \mapsto -1 \cdot g$ in these cases. (Notice also that the bundle $L_{\theta=0}$ is real.) We will see that this vacuum structure persists when we add matter. Thus parity symmetry (4) is spontaneously broken at $\theta = \pi$ for $G = U(k)$ and $G = SO(2\text{ell})$. This answers the third of our questions.

As the volume $V \to \infty$ the eigenvalues of (24) become widely separated. In particular, in the infinite volume limit there is no state of finite energy above the vacuum (or vacuua). So the physical Hilbert space in infinite volume consists only of the vacuum (or vacuua)—pure gauge theory on $\mathbb{R}^2$ is trivial.

Specialize to $G = U(1)$. Then the *vacuum energy density*, which is the minimum eigenvalue of the Hamiltonian (24) divided by the volume V, is

$$E_{\text{vac}}(\theta) = \frac{e^2}{2N} \min_{n\in\mathbb{Z}}(n - \frac{\theta}{2\pi})^2. \tag{26}$$

Notice that the derivative of E_{vac} has a discontinuity at $\theta = \pi$.

For $G = U(k)$ formula (26) is simply multiplied by k.

The partition function $Z_\Sigma(\theta)$ of the pure gauge theory for $\Sigma = [0, T] \times S^1_V$ has the Hilbert space interpretation

$$Z_\Sigma(\theta) \sim \langle\Omega|e^{-TH(\theta)}|\Omega\rangle \quad \text{as } T \to \infty,$$

where $H(\theta)$ is the Hamiltonian and Ω the vacuum. For the pure gauge theory, we obtain from (26) for $G = U(1)$

$$Z_\Sigma(\theta) \sim \exp(cTV - \frac{e^2TV}{2N} \min_{n\in\mathbb{Z}}(n - \frac{\theta}{2\pi})^2) \quad \text{as } T \to \infty. \tag{27}$$

Here c is a constant which represents the indeterminacy of the path integral, or equivalently the fact that we are free to add a constant (independent of θ) to the Hamiltonian. This is quite different than the prediction (3) from the instanton sum. Note that due to the classical conformal invariance of the σ-model, the instantons used in deriving (3) do not have a definite size; they can be rescaled. For that reason the instanton sum is not reliable at large distances. In any case (27) answers question 2 for pure gauge theory.

4.6 Classical Electromagnetism in Two Dimensions Before analyzing the quantum gauge theory with bosonic matter—the ϕ field in the action (20)—we discuss the classical physics. For the classical analysis we work in the $G = U(1)$ theory. The classical equations—Maxwell's equations—for pure gauge theory are

$$\frac{N}{e^2} d_A^* F_A = 0. \tag{28}$$

Since we are in two spacetime dimensions, this implies that the electric field $f = f_A = *F_A$ is a constant. Let 'x' denote the coordinate on space, which is simply a copy of $\mathbb{R}$. Add a point charge of charge 1 at $x = x_0$. Then the electric field f as a function of x satisfies (28) with a right hand side due to the charge:

$$\frac{N}{e^2}\frac{df}{dx} = -\delta(x - x_0).$$

Thus the value of the electric field jumps by $-e^2/N$ across a charge. (See Figure 4.) Allowing for many charges, and assuming all of them are multiples of the basic charge (which is a conclusion of the quantum theory), we see that

$$\theta = \frac{2\pi Nf}{e^2} \in \mathbb{R}/2\pi\mathbb{Z} \tag{29}$$

Figure 4: Change in electric field across a positive or negative charge

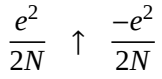


Figure 5: A one particle state for $\theta = \pi$

is a constant. So there is an angle in the classical theory (assuming charge quantization).

The classical energy density of an electric field f is computed from the action (23) (where $f = dA/dt$) to be $\frac{N}{2e^2}f^2$. As in the quantum theory, for a fixed value of θ in (29) there is a unique minimum obtained at some $f = f_0$ as long as $\theta \neq \pi$. Any valid configuration must have finite energy compared to the vacuum energy. Thus if $\theta \neq \pi$ we must have $f(x) \to f_0$ as $x \to \pm\infty$. Using the formula above for the jump of the electric field across a charge, we see that the total charge of a finite energy configuration is zero. This means that there is *confinement*—it is impossible to have a single charged particle or any other isolated collection of charges with nonzero total charge. On the other hand, for $\theta = \pi$ we have minimum energy density at $f_0 = \pm e^2/2N$, and so for any finite energy configuration $f(x) \to \pm e^2/2N$ as $x \to \pm\infty$. Thus there is a finite energy configuration with a single particle: the electric field satisfies $f(-\infty) = e^2/2N$, $f(\infty) = -e^2/2N$. (See Figure 5.) In this case there is no confinement. Also, in this case there are four components of finite energy configurations, depending on the value of f at $\pm\infty$.

Notice that whereas in three space dimensions the Coulomb potential between point charges separated by distance r is proportional to $1/r$, the Coulomb potential in one space dimension is proportional to r. This means the potential energy grows as the charges separate, which is another way of understanding that confinement occurs.

4.7 Quantum theory with matter Now we want to show that there is confinement in the quantum theory of the lagrangian (20) as long as $\theta \neq \pi$. This is the assertion that every finite energy configuration in the quantum theory has total charge zero.

As a preliminary, consider the theory of matter only (no gauge field). In the simplest case $k = N = 1$ there is a single free complex scalar field ϕ with mass M. The Hilbert space $\mathcal{H}_\phi$ of this theory is the completed symmetric algebra of $W \oplus \overline{W}$, where W is the scalar representation of Poincaré with mass M. There is a global $U(1)$ symmetry which rotates ϕ. The corresponding quantum operator Q_ϕ, the Noether charge, has value 1 on W and -1 on $\overline{W}$, so value $p - q$ on $\mathrm{Sym}^p W \otimes \mathrm{Sym}^q \overline{W}$. A state in this subspace represents p positively charged particles and q negatively charged particles. For arbitrary k, N we have kN copies of this picture. In particular, for $k = 1$ there is a global $SU(N)$ symmetry (beyond the $U(1)$ symmetry already discussed.)

Now add the gauge field. For $k = N = 1$ we have a $U(1)$ gauge theory with a single charged scalar field. The global $U(1)$ symmetry of the preceding paragraph is now a local symmetry. Consider

first the case $\theta = 0$. For small coupling e^2 we construct the quantum Hilbert space of the joint system by quantizing the symplectic manifold of classical solutions to (20). Thus take Σ to be the cylinder $\mathbb{R}\times S^1_V$. Then the space of classical solutions is a vector bundle over the cotangent bundle $T^*\mathcal{A}$ to the space $\mathcal{A}$ of connections on S^1_V; the fiber of this vector bundle at the trivial connection $A = 0$ is the real symplectic vector space underlying $W \otimes \overline{W}$. (Note we quantize by a complex polarization, which leads to the Hilbert space $\mathcal{H}_\phi$ described above.) To implement gauge symmetry we must take the symplectic quotient by the group $\mathcal{G}$ of gauge transformations. Fix a basepoint (infinity) on S^1_V. Then the subgroup $\mathcal{G}_0$ of gauge transformations which equal the identity at the basepoint acts freely on $\mathcal{A}$, and the quotient is identified with $U(1)$ via holonomy. In the pure gauge theory the symplectic quotient of $T^*\mathcal{A}$ by $\mathcal{G}_0$ is diffeomorphic to $T^*U(1)$; its quantization $L^2(U(1))$ was discussed previously. As the length $V \to \infty$ recall that the only state which remains of finite energy is the vacuum state. Classically, the vacuum corresponds to the zero section, a lagrangian submanifold of $T^*U(1)$. The subgroup $U(1) \subset \mathcal{G}$ of constant gauge transformations acts trivially on $\mathcal{A}$—hence trivially on $T^*\mathcal{A}$—so does not enter into pure gauge theory. In the theory with matter the symplectic quotient by $\mathcal{G}_0$ is a vector bundle over $T^*U(1)$ with fiber $W\oplus\overline{W}$ at the identity. Now the constant gauge transformations act nontrivially in the fibers by scalar multiplication. To implement the symmetry we have two choices: we can quantize the symplectic quotient or we can consider the subspace of Hilbert space annihilated by the corresponding quantum operator N_e. Pursuing first the latter, we see that in the infinite volume limit the quantum Hilbert space before implementing the $U(1)$ is simply $\mathcal{H}_\phi$, since the gauge field contributes only the vacuum state. The operator N_e is simply equal to Q_ϕ. Thus the Hilbert space of the theory is the subspace of states with total charge $Q_\phi = 0$. Therefore, just as in the classical theory we have confinement. There is a mass gap, and the smallest mass is $2M$. In the theory with small coupling, the qualitative picture is the same.

If instead we take the symplectic quotient of $W \oplus \overline{W}$ by $U(1)$, we are led to a singular space. The moment map is $\mu(w, \overline{w}') = |w|^2 - |\overline{w}'|^2$, and $\mu^{-1}(0)$ is singular at $(0, 0)$. In the quantization this singular point corresponds to the vacuum, and it is not hard to see heuristically that we are led to the same description as before.

Now allow $\theta \neq 0$. In the language of geometric quantization the prequantum line bundle over $T^*U(1)$ is now twisted by the pullback of $L_\theta \to U(1)$. Quantum states correspond to "Bohr-Sommerfeld" leaves of the given polarization, which are equally spaced parallel circles in the cylinder $T^*U(1)$. More precisely, if p is the (momentum) coordinate in the cotangent space, the circles occur at $p = n-\theta/2\pi$ for integral n. The energy of the corresponding quantum state (in pure gauge theory) is given in (26), and as $V \to \infty$ we only keep the closest circle(s) to $p = 0$, which corresponds to the vacuum state. If $\theta \neq \pi$ there is a unique such circle, and we have the same picture of confinement as above. For $\theta = \pi$ there are two such circles, corresponding to the two vacuum states. Thus in the theory with matter, before dividing out by the constant gauge transformations, we must quantize two disjoint copies of $W \oplus \overline{W}$ to obtain $\mathcal{H}_\phi \oplus \mathcal{H}_\phi$. If, as before, we assign zero charge to each of the vacuum states, then the operator N_e which corresponds to the $U(1)$ symmetry is $Q_\phi \oplus Q_\phi$. The subspace of $\mathcal{H}_\phi \oplus \mathcal{H}_\phi$ annihilated by $Q_\phi \oplus Q_\phi$ is simply two copies of the system seen previously, each copy with a vacuum. These are two "realizations" of the quantum theory, each with a mass gap of size $2M$. It turns out—and this is hard to explain from this viewpoint—that we can also assign *different* charges to the two vacuua. In that case the operator N_e is $(Q_\phi - 1) \oplus Q_\phi$ or $Q_\phi \oplus (Q_\phi + 1)$. The kernel in each case has one particle states of mass M, and there is no confinement. There is no vacuum state in either realization. The four realizations correspond to the classical picture of the previous section.

To justify these heuristic pictures we compute. From (20) we see that the Hamiltonian is

$$H = \int_{-\infty}^{\infty} dx\ (\frac{1}{2}|\pi_\phi|^2 + |d_A\phi|^2 + M^2|\phi|^2 + \frac{N}{2e^2}|f_A|^2), \tag{30}$$

where ϕ is a field on $\mathbb{R}$. Here π_ϕ is the conjugate momentum to ϕ, and $f = f_A$ is the Hodge star of the curvature as before. We derive an effective Hamiltonian for ϕ by plugging in the equation of motion of A. The latter is obtained by varying the lagrangian (20) with respect to A:

$$\frac{N}{e^2}\frac{df}{dx} = j, \tag{31}$$

where the current is

$$j = \phi\pi_\phi - \overline{\phi\pi_\phi}.$$

We integrate (31) to obtain

$$f(x) = \frac{e^2}{N}\left\{\int_{-\infty}^{\infty} dy\ G(x-y)j(y) + c\right\} \tag{32}$$

for some constant c, where

$$G(x-y) = \begin{cases} 1, & x \geq y; \\ 0, & x < y. \end{cases}$$

Note that

$$\lim_{x\to-\infty} f(x) = \frac{e^2 c}{N}. \tag{33}$$

Plugging into (30) we see

$$H = H_0 + \Delta H,$$

where H_0 is the Hamiltonian for the free scalar particle (if we compute at $A = 0$), and

$$\Delta H = \frac{e^2}{2N}\int_{-\infty}^{\infty} dx\ \left\{\int_{-\infty}^{\infty} dy\ G(x-y)j(y) + c\right\}^2.$$

The perturbation term ΔH is nonlocal and singular.

Now we must determine the subspace of states Ψ on which ΔH is finite. Consider first $c = 0$. Then $\langle\Psi|\Delta H|\Psi\rangle$ is finite if and only if

$$\lim_{x\to\infty}\langle\Psi|f(x)|\Psi\rangle = 0,$$

where $f(x)$ is defined by (32). Now

$$\lim_{x\to\infty}\int_{-\infty}^{\infty} dy\ G(x-y)j(y) = \int_{-\infty}^{\infty} dy\ j(y) = Q_\phi$$

is the charge operator in the Hilbert space $\mathcal{H}_\phi$. So if $c = 0$ we have confinement:

$$\langle\Psi|Q_\phi|\Psi\rangle = 0.$$

In general, c is related to θ through (33) and (29):

$$|c| = \min_n\left(\frac{\theta}{2\pi} + n\right).$$

This is the value of the electric field at $-\infty$. Also, we subtract an (infinite) constant from ΔH to account for the nonzero energy at ∞:

$$(\Delta H)_{\text{normalized}} = \frac{e^2}{2N}\int_{-\infty}^{\infty} dx \left[\left\{\int_{-\infty}^{\infty} dy\ G(x-y)j(y) + c\right\}^2 - c^2\right].$$

This is finite on states Ψ which satisfy

$$Q_\phi(Q_\phi + 2c)|\Psi\rangle = 0, \tag{34}$$

i.e., $Q_\phi|\Psi\rangle = 0$ or $Q_\phi|\Psi\rangle = -2c$. For $\theta \neq \pi$ the only possibility is $Q_\phi|\Psi\rangle = 0$ and we have confinement. For $\theta = \pi$ we have either $c = \pm 1/2$, and (34) is satisfied by states with $Q_\phi|\Psi\rangle = 0$ or $Q_\phi|\Psi\rangle = \mp 1$. As in the classical theory, this gives four sectors. We denote them $\mathcal{H}_{++}, \mathcal{H}_{+-}, \mathcal{H}_{-+}, \mathcal{H}_{--}$ according to the value of the electric field at $-\infty$ and $+\infty$. There is a vacuum state and confinement in $\mathcal{H}_{++}, \mathcal{H}_{--}$; there is neither a vacuum nor confinement in $\mathcal{H}_{+-}, \mathcal{H}_{-+}$.

In the confining cases the symmetry group $PSU(N)$ acts. In the sectors of the $\theta = \pi$ theory with no confinement the covering group $SU(N)$ acts.

The story for $k > 1$ is similar.

Finally, we make a remark about the electric charge. In the quantum picture it is an operator Q_e, and from Noether's theorem applied to (20) we compute the relation to N_e:

$$Q_e = \frac{e^2}{N}(N_e + \frac{\theta}{2\pi}).$$

The eigenvalues of N_e are integral, but those of Q_e are shifted from $\frac{e^2}{N}\mathbb{Z}$ if $\theta \neq 0$. Note the flow (monodromy) in the eigenvalues of Q_e as θ runs around the circle from $\theta = 0$ to $\theta = 2\pi$. We will encounter this phenomenon again in four dimensional gauge theory.

Lecture II-5: The Bose-Fermi correspondence and its applications

Edward Witten

Notes by Pavel Etingof and David Kazhdan

5.1. 2-dimensional gauge theories with fermions.

Today we will consider 2-dimensional gauge theories with fermions. We will work with Euclidean Lagrangians.

In 2 dimensions, the two spinor representations S_+, S_- are 1-dimensional and complex. They are defined by the formula $\theta \to e^{\pm i\theta/2}$, $\theta \in [0, 4\pi)$, where θ is the parameter on $Spin(2)$.

Let Σ be a Riemann surface with a specified Spin structure. Then S_+, S_- define holomorphic line bundles on Σ, which are called the Spin bundles, and denoted by the same letter. We have $S_+S_- = \mathbb{C}$, and $S_+^2 = T\Sigma, S_-^2 = T^*\Sigma$. Sections of ΠS_- (respectively, ΠS_+) will be called left (respectively, right) moving fermions, by analogy with the Minkowski picture.

We denote by $D_+ : S_- \to S_+$, $D_- : S_+ \to S_-$ the corresponding Dirac operators in Spin bundles.

We will do gauge theory with gauge group G (a compact Lie group). Let E_L, E_R be orthogonal, unimodular representations of G. Let P be a principal G-bundle on Σ, and let E_R, E_L denote the orthogonal vector bundles associated to the representations E_R, E_L. Let $\overrightarrow{\psi_+}, \overrightarrow{\psi_-}$ be sections of the bundles $E_R \otimes S_+$, $E_L \otimes S_-$.

The Lagrangian of a 2-dimensional gauge theory with fermions is

$$\mathcal{L} = \int d^2x \left(\frac{1}{4e^2} | * F_A|^2 + \frac{1}{4\pi} \overrightarrow{\psi_+}(D_-^A)\overrightarrow{\psi_+} + \frac{1}{4\pi} \overrightarrow{\psi_-}(D_+^A)\overrightarrow{\psi_-} \right), \tag{5.1}$$

where A is a connection in P, and $D_\pm^A$ are the corresponding Dirac operators.

In order for this theory to make sense quantum mechanically, the representations E_L, E_R have to satisfy an additional condition. To derive this condition, recall that the partition function of (5.1) is given by

$$Z = \int DA \int D\overrightarrow{\psi_+} D\overrightarrow{\psi_-} e^{-\mathcal{L}}. \tag{5.2}$$

(in this integral, we sum over all topological types of principal bundles). The fermion integral is easy to compute: it equals $I_A = Pf(D_-^A|_{E_R}) Pf(D_+^A|_{E_L})$, where Pf denotes the Pfaffian. The expression I_A is a section of the Pfaffian line bundle $B = PF(D_-^A|_{E_R}) PF(D_+^A|_{E_L})$ on the space of gauge classes of connections. In order for the A-integral to make sense, this expression should be a function, i.e. the bundle B has to be trivial. It is easy to show that this boils down to the condition

$$\mathrm{Tr}(\rho_L(t)\rho_L(t')) = \mathrm{Tr}(\rho_R(t)\rho_R(t')), \tag{5.3}$$

where $t, t' \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the Gauge group G, and $\rho_{L,R} : \mathfrak{g} \to SO(E_{L,R})$ are the representation maps. If G is simple, and E_L, E_R are irreducible, this condition means that the Casimirs of E_L, E_R are the same. Equation (5.3) is called the condition of cancelations of anomalies.

Today we will consider a simple case: $G = U(1)$, and E_L, E_R are irreducible 2-dimensional real representations. In this case (5.3) says that E_L, E_R are the same: $E_L = E_R = E$. We will take $E = \mathbb{C}$ with metric $|z|^2$, and $U(1)$ acting by multiplication (but remember that tensor products $S_\pm \otimes E$ are over $\mathbb{R}$). We decompose $S_\pm \otimes E$ in a direct sum of two 2-dimensional representations. Using this decomposition, we will write $\overrightarrow{\psi}_\pm = (\psi_\pm, \overline{\psi}_\pm)$, where for $z \in U(1)$ one has $z(\psi, \overline{\psi}) = (z\psi, \overline{z}\overline{\psi})$.

In the new notation, Lagrangian (5.1) has the form

$$\mathcal{L} = \int d^2x \left(\frac{1}{4e^2} | * F_A|^2 + \frac{1}{2\pi} \overline{\psi}_+ D_-^A \psi_+ + \frac{1}{2\pi} \overline{\psi}_- D_+^A \psi_- \right), \tag{5.4}$$

We can also add to this Lagrangian a topological term:

$$\mathcal{L}_\theta = \mathcal{L} - \frac{i\theta}{2\pi}\int F, \tag{5.5}$$

and a mass term:

$$\mathcal{L}_{\theta,m} = \mathcal{L}_\theta + \frac{m}{2\pi}\bar{\psi}_-\psi_+ + \frac{\bar{m}}{2\pi}\bar{\psi}_+\psi_-, \tag{5.6}$$

(here m is complex).

5.2. Chiral symmetry.

Chiral symmetry is a $U(1)$-symmetry of the classical Lagrangian (5.4) or (5.5) given by

$$\psi_+ \to \psi_+, \psi_- \to e^{i\gamma}\psi_-. \tag{5.7}$$

This symmetry is violated by the mass term. However, even if there is no mass term, this symmetry may be violated quantum-mechanically, for the following topological reason.

Let the spacetime be a closed Riemann surface. Then under the chiral symmetry, the "measure" $\mu = D\psi_- D\bar{\psi}_-$ transforms as $\mu \to e^{i\gamma I}\mu$, where I is the index of the operator $D_+^A : S_- \otimes_{\mathbb{C}} E \to S_+ \otimes_{\mathbb{C}} E$. "Proof": $I = dim(\{\psi_-\}) - dim(\{\bar{\psi}_-\})$; since $(\{\bar{\psi}_-\}) = (\{\psi_-\})^*$, we have $I = dim(\{\psi_-\}) - dim(\{\psi_+\}) = ind(D_+^A|_{S_+\otimes_{\mathbb{C}}E})$.

It is known that the index I equals $c_1(P) = \int \frac{F_A}{2\pi}$. Thus, since the bundle P may be nontrivial topologically, chiral symmetry is violated in the quantum theory.

This effect is called an anomaly: a classical symmetry does not hold quantum mechanically, because the measure is not invariant. The difference with spontaneous symmetry breaking, discussed in Lecture II-1, is that in the case of spontaneous symemtry breaking, the classical symmetry does exist in the quantum theory, but cannot be realized. Unlike spontaneous symmetry breaking, anomalies can occur even in 1 dimension.

Let us discuss the mechanism of chiral symmetry breaking in the language of currents. Classically, chiral symmetry is generated by the current

$$J_A = \bar{\psi}_-\psi_- \tag{5.8}$$

(this is a 1-form of type (0,1)), which is of course conserved: $dJ_A = 0$. Quantum mechanically, this differential equation may be deformed: $dJ_A = \mathcal{O}$, where $\mathcal{O}$ is some operator with values in 2-forms on the surface. It is not hard to show by listing all possible operators that there is only one operator (up to a factor) that can arise: it is the curvature operator F. Namely, it is enough to show that dJ_A is a functional of F, which can be seen by considering Feynman diagrams. Thus, $dJ_A = \alpha F$, where α is a constant, and the previous topological computation shows that $\alpha = 1/2\pi$.

Now consider Lagrangian (5.5), which depends on the theta-angle. The thing we learn from the above index computation is that the value of the correlation functions defined by the path integral with Lagrangian (5.5) depend on θ in a very trivial way. Namely, if $\mathcal{O}_i$ are any operators such that $\prod_i \mathcal{O}_i \to e^{in\gamma}\prod_i \mathcal{O}_i$ under chiral symmetry, then all nontrivial contributions to the correlator $\langle\prod \mathcal{O}_i\rangle$ are from bundles P with $c_1 = n$, so this correlator has the form $e^{in\theta}\langle\prod O_i\rangle_0$, where $\langle\prod O_i\rangle_0$ is θ-independent. For example, $\langle\psi_-\psi_+^*\rangle = Ce^{i\theta}$, where C is theta-independent.

If instead of (5.5) we consider Lagrangian (5.6) with the mass term, then this argument shows that

$$\langle\prod \mathcal{O}_i\rangle(m,\theta) = e^{in\theta}\langle\prod O_i\rangle_0(\tilde{m}), \tag{5.9}$$

where $\tilde{m} = me^{-i\theta}$. Thus, the really important parameter of the theory is $\tilde{m}$, which we will write as $m_* e^{-i\theta_*}$, $m_* \geq 0$. Our goal in this lecture to study this theory as a function of $\tilde{m}$.

5.3. Behavior of 2-dimensional gauge theory with massive fermions

Now we will describe the behavior of the theory defined by Lagrangian (5.6), and later will justify this conclusion.

First of all, we will see that for large m_* (i.e. $m_* >> e$) the theory is similar to the 2-dimensional gauge theory with massive bosons, considered in the previous lecture. In particular, it has a mass gap. It also has a unique vacuum for $\theta \neq \pi$, and two of them for $\theta = \pi$. The discrete symmetry of reversal of space orientation ($t \to t, x \to -x$), which acts by $m_* \to m_*$, $\theta_* \to -\theta_*$, is broken on the negative real axis (far away from 0), but not on positive real axis.

Next, we will show that for $\tilde{m} = 0$ the theory is in fact free, i.e. becomes a free massive theory after a change of variables. Thus the theory has a unique vacuum and a mass gap for small m_* (i.e. for $m_* << e$).

Unfortunately, we do not know for sure what happens in the region $m_* \sim e$. The most natural thing would be that the cut in the plane of m_*, representing points with symmetry breaking which starts at $-\infty$, ends at some point $-m_c$, $m_c = e/\lambda$, and λ is dimensionless. At $\tilde{m} = -m_c$, the theory should have no mass gap (since it is a point of transition from two vacua to one vacuum).

This is what is in fact believed. Furthermore, there is a conjecture that the theory at the critical value $\tilde{m} = -m_c$ is in fact conformal, and isomorphic to the theory of a free neutral fermion.

5.4. Heavy fermions.

In this section we will study the case $m_* >> e$. In this case we can regard our theory as a perturbation of a theory with $e = 0$ with perturbation parameter $\lambda = e/m_*$. At $\lambda = 0$, we have a direct product of a 2-dimenisonal pure gauge theory (which is free), and a free theory of massive fermions (to be safe here, we should introduce $B = A/e$; this makes sense, as for $e = 0$, only the trivial $U(1)$-bundle contributes to the path integral).

It turns out that the situation here is similar to the bosonic case. Namely, the small e perturbation of the free theory for $e = 0$ is singular. This means, the space of states of the deformed theory is actually smaller than that of the undeformed theory. More precisely, confinement of fermions takes place: the only allowed states (for $\theta \neq \pi$) are states of total charge 0 (here the charge of $\psi_\pm$ is 1 and of $\bar{\psi}_\pm$ is -1, and $\theta \in [0, 2\pi)$). In particular, the fermions can only exist in pairs, quadruples, and so on, and a single fermion cannot exist.

Like in the bosonic case, the theory has a mass gap by deformation argument (the fact that the deformation is singular does not invalidate this argument, since the Hilbert space does not increase but only becomes smaller). More precisely, we have one realization for $\theta \neq \pi$, and two of them for $\theta = \pi$ (as in the pure gauge theory), and any realization has a mass gap.

5.5. Bose-Fermi correspondence.

Before studying the case of light fermions, we will consider Bose-Fermi correpondence, which will be useful in studying the case of small m_*.

Consider two 2-dimensional free field theories

1. The fermionic theory defined by the Lagrangian

$$\mathcal{L}_f = \frac{1}{2\pi} \int d^2x (\bar{\psi}_- D_+ \bar{\psi}_- + \bar{\psi}_+ D_- \bar{\psi}_+). \tag{5.10}$$

2. The bosonic theory defined by the Lagrangian

$$\mathcal{L}_b = \frac{1}{4\pi R^2} \int d^2x |d\phi|^2, \tag{5.11}$$

These two theories are conformal (both classically and quantum-mechanically), and have Virasoro central charge 1. So we can suspect there may be some connection between them. And indeed, such a connection exists, and it is called the Bose-Fermi correspondence.

Remark. Conformal field theories with small central charge are very scarce. For instance, for $c < 1$ they are completely classified by Friedan, Qiu, Shenker (Ref ???). The answer is that there is no continuous parameters, and the theory is almost completely determined by c, which can take only a discrete sequence of values. For $c = 1$, such a classification is unavailable, but very few examples are known, and all of them have a construction in terms of a free Bose field.

Let us now take a look at the bosonic theory (5.11). As we remember from Lecture II-1, in order for this theory to make sense, ϕ has to be angle-valued, i.e. take values in the circle $\mathbb{R}/2\pi\mathbb{Z}$. The constant R in the Lagrangian has the meaning of the radius of this circle.

Remark. We always model the target circle as $\mathbb{R}/2\pi\mathbb{Z}$, but consider various Riemannian metrics on it, which are parametrized by values of the radius R.

Recall from Lecture II-1 that the Hilbert space of this theory (in its unique realization) is of the form $\mathcal{H}_b = \oplus_{k\in\mathbb{Z}}(F_b \otimes F_b^*)_k$, where F_b is the bosonic Fock space. The operator algebra A_b of the theory is generated by the derivatives of ϕ (but not ϕ itself), and $:e^{in\phi}:$ (for brevity in the future we will drop the colons). Operator product expansion is given by formula (3.9) of Lecture 3 in the fall, where $D(x-y) = -R^2 \ln|x-y|$. The action of A_b in $\mathcal{H}_b$: Derivatives of ϕ do not change k, and $:e^{in\phi}:$ maps $(F_b \otimes F_b^*)_k$ to $(F_b \otimes F_b^*)_{k+n}$.

Now consider the theory (5.12). The Hilbert space of this theory is $\mathcal{H}_f = F_f \otimes F_f^*$, where F_f is generated from the vacuum by holomorphic operators $\psi_+, \bar{\psi}_+$, and F_f^* is generated from the vacuum by $\psi_-, \bar{\psi}_-$. The operator algebra A_f is generated by $\psi_+, \bar{\psi}_+, \psi_-, \bar{\psi}_-$, with the standard OPE of the free theory.

Consider more closely the operator $:e^{in\phi}:$ for $n \in \mathbb{Z}$. As we know (Lecture 3 in the fall, Lecture II-1), this operator has holomorphic dimension $n^2R^2/4$ and antiholomorphic dimension $n^2R^2/4$ (the total of $n^2R^2/2$, as we saw in Lecture 3 in the fall).

Classically, the operator $e^{in\phi}$ locally factors as a product of a holomorphic one and an antiholomorphic one: $e^{in\phi} = e^{in\phi_+}e^{in\phi_-}$, where $\phi = \phi_+ + \phi_-$, and $\partial_-\phi_+ = \partial_+\phi_- = 0$. (Of course, here ϕ_+, ϕ_- are defined only up to adding a constant). There is no analogs of $e^{in\phi_\pm}$ in our operator algebra A_b. However, imagine for a second that the operators $e^{in\phi_\pm}$ make sense. Then we will find from the OPE for $e^{i\phi}$ that $e^{i\phi_+}$ has holomorphic dimension $R^2/4$ and antiholomorphic dimension 0, and

$$\langle e^{i\phi_+(x_1)}...e^{i\phi_+(x_n)}e^{-i\phi_+(y_1)}...e^{-i\phi_+(y_n)}\rangle = \frac{\prod_{1\le i<j\le n}(x_i-x_j)^{R^2/2}\prod_{1\le i<j\le n}(y_i-y_j)^{R^2/2}}{\prod_{1\le i,j\le n}(x_i-y_j)^{R^2/2}}. \tag{5.12}$$

where x_i, y_j are viewed as complex numbers. From this formula it is clear that in order for $e^{i\phi_+}$ to make any sense, we need $R^2/2 \in \mathbb{Z}$, so that the function on the R.H.S. of (5.12) is single-valued. Similarly, in order for $e^{in\phi_+}$ to make sense, $n^2R^2/2$ has to be an integer.

Let us consider the simplest case where $e^{in\phi_+}$ can make sense, i.e. $R = \sqrt{2}$. In this case, we have

$$\langle e^{i\phi_+(x_1)}...e^{i\phi_+(x_n)}e^{-i\phi_+(y_1)}...e^{-i\phi_+(y_n)}\rangle = \frac{\prod_{1\le i<j\le n}(x_i-x_j)\prod_{1\le i<j\le n}(y_i-y_j)}{\prod_{1\le i,j\le n}(x_i-y_j)}. \tag{5.13}$$

It is clear from (5.13) that $e^{in\phi_+}$ behaves like a fermion when n is odd, and like a boson when n is even. This makes us hope that it is at $R = \sqrt{2}$ that our theory is related to the theory of fermions.

Let us see why this is indeed the case. Let us compute the fermionic correlation function

$$\langle\psi_+(x_1)...\psi_+(x_n)\bar{\psi}_+(y_1)...\bar{\psi}_+(y_n)\rangle$$

. Using Wick's formula, it is easy to find that

$$\langle\psi_+(x_1)...\psi_+(x_n)\bar{\psi}_+(y_1)...\bar{\psi}_+(y_n)\rangle = \det(\frac{1}{x_i-y_j}). \tag{5.14}$$

The coincidence of the right hand sides of (5.13),(5.14) is a famous combinatorial identity, which follows from comparison of zeros and poles, and asymptotics at infinity.

Remark. In fact, the explicit form of (5.13),(5.14) is not relevant to the proof of the fact that they are equal. What is relevant is only the structure of zeros and poles, and the asymptotics at infinity, which in both cases are obvious from the OPE.

Let $\hat{A}_b$ be the operator algebra generated by A_b and $e^{in\phi_\pm}$. Equalities (5.13),(5.14) show that we have a homomorphism $\xi : A_f \to \hat{A}_b$ defined by $\xi(\psi_\pm) = e^{\pm i\phi_\pm}$, $\xi(\bar{\psi}_\pm) = e^{\mp i\phi_\pm}$, which preserves expectation values.

Using the OPE, it is easy to find

$$\xi(:\bar{\psi}_+\psi_+:) = \lim_{\varepsilon\to 0}\xi(\bar{\psi}_+(x+\varepsilon)\psi_+(x) - \frac{1}{\varepsilon}) = \lim_{\varepsilon\to 0}(e^{-i\phi_+}(x+\varepsilon)e^{i\phi_+}(x) - \frac{1}{\varepsilon}) = -i\partial_+\phi. \tag{5.15}$$

Likewise, $\xi(:\bar{\psi}_-\psi_-:) = i\partial_-\phi$. Also, $\xi(\psi_+\bar{\psi}_-) = e^{i\phi}$, $\xi(\psi_-\bar{\psi}_+) = e^{-i\phi}$. In fact, it is not difficult to see that this homomorphism is an isomorphism.

At the level of Hilbert spaces, ξ induces a bigraded isomorphism $F_f \to F_b \otimes l_2(\mathbb{Z})$ (bidegree=(quantum scaling dimension, charge)), where the charge of $e^{i\phi_+}$ and the charge of ψ_+ equal 1. Writing the corresponding character formula, we obtain

$$\frac{\sum_{n\in\mathbb{Z}} q^{n^2/2}z^n}{\prod_{n\geq 1}(1-q^n)} = \prod_{n\geq 1}(1+q^{n-1/2}z)(1+q^{n-1/2}z^{-1}), \tag{5.16}$$

which is the famous Jacobi triple product identity.

The correpondence ξ is called the Bose-Fermi correspondence.

Remark 1. If the radius of the circle is not $\sqrt{2}$, but 1, then the operator $e^{i\phi_+}$ is meaningless, as its 2-point function would be $(x-y)^{-1/2}$, which is not single-valued. However, the operator $e^{2i\phi}$ is defined, and its 2-point function is $\frac{1}{(x-y)^2}$. This indicates that $e^{2i\phi}$ behaves like a current of some symmetry. And indeed, it turns out that the corresponding model is equivalent to the $\widehat{SU(2)}$-WZW model with Kac-Moody central charge 1, so it has an $SU(2)$-symmetry. In fact, the Fourier components of the operators $e^{\pm 2i\phi_+}$ and $\partial_+\phi$ generate the left-moving $\widehat{SU(2)}$, and the Fourier components of the operators $e^{\pm 2i\phi_-}$ and $\partial_-\phi$ generate the right-moving $\widehat{SU(2)}$.

Remark 2. In fact, the Bose-Fermi correspondence is true not only locally (at the level of operators), but also globally (at the level of path integral). Namely, for any Riemann surface Σ one has the identity of partition functions

$$\int D\phi\, e^{-\frac{1}{8\pi}\int |d\phi|^2} = \sum_\varepsilon \int D\psi_+ D\bar{\psi}_+ D\psi_- D\bar{\psi}_- e^{-\mathcal{L}_f(\psi)}, \tag{5.17}$$

where $\mathcal{L}_f$ is the Lagrangian given by (5.10), and ε runs over spin structures on Σ (the same spin structure is taken for ψ_+ and ψ_-). There is a similar identity for correlation functions of operators, if the correspondence between operators is made as explained above.

5.6. Bose-Fermi correspondence for nonlinear theories.

We have established a correspondence between two free theories – the theory of a boson and the theory of fermions. A remarkable fact is that this correspondence generalizes to the case when the Lagrangian of one or both of the theories is not free. Consider examples of such situations.

1. Recall that under our correspondence $\mathcal{L}_f \to \mathcal{L}_b(\sqrt{2})$, where $\mathcal{L}_b(R)$ is given by (5.11). Since $\xi(\bar{\psi}_\pm\psi_\pm) = \mp i\partial_\pm\phi$, we find

$$\mathcal{L}_f + \frac{1}{2\pi}\int d^2x(g\bar{\psi}_+\psi_+\bar{\psi}_-\psi_-) \to \mathcal{L}_b(\sqrt{2}(1+g)^{-1/2}). \tag{5.18}$$

This shows that, to our surprise, the theory with the Lagrangian $\mathcal{L}_f + \frac{1}{2\pi}\int d^2x(g\bar{\psi}_+\psi_+\bar{\psi}_-\psi_-)$ is free. In particular, its β-function is zero. This is obvious when the theory is described in Bose variables, but not obvious in Fermi variables.

2. On the other hand, consider the theory of free massive fermions, with the Lagrangian $\mathcal{L}_f + \frac{1}{2\pi}\int d^2x(m\bar{\psi}_-\psi_+ + \bar{m}\bar{\psi}_+\psi_-)$. Using the fact that $\xi(\bar{\psi}_\pm\psi_\mp) = -e^{\mp i\phi}$, we get that under ξ, this free Lagrangian goes to

$$\mathcal{L}_b(\sqrt{2}) - \frac{1}{2\pi}\int d^2x(me^{i\phi} + \bar{m}e^{-i\phi}). \tag{5.19}$$

So we get another surprising fact that the nonlinear theory defined by (5.19) is in fact free.

Remark. It may appear that the second term in (5.19) has a wrong scaling dimension. This is not the case, because the operator $e^{\pm i\phi}$ has anomalous dimension 1. More precisely, (5.19) does not fix a theory but fixes a family of theories depending on a scale μ of momenta, which is introduced when the operators $e^{i\phi}$ are renormalized. This scale enters in front of the corresponding term in the Lagrangian and cancels the discrepancy in dimensions.

Using the symmetry $\phi \to \phi + \theta$ we can reduce (5.19) to the case of real m. In this case, (5.19) looks like

$$\mathcal{L}_b(\sqrt{2}) - \frac{m}{\pi}\int d^2x\cos\phi. \tag{5.20}$$

Thus, the classical equation of motion is $\Delta\phi = -4m\sin\phi$. This equation is called the sine-Gordon equation, and it is a well-known completely integrable soliton equation.

Now consider The Lagrangian

$$\mathcal{L}_b(R) - \frac{m}{\pi}\int d^2x\cos\phi. \tag{5.21}$$

This Lagrangian is proportional to (5.20), with $m' = mR^2/2$, so classically the two Lagrangians are equivalent. However, quantum mechanically, this is not the case, as the scale of Lagrangian is now relevant. In fact, the theory now essentially depends on R. If $R = \sqrt{2}$, the theory is free, but for a general R it is not. The map ξ shows that for a general R the theory is equivalent to the fermionic theory with the Lagrangian

$$\mathcal{L}_f + \int d^2x(m\bar{\psi}_-\psi_+ + m\bar{\psi}_+\psi_- + g\bar{\psi}_+\psi_+\bar{\psi}_-\psi_-), \tag{5.22}$$

where $R = \sqrt{2}(1+g)^{-1/2}$.

As we mentioned, the theory described by the Lagrangian (5.21) is not free for $R \neq \sqrt{2}$. However, it is solvable, in the sense that its S-matrix can be computed explicitly. This computation and the result are similar to the computation of Lecture II-3, for the sigma-model into the sphere. Solvability for this theory for large R (i.e. in the classical limit) is related to the complete integrability of the sin-gordon equation at the classical level.

Now we want to apply the Bose-Fermi correspondence to gauge theory. Consider the Lagrangian $\mathcal{L}_{\theta,m}$ given by (5.6). Define

$$\mathcal{L}^f_{\theta,m} = \mathcal{L}_{\theta,m} + \frac{1}{2\pi}\int d^2x g\bar{\psi}_+\psi_+\bar{\psi}_-\psi_-. \tag{5.23}$$

Let us rewrite it in Bose variables. Then we will get the Lagrangian

$$\mathcal{L}^b_{\theta,m} = \int d^2x\left(\frac{1+g}{8\pi}|d\phi|^2 - \frac{m}{2\pi}e^{i\phi} - \frac{\bar{m}}{2\pi}e^{-i\phi} + A_+(\frac{i\partial_-\phi}{2\pi}) + A_-(\frac{-i\partial_+\phi}{2\pi}) + \frac{|*F|^2}{4e^2} - \frac{i\theta}{2\pi}F\right). \tag{5.24}$$

For simplicity we assume that ϕ is a homotopically trivial map. (It is easy to generalize everything to the homotopically nontrivial case). Then the integral $\int(A_-\partial_+\phi - A_+\partial_-\phi)$ can be taken by parts, and it equals $\int \phi(\partial_-A_+ - \partial_+A_-)$, where ϕ is now understood as a lifting of the original ϕ to a map $\Sigma \to \mathbb{R}$. The expression $\partial_-A_+ - \partial_+A_-$ equals to the curvature F, so (5.24) is simplified:

$$\mathcal{L}^b_{\theta,m} = \int d^2x\left(\frac{1+g}{8\pi}|d\phi|^2 + \frac{m}{2\pi}e^{i\phi} + \frac{\bar{m}}{2\pi}e^{-i\phi} + \frac{|*F|^2}{4e^2} - \frac{i(\phi+\theta)}{2\pi}F\right). \tag{5.25}$$

From this equation it is clear that when $m = 0$, the theory is free, and there is no essential θ-dependence. This is the first thing we promised to show. Now, for $m \neq 0$, by changing ϕ to $\phi + \theta$ we find that theta can be absorbed in m.

Let us now see what happens for $m \neq 0$. It follows from Lecture II-4 that for an external field $\phi(x)$

$$\int DA\, e^{\frac{i}{2\pi}\int \phi F_A - \frac{1}{4e^2}\int |*F_A|^2} = Ce^{-\frac{e^2}{2\pi}\int d^2x \min_n (n-\frac{\phi(x)}{2\pi})^2}, \tag{5.26}$$

Therefore, the effective Lagrangian for ϕ for the Lagrangian (5.25) is

$$\mathcal{L}^{\theta_*,m_*}_{eff} = \int d^2x\left(\frac{1+g}{8\pi}|d\phi|^2 - \frac{m_*}{\pi}\cos(\phi-\theta_*) + \frac{e^2}{2\pi}\min_n(n-\frac{\phi}{2\pi})^2\right). \tag{5.27}$$

From this formula, it is seen, that the theory has a mass gap for small m_*: the potential

$$U(\phi) = -\frac{m_*}{\pi}\cos(\phi-\theta_*) + \frac{e^2}{2\pi}\min_n(n-\frac{\phi}{2\pi})^2 \tag{5.28}$$

has a unique global minimum (modulo 2π) with positive second derivative. This is the case for all m_* if $\theta \neq \pi$. However, if $\theta = \pi$ and m_* grows from 0 to ∞, the global minimum at $\phi = 0$ keeps flattening, and at some point splits in two symmetric minima, when the second derivative becomes zero. This should indicate that starting at some finite value of e/m_* there should be symmetry breaking. Of course, this proves nothing, because (5.28) is not the quantum effective potential for our system (it is the potential only classically). However, we hope that the true effective potential behaves similarly, and the picture is qualitatively the same.

For more on this model, see S.Coleman's article in Annals of Physics, vol. 101, p. 239-267 (1976).

Lecture II-6: Gauge theory in 2 dimensions with self-interacting bosons, the Wilson line operator, and confinement

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Notes by Pavel Etingof and David Kazhdan

In the last lecture we considered 2-d gauge theories with fermions. Today we will consider 2-dimensional gauge theories with bosons. As before, we will work with Euclidean Lagrangians.

6.1. Infrared behavior of $U(1)$ gauge theories with bosons in 2-dimensions.

We consider a $U(1)$ gauge theory with bosons in 2 dimensions.

Our spacetime is a Riemann surface Σ. Our fields are A – a $U(1)$ connection in some line bundle L, and $\phi_1, ..., \phi_N$ – complex scalar fields, which are sections of L.

Our Lagrangian is

$$\int d^2x \left(\frac{(*F)^2}{4e^2} + \frac{1}{2}|d\phi|^2 + \frac{\lambda}{4}(|\phi|^2 - a^2)^2 \right) - \frac{i\theta}{2\pi} \int F, \tag{6.1}$$

where $|\phi|^2 = \sum |\phi_i|^2, |d\phi|^2 = \sum |d\phi_i|^2$.

We want to understand the infrared behavior of this theory.

Clasisically, the "mass" of the bosons is given by ϕ_i is $m^2 = -\lambda a^2$. If $\lambda \to 0$ and $m^2 > 0$ remains fixed, this Lagrangian becomes the Lagrangian of a gauge theory with "free" massive bosons (i.e. bosons interacting with the gauge field only and not interacting with each other), which was considered in Lecture II-4.

The parameters of the theory are m, θ, λ, e (with $\lambda > 0$). We will now keep λ, e fixed, and vary m, θ, to see how the behavior of the theory depends on them.

First consider the situation $m^2 \to +\infty$ (i.e. $m^2 >> e^2$). In this situation the quartic term in the Lagrangian becomes unimportant, and the situation is essentially the same as in Lecture II-4. Namely, the theory has a mass gap, and the interaction between bosons is approximated by the 1-dimensional Coloumb potential, which causes their confinement: all states in the theory are of total charge zero. Also, the space reversal symmetry is preserved for $\theta = 0$, but broken for $\theta = \pi$, thus producing two vacua and a cut at $\theta = \pi$. For example, the vacuum energy density, which is a 2π-periodic function of θ given for $-\pi \le \theta \le \pi$ by formula (6.2) below, has a discontinuous derivative at $\theta = \pi$.

Now consider the situation when $m^2 \to -\infty$ (i.e. $m^2 << -e^2$). In this case, the situation is totally different: because of the presence of the quartic term, the classical potential for bosons has a minimum at $|\phi|^2 = a^2$, so the space of minima is S^{2N-1}. The gauge group $U(1)$ acts freely on this space, so the space of classical vacua is $\mathbb{C}P^{N-1} = S^{2N-1}/U(1)$. Thus, the low energy effective theory is the theory of maps from Σ to $\mathbb{C}P^{N-1}$ with the round metric (the sigma-model).

Remark. In Lecture 3 we saw that the theory with the Lagrangian density $\frac{1}{2}(d\phi)^2 + \lambda(\phi^2 - a^2)^2$ flows in the infrared to the sigma-model of maps to the sphere. We remarked that this means that this (superrenormalizable) theory is a good cutoff for the sigma-model to the sphere. Similarly, the superrenormalizable (and hence rigorously existing) theory defined by Lagrangian (6.1) is a good UV cutoff for the theory of maps to $\mathbb{C}P^{N-1}$.

For Large N, we found that the sigma-model of maps to $\mathbb{C}P^{N-1}$ behaves in the infrared as the gauge theory of massive bosons. Today we will consider in more detail the case $N = 1$. In this case $\mathbb{C}P^{N-1}$ is one point, so the theory defined by (6.1) has a mass gap and a unique vacuum.

6.2. The vacuum energy density.

Recall that for $m^2 >> e^2$ in Lecture II-4 we found the following value of the vacuum energy density:

$$E_{vac}(\theta) = \frac{e^2}{2}\left(\frac{\theta}{2\pi}\right)^2. \tag{6.2}$$

So the θ-derivative of the energy density is proportional to the first power of the coupling e^2.

In the case $m^2 << -e^2$, the situation is totally different. Namely, now the contributions to the path integral from nontrivial line bundles (relative to infinity) is exponentially small in e^2, since any section of such a bundle has to vanish somewhere, and this will have a big action due to the presense of the quartic potential. Thus, to any finite order of perturbation theory (in e^2), the path integral, and hence the vacuum energy density are independent of θ. More precisely, the theta-dependence is exponentially small, and comes from the sum over *instantons* – lowest action configurations of nontrivial first Chern class.

6.3. Instantons.

Let us compute the coefficient of $e^{i\theta}$ in the Fourier expansion of the path integral with Lagrangian (6.1). This is equivalent to taking the path integral over connections in line bundles with $c_1 = 1$.

As usual, the main contribution to the path integral comes from field configurations which are close to the configuration minimizing the action. Such a configuration (ϕ, A) is called an instanton.

Let (ϕ, A) be an instanton. In order for its action to be finite, we must have

$$|\phi(x)| \to a, F_A(x) \to 0, x \to \infty, \tag{6.3}$$

where F_A is the curvature of A. Moreover, since ϕ is classically massive, this convergence is exponentially fast. Thus, an instanton in our problem has to be a highly localized field configuration.

In our further discussion of instantons, we will assume that $\Sigma = \mathbb{R}^2$, and c_1 is measured with respect to trivialization at infinity. In this case, we expect that instantons are rotationally symmetric, with respect to some center of rotation x_0. Without loss of generality we can assume that $x_0 = 0$, so that the instanton has the form $\phi = ae^{i\alpha}f(r)$, $A = g(r)d\alpha$, where $f(r)$, $g(r)$ are some functions of the radius r (where r, α are polar coordinates on $\mathbb{R}^2$). It is easy to show that in this case one must have $f(0) = 0$, $f(\infty) = 1$, and $g(0) = 0$, $g(\infty) = 1$. Thus, we get a boundary value problem for ordinary differential equations. It can be proved by considering the corresponding ODE that this problem has a unique solution. Thus, the instanton is unique, up to translations. In particular, it has a definite size, and its action is a well-defined positive constant I.

Remark. In this respect, our instanton is different from the instantons of the $\mathbb{C}P^{N-1}$ model, which could be transformed by any conformal automorphism, and therefore had no definite size.

By dimensional analysis it is clear that I is of the form $a^2h(\frac{\lambda}{e^2})$, where h is a dimensionless function. It is possible to show that $h(z) \sim Cz$ as $z \to \infty$, where C is a constant, so for small e (compared to λ), $I \sim \frac{C\lambda a^2}{e^2}$. This calculation illustrates the fact that the contribution of the instanton to the path integral is exponentially small with respect to e^2.

6.4. Instanton gas.

Now we want to understand how to compute the contribution of the instantons to the path integral, and when such a computation gives a good approximation.

First of all consider line bundles with $c_1 = 2$. It can be shown that there is no instanton in this topological class if e^2/λ is sufficiently large. We will assume that there is no instanton for $c_1 > 1$, but what we will say can be generalized to the case when there is one (for small e^2/λ). In the case when there is no instanton with $c_1 > 1$, the problem of minimization of action in the case $c_1 = 2$ has no global minimum (the infinum is not attained). However, we can consider approximate instantons,

whose action approaches the infinum arbitrarily closely. More precisely, if we take $A(x) = A_*(x-x_1) + A_*(x-x_2), \phi(x) = \phi_*(x-x_1)\phi_*(x-x_2)/a$, where (A_*, ϕ_*) is the instanton centered at 0, then the action of (A, ϕ) equals $2I$, plus a correction which is of order $e^{-c|x_1-x_2|}$, where c is a positive constant. Thus, the infinum of actions for $c_1 = 2$ is $2I$.

Similarly, the infinum of actions for $c_1 = n$ is nI, for any $n \in \mathbb{Z}_+$. This follows from the fact that the action of the field configuration

$$\sum_{i=1}^{n} A_*(x-x_i), a\prod_{i=1}^{n}\frac{\phi_*(x-x_i)}{a}$$

has action which is exponentially close to nI when $|x_i - x_j|$ are big.

Now consider the situation when $c_1 < 0$. It is easy to see that $(-A_*, \bar{\phi}_*)$ is the instanton for $c_1 = -1$. It is called the antiinstanton. Thus, the situation for $n < 0$ is symmetric to the situation for $n \in \mathbb{Z}_+$: the field configuration

$$-\sum_{i=1}^{|n|} A_*(x-y_i), a\prod_{i=1}^{n}\frac{\bar{\phi}_*(x-y_i)}{a}$$

has action exponentially close to $|n|I$ when $|y_i - y_j|$ is big.

More generally, one can consider field configurations

$$\sum_{i=1}^{n} A_*(x-x_i) - \sum_{i=1}^{m} A_*(x-y_i), a\prod_{i=1}^{n}\frac{\phi_*(x-x_i)}{a}\prod_{i=1}^{m}\frac{\bar{\phi}_*(x-y_i)}{a},$$

with $c_1 = n - m$ which have action exponentially close to $(n+m)I$ when x_i, y_j are distant from each other.

Remark. Such field configurations are called "instanton gas". The term "instanton gas" refers to a gas with long range Coulomb forces (like the instantons we studied later in the Polyakov model in 2+1 dimensions). This gas with exponentially small forces at long range is more like an ordinary gas of atoms, for instance the air in the atmosphere. It is an almost ideal gas, the ideal gas law of thermodynamics is the case that the forces are exactly zero. Any real gas (hydrogen, oxygen) behaves as an almost ideal gas if the density is small enough, because then the particles are generally at big distances where the interactions are small. That is the case for the instantons in this model because the instanton action is big (and the instantons have a definite size)

6.5. Summing over instantons.

As we remarked, the perturbation series with respect to powers of e^2 for the path integral with Lagrangian (6.1) does not involve contributions from instantons, as they are exponentially small. Let us introduce a refined perturbation series, which will take instantons into account. For this purpose we will work on a Riemann surface of volume V, and introduce a new perturbation parameter $W = Ve^{-I}$. We will consider the perturbation expansion with respect to both W and e. The key fact is that in this refined perturbation expansion, the only contributions of finite order in W and e are from the instanton gas. Thus, the approximation to the partition function obtained this way has the form

$$Z(e^2, \theta, W) = e^{VP_0}\sum_{n,m=0}^{\infty}\frac{W^{m+n}P_+^n P_-^m e^{i\theta(n-m)}}{m!n!}, \tag{6.4}$$

where P_0, P_+, P_- are the perturbation series around the zero solution, the instanton, and the antiinstanton, respectively.

Here the term with indices m, n comes from a field configuration with n instantons and m antiinstantons, distant from each other. The factorials in the denominator arise from the fact that instantons and antiinstantons are not labeled, and their permutation does not change the configuration.

Summing (6.4) by ordinary calculus, we get

$$Z(e^2, \theta, W) = e^{VP_0 + W(P_+ e^{i\theta} + P_- e^{-i\theta})}. \tag{6.5}$$

In our case, $P_+ = P_- = P$, so we get

$$Z(e^2, \theta, W) = e^{VP_0 + 2WP\cos\theta}. \tag{6.6}$$

Now we can compute the energy density of the vacuum in this approximation. This can be done from the equality $Z = e^{-VE_{vac}}$, which is the definition of the vacuum energy density E_{vac}. Thus,

$$E_{vac} = -P_0 - 2Pe^{-I}\cos\theta. \tag{6.7}$$

As we expected, the theta-dependence is exponentially suppressed, but we were able to compute the main term of this dependence.

The next order correction to (6.7) is of order e^{-2I}, but it is hard to calculate. But if we had an instanton for $c_1 = 2$ with $I' < 2I$, then the correction would be of the form $e^{-2I'}\cos 2\theta$.

We see that unlike the case $m^2 >> 0$, where $E_{vac} = e^2\theta^2/8\pi^2$ is a non-smooth function of θ (it has a cut at $\theta = \pm\pi$), in the case $m^2 << 0$ the function $E_{vac}(\theta) = -P_0 + 2Pe^{-I}\cos\theta$ is smooth in the first approximation. Thus, for $m^2 << 0$ there is no cut at $\theta = \pm\pi$, and there is a mass gap and a unique vacuum for any θ.

Thus, the cut at $\theta = \pm\pi$ has to add at some point $m^2 = m_c^2 \in \mathbb{R}$. At this point the effective potential of the theory (if it makes sense) has a quartic critical point at the origin, so that the theory has no mass gap. It is conjectured that this theory is conformal, and is the continuous limit of the 2-dimensional Ising model.

6.6. The Wilson line operator.

In this and subsequent sections we will define the Wilson line operator, and try to understand its physical and formal properties.

Suppose we have a gauge theory with gauge group G on a spacetime M. Let A denote the corresponding gauge field with values in the Lie algebra $\mathfrak{g}$ of G. Let C be a closed loop in M, and $Hol(A, C) \in G$ be the holonomy of the connection A along C (it is only defined up to conjugation). Let R be a finite-dimensional representation of G. Define the classical Wilson line (or Wilson loop) functional to be

$$W_R(C)(A) = \mathrm{Tr}|_R(Hol(A, C)). \tag{6.8}$$

It is clear that this functional is gauge invariant.

If C is a union of disconnected loops C_i labeled with representations $R = (R_i)$, then by definition $W_R(C) = \prod W_{R_i}(C_i)$.

If G is abelian, and C is cotractible, then $W_R(C) = e^{i\int_D F}$, where F is the curvature of A, and D is a disk such that $\partial D = C$.

An important generalization of this is the following: $\hat{G}$ is the simply-connected cover of G, and R is a representation of the universal covering $\hat{G}$ of G. In this case it is easy to see that $W_R(C)$ is still well-defined when C bounds a disk D. (in the abelian case, it follows from the above integral representation of W).

Now we want to define an analogue of $W_R(C)$ in quantum theory. For this purpose we need to renormalize the classical functional $W_R(C)$. This can be done by expanding $W_R(C)$ in powers of A (like the exponential is expanded in Taylor series):

$$W_R(C) = \dim(R) + \int dl \mathrm{Tr}|_R(A(l)) + \frac{1}{2}\int dl dl' \mathrm{Tr}|_R(A(l)A(l')) + ..., \tag{6.9}$$

where l is some parameter on C, and $A(l)$ is the evaluation of the form A on the tangent vector $\frac{d}{dl}$ to C at the point l, with respect to this parametrization. Each term of this expansion is polynomial and can be renormalized as usual.

In fact, one can show in most cases that the operator $W_R(C)$ has only multiplicative renormalization. This follows from the fact that classically, $W_R(C)$ is the trace of the monodromy of the differential equation $x' = Ax$ along the loop. This equation is renormalized to $x' = (A + c)x$, where c is an operator in the theory, invariant under the same symmetries as A, in the adjoint representation of the gauge group, of dimension the same and lower than A. If there is no such operators except for constants (constants come up for the $U(1)$-case) then this equation will change to $x' = (A + c)x$ under renormalization, where c is a scalar operator. This shows that $W_R(C)$ will have multiplicative renormalization.

More precisely, one can show that in critical dimension 4, the divergent renormalization factor has the form $e^{L(C)\Lambda f(e^2)+o(\Lambda)}$, where Λ is the cutoff and $L(C)$ is the length of C, while in the super-renormalizable case (in less than 4 dimensions), the divergent factor has power growth with respect to Λ.

A more physical way of thinking of the Wilson loop operator is the following. We will use the Hamiltonian picture. Thus, $M = M_s \times \mathbb{R}$, where M_s is the space manifold. The Hilbert space of the theory is then $\mathcal{H}$ the space of functions on $\mathcal{A}$, where $\mathcal{A}$ is the space of gauge classes of connections on M_s. The space $\mathcal{H}$ has the form $\mathcal{H} = (\mathcal{H}_0)^{\tilde{G}}$, where $\mathcal{H}_0$ is the space of functions on all connections, and $\tilde{G}$ is the group of gauge transformations.

Now let C be a loop in M. Suppose first that the loop C is in the submanifold $t = 0$. In this case, the Wilson loop operator is obviously just the operator of multiplication by the function $W_R(C)$.

Now consider the situation when C is not in a horizontal section of the spacetime, but a general curve. In this case, $W_R(C)$ is no longer multiplication by a function.

Remark. We think of C as a worldline of a "charge" which transforms in a representation R. The best is to think of two charges of type R, R^* which are born at some time t_0 from nothing at the same point x_0, then fly around for a while (until t_1), and finally recombine at the time t_1, back into nothing, at a point x_1. We think of these charges as classical, external objects. That is, the expectations in the presence of C are conditional expectations, given that the worldlines of the charges form the loop C. In this setting, we can regard $e^{iH(t_1-t_0)}W_R(C)$ as the evolution operator from time t_0 to time t_1, in the system with presence of C (here H is the Hamiltonian of the system).

The operator $W_R(C)$ allows us to define the time ordered correlation functions in the presence of C for any set of local operators at points (x, t) such that $t \notin [t_0, t_1]$, simply as correlation functions with the insertion of $W_R(C)$ in the right place. But what should we do for $t_0 < t < t_1$?

The most natural method is to set the Hilbert space of states at t to be $\mathcal{H}(t) = (\mathcal{H}_0 \otimes R(x(t)) \otimes R^*(y(t)))^{\tilde{G}}$, where $R(x)$, $R^*(y)$ are the evaluation representations of the group $\tilde{G}$ at the points $x.y$, and $x(t), y(t)$ are the positions of the two charges at the time t. Define operators $W_R(C_+(t)) : \mathcal{H} \to \mathcal{H}(t)$, $W_R(C_-(t)) : \mathcal{H}(t) \to \mathcal{H}$ – the Wilson "open line" operators corresponding to the upper and lower half-loops $C_+(t), C_-(t)$ (the parts of C lying above and below time t). The definition of the "open line" operators similar to the definition of the closed loop operators. Now, the expectation value

of a local operator $O(x,t)$ in the presence of C is just the usual expectation value of the operator $W_R(C_+(t))\phi(x,t)W_R(C_-(t))$.

6.7. The path integral representation of the Wilson line operator.

It is convenient to represent the classical Wilson line functional as a 1-dimensional path integral. Such a representation will be deduced in this section.

Let R be an irreducible, finite-dimensional representation of a simple compact Lie group G. By Borel-Weil theory, we can think of R as $H^0(G/T, \mathcal{L}_R)$, where $\mathcal{L}_R$ is a suitable holomorphic line bundle. The bundle $\mathcal{L}_R$ has a natural Hermitian metric, and therefore a natural connection B. For any map $\phi : S^1 \to G/T$, define the action $I_R(\phi) = -\ln \mathrm{Hol}(\phi^*(B))$ (Hol is for holonomy). The number I_R is only defined up to $2\pi i n$, but $e^{-I_R(\phi)} = \mathrm{Hol}(\phi^*(B))$ is well defined.

Consider the path integral

$$\int D\phi e^{-I_R(\phi)}. \tag{6.10}$$

It can be represented in the form

$$\int D\phi e^{i\int_D \phi^*(F_B)}, \tag{6.11}$$

where F_B is the curvature of B, and D is a disk whose boundary is the image of S^1. The quantum theory defined by this path integral is the Hilbert space R with the zero Hamiltonian (as the path integral is invariant under diffeomorphisms).

Now let P be the total space of a principal G-bundle over the circle, with a connection A, and let P/T be the associated G/T bundle. Since $\mathcal{L}$ is a G-equivariant line bundle on G/T, it defines a complex line bundle on P, which we denote by $\mathcal{L}_P$. The hermitian connection B on $\mathcal{L}$ defines a connection on $\mathcal{L}_P$ in the vertical direction (along fibers of P). This connection can be naturally extended to a connection B_P on $\mathcal{L}_P$ using the connection A: in a local gauge where P is trivial and $A = 0$, B is extended by the condition that constant paths are horizontal paths.

For any smooth section ϕ of P/T, define the action

$$I(\phi, A) = -\ln \mathrm{Hol}(\phi^*(B_P)). \tag{6.12}$$

As before, I is defined up to $2\pi i n$.

Let $W_R(A)$ denote the trace in R of the holonomy of A around the circle. It is now easy to see that the functional $W_R(A)$ has the following integral representation:

$$W_R(A) = \int_{\phi\in\Gamma(P/T)} e^{-I(\phi,A)} D\phi. \tag{6.13}$$

Now let C is a loop in M, and A is a connection in a principal G-bundle P over M. Let $f : S^1 \to C$ be a parametrization. Then we have

$$W_R(C)(A) = \int_{\phi\in\Gamma(f^*P/T)} e^{-I(\phi,f^*A)} D\phi. \tag{6.14}$$

This gives an integral representation of the Wilson line operator. This representation allows us to represent the theory in the presence of C (for instance, for pure gauge theory) by a double path integral

$$\int DAD\phi e^{-\frac{1}{4e^2}\int \mathrm{Tr}|F^2| - I(\phi,A)}. \tag{6.15}$$

Remark. Note that the integrand of (6.13),(6.14) is the holonomy of an abelian connection, so it can be explicitly computed as the exponential of the integral of curvature. This, we have represented

the monodromy of an arbitrary linear differential equation on the circle as an explicit integral over infinitely many auxiliary variables. It is well known in classical analysis that such a reopresentation with finitely many auxiliary variables is, in general, impossible: it is only possible for some special equations of "hypergeometric" type.

6.8. The Higgs and the confinement regimes.

Consider a Wilson loop which is approximately a rectangle: two charges are born, move away from each other at distance L, stay at that distance from each other for a time $T >> L$, and then annihilate each other. Then The expectation value $\langle W_R(C)\rangle$ of the Wilson loop operator (in the Euclidean setting) has the following meaning: it is approximately equal to $Ce^{-TV(L)}$, where $V(L)$ is the energy of interaction of the charges at distance L, and C is a constant. This is clear from the interpretation of the Wilson loop operator as an evolution operator, which is given above.

So the asymptotics of $\langle W_R(C)\rangle$ depends on the asymptotics of $V(L)$ as $L \to \infty$.

Physicists believe that above 2 dimensions, in gauge theories with a mass gap, there are two possibilities:

1. Higgs regime: $VL \to \text{const}, L \to \infty$.

In this regime, charges can separate from each other.

2. Confinement regime: $V(L) \sim \text{const}L$ as $L \to \infty$, where the constant is positive.

In this regime charges are confined, and cannot separate without spending an arbitrarily large amount of energy.

Now let us consider a Wilson loop C of any shape, with circumference S and minimal area of the spanning surface A. There is a general conjecture that patterns 1 and 2, if they hold for $T >> L >> 0$, hold for an arbitrary C (large in all directions) in the following form: the Higgs regime corresponds to the asymptotics $\langle W_R(C)\rangle \sim e^{-wS}$ (the circumference law), and the confinement regime corresponds to the asymptotics $\langle W_R(C)\rangle \sim e^{-wA}$ (the area law), where $w > 0$.

Remark. Actually, confinement can only occur if R is a representation of the universal cover $\hat{G}$ of G and not of G itself. Indeed, if R is a representation of G, there are physical processes contributing to $\langle W_R(C)\rangle$ in which large portions of the Wilson line have zero charge (i.e. carry the trivial representation of G) (i.e. some particles have annihilated the charges on the Wilson line). These processes have an amplitude which is bigger than that predicted by the area law.

As a toy example, let us consider the 2-dimensional theory with Lagrangian (6.1), which we studied in the first part of the lecture. In this case, $G = U(1)$. Let R be the representation of $\hat{G}$ defined by $\lambda \in \mathbb{R}$: $x \to e^{i\lambda x}$. Then, classically, $W_R(C) = e^{i\lambda \int_D F_A}$. Thus, quantum mechanically

$$\langle W_R(C)\rangle = \int DAD\phi D\bar{\phi} e^{-\mathcal{L}} e^{i\lambda \int_D F}. \tag{6.16}$$

Let V be the volume of spacetime, and A_C be the area enclosed by C. We will split the path integral (6.16) into a product of two – the integral over values of fields inside the loop C and over the values outside. Observe that the last factor in (6.16) (the holonomy factor) is of the same type as the topological term in (6.1). Therefore, according to the results of the first part of this lecture, (6.16) yields

$$\langle W_R(C)\rangle = \frac{1}{Z} e^{-(V-A_C)E(\theta)-A_C E(\theta+\lambda)}, \tag{6.17}$$

up to boundary terms, which we will neglect here. (Here $E(\theta)$ is the vacuum energy density). Since $Z = e^{-VE(\theta)}$, we get

$$\langle W_R(C)\rangle = e^{A_C(E(\theta)-E(\theta+\lambda))}. \tag{6.18}$$

According to our calculations, for $\theta = 0$

$$E(\theta + \lambda) - E(\theta) = E(\lambda) - E(0) = 2(1 - cos\lambda)e^{-I/e^2}P. \tag{6.19}$$

This is always positive when λ is not a multiple of 2π. Thus, theory (6.1) for $\theta = 0$ exhibits the confinement regime.

In more than 2 dimensions, one expects that this theory obeys Higgs regime.

6.9. The confinement conjecture.

The following conjecture is central in quantum field theory.

Conjecture *Let G be a simple compact Lie group, and R be a representation of G which is not a representation of the adjoint group G_{ad}. In 3 and 4 dimensions, the pure gauge theory with gauge group G and Lagrangian $\int \mathrm{Tr} F \wedge *F$ exhibits confinement for charges with values in R and R^*.*

The physically interesting case of this theory is $G = SU(3)$, $R = \mathbb{C}^3$. This special case of the conjecture would explain confinement of quarks.

In 2 dimensions, this conjecture is true, as we saw in the previous lectures.

Lecture II-7: Abelian Duality

Edward Witten[*1]

1 Introduction

Today, we will discuss abelian duality in two and three dimensions (with a brief mention of four dimensions, which will be further developed in the next lecture). In two dimensions, abelian duality is often referred to as the "R goes to $1/R$" equivalence. In a certain supersymmetric version, it leads to a linear version of mirror symmetry.

In three dimensions, after studying the duality we will give an application to Polyakov's model of confinement.

Abelian duality in four dimensions will eventually have an application to Donaldson theory, that is, to $N = 2$ supersymmetric theories in dimension 4. This again gives a model of confinement. In fact, these two applications of duality are the most concrete models of the phenomenon of confinement which are known.

We begin with the classical statements of duality. In two dimensions, consider a theory which involves fields ϕ, σ, both obeying the Laplace equation

$$\nabla^2\phi = 0; \qquad \nabla^2\sigma = 0, \tag{1.1}$$

and which are related by

$$d\phi = *\, d\sigma. \tag{1.2}$$

Classically, either of these fields can be taken as the fundamental field of the theory. For example, if we begin with σ such that $\nabla^2\sigma = *\, d * d\sigma = 0$ then $d * d\sigma = 0$ so locally we can write $* d\sigma = d\phi$ for some field ϕ, i.e., σ determines ϕ (locally and up to an additive constant).

Likewise in three dimensions, consider a theory with a field ϕ obeying the Laplace equation, as well as a connection A on some line bundle $\mathscr{L}$ with curvature $F = dA$ obeying Maxwell's equations

$$dF = d * F = 0, \tag{1.3}$$

whose duality relationship is

$$* d\phi = F. \tag{1.4}$$

Again, as above, either of these fields can be taken as fundamental.

In four dimensions, the analogue is two connections A and B on line bundles $\mathscr{L}_A$ and $\mathscr{L}_B$, each satisfying Maxwell's equations, and related by

$$F_A = * F_B. \tag{1.5}$$

For abelian duality, we could keep going to higher dimensions if we wish, but we quickly run out of field theory applications. (There are some applications in string theory.)

[1]*Notes by David R. Morrison

2 Duality in two dimensions

We wish to study quantum versions of these classical statements. We begin with the two-dimensional case. We identify S^1 with $\mathbb{R}/2\pi\mathbb{Z}$, and use additive coordinates on the circle. We take ϕ to be a map $\phi : \Sigma \to S^1$ where Σ is a compact oriented surface equipped with a Riemannian metric $g_{\alpha\beta}$. (We will consider a variant later on, in which ϕ is not required to be defined at some specified points P_i, that is, ϕ will map $\Sigma - \{P_i\}$ to S^1.) Our Lagrangian[2] is

$$\mathcal{L}(\phi) = \frac{R^2}{4\pi} \int d^2x \ \sqrt{g}\, \partial_\alpha \phi\, \partial^\alpha \phi = \frac{R^2}{4\pi} \int d\phi \wedge * d\phi. \tag{2.1}$$

The equations of motion $dd\phi = d * d\phi = 0$ reproduce the classical theory discussed above.

We will study this theory in various ways. The usual trick is to introduce new variables with the property that integrating them out would lead back to the original theory, and then integrate out the old variables instead of the new ones in order to produce a dual formulation of the theory.

So we actually wish to study a different theory, one which will contain fields ϕ and A, with ϕ a section of a trivial S^1-bundle $\mathscr{S}$ and A a connection on $\mathscr{S}$. Choosing a trivialization ϕ_0 of $\mathscr{S}$ (so that $A = \phi_0^* \tilde{A}$ for some 1-form $\tilde{A}$ on the total space of the bundle), we can define a covariant derivative

$$D_A \phi = d\phi + A, \tag{2.2}$$

and introduce a new Lagrangian

$$\mathcal{L}(\phi, A) = \frac{R^2}{4\pi} \int d^2x \ \sqrt{g}(\partial_\alpha \phi + A_\alpha)(\partial^\alpha \phi + A^\alpha) = \frac{R^2}{4\pi} \int D_A\phi \wedge * D_A\phi. \tag{2.3}$$

While we can recover the old Lagrangian by setting A to zero, there is no mechanism which enforces this, unless we introduce yet a third field σ which plays the role of a Lagrange multiplier. We take σ to be a map from Σ to S^1, and write a Lagrangian

$$\mathcal{L}(\phi, A, \sigma) = \frac{R^2}{4\pi} \int D_A\phi \wedge * D_A\phi - \frac{i}{2\pi} \int \sigma \wedge F_A, \tag{2.4}$$

where F_A is the curvature of A.

This last term requires some comment. Since we are assuming that the circle bundle $\mathscr{S}$ is trivial, we can globally write $F_A = dA$ and the last term should be interpreted as $\frac{i}{2\pi} \int d\sigma \wedge A$ (after integration by parts); this step is needed because σ is not single-valued. More generally, to define such a term even when $\mathscr{S}$ is nontrivial, one can use a bit of topology to define $\exp(\frac{i}{2\pi} \int \sigma \wedge F_A)$, similar to defining a Chern–Simons form.

The point of writing this Lagrangian is that $\mathcal{L}(\phi, A, \sigma)$ is equivalent to $\mathcal{L}(\phi)$, as we will now show. A na¨ive analysis goes as follows: σ appears without derivatives, and its equation of motion is $F_A = 0$; imposing this, we can then go to a gauge where $A = 0$ and recover the original theory.

More globally, we consider the path integral

$$Z = \frac{1}{\mathrm{vol}(G)} \int \mathcal{D}\phi\, \mathcal{D}A\, \mathcal{D}\sigma \ \exp\left[-\frac{R^2}{4\pi} \int D_A\phi \wedge * D_A\phi + \frac{i}{2\pi} \int \sigma \wedge F_A \right]. \tag{2.5}$$

(We focus on the partition function for now, but path integrals with operator insertions can also be treated this way, as we will discuss later.) In light of the standard formula

$$\int \frac{dx}{2\pi} e^{ixy} = \delta(y), \tag{2.6}$$

[2] All Lagrangians in this lecture are written in Euclidean signature.

we would like to say that doing the σ-integral will set F_A to zero. We should treat this statement with care, since we are studying circle-valued functions.

Bearing in mind our identification of S^1 with $\mathbb{R}/2\pi\mathbb{Z}$, our map $\sigma : \Sigma \to S^1$ can be locally written as a real-valued function, but it might not be globally single-valued; however, $d\sigma$ will be a (single-valued) real 1-form on Σ. Choose a circle-valued function $\sigma_h : \Sigma \to S^1$ such that $d\sigma_h$ is the *harmonic* representative in the de Rham cohomology class of $d\sigma$. (We normalize the choice of σ_h by picking some point $P \in \Sigma$ and demanding that $\sigma_h(P) \in 2\pi\mathbb{Z}$.) Then we can write $\sigma = \sigma_h + \sigma_{\mathbb{R}}$, with $\sigma_{\mathbb{R}}$ a single-valued real function.

Notice that $\frac{1}{2\pi} d\sigma$, or equivalently $\frac{1}{2\pi} d\sigma_h$, must have integral periods. In particular, if we choose a basis λ_j of integral harmonic 1-forms, and write $d\sigma_h = \sum 2\pi m_j \lambda_j$, then $m_j \in \mathbb{Z}$.

Now we compute:

$$\int \mathcal{D}\sigma \, e^{\frac{i}{2\pi}\int \sigma \wedge F_A} = \int \mathcal{D}\sigma_{\mathbb{R}} \, e^{\frac{i}{2\pi}\int \sigma_{\mathbb{R}} \wedge F_A} \sum_{d\sigma_h \in H^1(\Sigma, 2\pi\mathbb{Z})} e^{-\frac{i}{2\pi} d\sigma_h \wedge A} \tag{2.7}$$

$$= \delta(F_A) \prod_j \left(\sum_{m_j \in \mathbb{Z}} e^{-i m_j \int \lambda_j \wedge A} \right). \tag{2.8}$$

The first factor tells us that A is a flat connection. Moreover, among flat connections the gauge equivalence classes $[A]$ are labeled by the holonomies, or equivalently by the quantities $\int \lambda_j \wedge A$. Since the remaining part of the integrand is gauge invariant, we can gauge fix (omitting the factor of $(\det G)^{-1}$) and integrate over the space of gauge equivalence classes (a finite dimensional integral):

$$Z = \int \mathcal{D}\phi \, \mathcal{D}[A] \, e^{-\frac{R^2}{4\pi}\int D_A\phi \wedge * D_A\phi} \delta(F_A) \prod_j \left(\sum_{m_j \in \mathbb{Z}} e^{-i m_j \int \lambda_j \wedge A} \right) \tag{2.9}$$

$$= \int \mathcal{D}\phi \, e^{-\frac{R^2}{4\pi}\int D_A\phi \wedge * D_A\phi} \delta(F_A) \, \delta\Big(\int \lambda_i \wedge A = 0 \mod 2\pi\mathbb{Z} \Big). \tag{2.10}$$

So there are delta functions setting the holonomies as well as curvature to zero. So A is zero modulo gauge transformations, and our new theory is indeed equivalent to the original theory, with partition function

$$Z = \int \mathcal{D}\phi \, e^{-\frac{R^2}{4\pi}\int d\phi \wedge * d\phi}. \tag{2.11}$$

Now let us integrate out in the opposite order, integrating out ϕ and A but keeping σ. To integrate out ϕ, we fix the gauge in such a way that $\phi = 0$, and suppress the factor of $(\mathrm{vol}(G))^{-1}$ from the path integral. Then the path integral (2.5) reduces to

$$\int \mathcal{D}A \, \mathcal{D}\sigma \, \exp\left[-\frac{R^2}{4\pi} \int A \wedge * A + \frac{i}{2\pi} \int \sigma \wedge F_A \right]. \tag{2.12}$$

An exercise you might enjoy is verifying that the Faddeev-Popov determinant associated with this gauge fixing is

$$\int \mathcal{D}c \, \mathcal{D}\bar{c} \, \exp\left[-\frac{R^2}{4\pi} \int d^2x \, \bar{c}c \right]. \tag{2.13}$$

To do the integral over A, we need to complete the square, thinking of the second term in the exponent as $-\frac{i}{2\pi}\int A \wedge d\sigma$, i.e., it is the term linear in A. We make a change of variables $A' = A + \frac{i}{R^2} * d\sigma$; the result, including the Faddeev-Popov integral, is

$$\left(\int \mathcal{D}c \, \mathcal{D}\bar{c} \, e^{-\frac{R^2}{4\pi}\int d^2x \, \bar{c}c} \right) \left(\int \mathcal{D}A' e^{-\frac{R^2}{4\pi}\int A' \wedge * A'} \right) \left(\int \mathcal{D}\sigma e^{-\frac{1}{4\pi R^2}\int d\sigma \wedge * d\sigma} \right). \tag{2.14}$$

(The Gaussian integral over A', like the Faddeev-Popov determinant, can be thought of as a normalization factor.)

The Faddeev-Popov integral gives a factor of $(\int dt\, e^{-tR^2/4\pi})^{-1}$ for each 0-form and each 2-form on Σ; the integral over A' gives a factor of $\int dt\, e^{-tR^2/4\pi}$ for each 1-form on Σ. Thus, if n_j denotes the "number of j-forms on Σ", those terms combine to give an overall factor of

$$(\sqrt{\pi/(R^2/4\pi)})^{(-n_0+n_1-n_2)} = (R/2\pi)^{\chi(\Sigma)}. \tag{2.15}$$

That is, the transformation rule which relates these dual formulations is

$$\int \mathcal{D}\phi\, \exp\left[-\frac{R^2}{4\pi}\int d\phi \wedge *\, d\phi\right] = (R/2\pi)^{\chi(\Sigma)} \int \mathcal{D}\sigma \exp\left[-\frac{1}{4\pi R^2}\int d\sigma \wedge *\, d\sigma\right]. \tag{2.16}$$

(The factor of $(R/2\pi)^{\chi(\Sigma)}$ did not show up in previous explicit calculations we have done because they were done in genus 1.) A coupling of R in the first theory maps to a coupling of $1/R$ in the second theory, which is why this duality is sometimes referred to as "R goes to $1/R$."

Now we would like to follow the operators in this theory through the duality transformation, and determine the effect on correlation functions as well as the partition function. The easy case is $d\phi$, which we expect to map to $*\, d\sigma$, as in the classical theory (actually it will map to a multiple of $*\, d\sigma$, when normalizations are taken into account). We need to repeat the above calculation with an insertion of operators $\prod O_i(\phi)$, one of which is $d\phi$. In order to do this, we need a gauge-invariant extension of the operator $d\phi$, which is provided by the covariant derivative:

$$d\phi(x) \text{ in first theory} \longrightarrow D_A\phi(x) \text{ in big theory.} \tag{2.17}$$

One then easily checks that this maps back to $d\phi$ when we integrate out σ (and gauge fix A to zero).

The difficult thing about mapping operators in general will be finding the appropriate extension to an operator in the larger theory, which reduces to the original operator when σ has been integrated out.

How will our calculation be modified? Earlier, when we completed the square, we made a change of variables $A' = A + \frac{i}{R^2} * d\sigma$. This means that if $D_A\phi$ has been inserted in the path integral, it now becomes $D_{A'}\phi - \frac{i}{R^2} * d\sigma$. Thus, when A' is integrated out, we are left with an insertion of $-\frac{i}{R^2} * d\sigma$ in the dual theory.

Some care must be used in manipulating these mappings between operator insertions. For example, even though $d\phi$ maps to $-\frac{i}{R^2} * d\sigma$, $(d\phi)^2$ will *not* map to $(\frac{i}{R^2} * d\sigma)^2$ in the dual theory, due to nonlinearities introduced when we complete the square.

What made this case easy was that the covariant derivative is a natural gauge-invariant extension of the ordinary derivative.

The hard operator insertion to deal with is $e^{i\phi}$. There is no gauge-invariant version of this. On the other hand, we don't really need it, because $\langle e^{i\phi}(P)\rangle = 0$. On the other hand, the two-point correlators $\langle e^{i\phi}(P)\, e^{-i\phi}(Q)\rangle$ are not zero in general, so we should try to dualize such insertions, when $P \neq Q$.

We are going to do something rather strange. It is impossible to construct a gauge-invariant extension *locally* for this pair of operator insertions, so we work non-locally during the duality (i.e., in the intermediate theory), introducing a term in the path integral of the form

$$e^{i\phi}(P)e^{-i\phi}(Q)e^{\frac{i}{2\pi}\int \theta\wedge A} \tag{2.18}$$

where θ is a 1-form such that

$$\frac{1}{2\pi}d\theta = \delta_P - \delta_Q \tag{2.19}$$

and all periods of θ are 0 modulo $2\pi\mathbb{Z}$.

To find such a θ, take a path ell from P to Q and let θ be Poincaré dual to ell (regarded as a distribution). We can rewrite our term (2.18) as

$$e^{i\phi}(P)e^{\frac{i}{2\pi}\int_{\rm ell} A}e^{-i\phi}(Q), \tag{2.20}$$

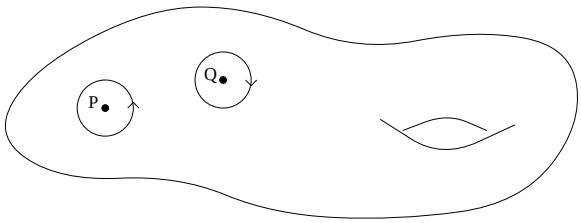
from which we see that this is a gauge-invariant expression. (This is an example of a choice of θ, but we allow more general θ's than that.) We will eventually think of θ as a connection form on a trivial S^1-bundle on $\Sigma - P - Q$.

Thus, we have found a gauge-invariant nonlocal object (2.18). Suppose we try to insert this into the path integral. First, if we set A to zero, we recover our original operator insertion $e^{i\phi}(P)e^{-i\phi}(Q)$, so we have correctly extended this pair of operators to the gauge theory.

To integrate out the other way, we gauge fix ϕ to zero, which reduces (2.18) to $e^{\frac{i}{2\pi}\int\theta\wedge A}$. This is a new contribution to the term linear in A in the exponent of the path integral, of the form $-\frac{i}{2\pi}\int A\wedge\theta$, so to complete the square we now must make the change of variables $A' = A + \frac{i}{R^2}*(d\sigma+\theta)$, and this makes our dualized path integral take the form

$$\int \mathcal{D}\sigma\, \exp\left[-\frac{1}{4\pi R^2}\int (d\sigma+\theta)\wedge *(d\sigma+\theta)\right]. \tag{2.21}$$

That is, we have obtained the same theory as before, but now formulated on $\Sigma - P - Q$, and now performing the path integral over maps to S^1 that do *not* extend over P or Q. That is, we write $\widetilde{\sigma} = \sigma + \alpha$, with $d\alpha = \theta$, α being a map from $\Sigma - P - Q$ to S^1, and then $\widetilde{\sigma}$ is the map of $\Sigma - P - Q$ to S^1 that does not extend over P or Q. In fact, it has winding numbers 1 and -1 around P and Q, respectively.



There are two kinds of operators now, one local in one picture, and the other local in the other picture. The operator $e^{in\phi}(P)$ in one description is mapped to an instruction "delete P from Σ, and perform the path integral over σ's that have winding number n around P" in the other description. Conversely, there is an ordinary local operator $e^{in\sigma}(P)$ in the second description, that corresponds to a nonlocal recipe in the first description.

3 Duality in three dimensions

We now turn to three-dimensional theories. Our initial theory is the theory of an S^1-valued function ϕ on a (fixed) compact 3-manifold M (possibly with boundary), governed by the Lagrangian

$$\mathcal{L} = \frac{\Lambda}{4\pi}\int d^3x\ \sqrt{g}\,\partial_\alpha\phi\,\partial^\alpha\phi = \frac{\Lambda}{4\pi}\int d\phi\wedge *\,d\phi. \tag{3.1}$$

One novel feature of three dimensions is that the prefactor Λ has the dimensions of mass. Since Λ is a dimensionful quantity rather than a constant, it will be impossible for anything special to happen at a numerical value of Λ.

We reinterpret ϕ as a section of a trivial S^1-bundle $\mathscr{L}_B$ with a connection B, and write a Lagrangian which includes the covariant derivative $D_B\phi = d\phi + B$:

$$\mathcal{L}(\phi, B) = \frac{\Lambda}{4\pi}\int d^3x\,(\partial_\alpha\phi + B_\alpha)(\partial^\alpha\phi + B^\alpha) = \frac{\Lambda}{4\pi}\int D_B\phi \wedge *\, D_B\phi \tag{3.2}$$

This would be a trivial theory if left as it is. To get somewhere, we introduce a line bundle $\mathscr{L}$ with connection A, and the Lagrangian

$$\mathcal{L}(\phi, B, A) = \frac{\Lambda}{4\pi}\int D_B\phi \wedge *\, D_B\phi - \frac{i}{2\pi}\int A \wedge F_B, \tag{3.3}$$

interpreting the last term as $\frac{-i}{2\pi}\int F_A \wedge B$ (after integrating by parts) which makes sense since $\mathscr{L}_B$ is trivial. As before, the extension to the case of $\mathscr{L}_B$ not being trivial is a term in the path integral $e^{-\frac{i}{2\pi}\int A\wedge F_B}$, the Chern–Simons form for the structure group $U(1)\times U(1)$ of the bundle $\mathscr{L}\oplus\mathscr{L}_B$.

In order to carry out the duality transformation, we need to sum over line bundles $\mathscr{L}$, producing a path integral

$$\frac{1}{\mathrm{vol}(G)\,\mathrm{vol}(G')}\sum_{\mathscr{L}}\int \mathcal{D}\phi\,\mathcal{D}A\,\mathcal{D}B\,\exp\left[-\frac{\Lambda}{4\pi}\int D_B\phi\wedge *\, D_B\phi + \frac{i}{2\pi}\int F_A\wedge B\right]. \tag{3.4}$$

where G and G' are gauge groups for $\mathscr{L}$ and $\mathscr{L}_B$.

We would like to integrate out A. The crude statement is that we get $\delta(F_B)$, which implies that B is a flat connection. As in the computation we did in two dimensions, the complications arise from the possibility of nontrivial holonomies for B. In fact, if $F_B = 0$ we can regard B as an element of $H^1(M, \mathbb{R}/\mathbb{Z})$ and define for each $x = c_1(\mathscr{L})$, the quantity $e^{i\int x\wedge[B]}$. When this is summed over $[x]$ in doing the path integral, the result is $\delta([B])$, showing that B is gauge-equivalent to the trivial flat connection. Thus, we reduce back to the original theory; we've learned that the extended theory with A and $\mathcal{L}$ is equivalent to the original theory.

Now we do the integral in the opposite direction, by doing the ϕ and A integrals. As in the two-dimensional case, we set ϕ to zero by a gauge transformation, removing the normalization factor $(\mathrm{vol}(G))^{-1}$. (Unlike in two dimensions, the Faddeev-Popov determinant does not contribute anything interesting, so we will not explicitly include it.) The square can be completed in the resulting path integral

$$\frac{1}{\mathrm{vol}(G')}\sum_{\mathscr{L}}\int \mathcal{D}A\,\mathcal{D}B\,\exp\left[-\frac{\Lambda}{4\pi}\int B\wedge *\, B + \frac{i}{2\pi}\int B\wedge F_A\right] \tag{3.5}$$

$$= \frac{1}{\mathrm{vol}(G')}\sum_{\mathscr{L}}\int \mathcal{D}A\,\mathcal{D}B'\,\exp\left[-\frac{\Lambda}{4\pi}\int B'\wedge *\, B' - \frac{1}{4\pi\Lambda}\int F_A\wedge *\, F_A\right], \tag{3.6}$$

where we shifted $B' = B - \frac{i}{\Lambda} * F_A$.
The integral in B' is Gaussian, and can be absorbed into the normalization of the path integral, leaving us with

$$\frac{1}{\mathrm{vol}(G')}\sum_{\mathscr{L}}\int \mathcal{D}A\,\exp\left[-\frac{1}{4\pi\Lambda}\int F_A\wedge *\, F_A\right], \tag{3.7}$$

the familiar path-integral from Yang–Mills theory! (With gauge coupling $\mathrm{e} = \sqrt{\Lambda}$.)

Now we want to map operators. An analogous calculation to the one we did in two dimensions shows that $d\phi$ extends to the covariant derivative $D_B\phi = d\phi + B$ in the larger theory, and becomes

$\frac{i}{\Lambda} * F_A$ in the dual theory. This is the meaning in the quantum theory of the duality between solutions of the Laplace equation and solutions of Maxwell's equation.

What about operators such as $e^{i\phi}(P)$? We will not do this, though I will state the answer. $e^{in\phi}(P)$ is mapped, in the dual gauge theory, to the instruction: delete the point P from the 3-manifold M and consider the path integral for connections on a line bundle $\mathcal{L}$ whose first Chern class evaluates to n on a small sphere around P. The argument leading to this is quite along the lines of what we did in two dimensions.

I will not present that argument (which I recommend as an exercise) and instead discuss what is roughly the reverse. We consider Wilson loop operators in the gauge theory, and try to map those back to the scalar field theory.

Let C be a circle in M which is a boundary, and let $\lambda \in \mathbb{R}$. We have an operator $\exp(i\lambda \oint_C A)$ in the gauge theory. Let us insert this into the path integral of the big theory (the one with Lagrangian $\mathcal{L}(\phi, B, A)$). (More generally, we can do this even if C is not a boundary, but then λ must be restricted to be an integer. C being a boundary, say of a two-surface D, lets us write $\exp(i\lambda \oint_C A) = \exp(i\lambda \int_D F)$, showing gauge-invariance for any λ.)

When we integrate out A with this insertion, instead of getting $\delta(F_B = 0)$ we get $\delta(F_B = 2\pi\lambda[C])$ with $[C]$ being the Poincaré dual to C. In other words, performing the A integral determines B to be such that $F_B = 2\pi\lambda[C]$, and the modified Lagrangian after integrating out is $\int D_B\phi \wedge * D_B\phi$. Here B is a flat $U(1)$-connection with monodromy $e^{2\pi i\lambda}$ around the circle. So the Wilson loop operator for a circle C is dual to a recipe "delete C and interpret ϕ as a section of a flat bundle with monodromy around C." In other words, we modify ϕ to $\widetilde{\phi}$ such that $d\widetilde{\phi} = d\phi + B$, and interpret $\widetilde{\phi}$ as a section of a trivial S^1-bundle on $M - C$ with a flat connection of monodromy $e^{2\pi i\lambda}$ around C. (To get such a trivial S^1-bundle, we either need to assume that C is a boundary, or that λ is an integer.)

4 Application to the Polyakov model

We will describe a model constructed by Polyakov around 1975. It was the first use of duality in a nonlinear relativistic theory. The model exhibits the phenomenon known as confinement.

We work in three dimensions, and study $SO(3)$ gauge theory with a scalar field s in the 3-dimensional representation, governed by the Lagrangian

$$\mathscr{L} = \frac{1}{4\pi e^2} \int d^3x \ \mathrm{Tr}\,|F|^2 + \int d^3x\, (D\vec{s})^2 + \int d^3x\, \lambda(\vec{s}^2 - a^2)^2. \tag{4.1}$$

This model exhibits the Higgs mechanism at tree level.

Classically, the vacuum can be rotated to

$$\langle \vec{s} \rangle = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \tag{4.2}$$

by a gauge transformation. In the vacuum state, the $SO(3)$ symmetry of the Lagrangian is broken to an $SO(2) = U(1)$ symmetry, and the only massless field is the $U(1)$-connection. (This is the so-called Higgs mechanism.) The low-energy theory looks like the $U(1)$ theory in three dimensions.

First Step. This theory looks classically, at low energies, like a $U(1)$ gauge theory with Lagrangian

$$\mathcal{L} = \frac{1}{4e^2} \int F \wedge * F. \tag{4.3}$$

Is that the answer? Is the Gaussian fixed point of the free $U(1)$ theory stable?

One possible source of instability is a Chern–Simons interaction

$$\frac{-in}{2\pi}\int A\wedge F_A \tag{4.4}$$

which could be added to the original Lagrangian $\mathcal{L}$. (In the flow to the infrared, this term is more relevant than those appearing in $\mathcal{L}$, hence the instability.)

There are two obstacles to this being a source of instability in our problem. The first is that the coefficient n in the Chern–Simons term must be an integer. This implies that even if you can only approximately calculate the theory, you can determine n if the approximation can be made arbitrarily accurate. One can, in particular, calculate n in perturbation theory, and – as higher loop terms would involve positive powers of e – it could only arise from a one-loop term. So the effective n in any three-dimensional gauge theory can be determined by an explicit one-loop computation. In the specific example we are considering here, one can simply notice that the Chern–Simons term is odd under parity (i.e., it depends on a choice of orientation of the 3-manifold) whereas our original theory was not. The parity-invariance is a symmetry of the one-loop determinants (whether or not parity ultimately is spontaneously broken at low energies) and ensures that $n = 0$.

Thus, anything which could make this theory unstable will be hard to describe in terms of A. However, we know that the $U(1)$ gauge theory is equivalent (dual) to a scalar theory

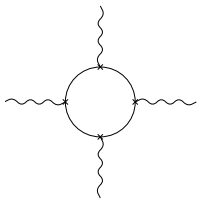
$$\mathcal{L} = \frac{\mathrm{e}^2}{4\pi}\int d^3x\,\partial_\alpha\phi\partial^\alpha\phi \tag{4.5}$$

In this theory, we could add a term

$$g\int d^3x\,\cos n\phi \tag{4.6}$$

for some n. We have chosen the potential to be periodic as ϕ is really a map to a circle. Notice that the theory with $g \neq 0$ has a mass gap, the theory with $g = 0$ does not. Thus, we have identified something which qualitatively changes the physics, but can only be conveniently interpreted by means of the dual variables. This is strange, because doing the duality requires going to a low energy description to begin with!

How do we see this effect in the original $SO(3)$ theory? Consider the Feynman diagrams of the $SO(3)$ theory. We represent these with massless modes given by wavy lines, massive ones by solid lines. A typical Feynman diagram such as



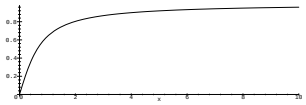
will be completely tame: we have massive propagators, which are analytic in the momenta at low momentum. Such diagrams merely give corrections to the effective action for the $U(1)$ gauge field A which can be described as additional local, gauge-invariant terms in the Lagrangian.

We need something completely different: *instantons*. Let us work on $\mathbb{R}^3$. We have an $SO(3)$ bundle on $\mathbb{R}^3$ whose structure group has been reduced to $U(1)$ at infinity. On the S^2 at infinity, a $U(1)$ bundle $\mathscr{L}$ can have a nontrivial first Chern class, $c_1(\mathscr{L}) \neq 0$.

Pick a trivialization of the $SO(3)$ bundle, so that we can identify the spatial $\mathbb{R}^3$ with the bundle $\mathbb{R}^3$, and simply treat our section s as a map $s : \mathbb{R}^3 \to \mathbb{R}^3$. To construct such a map with nontrivial topology on the S^2 at infinity, we take s of the form

$$s(\vec{x}) = a\frac{\vec{x}}{|x|}f(|x|), \tag{4.7}$$

with $f(|x|)$ an increasing function satisfying $f(0) = 0$, $\lim_{|x|\to\infty} f(|x|) = a$, e.g.,



Such a section is invariant under combined rotations of space and gauge rotations.

We would like the minimum action solution s in this class. It should be spherically symmetric, with asymptotic behavior $Ds \sim \frac{1}{|x|^2}$ for $|x| \to \infty$. This leads to an ODE, which has a unique solution. The action can be written as (I/e^2) for some constant I, so it will diverge as the coupling goes to 0.

Note that since $\vec{s}$ vanishes precisely at the origin, the structure group is reduced to $U(1)$ away from the origin, but this reduction does not extend over the origin. In fact, over a sphere surrounding the origin, the line bundle has a nonzero first Chern class (which actually is 2 if we work in $SO(3)$; that is, the adjoint bundle of $SO(3)$ decomposes over a two-sphere surrounding the origin as $\mathcal{O} \oplus \mathcal{L} \oplus \mathcal{L}^{-1}$ where $\mathcal{L}$ has degree 2).

From this point of view, the difference between a pure $U(1)$ theory and a theory that looks like a $U(1)$ theory only at long distances is that the latter can have bundles that over a large $\mathbf{S}^2$ at infinity have a nonzero first Chern class. The Chern class can be any multiple of the 2 that was found in the explicit solution that we just described.

Note that in abelian gauge theory, it was not possible to use a bundle $\mathscr{L}$ for which $c_1(\mathscr{L}) \neq 0$ over a sphere around the origin. The qualitative difference between an $SO(3)$ theory broken to $U(1)$ and a $U(1)$ theory is that the former admits singularities where the $U(1)$ description breaks down in this way.

If we did a similar thing in $3 + 1$ dimensions, we would find a time-independent solution of finite energy (rather than finite action). This solution looks like a particle sitting there, and in fact is a magnetic monopole. (The nonzero magnetic charge comes from the fact the $c_1(\mathscr{L}) \neq 0$ which implies that the magnetic field integrates to a nonzero amount.) The point of making this analogy is that we can think of the instanton as behaving like a zero-time slice of a magnetic monopole.

What do instantons look like at long distances? A monopole has a field which behaves like

$$F = \text{const}\,\frac{\vec{x}}{|x|^3} = \text{const}\,\frac{\hat{x}}{|x|^2} \tag{4.8}$$

(by Maxwell's equations).

The contribution to the path integral from each instanton is e^{-I/e^2}, which is small.

To return to our theory, we now ask how it behaves at long distance. For example, how does a two-point function $\langle F(x)\,F(0)\rangle$ behave as $|x| \to \infty$?

The answer in the free theory, by dimensional analysis, is $-1/|x|^3$. (One sees that the curvature F has dimension $3/2$ in three-dimensional theories by considering the basic term $\frac{1}{4\pi\mathrm{e}^2}\int F\wedge *F$ in the Lagrangian.) A bit more precisely, we are asserting that

$$\langle F_{ij}(x)\,F_{k\mathrm{ell}}(0)\rangle = \frac{1}{|x|^3}(\delta_{ik}\delta_{j\mathrm{ell}} + \dots). \tag{4.9}$$

The Feynman diagrams don't affect this asymptotic behavior: there is a slight renormalization of e^2, plus other corrections which are unimportant at big distances.

How does an instanton affect this analysis? Consider an instanton localized near y, and its effect on the two-point function between x and 0.

The leading approximation to the path integral in the instanton sector is

$$e^{-I/e^2} \int d^3y \, \frac{(\vec{x}-\vec{y})}{|x-y|^3} \, \frac{\vec{y}}{|y|^3}. \tag{4.10}$$

Thus, the overall behavior is

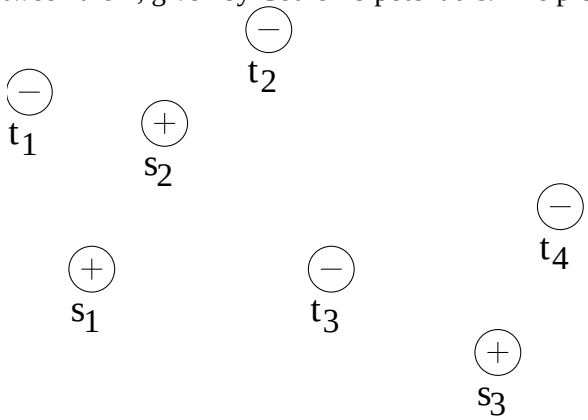
$$\langle F(\vec{x}) F(\vec{0}) \rangle \cong \frac{1}{Z} \left(\text{pert. theory} + e^{-I/e^2} (\text{instanton sector}) \right) \tag{4.11}$$

$$\cong \frac{1}{Z} \left((1 + \dots) \frac{1}{|x|^3} + e^{-I/e^2} (\frac{1}{|x|} + \dots) \right). \tag{4.12}$$

Since the instanton contribution is more important in the infrared than the free theory term ($1/|x|$ compared to $1/|x|^3$), the instanton triggers an instability.

To see in more detail what is happening in the infrared, we first note that instantons are *rare*: the probability to have an instanton in a small volume V_0 is proportional to $V_0 e^{-I/e^2}$, so the volume of space per instanton is given by $V \sim e^{I/e^2}$, and the spatial separation between two of them is $R \sim V^{1/3} \sim e^{I/3e^2}$.

In the infrared, though, if we consider a large enough volume of space, we will get lots of instantons, which we can treat as a gas of particles of definite size. The particles are *charged*, so we can't ignore the interactions between them, given by Coulomb potentials. The picture is as follows



We have labeled the positions of instantons (which are positively charged) with s_i's, and the positions of anti-instantons (which are negatively charged) by t_j's. The sum over all of these takes the form

$$\sum_{n,m=0}^{\infty} \frac{1}{n!m!} \int d^3 s_i |_{i=1}^{n} \int d^3 t_j |_{j=1}^{m} e^{-\frac{I}{e^2}(n+m)} e^{\sum_{i<j} \left(\frac{1}{|s_i - s_j|} + \frac{1}{|t_i - t_j|} \right) - \sum_{i,j} \frac{1}{|s_i - t_j|}}. \tag{4.13}$$

We also need an operator insertion, of

$$F(x) = \sum \frac{x - s_i}{|x - s_i|^3} - \sum \frac{x - t_j}{|x - t_j|^3}, \tag{4.14}$$

and similarly for $F(0)$.

The physics involved is the classical statistical mechanics of a plasma in space, with chemical potential I/e^2; energy E given by the Coulomb potential between the instantons and anti-instantons, and temperature $T = 4\pi\mathrm{e}^2$.

The phenomenon we need is known as "Debye screening"—a plasma screens external charges. As a result of this screening, the system will have a mass gap.

Here is a quick mathematical derivation of Debye screening (in this context). We go back to our low-energy theory, writing in dual variables—a scalar theory with $F = *\,d\phi$. We will add a term to the Lagrangian to account for the instanton effect: the term we will use (justified by the results of the calculation to come) is $\int e^{-I/\mathrm{e}^2}(e^{2i\phi} + e^{-2i\phi})$. (The 2 is present in the exponent because the basic instanton has first Chern class 2.) The operator insertions of F become insertions of $*\,d\phi$, and the quantity we are calculating can be written:

$$\Omega = \int \mathcal{D}\phi\ (*\,d\phi(x))\,(*\,d\phi(0)) \exp\left[-\frac{1}{4\pi\mathrm{e}^2}\int |d\phi|^2 + \int e^{-I/\mathrm{e}^2}(e^{2i\phi} + e^{-2i\phi})\right]. \tag{4.15}$$

First we want to show that this is a correct description in the dual variables, then we will analyze this version.

We will expand Ω in perturbation theory:

$$\Omega = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \int \mathcal{D}\phi\, e^{-\frac{1}{4\pi\mathrm{e}^2}\int d\phi \wedge *\,d\phi}\,(*\,d\phi(x))\,(*\,d\phi(0)) \left(\int e^{-I/\mathrm{e}^2} e^{2i\phi(y)} d^3y\right)^n \left(\int e^{-I/\mathrm{e}^2} e^{-2i\phi(z)} d^3z\right)^n. \tag{4.16}$$

We expand further, using the principle that $\left(\int dy\, f(y)\right)^n = \int dy_1 \dots dy_n\, f(y_1)\dots f(y_n)$. Thus, we can rewrite (4.16) as

$$\Omega = \sum_{n,m=0}^{\infty} \int \mathcal{D}\phi \int d^3s_i|_{i=1}^n d^3t_j|_{j=1}^m\,(*\,d\phi(x))\,(*\,d\phi(0))\, e^{-\frac{I}{\mathrm{e}^2}(n+m)} e^{2i\sum(\phi(s_i)-\phi(t_j))} e^{-\frac{1}{4\pi\mathrm{e}^2}\int |d\phi|^2}. \tag{4.17}$$

The ϕ-integral is Gaussian after the square has been completed; that integral will contribute

$$e^{\sum_i \sum_j G(s_i,t_j)} \tag{4.18}$$

to the overall answer, where $G(s,t) = 1/|s-t|$ is the Green's function for the Laplacian. We thus recover the previous formula. Note that the $i = j$ terms contributed a divergence which is renormalized, so they do not appear.

The lesson we have learned is this: our original problem ($SO(3)$ gauge theory) can be described in the infrared, using dual variables, by means of the Lagrangian

$$\frac{1}{4\pi\mathrm{e}^2}\int |d\phi|^2 + e^{-I/\mathrm{e}^2}\int d^3x\, \cos 2\phi. \tag{4.19}$$

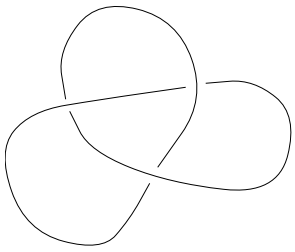
The second term in unrenormalizable, but is well-behaved in the infrared. Expanding around the minimum of the potential, we see that ϕ has a mass. Thus, what used to be a massless $U(1)$ photon

was (i) dualized and reinterpreted as a scalar, and (ii) got a mass that was more easily described in that language.

Here is another way to understand the above derivation. Start with the massless scalar field and dualize it, as in section 3 of this lecture, to a massless gauge field A. Now perturb the theory of the massless scalar by adding a weak perturbation $\int d^3x\epsilon(e^{22i\phi} + e^{-2i\phi})$. In terms of the gauge field, the operator $e^{2i\phi(P)}$ becomes, as we noted at the end of section 3, an instruction "delete the point P, and consider bundles with first Chern class 2 on a small sphere around P." In other words, from the point of view of an $SO(3)$ that looks like $U(1)$ at low energies, the instruction is "include an instanton at P." If we take the interaction term $\int d^3x\epsilon(e^{2i\phi} + e^{-2i\phi})$, and expand in powers of ϵ, we simply generate the instanton gas, with each insertion of $e^{2i\phi}$ or $e^{-2i\phi}$ corresponding to an instanton or antiinstanton.

Now we want to study confinement. An important preliminary is to note the symmetries of the problem. In general, in the duality from an abelian gauge theory, ϕ is an angular variable, a map to a circle, so ϕ is equivalent to $\phi + 2\pi$. However, because the instanton-induced interaction is a trigonometric function of 2ϕ, there is a symmetry under $\phi \to \phi + \pi$.

In the original description, we could consider a curve $C \subset M$



and the associated Wilson line operator

$$\langle \mathrm{Tr}_R \, \mathrm{Hol}(A, C)\rangle \tag{4.20}$$

where R is the 2-dimensional representation of $SU(2)$ and Hol denotes the holonomy. What does this operator translate to in our infrared description? As we have seen at the end of section 3, it translates into a recipe "delete C from spacetime and interpret ϕ as a section of a flat circle bundle with monodromy around C." The monodromy is given by the angle $2\pi\lambda$ where λ, introduced in the discussion in section three, is the $U(1)$ charge appearing in the Wilson loop, modulo 1. We have chosen a Wilson line in the two-dimensional representation of $SU(2)$; the $U(1)$ charges are $\pm 1/2$ in units of the weight lattice of $SO(3)$. Hence our example has $\lambda = 1/2$. The monodromy around the circle is thus $\phi \to \phi + \pi$, which, as we noted a moment ago, is a symmetry of the theory even with the instanton-induced interaction.

The problem is now classical: we want to find the minimum of the action

$$\frac{1}{4\pi \mathrm{e}^2} \int |d\phi + B|^2 - e^{-I/\mathrm{e}^2} \int d^3x \, \cos 2\phi \tag{4.21}$$

with ϕ required to have monodromy π around C. Thus, in particular, ϕ cannot be constant. In fact, the least action solution will be given approximately in terms of a surface Σ with $\partial\Sigma = C$ of least area; ϕ will be constant far from Σ, and will jump by π in crossing Σ.

The easiest case is when C is a large curve in the plane, say given by $x = 0$, with x a linear function on $\mathbf{R}^3$. We want ϕ to jump by π in going from negative x to positive x. In the limit that C is very large, ϕ, if observed somewhere deep in the interior of C, becomes a function of x only. What needs to be minimized is then

$$\int dx \, ((\frac{d\phi}{dx} + B(x))^2 - (\cos 2\phi - 1)) \tag{4.22}$$

with the boundary conditions that $\phi \to 0$ for $x \to -\infty$ and $\phi \to \pi$ for $x \to +\infty$. This variational problem, with the boundary conditions, has a solution that is unique up to translation of x, with ϕ approaching its asymptotic value exponential fast (because of the mass gap) and with some action L.

Now in general, if D is the minimal area surface with boundary C, and $A(D)$ is its area, the minimum action ϕ that is a section of the appropriate flat bundle has the property that ϕ is very near zero or π except near D, jumps by π in crossing D, and looks in profile near D just like the solution of the idealized one-dimensional problem discussed in the last paragraph. Its action is very nearly $A(D)L$, so the expectation value of the Wilson line is approximately $e^{-A(D)L}$, showing the area law and confinement.

Lecture II-8, Part I: Solitons

Edward Witten
Notes by Pavel Etingof and David Kazhdan

8.1. What is a soliton?

In classical mathematical physics, by a soliton one usually means a "traveling wave" solution of a nonlinear PDE $u_t = F(u, u_x, ...)$, i.e. a solution of the form $u(x,t) = f(x - vt)$. Solitons play a very important role in the theory of integrable systems, where any solution can be approximated by a superposition of solitons, moving at different velocities. As a result, the theory of integrable systems is sometimes called soliton theory.

Today we will be interested in solitons arising in field theory (as traveling wave solutions of the classical field equations) and primarily in the role they play in quantization of field theories. This is a different point of view from the one in soliton theory. In particular, no claim is made about nonlinear superposition of solitons, and the models we consider will not, in general, be exactly integrable.

We will consider solitons for Poincare invariant field theories on Minkowski space. By a soliton for a particular field theory we will mean a traveling wave solution of the field equations (i.e. a solution which depends on $x - vt$), which is localized in space and has finite energy. By Poincare invariance, we can always assume that $v = 0$, i.e. that the solution is time-independent. We will be mostly interested in solitons which provide the global minimum for the energy in the corresponding homotopy class.

Remark. It is important to distinguish solitons from instantons. Instantons are localized in Euclidean spacetime (i.e. only exist for an instant) and have finite action, while solitons are localized in space (of Minkowski spacetime), exist eternally, and have finite energy.

8.2. Solitons and components of the space of classical solutions

Classically, existence of solitons is related with existence of different components of the space of classical solutions of finite energy. In a connected component which does not contain a zero energy solution, the minimum of energy is often attained at a soliton. As an example of this, you may recall the situation discussed in Lecture II-1: a 2-dimensional scalar field theory with the potential $U = (\phi^2 - a^2)^2$. In this case, the space of t-independent classical solutions of finite energy has 4 components: $X_{++}, X_{-+}, X_{+-}, X_{--}$, where $X_{+-} = \{\phi : \phi(-\infty) = a, \phi(\infty) = -a\}$ etc. On two of these components, X_{+-} and X_{-+}, the energy is strictly positive, and its minimum is attained at 2 solitons, $\phi = f(x)$ and $\phi = -f(x)$, where $f(x)$ is the solution of the Newton equation $f'' = U'(f)$ for the potential $-U$, with boundary values a at $-\infty$ and $-a$ at ∞ (such a solution is defined uniquely up to translations).

Notice that solitons are not invariant under the Poincare group. But one often encounters solitons which have rotational symmetry in space, around some "center of mass". In a scalar field theory, this would mean that the group of symmetry of such a soliton is $P_s = SO(d-1) \times \mathbb{R}$. Since the Poincare group is $P = SO(d-1,1) \times \mathbb{R}^{d-1,1}$, the P-orbit $\mathcal{O}$ of the soliton in the space of solutions, i.e. the quotient P/P_s, is a $d-1$-dimensional vector bundle over the upper part of a 2-sheeted hyperboloid. It is easy to check the following.

1. This bundle is naturally isomorphic to the cotangent bundle.

2. The restriction of the symplectic structure on the space of solutions $\mathcal{O}$ is nondegenerate, and thus defines a symplectic structure on $\mathcal{O}$.

3 (normalization of symplectic form) Let m be the mass (i.e. the energy in the center of mass frame) of the soliton. Then there exists a P-equivariant symplectic diffeomorphism $\mathcal{O} \to T^*\mathcal{O}_m^+$, where $\mathcal{O}_m^+$ is the upper part of the hyperboloid $x^2 = -m^2$ in $\mathbb{R}^{d-1,1}$.

8.3. Solitons and quantization

Since classically solitons correspond to components in the space of solutions, quantum mechanically they should correspond to direct summands in the Hilbert space. As an example, consider a scalar field theory with the space X of classical solutions of finite energy, in which there is a component X_s containing a soliton $s \in X_s$. Then we have a symplectic embedding $T^*O_m^+ \to X_s$, where m is the mass of s. Let us assume that the minimum of energy at the image of this map is nondegenerate, in the sense that there is no other solutions of energy m, and the second derivative of the energy in a direction transversal to the image is positive.

In this case, in the weak coupling region we should expect that

1. The component $\mathcal{H}_s$ of the Hilbert space $\mathcal{H}$ of the quantum theory has no vacuum (because classically there is no P-invariant solution).

2. The Hamiltonian H on $\mathcal{H}_s$ satisfies the inequality $H \geq m'$, where m' is some positive mass parameter, such that $m' \to m$ in the weak coupling limit (in general, we should expect m to get quantum corrections). If the theory has a mass gap, we should expect that the states with $H = m$ form a space which is a quantization of $T^*O_m^+$, i.e. the irreducible representation of P of the form $L^2(O_m^+)$. We should also expect that the spectrum near m is discrete since the second derivative is positive.

Note that if the action (or energy) functional of the theory is multiplied by a constant C, the mass of the soliton is also multiplied by C, while masses of usual particles do not change. This means that in the classical approximation ($C \to \infty$), solitons are much heavier than usual particles. Therefore, a soliton cannot be seen in perturbation theory: the contribution to the correlation functions of an intermediate state containing a soliton is exponentially small (compared to the coupling constant) in the weak coupling limit.

8.4. Solitons in theories with fermions

In theories with fermions, the orbit of a soliton under P is often not the whole space of lowest energy states in the corresponding connected component of the space of solutions. For example, if the model is supersymmetric, it is clear apriori that the orbit is not the whole space of lowest energy solutions: the space of lowest energy states is the orbit of the superPoincare and not just the Poincare group.

Let us consider an example of such a situation. Consider the theory in 2 dimensions with a scalar and a pair of fermions:

$$\mathcal{L} = \frac{1}{2\lambda}\int d^2x(|d\phi|^2 + (\phi^2 - a^2)^2) + i\int d^2x(\psi_+\partial_-\psi_+ + \psi_-\partial_+\psi_- - g\phi\psi_+\psi_-). \tag{8.1}$$

Remark 1. For a suitable value of g this model is supersymmetric.

Remark 2. This model has a chiral symmetry $\phi \to -\phi$, $\psi_\pm \to \pm\psi_\pm$, which prohibits a mass term $m\psi_+\psi_-$ for the fermions.

Consider the soliton $\phi(x,t) = f(x)$ for the bosonic part of the theory. Fermionic extensions of this solution are functions of finite energy $\psi(x) = \begin{pmatrix}\psi_+(x)\\ \psi_-(x)\end{pmatrix}$ satisfying the Euler-Lagrange equations

$$\left[\partial_x\psi + \begin{pmatrix}0 & -gf(x)\\ -gf(x) & 0\end{pmatrix}\right]\psi = 0. \tag{8.2}$$

Thus,

$$\psi = \begin{pmatrix}\varepsilon\\ \varepsilon\end{pmatrix}e^{g\int_0^x f(y)dy} + \begin{pmatrix}\varepsilon'\\ -\varepsilon'\end{pmatrix}e^{-g\int_0^x f(y)dy}, \tag{8.3}$$

where $\varepsilon, \varepsilon'$ are odd variables. It is easy to see that only the solutions with $\varepsilon' = 0$ are in L^2, so the space of solutions of finite energy is 1-dimensional.

Thus, each pair of fermions ψ_+, ψ_- interacting with ϕ via (8.1) creates a fermionic degree of freedom in the space of configurations of minimal energy in the connected component of $f(x)$. Namely, if the number of such pairs is n then the space of minimal energy configurations is not $T^*O_m^+$, but the supermanifold $T^*O_m^+ \times \mathbb{R}^{0|n}$.

Therefore, if n is even, in quantum theory the space of lowest energy states in the corresponding component of the Hilbert space is $L^2(O_m^+) \otimes S$, where $S = S_+ + S_-$ is the spin representation of $Spin(n)$.

8.5. Solitons in 2+1 and 3+1 dimensions.

In spite of the difference between instantons and solitons, there is a connection between them. Namely, often an instanton in a Euclidean field theory in $d-1$ dimensions gives rise to a soliton in the Minkowski version of the same theory in d dimensions. Consider for example the 2+1-dimensional $U(1)$ gauge theory with a complex scalar:

$$\mathcal{L} = \int d^3x \left(\frac{F_A^2}{e^2} + \frac{1}{\lambda}(|d_A\phi|^2 + (|\phi|^2 - a^2)^2)\right). \tag{8.4}$$

In Lecture II-6 we saw that the 2-dimensional version of this theory has instantons with Chern classes 1 and -1. In the 2+1-dimensional theory, these instantons become solitons, and govern the lowest energy modes in the corresponding components of the Hilbert space as described above.

Now consider nonabelian gauge theory in 3+1 dimensions. Namely, consider an $SO(3)$ gauge theory with a boson in the 3-dimensional representation, and the Lagrangian

$$\mathcal{L} = \int d^4x \left(\frac{F_A^2}{4e^2} + \frac{1}{2\lambda}(|d_A\phi|^2 + (|\phi|^2 - a^2)^2)\right). \tag{8.5}$$

We considered in Lecture II-7 the 3-dimensional version of this theory. For convenience, we identified the spacetime $\mathbb{R}^3$ with the Lie algebra of the gauge group and with the space of values of ϕ (the bracket in the Lie algebra is the cross-product). This allows to write the scalar field ϕ and infinitesimal gauge transformations as vector fields on $\mathbb{R}^3$.

We found that in 3 dimensions this theory has an instanton in which ϕ is of the form $\phi = \frac{x}{r} f(r)$, $r = |x|$ (as we have explained, we identify the spacetime and the space of values of ϕ). In 4 dimensions this instanton will become a soliton. Such solitons are called magnetic monopoles.

In fact, we have not one soliton, but infinitely many, since the center of the soliton can be any point in $\mathbb{R}^3$. So the space of time-independent solitons is at least $\mathbb{R}^3$. In fact, this space is not $\mathbb{R}^3$ but $\mathbb{R}^3 \times S^1$. The reason is that there are some gauge symmetries compatible with spherical symmetry, which allow to produce new solitons out of old ones. Let us see how it happens.

First of all, recall that any field configuration (A, ϕ) of finite energy defines an integer topological invariant – "the first Chern class at infinity" c_1. Indeed, in order for the energy to be finite, we must have $|\phi| = a$ at infinity, so ϕ defines a map from a sphere at infinity in $\mathbb{R}^3$ to the sphere of radius a, and c_1 is the degree of this map. Another definition of c_1: the section ϕ defines at infinity a splitting of our 3-dimensional vector bundle into a direct sum $\phi \oplus \phi^\perp$ of a 1-dimensional and a 2-dimensional vector bundle. Thus, ϕ defines a reduction of the structure group to $SO(2) = U(1)$ at infinity. The number c_1 is the first Chern class of this bundle restricted to the infinite 2-sphere in $\mathbb{R}^3$.

For example, the soliton configuration discussed above has $c_1 = 1$.

In physical language, this topological phenomenon means that at spacial infinity the $SO(3)$ gauge symmetry is broken to $U(1)$. However, the remaining group $U(1)$ of transformations at infinity acts nontrivially on the space of classical solutions. In particular, it produces new solitons. To understand this action, let us consider the solitons (A, ϕ) with $c_1 = 1$, discussed above. Let us represent the infinitesimal operator of the group $U(1)$ by a spherically symmetric gauge symmetry: $\varepsilon = \frac{x}{r} g(r)$,

where g is some function. If the function $g(r)$ satisfies $g(r) \sim c_0 r$ at $r \to 0$ and $g(+\infty) = c$, then this formula defines a smooth gauge transformation which is "constant" at infinity (with respect to the reduction of structure group defined by the soliton). Then the action of ε is

$$A \to A - d_A\varepsilon, \phi \to \phi. \tag{8.6}$$

It is clear that $d_A\varepsilon$ cannot be identically zero, since the soliton connection A is not flat. On the other hand, if $c \in 2\pi\mathbf{Z}$, the connection $A - d_A\varepsilon$ is equivalent to A by the gauge transformation $e^{i\varepsilon}$ which vanishes at infinity, so the solutions (A, ϕ) and $(A - d_A\varepsilon, \phi)$ are the same in this case.

This shows that the space of time-dependent solitons with Chern class 1 at infinity is at least $\mathbb{R}^3 \times S^1$ (with no canonical zero on either $\mathbb{R}^3$ or S^1). One can show that in fact it is exactly that. We will denote the circle coordinate on this space by α.

By using the Poincare group transformations, we can generate solutions which are time-dependent and propagate at a constant speed. We can also perform a time dependent gauge transformation, which will make the α-coordinate time dependent, i.e. $\alpha = \alpha_0 + st$, $s \in \mathbb{R}$. As a result, the space of time dependent solitons is the product $T^*O \times T^*S^1$.

8.6. The 3+1-dimensional theory with the θ-angle

Consider the 3+1-dimensional theory of the previous section with the θ-angle, i.e. let us add to the (Minkowski) Lagrangian a term

$$-\frac{\theta}{16\pi^2}\int \mathrm{Tr}(F \wedge F). \tag{8.7}$$

To give this term a topological interpretation, let us compactify the time and consider the theory on the spacetime $\mathbb{R}^3 \times S^1$. In this case, for any field configuration of finite energy, besides the first Chern class c_1 at infinity (which is also called the monopole number, or the hedgehog number), we can define another integer topological invariant – the second Chern class c_2, which is given by the integral $\frac{1}{8\pi}\int Tr(F \wedge F)$. Thus, classically, term (8.7) just counts the second Chern class of the bundle. Therefore, quantum mechanically, it weights the contribution from bundles with $c_2 = k$ to the path integral with $e^{ik\theta}$.

It is clear that this theory has the same time-independent solitons as the theory without θ, since the added term is topological. However, since time has been compactified, the time dependent solitons which were discussed in the previous section have to satisfy the equality $\alpha = \alpha_0 + \frac{n}{T}t$, where n is an integer, which we will call the winding number, and T is the circumference of the compactified time axis. In other words, the parameter s defined in the previous section has to have the form $s = n/T$.

Claim *If the monopele number of a soliton configuration is k and the winding number is n then $c_2 = kn$.*

The proof is by a direct calculation.

Now consider 3 operators in our theory:

1. $\frac{\partial}{\partial\alpha}$. (This operator is the generator of the Lie algebra of the unbroken $U(1)$).
2. The electric charge

$$Q_{el} = \frac{1}{ae^2}\int_{\mathbb{R}^3} Tr(d_A\phi \wedge *F_A). \tag{8.8}$$

3. The magnetic charge

$$Q_{mag} = \frac{1}{4\pi a}\int_{\mathbb{R}^3} Tr(d_A\phi \wedge F_A). \tag{8.9}$$

In perturbation theory, $Q_{mag} = 0$ (as nonzero c_2 requires big action), and $Q_{el} = -i\frac{\partial}{\partial\alpha}$. Thus, in perturbation theory Q_{el} has integer eigenvalues.

Let us see, however, what happens nonperturbatively, i.e. when we take into account field configurations with nonzero c_2.

We compute in the gauge where $A_0 = 0$. It is easy to compute that classically we have $\frac{\partial}{\partial\alpha} = \int Tr(d_A\phi\frac{\delta}{\delta A})$. Therefore, the operator $-i\frac{\partial}{\partial\alpha}$ is the charge for the current $J = Tr(d_A\phi \wedge \pi_A)$, where π_A is the conjugate (momentum) variable for A.

Now compute π_A:

$$\pi_A = \frac{\delta\mathcal{L}}{\delta A_t} = \frac{A_t \wedge dt}{e^2} - \frac{\theta}{8\pi^2}F_A. \tag{8.10}$$

Thus, we get

$$-i\frac{\partial}{\partial\alpha} = Q_{el} - \frac{\theta}{2\pi}Q_{mag}. \tag{8.11}$$

Since the spectrum of $-i\frac{\partial}{\partial\alpha}$ is integer, we get $Q_{el} = \frac{\theta}{2\pi}Q_{mag} \bmod \mathbb{Z}$. But Q_{mag} is also an integer, since it is a topological invariant classically. Thus, Q_{el} need not be an integer beyond perturbation theory. In other words, there exist states (of very high energy $\sim 1/\hbar$) on which Q_{el} is not an integer. So the electric charge is discretized, but in presence of magnetic monopoles we do not expect an integral electric charge.

It is not hard to show that operators Q_{el}, Q_{mag} commute. (Classically, it is obvious, as Q_{mag} is a locally constant function). Thus, the joint spectrum of them is a lattice in $\mathbb{R}^2$. If we consider the family of theories parametrized by $\theta \in S^1$, the monodromy transformation of this lattice around the circle is given by the matrix $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ in the basis Q_{mag}, Q_{el}.

In the second part of this lecture we will explain (in the free theory) how to extend this monodromy representation to an action of $SL_2(\mathbb{Z})$. I.e. how to construct a family of theories parametrized by a complex parameter $\tau = \frac{\theta}{2\pi} + ir$ modulo modular transformations, such that the monodromy representation is the standard action of $SL_2(\mathbb{Z})$ on a 2-dimensional lattice.

Lecture II-8, part II: Abelian Duality in Four Dimensions and $\mathrm{Sl}(2,\mathbb{Z})$

Edward Witten[*1]

1 Duality and $\mathrm{Sl}(2,\mathbb{Z})$

In this second part of lecture II-8, we discuss abelian duality in four dimensions, and give an application to an $\mathrm{Sl}(2,\mathbb{Z})$ symmetry of the free $U(1)$ theory in four dimensions. We postpone discussion of $\mathrm{Sl}(2,\mathbb{Z})$ symmetries of non-free theories to a later lecture, since all known examples of that involve supersymmetry.

We work with a $U(1)$ bundle $\mathscr{L}$ on a 4-manifold M, and a connection A on $\mathscr{L}$, whose curvature is $F{=}F_A$. The gauge theory Lagrangian (in Euclidean signature) including the topological term is

$$\begin{aligned}\mathscr{L}(A) &= \frac{1}{4\mathrm{e}^2}\int d^4x\ \sqrt{g}F_{mn}F^{mn} + \frac{i\theta}{16\pi^2}\int d^4x\ \sqrt{g}\epsilon_{mnpq}F^{mn}F^{pq}\\ &= \frac{1}{2\mathrm{e}^2}\int F_A\wedge *F_A + \frac{i\theta}{4\pi^2}\int F_A\wedge F_A.\end{aligned} \tag{1.1}$$

We have used the standard normalization on the kinetic term, and have normalized the topological term so that replacing θ by $\theta+2\pi$ does not change the physics. (This property of the topological term derives from the fact that $c_1(\mathscr{L})^2 = \int (F_A/2\pi)\wedge(F_A/2\pi)$ is always an integer. Notice that on a spin manifold, $c_1(\mathscr{L})^2$ is always an *even* integer, and we gain an additional equivalence under replacement of θ by $\theta+\pi$.)

Let $\tau = \frac{\theta}{\pi} + \frac{2\pi i}{e^2} \in \mathfrak{h}$. As we have just observed, $\tau\mapsto\tau+2$ is a symmetry of this theory, and $\tau\mapsto\tau+1$ is a symmetry when working on a spin manifold. To extend this to an $\mathrm{Sl}(2,\mathbb{Z})$ action (in the spin manifold case) we also need a symmetry which maps τ to $-1/\tau$; this will be given by a *duality* transformation $F_A\leftrightarrow *F_C$ (with C being a new "dual" connection).

The computations for this duality transformation are similar to those in lecture II-7. We begin by defining $F_\pm = \frac{1}{2}(F_A \pm *F_A)$, and rewriting our Lagrangian (1.1) as

$$\begin{aligned}\mathscr{L}(A) &= \frac{i\overline{\tau}}{4\pi}\int F_+\wedge F_+ + \frac{i\tau}{4\pi}\int F_-\wedge F_-\\ &= \frac{i\overline{\tau}}{4\pi}\int \|F_+\|^2 - \frac{i\tau}{4\pi}\int\|F_-\|^2.\end{aligned} \tag{1.2}$$

Letting $\mathscr{G}$ denote the gauge group associated to A, the partition function for this theory can be written as

$$Z(\tau) = \frac{1}{\mathrm{vol}(\mathscr{G})}\sum_{\mathscr{L}}\int \mathscr{D}A\, e^{-\frac{i\overline{\tau}}{4\pi}\int\|F_+\|^2+\frac{i\tau}{4\pi}\int\|F_-\|^2}. \tag{1.3}$$

Our earlier examples of duality began with a theory of a scalar field ϕ which entered into the Lagrangian only through its derivative $d\phi$ so that the theory had a symmetry under $\phi\mapsto\phi+c$ (with c constant); the first step in the duality transformation was to gauge this symmetry, introducing also an appropriate Lagrange multiplier field.

The present theory is already a gauge theory, being a theory of a connection A which enters into the Lagrangian only through its curvature F_A, so that there is a symmetry under $A\mapsto A+B$ (with B a flat connection). We want to do the analogue of gauging this symmetry, by allowing B to be an

[1*] Notes by David R. Morrison

arbitary connection on an arbitrary bundle, introducing a kind of "exotic gauge field" G which is a 2-form field, and extending the symmetry to

$$\begin{aligned} A &\to A + B \\ G &\to G + F_B. \end{aligned} \tag{1.4}$$

Then $\mathcal{F} := F_A - G$ plays the role of the "gauge-invariant" quantity, analogous to the covariant derivative of a scalar field. It is to be stressed that two G fields will be considered gauge-equivalent if they differ by $G \to G + F_B$ for F_B the curvature of any connection on any line bundle. In our analysis, we will assume for simplicity that there is no torsion in $H^2(M)$.

We need a "gauge-invariant" extension of our Lagrangian. We might try

$$\mathcal{L}(A, G) = \frac{i\overline{\tau}}{4\pi} \int \|\mathcal{F}_+\|^2 - \frac{i\tau}{4\pi} \int \|\mathcal{F}_-\|^2, \tag{1.5}$$

but this is too simple (because, for example, we could gauge $\mathcal{F}$ to zero). To improve this, we introduce a new connection C on a line bundle $\mathscr{N}$, with curvature F_C, and consider the Lagrangian

$$\mathcal{L}(A, G, C) = \frac{i\overline{\tau}}{4\pi} \int \|\mathcal{F}_+\|^2 - \frac{i\tau}{4\pi} \int \|\mathcal{F}_-\|^2 - \frac{i}{2\pi} \int F_C \wedge G. \tag{1.6}$$

The partition function for this new theory can be represented as a path integral, which includes sectors associated to all choices of bundles $\mathscr{L}$ and $\mathscr{N}$:

$$\frac{1}{\mathrm{vol}(\widetilde{\mathcal{G}})} \frac{1}{\mathrm{vol}(\mathcal{G})} \frac{1}{\mathrm{vol}(\mathcal{G}_C)} \sum_{\mathscr{L},\mathscr{N}} \int \mathcal{D}A\, \mathcal{D}G\, \mathcal{D}C\, e^{-\frac{i\overline{\tau}}{4\pi}\int \|\mathcal{F}_+\|^2 + \frac{i\tau}{4\pi}\int \|\mathcal{F}_-\|^2 + \frac{i}{2\pi}\int F_C \wedge G}, \tag{1.7}$$

where $\mathcal{G}$ and $\mathcal{G}_C$ denote the gauge groups associated to A and C, respectively, and $\widetilde{\mathcal{G}}$ denotes the "exotic" gauge group.

To see that this new theory is equivalent to the original one, we first do the C-integral in (1.7): write $C = C_0 + C'$, for C_0 a fixed connection on the line bundle $\mathscr{N}$. Then the C' integral is

$$\frac{1}{\mathrm{vol}\,\mathcal{G}_C} \int \mathcal{D}C'\, e^{\frac{i}{2\pi}\int C' \wedge dG} = \delta(dG). \tag{1.8}$$

Thus, when we sum over $\mathscr{N}$ we find

$$\frac{1}{\mathrm{vol}\,\mathcal{G}_C} \sum_{\mathscr{N}} \int \mathcal{D}C e^{-\frac{i}{2\pi}\int F_C \wedge G} = \sum_{x \in H^2(M)} e^{i(x,G)} \delta(dG) = \delta\Big(\Big[\frac{G}{2\pi}\Big] \in \mathbb{Z}\Big)\delta(dG). \tag{1.9}$$

The conditions that $dG = 0$ and that $[G/2\pi]$ is an integral class precisely mean that G is of the form F_B for some connection on some line bundle and hence that G can be gauged to zero. After doing this, it follows that the partition function (1.7) coincides with $Z(\tau)$, and we recover the original theory.

Alternatively, we can evaluate the partition function (1.7) by gauging A to 0, using the "exotic" gauge invariance (which has an ordinary gauge invariance as an ambiguity). This leaves the path integral

$$\frac{1}{\mathrm{vol}\,\mathcal{G}_C} \sum_{\mathscr{N}} \int \mathcal{D}G \int \mathcal{D}C\, e^{-\frac{i\overline{\tau}}{4\pi}\int \|G_+\|^2 + \frac{i\tau}{4\pi}\int \|G_-\|^2 + \frac{i}{2\pi}\int F_C \wedge G}. \tag{1.10}$$

To evaluate the G integral, we complete the square, bearing in mind that

$$\int F_C \wedge G = \int (F_{C+} \wedge *G_+ - F_{C-} \wedge *G_-). \tag{1.11}$$

In fact, if we define $G' = G - \frac{1}{\overline{\tau}}F_{C+} + \frac{1}{\tau}F_{C-}$, then we can write the exponent from eq. (1.10) as

$$-\frac{i\overline{\tau}}{4\pi}\int \|G'_+\|^2 + \frac{i\tau}{4\pi}\int \|G'_-\|^2 + \frac{i}{4\pi\overline{\tau}}\int \|F_{C+}\|^2 - \frac{i}{4\pi\tau}\int \|F_{C-}\|^2. \tag{1.12}$$

When we carry out the G' integral, the first two terms give a Gaussian integral which contributes to the overall normalization; integrating out G' leaves the path integral

$$\frac{1}{\mathrm{vol}\,\mathcal{G}_C}\sum_{\mathscr{N}}\int \mathcal{D}C\, e^{\frac{i}{4\pi\overline{\tau}}\int \|F_{C+}\|^2 - \frac{i}{4\pi\tau}\int \|F_{C-}\|^2}. \tag{1.13}$$

This is the same as the original path integral, but with τ replaced by $-1/\tau$, precisely what we wanted to show.

As we did in the case of two dimensions, it is possible to analyze the τ-dependence of the normalization of the path-integral, and obtain further interesting results. Some hint of the flavor of the results to be obtained this way is seen if we evaluate the Gaussian integral indicated above, which yields

$$\left(\frac{2\pi}{\sqrt{i\overline{\tau}}}\right)^{n_{2+}}\left(\frac{2\pi}{\sqrt{-i\tau}}\right)^{n_{2-}}, \tag{1.14}$$

where $n_{2\pm}$ denote the numbers of self-dual and anti-self-dual 2-forms. Of course, these numbers are infinite, so there must be some cancellation against other normalization factors. When this is worked out in detail,[2] the result is found to be

$$Z(\tau) = \tau^{-\frac{\chi+\sigma}{4}}\overline{\tau}^{-\frac{\chi-\sigma}{4}}Z(-1/\tau), \tag{1.15}$$

where χ and σ are the Euler number and signature of M, respectively. Thus, the partition function $Z(\tau)$ is actually a modular form for $\mathrm{Sl}(2,\mathbb{Z})$ (or for a subgroup, when the manifold is not spin) of weight $(\frac{\chi+\sigma}{4}, \frac{\chi-\sigma}{4})$.

We can also follow certain operator insertions through the duality transformation, as we did in lower dimensions. An insertion of $F_\pm$ in the original theory can be realized by inserting $\mathcal{F}_\pm = F_\pm - G$ in the extended theory, which can be written

$$\mathcal{F}_+ = F_+ - G'_+ - \frac{1}{\overline{\tau}}F_{C+}, \text{ or } \mathcal{F}_- = F_- - G'_- + \frac{1}{\tau}F_{C-}, \tag{1.16}$$

respectively, after making the change of variables to G'. Thus, when we gauge A to zero, and integrate out G', we are left with operator insertions proportional to $F_{C\pm}$, namely:

$$F_+ \mapsto (-1/\overline{\tau})F_{C+}, \text{ and } F_- \mapsto (1/\tau)F_{C-}. \tag{1.17}$$

Notice that as a consequence of the τ-dependence of these mappings, a correlation function involving insertions of F_+ and F_- will have a different modular weight than that of the partition function.

2 The Hamiltonian formalism

Returning to the case that the gauge group is $U(1)$, let us briefly discuss abelian four-dimensional duality in a Hamiltonian framework. Take a 4-manifold of the form $M_3 \times \mathbb{R}$, where $\mathbb{R}$ is a timelike direction. Note that this is a spin manifold, so we expect full $Sl(2,\mathbb{Z})$ symmetry. For simplicity we

[2] E. Witten, *On S-duality in abelian gauge theory*, Selecta Math (N.S.) **1** (1995), 383–410.

suppose that there is no torsion in $H_1(M_3)$. Each class $x \in H^2(M_3)$ determines a complex line bundle $\mathscr{L}_x$ on the 3-manifold M_3 (satisfying $c_1(\mathscr{L}_x) = x$). The Hilbert space for our theory on the 3-manifold M_3 can be written in the form

$$\mathcal{H}_\tau(M_3) = \bigoplus_{x \in H^2(M_3,\mathbb{Z})} \mathcal{H}_x, \tag{2.1}$$

where $\mathcal{H}_x$ is the Hilbert space which comes from quantizing connections on $\mathscr{L}_x$. (On the left, we have explicitly indicated the dependence on the coupling constant τ.) To construct $\mathcal{H}_x$, write an arbitrary connection in the form $A = A_0 + \beta$, where A_0 is a harmonic connection (a connection whose curvature is a harmonic two-form) and β is a 1-form which is co-closed. Let $\mathcal{T}_x$ be the space of harmonic connections on the line bundle $\mathcal{L}_x$. Then the quantization yields

$$\mathcal{H}_x = \mathcal{H}_\beta \otimes L^2(\mathcal{T}_x). \tag{2.2}$$

Here $\mathcal{H}_\beta$ is a Hilbert space obtained by quantizing the space of β's, and $L^2(\mathcal{T}_x)$ is just the space of L^2 functions on $\mathcal{T}_x$.

Note that the factor $\mathcal{H}_\beta$ is independent of x, since the space of co-closed one-forms is defined with no reference to x. Duality maps $\mathcal{H}_\beta$ to itself while acting separately on $\mathcal{H}' = \oplus_x L^2(\mathcal{T}_x)$. The duality action on $\mathcal{H}_\beta$ follows from the operator mapping in (1.17).

The action of duality on $\mathcal{H}'$ can be described as follows. Note that the $\mathcal{T}_x$'s are all principal homogeneous spaces acted on by the torus $H^1(M_3, \mathbb{R}/\mathbb{Z})$, which parametrizes flat line bundles on M_3; the action is defined by tensoring any given line bundle with connection by a flat line bundle determined by a point in $H^1(M_3, \mathbb{R}/\mathbb{Z})$. Let y denote a character of the abelian group $H^1(M_3, \mathbb{R}/\mathbb{Z})$. There is a decomposition $L^2(\mathcal{T}_x) = \oplus_y \mathcal{T}_{x,y}$, where $\mathcal{T}_{x,y}$ is the subspace of $L^2(\mathcal{T}_x)$ transforming in the character y. Each $\mathcal{T}_{x,y}$ is one-dimensional. Hence

$$\mathcal{H}' = \oplus_{x,y} \mathcal{T}_{x,y} \tag{2.3}$$

Note that, by Poincaré and Pontryagin duality, the character group of $H^1(M_3, \mathbb{R}/\mathbb{Z})$ is $H^2(M_3, \mathbb{Z})$. Thus, x and y take values in the same space. It is hence possible to exchange them, and this is what the $\tau \to -1/\tau$ transformation does (more precisely, it acts by $(x, y) \to (-y, x)$). Thus duality exchanges a classical notion – the decomposition with respect to x – with a quantum notion – the decomposition with respect to y. The claim about how the duality acts will be justified below where we introduce the operators Q_E and Q_M.

Upon quantization—and suppressing θ for a moment—one writes the four-dimensional curvature as $F'_A + e^2 \pi_A dt$, where F'_A is a two-form on M_3 and π_A—a one-form on M_3—is the momentum conjugate to the connection A. The Hamiltonian becomes

$$H = \frac{1}{2\mathrm{e}^2} \int F_{A_0}^2 + \frac{\mathrm{e}^2}{2} \nabla_{A_0} + H(\beta). \tag{2.4}$$

Here $H(\beta)$ is the part of the Hamiltonian that acts on $\mathcal{H}_\beta$. The other terms act on $\mathcal{H}'$. The first term is the magnetic energy of the harmonic connection A_0; it comes from the part of the Lagrangian quadratic in F'_A and is a multiple of $\int_{M_3} x \wedge * x$. The second term, which comes from the part of the Lagrangian quadratic in π_A, is the electric energy, the Laplacian on $\mathcal{T}_x$; it is a multiple of $\int_{M_3} y \wedge * y$.

Including the θ term shifts the quantization. In fact, the canonical momentum $F_A^\vee = * \pi_A$ as determined from the original Lagrangian (1.1) is

$$F_A^\vee = 2\pi i \frac{\delta S}{\delta F_A} = \frac{2\pi i}{\mathrm{e}^2} * F_A - \frac{\theta}{\pi} F_A \tag{2.5}$$

At non-zero θ, one has not $\mathcal{H}' = \oplus_x L^2(\mathcal{T}_x)$ but $\mathcal{H}' = \oplus_x \Gamma_{L^2}(\mathcal{T}_x, \mathcal{S}_\theta)$, where $\mathcal{S}_\theta$ is a certain flat line bundle over $\mathcal{T}_x$ and Γ_{L^2} is the space of L^2 sections. I leave it as an exercise to the reader to identify $\mathcal{S}_\theta$. Of course, $\mathcal{S}_\theta$ is trivial at $\theta = 0$ and only depends on θ modulo 2π.

Rewriting the formula for $F_A^\vee$ in terms of τ, we can determine how this operator transforms under $\tau \to -1/\tau$. Indeed, under the operator mapping (1.17) one gets

$$F_A^\vee = -\overline{\tau} F_+ + \tau F_- \mapsto F_{C+} + F_{C-} = F_C. \tag{2.6}$$

We will also need the "dual" version of this computation:

$$F_A = F_+ + F_- \mapsto \overline{(-1/\tau)} F_{C+} - (-1/\tau) F_{C-} = -F_C^\vee. \tag{2.7}$$

In the Hamiltonian formalism, for any 2-cycle $\Sigma \subset M_3$ we can define associated "electric" and "magnetic" operators on the Hilbert space $\mathcal{H}_\tau(M_3)$ for any 2-cycle $\Sigma \subset M_3$, by

$$\begin{aligned} Q_E(\Sigma) &= \int_\Sigma \frac{F_A^\vee}{2\pi} = \int_\Sigma \frac{F_C}{2\pi} \\ Q_M(\Sigma) &= \int_\Sigma \frac{F_A}{2\pi} = -\int_\Sigma \frac{F_C^\vee}{2\pi}. \end{aligned} \tag{2.8}$$

(These operators only depend on the class of Σ in $H_2(M_3, \mathbb{Z})$.) Clearly, under $\tau \to -1/\tau$, that is under $A \to C$, one has $Q_E \to Q_M$, $Q_M \to -Q_E$. Since x and y are the eigenvalues of Q_M and Q_E, this means that $\tau \to -1/\tau$ acts by $(x, y) \to (-y, x)$. Moreover, from the explicit formula (2.5) for $F_A^\vee$, one sees that when θ is increased by 2π (an operation which leaves the Hilbert space unchanged), the operator Q_M is unaltered, but the operator Q_E maps to $Q_E + Q_M$.

The statements made in the last paragraph can be combined to the following: $Sl(2, \mathbb{Z})$ acts on $\mathcal{H}'$ via the natural action of $Sl(2, \mathbb{Z})$ on the pair (x, y).

Lecture II-9: Wilson loops,'t Hooft's loops, and 't Hooft's model of confinement

Edward Witten
Notes by Pavel Etingof and David Kazhdan

9.1. 't Hooft loop operator Let us recall abelian duality in 4 dimensions, which we discussed two lectures ago. Consider a free $U(1)$ gauge theory with a θ-angle. Thus we have two dimensionless couplings e, θ which combine into a single complex coupling $\tau = \frac{2\pi i}{e^2} + \frac{\theta}{\pi}$, and the Lagrangian is

$$\mathcal{L} = \int_M \Big(\frac{i\overline{\tau}}{4\pi} F_+^2 - \frac{i\tau}{4\pi} F_-^2 \Big), \tag{9.1}$$

where F is the curvature of a $U(1)$ connection A and F_+, F_- are the selfdual and antiselfdual parts of the curvature. We have seen that if the spacetime M is a spin manifold then this theory is "modular invariant" as a function of τ. One modular symmetry $\tau \to \tau + 1$ is obvious, as it corresponds to shifting the θ-angle by π, which does nothing because c_1^2 for a spin manifold is even. (On a manifold that is not a spin manifold, the symmetry would be only $\tau \to \tau + 2$.) The symmetry under the second generator of the modular group, $\tau \to -1/\tau$, is more interesting and corresponds to electromagnetic duality discovered by Maxwell. More precisely, this means that the theory of a connection A with coupling constant τ is identical both classically and quantum mechanically to the same theory with coupling $-1/\tau$ and the connection B such that $dA = const * dB$. Now, like two lectures ago, we want to see what happens to operators under this duality. In particular, we want to know what happens to the Wilson loop operator.

Recall that the Wilson loop operator has the form $W_\gamma(C) = e^{i\gamma Hol_C(A)}$, where $Hol_C(A)$ denotes the integral of the connection A along a closed oriented curve C in the spacetime M. This operator is gauge-invariant and well-defined if γ is an integer, or for any real γ if the curve C is homotopically trivial in M. (More generally, there could be several components C_i with real numbers γ_i, and the condition is that $\sum_i \gamma_i C_i$ should be an integral class in $H_1(M)$.) Matrix elements of this operator are computed, as usual, by inserting the above exponential into the path integral. Similarly to what we found in similar problems in two and three dimensions, we should get that the dual description of the Wilson loop is a recipe which says that rather than insert in the path integral an object living on C, we should integrate over connections having a singularity along C.

The precise answer is the following. For any curve C the expectation value $\langle W_\gamma(C) O_1 ... O_n \rangle$ equals to $\int e^{-\mathcal{L}(B)} O_1 ... O_n DB$, where the integral is taken over connections on $M \setminus C$ such that the integral of the curvature of B over a small 2-sphere S in a normal 3-space to C at any point equals $2\pi\gamma$.

Let us prove this. We assume that C is a boundary. Let D be a 2-chain whose boundary is C. Recall the calculation from the lecture on Abelian duality: our fields are A – the original connection, G- the 2-form, and B – the dual connection. We have

$$\int e^{-L(A)} W_\gamma(C) DA = \int DA\, DG\, DB e^{\frac{-i\overline{\tau}}{4\pi}\int \mathcal{F}_+^2 + \frac{i\tau}{4\pi}\int \mathcal{F}_-^2} e^{\frac{i}{2\pi}\int G\wedge F_B} e^{i\gamma\int_D (F_A - G)}, \tag{9.1}$$

where $\mathcal{F} = F_A - G$, and the last factor corresponds to the insertion of the Wilson loop (recall from the abelian duality discussion that the Wilson loop classically is $e^{i\gamma\int_D F_A}$, and F_A is to be replaced with $F_A - G$ in the extended theory). Gauging A to 0, we get

$$\begin{gathered} \int e^{-L(A)} W_\gamma(C) DA = \\ \int DG\, DB e^{\frac{-i\overline{\tau}}{4\pi}\int G_+^2 + \frac{i\tau}{4\pi}\int G_-^2} e^{\frac{i}{2\pi}\int G\wedge F_B} e^{-i\gamma\int_D G} = \int DG\, DB e^{\frac{-i\overline{\tau}}{4\pi}\int G_+^2 + \frac{i\tau}{4\pi}\int G_-^2} e^{i\int G\wedge(\frac{1}{2\pi}F_B - \gamma[D])}, \end{gathered} \tag{9.2}$$

where $[D]$ is the delta-function of D. Now we define $\tilde{F}_B := F_B - 2\pi\gamma[D]$. Then after integrating out G, (9.2) can be written as

$$\int e^{-L(A)} W_\gamma(C) DA = \int DB e^{\frac{i}{4\pi\tau}\int (\tilde{F}_B)_+^2 + \frac{-i}{4\pi\tau}\int (\tilde{F}_B)_-^2}, \tag{9.3}$$

Thus, the effect of the insertion $W_\gamma(C)$ is that F_B is replaced in the final answer by $\tilde{F}_B$. So B is now a connection on a line bundle with singularity along C, as discussed above.

Note that if C is not a boundary then $\langle W_\gamma(C)\rangle = 0$ (even with insertion of any number of local operators). Indeed, we have symmetry $A \to A + d\phi$, and ϕ does not have to be globally defined as a map to a circle; in fact, $d\phi$ can be any closed one-form. So if C is not a boundary then we can choose $d\phi$ in such a way that the operator $W_\gamma(C)$ will multiply by $e^{i\alpha}$ for some nonzero α. Hence its expectation value (even with inclusion of local operators, which are invariant under this transformation) vanishes. More generally, the correlator $\langle W_{\gamma_1}(C_1)...W_{\gamma_n}(C_n)\rangle$ (with any local operators) is zero if $\sum \gamma_i C_i \neq 0$ in $H_1(M)$, where M is the spacetime. As noted before, for the product of operators in question to be well-defined, we only need $\sum \gamma_i C_i$ to be an integral class.

From our construction so far, for any curve C we have two operators:

1) $W_\gamma(C) = e^{i\gamma\int_C A}$

2) $T_\gamma(C) = e^{i\gamma\int_C B}$.

The second operator, which is dual to the Wilson loop, is called the 't Hooft loop operator.

9.2. Hilbert space interpretation of the 't Hooft loop operator Now let us consider this picture from the Hamiltonian point of view. Then the spacetime M has the form $M = M^3 \times \mathbb{R}$ with Minkowski metric. Let C, C' be two nonintersecting closed simple curves in M^3. They define operators $W_\gamma(C)$ and $T_{\gamma'}(C')$ on the Hilbert space $\mathcal{H}$. The following commutation relation for these operators is due to 't Hooft:

$$W_\gamma(C) T_{\gamma'}(C') = e^{2\pi i\gamma\gamma' l(C,C')} T_{\gamma'}(C') W_\gamma(C). \tag{9.4}$$

Let us prove this formula. Let us work in terms of the original connection A. Then the Hilbert space consists of wave functions $\Psi(A)$. In this realization, the Wilson loop operator $W_\gamma(A)$ is simply the operator of multiplication by $e^{i\gamma\int A}$.

However, the 't Hooft loop operator is a bit harder to define. To do this, consider the homomorphism $\pi_1(M^3 \setminus C', x_0) \to \mathbb{Z}$ given by the linking number with C'. Let π_1^0 be the kernel of this homomorphism and X be the $\mathbb{Z}$-cover of $M_3 \setminus C'$ corresponding to π_1^0. Let ϕ be a function $X \to U(1)$ such that the monodromy corresponding to the generator of $\mathbb{Z}$ is $e^{i\gamma}$. Any two such functions differ by a gauge transformation, but ϕ itself is not an honest gauge transformation. Then it is not difficult to check that the 't Hooft loop operator $T_{\gamma'}(C')$ is just the "illegal" gauge transformation by ϕ:

$$(T_{\gamma'}(C')\Psi)(A) = \Psi(A^\phi). \tag{9.5}$$

Note that this is well-defined since any two such ϕ's differ by an honest gauge transformation.

Now formula (9.4) is clear since $TWT^{-1} = e^{i\gamma_C\int d\phi} W$, because of the way that T transforms the connection in the definition of W.

9.3. The 2+1-dimensional analogue of the 3+1-dimensional picture Consider the 2+1-dimensional analogue of this picture. As we saw before, in 2+1 dimensions the theory of a scalar field ϕ is dual to a gauge theory of the dual field A. The path integral in ϕ with insertion of $e^{i\phi(x)}$ is the same as path

integral in A where A is a connection on $M \setminus x$ which has $\int F = 2\pi$, where the integral is over a small sphere around x. Thus, the operator $e^{i\phi}$ corresponds to a magnetic monopole in gauge theory.

Now consider the 3-dimensional cosine theory, defined by the path integral

$$\int D\phi e^{-\int (|d\phi|^2 + \varepsilon(e^{i\phi}+e^{-i\phi}))}. \tag{9.6}$$

Decomposing this path integral in a power series, and passing to the dual variable A, we get the sum

$$\sum_{m,n} \varepsilon^{m+n} \int \frac{dx_1 ... dx_m}{m!} \int \frac{dy_1 ... dy_n}{n!} \int_{\mathcal{A}_{x,y}} e^{-\int F_A^2}, \tag{9.7}$$

where $\mathcal{A}_{x,y}$ is the space of connections with monopoles at x_i and antimonopoles at y_j. Thus the cosine theory maps to the theory with monopoles. We saw this more computationally when we discussed the Polyakov model two lectures ago.

9.4. The model of confinement Now we discuss a picture of confinement developed by 't Hooft. In general we don't assume that the gauge group is abelian. Recall the definition of confinement. We have a gauge group G and with universal cover $\widehat{G}$. We assume that G is the quotient of $\widehat{G}$ which acts faithfully on all fields in the Lagrangian. We let R be a representation of $\widehat{G}$. As we discussed before, if there is a mass gap, there are two usual patterns of decay of the expectation value $\langle W_R(C)\rangle$ of the Wilson line operator corresponding to the representation R as C gets big:

Pattern 1:

$$\langle W_R(C)\rangle \sim e^{-\lambda \,\mathrm{Length}(C)} \tag{9.8}$$

Pattern 2:

$$\langle W_R(C)\rangle \sim e^{-\lambda \,\mathrm{Area}(C)} \tag{9.9}$$

(here the parameter γ is a fixed nonzero number and the area of C means the minimal area of the spanning surface). The first regime is called the Higgs regime (the length law) and the second one is called the confinement regime (the area law).

As we discussed in the previous lecture on confinement, the first regime is the case when R is a representation of G itself, and to see confinement one needs to consider the case when R is a representation of $\widehat{G}$ but not of G. Thus interesting $W_R(C)$ correspond to elements of $\pi_1(G)^*$.

Now let us consider the 't Hooft loop operator $T_\gamma(C)$. It is defined for any G by analogy with the definition in the abelian case. We fix an element $\gamma \in \pi_1(G)$. Recall that G-bundles on a two-sphere S^2 are classified by a characteristic class that takes values in $H^2(S^2, \pi_1(G))$, which is canonically isomorphic to $\pi_1(G)$. The choice of γ therefore canonically determines an isomorphism class of G-bundles on S^2. We can now define the 't Hooft operator: a path integral with insertion of $T_\gamma(C)$ is computed by integrating over connections on $M \setminus C$ which have the property that when restricted to a small sphere S that links C, the bundle has characteristic class γ.

't Hooft's idea was to consider $T_\gamma(C)$ instead of $W_R(C)$ and find conditions under which there is an area law for its expectation value. This occurs, as he showed, for certain Higgs theories. Then, 't Hooft proposed (following earlier ideas of Nambu, Mandelstam, and others) that confinement would be related to the Higgs mechanism by a duality that maps 't Hooft loop operators into Wilson loop operators. This does not explain confinement, but it reformulates the problem: to reduce the mysterious phenomenon of confinement to the much more easily understood Higgs phenomenon, one must understand the nonlinear duality that exchanges 't Hooft and Wilson loop operators.

To illustrate the area law for the 't Hooft loop in Higgs theories, we consider a familiar example: the $U(1)$ gauge theory with a charged complex scalar ϕ (of charge 1). The Lagrangian is

$$\int (\frac{|F|^2}{4e^2} + |D_A\phi|^2 + V(\phi\bar{\phi})), \tag{9.10}$$

where V is a (quartic) potential. We will study the 't Hooft loop $T_\gamma(C)$, where $\gamma \in \mathbb{Z}$. Thus A is a connection and ϕ is a section for a hermitian line bundle over $M \setminus C$ such that it has first Chern class γ when restricted to a small sphere linking C. We will compute $\langle T_\gamma(C)\rangle$ for two classes of V:

1) $V = \lambda(\phi\bar{\phi} + a^2)^2$;

2) $V = \lambda(\phi\bar{\phi} - a^2)^2$.

This theory was considered in Lecture 2. Recall the results of this consideration.

Case 1. In the infrared the theory behaves like the product of the theory of a free massive field with a free gauge theory. In particular, there is no mass gap. Thus, we can calculate $\langle T_\gamma(C)\rangle$ for large C using the free theory. But in the free theory this expectation value is the same as $\langle W_\gamma(C)\rangle$ in the dual theory. It is easy to see that the expectation values of both W_γ and T_γ behave according to the length law, because of Coulomb law of charge interaction. A theory behaving in this way is said to be in the "Coloumb phase."

Case 2. In the infrared this theory has breaking of gauge symmetry and a Higgs mechanism. In particular, speaking classically, we have a circle of vacua, and at each of these vacua the low energy part of the Hamiltonian spectrum contains a massive vector and a real massive scalar. So there is a mass gap. This theory is not believed to exhibit confinement, i.e. it is believed that it exhibits the length law for the Wilson loop. This is certainly what one computes in perturbation theory.

In case 2, we will show that there is an area law for the 't Hooft loop operator, because of the Higgs mechanism. This happens for topological reasons, as explained below.

As a warmup consider a closed spacetime M and a line bundle $\mathcal{L}$ with a nontrivial c_1. Let us consider the path integral for our theory over sections of this bundle. It turns out that the action of all field configurations in this integral has to be very large: it is bounded below by a constant (which is independent of M) times the area of the minimal 2-surface which represents a cycle Poincare dual to $c_1(\mathcal{L})$.

Indeed, if ϕ is a section of $\mathcal{L}$ then ϕ has to vanish on a 2-cycle Σ which is dual to $c_1(\mathcal{L})$. If we fix Σ, we can look at the configuration of minimal action with such a zero. For this, we can (if the metric of M is scaled up) reduce to the case that $M = \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, with Σ equal to the first factor. We can assume that ϕ and A are invariant under translations of the first factor in M. In the second factor, we want ϕ to vanish at the origin and to approach the vacuum at infinity (up to gauge transformation), such that the first Chern class of the bundle, relative to the trivialization at infinity given by ϕ, equals 1. The same problem appeared in lecture 2 (in the guise of finding an instanton solution of the two-dimensional version of the same model), and we discussed qualitative properties of the solution. Anyway, let I be the action of this solution in the two-dimensional sense (that is, integrated over just the second factor in M). Going back to a global $\Sigma \subset M$ of smallest area representing the first Chern class, the minimum action field looks in the normal directions to Σ like the instanton just described; its action is approximately $I \cdot \mathrm{Area}(\Sigma)$.

Now let us come back to the 't Hooft loop in $\mathbb{R}^4$. In this case the bundle is over $M \setminus C$, where M is the spacetime. If D is a 2-chain in M whose boundary is C then D plays the role of the Σ of the previous discussion. Indeed, if $\mathcal{L}$ is a line bundle over $M \setminus C$ with Chern class 1, and ϕ its section then ϕ must vanish on a 2-surface whose boundary is C. Thus, the same argument as above shows that $\langle T_1(C)\rangle \sim e^{-\lambda\ \mathrm{area(D)}}$, where area($D$) is the smallest area of a disk spanning C. This is the area law which we wanted to demonstrate.

This behavior is characteristic of what is called the Higgs regime, or phase.

Now let us discuss in more detail the relation of the established behavior of the 't Hooft loop operator with confinement. It is believed that if $\pi_1(G) \neq 0$ there are at least three possible phases:

1) Coulomb: no mass gap, gauge bosons in the infrared, W and T behave like in the free theory and exhibit the length law.

2) Higgs: mass gap, length law for W, area law for T.

3) Confinement: mass gap, area law for W, length law for T.

As already suggested, 't Hooft's idea was that there should be a nonabelian analogue of duality which interchanges W with T, the Higgs and the confinement regimes, and maps the Coloumb phase to itself. Thus, the area law for T in a theory implies confinement in the dual theory. This is what happens for some supersymmetric theories, e.g. the theory relevant to Donaldson theory.

In fact, 't Hooft showed that if R and γ are such that $\gamma|_R \neq 1$ then either W_R on T_γ exhibit the area law. More specifically, he proved an even stronger statement, namely that the set H of all $(c, \gamma) \in \pi_1(G)^* \times \pi_1(G)$ such that for some representation R with central character c the operator $W_R(C)T_\gamma(C)$ (suitably renormalized) does not exhibit the area law, is an isotropic subgroup of $\pi_1(G)^* \times \pi_1(G)$ with respect to the natural symplectic form. The reason for this, very roughly, is the following. If $A(C) = W_{R_1}T_{\gamma_1}(C)$ and $B(C) = W_{R_2}T_{\gamma_2}(C)$ exhibit the length law then, when acting on the vacuum, they produce only effects that are localized along C (or there would be an area law instead of a length law). So

$$\langle A(C)B(C')\rangle = \langle A(C)\rangle\langle B(C')\rangle(1 + o(1)), d \to \infty \tag{9.11}$$

where d is the distance between C and C'. This shows that if $AB = qBA$ where q is a constant then q must be equal to 1. By 't Hooft's formula (9.4) (which is clearly valid in the nonabelian case as well), this implies that H is isotropic. By further physical arguments, one shows that if there is a mass gap, then H is maximal isotropic. This leads to a more refined classification of massive phases than was stated above: associated to each massive phase is a maximal isotropic subgroup of $\pi_1(G)^* \times \pi_1(G)$. All possibilities can arise, in general.

Remark. The argument showing that loop operators A and B must commute fails if one of the operators, say A, exhibits the area law. In this case, acting on the vacuum with this operator produces an effect that is not in any way localized near C; it rather has an effect which is localized near a minimal area disk D whose boundary is C; such a disk will always intersect C' when the linking number is not zero. Formula (9.11) is now valid only if d is the distance from C' to D, which is always 0, so the formula does not tell us anything.

Remark. The area law for the 't Hooft operator in a Higgs phase has many physical and mathematical applications. For example, with some small adjustments, what we said above in analyzing the behavior of the Higgs phase with a bundle of nonzero first Chern class could serve as an explanation of the Meissner effect, the fact that a superconductor (which is described approximately by the abelian Higgs model that we examined) expels magnetic flux. Perhaps the reader has, at a science fair, seen a demonstration of a magnet floating above a superconductor; this effect has the same origin. Mathematically, C. Taubes's analysis of the Seiberg-Witten invariants of symplectic four-manifolds made use of the same facts: the localization (in a closely analogous system of equations) of the zeroes of ϕ on a surface of smallest area.

Lecture II-10: Quantum gauge theories in two dimensions and intersection theory on moduli spaces

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10.1. The partition function in two dimensional Yang-Mills theory.

We consider gauge theories in two dimensions with a simple gauge group G. The spacetime is a compact Riemann surface Σ of genus g with no boundary. To apply our methods to intersection theory on moduli spaces, we shall need to consider the case $G = SU(n)/\mathbb{Z}_n$ and consider bundles on Σ for which the transition functions do not lift to $SU(n)$: in mathematical terms we are considering the moduli space of holomorphic vector bundles on Σ with rank n and degree d.

The theory we consider is *physical Yang-Mills theory*. The fields are a G connection A on a G bundle P on Σ, with curvature F_A; the Lagrangian is

$$\mathcal{L} = \frac{1}{4e^2}\int d^2x\left(|*F_A|^2\right). \tag{1}$$

The partition function is thus

$$Z = \frac{1}{\mathrm{Vol}(G)}\int DAe^{-\mathcal{L}}. \tag{2}$$

We introduce a scalar field ϕ with values in $\mathrm{ad}(P)$ and rewrite the path integral for the partition function as

$$Z = \frac{1}{\mathrm{Vol}(G)}\int DAD\phi\exp\{i\int \mathrm{Tr}\phi F - \frac{e^2}{2}\int d\mu \mathrm{Tr}\phi^2\}. \tag{3}$$

This theory is invariant under area preserving diffeomorphisms since it has no explicit dependence on a Riemannian metric on Σ but only on the *measure* $d\mu = *(1)$. It is clear that the theory depends only on the coupling constant e and the area $\mathbf{a}$ of Σ only through the combination $e^2\mathbf{a}$. The path integral (3) is well behaved under taking the limit as $e \to 0$.

We make a further modification to the path integral for the partition function by introducing a fermionic variable ψ which is a one-form in the adjoint representation (in other words ψ is an element of of $\Omega^1(\Sigma)\otimes\Gamma(\mathrm{ad}P)$. The field ψ should be thought of as lying in the tangent space to the space $\mathcal{A}$ of connections on Σ. The path integral becomes

$$Z = \frac{1}{\mathrm{Vol}G}\int DAD\phi D\psi\exp\{i\int_\Sigma \mathrm{Tr}(\phi F) + \frac{1}{2}\int \mathrm{Tr}(\psi\wedge\psi) - \frac{e^2}{2}\int d\mu\mathrm{Tr}(\phi^2)\}. \tag{4}$$

Since the Lagrangian in (4) contains no terms which involve both ψ and the other fields ϕ and A, we can integrate out ψ and recover the earlier expression (3).

We can now define a supersymmetry operation δ on the space of fields:

$$\delta A = i\psi, \tag{5}$$

$$\delta\psi = -d_A\phi \tag{6}$$

$$\delta\phi = 0 \tag{7}$$

It follows that $\delta^2 A = -id_A\phi$, in other words $\delta^2 = 0$ up to the action of a gauge transformation. (Here, d_A refers to the de Rham differential d twisted by a connection A on Σ.) We may check the invariance of the action under δ: we have

$$\delta\int \mathrm{Tr}(\phi F) = -i\int \mathrm{Tr}(\phi D_A\psi), \tag{8}$$

while

$$\delta \int \mathrm{Tr}(\psi \wedge \psi) = - \int \mathrm{Tr}(D_A \phi \wedge \psi). \tag{9}$$

After integrating by parts, we find that the action appearing in (4) is invariant.

10.2. A finite dimensional analogue: the Cartan model.

Our path integral is a path integral over the infinite dimensional space $\mathcal{A}$ of connections which is acted on by the gauge group $\mathcal{G}$ with Lie algebra Lie($\mathcal{G}$). The *Cartan model* is used to treat a finite dimensional analogue of this situation: for a more detailed description see Chapter 7 of **[BGV]**. The analogue of $\mathcal{A}$ is a finite dimensional manifold M equipped with the action of a compact Lie group H (which is the analogue of $\mathcal{G}$). Functions of A and ψ correspond to differential forms on M, while the analogue of functions of A, ψ and ϕ are elements of $\Omega^*(M) \otimes \mathrm{Fun}(\mathbf{h})$. In fact we restrict to the H-invariant subspace $\left(\Omega^*(M) \otimes \mathrm{Fun}(\mathbf{h})\right)^H$ where H acts in the obvious way on $\Omega^*(M)$ and acts on $\mathbf{h}$ (and hence on Fun($\mathbf{h}$)) via the adjoint action. Here Fun($\mathbf{h}$) denotes an appropriate class of functions on $\mathbf{h}$: in the literature one most usually restricts to polynomial functions Pol($\mathbf{h}$) = $S(\mathbf{h}^*)$ (the symmetric algebra on $\mathbf{h}^*$).

The set

$$\Omega^*_H(M) = \left(\Omega^*(M) \otimes S(\mathbf{h}^*)\right)^H$$

has a natural grading: it is the differential form grading plus two times the polynomial grading (in other words a linear function on $\mathbf{h}$ is assigned grading 2). With this grading one sees that both terms in the operator[3]

$$D = d - i\iota_{V(\phi)} \tag{10}$$

increase the grading by 1. (Here d is the de Rham differential and $\iota_{V(\phi)}$ is the interior product with the vector field $V(\phi)$ induced on M by the action of $\phi \in \mathbf{h}$.) We may write

$$\iota_{V(\phi)} = \sum_a \phi^a \iota_{V^a},$$

introducing a basis $\{\phi^a\}$ for $\mathbf{h}$. It is easy to check (since we have restricted to H-invariant elements of $\Omega^*(M) \otimes S(\mathbf{h}^*)$) that $D^2 = 0$, so one may take the cohomology with respect to D: this cohomology is identified with the H-equivariant cohomology $H^*_H(M)$ of M.[4] If H acts freely on M the topological quotient M/H is a manifold (and M is a principal H-bundle over M/H), and the equivariant cohomology is identified with the ordinary cohomology $H^*(M/H)$ of the quotient.

We shall start with classes in the D-cohomology of $\Omega^*_H(M)$. One type of classes come from $(S(\mathbf{h})^*)^H$ (in other words, the polynomials on the Lie algebra $\mathbf{h}$ which are invariant under the adjoint action): this is identified with the H-equivariant cohomology of a point, or in other words with the cohomology $H^*(BH)$ of the classifying space BH. If M is a principal H-bundle over M/H (or equivalently if the action of H on M is free) then each invariant polynomial on $\mathbf{h}$ corresponds to a characteristic class of principal bundles with structure group H. Under the isomorphism $H^*_H(M) \cong H^*(M/H)$, the invariant polynomial S on $\mathbf{h}$ is identified with the corresponding characteristic class of the principal H-bundle M. (This is given in Chern-Weil theory as $S(F_A)$ where A is a connection on the bundle $M \to M/H$ and F_A is its curvature, which is a 2-form on M with values in $\mathbf{h}$).

10.3 Infinite dimensional Cartan: the descent equations

We shall now outline the analogue of the Cartan model in our infinite dimensional situation: this material is covered in Section 3.3 of **[W2]**. The space M is the infinite dimensional vector space $\mathcal{A}$

[3]Mathematicians normally use a convention in which the i in (10) is omitted.

[4]Throughout this lecture all cohomology groups will be assumed to have complex coefficients.

of connections on a G bundle P over a Riemann surface Σ. We start with an Ad-invariant polynomial S on the Lie algebra $\mathfrak{g}$; from this we shall construct an operator $\mathcal{O}_S^{(0)}$ in two dimensional Yang-Mills theory, which corresponds to a cohomology class in the moduli space $\mathcal{M}$. Recall that the field theory contained a field ϕ with values in $\mathfrak{g}$. We define

$$\mathcal{O}_S^{(0)} = S(\phi). \tag{11}$$

The object $\mathcal{O}_S^{(0)}$ is thus a function on Σ: we shall see that up to the supersymmetry differential δ, the operator $\mathcal{O}_S^{(0)}(p) = S(\phi(p))$ is independent of the choice of a point $p \in \Sigma$. We decompose the field ϕ into components $\{\phi^b\}$ (where b indexes a basis for $\mathfrak{g}$). Then

$$d\mathcal{O}_S^{(0)} = \sum_b \frac{\partial S}{\partial \phi^b} d_A \phi^a \tag{12}$$

$$= i\delta(\sum_b \frac{\partial S}{\partial \phi^a}\psi^b), \tag{13}$$

in terms of the supersymmetry operator δ. Thus if we define

$$\mathcal{O}_S^{(1)} = -\sum_b \frac{\partial S}{\partial \phi^b}\psi^b,$$

we have proved

$$d\mathcal{O}_S^{(0)} = -i\delta\mathcal{O}_S^{(1)}. \tag{14}$$

Here $\mathcal{O}_S^{(1)}$ should be viewed as a 1-form on Σ.

We can iterate this procedure: we find that

$$d\mathcal{O}_S^{(1)} = \sum_{a,b} \frac{\partial^2 S}{\partial\phi^a\partial\phi^b} D_A\psi^a \wedge \psi^b + \sum_a \frac{\partial S}{\partial\phi^a} D_A\psi^a. \tag{15}$$

Observing that $D_A\psi = \delta F_A$, we can convert (15) into $d\mathcal{O}_S^{(1)} = -i\delta\mathcal{O}_S^{(2)}$, where we have defined

$$\mathcal{O}_S^{(2)} = \frac{1}{2}\sum_{a,b} \frac{\partial^2 S}{\partial\phi^a\partial\phi^b}\psi^a \wedge \psi^b + i\sum_a \frac{\partial S}{\partial\phi^a} F_a. \tag{16}$$

These equations may be summarized as follows:

$$(d + i\delta)(\mathcal{O}_S^{(0)} + \mathcal{O}_S^{(1)} + \mathcal{O}_S^{(2)}) = 0. \tag{17}$$

(The point is that in an appropriate double complex the total differential is $d + i\delta$ and $\mathcal{O}_S^{(0)} + \mathcal{O}_S^{(1)} + \mathcal{O}_S^{(2)}$ is closed.)

We shall now use this to construct cohomology classes on the moduli space

$$\mathcal{M} = \mathcal{A}^{\text{flat}}/\mathcal{G}. \tag{18}$$

If we choose a q-cycle C in Σ, we find (using Stokes' theorem) that

$$Q_S^{(q)}(C) \stackrel{\text{def}}{=} \int_C \mathcal{O}_S^{(q)}$$

satisfies

$$\delta Q_S^{(q)}(C) = 0, \tag{19}$$

and if $C = \delta B$ is a boundary, then

$$Q_S^{(q)}(C) = \delta T \text{ for some } T. \tag{20}$$

Provided that the group $\mathcal{G}$ of gauge transformations acts freely on $\mathcal{A}$, the $Q_S^{(q)}(C)$ correspond to the generators of the cohomology of $\mathcal{M}$: they are cohomology classes on $\mathcal{A}/\mathcal{G}$ which we will restrict to the moduli space $\mathcal{A}_{\text{flat}}/\mathcal{G}$. These generators are given in Section 2 of **[AB]**: they are produced by taking the slant product of the characteristic classes of the *universal bundle* over $\mathcal{M} \times \Sigma$ with classes in the homology of Σ. The quantum field theory will compute a generating functional

$$\int_{\mathcal{M}} \exp\Big\{\alpha Q_S^{(0)}(p) + \sum_{j=1}^{2g} \beta_j Q_{S_j'}^{(1)}(C_j) + \gamma Q_{S''}^{(2)}(\Sigma)\Big\} \tag{21}$$

which will encode all intersection numbers in the cohomology of $\mathcal{M}$. A mathematical proof of these formulas for intersection numbers is given in **[T]** for the case $G = SU(2)$ and in **[JK2]** in the case $G = SU(n)$ (in those cases where the moduli space $\mathcal{M}$ is smooth). Here, p is a point in Σ, the C_j are $2g$ cycles in Σ corresponding to the homology $H_1(\Sigma)$, and the S, S_j' and S'' are arbitrary invariant polynomials on $\mathfrak{g}$ (which are in general distinct). For simplicity we shall mostly treat the case $G = SU(2)$, for which the ring of invariant polynomials on $\mathfrak{g}$ is a polynomial ring on one generator S given by

$$S(\phi) = \text{Tr}(\phi^2).$$

10.4 Equivariant integration and localization

We now return to the finite-dimensional situation of Section 10.2. We would like to define a map $\mathcal{I} : Z_H^*(M) \to \mathbb{C}$ (where $Z_H^*(M)$ are the D-closed elements in $\Omega_H^*(M)$) by integrating over $M \times \mathbf{h}$. In order to define a convergent integral, we introduce a convergence factor $e^{-\varepsilon \text{Tr}(\phi^2)}$. (In fact in the mathematical treatment of this convergence factor one may replace the Gaussian $e^{-\varepsilon \text{Tr}(\phi^2)}$ by any collection of rapidly decreasing functions $\{f_\varepsilon\}$ on $\mathbf{h}$ which (as $\varepsilon \to 0$) represent the Dirac delta distribution: see **[JK1]** for a mathematical treatment of equivariant integration in the Cartan model.) For $\alpha \in Z_H^*(M)$ we define

$$\mathcal{I}(\alpha) = \int_{\phi \in \mathbf{h}} \int_M d\phi_1 \dots d\phi_s e^{-\varepsilon \text{Tr}(\phi^2)} \alpha(\phi). \tag{22}$$

In fact in order to ensure convergence of the integral (22) we must place some hypotheses on α. A useful class of equivariantly closed forms are obtained in the special case when M is a symplectic manifold with symplectic form ω, and the action of H on M is Hamiltonian with moment map $\mu : M \to \mathbf{h}^*$: We see readily that

$$\tilde{\omega} = -i\text{Tr}(\phi\mu) + \omega \tag{23}$$

satisfies

$$D\tilde{\omega} = 0$$

(where D is the Cartan model differential defined by (10)).

The computation of the integral (22) will be governed by a localization principle similar to that which leads to the Duistermaat-Heckman formula for an oscillatory integral over a symplectic manifold with a Hamiltonian torus action. We choose an H-invariant 1-form λ for which

$$D\lambda = d\lambda + -i(V(\phi), \lambda); \tag{24}$$

where $V(\phi)$ is the vector field given by the action of ϕ (which of course depends linearly on ϕ). The integral (22) is not changed by replacing $\exp(-\varepsilon \mathrm{Tr}(\phi^2))$ by $\exp(-\varepsilon \mathrm{Tr}(\phi^2) + itD\lambda)$ since $\exp(itD\lambda)$ expands as $1 + itD\lambda + \ldots$ Because $-tD\lambda = it(V(\phi), \lambda) - t(d\lambda)$, as $t \to \infty$ the integral localizes around points where $(V(\phi), \lambda) = 0$. We may see this by alternatively computing the partition function by completing the square and performing the integral over ϕ: to leading order in t this gives a factor

$$\exp -\frac{t^2}{4\varepsilon} \sum_a |(V_a, \lambda)|^2,$$

so by considering the behaviour for $\epsilon \to 0$ we see that the integral localizes around points where $(V_a, \lambda) = 0$ for all a.

We now specialize to the case where M is a symplectic manifold equipped with the Hamiltonian action of a compact group H: the moment map will be denoted μ. We define $I = |\mu|^2 : M \to \mathbb{R}$, the norm squared of the moment map. We may choose an almost complex structure $J \in \mathrm{End}(TM)$ compatible with the symplectic structure, and let $\lambda = J(dI)$ be the invariant one-form defined above. The points where $(V_a, \lambda) = 0$ are readily identified with the critical points of I. We see this because

$$dI = 2 \sum_a \mu_a d\mu_a,$$

so since $V_a = \omega^{-1} d\mu_a$ (regarding the symplectic form ω as an isomorphism $T_x M \to T_x^* M$ for each $x \in M$, with inverse $\omega^{-1} : T_x^* M \to T_x M$) we have

$$\omega^{-1} dI = 2 \sum_a \mu_a V_a.$$

Here $dI \in \Omega^1(M)$, so $\omega^{-1} dI$ should be regarded as an element of $\Gamma(TM)$. Thus $(V_a, \lambda) = 0$ implies $(\omega^{-1} dI, \lambda) = 0$, in other words

$$(\omega^{-1} dI, J(dI)) = 0$$

(where both sides of this equation are in $\Gamma(TM)$). This implies $dI = 0$ since ω and J combine to give a nondegenerate symmetric bilinear form g on the cotangent space, defined by

$$g(X, Y) = \omega^{-1}(X, JY)$$

(for $X, Y \in \Gamma(T^* M)$). Thus our integral formally localizes on the critical points of I. One obvious source of such critical points is $\mu^{-1}(0)$, but there are other critical points where $I \neq 0$.

We now give a derivation of (one version of) the Duistermaat-Heckman formula. As noted above, the form (23) $\tilde{\omega} = -i\mathrm{Tr}(\phi\mu) + \omega$ is equivariantly closed. We may thus compute

$$\frac{1}{\mathrm{Vol} H} \int_{\phi \in \mathbf{h}} d\phi_1 \ldots d\phi_n \int_M \exp\Big(-i\mathrm{Tr}(\phi\mu) + \omega\Big). \tag{25}$$

(Note that if we did not include the integration over ϕ in the formula (25), the oscillatory integral over M would correspond to a sum over the components of the fixed point set of T: see for instance Chapter 7 of **[BGV]**.) If however we first integrate over ϕ we obtain a delta function $\delta(\mu)$, so the formula becomes

$$\frac{1}{\mathrm{Vol} H} \int_M \exp \omega \delta(\mu) = \int_{\mu^{-1}(0)/H} e^{\omega}$$

in other words the symplectic (Liouville) volume of the symplectic quotient $\mu^{-1}(0)/H$.

We wish, however, also to include a factor $\exp(-\varepsilon \mathrm{Tr}(\phi^2))$ in the integrand. Thus we wish to compute

$$Z(\varepsilon) \stackrel{\text{def}}{=} \frac{1}{\mathrm{Vol}H} \int_{\phi \in \mathbf{h}} d\phi_1 \dots d\phi_n \int_M \alpha \exp\Big(-i\mathrm{Tr}(\phi\mu) + \omega - \varepsilon \mathrm{Tr}(\phi^2)\Big), \tag{26}$$

where $\alpha = \alpha(\phi)$ is an equivariantly closed form on M which has *polynomial* dependence on ϕ. We assume H acts freely on $\mu^{-1}(0)$; near $\mu^{-1}(0)$, the equivariantly closed class $\mathrm{Tr}(\phi^2)$ (which comes from the H-equivariant cohomology of a point in the Cartan model) is the pullback of a class Θ on the symplectic quotient $\mu^{-1}(0)/H$, which is the class which would be evaluated in Chern-Weil theory as $\mathrm{Tr}(F_B^2)$ if B is a connection on the principal H-bundle $\mu^{-1}(0) \to \mu^{-1}(0)/H$ and F_B is its curvature. Similarly, near $\mu^{-1}(0)$, α restricts to the pullback of a class $\hat{\alpha}$ on $\mu^{-1}(0)/H$.

We find that

$$Z(\varepsilon) = \int_{\mu^{-1}(0)/H} \hat{\alpha} \exp(\omega - \varepsilon\Theta) + O(e^{-b/\varepsilon}),$$

where b is the smallest nonzero critical value of I. To see heuristically why the nonzero critical value b of I should give a contribution to the integral (26) which is of order $e^{-b/\varepsilon}$, we compute (26) by first integrating out ϕ (assuming for simplicity that $\alpha = 1$). We compute the integral over ϕ by completing the square in ϕ, which gives

$$Z(\varepsilon) = \int_M \exp\left(\omega - \frac{1}{\varepsilon}\mathrm{Tr}(\mu^2)\right).$$

10.5 Equivariant integration: the infinite dimensional case

We now pass to the infinite dimensional case and study the path integrals corresponding to equivariant integration. Recall that we had defined the generating functional (21). We restrict to $G = SU(2)$ or $SO(3)$; we will pick the invariant polynomials S, S'_j and S'' to all be equal to the generating polynomial (denoted S), namely $S(\phi) = \mathrm{Tr}(\phi^2)$. (We have picked an Ad-invariant inner product on $\mathfrak{g}$, which is denoted Tr; a corresponding measure is defined on G.) We assume $\gamma = 1$ and $\alpha = -e^2/2$ in (21), and observe that for $S = \mathrm{Tr}(\phi^2)$ we have

$$Q_S^{(1)}(C_j) = \int_{C_j} \mathrm{Tr}(\phi\psi).$$

Thus the path integral that gives rise to (21) becomes (where p is a point in Σ)

$$Z(e, \{\beta_j\}) = \frac{1}{\mathrm{Vol}G} \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\phi e^{\{-\frac{e^2}{2}\mathrm{Tr}(\phi^2)(p) + \sum_{j=1}^{2g}\beta_j \int_{C_j}\mathrm{Tr}(\phi\psi) + \int_\Sigma Q_S^{(2)}\}}. \tag{27}$$

(Note that when $S(\phi) = \mathrm{Tr}(\phi^2)$, the object $\int_\Sigma Q_S^{(2)}$ corresponds to the symplectic form on $\mathcal{A}$, and becomes identified with the standard symplectic form on the moduli space $\mathcal{M}$.)

We can now see how the integral (27) can be rewritten in such a way that the generating functional (21) involving the $Q_S^{(q)}$ for $q = 0, 1, 2$ is equated with a corresponding generating functional involving only $Q_S^{(q)}$ for $q = 0, 2$. This is accomplished by performing the integral over ψ, which is Gaussian: completing the square to perform this integral we see that we pick up a term

$$\frac{1}{\mathrm{Vol}G} \int \mathcal{D}A\mathcal{D}\phi \exp\Big\{i \int_\Sigma \mathrm{Tr}(\phi F_A) - \varepsilon \mathrm{Tr}(\phi^2)(p), \Big\} \tag{28}$$

where $\varepsilon = e^2/2 - \sum_{i<j}\beta_i\beta_j \#(C_i \cap C_j)$. In other words we can eliminate the odd-dimensional generators of the cohomology ring of the moduli space corresponding to $Q_S^{(1)}$ (the non-algebraic cycles in the moduli space). A mathematical version of this argument is given in the work of Thaddeus **[T]**.

10.6 The partition function of Yang-Mills theory

The following material is covered in Section 4 of **[W2]**. We have now reduced the formula we need to compute to the partition function (2) of Yang-Mills theory without the fermionic variable ψ included. To compute the partition function we quantize the theory. This is accomplished by replacing the Riemann surface by a cylinder $C \times [0, T]$ of length T (where $C \cong S^1$ should be regarded as space and is assumed to have length L, while $[0, T]$ is a time interval of length T). The space of fields is equal to the space of connections $\mathcal{A}_C$ on the circle C, so the Hilbert space is

$$\mathcal{H} = C^\infty(\mathcal{A}_C)^{\mathcal{G}},$$

the gauge invariant functions on $\mathcal{A}_C$. Since up to gauge equivalence all connections are classified by their holonomy around the curve C, we find that the Hilbert space is identified with

$$\mathcal{H} = C^\infty(G)^G,$$

the ring of functions on G which are invariant under the adjoint action (or equivalently the functions on the maximal torus T which are invariant under the action of the Weyl group). The Hilbert space is thus identified with the irreducible (complex) representations R of G (or equivalently with their characters which are denoted χ_R or Tr_R): we take the characters χ_R to form a basis of $\mathcal{H}$, or

$$\mathcal{H} = \oplus_R \mathbb{C}_{(R)}.$$

For every irreducible representation R the basis element χ_R is identified with a function Ψ_R on $\mathcal{A}_C$, namely

$$\Psi_R(A) = \chi_R(\mathrm{Hol}_C A).$$

The Hamiltonian of the theory is

$$H = -\frac{e^2}{2}\int_C \mathrm{Tr}|F_{01}|^2$$

(where F_{01} is the curvature of the connection), which becomes (identifying the Hamiltonian as an operator on $\mathcal{H}$ in terms of the position variables $A^b(\theta)d\theta$ on $\mathcal{A}_C$ and the corresponding momentum variables $(\partial A^b(\theta)/\partial t)d\theta$ which are identified with $-i\partial/\partial A^b(\theta)$ under canonical quantization)

$$H = \frac{e^2}{2}\int_C \mathrm{Tr}(\frac{\delta}{\delta A})^2.$$

The action of the operator $\delta/\delta A^a(\theta)$ on Ψ_R gives

$$\frac{\delta}{\delta A^a(\theta)}\mathrm{Tr}_R \mathrm{Hol}_C(A) = \mathrm{Tr}_R T^a \mathrm{Hol}_C(A)$$

in terms of a basis T^a for the Lie algebra $\mathfrak{g}$ of G. We thus see that

$$\sum_a \frac{\delta}{\delta A^a(\theta)}\frac{\delta}{\delta A^a(\theta)}\mathrm{Tr}_R \mathrm{Hol}_C(A) = \mathrm{Tr}_R(\sum_a T^a T^a \mathrm{Hol}_C(A)). \tag{29}$$

The element $\sum_a T^a T^a$ is the *quadratic Casimir* of G, whose evaluation in the representation R is denoted

$$\mathrm{Tr}_R \sum_a T^a T^a \stackrel{\mathrm{def}}{=} c_2(R).$$

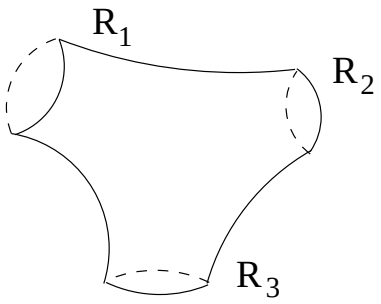

Figure 1

Thus we see that

$$H\Psi_R = \frac{e^2 c_2(R) L}{2}\Psi_R. \tag{30}$$

We thus see that

$$< \Psi_R | e^{-HT} | \Psi_R >= \exp(-e^2 c_2(R) LT/2) = \exp(-e^2 \mathbf{a} c_2(R)/2), \tag{31}$$

in terms of the area $\mathbf{a} = LT$ of the cylinder. The answer depends only on the combination $e^2\mathbf{a}$.

In fact the effect of renormalization in this quantum field theory is to introduce a factor of the form

$$\exp(\alpha \int_\Sigma d\mu \mathcal{R} + (e^2/2)\beta \int_\Sigma d\mu) \tag{32}$$

into the integrand (where $\mathcal{R}$ is the Riemann curvature and α and β are appropriately chosen real-valued constants). The effect of the term involving β is to change the quadratic Casimir c_R to a renormalized value $\tilde{c}_R$ which differs from c_R by addition of a constant. The factor α multiplies the Euler characteristic of Σ: it will eventually be adjusted to ensure that the overall normalization of the partition function and its dependence on the genus g agrees with that found by other methods (for instance by computing the partition function when $\varepsilon = 0$ using Reidemeister-Ray-Singer torsion, as in Section 4 of **[QYM]**).

We must also consider the field theory on the Riemann surface $\mathcal{P}$ which is a sphere with three disks removed (the trinion or pair of pants: see Figure 1). If we insert the operator $\hat{Q_S^{(0)}}(p)$ corresponding to the observable $Q_S^{(0)}(p)$ (where p is some point in $\mathcal{P}$) we see using (30) and its generalizations that the state Ψ_R is an eigenstate of $\hat{Q_S^{(0)}}(p)$ with some eigenvalue $C(S,R)$:

$$\hat{Q_S^{(0)}}(p)\Psi_R = C(S,R)\Psi_R. \tag{33}$$

Here, if S is an invariant polynomial of degree l on $\mathfrak{g}$, $C(S,R)$ is the corresponding l-th order Casimir of the representation R. By considering the one point function determined by the Riemann surface $\mathcal{P}$ with the operator $\hat{Q_S^{(0)}}(p)$ inserted near the j-th boundary component (see Figure 3), we find that this one point function is equal to $C(S,R_j)W_{R_1R_2R_3}$ (for $j = 1,2,3$) if $W_{R_1R_2R_3}$ is the partition function for $\mathcal{P}$ with states $\Psi_{R_1}, \Psi_{R_2}, \Psi_{R_3}$ along the three boundary components. This does not depend on the boundary component j (since this field theory is invariant under area preserving diffeomorphisms): it follows that the Casimirs $C(S,R_j)$ are all equal, and since this is true for all invariant polynomials S, we must have $R_1 = R_2 = R_3$ if $W(R_1,R_2,R_3) \neq 0$.

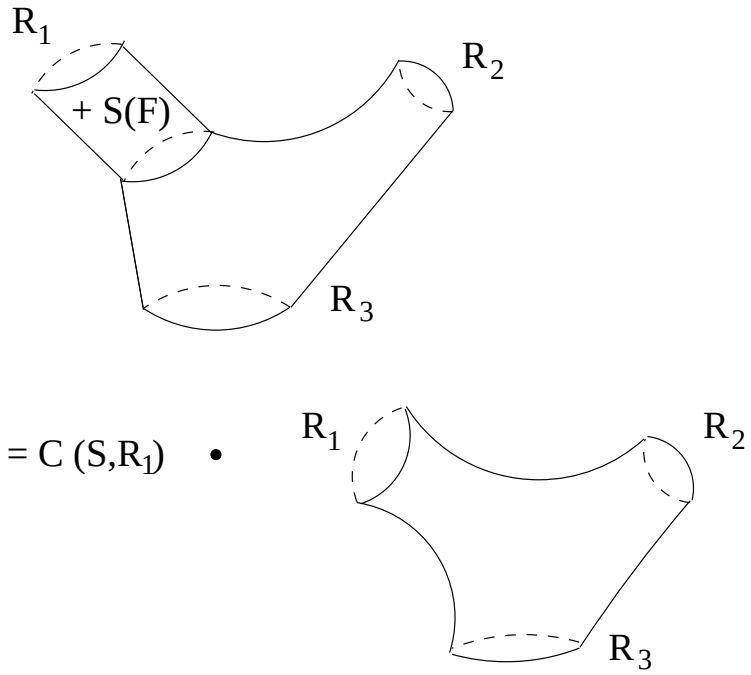

Figure 3

We are thus reduced to computing the partition function of $\mathcal{P}$ with boundary conditions determined by the external state Ψ_R along all three boundary components. Denote the value of this partition function in the limit of zero area by w_R: more generally for a pair of pants $\mathcal{P}$ of area **a** one obtains

$$Z_R = w_R \exp(-\tilde{c}_2(R)\frac{\mathbf{a}e^2}{2}). \tag{34}$$

We now consider the problem of computing the partition function for a Riemann surface of genus g with no boundary. Such a surface may be formed by gluing together $2g-2$ copies $\mathcal{P}_j$ of $\mathcal{P}$ along $3g-3$ boundary circles C_γ. We may factor the path integral for the partition function according to the values of the fields restricted to the boundary circles C_γ. If A' denotes the value of a connection on the boundary circles $\coprod_\gamma C_\gamma$ of the $\mathcal{P}_j$, and $\mathcal{A}_{A'} = \{A \in \mathcal{A} : A|_{\coprod_\gamma C_\gamma} = A'\}$ is the set of all connections which restrict to a given boundary value A', we have

$$\int \mathcal{D}A e^{-\mathcal{L}} = \int \mathcal{D}A' \int_{\mathcal{A}_{A'}} e^{-\mathcal{L}} \tag{35}$$

(cf. **[QYM]**, Section 4.5). Once we have fixed the boundary values A', the space $\mathcal{A}_{A'}$ is the product of $2g-2$ copies of the space of connections on $\mathcal{P}$ (with prescribed boundary values). In quantization the partition function of $\mathcal{P}$ is

$$Z(\mathcal{P}, A|_{\partial\mathcal{P}}) = \sum_R w_R \prod_{\gamma=1}^{3} \Psi_R(A|_{C_\gamma}). \tag{36}$$

To recover the partition function for the closed Riemann surface Σ of genus g, we multiply $2g-2$ copies of $Z(\mathcal{P})$ and integrate over the boundary values A'. Using the orthogonality relations for the group characters which correspond to the states Ψ_R, we find

$$Z(\Sigma) = \sum_R w_R^{2g-2} \exp(-\frac{e^2\mathbf{a}\tilde{c}_2(R)}{2}). \tag{37}$$

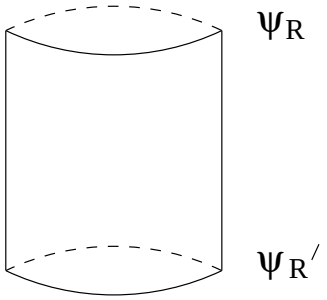

Figure 2

We now explain how to compute the w_R. We observe that the partition function for a disk D of area $\mathbf{a}$ with an external state Ψ_R on the boundary is

$$Z(D) = v_R \exp(-\frac{e^2 \mathbf{a} \tilde{c}_2(R)}{2}). \tag{38}$$

The partition function corresponding to a cylinder with boundary conditions determined by wavefunctions Ψ_R (resp. $\Psi_{R'}$) on the two boundary components is equal to 0 if $R \neq R'$, since Ψ_R is an eigenfunction of the Hamiltonian H. (See Figure 2.) The partition function of a cylinder $S^1 \times I$ of area $\mathbf{a}$ with external states Ψ_R on both boundary components is

$$Z(S^1 \times I) = 1 \cdot \exp(-\frac{e^2 \mathbf{a} \tilde{c}_2(R)}{2}) \tag{39}$$

(since the Ψ_R are eigenstates of the Hamiltonian H). We may decompose the cylinder as in Figure 4: this yields the partition function of the cylinder with external states Ψ_R on both boundary components as the product of the partition function of $\mathcal{P}$ (with external states Ψ_R on all boundary components) and that of D, so that

$$w_R \cdot v_R = 1. \tag{40}$$

To determine v_R, it suffices to consider a disk D of very small area. If we fix the holonomy of a connection around the boundary of D to take the value $U \in G$, the partition function for the disk (restricting to connections on D with boundary holonomy U) is (via quantization)

$$Z(U) = \sum_R v_R \chi_R(U) \tag{41}$$

(where as above χ_R denotes the character of the representation R). If we instead compute the partition function via the path integral, we write the action as

$$I(U) = \int_A \mathrm{Tr}(|F|^2), \tag{42}$$

and

$$Z(U) = \int DA \exp(-\frac{1}{e^2} \int \mathrm{Tr}(|F|^2)), \tag{43}$$

(where we have restricted to connections for which the boundary holonomy is U). This gives

$$Z(U) = \exp \alpha \delta(U - 1), \tag{44}$$

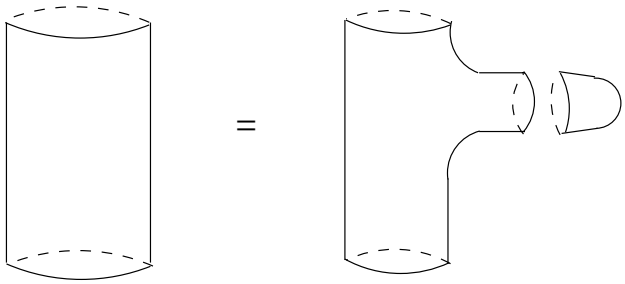

Figure 4

(since the Euler-Lagrange equation implies that the dominant contribution comes from flat connections, which necessarily have trivial holonomy around the boundary of D). Here, α is the (as yet undetermined) constant which appeared in (32) multiplying the Euler characteristic of the Riemann surface. Hence we see that

$$\exp(\alpha)\delta(U-1) = \sum_R v_R \chi_R(U). \tag{45}$$

If we multiply both sides of this equation by $\chi_{R'}(U)$ and integrate over U, using the orthogonality relations for group characters we find

$$v_R = \exp(\alpha)\dim R, \tag{46}$$

and hence

$$w_R = \frac{\exp(-\alpha)}{\dim R}. \tag{47}$$

Thus the partition function is

$$Z(\Sigma^g, e^2\mathbf{a}) = \sum_R w_R^{2g-2} \exp(-\frac{\tilde{c}_2(R)e^2\mathbf{a}}{2}) \tag{48}$$

$$= \sum_R \frac{e^{-\alpha(2g-2)}\exp(-\tilde{c}_2(R)\varepsilon/2)}{(\dim R)^{2g-2}} \tag{49}$$

where we have introduced $\varepsilon = e^2\mathbf{a}$.

The computation we have just performed gives the sum of partition functions corresponding to bundles of all possible topological types (recall that if G is not simply connected there will in general be several topological types of bundles over Σ). To study the contribution of one particular topological type, we take G to be simply connected and pick an element $\zeta \in Z(G)$. Choosing a point $p \in \Sigma$ we restrict to connections on $\Sigma - \{p\}$ such that the holonomy around p is equal to ζ. (Such a connection will descend to a flat connection on a quotient bundle with structure group $G/Z(G)$, whose topology is determined by ζ.) In this setting we repeat the analysis above: we find that the path integral over connections with holonomy ζ around p is given (via quantization) by

$$\sum_R u_R \chi_R(U) \tag{50}$$

and (via the path integral) by

$$\exp(\alpha)\delta(U-\zeta). \tag{51}$$

If we multiply by $\chi'_R(U)$ and integrate over U we find

$$u_R = \exp(\alpha)\chi_R(\zeta) = \exp(\alpha)(\dim R)\hat{\chi}_R(\zeta) \tag{52}$$

where $\hat{\chi}_R(\zeta) \in U(1)$ is the normalized character evaluated at ζ. Thus the partition function becomes (since the Riemann surface is now decomposed into $2g-1$ copies of $\mathcal{P}$ and one copy of the disk D)

$$Z(\varepsilon, g, \zeta) = \sum_R w_R^{2g-1} e^{-\tilde{c}_2(R)e^2\mathbf{a}/2} u_R \tag{53}$$

$$= \sum_R \frac{e^{-\alpha(2g-2)} e^{-\varepsilon\tilde{c}_2(R)/2}}{(\dim R)^{2g-2}} \hat{\chi}_R(\zeta). \tag{54}$$

This is the partition function for a particular class of G bundles corresponding to flat connections on a Riemann surface of genus g with one boundary component and with holonomy ζ around this boundary component. The partition function for the corresponding class of $G/Z(G)$ bundles on a Riemann surface of genus g with no boundary is obtained by dividing (54) by a factor $(\#Z(G))^{2g-2}$, since this component of the moduli space of flat $G/Z(G)$ connections is an unbranched cover (of order $(\#Z(G))^{2g-2}$) of the moduli space of flat G connections on a Riemann surface of genus g with one boundary component around which the holonomy is constrained to take the central value ζ.

We conclude this section by studying the example $G = SU(2)$, $\zeta = -1$. For each integer $n \geq 1$ there is a unique representation R_n of dimension n with $\hat{\chi}_n(\zeta) = (-1)^{n+1}$. The quadratic Casimir of the representation R_n is

$$c_2(R_n) = (n^2 - 1)/2;$$

this is renormalized to

$$\tilde{c}_2(R_n) = n^2/2.$$

The partition function corresponding to an $SO(3)$ bundle of $w_2 \neq 0$ is given (up to an overall multiplicative normalization constant independent of the genus g) by

$$Z(\varepsilon) = e^{-\alpha(2g-2)} \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^{2g-2}} e^{-\varepsilon n^2/4}. \tag{55}$$

Let us now verify that this is the sum of a term which is a polynomial in ε (the polynomial dependence being expected from (21)) and a second term which is exponentially decaying of order $e^{-b/\varepsilon}$ for some positive constant b. We do this by differentiating the series to find that

$$\left(e^{\alpha(2g-2)} 4^{g-1} \frac{\partial}{\partial\varepsilon}\right)^{g-1} Z(\varepsilon) = F(\varepsilon), \tag{56}$$

where we have defined

$$F(\varepsilon) = \sum_{n\geq 1} (-1)^{n+1} e^{-\varepsilon n^2/4}.$$

We have that

$$2F(\varepsilon) - 1 = -\sum_{n\in\mathbb{Z}} (-1)^n e^{-\varepsilon n^2/4},$$

and using the Poisson summation formula this is given by

$$2F(\varepsilon) = 1 - \sqrt{4\pi/\varepsilon} \sum_{m\in\mathbb{Z}} e^{-(2\pi)^2(m+1/2)^2/\varepsilon}. \tag{57}$$

(The terms in the sum indexed by m correspond to the contribution of the critical points of the Yang-Mills functional arising from unstable bundles of the form $\mathcal{L}_m \oplus \mathcal{L}_{-m+1}$, where $\mathcal{L}_m$ is a line bundle over Σ of degree m.) We may now apply (56) to integrate (57), showing that $Z(\varepsilon)$ is the sum of a polynomial in ε (of degree $g-1$) plus a term exponentially decaying like $G(\varepsilon)e^{-b/\varepsilon}$ for a positive constant b (where $G(\varepsilon)$ is a polynomial in $\sqrt{\varepsilon}$ and $1/\sqrt{\varepsilon}$).

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Lecture II-11: Supersymmetric field theories

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Notes by Pavel Etingof and David Kazhdan

11.1. General remarks on supersymmetry

Starting from today we study field theories with supersymmetry, i.e. theories whose symmetry group has a nontrivial extension to a supergroup. As usual, by even and odd (infinitesimal) symmetries we mean even, respectively odd, elements of the Lie superalgebra of this supergroup.

It often happens that a solution to the classical field equations has nontrivial odd symmetries. Examples:

gradient flowlines in Morse theory
holomorphic curves
instantons
monopoles
Seiberg-Witten solutions
hyperKahler structures
Calabi-Yau metrics
Metrics of G_2 and $Spin_7$ holonomy

In the next sections we will consider some of these examples.

11.2. Supersymmetric solitons (BPS states).

First consider a case in which the gradient flowlines of Morse theory will appear. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a Morse function (i.e. its critical points are nondegenerate and $|\nabla h|$ grows at infinity). A Morse function always has finitely many critical points.

Consider (in Minkowski signature) the theory of maps $\Phi : \mathbb{R}^{2|2} \to \mathbb{R}^n$ with the Lagrangian

$$\mathcal{L} = \int d^2x d^2\theta (\frac{1}{2} D_+\Phi D_-\Phi - h(\Phi)), \tag{11.1}$$

where $D_\pm = \frac{\partial}{\partial\theta_\pm} - \theta_\pm\partial_\pm$, $\partial_\pm = \frac{\partial}{\partial x_\pm}$, $x_\pm = \frac{1}{2}(t \pm x)$. This model has an obvious supersymmetry under $Q_\pm = \frac{\partial}{\partial\theta_\pm} + \theta_\pm\partial_\pm$. These supersymmetry operators satisfy the obvious commutation relations $Q_\pm^2 = \partial_\pm$, $\{Q_+, Q_-\} = 0$.

We have $(Q_+ \pm Q_-)^2 = 2\frac{\partial}{\partial t}$. So if we look for classical solutions with time translational symmetry (i.e. for solitons), we may in particular look for those of them which are invariant under one of the supersymmetry, say $Q_+ \pm Q_-$.

We have

$$Q_+ \pm Q_- = (\frac{\partial}{\partial\theta_+} \pm \frac{\partial}{\partial\theta_-}) + (\theta_+ \pm \theta_-)\frac{\partial}{\partial t} + (\theta_+ \mp \theta_-)\frac{\partial}{\partial x}. \tag{11.2}$$

Thus, the supersymmetry condition for time-independent solutions is

$$[(\frac{\partial}{\partial\theta_+} \pm \frac{\partial}{\partial\theta_-}) + (\theta_+ \mp \theta_-)\frac{d}{dx}]\Phi = 0. \tag{11.3}$$

Let us look for even solutions. It is easy to show that such solutions are of the form $\Phi = \phi + \theta_+\theta_-\nabla h(\phi)$. For them, the supersymmetry condition is

$$\frac{d\phi}{dx} \mp \nabla h(\phi) = 0, \tag{11.4}$$

which is the condition for the gradient flowline. Thus, supersymmetric solitons are the flowlines of the gradient flow.

Notice that these 1-st order equations imply the 2-nd order equations of motion. Indeed, it is easy to show that the equations of motion are

$$\partial_+\partial_-\phi + \nabla(\nabla h)^2(\phi) = 0, \tag{11.5}$$

or for time-independent solutions

$$\frac{d^2\phi}{dx^2} = \nabla(\nabla h)^2(\phi), \tag{11.6}$$

which can be obtained by differentiation of (11.4) with respect to x.

Another way to see this: the Lagrangian for time-independent fields (i.e the Hamiltonian) is

$$H(\phi) = \int dx(\frac{1}{2}(\frac{d\phi}{dx})^2 + \frac{1}{2}(\nabla h(\phi))^2). \tag{11.7}$$

Rewriting H, we get

$$H(\phi) = \frac{1}{2}\int dx(\frac{d\phi}{dx} \mp \nabla(\phi))^2 \pm \int_{-\infty}^{\infty} dh(s). \tag{11.8}$$

Since the last term is locally constant on the space of fields of finite energy, a supersymmetric solution provides the global minimum for the energy in each connected component of the space of fields. The value of energy at this minimum is $S = |\int dh|$.

Definition *Supersymmetric solitons, that is classical solutions invariant under some supersymmetries, are called classical BPS states.*

11.3. The role of BPS states in quantum theory.

We have mentioned above and used the fact that the vector fields Q_+, Q_- commute. Since the space of solutions of the classical field equations is a symplectic supermanifold, these vector fields must be (at least locally) generated by some Hamiltonian functions $\tilde{Q}_+, \tilde{Q}_-$ (defined up to adding a locally constant function). But these functions need not Poisson commute: their Poisson bracket has to be a locally constant function, not necessarily equal to zero.

In our case, the functions $\tilde{Q}_+, \tilde{Q}_-$ are easy to write down: if $\Phi = \phi + \theta_+\psi_+ + \theta_-\psi_- + \theta_+\theta_-F$, then

$$\tilde{Q}_+ = \int dx(\psi_+\partial_+\phi + \psi_-\nabla h(\phi)), \tilde{Q}_- = \int dx(\psi_-\partial_-\phi - \psi_+\nabla h(\phi)). \tag{11.9}$$

The computation of the Poisson bracket gives $\{\tilde{Q}_+, \tilde{Q}_-\} = -2S$, $S = \int dh = h(\phi(\infty)) - h(\phi(-\infty))$.

From this picture it is clear what will happen with operators Q_+, Q_- in quantum theory. We will have $Q_+^2 = P_+, Q_-^2 = P_-$ (where P_+, P_- are the corresponding momentum operators), and $(Q_+ \pm Q_-)^2 = 2(H \mp \hat{S})$, where $H = (P_+ + P_-)/2$ is the Hamiltonian, and $\hat{S}$ commutes with local operators.

In quantum theory, to every connected component X_a of the space X of fields of finite energy there corresponds a summand $\mathcal{H}_a$ of the Hilbert space. We expect that, if there is no breaking of supersymmetry, to every supersymmetric soliton $\Phi \in X_a$ there corresponds a state Ψ in $\mathcal{H}_a$ which is also supersymmetric: $(Q_+ \pm Q_-)\Psi = 0$. Then $(H \mp S_a)\Psi = 0$, where $S_a = \hat{S}|_{\mathcal{H}_a}$ is a scalar. Therefore, since $H \geq 0$, we have $H\Psi = |S_a|\Psi$. In particular, for every connected component of X there is only one supersymmetry (out of the two), for which there can be supersymmetric states in this component.

In general, on $\mathcal{H}_a$ we have $H \geq |S_a|$.

Now we want to determine whether there is a supersymmetric quantum state, that is a quantum state annihilated by $Q_+ \pm Q_-$, corresponding to the supersymmetric classical state. In the lowest order of perturbation theory one finds no bosonic zero mode except the translations and no fermionic zero

mode. Hence, in that approximation the lowest energy state of given momentum is unique and non-degenerate. Also, the theory in the vacuum sector has a mass gap (classically and therefore for weak enough coupling) so the unique ground state that is found in the leading approximation is isolated from any continuum. Hence the quantum theory for weak enough coupling has an isolated and unique ground state in this sector of the Hilbert space, and it follows from the supersymmetry algebra that this state must be annihilated by $Q_+ \pm Q_-$. This massive state is called a quantum BPS state.

However, if (in a family of theories) two critical points of h collide and become a degenerate critical point, the mass of BPS paths between them goes to zero, so we can expect that the corresponding sector of Hilbert space loses its mass gap, and massless particles appear. At such a point, supersymmetric states can appear or disappear in the quantum theory. We will make this more explicit later in the context of $N = 2$ supersymmetry.

11.4. N=2 supersymmety in 2 dimensions.

Consider the space $\mathbb{R}^{2|4}$ with coordinates $x_+, x_-, \theta_+, \theta_-, \bar\theta_+, \bar\theta_-$. This space admits an action of the N=2 supersymmetry algebra, with supersymmetry generators

$$\begin{aligned} Q_+ &= \frac{\partial}{\partial\theta_+} + \bar\theta_+\frac{\partial}{\partial x_+}, \bar Q_+ = \frac{\partial}{\partial\bar\theta_+} + \theta_+\frac{\partial}{\partial x_+}, \\ Q_- &= \frac{\partial}{\partial\theta_-} + \bar\theta_-\frac{\partial}{\partial x_-}, \bar Q_- = \frac{\partial}{\partial\bar\theta_-} + \theta_-\frac{\partial}{\partial x_-}, \end{aligned} \tag{11.10}$$

Let us also introduce vector fields

$$\begin{aligned} D_+ &= \frac{\partial}{\partial\theta_+} - \bar\theta_+\frac{\partial}{\partial x_+}, \bar D_+ = \frac{\partial}{\partial\bar\theta_+} - \theta_+\frac{\partial}{\partial x_+}, \\ D_- &= \frac{\partial}{\partial\theta_-} - \bar\theta_-\frac{\partial}{\partial x_-}, \bar D_- = \frac{\partial}{\partial\bar\theta_-} - \theta_-\frac{\partial}{\partial x_-}, \end{aligned} \tag{11.11}$$

which commute with the supersymmetry generators. Therefore, any Lagrangian which is written in terms of the D's is supersymmetric.

Recall that a chiral function (or superfield) on $\mathbb{R}^{2|4}$ is a function satisfying the equations $\bar D_+\Phi = \bar D_-\Phi = 0$. A general solution to these equations has the form

$$\begin{aligned} \Phi = \phi - \theta_+\bar\theta_+\partial_+\phi - -\theta_-\bar\theta_-\partial_-\phi + \theta_+\bar\theta_+\theta_-\bar\theta_-\partial_+\partial_-\phi + \theta_+\theta_- F + \\ \theta_+\psi_+ + \theta_-\psi_- - \theta_+\bar\theta_+\theta_-\partial_+\psi_- - \theta_-\bar\theta_-\theta_+\partial_-\psi_+. \end{aligned} \tag{11.12}$$

You can read more about chiral functions in the superhomework.

Consider the theory of chiral maps Φ of $\mathbb{R}^{2|4}$ into $\mathbb{C}^n$, with the Lagrangian

$$\frac{1}{2}\int d^2x d^4\theta\Phi\bar\Phi + \int d^2x d\theta_+ d\theta_- W(\Phi) + \int d^2x d\bar\theta_+ d\bar\theta_- \bar W(\Phi), \tag{11.13}$$

where W is a holomorphic function on $\mathbb{C}^n$, called the superpotential. This Lagrangian is N=2 supersymmetric. In components (for $x_+ = z, x_- = \bar z$), it looks like

$$\frac{1}{2}\int d^2x(|d\phi|^2 - |F|^2 + W'(\phi)F + \bar W'(\phi)\bar F + \text{terms with fermions}). \tag{11.14}$$

Setting all fermions to zero and the "dummy" field F to the stationary point $F = \bar W'(\phi)$, we get the bosonic energy functional

$$H = \frac{1}{2}\int d^2x(|d\phi|^2 + |W'(\phi)|^2). \tag{11.15}$$

It is easy to see that this functional coincides with (11.7), for the function $h = \mathrm{Re}(e^{-i\alpha}W)$, where α is any real number. Thus, at the classical level we are doing a special case of the previous problem. However, now we have more supersymmetry and therefore a more interesting theory.

11.5. N=2 BPS states

In the theory we are considering, there is an important symmetry called the R-symmetry. It acts according to $\theta_+ \to e^{i\beta}\theta_+, \theta_- \to e^{-i\beta}\theta_-$. If we require that Φ is unchanged under this symmetry (i.e. Bose fields are unchanged and $\psi_+ \to e^{-i\beta}\psi_+, \psi_- \to e^{i\beta}\psi_-$), then Lagrangian (11.13) is obviously invariant under the symmetry.

The commutation relations for the supersymmetry Hamiltonians are

$$\begin{gathered} \{Q_\pm, \bar{Q}_\pm\} = 2P_\pm, \\ \{Q_+, \bar{Q}_-\} = \{\bar{Q}_+, Q_-\} = 0 \text{ (by R-symmetry)}, \\ \{Q_+, Q_-\} = T, \ \{\bar{Q}_+, \bar{Q}_-\} = \bar{T}, \end{gathered} \tag{11.16}$$

where T is analogous to S in Section 11.2 – it is a locally constant function on the space of classical solutions (which is, unlike S, not necessarily real), and for brevity we drop twiddles over Q's. Also, the squares of all the Q's are zero.

In fact, the function T is easy to compute, like the function S in the previous problem. Namely,

$$T = W(\phi(\infty)) - W(\phi(-\infty)). \tag{11.17}$$

Now we will look at supersymmetric states. Choose a real number α and look for states which are invariant under two supersymmetries $Q_1(\alpha) == Q_+ + e^{i\alpha}\bar{Q}_-$, $Q_2(\alpha) = \bar{Q}_+ - e^{-i\alpha}Q_-$.

We have

$$\{Q_1, Q_2\} = 2i(H - \mathrm{Re}(e^{-i\alpha}T)), \tag{11.18}$$

which implies that supersymmetric classical states have to be time-independent.

The equation $Q_1\Phi = 0$ for an even function Φ gives (in the time-independent case):

$$\frac{\partial\phi}{\partial x} = e^{-i\alpha}\overline{W'(\phi)}, \tag{11.19}$$

and the second equation gives the same result. This implies, in particular, that

$$\bar{T} = \bar{W}(\phi(\infty)) - \bar{W}(\phi(-\infty)) = \int \bar{W}'(\phi)\frac{d\phi}{dx}dx = e^{-i\alpha}\int_\infty^\infty |W'(\phi)|^2 dx, \tag{11.20}$$

which implies that $\alpha = \arg T$. Thus, from equation (11.28) we get that for a supersymmetric solution of the classical equations we have

$$H = Re(e^{-i\alpha}T) = |T|. \tag{11.21}$$

For other states in the connected component of this solution we have $H \geq |T|$.

Now, what are the supersymmetric solutions (of finite energy) geometrically? It is clear from equation (11.19) that they are separatrices between critical points of W for the gradient flow of $h = \mathrm{Re}(e^{-i\alpha}W)$.

Now let us turn to quantum theory. From classical considerations we saw that in our theory $H \geq |T|$, and in a nondegenerate case $H = |T|$ only for BPS states. Therefore, we should hope that in quantum theory the same situation takes place, apriori with a corrected value of T.

Consider the generic situation when all zero-modes of the Hamiltonian near a classical BPS state arise from the superPoincare group. This is the case if for any 3 critical points a, b, c of the potential

we have $|T_{ac}| < |T_{ab}| + |T_{bc}|$, where T_{ab} is the value of T on the component of the space of solutions which go from a to b. In other words, this is the case when the gradient flowline between a, b never passes through c.

In this case, in quantum theory, for small values of the coupling, we expect that the point $|T|$ in the spectrum of H occurs discretely. From our classical computations, we expect that the eigenspace corresponding to this eigenvalue is finite-dimensional, and is an irreducible representation of the odd part of the superPoincare algebra, in which Q_1, Q_2 act by zero. This representation is nothing but the space of sections of the equivariant vector bundle on the upper part of the hyperboloid, where the fiber is the standard 2-dimensional irreducible representation of the 4-dimensional Clifford algebra generated by the two remaining supersymmetries.

Notice that all other superPoincare representations occuring in this theory have to be not 2-dimensional but rather 4-dimensional over the ring of functions on the hyperboloid, since for levels of energy above T the Clifford algebra satisfied by the supersymmetry operators corresponds to a nondegenerate quadratic form, and the only irreducible represenation of this algebra is 4-dimensional.

11.6. N=1 Supersymmetry in 4 dimensions.

Now consider supersymmetry in 4 dimensions. We start with $N = 1$ supersymmetry. In this case the odd part of the supersymmetry algebra is similar to the $N = 2$ case in two dimensions. It is generated by $Q_\pm$, $\bar{Q}_\pm$, with relations

$$\begin{aligned} \{Q_\alpha, \bar{Q}_\beta\} &= 2P_{\alpha\beta}, \\ \{Q_\alpha, Q_\beta\} &= 0, \\ \{\bar{Q}_\alpha, \bar{Q}_\beta\} &= 0. \end{aligned} \tag{11.22}$$

where $\alpha, \beta \in \{+, -\}$, and $P_{\alpha\beta}$ is a basis of the space of complex linear functions on the spacetime (These relations exhibit the isomorphism of Poincare representations $S_+ \otimes S_- \to V_{\mathbb{C}}$, where $S_\pm$ are the spinor representations.) A central extension like in (11.16) cannot arise here because it is prohibited by the Poincare symmetry (this central extension would have to be in the representation $S^2\mathbb{C}^2$ of $SU(2)$ (where $\mathbb{C}^2$ is the standard representation), which is the spin 1 representation and contains no invariants).

The determinant of the quadratic form corresponding to the Clifford algebra (11.22) is $(P^2)^2$. The rank of this form if $P^2 = 0$ is 2. Therefore, massless representations of the superPoincare correspond to 2-dimensional representations of the Clifford algebra, and massive representations correspond to 4-dimensional representations of the Clifford algebra.

More precisely, consider a representation W of the superPoincare and the subspace W_p on which $P = p$, where $p \in \mathbb{R}^{1,3}$. Let G_p be the stabilizer of p in the group of rotations, H_p the maximal torus in G_p (always isomorphic to $U(1)$). With respect to H_p, W has a decomposition in a direct sum of representations of integer and half-integer spins. These spins are called helicities of W, and each helicity has a multiplicity.

It is easy to see that the supersymmetry operators $Q_\alpha, \bar{Q}_\alpha$ raise helicity by $1/2$ (since they live in the spinor representation of the Poincare). Thus, the massless representations have helicities $j, j + 1/2$ with multiplicity 1, and the massive representations have helicities $j, j + 1/2, j + 1$ with multiplicities $1, 2, 1$.

Now let us consider massless particle multiplets which are allowed by $N = 1$ supersymmetry. There are two such basic multiplets.

1. Vector multiplet: a gauge field A and a chiral spinor λ in the adjoint representation. This is the N=1 analogue of the gauge field. In particular, in the $U(1)$ case the theory is free. In the infrared, it generates a massless vector and a massless spinor, so the helicities are $-1, 1$ (for vector) and $-1/2, 1/2$ (for spinor). In particular, the massless representation of the SuperPoincare arising in this theory is

reducible, and splits into two: the one with helicities $-1, -1/2$ and the one with helicities $1/2, 1$. However, over $\mathbb{R}$ this splitting does not exist and our representation is irreducible. A more physical version of this statement is to say that the helicity $-1, -1/2$ states are related to helicity $1/2, 1$ states by CPT conjugation, so that one pair must be present if the other is. Thus, this field configuration is the minimal supersymmetric one which contains a gauge boson.

2. Chiral multiplet: A massless complex scalar ϕ and a massless chiral spinor χ. In this case the story is analogous. The obtained massless representation has a spinor and a scalar, so it has helicities $0, 0$ (for scalar) and $-1/2, 1/2$ (for spinor). Thus this representation is again a sum of two, with helicities $-1/2, 0$ and $0, 1/2$. This decomposition is only valid over $\mathbb{C}$ and not over $\mathbb{R}$; over $\mathbb{R}$, the representation is irreducible, so this combination is the smallest supersymmetric one containing a scalar.

The massive versions of these multiplets are as follows.

1. Massive vector multiplet (or hypermultiplet). The minimal real supersymmetric representation containing a massive vector has helicities $-1, -1/2, 0, 1/2, 1$ with multiplicities $1, 2, 2, 2, 1$. This involves a massive vector, two massive spinors (chiral and antichiral), and a real massive scalar. The corresponding 8-dimensional representation of the Clifford algebra is (over $\mathbb{C}$) a sum of two 4-dimensional representations.

2. Massive chiral multiplet. Same as massless chiral multiplet (we could consider the same fields with masses, such that the mass of bosons equals the mass of fermions).

11.7. N=2 Supersymmetry in 4 dimensions.

In the N=2 case, we have two copies of operators Q: $Q^{(1)}$ and $Q^{(2)}$, and they commute in the same way as before if the indices are equal, and give zero commutator if they are not equal (as vector fields). However, now there is a possibility for a central extension: it is no longer prohibited by the Poincare symmetry.

Consider first the case when the central term is zero. In this case the quadratic form of the Clifford algebra is nondegenerate (in 8 dimensions) in the massive case, and has rank 4 in the massless case. Thus, an irreducible massive representation should be 16-dimensional, and an irreducible massless representation should be 4-dimensional. The helicities for representations are obained like in $N = 1$ case. In particular, in a massive representation the helicities are in groups of the form (j,j+1/2,j+1,j+3/2,j+2) with multiplicities (1,4,6,4,1), and in a massless representation they are in groups (j,j+1/2,j+1) with multiplicities (1,2,1).

Now let us consider the simplest $N = 2$ supersymmetric theory. It is obtained by combining fields from a (classically) massless vector multiplet and a massless chiral multiplet in adjoint representation. Thus the fields are: from vector multiplet – (A, λ), from chiral multiplet – (ϕ, χ). The Lagrangian is the minimal $N = 2$ supersymmetric Lagrangian on these fields. Such a Lagrangian exists and is uniquely determined by the minimality condition. In the $U(1)$ case the theory is free, but in the nonabelian case there are nontrivial interactions.

What does this theory do in the infrared? In the $U(1)$ case, the answer is simple: we should add together the representations for the vector and chiral multiplets. We get helicities (-1,-1/2,0,1/2,1) with multiplicities (1,2,2,2,1). This is the sum of two complex conjugate massless representations of the N=2 Clifford algebra: the one with helicities (0,1/2,1) and multiplicities (1,2,1) and the one with helicities (-1,-1/2,0) and multiplicities (1,2,1).

But now let us consider the nonabelian case (say the gauge group G is simple). Then the bosonic fields are the gauge field A and a scalar ϕ in the complexified adjoint representation $\mathfrak{g}_{\mathbb{C}}$. The bosonic part of the Lagrangian is

$$L_{bosonic} = \int (\frac{1}{4e^2} F^2 + |d_A \phi|^2 + |[\phi, \bar{\phi}]|^2). \tag{11.23}$$

Thus the bosonic part of the space of classical vacua is the set of solutions of the equations $[\phi, \bar{\phi}] = 0$ modulo the action of G. This space can be identified with $\mathfrak{t}_{\mathbb{C}}/W$, where $\mathfrak{t}$ is the maximal commutative subalgebra in $\mathfrak{g}$ and W the Weyl group. In the case $G = SU(2)$, this quotient is identified with $\mathbb{C}$ by introducing the global coordinate $u = Tr(\phi^2)$ (the u-plane).

Recall from Lecture 2 that in this situation we have gauge symmetry breaking from G to the centralizer H of ϕ. In particular, for $SU(2)$ near $u \neq 0$ the gauge symmetry is broken classically from $SU(2)$ to $U(1)$ (Higgs mechanism). The components of the gauge field which are charged nontrivially with respect to the surviving $U(1)$ will become massive. By N=2 supersymmetry, the same will happen for the corresponding components of λ, ϕ, ψ. Thus, in the charge 2 and -2 sectors of the Hilbert space the lowest energy states will be in a massive representation with helicities as before: (-1,-1/2,0,1/2,1) with multiplicities (1,2,2,2,1). However, we know that there is no such representation without central extension. Thus, without central extension we get a contradiction, and hence the central extension must appear.

The central extension will show up in the commutation relations between $Q_\alpha^{(1)}$ and $Q_\beta^{(2)}$:

$$\{Q_\alpha^{(1)}, Q_\beta^{(2)}\} = \varepsilon_{\alpha\beta} Y, \tag{11.24}$$

where Y is an operator which commutes with all local operators (central charge). Before the central charge, the algebra had a $U(2)$ R-symmetry (action on indices 1 and 2), but the central charge breaks this symmetry down to $SU(2)$ (the chiral $U(1)$ symmetry is anomalous in our theory, because of the index problems, and it is broken to $\mathbb{Z}/2\mathbb{Z}$).

It is easy to see that in this case the massive particles considered above must have mass exactly $|Y|$ (this is where the quadratic form of the Clifford algebra becomes degenerate).

It turns out that classically there are BPS states which correspond to these massive particles. Namely if we are at the vacuum $u \in \mathbb{C}$, we have

$$\phi \sim \frac{1}{\sqrt{2}} \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}. \tag{11.25}$$

where $a = \pm u^{1/2}$. Let $a = \rho e^{i\alpha}$. We should look for BPS states (i.e. states which are time-independent and invariant under half of the supersymmetry) which satisfy the condition (11.25) at infinity. If $Y = |Y| e^{i\beta}$ then the equation of being invariant under half of the supersymmetry is

$$F = e^{-i\beta} * d_A \phi, \tag{11.26}$$

where $*$ is in $\mathbb{R}^3$. (the BPS monopole equations). Since F is real, we must have $\alpha = \beta$ modulo π.

It can be shown that such BPS states exist in sectors with charges 2 and -2 (charges are the eigenvalues of the infinitesimal operator corresponding to the unbroken $U(1)$-gauge symmetry). When these solutions are quantized, we will get the same result as in 2 dimensions. Namely, in quantum theory, we will get a representation of the superPoincare with helicities coming in groups (j,j+1/2,j+1) with multiplicities (1,2,1). Adding two copies of such groups with $j = -1$ and 0, we will get the massive hypermultiplet (=massive vector multiplet); this multiplet has the right helicities, which we found by considering the Higgs mechanism. Thus, in presence of central charge we get no contradiction.

Remark. Computing the commutators of the Q-s using currents, one can show that classically

$$Y = \int_\Sigma (\phi * F + \frac{1}{e^2} \phi F), \tag{11.27}$$

where Σ is a distant sphere in the space cycle. So it is a combination of the electric and magnetic charge for the effective $U(1)$ theory.

Lecture II-12: $N = 2$ SUSY theories in Dimension Two: Part I

Edward Witten[1]

1 Introduction

In this lecture and the next two we consider $N = 2$ supersymmetric field theories in dimension two. Today, after some general introductory remarks about such theories, we consider two-dimensional $U(1)$-gauge theories with complex-valued chiral superfields charged under the gauge group. These theories depend on two functions – a superpotential W and a twisted superpotential $\widetilde{W}$. The superpotential W is a complex polynomial in the superfields. This polynomial must satisfy a certain 'weight' conditions, but is otherwise free. For gauge group $U(1)$, the twisted superpotential $\widetilde{W}$ of the sort of model we will look at is determined by a single complex parameter $t = -ir + \frac{\theta}{2\pi}$, or more precisely by $\exp(-2\pi i t)$. Today we will keep the superpotential is fixed and we allow the twisted superpotential to vary. Thus, for each (generic) complex polynomial W satisfying the weight conditions, we get a family of theories parameterized by the point $\exp(2\pi i t)$ in the cylinder $\mathbf{C}/\mathbf{Z}$. We are interested in what happens at the ends of the cylinder $\mathrm{Im}(t) \mapsto +\infty$ and $\mathrm{Im}(t) \mapsto -\infty$. In the cases we consider, the limit as $\mathrm{Im}(t) \mapsto -\infty$ is described in terms of a σ-model on the projective variety X defined by the polynomial W giving the superpotential. In this region, $-\mathrm{Im}(t)$ is the Kahler class of X, and the fact that there is only one parameter in the superpotential is related to the fact that $H^{1,1}(X)$ (or at least the part of it that is pulled back from a certain ambient weighted projective space) is one-dimensional. The limit as $\mathrm{Im}(t) \mapsto +\infty$ is described in terms of an orbifold version of a Landau-Ginzburg model.

When $c_1(X) = 0$, we will find that the beta function vanishes for all t, and we get a family of conformally invariant theories. For $c_1(X) \neq 0$, we get instead a non-trivial renormalization group flow in t. The flow increases $\mathrm{Im}\, t$ when $c_1(X) > 0$ and decreases $\mathrm{Im}\, t$ when $c_1(X) < 0$. (Furthermore, in these latter two cases there are "extra" vacua at the end of the cylinder to which one flows in the infrared.) This allows us to see various Landau-Ginzburg models as infra-red limits of σ-models and other σ-models as infra-red limits of Landau-Ginzburg models. This is interesting because the descriptions of these types of models are very different at the classical level. Most of today's lecture will deal with special cases (the cases that $c_1(X) = 0$, and some of the cases with $c_1(X) > 0$ with flow to a massive infrared theory) that have been extensively studied, but the general picture that will be presented at the end of the lecture is actually new, as far as I know.

2 Generalities on $N = 2$ SUSY Theories in Two Dimensions

2.1 The β-function of $N = 2$ SUSY σ-models

The first remark concerns $N = 2$ supersymmetric σ-models with target a compact Kähler manifold X, which we assume has a Kähler-Einstein metric. In Gawedski's lectures we have seen that the one-loop β-function is determined by the sign of the Ricci curvature of the metric and in fact when the Ricci tensor is positive or negative-definite, the sign of the one-loop β-function is opposite that of the Ricci curvature. Thus, if $c_1(X) > 0$ then the one-loop β-function is negative. Thus, at small coupling the β-function is negative so that the σ-model under consideration is a 'good' fundamental theory which is asymptotically free. Such a theory is well-defined at the quantum level. The question arises of what it flows to in the infrared. Today we shall see examples of such X for which the infra-red theory is

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massive and trivial, and also examples with flow to a non-trivial infra-red fixed point. In the latter case, the limiting superconformal theories that we will get can be described in terms of what are known as supersymmetric Landau-Ginzburg theories. The Landau-Ginzburg models are sometimes explicitly soluble by algebraic methods.

On the other hand, if $c_1(X) < 0$, then the one-loop β-function is positive and hence β is positive at weak coupling. Thus, we have no reason to expect σ-models for such X to be 'good' fundamental theories. Rather in this case the σ-model is an effective theory, free in the infra-red, and one should ask whether one can describe a concrete, well-defined ultraviolet theory that flows to these sigma models in the infrared. We will solve this problem for this class of sigma models by taking $\mathrm{Im}t \to \infty$, where we will find a Landau-Ginzburg theory (which is superconformal), which after a relevant perturbation (to $\mathrm{Im}\, t$ large and positive but not ∞) flows to the sigma model with target space X (plus some extra massive vacua).

The case when $c_1(X) = 0$, i.e., the case when X is Calabi-Yau, is different. Here, the one loop β-function vanishes and one-loop contributions do not change the Kähler metric. It turns out that there are higher loop corrections to the β-function so that it is not identically zero and hence the σ-model on X (with its Ricci-flat metric) is not conformal. Nevertheless, standard invariant theory arguments show that there is no higher loop correction to the Kähler class of the metric. Using Yau's theorem about existence of Kähler metrics on X with prescribed Ricci curvature ($\partial\overline{\partial}$-exact) we can modify (at least order-by-order in perturbation theory) the original Ricci-flat Kähler metric without changing the complex structure or the Kähler class until the σ-model for the new Kähler structure on X has vanishing β-function. This process will converge if the Kähler class is sufficiently ample and far from the boundaries of the Kähler cone; for today's models these conditions simply mean that $-\mathrm{Im}\, t$ should be large enough. What we will be exploring today is really the question of what happens when $-\mathrm{Im}\, t$ is not large.

Today we shall see examples of X with $c_1(X) = 0$ for which there is a family of theories, parametrized by a punctured cylinder that approaches the σ-model on X as $\mathrm{Im}t \mapsto -\infty$ and approaches a Landau-Ginzburg model as $\mathrm{Im}t \mapsto +\infty$. To get such a family, the cylinder must be punctured at one point, where the theory has a sort of pole. It is believed that the punctured cylinder parametrizes a smooth family of theories, which flow in the infrared to a punctured cylinder of superconformal field theories. As one check of this belief, we argue that the central charge of the Landau-Ginzburg theory at $+\infty$ and the central charge of the σ-model at $-\infty$ are equal. We will present more evidence for the scenario described today in a later lecture.

2.2 Reasons for considering $N = 2$ SUSY theories

One advantage of studying $N = 2$ supersymmetric theories is that they are much more rigid than theories with less (or no) supersymmetry. For example if we are considering two-dimensional σ-models into a compact Riemannian manifold X, then without supersymmetry we can add terms to the Lagrangian of the form

$$\int d^2y\, \varphi^*(h)$$

for a potential function $h\colon X \to \mathbf{R}$. (Here, $\varphi\colon \Sigma \to X$ is the basic field in the Lagrangian.) In $N = 1$ supersymmetric σ-models, such a term is not possible, but there can still be a term

$$\int d^2y\, d^2\theta\, \Phi^*(h)$$

where again h is a real-valued function on X (called the superpotential) and Φ is the basic superfield in the Lagrangian. In both cases this is a relevant perturbation. When we require $N = 2$ supersymmetry,

the Riemannian manifold must be Kähler and the possible potential term can still exist but must be of the form $h = \mathrm{Re}(f)$ for f a holomorphic function. Since our Riemannian manifold is assumed to be compact, this means that f is constant and hence, after performing the θ integrals, this term vanishes. Simple power counting shows that at the classical level this is the only possible relevant perturbation over a 2|4-superspace.

2.3 Revelant and Marginal Perturbations of the conformal two-dimensional σ-model

Let us study the marginal perturbations of these conformal σ-models. Part of the data of the σ-model is the complex structure on X. Varying this is a marginal perturbation of the theory. Let us examine this perturbation in local coordinates. The metric tensor on X is a $(1,1)$-tensor of the form $h_{i\bar{j}}dz^i \otimes d\bar{z}^{\bar{j}}$. A change in the complex structure corresponds to a perturbation of the metric by a term of the form

$$\delta h_{\bar{i}\bar{j}} + \delta h_{ij}$$

of type $(2,0) + (0,2)$. Of course, this form must be real in order that the new form be a metric tensor, and $\delta h_{\bar{i}\bar{j}}$ must be $\overline{\partial}$-closed in order that the new almost complex structure is integrable. Let us compute the change in the Lagrangian that goes with this perturbation of the Kähler metric. We write things in local coordinates using bosonic coordinates $y^\alpha, \alpha = 1,2$ on the Riemann surface and fermionic coordinates $\theta^\pm, \overline{\theta}^\pm$. We write superfields as $X = X(y,\theta,\overline{\theta})$, chiral superfields as $\Phi(y,\theta)$ and anti-chiral superfields as $\overline{\Phi}(y,\overline{\theta})$. The change in the Lagrangian is given by

$$\Delta\mathcal{L} = \left(\int d^2y\, d^2\overline{\theta}\delta h_{\bar{i}\bar{j}}\overline{D}_+\overline{X}^{\bar{i}}\overline{D}_-\overline{X}^{\bar{j}}\right) + \text{c. c..}$$

It is a nice exercise to verify that the argument of the $\overline{\theta}$ integral in $\Delta\mathcal{L}$ is anti-chiral. (This uses the fact that $\delta h_{\bar{i}\bar{j}}$ is $\overline{\partial}$-closed.) This is one of the two possible types of marginal perturbations.

The other type of marginal perturbation is to fix the complex structure X and vary the complexified Kähler class of the Kähler metric. One piece of this deformation is an ordinary variation $\delta h^{\mathrm{herm}}_{\bar{i}j}d\bar{z}^{\bar{i}} \wedge dz^j$ by a hermitian symmetric closed two-form. This ordinary metric perturbation is the real part of a complex perturbation whose imaginary part is called the "B-field" perturbation (which, as we shall see, is analogous to the θ-angle which appears in gauge theories). Thus, we consider a perturbation $\delta h = \delta h_{\bar{i}j}d\bar{z}^{\bar{i}} \wedge dz^j$ where δh is a closed complex two-form. This leads to a perturbation of the Lagrangian of the form

$$\Delta\mathcal{L} = \left(\int d^2y d\overline{\theta}^+ d\theta^- (\delta h_{\bar{i}j}\overline{D}_+\overline{X}^{\bar{i}}D_-X^u\right) + \text{c. c..}$$

Other than the marginal deformations of the theory described above, there can only be deformations of the form

$$\int d^2y d^4\theta(\text{something}).$$

For these the θ-integration has weight 2, and hence all such terms are marginal (of dimension two) classically. (Notice this count requires four θ's and hence uses the fact that we have $N = 2$ SUSY). But at the quantum level, they have anomalous dimensions (proportional to nonzero eigenvalues of the Laplacian and so strictly positive) and are irrelevant. These types of deformations can affect the Kähler metric without changing the complex structure or Kähler class.

These computations show that for the σ-model with $c_1(X) = 0$, the theory has no relevant perturbations and the marginal perturbations are obtained by deforming the complex structure and complexified Kähler metric.

Let us begin with such a σ-model with a Kahler metric that is smooth and has a large radius of curvature close to, but not equal to, the metric that actually gives zero beta function. The renormalization group flow will bring us to a metric that gives conformal invariance, as all other deformations are irrelevant. Were there relevant perturbations these would dominate in the infrared limit and the limit would be in general impossible to control.

2.4 Central Charge

Let us review some material on central charges in conformal and super-conformal field theories in dimension two. There are two basic types of $N = 0$, i.e., of ordinary, conformal field theories – those with central charge $c < 1$, and those with central charge $c \geq 1$. The former are called minimal models, exist only for a discrete set of values of c, and are explicitly algebraically described in terms of representation theory of the Virasoro algebra. The theories with $c = 1$ can also be explicitly described, in terms of a free boson. When $c \leq 1$, the Virasoro algebras of left- and right- movers act almost irreducibly in the quantum Hilbert space, which is why these theories are known rather explicitly. For $c > 1$, the Virasoro action is very far from being irreducible (the number of highest weight vectors grows exponentially as one goes to higher energies), and conformal field theory has a completely different flavor.

When we go to the $N = 1$ super-conformal algebra in dimension two, the central charge again lies either in a discrete set or in a continuous part. The lowest value of the central charge in the continuous part is realized by the free scalar super-field. Being the sum of a free scalar boson and a free scalar fermion, considered as a representation of the ordinary conformal algebra this field has central charge $c = 3/2$. For this reason one usually defines $\widetilde{c} = 2c/3$ so that the continuous values for $\widetilde{c}$ begin at $\widetilde{c} = 1$.

The story is similar for the $N = 2$ super-conformal algebra in dimension two. Here the lowest value in the continuous part is realized by the free chiral super-field. Since this field consists of a complex scalar boson and a complex fermion, its $N = 0$ central charge is 3. Hence, we set $\widehat{c} = c/3$ so that once again the lowest value in the continuous part is $\widehat{c} = 1$. The discrete set of $N = 2$ theories with $\widehat{c} < 1$ are the "$N = 2$ minimal models" which we will consider in detail later on. Their central charges take the form $\widehat{c} = 1 - \frac{2}{n}$ for $n = 3, 4, 5, \ldots$, and they can be described algebraically and studied in a variety of attractive ways.

A conformal $N = 2$ supersymmetric σ-model with target space a k-dimensional complex manifold X of $c_1(X) = 0$ has central charge $\widehat{c} = k$. This is proved by using the fact that $\widehat{c}$ is constant in a family of conformal field theories, and expanding the metric until X can be approximated, in a local region, to any desired accuracy, to $\mathbf{C}^k$ with a flat metric. Thus, this sigma model has the same $\widehat{c}$ as a sigma model with target a flat $\mathbf{C}^k$. That is a free theory with k chiral superfields, and so has $\widehat{c} = k$, as claimed.

3 The $U(1)$ Theories

Let us turn now to the study of $U(1)$-gauge theory with complex-valued chiral superfields $A_1, \ldots, A_n$. These superfields are charged under the $U(1)$ with charges $q_1, \ldots, q_n \in \mathbf{Z}$. Recall from the super-homework that connections we use on 2|4-space are constrained to be flat in all pairs on odd directions except the pair $\{\overline{\theta}^+, \theta^-\}$ (and its complex conjugate). The basic invariant of such a connection is

$$\Sigma = \{\overline{\mathcal{D}}_+, \mathcal{D}_-\}$$

which is a section of the complexification of the adjoint bundle over 2|4-space. The minimal pure gauge theory Lagrangian that we can write down is

$$\int d^2y d^4\theta \frac{1}{2e^2}\overline{\Sigma}\Sigma.$$

Adding in a kinetic term for the n chiral superfields, the minimal Lagrangian becomes

$$\mathcal{L}_0 = \int d^2y d^4\theta \left(\frac{1}{2e^2}\overline{\Sigma}\Sigma + \sum_{i=1}^{n} \overline{A}_i A_i \right). \tag{3.1}$$

Eventually, we will add a superpotential and a twisted superpotential term to this Lagrangian.

3.1 *R*-symmetries

The above $N = 2$ supersymmetric gauge theory in two dimensions (or 2|4 supertheory) comes by dimensionally reducing from $N = 1$ gauge theory in four-dimensions (or 4|4 supertheory). As such, there is an R-symmetry for the $N = 2$ two-dimensional theory induced by spatial rotation in the remaining two dimensions in four-space. But, in addition, the $N = 1$ supersymmetric version of $U(1)$-gauge theory in four-dimensions itself has an R-symmetry and that symmetry continues to exist even if we add a superpotential. This R-symmetry from dimension four dimensionally reduces to give a second R-symmetry for our 2-dimensional theory.

Independent of their sources, it is easy to see directly in the 2|4-theory what the $U(1)\times U(1)$-group of R-symmetries is. We have J_R, the circle action

$$\theta^+ \mapsto e^{i\alpha}\theta^+$$

$$\overline{\theta}^+ \mapsto e^{-i\alpha}\overline{\theta}^+$$

with θ^- left fixed, and we have J_L, the circle action on θ^- with θ^+ left fixed. The R-symmetry that comes from the R-symmetry of four-dimensional gauge theory is the product of these two (i.e., it acts by the standard representation on both $\theta^{\pm}$). Spatial rotation in the omitted two dimensions induces the circle action which is the product of the first times the inverse of the second of these. Potentially there is an anomaly in one of these R-symmetries. To have a super-conformal theory this anomaly must vanish (or be cancelled) since these symmetries are part of the super conformal algebra.

Let us make more explicit the earlier remark that the anomaly of the σ-model is proportional to the first Chern class $c_1(X)$. The point is that when we write the Lagrangian out in coordinates we have an expression of the form

$$\mathcal{L} = \int d^2y \left(g_{i\bar{j}}\partial X^i \partial \overline{X}^{\bar{j}} + \overline{\psi}_-^{(0,1)}\partial\!\!\!/_+ \overline{\psi}_-^{(1,0)} + \psi_+^{(0,1)}\partial\!\!\!/_- \psi_+^{(1,0)} \right)$$

(plus a four-fermion term that will not affect the present discussion). One of the basic R-symmetries J_L rotates the $\psi_-^{(0,1)}$ and $\psi_-^{(1,0)}$ and the other J_R rotates in the same manner $\psi_+^{(0,1)}$ and $\psi_+^{(1,0)}$. Thus, under a chiral rotation generated by $a_R J_R + a_L J_L$, the measure of integration in the path integral $\mathcal{D}\psi_+\mathcal{D}\psi_-$ changes by an amount proportional to

$$a_R \text{index}(\partial\!\!\!/_+) + a_L \text{index}\partial\!\!\!/_-.$$

The operators $\partial\!\!\!/_\pm$ are Dirac operators of positive or negative chirality acting on the pullback to the Riemann surface of the holomorphic tangent bundle $T^{(1,0)}(X)$. By the index theorem both of these

operators have index given by $c_1(X)$ evaluated on the fundamental class of the Riemann surface. Thus, we see that the R-symmetry induced from the R-symmetry in four-dimensions is not anomalous, since $\text{index}\partial\!\!\!/_+ - \text{index}\partial\!\!\!/_- = 0$, but that the R-symmetry coming from the spatial rotation has an anomaly: $\text{index}\partial\!\!\!/_+ + \text{index}\partial\!\!\!/_- = 2\text{index}\partial\!\!\!/_+$.

We have just given a computation for the anomaly for the σ-model. Let us make an analogous computation in $U(1)$-gauge theories with chiral superfields A_i, $1 \leq i \leq n$. The superfield A_i is a section of the q_i-power of the line bundle associated to the connection on 2|4-space. It can be expanded in terms of ordinary fields, both bosons and fermions. The R-symmetries act as chiral symmetries on the fermions. So the same type of anomaly-index computation as above shows that the anomaly of J_R is given by

$$\sum_i q_i \int d^2y \frac{F_A}{2\pi}$$

where F_A is the curvature of the ordinary connection on the Riemann surface. (The anomaly for J_L is the negative of this number.) Thus, in order to get a superconformal field theory we need $\sum_i q_i = 0$.

4 One Example

To make things as simple and concrete as possible let us focus on one special case of our $U(1)$-gauge theory: $n+1$ chiral superfields

$$A_1, \ldots, A_n \text{ of weight } q = 1$$
$$A_{n+1} = P \text{ of weight } q = -n.$$

This choice of weights assures us that our R-symmetries are anomaly-free. Now it is time to supplement the minimal Lagrangian $\mathcal{L}_0$ of Equation 3.1 by a superpotential and a twisted superpotential. That is to say we consider Lagrangians of the form

$$\mathcal{L} = \mathcal{L}_0 + \left(\int d^2y d\theta^+ d\theta^- W(A, P) + \text{c. c.} \right) + \left(\int d^2y d\bar{\theta}^+ d\theta^- \widetilde{W}(\Sigma) + \text{c. c.} \right). \tag{4.1}$$

Here, W is called the superpotential; it must be a holomorphic function on $\mathbf{C}^{n+1}$. The function $\widetilde{W}(\Sigma)$ is called the twisted chiral superpotential term.

Let us examine the consequence of assuming that each of these two terms in the Lagrangian is invariant under the R-symmetries. We consider the twisted chiral superpotential term $\widetilde{W}$ first. Here, it is important to use the fact that our theory is obtained by dimensionally reducing a gauge theory on 4|4-space. Starting with a connection $A = \sum_{\mu=0}^{3} A_\mu dx^\mu$ on 4|4-space we form

$$\sum_{\alpha=0}^{1} dy^\alpha + \sigma(dx^2 + i dx^3) + \text{c. c.}.$$

This shows us how Σ transforms under the R-symmetries. For the extra term in Equation 4.1 coming from the twisted superpotential to be invariant, it must have the same transformation law under the R-symmetries as Σ does. That is to say $\widetilde{W}(\Sigma)$ must be a linear function of Σ, i.e., $\widetilde{W}(\Sigma) = t\Sigma$ for some $t \in \mathbf{C}$. Writing

$$t = \frac{\theta}{2\pi} - ir,$$

we have

$$\left(\int d^2y d\bar{\theta}^+ d\theta^- t\Sigma \right) + c.\, c. = \theta \int \frac{F}{2\pi} + r \int D.$$

Thus, we see that adding such a linear twisted superpotential produces a Fayet-Iliopulous term $r \int D$ plus a θ-angle term which is θ times an integral topological quantity. In this case the Fayet-Iliopulous term corresponds to adding a constant to the moment map for the $U(1)$-action on $\mathbf{C}^{n+1}$.

Now let us turn to the superpotential $W(A, P)$. For the term

$$\int d^2y d\theta^+ d\theta^- W(A, P)$$

to be invariant, we need $W(A, P)$ to transform with weight two under the R-symmetry induced from the R-symmetry in four dimensions and be invariant under the R-symmetry induced by spatial rotation in four-space. In order to arrange this, we choose the action of the R-symmetry groups on the line bundles of which the A_i and P are sections as follows: the line bundles L_i with sections A_i are invariant under the R-symmetries and the line bundle L_{n+1} with section P is invariant under the R-symmetry induced by spatial rotation and has weight two under the R-symmetry coming from the R-symmetry of four-space. With this choice of R-symmetry actions any superpotential of the form

$$W = P \cdot F(A_i)$$

for F an arbitrary polynomial of $A_1, \ldots, A_n$, is invariant under the R-symmetries. There is of course, one extra condition, namely that W must be gauge invariant. This implies that F must be homogeneous of degree n. Thus, any W of this form will produce a gauge invariant term in the Lagrangian which is invariant under the R-symmetries. This is the most general interaction with these symmetries.

4.1 Classical analysis

Now that we have specified our theory in detail, we want to determine what this theory is, particularly in the limits $r \to \pm\infty$. We begin with a classical analysis. Write each of the superfields appearing in our Lagrangian in components

$$A = a + \theta(\cdots) + \cdots$$

$$P = p + \theta(\cdots) + \cdots$$

$$\Sigma = \sigma + \theta(\cdots) + \cdots$$

The superfield Σ includes auxiliary fields D and F which can be eliminated by means of their equations of motion.

The bosonic potential is a function $V(a, p, \sigma)$ of the bosonic components of these fields. For the Lagrangian we have described, the potential takes the form

$$V(a, p, \sigma) = \frac{1}{2e^2} D^2 + |dW|^2 + |\sigma|^2 \left(\sum |a_i|^2 + n^2 |p|^2 \right). \tag{4.2}$$

The last term in eq. (4.2) arises as follows: in four dimensions, a chiral field $\Phi = \phi + \theta(\cdots) + \cdots$ leads to a term in the potential of the form

$$\sum_{\mu=0}^{3} \left| \frac{D\phi}{Dx^\mu} \right|^2$$

which upon reduction to two dimensions becomes

$$\sum_{\alpha=0}^{1} \left| \frac{D\phi}{Dx^\alpha} \right|^2 + n^2 |\sigma|^2 |\phi|^2$$

if ϕ has charge n.

In general, the "D-term" (the auxiliary field in Σ), is equal by its equations of motion to $\sum q_i|\phi_i|^2 - r$. This is actually a familiar function mathematically; it is the "moment function" generating the $U(1)$ gauge action on the flat Kahler manifold (a copy of $\mathbf{C}^s$, with s the number of chiral superfields) in which the chiral superfields take their values. In our example, this becomes

$$D = \sum |a_i|^2 - n|p|^2 - r. \tag{4.3}$$

Since $W = P \cdot F(A)$, we also have

$$|dW|^2 = |F|^2 + |p|^2|dF|^2. \tag{4.4}$$

We will focus on the case in which F is *transverse*, i.e., $F = dF = 0$ only at the origin.

We want to find all classical zero-energy solutions, in other words, to solve $V = 0$. Thanks to the form of the potential, this requires that (i) $D = 0$, (ii) $|dW|^2 = 0$ and (iii) either $\sigma = 0$ or $\sum |a_i|^2 + n^2|p|^2 = 0$. We will also need to take the gauge equivalence under $U(1)$ into account.

First, note that thanks to eq. (4.4), setting $|dW|^2 = 0$ when $p \neq 0$ implies $F = dF = 0$ so by our transversality condition, all $a_i = 0$. Thus, there are two cases: $p = 0$ or $a_i = 0$ for all i. On the other hand, if $r \neq 0$, then by eq. (4.3) since $D = 0$ we cannot have both $p = 0$ and $a_i = 0$ for all i. (In fact the sign of r will determine which one of these holds.) Thus, when $r \neq 0$ we have $\sum |a_i|^2 + n^2|p|^2 \neq 0$ and so σ must be 0.

Consider now the case $r > 0$. Our equations for classical vacua become $p = 0$, $\sum |a_i|^2 = r$, and $F = 0$, and we must divide by the action of the gauge group $U(1)$. This gives the hypersurface X defined by the equation $F = 0$ in $\mathbf{CP}^{n-1}$, with Kähler class r. Thus, classically our theory can be described as a nonlinear σ-model whose target space is this hypersurface X, using a metric whose Kähler class has volume proportional to r^{n-2}. As we noted above, such a theory has central charge $\widehat{c} = n - 2$.

This classical description will be a good approximation to the quantum theory when $r \tilde{\gg} 0$, since the nonlinear σ-model is weakly coupled there. In fact, since the nonlinear σ-model is stable (in the sense of having no relevant perturbations), and marginal perturbations merely vary the complexified Kähler and complex structures, we should expect our quantum theory to coincide with one member of this family of σ-models.

On the other hand, when $r < 0$, the space of classical vacua satisfies $a_i = 0$ and $n|p|^2 = -r$. We can use a gauge transformation to fix $p = \sqrt{-r/n}$, leaving a residual gauge invariance of $\mathbf{Z}_n$, i.e., the original $U(1)$ gauge group is broken to $\mathbf{Z}_n$. This will therefore be what is known as an "orbifold" theory, in which the $\mathbf{Z}_n$ shows up in a global analysis of the theory (in a manner which we shall describe later). The local description of the theory (for which we can ignore the orbifolding issue) is this: for $r \ll 0$, the field P has a large mass and can be integrated out, leaving an effective theory of n massless chiral superfields $A_1, \ldots, A_n$ with an effective interaction

$$W_{\text{eff}} = \text{constant} \cdot F(A_1, \ldots, A_n),$$

where F is a homogenous polynomial of degree n.

Such a theory of n massless fields with a polynomial interaction, is called a *Landau–Ginzburg theory*. It should apparently flow in the IR to a conformal field theory with $\widehat{c} = n - 2$. (We expect this since that is what happened for $r \tilde{\gg} 0$, and neither the conformality nor the central charge of the IR theory should change when we vary r.)

Let's check this prediction, following work of Greene, Vafa, Warner, and Martinec among others. Consider a special F of "Fermat type:"

$$F(A_1, \ldots, A_n) = \sum A_i^n.$$

The Landau–Ginzburg theory with this interaction will (at least locally) factor as a product of n identical theories, each with a simple A^n interaction. Our prediction implies that each of these theories must have $\widehat{c} = \frac{n-2}{n}$. But that is exactly the central charge of a minimal model! So if we can identify the minimal model with this Landau–Ginzburg theory, we will have established the result. Note that we would have been in difficulties if we had obtained a value of $\widehat{c}$ less than one and not of the form $1-2/k$ for some integer k! Greene, Vafa, Warner and Martinec presented evidence that the Landau–Ginzburg theory with superpotential A^n does flow in the infrared to an $N = 2$ minimal model. More arguments have been found subsequently. We will see some of the evidence for this claim in Lecture II-14.

4.2 Orbifolding

As we pointed out above, the original gauge symmetry group of $U(1)$ of our theory was broken to $\mathbf{Z}_n$ in the effective theory when $r \ll 0$. Thus, if we consider the path integral for the low energy effective theory on a Riemann surface Σ, the $U(1)$ gauge group is reduced to $\mathbf{Z}_n$, and we need only sum over flat $\mathbf{Z}_n$ bundles. On any surface Σ, a flat $\mathbf{Z}_n$ bundle can be specified by its holonomies $\gamma_1, \ldots, \gamma_k \in \mathbf{Z}_n$ around various loops in Σ. The process of summing over all flat $\mathbf{Z}_n$ bundles in a theory with a $\mathbf{Z}_n$ symmetry is known as *orbifolding* the theory.

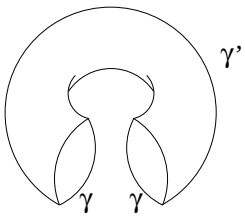
In a Hamiltonian approach, formulated on a circle S^1, we need to specify the holonomy $\gamma \in \mathbf{Z}_n$ around the circle as part of the data determining a state. Let $\mathcal{H}_\gamma$ be the Hilbert space of states whose holonomy is γ. (Then Riemann surfaces with boundary on which flat $\mathbf{Z}_n$ bundles have been specified will determine operators mapping among these various Hilbert spaces, depending on the holonomies on the boundary circles.) $\mathbf{Z}_n$ acts on each $\mathcal{H}_\gamma$, since as the group $\mathbf{Z}_n$ is abelian, twisting as one goes around the circle by an element of $\mathbf{Z}_n$ is an operation that commutes with $\mathbf{Z}_n$.

We claim that the Hilbert space for the orbifolded theory is

$$\mathcal{H} = \bigoplus_\gamma (\mathcal{H}_\gamma)^{\mathbf{Z}_n}.$$

To see this, consider a cylinder with holonomy γ on the ends:

There is still as we have noted an action of $\mathbf{Z}_n$ on each sector $\mathcal{H}_\gamma$; let $\gamma' \in \mathbf{Z}_n$ determine an operator $\gamma' : \mathcal{H}_\gamma \to \mathcal{H}_\gamma$. The trace of this operator can be evaluated as a partition function $Z_{\gamma'\gamma}$ on a torus



constructed by using γ' to twist the identification of bundles on the boundary circles of the cylinder, i.e.,

$$Z_{\gamma'\gamma} = \mathrm{Tr}_{\mathcal{H}_\gamma} \gamma' q^H.$$

If we sum the operators γ', the result is $n\pi$ where π is the projection onto the $\mathbf{Z}_n$-invariant subspace:

$$\frac{1}{n}\sum_{\gamma'} Z_{\gamma'\gamma} = \mathrm{Tr}_{\mathcal{H}_\gamma}\pi q^H = \mathrm{Tr}_{(\mathcal{H}_\gamma)^{\mathbf{Z}_n}} q^H.$$

Now the partition function for the $\mathbf{Z}_n$ gauge theory[2] is obtained by summing further on γ:

$$\frac{1}{n}\sum_{\gamma,\gamma'} Z_{\gamma'\gamma} = \mathrm{Tr}_{\oplus(\mathcal{H}_\gamma)^{\mathbf{Z}_n}} q^H,$$

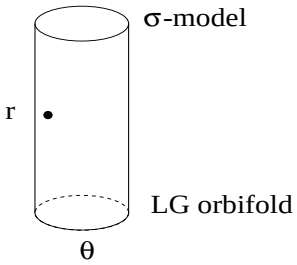
from which we conclude that the Hilbert space for our problem is indeed $\mathcal{H} = \bigoplus_\gamma (\mathcal{H}_\gamma)^{\mathbf{Z}_n}$.

4.3 Interpolation from positive to negative r

Our classical analysis can be summarized as follows: we needed to set $D = dW = 0$ and divide by $U(1)$. The step of setting $D = 0$ and dividing by $U(1)$ is the familiar mathematical operation of *symplectic reduction*, in which $D = 0$ defines a level set for the moment map of the $U(1)$ action (with the choice of r specifying the level). There is another mathematical interpretation of this process, as a quotient in the sense of Geometric Invariant Theory (GIT): we complexify the group $U(1)$ to $\mathbf{C}^*$, and consider the action of $\mathbf{C}^*$ on $\mathbf{C}^{n+1}$ with the same weights as before (the A_i's have weight 1 and P has weight $-n$). There are two possible quotients (topologically): for $r > 0$ the quotient can be interpreted as the total space of the bundle $\mathcal{O}_{\mathbf{P}^{n-1}}(-n)$ (in which p serves as a fiber coordinate), while for $r < 0$ the quotient is $\mathbf{C}^n/\mathbf{Z}_n$.

In either case we must still impose $dW = 0$: in the $r > 0$ case, $W = PF(A_1, \ldots, A_n)$ is a nondegenerate function (in a generalized sense introduced by Bott) and the the space of critical points in $\mathcal{O}_{\mathbf{P}^{n-1}}(-n)$ is the variety in $\mathbf{P}^{n-1}$ defined by the equation $F = 0$; in the $r < 0$ case we have a homogenous polynomial $F(A_1, \ldots, A_n)$ with a highly degenerate critical point at the origin of $\mathbf{C}^n/\mathbf{Z}_n$.

The natural parameter in our Lagrangian is $t = -ir + \frac{\theta}{2\pi}$. It is the possibility of going to $\theta \neq 0$ that will enable us to interpolate from positive r to negative r without meeting the singularity that one would find in classical geometry.



Classically, there is a singularity at $r = 0$ with arbitrary θ, and interpolation is not possible. Quantum mechanically, we claim that the singularity will be isolated, located at $r = r_0$ (for some r_0)

[2] In performing the path integral in a gauge theory, one is supposed to divide by the volume of the gauge group. When the gauge group is a Lie group of positive dimension, the group of gauge transformations is infinite-dimensional and making sense of "dividing by the volume of the gauge group" requires the Faddeev–Popov construction and introduction of ghosts. In the present case, we are considering a low energy effective theory with gauge group $\mathbf{Z}_n$. The only gauge transformations on Σ are constant transformations by elements of $\mathbf{Z}_n$; the volume of the gauge group is therefore n, the number of elements of $\mathbf{Z}_n$. The only gauge connections are the flat connections. So in the formula below, the sum over γ and γ' is the path integral over the $\mathbf{Z}_n$ connections, and the factor of $1/n$ results from dividing by the volume of the gauge group.

and $\theta = 0$. (Classically, the singularity is at $r = 0$, but quantum mechanically, it is shifted away from $r = 0$ by a one-loop correction.) It is believed that there are no other singularities; we will not prove this claim rigorously, but only explain some of the reasons physicists believe it to be true. The more rigorous part of our analysis will be the demonstration that there *is* a singularity at this location; the less rigorous part will be the argument that this is the only singularity.

To make this argument, we again work on a circular space S^1 of finite radius. The compactness means that the story is rather similar to ordinary quantum mechanics in zero space dimensions, so let us first recall what happens in ordinary quantum mechanics of a point particle. We suppose that the particle is moving on a manifold X with a potential function V. If X is compact, or if V grows at the "ends" of X, then one can vary the parameters upon which X and V depend without meeting a singularity in the ground state of the quantum mechanics. However, one will get a singularity if one varies in the parameters to a point at which V no longer grows at infinity. For instance, if $X = \mathbf{R}$, and $V = \frac{1}{2}kx^2$ (with x a linear function on $\mathbf{R}$), then the ground state wave is a smooth function of k as long as $k > 0$, but develops a singularity at $k = 0$. The $k = 0$ problem has no normalizable ground state wave function.

Something like that will happen in our problem on a circle. Recall that the form of our potential is

$$V = |P|^2|F|^2 + |dF|^2 + \frac{e^2}{2}\left(\sum |A_i|^2 - n|P|^2 - r\right)^2 + |\sigma|^2\left(\sum |A_i|^2 + n^2|P|^2\right).$$

For generic r, this theory has the property that the potential grows at infinity. This ensures that the theory has a discrete spectrum, and a ground state wave function that varies smoothly with the parameters. However, at $r = 0$, we see that one can go to infinity in σ, at no cost in energy, as long as A_i and P vanish. This suggests that at $r = 0$ there might be a continuous spectrum and a singularity of the ground state wave function.

To explore this question in more detail, we need to understand how the theory behaves in the dangerous region, that is, very large σ with other fields small. When A and P are very close to zero, and σ is large, A and P have large masses (on the order of $|\sigma|^2$) and they can be integrated out, leaving us with a pure gauge theory with an effective bosonic potential which at the classical level is merely a constant

$$V_{\rm eff} = \frac{e^2}{2}r^2. \tag{4.5}$$

If this is the correct answer, quantum states decay exponentially in the large σ region if their energy is less than $e^2r^2/2$. If so, this region could be dangerous for a supersymmetric ground state of zero energy – and could lead to a singularity in the wave function of such a state – only if $r = 0$.

To understand the situation better, we must make the analogous argument quantum mechanically. For this, we consider the superfield to which σ belongs, i.e., we write

$$\Sigma = \sigma + \theta\lambda + \theta\bar{\theta}(F + iD).$$

The action for the supersymmetric gauge theory is

$$\frac{1}{2e^2}\int d^4\theta\,\overline{\Sigma}\Sigma + \left(\int d\overline{\theta}^+\,d\theta^-\,t\Sigma + \text{c.c.}\right). \tag{4.6}$$

If we perform the θ-integrals in the effective Lagrangian (4.6), we get

$$\frac{1}{e^2}\int d^2x\left(\frac{F^2}{2} + |d\sigma|^2 + \overline{\lambda}\partial\!\!\!/\lambda\right) + \int d^2x\left(\frac{e^2r^2}{2}\right) + i\theta\int_\Sigma \frac{F}{2\pi},$$

which includes a curvature term (multiplied by a "θ-angle") as well as the bosonic potential previously discussed.

Luckily, we know the θ-dependence of energy in $U(1)$ gauge theories in two dimensions as computed in lecture II-4:

$$\frac{e^2}{2} \min_{n \in \mathbf{Z}} \left(n - \frac{\theta}{2\pi} \right)^2 .$$

So as $\sigma \to \infty$, we have $V(\sigma) \sim \frac{e^2}{2} |\widetilde{t}|^2$ where $\widetilde{t} \equiv -ir + \frac{\theta}{2\pi} \pmod{\mathbf{Z}}$. In other words, while the classical energy at large σ is $e^2 r^2/2$, quantum mechanically r^2 is replaced by $|\widetilde{t}|^2$. Thus, while classically the singularity is at $r = 0$ and θ is invisible, quantum mechanically the singularity is at $\widetilde{t} = 0$. For any other value of $\widetilde{t}$, there is a positive energy for $\sigma \to 0$, and this region is not dangerous for supersymmetric ground states.

Their is actually one more quantum effect of relevance: a one loop correction gives a finite renormalization of r, and shifts the position of the singularity from $\widetilde{t} = 0$ to $\widetilde{t} = -ir_0$ for a certain constant r_0. This correction is important for comparing to certain predictions of classical geometry, but not important for what we will say today.

5 Another example: flops

As a second example, we consider a model with gauge group $G = U(1)$, chiral fields A_1 and A_2 of charge 1, chiral fields B_1 and B_2 of charge -1, with *no* superpotential, and no "P." The Lagrangian takes the form

$$\int d^2x\, d^4\theta \left(\frac{1}{2e^2} \overline{\Sigma}\Sigma + \sum (|A_i|^2 + |B_i|^2) \right) + \left(t \int d\overline{\theta}^+ \, d\theta^- \, \Sigma + \text{c.c.} \right).$$

The D term is

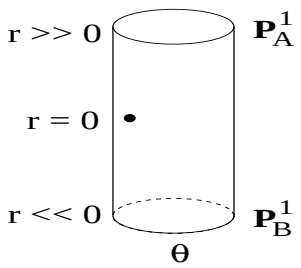
$$D = \sum (|A_i|^2 - |B_i|^2) - r,$$

and the bosonic potential also contains a term

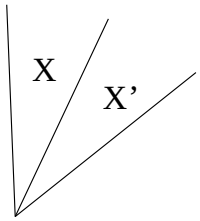
$$|\sigma|^2 \sum (|A_i|^2 + |B_i|^2).$$

As before, we must set $D = 0$ and divide by $U(1)$. When $r > 0$ this yields the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbf{P}^1_A$, whereas when $r < 0$ this yields the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbf{P}^1_B$. Here $\mathbf{P}^1_A$ is a copy of $\mathbf{P}^1$ obtained by requiring A_1, A_2 to be not both zero, and dividing the pair (A_1, A_2) by $\mathbf{C}^*$. Likewise $\mathbf{P}^1_B$ is a copy of $\mathbf{P}^1$ obtained by projectivizing (B_1, B_2).

The transformation from the bundle over $\mathbf{P}^1_A$ to the bundle over $\mathbf{P}^1_B$ is a simple model of a birational transformation known as a "flop." As in the previous example, the singularity only occurs at $r = \theta = 0$ and one can interpolate between these two models. This contrasts with classical geometry, where one passes through a singularity in going from one model of the quotient $\mathbf{C}^4/\mathbf{C}^*$ to the other.



More generally, consider two Calabi–Yau manifolds X and X' which are birationally equivalent, in such a way that their ample cones meet.



For simplicity assume $h^{2,0} = 0$, and note that birational transformations won't affect the value of $h^{p,0}$. Classically, to pass from X to X', one goes through a singularity on the wall of the Kahler cone. The singularity is rather similar to the singularity just found in our "flop" example at $r = 0$. In quantum field theory, one can go around the singularity by taking $\theta \neq 0$; thus one can smoothly continue from X to X' by varying the parameters of the conformal field theory.

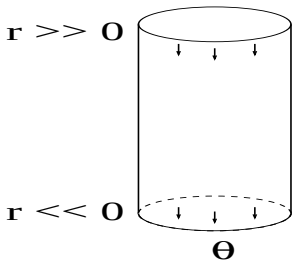
For example, we might construct such a pair of Calabi–Yau manifolds with $h^{1,1} = 2$ by starting with a gauge group $G = U(1) \times U(1)$, and a superpotential of the form $W = PF(A_i)$. If one sets things up correctly, one of the $U(1)$'s puts us in the world of projective varieties X and X', and by varying the moment map for the other $U(1)$, we can make a "flop" between X and X' that is quite like our above discussion with the four chiral superfields fields A_i and B_j.

6 Cases in which $c_1 \neq 0$

We now wish to consider linear sigma models in which the R-symmetry is anomolous, so that we expect a nontrivial renormalization group flow.

6.1 Negative β-function

We begin with the case of negative β-function, which for nonlinear sigma models corresponds to target spaces X for which $c_1(TX) > 0$. We can build examples of these exactly as before, with $U(1)$ acting on $n + 1$ fields A_i, P, but this time we give P charge $-k$ and take a superpotential $W = P \cdot F$ with F homogeneous of degree k. The beta function of the sigma model is negative, zero, or positive for $k < n$, $k = n$, or $k > n$. For $k \neq n$, even though there is still classically a singularity at $r = 0$, the quantum theory has no singularity on the t-cylinder. (This is shown by analyzing the behavior at large σ, as we do presently.) For $k \neq n$, there is a nontrivial renormalization group flow on the cylinder. The flow is holomorphic and singularity-free, so it is a constant flow "downward" or "upward" on the cylinder, depending on the sign of $n - k$.



When we repeat the previous analysis, we find that the R-symmetry which comes from the four-dimensional R-symmetry is anomalous, although the one which comes from rotations in the missing directions is anomaly-free.

The classical analysis can also be repeated: for $r \tilde{g} 0$ we find a space of classical vacua desribed by $\{F = 0\}$ in $\mathbf{P}^{n-1}$, and we expect the nonlinear sigma model on this hypersurface to be a good approximation to our theory. This hypersurface has positive first Chern class if $k < n$, so we expect a well-defined quantum field theory from looking at the nonlinear sigma model. On the other hand, since $k < n$, the sigma model is infrared-unstable. The infrared flow will take us away from the sigma model in the infrared and toward the theory we will find at the $r \to -\infty$ limit. Note that the sigma model has an $n-2$ dimensional target space, so for $k < n$ the effective central charge in the ultraviolet limit is $\widehat{c} = n - 2$.

On the other hand, if $r \ll 0$ then on the space of classical vacua $P \neq 0$ but $A_i = \sigma = 0$. Giving a nonzero expectation value to P breaks $U(1)$ to $\mathbf{Z}_k$, and (in the $r \to -\infty$ limit) we get a Landau-Ginzburg orbifold with an effective interaction

$$W_{\text{eff}}(A_i) = F(A_i).$$

The central charge of this model is

$$\widehat{c} = n(1 - \frac{2}{k}).$$

This is less than $n - 2$ if $k < n$. We will argue shortly that, for $k < n$, the Landau-Ginzburg theory is the infrared limit of the renormalization group flow from the sigma model; and the fact that its central charge is smaller than that of the sigma model illustrates a general theorem by Zamolodchikov, which asserts that the central charge always diminishes along a renormalization group flow: This infrared theory is behaving as if it had fewer degrees of freedom

$$c_{\text{IR}} \leq c_{\text{UV}}.$$

The intuition behind this theorem is that the central charge measures the total number of degrees of freedom of the theory. As one flows toward the infrared, massive degrees of freedom are "integrated out," and no longer contribute to the central charge, which therefore can only diminish.

As we will see, the Landau-Ginzburg model is only one possibility for what the infrared flow can lead to for $k < n$. There are in fact $n - k$ other vacuum states at $r = -\infty$, with a mass gap, which can't be seen classically but will require a 1-loop calculation. Being massive, they have $\widehat{c} = 0$, which is certainly less than $n - 2$.

6.2 The $\mathbf{CP}^{n-1}$ model

An extreme case of the situation we are considering is the case $k = 0$, i.e., the case of no P field and no superpotential. This is a $U(1)$ gauge theory with n free charged superfields $A_1, \ldots, A_n$ (all of charge 1). The D-term takes the form

$$D = \sum |A_i|^2 - r,$$

and we find the supersymmetric $\mathbf{CP}^{n-1}$ model which we studied earlier. It will be very instructive to re-examine this case in detail before going back to the general case.

At first glance, there would appear to be no supersymmetric vacua at all in the $r \to -\infty$ limit of this $\mathbf{CP}^{n-1}$ model, since the moment map is strictly positive. However, it cannot be so that there are

no supersymmetric vacua at all in this limit. An obstruction is provided by the supersymmetric index $\mathrm{Tr}\,(-1)^F$. This, when computed for large positive r, is seen to coincide with the Euler characteristic of $\mathbf{CP}^{n-1}$, which is n. The supersymmetric index must be invariant under deformations of r, so there must be n supersymmetric vacua even if r is large and negative. We must find them somewhere!

To see what is happening, we must consider the quantum mechanics more carefully. The classical treatment of the $r \to -\infty$ limit is valid in any compact region of field space, so if there are quantum mechanical vacua for arbitrarily negative r that cannot be seen classically, they must disappear from the classical field of view by going off to infinity in field space for $r \to -\infty$. So again we must consider the quantum-mechanical behavior at large σ. For our CP^{n-1} model, the bosonic potential takes the form

$$V = \frac{e^2}{2} D^2 + |\sigma|^2 \sum |A_i|^2.$$

The theory is weakly coupled for $r \ll 0$.

What happens if $|\sigma| \to \infty$? As earlier, the fields A_i will then be massive and can be integrated out to yield an effective theory which is an abelian gauge theory with Lagrangian

$$\mathcal{L}_{\mathrm{eff}} = \int d^2x\, d^4\theta\, \frac{1}{2e^2}\, \overline{\Sigma}\Sigma + \left(\int d^2x\, d\overline{\theta}^+\, d\theta^-\, \widetilde{W}_{\mathrm{eff}}(\Sigma) + c.c. \right).$$

The effective twisted superpotential depends on the renormalization mass scale μ, and we claim that it takes the unusual (R-symmetry violating) form

$$\widetilde{W}_{\mathrm{eff}}(\Sigma) = t\Sigma + \frac{in}{2\pi} \left(\Sigma - \Sigma \ln(\Sigma/\mu) \right) \tag{6.1}$$

which we shall verify by making a 1-loop calculation below. Notice that a change in the choice of logarithm in eq. (6.1) can be compensated for by changing the value of t by an integer; such a change has no effect on the physics. Of course, we must keep Σ away from zero in order to define this term, but that is consistent with our assumption that σ is large.

With this new term in the twisted superpotential, the equation for a critical point of $\widetilde{W}_{\mathrm{eff}}$ is

$$0 = t - \frac{in}{2\pi} \ln(\Sigma/\mu),$$

or in components,

$$0 = -ir + \frac{\theta}{2\pi} - \frac{in}{2\pi} \ln(\sigma/\mu) \pmod 1.$$

This has n solutions

$$\sigma = \mu \exp\left(-\frac{2\pi r}{n} - \frac{i\theta}{n} + \frac{2\pi i k}{n} \right), \tag{6.2}$$

$k = 0, \ldots, n-1$, corresponding to n different vacua in the $r \to -\infty$ limit. n is of course, the expected number, the Euler characteristic of $\mathbf{CP}^{n-1}$. Notice that $|\sigma|$ is indeed large when $r \to -\infty$, so the assumptions made in deriving these vacua hold. The fact that the n vacua go off to infinity for $r \to -\infty$, along with the fact that their existence depends on the one-loop quantum correction, explains why one cannot see them classically.

Another very important consequence of the logarithm is that it means that for $\sigma \to \infty$, the energy in the quantum theory grows like $|\ln \sigma|^2$. Hence, in particular, low energy states are limited to a bounded region of field space, and there is no singularity in the vacuum behavior for any value of t on the cylinder. As we will see, the models with $n \neq k$ all have such a logarithm, and hence they are all nonsingular throughout the t cylinder.

6.3 The 1-loop calculation

To understand the origin of the term $\frac{in}{2\pi}(\Sigma - \Sigma\ln(\Sigma/\mu))$ in the effective twisted superpotential, note that its presence is equivalent to the effective Lagrangian having a term

$$\int d^2x\, r_{\text{eff}}\, D,$$

where

$$r_{\text{eff}} = r + \frac{n}{2\pi}\ln(|\sigma|/\mu).$$

In this theory, we have $D = \sum \overline{A}_i A_i - r$. Classically, $A_i = 0 = \overline{A}_i A_i$ for large σ, but quantum mechanically $\overline{A}_i A_i$ has an expectation value at large σ; this expectation value can be interpreted in an effective classical description as a shift in $-r$. So we need to calculate the expectation value of $\overline{A}A$ by means of a 1-loop diagram

which contributes

$$\int \frac{d^2k}{(2\pi)^2}\,\frac{1}{k^2+m_A^2} = \int \frac{d^2k}{(2\pi)^2}\,\frac{1}{k^2+|\sigma|^2}$$

where we got the mass m_A from the term $|\sigma|^2|A|^2$ in the potential V. Cutting off the divergent integral, we find

$$\int^{\Lambda} \frac{d^2k}{(2\pi)^2}\,\frac{1}{k^2+|\sigma|^2} = -\frac{1}{2\pi}\ln\frac{|\sigma|}{\mu}.$$

This is the shift in $-r$, and accounts for the claimed formula for r_{eff}. There are, by the way, no higher loop corrections to this formula. This can be proved using holomorphy, or by noting that, in this superrenormalizable theory, higher loop diagrams would involve negative powers of σ and would vanish for large σ.

Let us now repeat this for a general case with chiral superfields B_i of charges q_i. (The B_i are to include all of the chiral superfields, including the P field of the sigma model discussion, if it is present.) The D-term is $\sum q_i|B_i|^2$ and for its expectation value at large σ we get the 1-loop correction

$$\sum q_i \int^{\Lambda} \frac{d^2k}{(2\pi)^2}\,\frac{1}{k^2+q_i^2|\sigma|^2} = -\frac{1}{2\pi}\sum q_i \ln|q_i| - \frac{1}{2\pi}\Big(\sum q_i\Big)\ln\frac{|\sigma|}{\mu},$$

leading to an effective twisted superpotential

$$\widetilde{W}_{\text{eff}}(\Sigma) = \left(t - \frac{i}{2\pi}\sum q_i\ln|q_i|\right)\Sigma + \frac{i\sum q_i}{2\pi}(\Sigma - \Sigma\ln(\Sigma/\mu)).$$

In the Calabi-Yau case, $\sum_i q_i = 0$ and the coefficient of the logarithm vanishes. Hence $\widetilde{W}_{\text{eff}}$ is linear in Σ, as a result of which its contribution to the potential energy is independent of σ. As a result of this, it is possible for σ to go to infinity at no cost in energy if t has the correct value. At this value of t, one gets a singularity. The correct value is the value at which the coefficient of Σ in the twisted chiral superpotential vanishes; the condition is not $\text{Im}\, t = 0$, as one would expect classically, nor even $t = 0$, but rather $t - (i/2\pi)\sum_i q_i \ln|q_i| = 0$. This is the shift in the position of the singularity that

was mentioned in our discussion of the Calabi-Yau case. In order to compare the present discussion to, for instance, the counting of rational curves on the Calabi-Yau manifold, it is important to know about this shift. When $\sum_i q_i \neq 0$, the logarithm prevents σ from going to infinity, and hence prevents a singularity on the t cylinder, as we have already asserted.

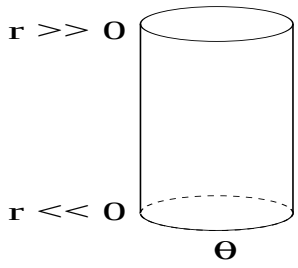
On the other hand, if $\sum q_i \neq 0$, then there are $|\sum q_i|$ extra vacua (as in the $\mathbf{CP}^n$ model) with σ-values

$$\sigma = \mu \exp\left(-\frac{2\pi r}{\sum q_i} - \frac{i\theta}{\sum q_i} - \frac{\sum q_i \ln q_i}{\sum q_i} + \frac{2\pi i k}{\sum q_i}\right), \tag{6.3}$$

However, the prediction of these vacua is only valid if they occur at large σ. Hence, these vacua are only trustworthy for $r \to -\infty$ if $\sum_i q_i > 0$, or for $r \to +\infty$ if $\sum_i q_i < 0$.

6.4 Hypersurfaces

Let us now return to the situation of a hypersurface defined by a homogeneous polynomial F of degree k in n variables. Thus $\sum_i q_i = n - k$. Hence, vacua at large σ are present for $r \to -\infty$ if $n > k$, or at $r \to +\infty$ if $n < k$. The vacua at large σ are the ones that one cannot see classically. In addition to those, one has vacua that one can see classically – a sigma model for $r >> 0$ and a Landau-Ginzburg orbifold for $r << 0$.



Let us now consider the two cases in detail. For $k < n$, the sigma model has for its target space a Fano variety, a hypersurface in $\mathbf{CP}^{n-1}$ of degree $k < n$. This is a good description in the ultraviolet, and gives a well-posed quantum field theory whose infrared behavior we would like to determine. The infrared flow brings us to $r \ll 0$, where we find two types of vacua: (1) vacua which are conformal field theories of central charge $\widehat{c} = n(1 - \frac{2}{k})$, and (2) $n - k$ new vacua at large σ. The corresponding effective twisted superpotential for large σ is

$$\widetilde{W}_{\text{eff}} = t\Sigma + \frac{i}{2\pi}(n-k)(\Sigma - \Sigma \ln \Sigma).$$

Both of these types of vacua have central charge less than the value $\widehat{c} = n - 2$ of the UV theory. Thus, in this example, a single quantum field theory in the ultraviolet can flow to quite different possibilities in the infrared. Renormalization group flow for $N = 2$ sigma models in two dimensions has many invariants (such as $\mathrm{Tr}\,(-1)^F$, the elliptic genus, and some of their cousins). To reproduce the invariants of the UV sigma model with Fano target, one must sum over the infrared vacua of types (1) and (2).

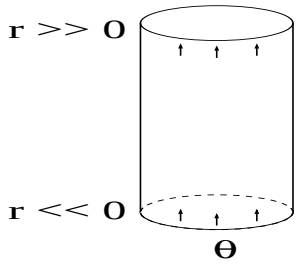
6.5 The case $c_1 < 0$

Consider now the opposite case $k > n$. A hypersurface of degree $k > n$ in $\mathbf{CP}^{n-1}$ is an algebraic variety of general type. In this case $\beta > 0$, and there is not a well-defined quantum field theory of maps from spacetime to this hypersurface. However, the sigma model with this hypersurface as target

makes sense as an effective infrared theory. One can ask what well-defined ultraviolet theory can flow to this effective theory in the infrared.

For this question, we will now give an answer. For $k > n$, the ultraviolet stable end of the cylinder is the Landau-Ginzburg end. The Landau-Ginzburg model has central charge $\widehat{c} = n(1 - 2/k)$, which for $k > n$ is greater than the value $\widehat{c} = n - 2$ of the sigma model. Thus the Zamolodchikov theorem permits a renormalization group flow from the LG model to the sigma model. That is precisely what happens, since for $n < k$ the renormalization group flow on the cylinder is "upwards," away from the LG model and towards the sigma model.

If $k > n$, there are no large σ vacua at the LG end of the cylinder. Instead, $k - n$ such vacua accompany the sigma model. Thus, a well-defined ultraviolet fixed point, given by the Landau-Ginzburg model, can flow in the infrared either to the sigma model of the hypersurface, or to a nonclassical massive vacuum at large σ. Just as for $k < n$, to equate renormalization group invariants such as $\mathrm{Tr}\,(-1)^F$ between the ultraviolet and infrared theories, one must sum over both types of vacua in the infrared limit.



Note that, regardless of the sign of $n - k$, the nonclassical vacua always appear in the infrared, not the ultraviolet. There is a good intuitive reason for this. In the ultraviolet, where we are defining our theory, we can select what theory we wish to study (from all of the possible UV fixed points), so the UV theory has a vacuum of just one type. Going to the infrared means solving the equations, and at this stage things are out of our hands: a definite quantum field theory may have more than one solution for its infrared behavior. That is what we have found today. In fact, we have seen that a definite UV field theory (based on a Fano variety if $k < n$ or a Landau-Ginzburg model if $k > n$) can flow to quite different possibilities in the infrared.

Both of the $k \neq n$ cases that we have studied today are of considerable methodological interest. For $k < n$, we have seen how the quantum theory with a Fano target gives, in general, an answer which is a sort of mixture of the behavior of a Calabi-Yau manifold and the behavior of complex projective space. For $k > n$, we have found, in a concrete situation, a well-defined UV fixed point with flow to the effective theory based on an algebraic variety of general type.

Lecture II-13: $N = 2$ SUSY theories in Dimension Two, Part II: Chiral Rings and Twisted Theories

Edward Witten[1]

This lecture is again concerned with $N = 2$ supersymmetric theories in dimension 2. Using one of the supersymmetries Q, we shall define the chiral ring of local operators and the Q-cohomology of states for theories defined in flat Minkowski or Euclidean space. We then discuss how to twist theories so that the supersymmetry Q has global meaning at least over a large class of 2|4 supermanifolds. Once Q has been globalized in a theory over Σ we have the notions of the ring of chiral local operators for that theory. We can define what are called descendants of Q-invariant operators, and we show that the integrals of these descendants over cycles in the manifolds lead to correlation functions which have topological meaning. To compute these functions one does integrals over the Q-invariant fields.

We then consider in detail two examples of twisted theories with a global supersymmetry Q. The first is pure $N = 2$ gauge theory in dimension 2 with a simple Lie group G. Here, we see that the ring of chiral functions is identified with the invariant polynomials on the Lie algebra. Classically, the space of Q-invariant states is the usual space of flat connections (plus an extra parallel section of the adjoint bundle in the case of reducible connections). The correlation functions of the Q-invariant operators are identified with the integrals over the moduli space of flat connections of the usual classes derived from the invariant polynomials in the Lie algebra. (There are extra complications from reducible connections.)

The second example is $N = 2$ supersymmetric two-dimensional σ-models into a Kähler manifold. It turns out that there is one way of twisting that leads to a chiral ring which is the topological cohomology of the Kähler manifold. The space of Q-invariant states is the space of holomorphic maps from the riemann surface into X. In this twisted theory the path integrals which commute the topological correlation functions are the integrals in quantum cohomology. When the Kähler manifold is Calabi-Yau, there is a second, inequivalent way of twisting which leads to a different theory. In this theory the chiral ring is $H^*(X, \wedge^* T^{1,0}(X))$ and the Q-invariant fields are constant maps from the riemann surface to X. Thus, the path integrals which compute the topological correlation functions are integrals of these objects over X – these integrals are related to variation of Hodge structures on X.

1 R-symmetry revisited

Most of the $N = 2$ supersymmetric examples we shall consider will have (at least classically) both left- and right-moving R-symmetries – the R-symmetries denoted by J_L and J_R in the last lecture. It is possible to carry out today's discussion with only one of these R-symmetries, but we must have at least one. But we will suppose that the group of R-symmetries is $U(1) \times U(1)$ so that we have both J_R and J_L. The supermanifold on which the theory is defined in $\mathbf{R}^{2|4}$. We shall also eventually allow a certain class of split 2|4 super-riemann surfaces with a Euclidean metric (and 2|4-supermanifolds with a flat Minkowski metric). The space of odd directions is a copy of the two-dimensional representation for J_L and the two dimensional representation for J_R. We expect a symmetry between J_L and J_R obtained by reversing the orientation of the worldsheet.

One operation we can attempt to do to a symmetry of a Lagrangian is to gauge it. If the symmetry group to be gauged is H, then the new Lagrangian will involve gauge fields with gauge group H times the gauge group (if any) of the original theory. Intuitively, the process of gauging involves replacing fields with values in a constant space by fields with values in an H-bundle and adding a new field – the

[1] Lecture notes by John Morgan

H-connection. In order to carry out this program the original Lagrangian must be a local expression in the fields and the symmetry must be also be local in the sense that it maps local fields to local fields at the same point. With these reasonable assumptions on the Lagrangian and its symmetry, it is always possible to form a new, gauged, Lagrangian. We replace ordinary derivatives in the Lagrangian by covariant derivatives. Quite often there is a natural way to carry out this process, but in any case different choices for gauging the symmetry lead to Lagrangians which differ by gauge invariant terms up to a total derivative. There is however the issue of whether the gauged theory makes sense as a quantum theory. If the original theory has this property then the necessary and sufficient condition that the gauged theory does as well is that the symmetry being gauged be anomaly-free.

In our case it is never possible to gauge the entire $U(1) \times U(1)$ group of R-symmetries to produce a theory that makes sense as a quantum theory. In fact the only combinations of J_L and J_R that can be gauged in this way are $J_L \pm J_R$. These are referred to as the vector and axial currents

$$J_V = J_L + J_R$$

$$J_A = J_L - J_R.$$

If one of these is gauged, then the other develops anomalies and stops being a symmetry of the quantum theory. To explain this, we need some general observations about anomalies. The statement that a theory has a symmetry, generated by a current J_L, means that correlation funcions obey the identity

$$d_x \langle J_L(x) \mathcal{O}(y_1) \dots \rangle = 0 \quad (*)$$

as long as x is distinct from the y_i. Here d_x is the exterior derivative in the x variable. If one extends the correlation functions (as distributions) across the diagonal, this identity will pick up extra terms supported on the diagonal. A very important special case is the two point function of the current itself (or, in $2k$ spacetime dimensions, the $k+1$ point function). For this the general structure is

$$d_x \langle J_L(x) J_L(y) \rangle = \frac{k}{2\pi} d_y \delta(x-y) \quad (**)$$

with some constant k, which may or may not be zero. The condition that the global symmetry generated by J_L can be gauged is precisely that $k = 0$. To see this, let us try to gauge the symmetry by adding to the theory a $U(1)$ gauge field A. The Lagrangian obtains an extra term $\int A \wedge J_L$. (There may be more terms, higher order in A, if J_L itself contains derivatives of charged fields.) Now, let us try to determine whether the theory with this coupling actually has the symmetry generated by J_L, that is whether the identity of equation $(*)$ holds for this theory. For this we must look, in the original theory, at

$$d_x \langle J_L(x) \mathcal{O}(y_1) \dots \exp\left(\int A \wedge J_L \right) \rangle$$

The exponential is included to express correlation functions of the new theory in terms of the old theory. If we expand this in powers of A, the term linear in A will contribute, upon using $(**)$:

$$\langle \mathcal{O}(y_1) \cdots \cdot kF(x) \rangle$$

with F the curvature. This can be expressed as an operator relation

$$dJ_L = \frac{k}{2\pi} F,$$

asserting that after gauging J_L no longer generates a symmetry.

More generally, if there are several currents $J^1, \dots, J^s$, generating a Lie group G of dimension s, with Lie algebra $\mathcal{G}$, then the two point functions obey

$$d_x \langle J^a(x) J^b(y) \rangle = \frac{k^{ab}}{2\pi} d_y \delta(x-y)$$

with k^{ab} some invariant quadratic form on $\mathcal{G}$. The quadratic form k has no positivity (since for instance left and right moving massless fermions make contributions of opposite signs) but does obey an integrality condition. If we gauge a subgroup H of G, with Lie algebra $\mathcal{H}$, then by a reasoning along the lines of the above, the quantum theory only makes sense if the quadratic form k vanishes when restricted to $\mathcal{H}$. If this is so, then the "global symmetries" are generated by currents in $\mathcal{G}$, not in $\mathcal{H}$, that are orthogonal to $\mathcal{H}$ with respect to k.

In our problem, with the two currents J_L, J_R, the structure of k is typically that $k^{LL} = -k^{RR} \neq 0$, $k^{LR} = 0$. This quadratic form has two null vectors, $J_L \pm J_R$, so either of those two linear combinations of the R-symmetries can be gauged. If one is gauged, the second becomes "anomalous" – it ceases to be conserved – since the two null vectors are, of course, not orthogonal to each other. "Luckily," these options agree with the choices that we will need below for other reasons.

What we are really interested in is not just gauging but twisting theories by R-symmetries. This process requires first gauging the R-symmetry and then twisting it using a homomorphism from the orthogonal group determined by the metric to the symmetry group. It turns out that there will be two equivalent ways of going about this process. But before we embark on twisting, we must understand the local story. That is the subject of the next section.

2 Q-cohomology of operators

We consider $\mathbf{R}^{2|4}$ with its standard Euclidean or Minkowski metric. Suppose that our $N = 2$ supersymmetry theory in dimension two has the following supersymmetries:

characters of $U(1)_L$	characters of $U(1)_R$
Q_-	Q_+
$\overline{Q}_-$	$\overline{Q}_+$.

These supersymmetries obey the relations

$$Q_+^2 = \overline{Q}_+^2 = 0$$

$$\{Q_+, \overline{Q}_+\} = P_+$$

with symmetric relations with + replaced by −. Also,

$$\{Q_+, Q_-\} = \{\overline{Q}_+, Q_-\} = \{Q_+, \overline{Q}_-\} = \{\overline{Q}_+, \overline{Q}_-\} = 0.$$

We will consider at the same time four closely related possibilities. We let Q be one of the following:

$$Q_+ + Q_-, \;\; Q_+ + \overline{Q}_-, \;\; \overline{Q}_+ + Q_-, \;\; \overline{Q}_+ + \overline{Q}_-.$$

(The first two possibilities are inequivalent and the last two are hermitian conjugate to the first two and would give equivalent theories.) To fix notation let us choose the generators so that $Q = Q_+ + Q_-$. Letting $\mu = \pm$ we see that P_μ can be written as a commutator

$$P_\mu = \{Q, T_\mu\} \tag{2.1}$$

where T_μ is a linear combination of $Q_\pm, \overline{Q}_\pm$. We are interested in the cohomology of Q acting as a differential (by superbracketing) on the space of local operators (and more generally on products of such operators). Thus, a local operator $\phi(x)$ will define a Q-cohomology class if $\phi(x)$ is Q-closed, i.e., if $\{Q, \phi(x)\} = 0$. A simple computation using Equation 2.1 shows that for a Q-closed operator $\phi(x)$ we have

$$d_\mu \phi = \{Q, [T_\mu, \phi]\}.$$

(This derivative is interpreted to be the covariant derivative if the super-riemann surface is not flat.) This equation means that the operator $d\phi$ is Q-exact, and hence that the class of $d\phi$ in the space of Q-cohomology classes of operators is trivial. This means that as we vary the point x in the riemann surface the Q-cohomology class of the local operator $\phi(x)$ is unchanged. It then follows that in Q-cohomology such Q-closed local operators commute – simply move the points to be space-like separated. That is to say, the space Q-cohomology of operators is a $\mathbf{Z}/2\mathbf{Z}$-graded commutative ring. We can always represent these operators by 0-form operators. This is an easy consequence of the independence of the Q-cohomology class on the metric and diffeomorphism invariance of the Q-cohomology class. These facts will be established later.

Let us recall a construction that we have seen before. Given a Q-invariant 0-form local operator $\phi = \phi^{(0)}$ we define a 1-form local operator by Cross reference?

$$\phi^{(1)}_\mu = [T_\mu, \phi^{(0)}]$$

and

$$\phi^{(1)} = \phi^{(1)}_\mu dx^\mu.$$

The same computation as we did in the previous paragraph shows that

$$d\phi^{(0)} = \{Q, \phi^{(1)}\}.$$

This means that if Σ_1 is a one-cycle in the super-riemann surface then

$$\int_{\Sigma_1} \phi^{(1)}$$

is Q-invariant. Let $\phi^{(2)} = [T, \phi^{(1)}]$. Then

$$d\phi^{(1)} = \{Q, \phi^{(2)}\}.$$

So in this way, beginning with a Q-invariant 0-form local operator $\phi^{(0)}$, we can build a tower of k-form local operators $\phi^{(k)}$, called the descendants of $\phi^{(0)}$, and hence produce more general Q-invariant operators of the form

$$\int_{\Sigma_k} \phi^{(k)}.$$

Our computations show that for any Q-invariant local operator $\phi^{(0)}$ the operator $\phi^{(0)} + \phi^{(1)} + \phi^{(2)}$ is closed for the total differential $d - Q$.

3 Twisting the theory to give it global meaning

So far we have been working on $R^{2|4}$ with fixed coordinates. If we rotate space, then $Q = Q_+ + Q_-$ is not invariant. Our goal is to modify things until Q is invariant under rotations and hence can be defined

on more general Euclidean 2|4-manifolds. Let us denote by $SO(2)_E$ the Euclidean rotation group. The entire group of symmetries at our disposal for the Euclidean theory is $SO(2)_E \times U(1)_L \times U(1)_R$. This group has the obvious projection onto $SO(2)_E$ with a splitting back into the first factor. We wish to find another splitting of this projection map, i.e., a group $SO(2)'_E$ inside the product projecting isomorphically onto $SO(2)_E$ but which has the extra property that it commutes with Q. It is easy to see that there exists a unique such splitting and it is given on the Lie algebra level by

$$K' = K \pm \frac{1}{2}J_L \pm \frac{1}{2}J_R$$

where K is the generator for $SO(2)_E$ and the signs depend on which of the four possibilities we have chosen for Q. In the case that we are concentrating on when $Q = Q_+ + Q_-$ we have

$$K' = K + \frac{1}{2}J_L - \frac{1}{2}J_R.$$

The fact that $1/2$ appears in these formulas means that the twisting actually involves the spin form of $SO(2)_E$.

Notice that in order to make such twisting in dimension n there must be non-trivial homomorphisms from $Spin(n)$ to the R-symmetry group. In dimension two it is possible to do this when the R-symmetry group is $U(1)$, e.g., for $N = 1$ supersymmetry. But for $n = 4$ it is necessary to have $N = 2$ supersymmetry so that the R-symmetry group is $SU(2) \times U(1)$ which has a non-trivial homomorphism from $Spin(4)$.

With this preliminary discussion in place, we are now ready to globalize our theory of Q-cohomology classes of local operators. Suppose that we have a 2|4 super-riemann surface locally modeled on $\mathbf{R}^{2|4}$. The hypothesis we have in mind for the super-riemann surface is that it be a split 2|4-manifold complex with the odd directions being

$$\left(\Pi K^{1/2} \oplus \Pi \overline{K}^{1/2}\right) \oplus \left(\Pi \overline{K}^{1/2} \oplus \Pi K^{1/2}\right).$$

Here $\theta^+, \overline{\theta}^+$ are sections of the first two line bundles and $\theta^-, \overline{\theta}^-$ are sections of the last two. The process of twisting changes the odd part of the underlying supermanifold of the theory. We twist the spinor bundles containing $\theta^\pm$ by the square root of the canonical bundle to the power $\pm 1/2$. The result of twisting replaces the above line bundles by

$$(\Pi \mathcal{O} \oplus \Pi K) \oplus \left(\Pi \mathcal{O} \oplus \Pi \overline{K}\right).$$

and makes the supersymmetry

$$Q = \frac{\partial}{\partial \theta^+} + \frac{\partial}{\partial \theta^-}$$

which is a global symmetry. Locally, this symmetry agrees with the symmetry Q on $\mathbf{R}^{2|4}$ that we have been discussing.

Once we have globalized the supersymmetry Q, all the previous discussion globalizes. In particular, for a Q-closed local operator $\mathcal{O}(p)$ the Q-cohomology class of $\mathcal{O}(p)$ is independent of the point p. Thus, the Q-cohomology classes of local operators on this global super-riemann surface generate a $\mathbf{Z}/2\mathbf{Z}$-graded commutative ring called the *chiral ring* of the theory.

3.1 Certain Correlation Functions and Their Basic Properties

We will consider path integrals of the following types of path integrals:

$$Z_{(\mathcal{O}_\alpha)} = \int \mathcal{D}\Phi e^{-\mathcal{L}} \prod_{\alpha=1}^{n} \int_{\Sigma_\alpha} \mathcal{O}_\alpha^{(k_\alpha)}$$

where for each α the operator $\mathcal{O}_\alpha$ is a Q-closed local operator, $k_\alpha = 0, 1, 2$ and Σ_α is a k_α-cycle in the super-riemann surface Σ.

Here are the basic properties of $Z_{(\mathcal{O}_\alpha)}$:

1). It is independent of the metric on the super-riemann surface.

The reason is that as you vary the metric by δg you get an insertion of $\int_\Sigma \delta g \cdot T$, where T is the stress-energy tensor. But as we have seen before T is Q-exact in the sense that there is a field $S_{\mu\nu}$ cross ref? with $T_{\mu\nu} = \{Q, S_{\mu\nu}\}$. We then see that this insertion produces a term of the form $\{Q, \cdot\}$. The path integral of such a term is zero since since the fields are Q-invariant. (Notice also that it is necessary to assume that Q preserves the measure in the path integral. This will be so if the original theory was supersymmetric and the current $\frac{1}{2}J_L - \frac{1}{2}J_R$ was gaugeable. An important corollary of the fact that the correlation function is metric-independent is that the zero-form operators $\mathcal{O}_\alpha^{(0)}$ must be scalar operators, of spin zero; objects of higher spin would not be invariant under local rotations, and their insertion in a correlation could not be metric-independent.)

2). It is independent of most of the parameters in $\mathcal{L}$.

First, the correlation function is independent of any term $\int d^2y d^4\theta(\cdots)$. The reason is that to evaluate such a term we differentiate by all four of the Q's. For the same reason, only one of the terms

$$\int d^2y d\theta^+ d\theta^- W + \text{c. c.}$$

and

$$\int d^2y d\bar{\theta}^+ d\theta^- \tilde{W} + \text{c.c.}$$

can be non-trivial depending on which combination of $Q_+, Q_-, \overline{Q}_+, \overline{Q}_-$ we are using. In fact the correlation function is independent of one of these terms and varies holomorphically with the other.

3). We can calculate $Z_{(\mathcal{O}_\alpha)}$ by using a fixed point theorem applied to the action of Q. We consider Q as acting (infinitessimally) on the function space of all the fields in the theory. If this action has no fixed points, then the value of the path integral would be zero. In general, we can reduce the computation to an integral over the fixed point set of Q. These of course are the field configurations which are invariant under the supersymmetry Q.

3.2 Q-cohomology of States

Now we pass from the local operators, path integrals and correlation functions world to the Hamiltonian framework. Our riemann surface is now a cylinder $S^1 \times \mathbf{R}$. We trivialize all the relevant spin bundles over the S^1. We have the Hilbert space $\mathcal{H}$ of states of the quantum theory over the circle and the Hamiltonian operator H on $\mathcal{H}$ associated with time translation. Since Q is acting on the theory it produces an operator, which we also call Q, on $\mathcal{H}$ commuting with H. Also, from the commutation relations in the superalgebra we see that

$$H = \{Q, \tilde{Q}\}$$

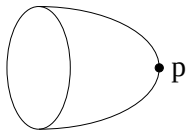
where $\tilde{Q}$ is the adjoint of Q. Of course, we still have the relation $Q^2 = 0$, so that we can take the cohomology of the action of Q on $\mathcal{H}$:

$$H^*(Q) = \ker Q / \mathrm{Im}\, Q.$$

If our theory is sufficiently well-behaved so that H has a discrete spectrum, then $H^*(Q)$ is identified with $\{\psi | H\psi = 0\}$. (This is not the case for the gauge theory example we discuss later today but it is true for the sigma models.) In general $H^*(Q)$ is bigraded by the action of the $U(1) \times U(1)$ R-symmetry group.

3.3 Comparison of Q-cohomology of local operators and Q-cohomology of states

We have seen two types of Q-cohomology – the first involving local operators and yielding the chiral ring and the second involving states on the circle. Let us construct a map between these Q-cohomologies. We consider a hemisphere



Let O be a local operator at p. Doing the path integral for the hemisphere over the space of fields with a given boundary value produces a number, and hence a function of the state on the boundary. This function is an element in the Hilbert space of the boundary. This then gives a map from local operators at p to $\mathcal{H}$. One sees easily that it commutes with the action of Q and hence induces a map from the Q-cohomology of local operators to the Q-cohomology of states on the circle:

$$\psi\colon (\text{chiral ring}) \to H^*(Q).$$

In examples we shall see that if the spectrum of Q on $\mathcal{H}$ (at least near zero) is discrete then ψ is an isomorphism.

If J_L and J_R are both R-symmetries classically, then the chiral ring and $\ker H$ are both bigraded by these symmetries. Because of the anomaly, the map ψ shifts the bigrading

$$\psi\colon H^{p,q}(Q) \to \ker H^{p \pm k, q \pm k},$$

where k is a measure of the anomaly. Notice that the bigrading need not be by integers, only by rational numbers whose denominators divide the order of the covering of $U(1) \times U(1)$ that actually acts.

4 A Gauge Theory Example

The first example we take is that of gauge theory of a simple Lie group G. We begin with $N = 1$ supersymmetric gauge theory in four dimensions and dimensionally reduce to an $N = 2$ supersymmetric theory in dimension two. As we saw in the last lecture, the basic field strength is the field

$$\Sigma = \{\overline{\mathcal{D}}_+, \mathcal{D}_-\},$$

which is a twisted chiral superfield with values in the adjoint bundle of the principal bundle. We use the expansion

$$\Sigma = \sigma + \cdots + \overline{\theta}^{+}\theta^{-}(F + iD).$$

The bosonic fields are the connection A_μ and $\sigma, \overline{\sigma}$. Remember that the connection on four-space is

$$A^{(4)} = \sum_{\mu=0}^{1} A_\mu dy^\mu + \sigma(dy^2 - idy^3) + \overline{\sigma}(dy^2 + idy^3).$$

The two R-symmetries are:

$$J_A = J_L - J_R; \;\; J_V = J_L + J_R.$$

The first is induced by rotation in the (y^2, y^3)-plane and the second comes from the chiral $U(1)$ R-symmetry in four dimensions. (This symmetry is anomalous in four dimensions but not in two dimensions – see the superhomework.) We now twist by J_V, so that $SO(2)'_E$ is generated by $K' = K \pm \frac{1}{2}J_V$. Then A_μ remains a connection one-form and $\sigma, \overline{\sigma}$ become 0-forms. In general, this twist eliminates spin bundles and turns sections of the spin bundles with differential forms. Then four spinor fields $\lambda_\pm, \overline{\lambda}_\pm$ get twisted into a 0-form $\eta = \psi_{(0)}$, a one form $\psi = \psi_{(1)}$ and a two-form $\chi = \psi_{(2)}$. (This new theory is of course a topological theory existing on a compact surface. It is not a physical theory in the sense that there is a Poincaré group action on the theory.)

In this example J_A is twice the rotation of (y^2, y^3)-space, normalized so that $[J_A, Q] = Q$. Let us give the J_A degrees of the various propagating fields in the Lagrangian, cf. Formulae 7.15 of the superhomework. (The reference to propagating fields, means that we have in mind using the equations of motion to set the auxiliary field $F + iD$ equal to some expression in the other fields (F is set equal to the curvature of A and D is set equal to zero since there are no matter fields.)

J_A-degree	-2	-1	0	1	2
field	$\overline{\sigma}$	$\psi_{(0)}, \psi_{(2)}$	A_μ	$\psi_{(1)}$	σ

Actually, the J_A-degree -1 and $+1$ terms could be reversed, but after appropriately normalizing things the degrees will be as indicated.

Recall from Lecture II-10[2] that the super symmetry Q lifts to a supersymmetry δ of the space of super connections. It is not the case that $\delta^2 = 0$, but only that δ^2 is a gauge transformation. In fact, $\delta^2 = [\sigma, \cdot]$. In terms of the coordinate fields it is given by:

$$\delta A = \psi_{(1)}$$

$$\delta\psi_{(1)} = -D_A\sigma$$

$$\delta\sigma = 0.$$

The symbol D_A of course refers to covariant derivative with respect to the connection A.

In Lecture II-10 using the analogous symmetry led us to the equivariant cohomology of the moduli space of connections, via the standard model of forms. We let $\mathcal{G}$ be the Lie algebra of the gauge group G and M the space on which this group acts. Then we have the complex

$$(\Omega^*(M) \otimes Func(\mathcal{G}))^G$$

with differential

$$D = d_M + i_{V(\phi)}.$$

[2] In that lecture we had a real field ϕ instead of the complex field σ here, but that changes nothing.

Here, we use another closely related model which is not as well known but should be. We replace the Lie algebra $\mathcal{G}$ by its complexification $\mathcal{G}_{\mathbf{C}}$ and we form the complex

$$\left(\Omega^*(M) \otimes \Omega^{0,*}(\mathcal{G}_{\mathbf{C}})\right)^G.$$

Here complex linear function ϕ on $\mathcal{G}_{\mathbf{C}}$ have degree 2, the functions $\overline{\phi}$ have degree -2 and the $(0,1)$-form $d\overline{\phi}$ has degree -1. The differential in this complex is

$$\tilde{D} = d_M + i_{\tilde{V}(\phi)} + \overline{\partial}_{\mathcal{G}_{\mathbf{C}}}$$

where $\tilde{V}$ is a vector field on $M \times \mathcal{G}_{\mathbf{C}}$ giving the action. To see that this complex also computes the equivariant cohomology we view $\tilde{D}$ as a sum of $(d_M + i_{\tilde{V}(\phi)})$ and $\overline{\partial}_{\mathcal{G}_{\mathbf{C}}}$. We apply the usual spectral sequence for the double complex, doing $\overline{\partial}_{\mathcal{G}_{\mathbf{C}}}$ first. Since this cohomology vanishes except in degree zero, where it produces the polynomial functions on $\mathcal{G}_{\mathbf{C}}$, we see that one recovers the same answer as for the first complex. (Notice that this second approach leads to a Hodge theory version of equivariant cohomology, using the operator $\tilde{D}^*\tilde{D}$ as a "Laplacian"; it would be interesting to understand the resulting $\mathbf{L}^2$ version of equivariant cohomology.)

Let us complete the table describing the action of the supersymmetry δ on the space of propagating fields. The above model of equivariant cohomology translates into the following formulae for the supersymmetry:

$$\delta A = \psi_{(1)}$$
$$\delta\psi_{(1)} = -D_A\sigma$$
$$\delta\sigma = 0$$
$$\delta\overline{\sigma} = \psi_{(0)}$$
$$\delta\psi_{(0)} = [\sigma, \overline{\sigma}]$$
$$\delta\psi_{(2)} = F_A.$$

It follows from the last equation that the fixed points of δ will be field configurations where the connection is flat, the section σ is covariantly constant and commutes with $\overline{\sigma}$. Next we see that the anomaly of J_A is $\dim G \cdot (-\chi(\Sigma)) = (2g-2)\cdot\dim(G)$. The reason for this is that the term that contributes to the anomaly in the measure of integration for the path integral is $\mathcal{D}\psi_{(0)}\mathcal{D}\psi_{(1)}\mathcal{D}\psi_{(2)}$. The symmetry J_A of the Lagrangian acts by -1 on $\psi_{(0)}$ and $\psi_{(2)}$ and by $+1$ on $\psi_{(1)}$. Thus, the index computation for the anomaly becomes the dimension of the adjoint representation times $-\chi(\Sigma)$. By invariance the correlation function

$$\langle \prod_\alpha \int_{\Sigma_\alpha} O_\alpha^{(k_\alpha)} \rangle = 0$$

unless $\sum_\alpha n_\alpha = (2g-2)\dim(G)$ where n_α is the J_A-degree of $O_\alpha^{(k_\alpha)}$. Let us consider a local operator which is of the form $P(\sigma)$ for some gauge invariant polynomial on the Lie algebra of G. The ring structure on these operators is just the usual ring structure on invariant polynomials. It is easy to see that if P is of degree r, then the operator $O_P^{(0)}$ has J_A-degree $2r$, and the descendants $O_P^{(k)}$ have J_A-degree equal to $2r-k$. Notice that these degrees agree with the usual dimension of the analogous classes in Donaldson theory.)

Now let us calculate $\langle \prod_\alpha \int_{\Sigma_\alpha} O_\alpha^{(k_\alpha)} \rangle$ via fixed point theory. Recall that the Lagrangian is

$$\mathcal{L} = \frac{1}{4e^2} \int d^2y d^4\theta \mathrm{Tr}\overline{\Sigma}\Sigma.$$

We will recover a theoretical formula for this correlation function. (We actually calculated the answer in Lecture II-10.) In coordinates we have

$$\mathcal{L} = \frac{1}{4e^2}\int d^2y\left(\mathrm{Tr}F^2 + \mathrm{Tr}D\sigma D\overline{\sigma} + \mathrm{Tr}([\sigma,\overline{\sigma}]^2) + \cdots\right). \tag{4.1}$$

By the fixed point theorem we need only calculate at points where the curvature is zero and $D_A\sigma = [\sigma,\overline{\sigma}] = 0$. There are two cases. If A is irreducible and $D_A\sigma = 0$ then $\sigma = 0$. These solutions give smooth points in the moduli space of classical solutions to the equations of motion and this part of the moduli space of classical δ-invariant solutions is simply the space of of irreducible representations of $\pi_1(\Sigma) \to G$. The other possibility is that A is reducible and $\sigma \neq 0$. These points produce singular points of the moduli space of classical solutions. Recall that σ takes values in the complexified adjoint bundle. Since $D_A\sigma = 0$, σ is covariantly constant, and it suffices to study its structure over one point of Σ. Since σ and $\overline{\sigma}$ commute, the σ can be diagonalized (conjugated into the Cartan subalgebra of the complexification of the Lie algebra of G) by an element of G.

Now let us examine how perturbation theory will compute these topological correlation functions. We will restrict to the smooth case where $\sigma = 0$. Of course, we are taking the coupling constant e in the Lagrangian, Equation 4.1 to be small. We are reduced to computing an integral over the moduli space, taken in the supersense, of classical solutions. Let us consider integrating out the odd variables at a point $[\rho]$ of the ordinary moduli space of flat connections. The fermion space is the space of zero modes of the $\psi_{(k)}$. These are the spaces of harmonic forms and hence are identified with the spaces $H^k(\Sigma, \mathrm{ad}(\rho))$. Since we are assuming that ρ is irreducible and that the moduli space is smooth at $[\rho]$ we have that $H^0(\Sigma,\mathrm{ad}(\rho)) = H^2(\Sigma,\mathrm{ad}(\rho)) = 0$ and $H^1(\Sigma,\mathrm{ad}(\rho))$ is the tangent space to the moduli space at $[\rho]$. So in fact the space of odd directions at $[\rho]$ is identified with $\Pi T\mathcal{M}_{[\rho]}$. Thus, at the open subset of irreducible connections, the supermanifold of classical solutions is $\Pi T\mathcal{M}$, the parity reversed tangent bundle over the moduli space of flat connection. As we have seen, the Berezinian of such a supermanifold is naturally trivialized so that one integrates functions over this supermanifold. We have also seen that a function on this supermanifold is the same thing as a differential form on $\mathcal{M}$ and integration of a function over $\Pi T\mathcal{M}$ is the same thing as the integral of the corresponding differential form over $\mathcal{M}$. (This requires an orientation of $\mathcal{M}$ which is determined by an orientation of Σ.)

Of course, we wish to compute correlation functions involving the operators $\int_{\Sigma_\alpha} \mathcal{O}_\alpha^{(k_\alpha)}$. These operators determine functions on $\Pi T\mathcal{M}$ and the correlation function of a product of them becomes the integral of the product of the functions. In this manner gauge invariant polynomials in σ are mapped to functions on $\Pi T\mathcal{M}$ and hence to differential forms on $\mathcal{M}$. A crucial step in the reduction from polynomials to differential forms is the observation that with fermions present, it is no longer the case that $\sigma = 0$. Rather the equations of motion give

$$\Box\sigma = -[\psi_{(1)},\psi_{(1)}].$$

This is important because the right hand side is directly interpreted as a two-form on moduli space (as the zero modes of $\psi_{(1)}$ represent one-forms on the moduli space), so the formula enables us to express the function σ, which is formally of degree two in the quantum field theory formalism, in terms of an ordinary two-form on the moduli space. One evaluates the correlation functions by integrating the differential forms obtained in this way over the moduli space. (Of course, these computations have to be augmented by computations at the components given by reducible connections and a non-zero but parallel field σ.)

In this gauge theory example, because of the presence of reducible connections, the Hamiltonian has a continuous spectrum beginning at zero energy. As a result the map from Q-cohomology of operators to Q-cohomology of states is not very useful.

5 A σ-model example

Let us consider the $N = 2$ supersymmetric σ-model

$$\Phi\colon \Sigma \to X$$

where X is a Kähler manifold and where Σ is a split 2|4 manifold as in the previous gauge theory example. With this assumption, the fermions of the σ-model lie in

$$K_\Sigma^{\pm 1/2} \otimes T^{\pm} X$$

where K_Σ is the canonical bundle of Σ, $T^+X = T^{1,0}X$ and $T^-X = T^{0,1}X$ are the holomorphic and anti-holomorphic tangent bundle of X. We denote these fermions by $\chi^{\pm,\pm}$. The first sign refers to whether the fermion is left- or right-moving (i.e., in whether it is a section of K_Σ^+ or K_Σ^-) and the second sign refers to the type of the section (holomorphic or anti-holomorphic) on X. Once again we twist the theory for it to exist globally on Σ replacing K by

$$K' = K \pm \frac{1}{2} J_L \pm \frac{1}{2} J_R.$$

Depending on the choice of signs in this twisting we get two possible cases called the A-model (using $-\frac{1}{2}J_L + \frac{1}{2}J_R$) or the B-model (using $\frac{1}{2}J_L + \frac{1}{2}J_R$).

5.1 The A-model

The zero-forms are $\chi^{++} \in K^{1/2} \otimes T^{1,0}X$ and $\chi^{--} \in K^{-1/2} \otimes T^{0,1}$. The ring of 0-form operators is

$$F(\Phi, \chi^{++}, \chi^{--}) = F(\Phi, \partial\phi, \bar{\partial}\phi).$$

This is simply the ring of functions on ΠTX which is identified with the ring of differential forms on the Kähler manifold. Furthermore, with this identification of a differential form F with a corresponding operator $\mathcal{O}_F$, we have

$$Q\mathcal{O}_F = \mathcal{O}_{dF},$$

so that Q becomes the ordinary differential. The chiral ring is then $H^*(X)$, the usual topological cohomology of X, and the bigrading induced by J_L and J_R is the usual Hodge bigrading. In the space of maps the Q-fixed points are the holomorphic curves

$$\phi\colon \Sigma \to X.$$

Thus, to compute correlation functions in this model we must do integrals over the spaces of holomorphic curves in X. This will lead, as we shall see in the next lecture, to quantum cohomology.

5.2 The B-model

It is possible to construct this model only in the case X is Calabi-Yau, for this is the only case when the symmetry $(J_L + J_R)$ by which we wish to twist is not anomalous.[3] We fix a non-zero holomorphic n-form ω on X. In this case the zero-forms are $\chi^{+-} \in K^{1/2} \otimes T^{1,0}X$ and $\chi^{--} \in K^{-1/2} \otimes T^{0,1}X$. The ring of 0-form operators is

$$F(\Phi, \chi^{+-}, \chi^{--})$$

which is the ring

$$\Omega^{0,*}(X) \otimes \Omega^{0,*}(X) = \Omega^{0,*}\left(X, \wedge^* T^{1,0}(X)\right).$$

Under this identification we have

$$QO_F = O_{\bar{\partial}F},$$

so that the chiral ring is

$$\oplus_{q,p} H^q(X; \wedge^p T^{1,0}X).$$

The Q-fixed points in the space of fields are the constant maps $\Sigma \to X$. Thus, the computations of correlation functions in this model will become integrals over X

$$\langle \prod_{i=1}^{s} O_{F_i} \rangle = \int_X \left(\wedge_{i=1}^s F_i\right) \cdot \omega^{\otimes 2}.$$

These integrals lie squarely in classical mathematics; they have to do with variation of Hodge structures under deformation of complex structure.

[3] In other words, if $c_1(X)$ is nonzero, then $J_L + J_R$ is simply not a symmetry of the quantum theory, even before gauging. The discussion of the obstruction to gauging in the first part of this lecture assumed that J_L and J_R both generated symmetries prior to gauging, and thus does not apply to this case.

Lecture II-14, part I: The Landau–Ginzburg Description of $N = 2$ Minimal Models

Edward Witten[1]

1 Landau–Ginzburg models

The topic for the first part of today's lecture is a more detailed discussion of the known evidence for the predicted equivalence between IR limits of certain Landau–Ginzburg models, and the algebraically constructible conformal field theories known as "minimal models."

Consider first the simplest Landau–Ginzburg model: a two-dimensional N=2 theory of a single chiral superfield Φ with Lagrangian

$$\mathcal{L} = \int d^2x\, d^4\theta\, \overline{\Phi}\Phi + \left(\int d^2x\, d\theta^+\, d\theta^-\, \Phi^k + \text{c.c.} \right).$$

Our claim in lecture II-12 was that this model flows in the IR to a conformal field theory with $\widehat{c} = 1 - \frac{2}{k}$. If true, then the theory it flows to must be an algebraically solvable theory, due to the known classification of representations of the N=2 superconformal algebra with $\widehat{c} < 1$. More precisely, fixing $\widehat{c} = 1 - \frac{2}{k}$, there turn out to be a finite number of irreducible representations $\mathcal{H}_\alpha$ of the algebra of central charge $\widehat{c}$ (up to isomorphism); a conformal field theory must be built as a modular-invariant combination

$$\bigoplus (\mathcal{H}_\alpha \otimes \overline{\mathcal{H}}_\beta)^{e_{\alpha\beta}}$$

of these representations. The simplest modular-invariant combination, and the one which will occur here, is the unweighted diagonal sum over all of the representations:

$$\bigoplus_\alpha \mathcal{H}_\alpha \otimes \overline{\mathcal{H}}_\alpha.$$

More general Landau–Ginzburg models involve several chiral superfields $\Phi_1, \dots, \Phi_n$ and a quasi-homogeneous polynomial $W(\Phi_1, \dots, \Phi_n)$ which serves as the superpotential of the theory. We should assume that the polynomial is transverse in the sense that $W = dW = 0$ holds only at the origin; the quasi-homogeneity condition can be expressed as the existence of positive rational numbers α_j such that

$$W = \sum \alpha_j \Phi_j \frac{\partial W}{\partial \Phi_j}.$$

The Lagrangian can be written

$$\mathcal{L} = \int d^2x\, d^4\theta\, \overline{\Phi}_j \Phi_j + \left(\int d^2x\, d\theta^+\, d\theta^-\, W(\Phi_j) + \text{c.c.} \right).$$

Intuitively, the quasi-homogenity is important because it allows the possibility of extending one of the R-symmetries, namely

$$\theta^+ \to e^{-i\varepsilon}\theta^+, \qquad \theta^- \to \theta^-,$$

to act on the chiral fields as $\Phi_j \to e^{i\varepsilon\alpha_j}\Phi_j$ so that the superpotential transforms as $W \to e^{i\varepsilon}W$ and hence the Lagrangian is invariant. If we *assume* that these microscopic R-symmetries *become* the

[1] Notes by David R. Morrison

R-symmetries of a limiting conformal field theory in the IR, then we can calculate the central charge $\widehat{c}$ of that limiting theory via operator product expansions.

This works as follows. Recall that the coefficient k which we encountered in lecture II-13 as measuring the anomaly which arises if one attempts to gauge the R-symmetry J_L appears in the operator product expansion as

$$J_L(x)\, J_L(x') = \frac{k}{(x^- - x'^-)^2} + \cdots .$$

The interpretation of k as the obstruction to gauging makes it clear that it is a topological invariant which can be reliably measured or computed at any scale. In particular, it can be calculated in the UV, where the superpotential doesn't matter. On the other hand, if J_L is the same R-symmetry which appears in the limiting $N = 2$ superconformal field theory, then the operator product expansion of the superconformal algebra asserts that

$$J_L(x)\, J_L(x') = \frac{\widehat{c}}{(x^- - x'^-)^2} + \cdots ,$$

where $\widehat{c}$ is the central charge in the $N = 2$ algebra. Thus, $k = \widehat{c}$ and we can measure the IR central charge by computing in the UV.

To compute k in the ultraviolet, one treats the fields as free fields. The obstruction to gauging comes from the fact that the R-symmetries are chiral symmetries for the fermions in the free supermultiplets. One gets $k = \sum_j (1 - 2\alpha_j)$. Assuming the flow to an $N = 2$ theory in the infrared, with the same R-symmetry appearing in the $N = 2$ algebra, we have $k = \widehat{c}$ and hence $\widehat{c} = \sum_j (1 - 2\alpha_j)$. Notice that in the case of a single field with $W(\Phi) = \Phi^k$, we must have $\alpha = 1/k$ and so we get the desired result $\widehat{c} = 1 - 2/k$ for the minimal model.

We now want to consider a more sophisticated argument in favor of this equivalence, one which reveals more of the structure of the (limiting) conformal field theory. Recall the structure of the N=2 algebra:

$$\{Q_\pm, \overline{Q}_\pm\} = P_\pm$$

$$\{Q_\pm, Q_\mp\} = 0.$$

We have $\overline{Q}_+^2 = 0$ so we can consider the cohomology of $\overline{Q}_+$, i.e., consider local operators $\mathcal{O} = \mathcal{O}(x)$ such that $\{\overline{Q}_+, \mathcal{O}\} = 0$. Then

$$\partial_+ \mathcal{O} = \left[\left\{Q_+, \overline{Q}_+\right\}, \mathcal{O}\right] = \left[\overline{Q}_+, \{Q_+, \mathcal{O}\}\right],$$

i.e., in Euclidean language (identifying (x^+, x^-) with $(z, \bar{z})$) we would say that the class of $\mathcal{O}(x)$ in $\overline{Q}_+$-cohomology varies holomorphically.

Taking two such operators, using holomorphic language, we write

$$\mathcal{O}(z)\mathcal{O}'(z') = \sum f_k(z - z')\mathcal{O}_k(z') + \{\overline{Q}_+, \cdots\},$$

with the f_k's being holomorphic functions. From this formula, we can see that the $\overline{Q}_+$-cohomology has the general struture of a conformal field theory in which the operators are "purely left-moving," that is, they vary holomorphically.

An important remark is that the $\overline{Q}_+$-cohomology is invariant under Weyl rescaling, and hence it depends only on the complex structure of the Riemann surface (not the metric). To see this invariance, we must show that $T_{z\bar{z}}$ is $\overline{Q}_+$-exact. A natural way to prove this is to modify the usual definition of scale invariance, employing the quasi-homogeneity of our polynomial, so that the superpotential term

is preserved under the modified scaling. Then the kinetic term in the action fails to be scale-invariant, but the change in the kinetic term is of the form $\int d^4\theta(\cdots)$ and hence is $\overline{Q}_+$-exact.

We have formulated this discussion in terms of $\overline{Q}_+$-cohomology in order to include very general N=2 theories. However, whenever there is a superspace realization of the theory (as is true of the Landau–Ginzburg theories we are studying), it is more convenient to study the $\overline{D}_+$-cohomology rather than the $\overline{Q}_+$-cohomology, where

$$\overline{D}_+ = e^{-\theta^+\overline{\theta}^+\partial_+}\,\overline{Q}_+\,e^{\theta^+\overline{\theta}^+\partial_+}.$$

In our Landau–Ginzburg theory, the equations of motion are

$$2\overline{D}_+\overline{D}_-\Phi^j = \frac{\partial W}{\partial\Phi^j}.$$

Consider the operator

$$\mathcal{T} = \sum_j \left(\left(\frac{1-\alpha_j}{2}\right) D_-\Phi^j\ \overline{D}_-\,\overline{\Phi}_j - i\alpha\overline{\Phi}_j\partial_-\Phi_j\right);$$

then $\overline{D}_+\mathcal{T} = 0$ so that this defines a $\overline{D}_+$-cohomology class. [2]

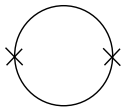
In the free theory, with $W = 0$, $\mathcal{T}$ is an operator of dimension $(1, 0)$. If we have a superconformal theory, we will be able to write

$$\mathcal{T} = J + \theta^- G + \overline{\theta}^-\overline{G} + \theta^-\overline{\theta}^- T.$$

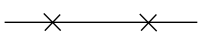
If we do arrive at an N=2 conformal field theory in the limit, then we should expect an OPE of the form

$$\mathcal{T}(x)\mathcal{T}(y) = \frac{\widehat{c}}{(x-y)^2} + \frac{\mathcal{T}}{(x-y)} + \text{holomorphic in } \mathcal{O}\text{'s}.$$

We can check this structure in the free field theory: the first term comes from a diagram of the form



with two propagators exchanged between the two operators and the second term comes from a diagram of the form



with exchange of only one propagator.

Note that, in computing these terms in the operator product expansion, we can set $W = 0$, since the superpotential term is too soft (superrenormalizable) to contribute to these singularities. Once this is done, the different superfields are decoupled, so the computation of $\widehat{c}$ is a sum over different superfields. So there is an immediate conclusion that $\widehat{c}$ is *additive* in the α_j's, that is, $\widehat{c} = \sum f(\alpha_j)$ for some function f.

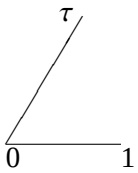
[2] We should worry about anomalies here, but in fact there is no quantum anomaly in the statement that $\overline{D}_+\mathcal{T} = 0$, though there are such anomalies in somewhat similar statements in two-dimensional models with $(0, 2)$ supersymmetry. See E. Silverstein and E. Witten, "Global $U(1)$ R Symmetry And Conformal Invariance Of $(0, 2)$ Models," Phys. Lett. **B328** (1994) 307.

In summary, even in our microscopic Lagrangian, we can see an operator $\mathcal{T}$, which at the level of $\overline{D}_+$ cohomology obeys at $N = 2$ superconformal algebra. So presumably this theory flows in the IR to a conformal field theory whose $\overline{D}_+$-cohomology contains the superconformal algebra.

If we have only one chiral superfield, so $\widehat{c} < 1$, the $N = 2$ algebra acts almost irreducibly in the quantum Hilbert space. Hence one should expect that $\mathcal{T}$ will generate the $\overline{Q}_+$-cohomology. For $\widehat{c} > 1$, the $N = 2$ algebra acts in a way that is far from irreducible, and one should expect to require additional generators.

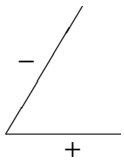
2 The elliptic genus

Our refined evidence for the equivalence of these theories will come from a computation of the *elliptic genus*. We work on a Riemann surface of genus 1 with period lattice spanned by 1 and τ.



We let $q = e^{2\pi i\tau}$, and compute $\mathrm{Tr} q^{(H-P)/2}\overline{q}^{(H+P)/2}(-1)^{F_R}$.

The insertion of $(-1)^{F_R}$ means that the right-moving fermions ψ_+, $\overline{\psi}_+$ should be given the *odd* spin structure (the one periodic in both directions) while the left-moving fermions ψ_-, $\overline{\psi}_-$ should be be given the *even* spin structure (anti-periodic in one direction):



For N=1 supersymmetry, that was the right thing to compute—it turns out to be an index

$$\mathrm{Tr} q^{(H-P)/2}\overline{q}^{(H+P)/2}(-1)^{F_R} = \mathrm{index}(Q_+ + \overline{Q}_+),$$

where $(Q_+ + \overline{Q}_+)^2 = 2P_+ = H + P$.

The story is more interesting in the case of N=2 supersymmetry, since there is the possibility of inserting $e^{i\gamma J_L}$, which commutes with both Q_+ and $\overline{Q}_+$.[3] We introduce the function

$$F(q,\gamma) = \mathrm{Tr} q^{(H-P)/2}\overline{q}^{(H+P)/2}e^{i\gamma J_L}(-1)^{F_R}.$$

The insertion of $e^{i\gamma J_L}$ means: when calculating the trace, glue the data on the torus using $e^{i\gamma J_L}$.

The first observation is that the quantity $F(q,\gamma)$ is independent of any continuously variable parameters of the theory that preserve the $N = 2$ structure. This is so as it is a linear combination of terms each of which is an index of an operator. For the same reason (or because the trace of the stress tensor of the theory is $\overline{Q}_+$-exact) $F(q,\gamma)$ is independent of the area of the torus, and only depends on the complex structure.

We want to calculate $F(q,\gamma)$ in two contexts: (i) an algebraic calculation for the known algebraically constructed models in the IR, and (ii) directly for the Landau–Ginzburg models in the UV. The comparison of these two will then provide additional evidence that the Landau–Ginzburg theory indeed flows to the "minimal model" in the IR.

[3] We use the same symbol J_L for the left R-current and the conserved charge derived from it.

2.1 Algebraic calculation in the IR

The Hilbert space of a "minimal model" takes the form $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha \otimes \overline{\mathcal{H}}_\alpha$, with the left-movers in $\mathcal{H}_\alpha$ and the right-movers in $\overline{\mathcal{H}}_\alpha$. Thus, the elliptic genus can be written

$$F(q,\gamma) = \sum_\alpha \left(\mathrm{Tr}_{\mathcal{H}_\alpha} q^{(H-P)/2} e^{i\gamma J_L}\right)\left(\mathrm{Tr}_{\overline{\mathcal{H}}_\alpha} \overline{q}^{(H+P)/2}(-1)^{F_R}\right).$$

The right-moving contribution $\mathrm{Tr}_{\overline{\mathcal{H}}_\alpha} \overline{q}^{(H+P)/2}(-1)^{F_R}$ evaluates to 1 if $\overline{\mathcal{H}}_\alpha$ has highest weight with $H + P = 0$ and 0 otherwise. This has the effect of selecting a distinguished subset from among the irreducible representations. Let us introduce a restricted sum $\sum'$ to denote the sum over that subset. Then our elliptic genus can be written

$$F(q,\gamma) = \sum_\alpha{}' \mathrm{Tr}_{\mathcal{H}_\alpha} q^{(H-P)/2} e^{i\gamma J_L}.$$

(Note that the eigenvalues of the charge J_L are not integers in general. Rather, the object $\exp(2\pi i J_L)$ commutes with the $N = 2$ algebra and is a constant in each representation; the constant depends on the R-charges of the chiral primary fields, which are rational numbers. So $F(q,\gamma)$ is periodic in γ, but the period is not 2π.)

2.2 Direct calculation in the UV

For the Landau–Ginzburg theory itself, we can try to directly calculate in the UV. We use the fact that $F(q,\gamma)$ is independent of the area of the torus. We take the area to be very small. Roughly, in the UV the interaction is negligible and we should be able to set $W = 0$. The only reason for this to fail has to do with the behavior for large fields. The bosonic part of the action is

$$\int |d\phi|^2 + \int \left|\frac{dW}{d\phi}\right|^2.$$

In the presence of the W term, the unique minimum of the action is $\phi = 0$. If we would set W to zero, the action would be minimized for any constant ϕ. One would have to integrate over the space of constant ϕ's even in a leading approximation to the path integral, and then when ϕ becomes sufficiently large, the W term, if its coefficient is not strictly zero, is important.

The conclusion is, then, that as long as ϕ is a *function* on the torus, the W term cannot quite be ignored. If, however, ϕ were not a function but a section of a nontrivial flat bundle, then even at $W = 0$, the $|d\phi|^2$ part of the action would have a unique and nondegenerate minimum. In that situation, it would be straightforward, in the limit of small area on the torus, to set W to zero.

In the path integral evaluation of $F(q,\gamma)$ for generic γ, we get just such a situation in which ϕ is a section of a nontrivial flat bundle. The reason is that J_L acts on ϕ by $\delta\phi = \alpha\phi$. So in evaluating the elliptic genus we are making the identification

$$\phi(P + \tau) = e^{i\gamma\alpha}\phi(P)$$

(where we regard the torus as the quotient of the complex plane by the lattice generated by 1 and τ, and P is any point on the complex plane). ϕ is a section of the flat bundle on the torus specified by this gluing. If γ is generic, the zero-section is isolated and moreover nondegenerate minimum of the action even if we set $W = 0$. The nondegeneracy makes computations straightforward. The calculation of

$F(q,\gamma)$ can then be carried out in detail, setting $W = 0$ and expressing F as a ratio of determinants. Explaining how this goes would take us too far afield, but the result[4] is

$$F(q,\gamma) = e^{-i\gamma k\alpha/2} \cdot \frac{1 - e^{i\gamma(k+1)\alpha}}{1 - e^{i\gamma\alpha}} \cdot \prod_{n=1}^{\infty} \frac{(1 - q^n e^{i\gamma(k+1)\alpha})(1 - q^n e^{-i\gamma(k+1)\alpha})}{(1 - q^n e^{i\gamma\alpha})(1 - q^n e^{-i\gamma\alpha})}.$$

When compared with the algebraic calculation in the IR, one can extract formulas for certain characters of the N=2 algebra. These formulas can be verified independently, giving strong support to the claim that the Landau-Ginzburg theory flows to the $N = 2$ minimal model in the infrared.

2.3 Connection to Calabi–Yau models

Consider now the $U(1)$ gauge theory with chiral superfields Φ_j of charge α_j and P of charge -1, such that $\sum \alpha_j = 1$. (We are no longer assuming that all of the α_j's are equal, and we are using a different normalization for the generator of $U(1)$ than in lecture II-12.) Denote the gauge supermultiplet by

$$\Sigma = \sigma + \cdots,$$

let $W(\Phi_1, \ldots, \Phi_n)$ be a transverse, quasi-homogeneous polynomial as above (i.e., satisfying $W = \sum \alpha_j \Phi_j \frac{\partial W}{\partial \Phi_j}$, with $W = dW = 0$ only at the origin), and consider the Lagrangian

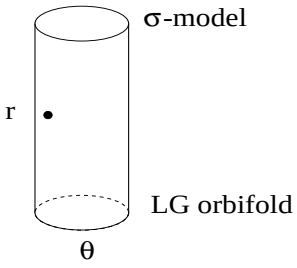
$$\mathcal{L} = \int d^2x\, d^4\theta \left(\frac{\overline{\Sigma}\Sigma}{4e^2} + \overline{\Phi}^j \Phi^j \right) + \left(\int d^2x\, d^2\theta\, PW(\Phi) + \text{c.c.} \right) + \left(\int d^2x\, d\theta^+\, d\theta^-\, t\Sigma + \text{c.c.} \right)$$

such that $t = \frac{\theta}{2\pi} - ir$.

The polynomial $PW(\Phi)$ is gauge invariant, and we can repeat our earlier story: for $r \gg 0$ we classically get a Calabi–Yau hypersurface $\{W = 0\}$ in weighted projective space $\mathbf{WCP}^{n-1}_{(\alpha_1,\ldots,\alpha_n)}$ (which may inherit some singularities from those of the weighted projective space); for $r \ll 0$ we get a Landau–Ginzburg orbifold whose superpotential is W. The quantum theory is singular only when $r = r_0$, $\theta = 0$. (r_0 is a constant computed as in lecture II-12.) The computation of the central charge in Landau–Ginzburg theory gives

$$\widehat{c} = \sum_j (1 - 2\alpha_j);$$

the computation for the σ-model gives $\widehat{c} = \dim_{\mathbf{C}} X$, and these agree since $\sum \alpha_j = 1$.



The elliptic genus should be independent of t since it is a topological invariant; this leads to a formula which relates the elliptic genus of this type of Calabi–Yau manifold to the explicit elliptic genus which can be computed via characters of minimal models on the Landau–Ginzburg side.

[4] E. Witten, "On the Landau–Ginzburg Description of N=2 Minimal Models," Int. J. Mod. Phys. A **9** (1994) 4783–4800; `hep-th/9304026`.

Although the elliptic genus is independent of t, one of the other quantities we have discussed for N=2 theories—the $\overline{Q}_t$-cohomology—varies holomorphically with t. The *instantons* in this problem—which are holomorphic curves in the σ-model—will contribute to the path integral in the form $e^{-i\theta\int\frac{F}{2\pi}}\cdot e^{-r\int d^2x\,D}$ where $D=\sum\alpha_j|\Phi_j|^2-r$, i.e., $e^{-2\pi it\int_\Sigma\Phi^*(c_1(\mathcal{L}))}$. (I have used the fact that the instanton equation of the $N=2$ model – the condition for the field to be invariant under some supersymmetry – gives $\int d^2xD=\int F/2\pi$.) We will explore the instanton sums in the second half of the lecture.

Lecture II-14, part II: Quantum Cohomology of Kähler Manifolds

Edward Witten[1]

1 Introduction

We return now to twisted $N = 2$ supersymmetric σ-models in dimension two with target a Kähler manifold X. Recall that there were four possibilities for twisting depending on which supersymmetry $Q \in \{\overline{Q}_+ + \overline{Q}_-, Q_+ + \overline{Q}_-, \overline{Q}_+ + Q_-, Q_+ + \overline{Q}_-\}$ is preserved. Whichever possibility one selects, the cohomology of Q as a vector space can be identified with the cohomology of X. But there is more structure; if we restrict to the case that the Riemann surface is a fixed $\mathcal{S}^2$, then the additional structure is a ring structure on the cohomology of Q. The resulting rings are called the (a, a), (c, a), (a, c), and (c, c) chiral rings.

With one of the choices, the classical version of the chiral ring turns out to be the usual ring structure on cohomology, but quantum mechanically there are corrections, instanton corrections, to this classical answer. For this reason the chiral ring of the σ-model is called the quantum cohomology of the Kähler manifold X. In this half of the lecture, we will discuss quantum cohomology generally and then make some explicit computations in the case $X = \mathbf{C}P^{n-1}$ and the case X is a Fano hypersurface. These computations will be done both mathematically, counting holomorphic rational curves with certain properties, and will be done from the physics point of view by using the renormalization group flow (discussed in Lecture II-12 and the first half of Lecture II-14) from these σ-models to Landau-Ginsburg theories with extra vacua.

2 The space of 0-energy states

We wish to compute some of these chiral rings, but before we deal directly with the operators, we compute the 0-energy states in the Hamiltonian framework. The reason for doing this computation is that as we showed in the last lecture there is a map

$$\psi\colon \text{chiral ring} \to \{0-\text{energy states}\}$$

given by inserting a local operator at a point on the hemisphere. In favorable circumstances this map is an isomorphism.

In the Hamiltonian framework the bosonic space underlying the space of fields is

$$\mathcal{W} = \text{Maps}(S^1, X).$$

The minimum energy configurations are the constant maps and hence the space of these is a copy of X. Since there are fermions in two copies of the spin bundle, when we quantize the resulting Hilbert space is the tensor product of two copies of the spin bundle over X, i.e., the space of differential forms on X. The Hamiltonian is the Laplacian and its space of ground states is then the space L^2-harmonic forms on X which is identified with $H^*_{L^2}(X)$. Of course, there are normal directions to the copy of X in $\mathcal{W}$ to consider, but supersymmetry implies that the quadratic form in the normal directions is non-degenerate, and hence these directions produce a tensor products of harmonic oscillators and have a unique ground state. Thus, in the end we find that the ground states for the entire theory are identified with the harmonic forms on X. We shall see that the computation of the chiral rings gives the same

[1] Notes by John Morgan

answer (additively), so that in the cases we consider today the map ψ is indeed an isomorphism. This is true generally when the classical theory has the property that the energy grows as the fields go to infinity.

3 Generalities on the chiral ring

In order to define a chiral ring we need a global supersymmetry. As we saw in Lecture II-13 the way to obtain these is to twist the usual form of the $N = 2$ supersymmetric theory. Today we shall only consider twists making $\overline{Q}_+ + Q_-$ global. This is what we referred to in Lecture II-13 as Case A. It is a twist that can be performed for any Kähler manifold – no assumption of vanishing first Chern class is necessary. We shall consider local operators which are functions of the basic bosonic field $\phi\colon S^1 \to X$ and fermions. Recall that one of the effects of twisting is to make the fermions sections of bundles of integral rather than half-integral spin. That is to say, they are differential forms instead of spinors. It will suffice in describing the Q-cohomology of operators, to consider only fermions of zero spin. The reason for this is that all of the Q-cohomology classes of operators have representatives that are constructed using only those fermions. To be more precise, the Q-cohomology classes have representatives that depend only on the bosonic fields and spin zero fermions and not their derivatives. [2] Thus, our local operators will be of the form

$$O = O(\phi, \psi_L, \psi_R)$$

where in the untwisted version of the theory, ψ_L is a section of $K^{-1/2} \otimes \phi^* T^{0,1}X$ and ψ_R is a section of $K^{1/2} \otimes T^{1,0}X$. After we twist, these fermion fields lie in $\phi^* T^{0,1}X$ and $\phi^* T^{1,0}X$, since the twisting cancels out the line bundle over Σ. In particular, it is possible to form the sum $\psi = \psi_L + \psi_R$ and consider operators $O = O(\phi, \psi)$.

3.1 Local Operators

For each differential form λ on X there is an operator $O_\lambda(x) = O_\lambda(\phi(x), \psi(x))$. As we remarked in the Lecture II-13, under the identifications of operators and forms, Q is identified with the usual exterior derivative:

$$\{Q, O_\lambda\} = O_{d\lambda}.$$

When, as in the case of these $N = 2$ supersymmetric σ-models, the operator-to-state correspondence is an isomorphism we can compute the ring structure of the chiral ring by computing correlation coefficients. Our goal here is to compute in more classical topological and geometric terms the correlation coefficient

$$\langle O_{\lambda_1}(x_1) \cdots O_{\lambda_s}(x_s) \rangle$$

for closed forms λ_i. As we have already remarked, the correlation coefficient is unchanged if we replace the λ_i by cohomologous forms. Also, to compute this correlation function, which means to compute the path integral

$$\int \mathcal{D}\phi \mathcal{D}\psi e^{2\pi i \mathcal{L}} \prod_i O_{\lambda_i}(x_i)$$

[2] This is the operator analog of the fact that in the Hamiltonian description, the zero energy states are constructed from differential forms on the target space with all oscillators in their ground state. Operators that contain derivatives of fields would correspond to states in which some oscillators are excited.

over the space of all fields, it suffices to compute the integral over the subspace of Q-invariant fields. We have already identified the bosonic part of the subspace of Q-fixed points: it is the space of holomorphic maps $\Sigma \to X$. For each i choose a cycle H_i Poincaré dual to $[\lambda_i]$. Then we can choose a representative for λ_i with support in an arbitrarily small neighborhood of H_i. To compute the correlation function one will consider the moduli space of holomorphic maps from the Riemann surface passing through all of the H_i at the points x_i. In the 'best' case when there are only finitely many holomorphic maps passing through the cycles, one simply counts each of these maps with a sign. The sign comes from evaluating the bosonic and fermionic determinants in an expansion around the classical solution. Then the result is weighted by the exponential of $2\pi i$ times the value of the action at this component.

Mathematically, there is a dimension count which must be satisfied for the differential form being integrated to be top dimensional on the moduli space of holomorphic curves, and hence for the answer to have a chance of being non-zero. From the physics point of view one sees the same dimension restriction coming from the anomaly of the R-symmetry. The measure of integration in the path integral has an anomaly under this symmetry, and one must use a set of operators whose product has precisely the cancelling anomaly for the path integral to have a chance to be non-zero. Not surprisingly, these two conditions are the same: namely, that the sum of the codimensions of the H_i in X must be equal to the dimension of the instanton moduli space of non-constant holomorphic maps from Σ to X.

If one wants to see just the classical cohomology ring, rather than the quantum cohomology, then one should consider just the constant maps to X. Then the dimension condition is

$$\sum_{i=1}^{s} \text{degree}\lambda_i = \dim X$$

and the classical answer is simply

$$\int_X \lambda_1 \wedge \cdots \wedge \lambda_s.$$

That is to say, the classical chiral ring is the cohomology ring of X. This answer gets corrected quantum mechanically by instanton corrections given by the integrals we were describing over the moduli space of non-constant holomorphic maps. Thus, the chiral ring is a quantum correction to the usual cohomology ring of X. For this reason the chiral ring is usually called the quantum cohomology ring of X.

4 More Details on the Ring Structure

Having given the general definition of quantum cohomology as the chiral ring and indicated the sorts of mathematics that go into computing it, it is now time to be more concrete and compute an example. We take the case of $\Sigma = S^2$ mapping into a compact Kähler manifold X. Let us begin with a correlation function with only two operators:

$$\langle O_{\lambda_1} O_{\lambda_2} \rangle,$$

with the degree of λ_1 plus the degree of λ_2 equal to the dimension of X. We of course know that the space of constant maps gives a contribution to this correlation function equal to

$$\int_X \lambda_1 \wedge \lambda_2.$$

In this case there is no quantum correction and this is also the answer in the chiral ring. The reason that there is no quantum correction is that the space of non-constant holomorphic maps of $S^2 \to X$ passing through a point of H_1 and H_2 has a free $\mathbf{C}^*$-action on it. For a component of this space to possibly give a non-zero correction to the correlation coefficient, it must be the case that its formal dimension is zero. But if it has zero dimension and has a free $\mathbf{C}^*$-action on it, then it must be empty. (Even for a possible nonzero component of this space, the free $\mathbf{C}^*$ action ensures that the contribution is zero.) Thus, we have shown that under the identification of the chiral ring with the cohomology of X, the two point correlation functions, which compute the inner product on the vector space underlying the chiral ring, give the usual inner product on cohomology or equivalently give the intersection form on homology.

Now let us consider the ring structure. Fix a basis $\{\lambda_i\}$ for $H^*(X)$. Let the inner product in this basis be given by η_{ij}:

$$\eta_{ij} = \langle O_{\lambda_i} O_{\lambda_j} \rangle.$$

Then the ring structure is determined by structure constants c_{ij}^k defined by

$$O_{\lambda_i} O_{\lambda_j} = c_{ij}^k O_{\lambda_k}.$$

Let ω_{ijk} be the three point function

$$\langle O_{\lambda_i} O_{\lambda_j} O_{\lambda_k} \rangle.$$

Since this three point function is also the inner product of the product of the first two local operators with the third, we see

$$\omega_{ijk} = \sum_r c_{ij}^r \eta_{rk}.$$

Since η_{ij} is non-degenerate, this implies that the two and three point correlation functions determine the ring structure of the chiral ring.

5 Calculations for $\mathbf{C}P^{n-1}$

We will now consider an example. We do the computation for $\mathbf{C}P^{n-1}$ by using a gauged linear σ-model with linear bosonic fields $\phi_1, \ldots, \phi_n, p$ as in Lecture II-12. To get all of $\mathbf{C}P^{n-1}$ we set the superpotential W equal to zero. We get a family of theories depending on a complex parameter $t = -ir + \frac{\theta}{2\pi}$, and as we have seen, for $\operatorname{Im} t \ll 0$ these are non-linear σ-models on X with some Kähler metric depending on t and Kähler class roughly proportional to $\operatorname{Im} t$. The only observable is O_H where H is the hyperplane section. We call this operator σ. Classically, in $\mathbf{C}P^{n-1}$ we have one relation, namely $\sigma^n = 0$. Let us see what happens quantum mechanically. We need to compute the three point function

$$\langle \sigma^a \sigma^b \sigma^c \rangle$$

where $a+b+c = n-1+dn = \dim \mathcal{M}_d$, the dimension of the moduli space of rational curves of degree d. Clearly, it suffices to consider the case when $a, b, c < n$. This implies that the only possibilities for d are $d = 0, 1$. Of course, $d = 0$ is the moduli space of constant maps and its contribution is the classical ring structure. For $d = 1$, we are considering straight lines in $\mathbf{C}P^{n-1}$. To calculate

$$\langle \sigma^{n-1}(x) \sigma^{n-1}(y) \sigma(z) \rangle$$

we must consider the space of straight lines through two points meeting a fixed hyperplane. There is exactly one such line. Since the instanton action is $e^{-2\pi i t}$, this lead to the relation

$$\langle \sigma^{n-1}(x) \sigma^{n-1}(y) \sigma(z) \rangle = e^{-2\pi i t}.$$

Unraveling the ring structure from this three point function yields

$$\sigma^n = e^{-2\pi i t}$$

in quantum cohomology.

We can make the same calculation physically. We are considering the σ-model $\mathbf{C}P^1 \to \mathbf{C}P^{n-1}$. At the end of Lecture II-12 we found that in the infra-red this flows to n vacua with a mass gap (since this is a case in which $c_1(X) > 0$, the flow is from the σ-model to the Landau-Ginsburg model). The field σ has different expectation values in each vacuum. Since the fields are massive, we simply set them equal to their expectation values. These vacuum expectation values are $\sigma = \mu e^{2\pi i(-t+k)/n}$, $k = 0, \ldots, n-1$. These give the idempotents of the quantum cohomology ring, from which we deduce the same answer as before,[3] namely that the ring is generated by σ modulo the relation $\sigma^n = e^{-2\pi i t}$.

6 Calculations for Fano Hypersurfaces

Now let us generalize these computations to the case of a hypersurface of degree $d < n$ in $\mathbf{C}P^{n-1}$, so-called Fano hypersurfaces.

According to [Collino-Jinzenji] the answer, computed mathematically, is that the quantum cohomology ring is generated by σ with one relation:

$$\sigma^{d-1}\left(\sigma^{n-d} - e^{-2\pi i t}\right) = 0. \tag{6.1}$$

Let us think about how this computation fits with the fact that again this σ-model flows in the infra-red to a Landau-Ginsburg model plus extra massive vacua. The roots of the equation $\sigma^{d-1}(\sigma^{n-d} - e^{-2\pi i t}) = 0$ are of two types: a root at $\sigma = 0$ of multiplicity $d-1$, and nondegenerate roots (that is, roots of multiplicity one) at nonzero sigma. The roots of the equation correspond to the vacua of the quantum field theory. Nondegenerate roots correspond to massive vacua, which give idempotents of the quantum cohomology ring. In analyzing the vacuum structure of this theory in lecture 12, we found $n-d$ massive vacua at nonzero sigma. These give the factor $(\sigma^{n-d} - e^{-2\pi i t})$ in the equation for the quantum cohomology. The other factor σ^{d-1} is the contribution of the Landau-Ginzburg vacua at the origin. This factor is present if $d > 1$ (for $d = 1$ we are discussing a projective space of codimension one, and there is no Landau-Ginzburg vacuum), and corresponds to a root at the origin of multiplicity greater than one if $d > 2$ (for $d = 2$, the central charge of the Landau-Ginzburg theory, which in general is $\widehat{c} = n(1 - 2/d)$, vanishes; in this case the Landau-Ginzburg theory is massive and infrared trivial). For general d, the multiplicity of the $\sigma = 0$ root of the quantum cohomology can be computed by evaluating $\mathrm{Tr}(-1)^F$ in the Landau-Ginzburg theory at $\sigma = 0$.

[3] Except for a factor of μ which has to do with a mismatch between our physical and mathematical conventions. In lecture 12, σ was defined as a field of dimension one and no effort was made to compare to any topological normalization. In our mathematical discussion today, σ was normalized topologically as a field related to a hyperplane section of projective space, and in particular has dimension zero. To compare the two definitions, one should, in the setting of lecture 12, integrate out the σ field in the region $r\tilde{g}0$ in favor of the massless fields of the sigma model. In this process, σ will turn into an operator of the low energy theory that can be derived from a two-form on $\mathbf{CP}^{n-1}$. The cohomology class of this operator will be a multiple C of the hyperplane section, and C is the factor by which the physical normalization of lecture 12 disagrees with the topological normalization that gives $\sigma^n = e^{-2\pi i t}$.

Lecture II-15. Four-dimensional gauge theories

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Notes by Pavel Etingof and David Kazhdan

In this lecture we will discuss the main known, believed, and conjectured results about 4-dimensional gauge theory, with or without supersymmetry.

15.1. Gauge theory without supersymmetry.

We start with no supersymmetry ($N = 0$). We consider a pure nonabelian gauge theory for a simple Lie group G, with Lagrangian

$$L = \frac{1}{4g^2}\int |F|^2.$$

We could add a topological term with the theta angle: $\frac{i\vartheta}{16\pi^2}\int F\wedge F$. This is a well defined, asymptotically free theory. The problem we are interested in is how this theory behaves in the infrared.

It is believed that this theory has a mass gap, and conjectured that it exhibits confinement (the area law for the Wilson loop operator, see Lecture 6). However, to justify the first and especially the second statement is still an important open problem.

Remark. As we will see, the benefit of supersymmetry is that in the supersymmetric gauge theory, both statements can be justified.

Now let us add matter. We consider the group $G = SU(N)$. The matter will be chiral spinors ψ_i, ψ_i^* with values in the vector representation V and the dual representation V^*, and the corresponding antichiral spinors $\bar{\psi}_i$, $\bar{\psi}_i^*$ with values in V^* and V (we are in Euclidean signature). Let the number of spinors of each kind be n_f, n_f^* (the number of flavors). We must have $n_f = n_f^*$ to avoid local anomalies (see the next lecture).

The Lagrangian for the theory with spinors (with zero bare masses) is

$$L = \frac{1}{4g^2}\int |F|^2 + \frac{i\vartheta}{16\pi^2}\int F\wedge F + \sum_i \int (\bar{\psi}_i, D\psi_i) + (\bar{\psi}_i^*, D\psi_i^*). \tag{15.2}$$

Remark. For $G = SU(3)$ and $n_f = 2$ or 3 this is the theory of strong interactions. In these cases it is necessary to add small bare masses to match the physical reality.

This theory is asymptotically free if n_f is not too large. So one is interested in its infrared behavior.

It turns out that this theory exhibits symmetry breaking. Namely, the Lagrangian has a global $U(n_f)\times U(n_f)$ symmetry permuting the flavors. However, quantum mechanically, this theory has a chiral anomaly (i.e. $D + A$ on chiral spinors may have a nonzero index). Therefore, quantum mechanically the symmetry of the operator algebra is broken to the subgroup $H = \{(A, B) \in U(n_f)\times U(n_f), det(A) = det(B)\}$ (the rest of the group does not fix the measure of path integration).

It is believed that the group H is spontaneously broken to the diagonal subgroup H_{diag} of $U(n_f)\times U(n_f)$. This breaking produces Goldstone bosons by Goldstone theorem. If $n_f \le 1$, H is already diagonal, and we don't get any Goldstone bosons. In this case it is believed that the theory has a mass gap. However, for $n_f > 1$ the group H is different from the diagonal subgroup, and there is no mass gap because of Goldstone bosons. In this case, if the theory is infrared free and has no other massless particles than Goldstone bosons, the low energy effective theory should be a sigma-model in the space of vacua H/H_{diag}.

Remark. The global symmetry breaking from H to the diagonal subgroup is more delicate than the one we considered in Lecture II1: in the present case this breaking is not seen classically, since the classical space of vacua is a point.

The infrared behavior of this theory is not completely understood, but some aspects are. For example, one can sometimes prove the absence of a mass gap, in the following way.

Let J^a, $a = 1, ..., dim(H)$ be the currents for H. Consider the 3-point function $\langle J^a(x)J^b(y)J^c(z)\rangle$. Since the currents are conserved classically, we get

$$d_x\langle J^a(x)J^b(y)J^c(z)\rangle = 0. \tag{15.1}$$

if x, y, z are distinct. However, if there is symmetry breaking, the l.h.s. of (15.1) may be a singular distribution supported on the set of non-distinct x, y, z. So let us go to the momentum space and compute (15.1) to 1-loop order. Let

$$F^{abc}_{\lambda\mu\nu}(k, q, r)\delta(k + q + r) = \langle \hat{J}^\lambda_a(k)\hat{J}^\mu_b(q)\hat{J}^\gamma_c(r)\rangle$$

be the 3-point function of J in momentum space. Then we get (to 1-loop accuracy)

$$k_\lambda F^{abc}_{\lambda\mu\nu} = \varepsilon_{\mu\nu\alpha\beta}q^\alpha r^\beta d^{abc}, \tag{15.2}$$

where $k + q + r = 0$ and d^{abc} are some numbers (ε is a symmetrization tensor).

It can be proved that this formula is exact, with no higher order corrections. This is proved, for instance, by introducing a regularization (such as adding higher derivative terms of a certain kind to the action) that guarantees that higher than one-loop contributions to $\partial_x < JJJ >$ are zero.

It is easy to show that no function F that is real-analytic near k=q=r=0 can obey (15.2) with nonzero d. Hence if d is not zero, F is singular near zero momentum. This implies that the theory has massless particles, so there is no mass gap.

Remark. This argument would not work if we considered a two-point function instead of the three-point function. Namely, it can be shown that in 2n dimensions, it is precisely the n+1 point function of currents for which there is a formula analogous to equation (15.2) – computable from one-loop but exact. Whenever one has such a formula, one can use it to prove the absence of a mass gap. In four dimensions, one uses a three point function; in two dimensions, one would use a two point function.

It can be seen from this calculation that the tensor d^{abc} is of the same nature as the tensor d^{abc} in Lecture 16 which corresponds to the local gauge anomaly. In particular, its presence has to do with the fact that our theory has fields with coefficients in the vector representation of the first copy of $U(n_f)$ and no fields in the dual representation (i.e. with the fact that the theory is chiral with respect to the first copy of $U(n_f)$). In fact, if we try to gauge the H-symmetry, we will get a local gauge anomaly like in Lecture 16.

Thus, in some cases we can see by a short distance calculation that we will have massless particles in the infrared.

In 4 dimensions, there can be three types of massless particles: scalars (spin 0), spinors (spin 1/2), and vectors (spin 1). Other particles are inconsistent with the existence of the energy-momentum tensor. The general principle about massless particles is that they are massless for a reason. Namely, it is believed that in infrared free theories spin 0 particles are always Goldstone bosons coming from a broken global symmetry, spin 1/2 particles are fermions for which mass is prohibited by an unbroken chiral symmetry, and spin 1 are gauge bosons. This allows us to analyse the origin of a nonzero tensor d^{abc}.

Let us look at the possibilities in the case when $d^{abc} \neq 0$.

Possibility 1. The theory is not free in the infrared.

Possibility 2. The theory is free in the infrared. Then Green's functions can be calculated using Feynman diagrams of light fields. So there must be diagrams which contribute to the non-analyticity of F. There may be two cases.

Case 1. The field $J(x)$ does not create from the vacuum a state of one massless particle (the two-point function of J has no pole at zero momentum). Then the tensor d^{abc} can come only from loop diagrams. Such diagrams are expected to be with massless fermions inside, maybe not in the same representation as the one in the classical Lagrangian, but producing a nonzero d tensor (massless spin 0 and spin 1 particles are not expected to contribute to the tensor d since, as we mentioned, such a contribution arises from chirality which has to do with spin 1/2). This possibility was pointed out by t'Hooft in the late seventies, but examples were only found in the last few years.

Case 2. There is no massless fermions, and no loop contributions to the singularity of F. Then the tensor d must come from the tree diagram for J. In this case, the two point function of J has a pole at zero momentum, corresponding to the fact that J creates from the vacuum a state consisting of one massless particle. This state is a massless boson π of spin 0 from the vacuum (see Lecture II1), such that $J = *d\pi$ in the infrared limit. This boson π is the Goldstone boson for the broken global H-symmetry.

Remark. Even in case 1, the two point function need not be analytic at the origin. If there is no mass gap, then the two point function of J is almost always nonanalytic at zero momentum: it has a branch cut beginning at zero momentum.

Thus the conclusion is: if the theory has no massless fermions, the H-symmetry is spontaneously broken to a group $H' \subset H$ such that $d|_{H'} = 0$.

This is what happens in nature. Namely, the diagonal subgroup of H has $d = 0$ since the representation in which the fermions live is real for this group, and thus all chiral fields are balanced by antichiral ones. This is the most obvious subgroup of H with this property.

Remark. Consider Case 2 from the point of view of effective Lagrangians. Consider the microscopic Lagrangian $L' = L + \int c^a \wedge J^a$, where J^a are the currents defined above and c^a are external sources which are 1-forms on the spacetime. For this Lagrangian, consider the effective Lagrangian in terms of the fields c^a. It will have the linear term, the quadratic term, etc. The linear term corresponds to one insertion of J, so it has the form $c^a \wedge *d\pi^a$, where π is the Goldstone boson. The 3-point function $\langle d * J^a(x) * J^b(y) * J^c(z)\rangle$ considered above corresponds to the quadratic term $dc^a \wedge c^b \wedge d\pi^c$ in the effective Lagrangian. The particles π and c can actually be observed in nature. In fact, it was in seeking to understand the decay of the π^0 meson to two photons that Adler, Bell, and Jackiw discovered anomalies around 1970.

In conclusion we will say what behavior in the infrared is expected in the theories we are considering. Recall that the condition for asymptotic freedom is $n_f < 11n/3$. If n_f is small compared to $11n/3$, it is believed that the theory is infrared free with the behavior described above. It is also expected that it exhibits confinement for sufficiently small n_f/n. However, if n_f is close to $11n/3$, it is expected that the theory has nontrivial IR stable fixed points. This is expected because the two-loop correction to the beta function of these theories is positive. Hence, if the one-loop beta function is made small by taking n_f close to but smaller than 11n/3, then the beta function has an infrared-stable zero close to the origin and within reach of perturbation theory.

15.2. N=1 supersymmetric pure gauge theory.

Now consider $N = 1$ supersymmetric theories. We start with "pure gauge theory". In terms of components, this theory has a gauge field A and a chiral spinor λ with values in the adjoint represen-

tation. The Lagrangian is

$$L = \frac{1}{4g^2}\int |F|^2 + \frac{i\vartheta}{16\pi^2}\int F\wedge F + \int \bar{\lambda} iD\lambda. \tag{15.3}$$

The symmetry of this theory classically is the chiral $U(1)$ symmetry $\lambda \to e^{i\delta}\lambda$. This chiral symmetry, however, does not exist quantum mechanically since the operator $D + A$ on spinors in the adjoint bundle may have a nontrivial index. However, one can show that this index for the sphere S^4 is always divisible by $2h$, where h is the dual Coxeter number of the Lie algebra (it equals $2hk$ for the bundle with second Chern class k). This shows that the group of symmetries of the measure of integration is $\mathbb{Z}/2h\mathbb{Z}$. This group acts on the quantum operator algebra.

In the infrared limit, the group $\mathbb{Z}/2h\mathbb{Z}$ could possibly be spontaneously broken to a subgroup. The smallest subgroup it can break down to is $\{-1, 1\}$ since -1 is the central element of the double cover of the Poincare group.

The standard expectations about this theory are:

1. Mass gap.

2. Confinement.

The first statement implies that fermion masses are generated dynamically. This suggests that $\mathbb{Z}/2h\mathbb{Z}$ is broken down to a subgroup that allows bare masses for the elementary fermions. The largest such subgroup is $\mathbb{Z}/2\mathbb{Z}$. So one expects that the unbroken group is precisely $\mathbb{Z}/2\mathbb{Z}$. The spontaneous breaking from $\mathbb{Z}/2h\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ implies that the theory has at least h vacuum states. It is believed that the number is precisely h.

It is remarkable that the statement about symmetry breaking can actually be checked. In order to do it, it is enough to show that the one point function $\langle\lambda\lambda\rangle$ of the operator $\lambda\lambda$ is not zero (by $\lambda\lambda$ we mean a scalar obtained by contraction of the Lie algebra dimensions using the Killing form and of the spinor dimensions using the skew form on spinors).

It is clear that $\langle\lambda\lambda\rangle = Ce^{2\pi ik/h}$ in the k-th vacuum, for some constant C, since the generator of the group $\mathbb{Z}/2h\mathbb{Z}$ sends the k-th vacuum to $k+1$-th and multiplies $\langle\lambda\lambda\rangle$ by $e^{2\pi i/h}$.

Now let us ask ourselves: how does the function $\langle\lambda\lambda\rangle$ depend on the theta-angle? Let J be the current of the $U(1)$ symmetry. Because of the chiral anomaly, $dJ \neq 0$ but equals to a multiple of the Chern-Weil form:

$$dJ = \frac{h}{8\pi^2}F\wedge F. \tag{15.4}$$

This implies that the rotation of λ by $e^{i\alpha}$ is equivalent to adding $2h\alpha$ to the theta-angle: $\vartheta \to \vartheta + 2h\alpha$ (i.e. these two actions of $U(1)$ on the space of theories coincide). Thus we should expect that $\langle\lambda\lambda\rangle = Ce^{i\vartheta/h}$, where C is real and independent of the theta-angle. Note that this function is multivalued and has h different values, corresponding to h vacua.

Since $\langle\lambda\lambda\rangle$ has h different values, it is convenient to consider its h-th power $\langle\lambda\lambda\rangle^h = C^h e^{i\vartheta}$. This is a single-valued function of ϑ.

The constant C depends only on the gauge coupling g, and is zero in perturbation theory, since the classical chiral symmetry prohibits masses in Feynman diagrams of all orders. Thus, if $C \neq 0$, it is a nonperturbative phenomenon (like in the Gross-Neveu model, see Gross' lecture 4).

Now let us compute $\langle\lambda\lambda\rangle^h$. We have

$$\langle\lambda\lambda\rangle^h = \lim_{|x_i - x_j|\to\infty}\langle\lambda\lambda(x_1)...\lambda\lambda(x_n)\rangle \tag{15.5}$$

(cluster decomposition). This statement is true at all the vacua.

We would like to show that $\langle\lambda\lambda(x_1)...\lambda\lambda(x_n)\rangle$ is independent of $x_1, ..., x_n$. To do this, recall the superspace formulation of our model (see the superhomework).

In the superspace formulation, we have the super-spacetime $\mathbb{R}^{4,4}$ with coordinates $x_0, ..., x_3$ and $\theta_+, \theta_-, \bar{\theta}_+, \bar{\theta}_-$. The Lagrangian of our model is equivalent to the Lagrangian

$$L' = \int d^4x(d^2\theta\tau W^2 + d^2\bar{\theta}\bar{\tau}\bar{W}^2),$$

where $\tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2}$, and W is the supercurvature: $W = \lambda + \Theta F_+ ...$, $\bar{W} = \bar{\lambda} + \Theta F_- + ...$ Here Θ is the canonical linear chiral function on $\mathbb{R}^{4,4}$ with values in S_+, F_+ is the self-dual part of the curvature. The terms ΘF_+ involves the $SU(2)$-map $\mathbb{C}^2 \otimes \mathbb{C}^3 \to \mathbb{C}^2$.

The supercurvature is a spinor-valued chiral superfield. Namely, let $\bar{D}$ be the spinor valued operator which annihilates chiral functions (we have $[\bar{D}, \bar{D}] = 0$). Then $\bar{D}W = 0$. This implies that $\bar{D}(WW) = 0$ (where WW is a scalar obtained by contracting spinors). Since $d = [D, \bar{D}]$, we get $d(WW) = \bar{D}D(WW)$.

Now recall that for any operator X the 1-point function $\langle\bar{D}X\rangle$ is zero. This implies that

$$\langle d(WW(x_1))WW(x_1)...WW(x_n)\rangle = 0,$$

which yields the independence of $\langle\lambda\lambda(x_1)...\lambda\lambda(x_n)\rangle$ of $x_1, ..., x_n$ after setting $\Theta = 0$.

This implies that we can compute $\langle\lambda\lambda\rangle^h$ at short distances. At short distances, $\langle\lambda\lambda\rangle^h$ can be computed using asymptotic freedom.

Our parameters are μ (the scale of renomalization), g (the gauge coupling), and ϑ (the theta angle). Thus $\langle\lambda\lambda\rangle = F(\mu, g, \vartheta)$. From dimensional arguments, $F(\mu, g, \vartheta) = \mu^{3h} f(g, \vartheta)$, where f is dimensionless (fermions have scaling dimension 3/2 in 4 dimensions). Also, we found that $\langle\lambda\lambda\rangle^h$ has the form $Ce^{i\vartheta}$, where C is independent of ϑ. This implies that $f(g, \vartheta) = \rho(g)e^{i\vartheta}$.

Now comes the main point. The main property of our theory, which follows from supersymmetry, is that if X is an operator such that $\bar{D}X = 0$ then $\langle X\rangle$ is a holomorphic function of τ, where τ is the "modular" parameter in the superspace formulation. This follows from the fact that $\frac{\partial L'}{\partial\bar{\tau}} = \int d^4x d^2\bar{\theta}\bar{W}^2 = \bar{D}Y$, where $Y = \bar{D}\bar{W}^2$. Therefore, F is holomorphic in τ. This implies that $f(g) = e^{-8\pi^2/g^2}$, i.e.

$$F(\mu, g, \vartheta) = C_0 e^{i\vartheta - \frac{8\pi^2}{g^2}}. \tag{15.6}$$

where C_0 is a constant.

The constant C_0 is tricky to find. To find C_0, one should notice that the only nontrivial contributions to $\langle\lambda\lambda\rangle^h$ can be from field configurations with instanton number (=second Chern class) equal to 1. The reason that C_0 can come only from instanton number 1 is that $\langle\lambda\lambda\rangle^h$ has charge $2h$ under the R symmetry, so the index of the Dirac operator must be 2h. An attractive further check is that $8\pi^2/g^2$ is the instanton action. Thus, the constant C_0 can be found by doing perturbation theory in the vicinity of a single instanton. In particular, it can be shown that $C_0 \neq 0$ (dynamical symmetry breaking).

Remark 1. Since $\bar{W} = \bar{\lambda} + \Theta F_- + ...$, and since instantons are defined by having self-dual curvature and $\lambda = 0$, we have $\bar{W} = 0$ for instantons and hence the Lagrangian is holomorphic in τ at the instantons (and only at them). This is an indication of why the function F should receive its contribution from instantons.

Remark 2. The function F is a physically meaningful quantity (it has to do with fermion masses), so it obeys the renormalization group equation

$$(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g})F = 0, \tag{15.7}$$

where β is the beta-function. Apriori we know only that $\beta(g) = -3hg^3 + O(g^5)$. Formula (15.6) implies that the 1-loop beta function is in fact exact:

$$\beta(g) = -3hg^3. \tag{15.8}$$

In other words, the parameter τ has an additive renormalization: $\tau(\mu) = \tau(\mu_0) + 48\pi^2 h \ln(\mu/\mu_0)$.

15.3. N=1 theories with chiral superfields.

Now let us consider the same theory with an adjoint-valued chiral superfield Φ. The new Lagrangian is

$$L'' = L' + \int d^4x d^4\theta \bar{\Phi}\Phi + (\varepsilon \int d^4x d^2\theta \Phi^2 + c.c). \tag{15.9}$$

(c.c. is "complex conjugate".) Here ε is a complex number which is a mass parameter for Φ. If ε is very large, we recover the previous theory as an effective theory after integrating out Φ. If $\varepsilon = 0$ we get Donaldson theory.

Remark. It can be deduced from supersymmetry that the case $\varepsilon \neq 0$ is the same as the case of large ε in terms of the qualitative behavior of the theory in the infrared. This is proved by redefining Φ by $\Phi \to \Phi/\sqrt{\varepsilon}$. This operation removes ε from the superpotential. The ε dependence appears now only in the kinetic energy of Φ, which is a term of the form $\int d^4\theta(...)$. Such terms are believed to be generally irrelevant in the infrared behavior of a supersymmetric theory, so one expects the qualitative behavior of this theory to be independent of ε.

For $G = SU(2)$, this theory has two vacua (as in the theory without the superfield Φ). Also, we will show in later lectures that this theory has a mass gap and confinement.

Now we will specialize to $G = SU(N)$ and do a superversion of QCD. So the Lagrangian will be the same as before (with $\varepsilon = 0$), except that the chiral superfield Φ will be in the representation $R = V^{\oplus n_f} \oplus V^{*\oplus n_f}$, where V is the vector representation of $SU(N)$. (As before, the multiplicity of V and V^* must be the same to avoid anomalies). Namely, $\Phi = (Q_1, ..., Q_k, Q_1^*, ..., Q_k^*)$, where $k = n_f$. This theory is asymptotically free iff $k < 3N$, since $\beta = -\frac{g^3}{\pi}(3N - k)$.

The dynamical behavior of this theory is very nontrivial. For fuller details see Seiberg's lectures. The following picture is believed to hold:

1. $k = 0$ – discussed above. There is a mass gap.

2. $k = 1, ..., N-1$ – discussed below. Low energy theory is a supersymmetric sigma-model with a nonzero superpotential.

3. $k = N, N+1$ – The theory is infrared free and has more massless fields than in case 2.

4. $N + 2 \leq k < 3N/2$ – The theory is infrared free, is described in the infrared by a gauge theory, but the IR gauge group is not a subgroup of the UV gauge group.

5. $3N/2 \leq k < 3N$ – the infrared limit is nontrivial and non-free.

Today we only discuss the case $1 \leq k \leq N-1$. We will try to explain why the superpotential is nonzero.

Let us compute the classical moduli space of vacua. According to lecture II2, this moduli space should be the symplectic quotient: $\mathbf{M} = R//G$ (since R is the set of zero energy states). Here the symplectic structure on R is induced by the Hermitian form on R. Since G acts holomorphically, we have $R//G = R/G_{\mathbb{C}}$. So we are down to classical invariant theory.

Recall that for $k < n$ all invariants of $SL(N)$ in R are functions of the inner products $M_{ij} = (Q_i, Q_j^*)$. These invariants are independent, and form a coordinate system on $R/G_{\mathbb{C}}$. Moreover, let us compute the stabilizer of a point with $det(M) \neq 0$. It is easy to see that this stabilizer is isomorphic to $SL(n-k)$. Thus, we have a Higgs phenomenon: the $SU(N)$ gauge symmetry is spontaneously broken to $SU(N-k)$ (see lecture II2). In particular, for $k = N-1$ it is broken completely. This says that for $k < N-1$

the classical theory has massless gauge fields of $SU(N-k)$ as well as massless chiral multiplets parametrizing the symplectic quotient. Quantum mechanically the story will be more complicated for several reasons.

Now, since the stabilizer of a point in $\mathbf{M}$ is $SU(N-k)$, we should expect that the low energy effective theory is an $N=1$ supersymmetric $SU(N-k)$-gauge theory. As we know, this theory has $N-k$ vacua, so we should expect that the quantum vacuum space will be an $N-k$-fold cover of $\mathbf{M}$. Call this cover $\tilde{\mathbf{M}}$. Thus the low energy effective theory involves a sigma-model into $\tilde{\mathbf{M}}$ and possibly a superpotential. It is shown similarly to the above arguments that this superpotential W must be holomorphic on $\tilde{\mathbf{M}}$ and as a function of the complexified gauge coupling τ. The superpotential should of course have the same symmetries as the theory itself.

Classically our theory has the $U(k)\times U(k)$ global symmetry acting on the superfield, as well as the R-symmetry, which is the $U(1)$ symmetry rotating λ. Some of these symmetries are anomalous (chiral anomaly). The anomaly free subgroup which acts on the quantum theory is generated by the group $H=\{(A,B)\in U(k)\times U(k); det(A)=det(B)\}$ and the group $U(1)=\{(\zeta,\zeta^{-\frac{N-k}{N}},\zeta^{\frac{N-k}{N}})\in U(1)\times U(k)\times U(k)\}$. From these symmetries (and remembering from the first part of the lecture how the R-symmetry acts on the theta angle), we see that the superpotential must equal

$$W(M)=c_{N,k}(detM)^{\frac{-1}{N-k}}. \tag{15.10}$$

So we do get a function on a cover $\tilde{\mathbf{M}}$ of $\mathbf{M}$.

Now we need to compute $c_{N,k}$; in particular see if it is zero. The key fact for doing this is: If $c_{N,k}\neq 0$ then $c_{N-1,k-1}$ is nonzero, and one can be determined from the other. Indeed, if we take M which has one very large eigenvalue, we will effectively be down to the theory with parameters $N-1,k-1$.

Thus if $k<N-1$, we can descend from (N,k) to $(N-k,0)$ and thus reduce the problem to the case of pure $N=1$ gauge theory. In pure gauge theory, the space $\mathbf{M}$ is a union of $N-k$ points, and the role of the superpotential W is played by the function $\langle\lambda\lambda\rangle$ that we have computed. Thus, $c_{N-k,0}\neq 0$.

Remark. It might seem that the superpotential is defined up to adding a locally constant function. However, it is in fact defined up to adding only a constant function. This is because in the case when the space is compact, there is only one vacuum and so the h points of the moduli space "know" about each other.

Now let $k=N-1$. Then the descent procedure will stop at $N=2,k=1$. In the case $N=2,k=1$ one has $W=c_{2,1}(detM)^{-1}$. More precisely, by dimension arguments $c_{2,1}=\tilde{c}_{2,1}\mu^5$, where $\tilde{c}_{2,1}$ is dimensionless. It is shown like for the pure gauge theory that instantons which contribute to $c_{2,1}$ have instanton number 1. Thus, by holomorphicity of W with respect to τ, we have

$$W=\tilde{c}_{2,1}\mu^5 e^{i\vartheta-8\pi^2/g^2}/det(M). \tag{15.11}$$

The computation of $\tilde{c}_{2,1}$ is tricky. The reason is that there are instantons of all sizes, which makes it more difficult to do perturbation theory around the instantons.

This was done by Affleck, Dine, and Seiberg. They showed $c_{2,1}$ is not zero. Thus $c_{N,N-1}$ is not zero.

Lecture II-16: BRST quantization of gauge theories

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In this lecture we will discuss quantization of gauge theory by using BRST cohomology. This approach is an improvement of the original Faddeev-Popov approach. An advantage of the BRST approach as opposed to the Faddeev-Popov method is that BRST makes explicit the independence of quantization of the choice of the gauge fixing procedure.

A similar approach can be used in gravity (see D'Hoker's lectures).

16.1. The general setup.

We start with a general setup, and then consider examples. In the general setup, we have a compact gauge group G with Lie algebra $\mathfrak{g}$, and the group $\hat{G}$, which is the group of gauge transformations of a principal G-bundle E over a spacetime M. Formally, we want to compute the path integral

$$\frac{1}{Vol(\hat{G})}\int DAD\phi e^{-L(A,\phi)}, \tag{16.1}$$

where $L(A,\phi)$ is a gauge invariant Lagrangian with a gauge field A and matter fields ϕ.

The difficulty with a perturbative treatment of this path integral is that its kinetic term F_A^2 for the gauge field is degenerate along orbits of $\hat{G}$. One way to deal with this difficulty is to replace the integrand in (16.1) by some expression that integrates to 1 on orbits of $\hat{G}$ – then (16.1) would equal to the integral of this expression (at least if everything were finite dimensional). For example, this expression could be the delta-function of some gauge, i.e. of some submanifold in the space of connections and matter fields which is a cross-section for the $\hat{G}$-action (this procedure is called gauge fixing). As we know (see Kazhdan's lectures on gauge theory and Faddeev's lectures), this introduces a determinant under the integral (the Faddeev-Popov determinant). The determinant is a nonlocal expression, so in order to work only with local expressions, one should replace this determinant with a Gaussian integral over the space of fields times two copies of the odd space of sections of the coadjoint bundle of E. Thus, one has to introduce additional fermionic fields $c, \overline{c}$ with values in $ad(E)$. These fermions are called ghosts, since they do not correspond to any physical particles and violate spin-statistics. After the introduction of ghosts the path integral can be treated as usual, e.g. by perturbation theory techniques.

16.2. The BRST differential.

Of course, any gauge fixing procedure by definition destroys gauge invariance. Therefore, in order to obtain a sensible quantization, we must make sure that in the final result the gauge symmetry is restored. In particular, we must explain what replaces the gauge symmetry in the ghost setting of the previous section.

It turns out that what replaces the gauge symmetry is a certain odd derivation of the algebra of local functionals, which we will now construct.

We will first consider the classical theory. Let us look what fields our theory has after introduction of ghosts. The basic fields are the connection A, the matter fields ϕ, and the ghosts $c, \overline{c}$, which are sections of the adjoint bundle to E. We will add an auxiliary scalar bosonic field h, whose significance will be seen below.

Let R be the algebra of local functionals. We want to define an odd derivation $\delta : R \to R$ such that $\delta^2 = 0$ (the BRST differential).

Recall that the algebra of local functionals R is the quotient $\tilde{R}/I$, where $\tilde{R}$ is the algebra of local expressions in the fields and I is the differential ideal generated by field equations.

We first define a derivation $\delta : \tilde{R} \to \tilde{R}$, and then make sure that the field equations are respected.

Define δ on generators by

$$\delta c = \frac{1}{2}[c,c], \delta\phi = \delta_c\phi, \delta A = -d_A c, \delta\bar{c} = h, \delta h = 0, \tag{16.2}$$

where $\delta_c\phi$ means the variation of ϕ along the infinitesimal gauge transformation c. It is easy to check that $\delta^2 = 0$.

Recall from Kazhdan's and D'Hoker's lectures that the Lagrangian with ghosts for our theory is

$$\tilde{L} = L(A,\phi) - \delta(\bar{c}(h/2 + \Lambda)) = L(A,\phi) - (h\Lambda + \frac{h^2}{2} - \bar{c}\delta\Lambda). \tag{16.3}$$

where $\Lambda = \Lambda(A,\phi)$ is a non-gauge invariant local expression (the gauge fixing function). This Lagrangian is of course equivalent to

$$\hat{L} = L(A,\phi) + (\frac{1}{2}\Lambda^2 + \bar{c}\delta\Lambda).$$

by eliminating h.

We want the Lagrangian to be invariant under the derivation δ. Since $L(A,\phi)$ is already invariant (because it is gauge invariant), it is enough to check that the expression $h\Lambda + \frac{h^2}{2} - \bar{c}\delta\Lambda$ is closed under δ, which follows from $\delta^2 = 0$.

Since δ preserves the Lagrangian, it preserves the set of its critical points and hence indeed defines a derivation of the algebra of local functionals.

As an example consider the case when E is the trivial bundle (and there is no matter fields). One of the possible gauge fixing conditions is the Feynman gauge condition $d^*A = 0$. So we take $\Lambda = d^*A$. Then one has $\delta\Lambda = -d^*d_A c$, so after elimination of h the Lagrangian for pure gauge theory is

$$L = \frac{1}{4e^2}\int F^2 + ((d^*A)^2 - \bar{c}d^*d_A c). \tag{16.4}$$

We see that this Lagrangian is nondegenerate, so one can do perturbation theory with it as usual. The derivation δ on R in this case is defined by

$$\delta c = \frac{1}{2}[c,c], \delta A = -d_A c, \delta\bar{c} = -d^*A. \tag{16.5}$$

16.3. The properties of the BRST derivation.

Thus, we have constructed a derivation δ. The main properties of δ are:

1. $\delta^2 = 0$.

2. δ is defined on $\tilde{R}$ apriori, without the use of the Lagrangian $L(A,\phi)$ and the gauge fixing term Λ. It preserves the Lagrangian with ghosts $\tilde{L}$ and therefore descends to R.

Now let us turn to quantum theory. In this case local functionals are replaced by local operators. It can be shown that there exists a renormalization procedure under which δ can be defined as above, and properties 1 and 2 hold. This is discussed below.

However, in order to use the BRST method for quantization, we will need another, purely quantum, property of δ. Namely, denote by $\mathcal{L}_{eff}$ the effective Lagrangian, i.e. the Lagrangian for which the classical theory is equivalent to the quantum theory for $\mathcal{L}$. This Lagrangian is of course nonlocal. The third property of δ that we need is

3. δ preserves the effective action $\mathcal{L}_{eff}$. That is, $\delta\mathcal{L}_{eff} = 0$.

This property may fail, and if it fails then one says that the theory is anomalous.

Remark. Although $\mathcal{L}_{eff}$ is nonlocal, it can be shown that $\delta\mathcal{L}_{eff}$ is always the integral of a local expression. It can be shown that the obstruction to making $\delta\mathcal{L}_{eff}$ zero by adding an auxiliary term to the Lagrangian (in a way that does not change the physics) is given by a 1-loop calculation. Thus, anomalies arise in the 1-loop order of perturbation theory, and don't have higher order corrections. We will see this at the end of the lecture.

16.4. Operators in gauge theory and BRST cohomology.

Assume properties 1-3 hold. Consider the path integral Z given by Lagrangian $\tilde{L}$:

$$Z = \int DADcD\bar{c}DhD\phi e^{-\tilde{L}} \tag{16.6}$$

(possibly with some gauge-invariant insertions). Properties 1-3 imply that Z is independent on the gauge fixing condition Λ. Indeed, for any local expression X we have

$$\int DADcD\bar{c}DhD\phi e^{-\tilde{L}}\delta X = \delta \int DADcD\bar{c}DhD\phi e^{-\tilde{L}}X = 0, \tag{16.7}$$

which implies the independence of Z on Λ.

Remark. In (16.7), we used that δ preserves the measure of integration $DADcD\bar{c}DhD\phi$. It is easy to see that this is equivalent to Property 3 (absence of anomalies).

The statement that Z is independent of Λ holds for operators (insertions) which are annihilated by δ; for example, for any gauge-invariant insertions into Z, depending only on A, ϕ. On the other hand, if $O = \delta O'$ then the integral of O is zero by (16.7). Thus, the space of "physical" quantum operators in our theory is the cohomology of δ. This cohomology is called the BRST cohomology.

The BRST cohomology comes with a natural $\mathbb{Z}$-grading. Namely, we have a grading in the space of local operators, in which gauge and matter fields have degree 0, c has degree 1 and $\bar{c}$ degree -1. This degree is called the ghost number. It is easy to see that δ raises the degree by 1. This allows to introduce a grading in cohomology: we denote by H^q_δ the cohomology in degree q.

The properties of this cohomology, which usually hold in this situation are:

1. H^q_δ vanishes for $q < 0$.

2. H^0_δ is the space of gauge invariant local operators depending only on A and ϕ.

This shows that δ plays the role of the gauge symmetry which was broken when ghosts were introduced. Thus, we have established a setting for gauge theory which works well in perturbation theory and in which the gauge symmetry does not die but rather appears in the form of δ.

16.5. Renormalization and BRST differential.

Now let us discuss the renormalizability and renormalization group equation in the BRST approach. We will restrict ourselves to 4 dimensions and pure nonabelian gauge theory.

Recall that in order for a theory to be renormalizable, all interactions have to have nonnegative dimension. To find out whether it is so for the Lagrangian with ghosts, we will compute the dimensions of fields (assuming that δ preserves the scaling dimensions). It is easy to see that the dimensions are as follows: $[c] = 0, [\bar{c}] = 2, [A] = 1$. This shows that all interactions in the Lagrangian with ghosts are renormalizable.

Remark. In this theory, dimensions of c and $\bar{c}$ are not uniquely determined; the only thing that is determined canonically is $[\bar{c}c]$, which equals 2. This does not lead to a contradiction, since all operators of nonzero ghost number have zero expectation value, and so their scaling dimension has no intrinsic meaning. This is why we needed to make an additional assumption that δ preserves dimensions to fix precise values of the dimensions. If we had assumed that δ raises dimension by k we would get a different answer, which would be equally good for our purposes.

Now let us look for critical couplings which will be renormalized. In the setting without ghosts, the usual thing to do is to write down renormalizable (non-gauge invariant) operators of dimension 4, which correspond to critical couplings: $[A,A]^2, [A,A]dA, (dA)^2$, and then argue that there is only one gauge invariant combination of these operators, so that the only coupling which is to be renormalized is the charge e. However, in the setting with ghosts, we also have to include operators of degree 4 involving ghosts: $\bar{c}cA^2, \bar{c}d^*dc, ...$ The gauge invariance condition is now replaced by the condition that δ is a symmetry, so we need to renormalize only delta-invariant interactions. This cuts down the number of operators to be renormalized, but still leaves us with two renormalizable couplings: the charge e and the gauge fixing parameter a, corresponding to the scaling of the gauge fixing term $\frac{\Lambda^2}{2} + \bar{c}\delta\Lambda$. Thus the renormalization group vector field looks like

$$W = \mu\frac{\partial}{\partial\mu} + \beta(e)\frac{\partial}{\partial e} + \tilde{\beta}(e,a)\frac{\partial}{\partial a}. \tag{16.7}$$

Here β is the beta-function of the theory, and $\tilde{\beta}$ is the ghost beta-function. The beta-function of the theory depends only on e and is physically meaningful; for example, the negativity of its leading term insures asymptotic freedom. However, the ghost beta-function $\tilde{\beta}$ has no physical meaning: it only matters for renormalization of operators and correlators containing c and $\bar{c}$, which don't make sense physically.

16.6. The Hamiltonian approach.

So far we have considered BRST method from the Lagrangian point of view. Now let us consider the connection of the BRST method with the Hamiltonian formalism.

Since ghosts violate spin-statistics (being scalar fermions), the "Hilbert space" of the theory with ghosts cannot be an actual Hilbert space. Namely, it is possible to construct a certain space $\tilde{\mathcal{H}}$ with a Hermitian form, which is analogous to the Hilbert space in actual physical theories, but the form will not be positive definite. However, on this space we have local quantum operators, obtained by quantization of classical operators in the usual way. In particular, we have the global charges – the Hamiltonian H as well as the BRST charge Q, obtained from ghosts as explained in D'Hokers lecture. We also have a grading of $\tilde{\mathcal{H}}$ by ghost number, obtained naturally from the quantization procedure.

The operator Q has the property $[Q,O] = \delta O$ for any operator O (not necessarily δ-closed) in the theory with ghosts. Also, $Q\Omega = 0$, where Ω is the vacuum, and the ghost number of Q is 1.

The operator Q has properties analogous to those of δ:

1. $Q^2 = 0$. This can be confirmed by a direct computation.

2. Q is defined apriori, without the use of L and Λ (by an explicit formula as in D'Hoker's lectures). In particular, if $\tilde{\mathcal{H}}$ is an irreducible representation of the operator algebra, then Q is completely determined by δ and the properties $[Q,O] = \delta O$, $Q\Omega = 0$.

3. If there is no anomalies, the element Q commutes with the Hamiltonian and with all gauge invariant local operators which involve no ghosts.

Remark. As in the Lagrangian setting, here the explicit expression for Q is independent on Λ only if one uses the operator h corresponding to the auxiliary field in the Lagrangian. This operator can be expressed via other operators in the theory, in a way which depends on Λ: $h = -\Lambda$. If one makes this substitution, the obtained formula for Q will involve Λ. Thus, property 2 should be understood as follows: there exists a formula for Q in terms of the fields (including h!) which is independent on $\mathcal{L}$ and Λ but depends only on field configuration.

Let H_Q^q be the cohomology of Q on $\tilde{\mathcal{H}}$, graded by ghost number.

The properties of Q which usually hold are

1. H_Q^q vanishes if $q < 0$.

2. The Hermitian form is degenerate on the kernel of Q in $\tilde{\mathcal{H}}^0$ (operators of ghost number zero); the kernel of this form is the image of Q. The induced form on H^0_Q is positive definite.

The space H^0_Q plays the role of the physical Hilbert space of the theory, so we denote it by H_{phys}.

In the space H_{phys}, we have an action of the Hamiltonian H and "physical" local operators $O \in H^0_\delta$. These operators no longer involve ghosts and correspond to actual observables of the theory.

Let us now compare the BRST and the "traditional" approaches to quantization of gauge theory. For simplicity, we consider pure gauge theory. Traditionally, a scheme of quantization would be as follows. Suppose that the space part of the spacetime is compact. In this case we have seen that classically the space of solutions to the equations of motion can be realized as $T^*\mathcal{A}$, where $\mathcal{A}$ is the space of connections on a space cycle modulo gauge transformations. Therefore we would define the Hilbert space as $L^2(\mathcal{A})$ (with respect to some measure). We call this Hilbert space the traditional Hilbert space.

We claim that these approaches give the same result, i.e. H_{phys} is isomorphic to H_{trad} as a representation of the operator algebra.

First of all, H_{phys} does not depend on the gauge fixing term Λ, and the Hamiltonian and the quantum operators in H_{phys} don't depend on it either. This follows from the fact that when Λ is varied, operators in the pseudo-Hilbert space $\tilde{\mathcal{H}}$ are changed by adding a δ-exact expression, so their action on δ-closed vectors is changed by a δ-exact expression.

To identify H_{phys} with H_{trad} we can use a convenient gauge fixing term Λ. It is enough to do it for one such term, but we will do it for two – just for fun.

Set $\Lambda = ud^*A + vA_0$, where A_0 is a time component of the connection (this uses the splitting of spacetime into space and time). Then we get a sensible theory unless both $u = 0$ and $v = 0$. Even $u = 0, v \neq 0$ gives a nice theory – this gauge fixing term is called "temporal gauge".

We first consider the case $u = 0$. Then the Lagrangian is

$$\hat{L} = \int (\frac{1}{4e^2}F^2 + \frac{1}{2}v^2A_0^2 - v\bar{c}\frac{Dc}{Dt}),$$

where D denotes covariant derivative. Replacing $\bar{c}$ with $-v\bar{c}$, and tending v to infinity (using the fact that nothing depends on v), we see that the path integral is localized to the hyperplane $A_0 = 0$, and in the limit we get a Lagrangian

$$\hat{L} = \int (\frac{1}{4e^2}F^2 + \bar{c}\frac{dc}{dt}), \tag{16.8}$$

Since A_0 is now zero, we get usual quantum mechanics where dynamical variables are a spacial connection, its time derivative, and the ghosts. Thus, $\tilde{\mathcal{H}}$ has the form $O(\tilde{\mathcal{A}}) \otimes \Lambda\tilde{\mathfrak{g}}^*$, where $\tilde{\mathcal{A}}$ is the space of connections on the space cycle, $O(\tilde{\mathcal{A}})$ is the space of functions on $\tilde{\mathcal{A}}$, $\tilde{\mathfrak{g}}$ is the Lie algebra of the group of gauge transformations on the space cycle, and $\Lambda\tilde{\mathfrak{g}}^*$ is the space of functions of $c \in \Pi\tilde{\mathfrak{g}}$. Moreover, the Hamiltonian for the ghosts vanishes since there is no nontrivial evolution on the space of classical solutions (the Euler-Lagrange equations for ghosts are simply $\frac{dc}{dt} = \frac{d\bar{c}}{dt} = 0$). Thus the Hamiltonian in our theory is the usual gauge theory Hamiltonian

$$H = \frac{1}{2e^2}\int d^3x F_A^2 + \frac{e^2}{2}\nabla_A^2, A \in \tilde{\mathcal{A}} \tag{16.9}$$

acting on the first component of the tensor product.

It is easy to see that the space $\tilde{\mathcal{H}}$ with the grading by ghost number is nothing but the space of the standard complex of the Lie algebra $\tilde{\mathfrak{g}}$ with coefficients in the module $O(\tilde{\mathcal{A}})$. Moreover, from the explicit formula for Q one gets that in this case Q is exactly the differential in the standard complex.

Thus, the physical Hilbert space H_{phys} which is by definition the 0-th cohomology of Q, is the 0-th the cohomology of $\tilde{\mathfrak{g}}$ with coefficients in $O(\tilde{\mathcal{A}})$. This is just the space of $\tilde{\mathfrak{g}}$-invariants in $O(\tilde{\mathcal{A}})$, i.e. the space of functions on $\tilde{\mathcal{A}/\mathcal{G}}$, which is by definition the traditional Hilbert space H_{trad}.

This shows that BRST cohomology is an infinite dimensional generalization of Lie algebra cohomology.

Now consider another gauge obtained by setting $v = 0, u = 1$: $\Lambda = d^*A$. Let us try to see the isomorphism between H_{phys} and H_{trad} using this gauge. We have,

$$\hat{L} = \frac{1}{4e^2}\int F^2 + (\frac{1}{2}(d^*A)^2 - \bar{c}D^*d_Ac). \tag{16.10}$$

In this case the equations of motion for ghosts are nontrivial and of second order, so the Hilbert space $\tilde{\mathcal{H}}$ consists of functions of $A_s, c, \bar{c}, A_0$, where A_s is a connection on the space cycle. In this case, one finds

$$Q = Q_{\text{Lie algebra cohomology}} + Q', \tag{16.11}$$

where $Q' = \int \pi_{\bar{c}}\pi_{A_0}$, and $\pi_{\bar{c}}$, π_{A_0} are the momentum operators for $\bar{c}, A_0$.

It is easy to check directly that the two summands in (16.11) anticommute, and that $(Q')^2 = 0$. It can also be checked that Q' is acyclic except in 0-th degree, where it has a 1-dimensional cohomology. Thus, by Kunneth formula, we again get $H_{phys} = H_{trad}$.

16.7. Anomalies.

Now let us recall conditions 1,2,3 which were necessary for the BRST construction, and analyze when they are satisfied. These conditions are

1. Lagrangian: $\delta^2 = 0$.

Hamiltonian: $Q^2 = 0$.

2. Lagrangian: δ is independent on Λ and $\tilde{\mathcal{L}}$.

Hamiltonian: Q is independent of $\Lambda, \tilde{\mathcal{L}}$.

3. Lagrangian: $\delta\mathcal{L}_{eff} = 0$.

Hamiltonian: $[Q, H] = 0$.

As we mentioned, properties 1 and 2 can always be attained.

However, as we also mentioned, Property 3 may fail if anomalies are present. So let us consider anomalies more closely.

Consider a 4-dimensional gauge theory with a chiral spinor ψ with values in a complex representation ρ of the gauge group G and antichiral spinor $\bar{\psi}$ with values in the dual representation $\bar{\rho}$. The basic gauge-invariant Lagrangian for such fields is

$$\mathcal{L} = \frac{1}{4e^2}\int F^2 + \int \bar{\psi}D_A\psi. \tag{16.12}$$

In quantum theory we are interested in the path integral $\int e^{-\mathcal{L}}$. Integrating out ψ in this path integral, we get

$$\int \det(D_A)e^{-L(A)}DA, \tag{16.13}$$

where $D_A : \Gamma(S_+) \to \Gamma(S_-)$ is the covariant Dirac operator and $L(A)$ the Lagrangian of the pure gauge theory.

Integral (16.13) may not have a gauge invariant regularization. What is worse, it may not even have a non-gauge-invariant regularization for which the gauge invariance is restored as the cutoff goes to infinity. In this case the gauge theory we are considering does not make sense quantum mechanically,

even in perturbation theory, because gauge symmetry cannot be restored. This phenomenon is called an anomaly.

A geometric reason for an anomaly is that although the operator D_A is gauge invariant, its determinant $\det(D_A)$, in general, fails to be gauge invariant. In other words, this determinant is not a function on the space of gauge classes of connections but rather a section of some line bundle over this space, called the determinant line bundle; this bundle comes with a canonical connection. If this canonical connection does not trivialize the bundle, this "function" cannot be sensibly integrated.

It is useful to distinguish two types of anomalies.

1. Local anomaly. The canonical connection has a nonzero curvature. In this case for suitable spacetime manifolds this curvature may represent a nontrivial second cohomology class, so that the determinant bundle is not trivial topologically.

2. Global anomaly. The canonical connection is flat but has a nontrivial monodromy (and possibly the bundle is not trivial).

Thus, both local and global anomalies can produce topological anomalies, but only the first one can be seen in perturbation theory (by computing of the curvature).

Here we will consider only local anomalies.

Remark 1. To analyze when we can expect local anomalies, one may consider the situation from the topological point of view. We assume that our spacetime M is compact and orientable (e.g. S^d), with a specified point ∞, and will consider bundles, connections, and gauge transformations which are trivial at infinity. In this case the space of gauge classes of connections can be regarded as a classifying space $B\hat{G}$ for the group of gauge transformations $\hat{G}$. Nontrivial line bundles on $B\hat{G}$ are classified by $H^2(B\hat{G})$.

Now, if M is compact and orientable, we have the transgression map $\tau : H_2(B\hat{G}) \to H_{d+2}(BG)$ defined as follows: given a two-dimensional homology class, pick a surface S in $B\hat{G}$ which represents it, and take the corresponding principal $\hat{G}$-bundle on S. Its transition functions can be considered as transition functions of a G-bundle on the 6-dimensional manifold $S \times M$, which defines an element $\tau([S])$ of $H_{d+2}(BG)$. Consider the dual map $\tau^* : H^{d+2}(BG) \to H^2(B\hat{G})$. It can be shown that the Chern class of our line bundle is $\tau^*(C)$, where C is a fixed $d+2$-dimensional characteristic class which does not depend on M, and is computed locally from the curvature. This class is exactly the local anomaly.

Thus for $d = 4$ local anomalies live in $H^6(BG)$, or $(S^3\mathfrak{g})^{\mathfrak{g}}$, where $\mathfrak{g}$ is the Lie algebra of G.

For example, in the standard model the gauge group is $SU(3)\times SU(2)\times U(1)$, and thus the space of anomalies $(S^3\mathfrak{g})^{\mathfrak{g}}$ is 4-dimensional: it equals to the sum of four subspaces $S^3_{inv}(su(3)), S^3_{inv}(u(1)), S^2_{inv}(su(2))\otimes u(1), S^2_{inv}(su(3))\otimes u(1)$, which are 1-dimensional (here "inv" denotes that we are taking invariant symmetric polynomials).

This discussion illustrates why anomalies don't arise in the case when all fermions are in a real representation of the gauge group. Indeed, in this case, the determinant bundle is real, and thus its Chern class must be zero.

Remark 2. Although the local anomaly can be considered from the above topological point of view, one should remember that it has a purely local nature, and has nothing to do with the macrostructure of the spacetime. If there is a local anomaly, the quantum theory will not make sense on any spacetime, even on $\mathbb{R}^d$. The problem is that even if the determinant bundle is topologically trivial, it will not have a flat connection defined in a local way: otherwise this flat connection would have been good for any simply connected spacetime, and no topological anomaly would arise. Thus, path integral (16.13) is not sensible even on $\mathbb{R}^d$.

Remark 3. In the standard model, the gauge group is $SU(3) \times SU(2) \times U(1)$. In particular, there is a possibility for local anomalies, and they do appear in reality. However, one can check that the anomalies coming from the different matter fields of the standard model miraculously cancel, in all four components of the space of anomalies. An explanation of this is that the representation of the

gauge group in the standard model extends (after adding some insignificant summands) to a spinor representation of $Spin_{10}$, for which $H^6(BG)$ vanishes.

Let us show how to analyse anomaly in perturbation theory. Our goal is to explain why, after possibly enlarging the space of fields, properties 1. and 2. of section 16.3 can always be assumed to hold (that is, $\delta^2 = 0$ and δ is defined independently of the choice of a particular Lagrangian) but one cannot assume that the effective Lagrangian is delta-invariant.

First, let us just try to make sense of integral (16.3) perturbatively. When we write down Feynman diagrams, we will find divergences in the 1-loop order which we cannot remove in a gauge invariant fashion. To fix the 1-loop order, we will regularize the path integral by adding another, very heavy matter field χ such that its determinant bundle is inverse to that for ψ. In favorable cases, our original theory should be recovered from this theory in the limit when the mass m of χ goes to infinity. In other cases, the procedure will exhibit why there is an anomaly.

To satisfy this condition, the matter field χ can be taken to be a bosonic field (χ_+, χ_-) with values in $(S_+ \oplus S_-) \otimes \rho$. In this case the complex conjugate field $\overline{\chi}$ is with values in $(S_+ \oplus S_-) \otimes \overline{\rho}$, where S_+, S_- are the spin representations of the Poincare (recall that both S_+ and S_- are self-dual and self-complex-conjugate in Euclidean signature). It is of course needless to say that these fields violate spin-statistics and therefore, like ghosts, don't make physical sense.

The natural Lagrangian term for the fields $\chi_\pm$ would be

$$L'(A, \chi_\pm) = \int d^4x((\overline{\chi}_+, D_A\chi_-) + (\overline{\chi}_-, D_A\chi_+) + m(\overline{\chi}_+, \chi_+) + m(\overline{\chi}_-, \chi_-)) \tag{16.14}$$

(Here the Dirac operator is skewselfadjoint).

Remark. The χ's are called Pauli-Villars regulator fields.

If we add expression (16.14) to the Lagrangian, we will get the squared absloute value of the determinant rather than the determinant itself, and will not fix the anomaly. Thus, we modify (16.14) in a way that breaks the gauge invariance: we let A_0 be a fixed connection and set

$$L''(A, \chi_\pm) = \int d^4x((\overline{\chi}_+, D_A\chi_-) + (\overline{\chi}_-, D_{A_0}\chi_+) + m(\overline{\chi}_+, \chi_+) + m(\overline{\chi}_-, \chi_-)) \tag{16.15}$$

Now consider the theory with the Lagrangian $\mathcal{L} + L''$. Integrating out the χ fields, we will get a factor $\det(D_A D_{A_0} - m^2)^{-1}$. For $m = 0$ this factor is gauge invariant up to a multiplicative constant, and cancels the determinant in the numerator, which is caused by anomaly. This shows that in this theory, we don't have a topological anomaly for any finite m (i.e. the appropriate determinant bundle is trivial). However, for $m > 0$, the gauge invariance fails. So we have to study the limit $m \to \infty$ (which is supposed to recover our original theory) and see whether the gauge symmetry reappears.

Differentiating the determinant ratio $\det(D_A D_{A_0} - m^2)/\det(D_A)$ in the direction of a gauge trasformation $t \in \hat{\mathfrak{g}}$, we obtain (using the path integral interpretation) that it is equal to $m\langle\int[(\overline{\chi}_+, t\chi_+) + (\overline{\chi}_-, t\chi_-)]\rangle$, where $\langle X\rangle$ denotes the expectation value of X. This expectation value has a decomposition in powers of $1/m$.

To see whether the failure of gauge invariance persists for $m \to \infty$, let us consider the two-point function of the curvature operator F. It is easy to see that the leading contribution (in $1/m$) to the derivative of this function in the direction of t is from the 1-loop diagram with a χ loop having the t operator inside and two outgoing F-edges. This contribution is of the 0-th order in $1/m$, and has the form $\sum d_{abc} t^a F^b F^c$, where d_{abc} is some tensor. So if $d_{abc} \neq 0$, the gauge-noninvariance remains in the limit.

Remark. In case the original fermions were in a real representation, d_{abc} is zero and the regularization in (16.14) is completely satisfactory. The problem arises when the original representation is complex. Then a regularization as in (16.14) doesn't work unless one gives up gauge invariance.

Remark. When $d_{abc} \neq 0$, one can choose a regularization scheme to remove all loop contributions to the non-gauge invariance except 1 loop.

Now let us consider anomalies from the BRST veiwpoint. If a local anomaly is present, we will have $U = \delta\mathcal{L}_{eff}(A) \neq 0$ (here $\mathcal{L}_{eff}(A)$ is the effective Lagrangian for A, with the ghosts integrated out). However, since the anomaly is local U must be the integral of a local expression of A and c which is linear in c. It is also clear that $\delta U = 0$. Furthermore, one can show that U involves only A and its first derivatives (and no matter fields).

On the other hand, if U is δN where N is the integral of some local expression of A then we can arrange $\delta\mathcal{L}_{eff} = 0$ by redefining the Lagrangian as $\mathcal{L} \to \mathcal{L}+N$. Thus, anomalies lie in the cohomology of δ on local functionals of degree ≤ 1 of A and c (linear in c) modulo complete derivatives.

Now let us show that such cocycles are in fact related to invariant symmetric tensors on the Lie algebra $\mathfrak{g}$ (or equivalently, the cohomology of $\mathfrak{g}$).

Let C be a G-invariant element in $S^{n+1}\mathfrak{g}$. To this element there corresponds a $2n+2$-dimensional characteristic class of G-bundles, namely $C(F^{n+1})$, where F is the curvature. The Chern-Simons form $CS_C(A)$ corresponding to C is the local $2n+1$ form such that $\frac{\tilde{\delta}C(F^{n+1})}{\tilde{\delta}A} = CS_C(A) \wedge \tilde{\delta}A$ modulo differentials of local forms (here $\tilde{\delta}$ denotes the variation to distinguish from the BRST differential δ).

The main property of the Chern-Simons form is the following. Although this form is not gauge invariant, its Lie derivative along an infinitesimal gauge transformation is a differential of a local form.

Now let M^{2n} be our spacetime. Let A be a connection on M^{2n}. We want to define a functional of the form $U(A) = \int W(A)$, where $W(A)$ is a $2n$-form on M^{2n} which is local in A but not gauge invariant, and such that $\delta W = 0$.

Let X^{2n+1} is a smooth manifold with boundary equal to M^{2n}. Choose an extension of the connection A to X^{2n+1} in any way (for simplicity we assume that there is no topological obstruction to the choices of X and the extension of A; this assumption is in fact inessential). Now set $V(A) = \int_X CS_C(A)$. This functional depends on the extension of A to X. However, by the main property of CS, the functional $\delta V(A) = W(A, c)$ (where δ is the BRST differential) does not depend on the extension and therefore is a local functional in A and c linear in c. One can show that it represents a nontrivial cohomology class in the local δ-cohomology. Thus, we get an injective map $S^{n+1}(\mathfrak{g})^{\mathfrak{g}} \to H_{\delta}^{1,local}$. For 4-dimensional theories $n = 2$ and the cocycles come from $(S^3\mathfrak{g})^{\mathfrak{g}} = H^6(BG)$ as we expected.

Recall (Lecture 2) that in a Hamiltonian approach to classical gauge theory, the phase space is obtained as a symplectic quotient of the cotangent bundle. The quantum analogue of taking symplectic quotient is taking invariants in the Hilbert space. If degeneracies are present,

In the future it will be convenient to eliminate h and use the Lagrangian $\hat{L}$. After elimination of h, we will have $\delta\overline{c} = h = -\Lambda$. However, in presence of h the operator δ is independent on the theory and on Λ and depends only on the field configuration, which is an important property of δ; this is the reason that h is introduced.

Lecture II-17: $N = 2$ Super-symmetric Yang-Mills theories in dimension four: Part 1

Edward Witten[1]

1 Introduction

Today we shall be studying the dynamics of $N = 2$ super-symmetric Yang-Mills theories in four-dimensions. The underlying manifold on which we shall study these theories will be Minkowski four-space $M^4 = \mathbf{R}^{3,1}$. We are mainly interested in the case when the gauge group G is $SU(2)$. Since low energy limits of $SU(2)$ theories can generate $U(1)$-theories, we shall begin by studying $U(1)$-Yang-Mills theory. One should note that these $U(1)$-theories are purely unrenormalizable (all terms are unrenormalizable) in dimension four. Thus, they don't make sense as fundamental theories, but they are free in the infra-red. But, by the same token, we can study their low energy limits by doing semi-classical calculations. Of course, when we get to the $SU(2)$-theory we will be dealing with an asymptotically free, renormalizable theory which has strong quantum effects. This is then a reasonable fundamental theory, but its low energy limit is hard to understand.

We shall eventually use the analysis of low energy limits of these gauge theories on Minkowski space as a tool in the analysis of low energy limits of these theories on compact Riemannian four-manifolds. Understanding these limits will give another mechanism for computing certain correlation functions associated to cycles in the four-manifold. These correlation functions can be computed in the low energy limit because one shows that they are invariant under deformation of the parameters. Computing in the high energy limit, one identifies these correlation functions with differentiable invariants, namely the Donaldson invariants of the four-manifold. In this way we obtain totally different expressions for the Donaldson invariants of a four-manifold. In most cases these invariants can be directly computed from a $U(1)$-monopole theory. This then produces the link between the Donaldson polynomial invariants and the Seiberg-Witten $U(1)$-monopole invariants. All this is the subject of the next several lectures. Today we study low energy limits of $U(1)$-gauge theories.

2 Low Energy $U(1)$ $N = 2$ super Yang-Mills Theories

2.1 The basic fields

Consider pure $N = 2$ super Yang-Mills theory in four-dimensions with gauge group $U(1)$. This means that the fields in the Lagrangian will form a single $N = 2$ vector super-multiplet. That is to say, we will have the gauge field, and its $N = 2$ super-symmetric partners. All the fields will be massless. Let us recall from the super-homework what the $N = 2$ massless vector super-multiplet looks like in dimension 4. Massless particles are associated to representations of the 'little' subgroup $Spin(2)$ in the Lorentz group $Spin(3,1)$ fixing a nontrivial vector $v \in R^{3,1}$ of norm zero. The 'helicity' of such a particle (i.e., its spin around its direction of motion) is the representation of its stabilizer. These are indexed so that the fundamental representation of $Spin(2)$ is labeled $1/2$. The irreducible representations of the Poincaré group that comprise the $N = 2$ vector super-multiplet in dimension four are arrayed as pictured below:

helicity	-1	$-1/2$	0	$\oplus$	0	$1/2$	1
multiplicity of representation	1	2	1		1	2	1.

[1]Notes by John Morgan

The action of CPT conjugation on these representations is to interchange the right and left halves of this table. This multiplet has a $U(2)_R$ symmetry, acting so as to preserve helicity. Thus, the $SU(2)_R \subset U(2)_R$ acts trivially except on those particles of helicities $\pm(1/2)$. On each of these it acts via the fundamental two-dimensional representation. The $U(1)_R$ acts trivially on the states of helicity ± 1 and acts by (two copies of) the standard representation on the states of helicity $\pm 1/2$ and by the ± 2 power of the fundamental representation on the states of helicity zero. The CPT involution normalizes the $U(1)_R$ and acts on it by conjugation. Thus, the representation of $U(1)_R$ on the other half of the multiplet is given by the negative of the above.

The fields of an $N = 2$ vector super-multiplet on $N = 1$ super-space in dimension four decompose into an $N = 1$ vector multiplet and an $N = 1$ chiral multiplet. The $N = 1$ vector super-multiplet is the gauge $N = 1$ super-multiplet $\mathcal{A}$. The field $\mathcal{A}$ is the super-connection and (after partial gauge fixing to Wess-Zumino gauge) consists of a triple

$$\mathcal{A} = (A, \lambda, D)$$

where A is the usual $U(1)$-gauge field, λ is a fermion field, and D is an auxiliary field. The curvature of $\mathcal{A}$ is a two-form on $N = 1$ super-space. The entire curvature form is determined by certain components W_α which are $(\frac{1}{2}, 0)$ spinor chiral super-fields. (Representations of $Spin(3,1) = SL_2(\mathbf{C})$ are indexed by an ordered pair (a, b) of half-integers. The pair $(1/2, 0)$ indexes the defining two-dimensional complex representation. It is chiral because the second half-integer is zero.) The only way that the super-connection $\mathcal{A}$ enters the action is through the field W_α. The other fields form an $N = 1$ super-multiplet called Φ. This is a (scalar) chiral super-multiplet. It consists of

$$\Phi = (\phi, \psi, F)$$

where ϕ is a complex-valued scalar field, ψ is a fermion, and F is an auxiliary field. The auxiliary fields D and F enter the action quadratically and without derivatives so that they can be integrated out using the equations of motion. Thus, they play no role in the analysis.

2.2 The Lagrangian

Now we are ready to write the Lagrangian for the pure $N = 2$ $U(1)$-gauge theory in four-dimensions. Though we require impose $N = 2$ super-symmetry, we choose to write the Lagrangian first as a Lagrangian for fields on $N = 1$ super-space and then study the consequences of the extra super-symmetry. The Lagrangian is:

$$\mathcal{L} = \left(\frac{i}{4\pi} \int d^4x d^2\theta \tau(\Phi) W_\alpha W^\alpha + \text{c.c.} \right) + \frac{1}{4\pi} \int_{R^{3,1|4}} d^4x d^4\theta K(\Phi, \overline{\Phi}). \tag{2.1}$$

As we remarked above, W is a piece of the curvature of the $N = 1$ super connection $\mathcal{A}$. It is a chiral super-field. The function $\tau(\Phi)$ is a holomorphic function on the complex plane (or more generally the complex curve) where Φ takes its values. Since Φ is a chiral super-field it follows that $\tau(\Phi)$ is also a chiral super-field. We write

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}.$$

Lastly, $K(\Phi, \overline{\Phi})$ is a Kähler potential on Φ-space.

We have encountered this Lagrangian before, with one essential difference. That difference is that here we are not assuming τ is constant. Rather we allow it to be a holomorphic function of Φ.

Briefly, the reason that we consider this generalization is that this theory will arise as a low energy limit of some $SU(2)$-theory. That theory will have a complex plane of classical vacua (i.e., the space of minima of the potential energy will be $\mathbf{C}$). We find terms in the low energy limit which involve functions on this space of classical vacua (cf, sigma-models arising as low energy limits). It is in this way that the non-constant holomorphic function τ arises. The main application we have in mind for this study is to the low energy limits of $N = 2$ super-symmetric pure $SU(2)$ gauge theory. Of course, in the low energy limit Φ will be slowly varying, i.e., almost constant. To the extent that Φ is almost constant the same will be true of τ, and we find ourselves expanding around a low energy effective theory with constant τ_{eff}.

Another difference of this Lagrangian from the general $N = 1$ super-symmetric Lagrangian is that there is no super-potential term. The reason is that one can have such a term in an $N = 1$ super-symmetric theory on $N = 1$ super-space in four-dimensions, but the extra conditions imposed by requiring the theory to have $N = 2$ super-symmetry imply that this term must be zero. Another related, but more delicate, consequence of the same idea is that $N = 2$ super-symmetry imposes a relation between τ and K. To see this, let us write out the Lagrangian in components. We get

$$\begin{aligned}\mathcal{L} &= \frac{1}{4\pi}\int_{R^{3,1}} d^4x \Big(i\left[\overline{\tau}(F^+ \wedge F^+) - \tau(F^- \wedge F^-)\right] + K_{\phi\bar{\phi}}\partial\phi\overline{\partial\phi} \\ &\qquad + \mathrm{Im}\tau\bar{\lambda}i\partial\!\!\!/\lambda + K_{\phi\bar{\phi}}\overline{\psi}i\partial\!\!\!/\psi\Big)\end{aligned} \tag{2.2}$$

One thing which is clear from this formula, which we have seen before, is that it is not the Kähler potential $K(\Phi, \overline{\Phi})$ *per se* that enters into the Lagrangian; only the Kähler metric $K_{\phi\bar{\phi}}\partial\phi\overline{\partial\phi}$ of the potential enters.

To see the relation of τ and K we use the $SU(2)_R$-symmetry which is a consequence of $N = 2$ super-symmetry. Notice that if $K_{\phi\bar{\phi}} = \mathrm{Im}\tau$, then the pair (λ, ψ) form a multiplet for the $SU(2)_R$ which is isomorphic to the standard representation. Conversely if there is an $SU(2)_R$-symmetry, then with an appropriate holomorphic redefinition of the coordinates the (λ, ψ) form a standard multiplet for the $SU(2)_R$ action. Thus, in these coordinates the coupling terms for λ, λ and ψ, ψ must have the same strength, and hence with this choice of coordinates we have $\mathrm{Im}\tau = K_{\phi\bar{\phi}}$. One immediate consequence of this is that $\mathrm{Im}\tau > 0$. This seems like a problematic condition. Locally it is not hard to ensure that it holds, but for a global holomorphic function on all of $\mathbf{C}$ it cannot be achieved. There are two possibilities to contemplate. One is that as τ approaches the real axis, for some reason the physics breaks down. One would then be faced with studying what happens at these 'bad' limits. Another possibility is that τ is multi-valued. After all, we already know that the transformation $\tau \to \tau + 1$ (i.e., increasing θ by 2π) does not affect the theory. So at least to that extent one must consider τ as multi-valued. Of course, this does not solve the problem we are addressing since any non-constant, entire holomorphic function τ to $\mathbf{C}/\{\tau \cong \tau + 1\}$ still cannot have $\mathrm{Im}\tau > 0$. To arrange this we need a larger group of indeterminacy for the values of τ. For example, recall that it is easy to construct global holomorphic functions into the upper half-plane if we permit them to be multi-valued, say under the standard action of $SL_2(\mathbf{Z})$. As we shall see, this is exactly what happens in this case.

2.3 Indeterminacy of τ

In order to make the notation consistent with that of [Seiberg-Witten ???] we set $\phi = a$. The term in the Lagrangian derived from the Kähler potential becomes

$$\frac{i}{4\pi}\int_{R^{3,1}} d^4x(\tau - \overline{\tau})\frac{\partial a}{\partial x^i}\frac{\partial\overline{a}}{\partial\overline{x}^i},$$

which means that the induced Kähler metric on the Φ-plane, now called U, is

$$ds^2 = \frac{i}{4\pi}(\tau da d\overline{a} - da\overline{\tau}d\overline{a}),$$

where a is a local holomorphic coordinate on the U. It is convenient to introduce another holomorphic function a_D (called *the dual of* a) locally on an open subset V of U with the property that $da_D = \tau da$. Given this, we can re-write the metric on V as

$$ds^2 = \frac{i}{4\pi}(da_D d\overline{a} - da d\overline{a}_D).$$

We denote by Ω the Kähler form of this metric:

$$\Omega = \frac{-1}{8\pi}\left(da_D \wedge d\overline{a} - da \wedge d\overline{a}_D\right).$$

We can view this more symmetrically in a and a_D by forming the complex two-space with coordinates (a, a_D). Then there is an embedding of the complex curve $f : V \to \mathbf{C}^2$ into this complex two-space so that the image is a solution curve for the differential equation $da_D = \tau da$. This embedding will be the graph of either of the variables as a function of the other. Let $\omega = \frac{-1}{8\pi}\left(da \wedge d\overline{a}_D - da_D \wedge \overline{a}\right)$. It is a closed, non-positive $(1,1)$-form on complex two-space. Of course, f must be such that $f^*\omega = \Omega > 0$.

The question is: How much freedom do we have in the representation of the theory in terms of the parameters a, a_D (and implicitly τ)? Said another way, In what ways can we find other Lagrangian's of this form with different a and a_D (and hence different τ) which represent the same theory? Just considering the Kähler potential term we see an obvious geometric change of coordinates that brings the Lagrangian back into the same form with different values of the parameters. Namely, we can act by an extended version $ISL_2(\mathbf{R})$ of $SL_2(\mathbf{R})$. We define

$$0 \to \mathbf{C}^2 \to ISL_2(\mathbf{R}) \to SL_2(\mathbf{R}) \to 0$$

to be an extension which is a semi-direct product with the natural action of $SL_2(\mathbf{R})$ on $\mathbf{C}^2$. An element $(M, \begin{pmatrix} c_D \\ c \end{pmatrix})$ in this group acts by

$$(M, \begin{pmatrix} c_D \\ c \end{pmatrix}) \cdot \begin{pmatrix} a_D \\ a \end{pmatrix} = M \cdot \begin{pmatrix} a_D \\ a \end{pmatrix} + \begin{pmatrix} c_D \\ c \end{pmatrix}.$$

Since $\tau = \frac{da_D}{da}$, one sees that the action of this group on τ is the usual fractional linear transformation of M on $\tau \in \mathbf{H}$:

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \cdot \tau = \frac{u\tau + t}{s\tau + r}.$$

In this way this change of variables brings the $d^4\theta$ term of the Lagrangian back into the same form with a different a, a_D, τ.

This computation with the Kähler metric leads us to ask whether the non-uniqueness in our description of the holomorphic function τ should be that we are allowed to compose τ with any $SL_2(\mathbf{R})$ fractional-linear transformation. But we must examine the other term in the Lagrangian where τ appears. That is in the term involving the gauge field (or photon). Recall from Lecture II-8 what happens for constant τ. Since $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$ and the action is given by $e^{i\mathcal{L}}$, if we consider strictly upper triangular matrices the only ones that leave invariant this part of the action are those of the form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

with $b \in \mathbf{Z}$. As we have seen in Lecture II-8, this action of the strictly upper triangular matrices in $SL_2(\mathbf{Z})$ extends to an action of $SL_2(\mathbf{Z})$ on representations of the theory. That is to say, if τ and τ' in $\mathbf{H}$ differ by the action of $SL_2(\mathbf{Z})$, then the pure $U(1)$-gauge theory with τ as parameter has another representation as a $U(1)$-gauge theory (with different basic fields) where the parameter is τ'. The transformation $\tau \to (-1)/\tau$ is the duality transformation in $U(1)$-gauge theory interchanging the electric and magnetic charges. The upshot of all this is that the full symmetry group of representations of the theory by Lagrangian's of the given form is

$$ISL_2(\mathbf{Z}) = \mathbf{C}^2 \times SL_2(\mathbf{Z}).$$

The way this group transforms the parameter τ is through composing with the usual fraction linear transformation action of $SL_2(\mathbf{Z})$ on the upper half-plane.

We now see how to describe our low energy effective theory globally. There is a complex curve U of vacua and there is a global holomorphic function $\overline{\tau}\colon U \to \mathbf{H}/PSL_2(\mathbf{Z})$. Furthermore, for each $u \in U$ there is a neighborhood V of u, a lifting of $\overline{\tau}$ to a holomorphic function $\tau\colon V \to \mathbf{H}$, and $N = 1$ super-fields $\mathcal{A}, \Phi$ such that there is a Lagrangian in the form given in Equation 2.1 (or Equation 2.2 in components) which represents the theory in V. As we pass from one of these open subsets to another, the representation of the low energy effective theory locally by Lagrangian's changes. The function τ changes by the natural action of $SL_2(\mathbf{Z})$ on $\mathbf{H}$. The chiral super-field changes by the action of $ISL_2(\mathbf{Z})$ described above, and the gauge super-field $\mathcal{A}$ changes by the duality action of $SL_2(\mathbf{Z})$ action on the representations of pure $N = 1$ super-symmetric $U(1)$-gauge theory as described in Lecture II-8. Because there are global holomorphic functions to $\mathbf{H}/SL_2(\mathbf{Z})$, this is a completely consistent picture of a global theory in which $\mathrm{Im}\tau$ never goes to zero.

2.4 The Family of elliptic curves associated to τ

We have now established all the relevant general features of the low energy effective theories we want to study. We have an $N = 2$ super-symmetric pure $U(1)$-gauge theory with a family of vacua parameterized by a complex curve U, where the scalar field $\phi = a$ takes its values. This curve has an open covering $\{V^\alpha\}$ and on each open subset there is a pair of dual holomorphic coordinates a^α, a_D^α so that $\tau^\alpha = \frac{da_D^\alpha}{da^\alpha}$ is the parameter appearing in the Lagrangian. We assume that the different local representations of the theory differ by automorphisms in $ISL_2(\mathbf{Z})$. We have just seen that the local holomorphic function τ on U which appears in the Lagrangian is in fact a global function of U to $\mathbf{H}/SL_2(\mathbf{Z})$. Thus, for each $u \in U$ it is natural to consider $\tau(u)$ as the j-invariant of an elliptic curve $E(u)$. Of course, it is possible that at a discrete set of points $\{p_i\}_i$ in U that τ goes off to infinity, or equivalently that the elliptic curves $E(u)$ develop a node as $u \to p_i$. At these points the physics changes and there are more massless fields, fields that become massive at nearby $u \in U$.

There is a natural way to fit these elliptic curves together. Namely, there is a complex analytic surface $\mathcal{E}$ together with a proper holomorphic map $\mathcal{E} \to U$ whose fibers form the family of elliptic curves $\{E(u)\}_{u \in U}$ The simplest way to construct such a family is to take the Weierstrass form. That is to say, there are functions A, B on U such that $\mathcal{E}$ is defined by

$$Y^2 = X^3 + AX + B,$$

with an appropriate completion of the missing point at infinity in each elliptic curve. (For globally non-trivial bases U, we will have a line bundle L over U and A and B will be sections of $L^{\otimes 4}$ and $L^{\otimes 6}$, respectively.) This family of elliptic curves has a natural section (the section at infinity in the Weierstrass description), and thus, it can be viewed as a family of one-dimensional Abelian varieties.

There are, of course, other families of holomorphically varying genus-one curves over the same base with the same j-invariant function but without a section. But given the information of the j-invariant function only, there is no distinguished family except the one with a section.

Suppose that we have a family of elliptic curves $\mathcal{E} \to U$ with a section σ. Furthermore, suppose that we have an open covering $\{V^\alpha\}_\alpha$ of U such that on each V^α there are a pair of dual holomorphic coordinates a^α, a_D^α such that $\tau^\alpha = \frac{da_D^\alpha}{da^\alpha}$ is a function to the upper half-plane. We suppose that these two sets of data are related as above. That is to say, $\mathcal{E}|_{V^\alpha}$ is isomorphic to the family $V^\alpha \times \mathbf{C}/(1, \tau^\alpha(v))$. This allows us to define one-cycles $E_1^\alpha(u), E_2^\alpha(u) \subset E(u)$ for all $u \in V^\alpha$ as the images of the arcs $\{u\} \times [0,1]$ and $\{u\} \times [0, \tau^\alpha(u)]$ in $\{u\} \times \mathbf{C}$. Then, there is a differential of the second kind λ^α (i.e., a meromorphic one-form with trivial residue along its polar locus) on $\mathcal{E}|_{V^\alpha}$ such that

$$\int_{E_1^\alpha(u)} \lambda^\alpha = a^\alpha(u)$$

$$\int_{E_2^\alpha(u)} \lambda^\alpha = a_D^\alpha(u).$$

In general, we can not fit these differentials λ^α of the first kind together to produce a global object. (We will see the precise conditions when this can be done soon.) But we claim that we can fit together the two-forms $d\lambda^\alpha$. Namely, we claim that we there is a unique global holomorphic two-form η on $\mathcal{E}$ such that on $\mathcal{E}|_{V^\alpha}$ we have

$$\int_{E_1^\alpha} \eta = da^\alpha$$

$$\int_{E_2^\alpha} \eta = da_D^\alpha.$$

Of course, the uniqueness assertion immediately implies that the restriction of any such global two-form η to $\mathcal{E}^\alpha$ is equal to $d\lambda^\alpha$.

Recall that $\mathcal{E}^\alpha$ is identified with $V^\alpha \times \mathbf{C}/(1, \tau^\alpha(u))$. On the universal covering space $V^\alpha \times \mathbf{C}$ we have the holomorphic one-form dz induced from the standard holomorphic coordinate on $\mathbf{C}$. This one-form does not descend to the quotient because τ is not a constant function of a^α. Nevertheless, the two-form $da^\alpha \wedge dz$ does descend to give a well-defined two-form on $\mathcal{E}^\alpha$. It is clear that the only holomorphic two-form on $\mathcal{E}|_{V^\alpha}$ which satisfies the first integral equation above is

$$\eta^\alpha = -da^\alpha \wedge dz.$$

This two-form η^α also clearly satisfies the second integral equation. Now we need to see that these local two-forms fit together to give a global form η. But this is clear from the uniqueness of the local forms.

Now let us compute $\pi_*(\eta \wedge \overline{\eta})$, where π is the projection from $\mathcal{E} \to U$. Let us restrict attention to $\mathcal{E}|_{V^\alpha}$. From our local description we have

$$\pi_*(\eta \wedge \overline{\eta}) = \int_{\text{fibers}} d\lambda^\alpha \wedge d\overline{\lambda}^\alpha.$$

It follows immediately from the formulas for the integrals of $d\lambda$ over E_1^α and E_2^α and the fact that $E_1^\alpha \cdot E_2^\alpha = 1$ in the homology of $E(u)$ that the result is

$$\pi_*(\eta \wedge \overline{\eta}) = da^\alpha \wedge d\overline{a}^\alpha \overline{\tau}^\alpha + da^\alpha \wedge d\overline{a}^\alpha \tau^\alpha = (\tau^\alpha - \overline{\tau}^\alpha) da^\alpha \wedge d\overline{a}^\alpha = 8\pi\Omega|_{V^\alpha}.$$

Since the two-form η is holomorphic, it integrates trivially on any fiber and on the section σ of the elliptic fibration. Thus, its homology class is determined by the integrals of η along two-tori which lie over circles in U.

The deRham cohomology class of η measures whether or not the automorphisms required in passing between our local descriptions in terms of various sets of (a, a_D, τ) lie in $SL_2(\mathbf{Z})$ or in the extended group $ISL_2(\mathbf{Z})$. We claim that if the deRham homology class of η is trivial, then the automorphisms we use lie in $SL_2(\mathbf{Z})$. The point is the following. We already know that the change in da, da_D is by elements in $SL_2(\mathbf{Z})$. We need to compute the changes in a, a_D. If we go around any loop γ in U with the property that E_1 comes back to itself, then the cycles $E_1(\gamma(t))$ fit together to make a torus and the integral around this torus of η is equal to $\int_\gamma da$. If this integral is trivial, then a can be analytically continued around this loop to a single valued function. Similar arguments apply to a_D. Thus, if all the periods of η are trivial, then a, a_D transform by $SL_2(\mathbf{Z})$.

In general, the deRham cohomology class of a holomorphic two-form such as η is trivial if and only if there is a global differential of the second kind λ on $\mathcal{E}$ with $d\lambda = \eta$. In the more general case, we can write the holomorphic two-form η as $d\lambda$ for some global differential λ of the third kind on $\mathcal{E}$ (i.e., λ is a meromorphic one-form on $\mathcal{E}$ with constant residue along its polar locus.) In turns out in our application to pure $SU(2)$-gauge theory, that the automorphisms between our various descriptions lie in $SL_2(\mathbf{Z})$, so that we shall not need to consider this more general case. The proof of this fact is the subject of the next section.

2.5 The BPS formalism

Our goal here is to show that if the low energy effective $U(1)$-gauge theory that we have been considering is the low energy limit of a pure $N = 2$ super-symmetric $SU(2)$ gauge-theory, then the transformations between the various local descriptions in terms of (a, a_D, τ) of the $U(1)$-theory lie in $SL_2(\mathbf{Z})$. We shall establish this by using the BPS formalism. In any $U(1)$-theory we have two conserved charges, the electric and magnetic charges:

$$Q_e = \frac{1}{4\pi}\int_{S^2} *F$$

$$Q_m = \frac{1}{4\pi}\int_{S^2} F.$$

Here, we are doing the integrals on the sphere at infinity in some time slice in Minkowski four-space. In pure $U(1)$-theory on four-space, these charges are both zero. But in the case that our $U(1)$-theory is the low energy limit of an $SU(2)$-theory these charges can be non-zero. Recall from Lecture II-9 that our $SU(2)$-theory has a $U(1)$-symmetry whose conserved charge is

$$N_e = Q_e + \frac{\theta Q_m}{2\pi}.$$

As a result, this operator has integral eigenvalues. It is called the electric charge. There is another conserved charge, the magnetic charge, given by

$$N_m = Q_m.$$

This is the first Chern class of the line bundle at infinity and hence is clearly has integral eigenvalues as well. Of course, as we saw in Lecture II-9 and II-9 these two charges are interchanged by duality. Thus, both charges come from $U(1)$-symmetries of the theory.

At least in the case of constant τ they are related to the Q_e, Q_m by

$$\begin{aligned} N_m &= Q_m \\ N_e &= Q_e + \frac{\theta Q_m}{2\pi} \end{aligned}$$

The $N = 2$ super-symmetry generators Q^i_α, $i = 1, 2$ satisfy the relations of the super Poincaré group

$$\{Q^i_\alpha, Q^j_{\dot\alpha}\} = \delta^i{}_j P_{\alpha\dot\alpha}$$

where $P_{\alpha\dot\alpha}$ is infinitesimal translation in the spatial direction $\{\alpha\dot\alpha\}$. Also, we have

$$\{Q^i_\alpha, Q^j_\beta\} = \epsilon^{ij}\epsilon_{\alpha\beta} U$$

where U is multiplication by a scalar, the central charge of the theory.

In general, there is the BPS bound on the mass of any state which says

$$M \geq |U|.$$

In addition, there is a good understanding of the BPS saturated states, i.e., those for which this inequality is an equality.

In our case we wish to compute the central charge U. To do this recall that the Lagrangian (for τ constant) is:

$$\mathcal{L} = \frac{1}{4\pi}\int d^4x d^2\theta i\tau W_\alpha W^\alpha + \int d^4x d^4\theta \mathrm{Im}\tau\overline{\Phi}\Phi.$$

From this one knows (from Lecture II-9) that for this value of τ

$$U = aN_e + a_D N_m$$

where $\Phi = a + \ldots$. (On dimensional grounds, this formula holds even if τ is not constant.) The interpretation is that different realizations of the low energy theory correspond to different values of a. In these realizations U depends on the values of a, a_D by the above formula. Recall from Lecture II-9 that (N_m, N_e) transform under $SL_2(\mathbf{Z})$ as we take different representations of our theory. Since U has to be unchanged, we see that if

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \mapsto M \cdot \begin{pmatrix} a_D \\ a \end{pmatrix},$$

then

$$\begin{pmatrix} N_m & N_e \end{pmatrix} \mapsto \begin{pmatrix} N_m & N_e \end{pmatrix} \cdot M^{-1}$$

for any $M \in SL_2(\mathbf{Z})$.

The first important point to notice is that we can not compensate for the operation of adding a constant to $\begin{pmatrix} a_D \\ a \end{pmatrix}$ by a change in (N_m, N_e). This shows that the automorphisms between the various representations of the $U(1)$-theory in terms of (a, a_D, τ) lie in $SL_2(\mathbf{Z})$. Thus, in this case we can construct a differential λ of the second kind on $\mathcal{E}$ such that for each α we have

$$\int_{E_1} \lambda = a^\alpha$$

$$\int_{E_2} \lambda = a^\alpha_D.$$

on $\mathcal{E}|_{V^\alpha}$.

More generally, if the $SU(2)$-theory has extra symmetries (as might be the case if we add matter), then the form of the central charge becomes

$$U = aN_e + a_D N_m + S$$

where S is some other conserved charge. In this case it is possible to compensate for the affine translations in (a_d, a). Namely, we can have

$$\begin{pmatrix} a_D \\ a \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} M & \begin{pmatrix} c_D \\ c \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} & 1 \end{pmatrix} \cdot \begin{pmatrix} a_D \\ a \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} N_m & N_e & S \end{pmatrix} \mapsto \begin{pmatrix} N_m & N_e & S \end{pmatrix} \cdot \begin{pmatrix} M^{-1} & \begin{pmatrix} * \\ * \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} & 1 \end{pmatrix}.$$

In all these transformations no derivatives of τ enter. In fact, what we did was to compute the transformation laws for τ constant and just extended them in the obvious way to the case when τ is holomorphic. The reason that we know that this mechanism works is that on dimensional grounds alone we know that no derivatives of τ can enter the formulas. While one can consider the low energy limits of these more complicated $SU(2)$-theories with matter, we shall restrict ourselves to the case when there are no extra symmetries and the transformations between the various representations of our low energy theory lie in $SL_2(\mathbf{Z})$.

We gave a derivation of the formula for the central charge above from the form of the Lagrangian. Let us give another derivation of the same formula (or at least a piece of it). Let us consider $U(1)$-gauge theory with a charged $N = 2$ vector hyper-multiplet. In the $N = 1$ language, this hyper-multiplet is a pair of chiral super-fields $(T, \tilde{T})$ of equal but opposite charges. Super-symmetry implies that the charges must be opposite, let us denote them by $\pm n_e$. In the Lagrangian we have added a super-potential

$$W = n_e \Phi T \tilde{T}.$$

Since $\Phi = a + \ldots$, we see that the mass of the pair $(T, \tilde{T})$ is aN_e. Thus, we see this term in the central charge of the theory from this point of view.

2.6 Jumping of the BPS spectrum

Let us return to our low energy $N = 2$ super-symmetric $U(1)$-gauge theory given locally by a, a_D. In such theories the BPS mass inequality is

$$M \geq |n_e a + n_m a_D|.$$

Since N_e and N_m are integers, this is a good bound in the sense that it implies that there is a positive constant which is a lower bound for M as long as the lattice in $\mathbf{C}$ generated by a and a_D is a honest lattice. Let us examine what happens as the lattice generated by a and a_D degenerates, i.e., when a and a_D become collinear in $\mathbf{C}$. At such points the triangle inequality is no long strict and this permits states to decay to other lower states. In this way BPS states can disappear (or appear), accounting for the jumping in the BPS spectrum.

Generically, we expect this to happen along a real curve in U. (Notice the lattice generated by $(1, \tau)$ is not degenerating, rather it is the lattice of values of a and a_D which is degenerating.) This phenomenon is similar to that in Vafa's lecture for $N = 2$ super-symmetric theories in dimension two. There the jumping occurred when the values of the central charge became collinear. In the example of interest, we shall obtain explicit information about the curve in U where the BPS spectrum jumps.

2.7 More about the application to $SU(2)$-gauge theory

As we have remarked in passing several times, we will study an $N = 2$ super-symmetric $SU(2)$-gauge theory with no matter hyper-multiplets. We shall do this by studying the theory as an $N = 1$ super-symmetric theory with Φ an adjoint-valued chiral super-field. The microscopic (or high energy) effective Lagrangian is given by:

$$\mathcal{L}_{\text{mac}} = \frac{1}{4\pi} \int d^4x d^2\theta \tau_0(\Lambda_0) \text{Tr} W_\alpha W^\alpha + \text{c.c.} + \frac{1}{4\pi} \int d^4x d^4\theta \text{Im}\tau_0(\Lambda_0) \text{Tr}\overline{\Phi}\Phi$$

where W is a piece of the super-curvature of the super-connection $\mathcal{A}$ on a principal $SU(2)$-bundle over Minkowski space, and $\Phi = \phi + \theta\psi + \cdots$ with ϕ being a field with values in the complexification of the adjoint bundle. The Λ_0 refers to the ultraviolet momentum cut-off that we have chosen to regulate the $SU(2)$-theory. Here, τ_0 is a constant (in the fields) which depends on the value of Λ.

The space of classical vacua is given by configurations consisting of trivial connections and constant (covariantly constant) sections ϕ of $\mathbf{C} \otimes su(2)$ which are zeros for the potential, up to gauge equivalence. The potential for this Lagrangian is the norm-squared of the moment map

$$V = \frac{\text{Im}\tau_0}{4\pi} \int d^4x \text{Tr}\left([\phi, \overline{\phi}]^2\right).$$

If ϕ is a zero of the potential function, then clearly ϕ and $\overline{\phi}$ commute. Writing $\phi = \gamma + i\delta$ with $\gamma, \delta \in su(2)$, we see that the condition that ϕ and $\overline{\phi}$ commute is simply that γ and δ commute. This means that γ and δ can be simultaneously diagonalized. That is to say, up to gauge transformation,

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

for $a \in \mathbf{C}$. There is a further conjugation, namely the action of the Weyl group, which in this case is $\mathbf{Z}/2\mathbf{Z}$ and acts by sending $a \mapsto -a$. Thus, the moduli space of classical vacua is a complex plane parameterized by the complex coordinate

$$u = \text{Tr}\phi^2 = a^2.$$

In the classical theory, for $u \neq 0$ the $SU(2)$-gauge symmetry is broken to $U(1)$ and the low energy theory is as we have discussed today. It has a single massless multiplet, which becomes Higgs'ed and acquires a mass. The low energy effective theory at the exceptional point $u = 0$ remains an unbroken $SU(2)$-gauge theory.

Let us examine it in light of our discussion today about low energy effective $U(1)$-theories. As we have just explained the space of classical vacua is the complex plane – the u-plane. Let us examine the metric on the u-plane. All points of this plane except zero correspond to a $U(1)$-theory as described in this lecture. The metric on the space of classical vacua is induced from the natural metric on the space of trivial connections and covariantly constant scalars in the complexified Lie algebra. This is simply

the ad-invariant metric on the complexified Lie algebra. Restricting to the diagonal matrices as above, we see that the induced metric is $da \wedge d\overline{a}$. Since $u = a^2$, this means that the metric on the u-plane is

$$ds^2 = \frac{du d\overline{u}}{|u|^2},$$

which has a singularity at the origin. This singularity reflects the fact that at the origin there are other masses fields so that the the low energy effective theory is not a pure $N = 1$ super-symmetric $U(1)$-gauge theory at that point.

Now we can describe the family of tori over the u-plane punctured at 0, and the holomorphic two-form η on the total space of this family. The function τ is constant, and hence the family of tori has monodromy contained in $\{\pm 1\} \subset SL_2(\mathbf{Z})$. Since the metric on the u-plane minus the origin is given by $\mathrm{Im}\tau da \wedge d\overline{a}$ we see that the parameter $a = \sqrt{u}$ is the parameter of the same name in the $U(1)$-theory (i.e., the first of the dual pair of local holomorphic coordinates on the base) and the dual parameter a_D is τa. This means that the monodromy around infinity (or equivalently around zero) sends a to $-a$ and a_D to $-a_D$. This means that the family of tori over the puncture u-plane is obtained from the product family $(\mathbf{C} - \{0\}) \times E$ over the punctured a-plane by dividing out by the involution $(a, z) \mapsto (-a, -z)$. (Here, E is the elliptic curve $\mathbf{C}/L(1, \tau)$, the quotient of the complex plane by the lattice generated by 1 and τ.) The holomorphic two-form η is the image of $da \wedge dz$ where dz is a global holomorphic one-form on E.

This description is of the classical theory. In the next lecture we shall discuss the quantum version of this theory. It turns out that things change somewhat in going to the quantum theory, but the changes are in some ways not as drastic as one might imagine. For example the space of quantum vacua will be the same u-plane, though the singular points will be different, and they will correspond to a different type of object becoming massless. Also, the monodromy will be different.

Lecture II-18: $N = 2$ Super-symmetric Yang-Mills theories in dimension four: Part II

Edward Witten[1]

1 Review of material from the last lecture

We are considering pure $N = 2$ super-symmetric gauge theory (i.e., without hyper-multiplets) in four-dimensional Minkowski space. The gauge group is either $SU(2)$ or the closely related group $SO(3)$. (In fact, working as we are today over Minkowski four-space, which is contractible, we will not see the distinction between these two gauge groups.) We write the theory in the $N = 1$ super-symmetric notation. The fields which form a single $N = 2$ vector multiplet decompose under the $N = 1$ super Poincaré group in four dimensions as a vector multiplet $\mathcal{A}$ (the gauge fields) and a chiral super-field

$$\Phi = \phi + \theta\psi + \cdots$$

where ϕ is a scalar field (under the Poincaré group). It is a section of the complexification of the adjoint bundle. The microscopic Lagrangian is

$$\mathcal{L} = \int d^4x d^2\theta \tau_0(\Lambda_0)\mathrm{Tr}W_\alpha W^\alpha + \text{c.c.} + \int d^4x d^2\theta \mathrm{Im}\tau_0(\Lambda_0)\mathrm{Tr}(\overline{\Phi}\Phi)$$

where W is a certain piece of the super-connection of the super-connection $\mathcal{A}$. (Here Λ_0 is a cut-off used to render finite the Feynmann path integrals. We vary τ_0 as a function of Λ_0 in order to renormalize, i.e., render the expectation values in the theory finite as $\Lambda_0 \mapsto \infty$. In order to decide how to renormalize we must fix a so-called 'subtraction point' Λ at which to fix the value of one parameter in the theory. We shall see later a convenient normalization scheme for this particular theory which involves the details of the theory (the u-plane, etc). But for now we leave vague the actual renormalization scheme. Once we have fixed the value of the extra parameter Λ, the coupling constant τ_0 is a function of Λ_0.) While τ depends on the cut-off, it is independent of the fields.

Our object today is to solve this theory by explicitly understanding the moduli space $\mathcal{M}$ of quantum vacua, as well as a description of the theory near each vacuum state. As we established last time, the space of classical vacua of this theory consists of gauge equivalence classes of constant matrices in $su(2)$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

modulo the action of the Weyl group, which is a group of order two switching a and $-a$. This space is analytically isomorphic to $\mathbf{C}$ and the natural local holomorphic coordinate to use on this plane is $u = \mathrm{Tr}\phi^2 = a^2$. While the analytic space of classical vacua is smooth, there is one point in the moduli space of classical vacua where the Kähler metric becomes singular. This is the origin. The singularity is a consequence of the fact that at this point there are more massless fields (in fact a super-symmetric $SU(2)$-theory is the lower energy effective theory). Furthermore, as we discussed last time, away from $u = 0$ the low energy effective theory is a pure $N = 1$ super-symmetric $U(1)$-gauge theory. Thus, away from $u = 0$, the nature of this effective theory is described by a family of tori over the punctured u-plane. In the case of the moduli space of classical vacua this is a family of tori with constant j

[1]Notes by John Morgan

invariant given by the choice of τ in the Lagrangian. The monodromy around infinity for this family is -1. The Kähler metric on the u-plane is

$$ds^2 = \mathrm{Im}\tau \frac{du d\overline{u}}{4|u|^2}.$$

(Notice that there is a singularity in the metric at $u = 0$ as predicted by the fact that at this vacuum there are extra massless fields classically.)

1.1 First results about the moduli space $\mathcal{M}$ of quantum vacua

Let $\mathcal{M}$ be the space of quantum vacua. Notice that, because we are assuming that our theory is $N = 2$ super-symmetric, the space of quantum vacua has a Kähler metric (from the weaker $N = 1$ super-symmetry) and over this space of vacua there is a holomorphic family of tori. The field u, which is a coordinate on the space of classical vacua, is a chiral field in the theory, and thus its expectation values in the various quantum vacua defines a holomorphic function on $\mathcal{M}$. We denote this function by $u\colon \mathcal{M} \to \mathbf{C}$:

$$u(\Omega) = \langle \Omega | u | \Omega \rangle, \quad \text{for all } \Omega \in \mathcal{M}.$$

The part of $\mathcal{M}$ that is most closely related to the u-plane of classical vacua is the region where u is large. More precisely, suppose that we have fixed the mass parameter, Λ, in the renormalization scheme which determines the specific quantum theory we are studying. Suppose that we are studying the theory near a classical vacuum Ω with $u(\Omega) >> \Lambda^2$. (Notice that since u has dimensions of mass squared it naturally compares to Λ^2.) Because our $SU(2)$-theory is asymptotically free, and the $U(1)$-theory to which it is limiting is free in the infra-red, if $u >> \Lambda^2$, then the quantum theory is well-approximated near Ω by the classical theory. This means that the low energy quantum theory near Ω is described by a sigma model on the u-plane, perhaps perturbed by some super-potential. But $N = 2$ super-symmetry means that any such super-potential must be a constant, and hence cannot lift the vacuum degeneracy near Ω. (There is also the possibility of a Fayet-Iliopoulos θ-term to consider as well, but the $SU(2)_R$-symmetry rules these out.) What this argument shows is that for large u there is a unique quantum vacuum for each classical vacuum. That is to say, there is a constant $C(\Lambda) > 0$ and an open subset $\mathcal{U} \subset \mathcal{M}$ with the property that $u\colon \mathcal{U} \to (\{z \in \mathbf{C} | |z| > C\})$ is a holomorphic isomorphism.

In Seiberg's lectures we have seen other examples of gauge theories where near infinity in the space of classical vacua the gauge group $SU(N_c)$ is broken down to a non-abelian gauge group $SU(N_c - N_f)$. Since the gauge theory for this smaller group is not free in the infra-red, we cannot conclude that the space of quantum vacua near infinity can be identified with the subspace of critical points of a superpotential on the space of classical vacua. In fact, in these examples the moduli space of quantum vacua is a finite covering space of the space of classical vacua near infinity. To repeat, this is possible because the low energy theory is not free in the infra-red. When it is free, as in the case under discussion now, and when $N = 2$ super-symmetry rules out the existence of super-potentials, the classical and quantum moduli spaces can be identified over a neighborhood of infinity in the space of classical vacua.

One assumption that we shall make without any justification is that the moduli space $\mathcal{M}$ is complete in an appropriate sense so that it corresponds to an open complex algebraic variety.

We have found one component of $\mathcal{M}$ which has an end isomorphic under u to a neighborhood of infinity in $\mathbf{C}$. Two other questions present themselves:

- Are there other 'ends' of $\mathcal{M}$, ends that are not seen classically?

- Are there other irreducible components of $\mathcal{M}$ not seen classically?

The answer to the first question is 'No.' The point is that all ends of the quantum moduli space must be visible classically, i.e, they must be related to ends of the classical moduli space. The reason is that quantum effects are bounded and hence cannot effect a non-proper change in the passage from the space of classical vacua to the space of quantum vacua.

As to the second question, there are no examples known where there are branches of the quantum space of vacua which are not seen classically. Still in this generality, it has not been established that such components do not exist. Nevertheless, in our case we can give arguments showing that there are no other components. The first thing to note is that, by the answer to the first question, any other component of $\mathcal{M}$ must be compact. To exclude the possibility of other compact branches we use the fact that the theory has a relevant $N = 1$ super-symmetric perturbation. That is the perturbation

$$\Delta\mathcal{L} = \int d^4x d^2\theta \epsilon u + \text{c.c.}$$

where ϵ is a complex constant. As we saw in Lecture II-??, with this perturbation the resulting $SU(2)$-theory has two vacua, each with a mass gap and each with confinement. If we examine what happens to these perturbed theories in the low energy limit, then of course they look the same: they have two vacua each with a mass gap and confinement. Thus, this perturbation would have to lift the vacuum degeneracy along a compact component $\mathcal{N}$ of $\mathcal{M}$ by adding a super-potential. On the other hand, since u is a holomorphic function along this branch, it is constant. Now the super-potential term that we have added to the microscopic Lagrangian is simply ϵu which is constant along this branch. One can see that this persists in the low energy theory, so that the low energy effective super potential is also a multiple of u, and hence is constant. This means that the addition of the above super-potential cannot lift the vacuum degeneracy along this branch. Since we know the resulting low energy theory has only two vacua, this implies that the other branch is in fact not present. (This argument still allows for the possibility of isolated points (at most two), but we shall see later by counting singularities, that they do not occur either.)

We have now argued that $\mathcal{M}$ has a single branch (plus possibly at most two isolated points) and that this branch has a single neighborhood of infinity which is mapped by u isomorphically to a neighborhood of infinity in the u-plane. By our completeness assumption, we know that $\mathcal{M}$ is a possibly singular, irreducible complex curve with one puncture. The existence of the holomorphic function u with a only one pole, that being the simple pole at infinity of $\mathcal{M}$, implies that $\mathcal{M}$ is actually the complex plane and that u is a global holomorphic parameter on $\mathcal{M}$. Following the analysis of the last lecture, our goal is to solve the theory by finding the family of elliptic curves (in Weierstrass form) over $\mathcal{M}$ minus the singular points

$$y^2 = x^3 + A(u)x + B(u)$$

and a holomorphic two-form η on the total space of this family

$$\eta = f(u)du\frac{dx}{y}$$

whose periods determine the differentials da and da_D of the dual pair of local holomorphic coordinates.

1.2 The nature of infinity in $\mathcal{M}$

As we have already remarked, we obtain information near infinity in $\mathcal{M}$, i.e., where u is large using asymptotic freedom of the $SU(2)$-theory. Our goal here is to compute the monodromy at infinity

of the family of elliptic curves over the moduli space of quantum vacua. We begin by studying the monodromy at infinity for the space of classical vacua. Recall that the space of classical vacua is the space of matrices

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

up to $a \mapsto -a$. The dual coordinate a_D is given by $a_D = \tau_{\text{cl.}} a$ where $\tau_{\text{cl.}}$ is a constant (denoted by τ_0 earlier) with large imaginary part. The global coordinate on the u-plane is $u = \text{Tr}(\phi^2) = a^2$, so that $a = \sqrt{u}$. From this it is clear that the monodromy at infinity in this classical family is -1.

The monodromy around infinity for the space $\mathcal{M}$ of quantum vacua is different. We give two arguments calculating it. The first is based on asymptotic freedom and the second is based on the $U(1)_R$-anomaly. To compute $\tau_{\text{eff}}(u)$ for large u we consider the 1-loop formula. With an appropriate choice of the mass parameter Λ, the formula is:

$$\text{Im}\tau = \frac{4\pi}{g^2_{\text{eff}}} = 4\pi\left(2b_0 \ln(\sqrt{|u|/\Lambda})\right), \tag{1.1}$$

where b_0 is the 1-loop β-function coefficient.

One loop computations identify

$$b_0 = \frac{1}{4\pi^2}.$$

Since τ is holomorphic, knowing its imaginary part determines it uniquely. We see that for u large we have

$$\tau_{\text{eff}} = \frac{2i\ln(u/\Lambda^2)}{2\pi}. \tag{1.2}$$

Thus, we see that monodromy around infinity shifts τ by -2 and hence shifts θ by -4π. This implies that the monodromy around infinity in $\mathcal{M}$ is given by

$$\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}.$$

Said another way we are considering the system

$$\begin{aligned} u^2 &= a \\ \frac{da_D}{da} &= \tau = \frac{2i\ln(u/\Lambda^2)}{2\pi} \end{aligned} \tag{1.3}$$

as we turn once around infinity. Clearly, a comes back to $-a$ and τ comes back to $\tau - 2$. The above form for the monodromy then follows easily.

Now let us compute this monodromy using the anomalous $U(1)_R$ symmetry. Classically, the $U(1)_R$-charges of the component fields in the gauge super-multiplet are given as follows: the charge of A is zero, the charges of λ, ψ are 1 and the charge of a is two. This means that $u = a^2$ has charge 4. (This is related to the fact that in Donaldson theory $\mu(pt)$ is four-dimensional.) This means that under the $U(1)_R$ symmetry $e^{i\beta}$ the variable u transforms by $u \mapsto ue^{\pm 4i\beta}$. In fact, the sign is negative. On the other hand, as we know, there is a quantum anomaly breaking this $U(1)_R$ down to $\mathbf{Z}/8\mathbf{Z}$ (which is the dimension of the one-instanton moduli space on $\mathbf{R}^4$). This is an index computation which uses the fact that the Casimir of the adjoint representation of $su(2)$ is 4 times the Casimir of the defining representation. This means that under the $U(1)_R$ symmetry $e^{i\beta}$ the angle θ transforms to $\theta - 8\beta$. Thus, as we transverse a loop around infinity in the u-plane the argument of u changes by 2π, β increases by $2\pi/4 = \pi/2$ and θ decreases by 4π. This means that $\tau \mapsto \tau - 2$ under the monodromy around infinity.

Of course, these are not independent computations. Super-symmetry imposes a relation between the $U(1)_R$ anomalies and the β-function.

Actually, we can say more about the nature of τ at infinity. For $|u| >> \Lambda^2$ in perturbation theory we have a series

$$\tau(u) = \frac{2i\ln(u/\Lambda^2)}{2\pi} + \tau^{\geq 2} \tag{1.4}$$

where the first term is the sum of the classical value and the one-loop correction (as calculated above) and the second term is the sum of the corrections for higher loops. These higher loop corrections produce a power series in g^2 (k-loop corrections produce the power g^{2k-2}). Since there are no loop corrections to θ and since τ is a holomorphic function of u, it follows immediately that all the higher loop corrections vanish. Thus, in perturbation theory the one-loop correction is exact. There are however non-zero corrections to the perturbation expansion coming from instantons. Of course, these can not be seen in perturbation theory. Let's study their nature. We know that τ is invariant under the $U(1)_R$-symmetry. An instanton breaks the $U(1)_R$-symmetry by $8k$ where $8\pi^2 k$ is its total energy. Since u has $U(1)_R$-charge 4, the k instantons go with even u^{2k}. Of course, the dimensionless expression is Λ^{4k}/u^{2k}. Thus, the instanton action for an instanton of total energy $8\pi^2 k$ is of the form

$$f_k \exp(-8k\pi^2/g^2)e^{ki\theta} = f_k \Lambda^{4k} e^{2k\pi i\tau} = f_k \frac{\Lambda^{4k}}{u^{2k}}$$

for a dimensionless constant f_k. Thus, the full expression for τ at $|u| >> 1$ is of the form

$$\tau(u) = \frac{2i\ln(u/\Lambda^2)}{2\pi} + \sum_{k\geq 1} f_k \frac{\Lambda^{4k}}{u^{2k}}. \tag{1.5}$$

1.3 BPS states and singularities in $\mathcal{M}$

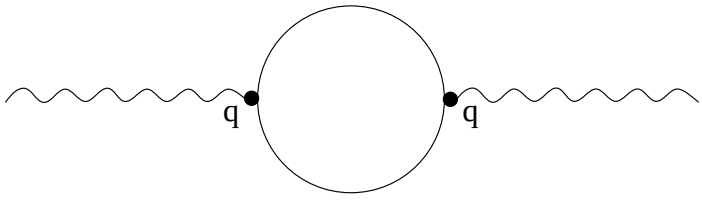
Next, we wish to understand the singularities in $\mathcal{M}$. By a *singularity* we mean a point of the moduli space where the physics changes. This may or may not be accounted for by a topological singularity in the space. As a general principle, we expect that the singularities have to do with particles becoming massless. In this case since the particles are to add singularities to the low energy limit which is a $U(1)$-theory, they must be charged under the $U(1)$. It is also quite reasonable to expect that if any particles become massless, then some BPS states (which are the states of lowest energy) become massless. In our case we have two types of BPS states near infinity in the u-plane. We have the vector multiplet (i.e., the gauge boson) which has $n_e = \pm 1$ and $n_m = 0$ and we have the magnetic monopoles with $n_m = 1$ and any value of n_e. (Of course, as we increase θ by 2π a particle with $n_m \neq 0$ has n_e which changes by ± 2, so that having one BPS state with $n_m = \pm 1$ gives rise to BPS states with all even or all odd values of n_e. Our claim here is that there are two families of these BPS states, one with even n_e and one with odd n_e.) Notice that this BPS spectrum is invariant under the monodromy at infinity since under this monodromy $n_e \mapsto -n_e + 2n_m$ and $n_m \mapsto -n_m$. On the other hand we claim that this spectrum can not be invariant under the entire monodromy of our family. The reason is that as we shall see the image of the monodromy on all of $\mathcal{M}$ is a non-abelian subgroup of SL_2, under which the above-described BPS spectrum is not invariant. This means that the BPS spectrum cannot be continuous throughout $\mathcal{M}$, that is to say there must be a jumping locus, which is a real curve in $\mathcal{M}$ along which extra BPS states become massless. As we saw in the last lecture, this happens when a/a_D becomes real.

As we have seen $\mathcal{M}$ is a complex analytic space with exactly one irreducible component. Furthermore, the holomorphic map $u\colon \mathcal{M} \to \mathbf{C}$ is proper and is an isomorphism in a neighborhood of infinity

of $\mathcal{M}$. Next, we wish to analyze the singularities of $\mathcal{M}$. We are assuming that $\mathcal{M}$ is in fact a complex algebraic variety. There must be singularities in $\mathcal{M}$ since the monodromy at infinity is non-trivial. As we have argued singularities in $\mathcal{M}$ are associated with a certain super-symmetric multiplet consisting of charged BPS states becoming massless. We have two types of BPS multiplets – those where the maximal spin in the multiplet is 1 (the vector multiplet) and those where the maximal spin is $1/2$, (the hyper-multiplet). A vector multiplet becoming massless corresponds to restoring a non-abelian gauge group ($SU(2)$). There are several reasons one does not expect this to happen. First of all, with unbroken $SU(2)$ one has a theory which is unstable in the infra-red. This is exactly what happens at the singularity in the classical moduli space. Since the β-function is negative, we have flowed away this theory, so we don't expect to see it occurring in the low energy limit which we are assuming to be free in the infra-red. Of course, if our gauge theory has enough hyper-multiplets becoming massless to make $\beta > 0$ then we might see such a low energy limit. But at these points we have multiplets with maximal spins both 1 and $1/2$ becoming massless. So the simplest thing that we can do is to require that only a hyper-multiplet with maximal spin $1/2$ become massless. Let us analyze this possibility.

Suppose that the singularity in $\mathcal{M}$ is associated to a multiplet whose maximal spin is $1/2$ becoming massless. Recall that a massive $N = 2$ super-symmetric multiplet of spin at most $1/2$ is automatically BPS saturated. Thus, we might expect that what is happening is that one of the BPS states that exist in theory for $|u| >> \Lambda^2$ is simply becoming massless at the singularity. Of course, more complicated phenomena could occur. For example, as we come in from infinity in $\mathcal{M}$ we could cross a jumping line of the BPS spectrum before we reach the singularity, and then at the singularity it is one of the 'new' BPS saturated states that becomes massless, rather than one of the original states that exist near $u = \infty$.

Let us assume that a hyper-multiplet becomes massless. Denote its charge by (n_e, n_m) under the electric and magnetic charge. We have already argued that this hyper-multiplet must be charged, so that $q = \text{g.c.d}\,(n_e, n_m)$ is a well-defined positive number. After an $SL_2(\mathbf{Z})$ change in the representation of the theory by a Lagrangian, we can arrange that $n_e = q$ and $n_m = 0$. Thus, near the singular point $u_0 \in \mathcal{M}$ our theory is a $U(1)$-theory with a charged particle which is becoming massless at $u = u_0$. (But notice that it may well be the case that $n_m \neq 0$. If that happens then the $U(1)$-gauge fields in the representation of the theory near $u = u_0$ are different from the $U(1)$-gauge fields that we naturally have at infinity.) Unlike the spin-one case, this theory is stable in the infra-red, i.e., $\beta > 0$. Since the behavior of the theory near $u = u_0$ is infra-red free, we can compute g_{eff} and θ_{eff} in this region using perturbation theory. We see that g_{eff} vanishes logarithmically and is given by the one-loop β-function. The relevant one-loop Feynmann diagram is:



Clearly, this diagram gives a multiplicative factor of q^2. It also diverges logarithmically for $u \mapsto u_0$. The result is that for u near u_0 we have:

$$\tau(u) = 2q^2 \frac{\ln(u - u_0) + \text{constant}}{2\pi i}.$$

This means that the monodromy of $\tau(u)$ in $PSL_2(\mathbf{Z})$ around this singularity is

$$\pm\begin{pmatrix}1 & 2q^2\\ 0 & 1\end{pmatrix}. \tag{1.6}$$

Of course, the monodromy is actually an element in $SL_2(\mathbf{Z})$ and is only determined up to sign by the monodromy in $PSL_2(\mathbf{Z})$ To pin down the sign we argue as follows. The physics near $u = u_0$ is described by a $U(1)$-gauge theory with gauge fields $(\tilde{A}, \tilde{a})$ and a charged hyper-multiplet (T, T') which is becoming massless at u_0. Thus, the Lagrangian is of the form of the gauge couplings plus a super-potential $\tilde{a}qTT'$. The scalar field $\tilde{a}$ is a local coordinate on the u-plane and in fact is related to a, a_D at infinity by an $ISL_2(\mathbf{Z})$ transformation. (The image of this transformation in $SL_2(\mathbf{Z})$ is determined by the charge (n_e, n_m) of the hyper-multiplet (T, T') with respect to the $U(1)$ at infinity in the u-plane.) Since the $U(1)$-theory exists throughout this neighborhood even at the singularity, we see that under monodromy around u_0 the field $\tilde{a}$ comes back to itself. This implies that there is an invariant vector under the $SL_2(\mathbf{Z})$-monodromy around u_0 and hence shows that the monodromy is as written in Equation 1.6 with a plus sign.

We can generalize this picture. Suppose that at u_0 there are several hyper-multiplets of charge q_i becoming massless. Then the local form of τ would be

$$\tau = 2\sum_i q_i^2 \frac{\ln(u - u_0) + \text{constant}}{2\pi i}$$

and the monodromy around u_0 would be

$$\begin{pmatrix}1 & \sum_i 2q_i^2\\ 0 & 1\end{pmatrix}.$$

Notice that the monodromy around a singularity where a hyper-multiplet is becoming massless has trace 2 whereas the monodromy at infinity has trace -2. This means in particular, that there cannot be only one singularity in the u-plane if the singularity is of this type. (This of course also follows from the positivity of τ in the complement of the singularities.)

1.4 The number of singularities in $\mathcal{M}$

To understand the number of singularities in $\mathcal{M}$, let us begin with a microscopic $N = 2$ super-symmetric $SU(2)$-theory and break it to an $N = 1$ super-symmetric theory by adding a perturbation of the form

$$\Delta\mathcal{L} = \int d^4x d^2\theta mu + \text{c.c}, \tag{1.7}$$

with m a complex constant. (Recall that $u = \text{Tr}\phi^2$.) This theory has two vacua, each with a mass gap and with confinement.

Let us suppose that at a point in the space of quantum vacua we have N_h hyper-multiplets becoming massless, say (T_i, T_i') of charge q_i. So near this point the physics is described by a super potential

$$W = \tilde{a}\left(\sum_i q_i T_i T_i'\right) + mu(\tilde{a}).$$

where, as before $\tilde{a}$ is a local parameter on the u-plane and is related by an $SL_2(\mathbf{Z})$ transformation to a, a_D at infinity. Let us find the space of vacua in the macroscopic theory. That is to say we solve

$dW = 0$ and divide by the $\mathbf{C}^*$-action. Clearly, the equation $dW = 0$ is equivalent to

$$\tilde{a}T_i = \tilde{a}T_i' = 0 \quad \text{for all } i.$$

$$m\frac{du}{d\tilde{a}} + \sum_I q_i T_i T_i' = 0.$$

If $N_h > 0$ this gives us a positive dimensional space of vacua. This is impossible since the microscopic theory has only two vacua. This means that (as indicated above) at each singularity we have only one hyper-multiplet becoming massless. Furthermore, since $\frac{du}{d\tilde{a}} \neq 0$ we see from the second equation that $T, T' \neq 0$. By the first equation, this means that $\tilde{a} = 0$. Since there is only one hyper-multiplet, the equation becomes

$$m\frac{du}{d\tilde{a}}|_{\tilde{a}=0} + qTT' = 0.$$

This determines the vacuum uniquely up to the action of $\mathbf{C}^*$. Thus, we see that there is a unique vacuum associated to this singularity. Having one vacuum for each singularity in the macroscopic theory, and having only two vacua in the microscopic theory, it follows that there are at most two singularities in $\mathcal{M}$. Notice also that the critical points for u on $\mathcal{M}$ also give vacua in the macroscopic theory. Thus, we see

$$N_c + N_h \leq 2$$

where N_c is the number of critical points in $\mathcal{M}$ of the analytic function u and N_h is the number of points at which a hyper-multiplet becomes massless.

As we have already remarked, the monodromy around a point where a hyper-multiplet is massless and the monodromy at infinity have opposite traces and hence are not conjugate in $SL_2(\mathbf{Z})$. At a critical point of u there is no monodromy. Thus, these facts about monodromy imply that $N_h \geq 2$. It follows that $N_h = 2$ and $N_c = 0$. That is to say $u\colon \mathcal{M} \to \mathbf{C}$ is everywhere a local analytic isomorphism. Since it is an isomorphism between neighborhoods of infinity, it follows that $u\colon \mathcal{M} \to \mathbf{C}$ is an isomorphism. (Recall that we established this fact directly from the fact that u is an isomorphism at infinity. This gives a consistency check on our model.) Even though the spaces of classical and quantum vacua are identified, there are important differences. Whereas classically, there is only one singular point in the u-plane and at this singular point the $SU(2)$-theory is restored as the low energy limit, quantum mechanically there are two singularities in the u-plane (at as yet to be identified points), and at each of these a charged spin $1/2$ hyper-multiplet becomes massless. Furthermore, unlike the classical case, the monodromy in the quantum family is non-trivial in $PSL_2(\mathbf{Z})$. This implies that, unlike the classical case, the family of tori over the space of quantum vacua do not have constant j-invariant.

Notice that now that we have found two singularities in the u-plane, each contributing a vacuum when we add the perturbation in Equation 1.7. Since there are at most two vacua in the perturbed theory, it follows that there cannot be isolated points of the space of quantum vacua; these isolated points would also produce quantum vacua of the perturbed theory.

The $U(1)_R$-symmetry is broken down to a $\mathbf{Z}/8\mathbf{Z}$-symmetry which since u has charge four under $U(1)_R$ acts on the u-plane as a $\mathbf{Z}/2\mathbf{Z}$-symmetry. It is the symmetry $u \mapsto -u$. This implies that the two singularities in $\mathcal{M}$ are at points $\pm u_0$ for some $u_0 \neq 0$. As we have already seen dimensional analysis tells us that u goes like Λ^2, the square of the mass parameter in the theory. This means that we can make a choice of the mass parameter Λ so that the quantum theory labeled by Λ has singularities at $u = \pm\Lambda^2$. In this way we are adopting the precise definition of the mass scale parameter Λ promised in the introduction. Notice that this choice of Λ is different from the one that we have used until now. The only affect of this change in the choice of Λ simply is to introduce an additive constant into Formulas 1.1 through 1.5.

1.5 The new massless particles

The next step is to identify what is becoming massless. Let (n_e, n_m) be the charge of the hypermultiplet becoming massless at Λ^2 when we express things in the basis in which the monodromy at infinity is upper triangular. If $n_m = 0$, then the monodromy at Λ^2 commutes with the monodromy at infinity. Since the total space of the non-singular part of the family is the three-times punctured sphere, the monodromies at Λ^2 and ∞ generate the entire monodromy image. Thus, if the monodromy at ∞ and Λ^2 are simultaneously upper triangular, then the entire monodromy image is upper triangular and hence commutative. This would mean that $\mathrm{Im}\tau$ could not be positive everywhere. Our conclusion is that $n_m \neq 0$. That is to say, the particle becoming massless at $u = \Lambda^2$ is magnetically charged. Exactly the same analysis holds for the singularity at $-\Lambda^2$. This is a very interesting phenomenon. Particles which started life when $|u| >> \Lambda^2$ as massive solitons are becoming massless as u approaches $\pm\Lambda^2$. Also, it means that our description of the theory near $\pm\Lambda^2$ as a $U(1)$-theory is a different representation from the $U(1)$-representation of the theory near infinity in the u-plane.

Now we wish to calculate the magnetic charge $\pm n_m$ of these particles. Notice that as we come in from infinity $\pm n_m$ is well-defined independent of the path we choose from infinity to $\pm\Lambda^2$ but the electric charge depends on the path – if we wind around the pair of singularities $\pm n_e$ changes by twice the magnetic charge, since the monodromy around infinity is $n_e \mapsto -n_e + 2n_m$.

The model we are discussing today has various mathematical applications, but just in terms of physics one of its most striking applications was to give a new model of confinement. We recall that confinement is the statement that there is a linear potential between external electric charges or alternatively that there is area law decay for a Wilson loop in the an appropriate representation (in the present example, the two-dimensional representation of $SU(2)$). We recall that in a Higgs phase, where a charged field gets a vacuum expectation value[2], one can explicitly calculate by a topological argument that the 't Hooft loop gets area law decay. If one could just naively exchange electricity and magnetism, then Wilson loops would be exchanged with 't Hooft loops and we would conclude that confinement will arise if a magnetically charged field gets a vacuum expectation value. This was argued heuristically in the 1970's by 't Hooft, Mandelstam, Nambu and others but with great difficulty in exhibiting actual models, since usually one does not have magnetically charged fields in the formalism and magnetically charged objects (which arise as classical solutions) are heavy in the weakly coupled regime where one understands them concretely. Thus it was generally rather hard to see how one could actually exhibit a phenomenon in which a magnetically charged field gets a vacuum expectation value.

In the present context, this happens. We have already seen that after adding the *mu* perturbation to the superpotential, which is expected to give confinement, the vacuum near $u = \Lambda^2$ has $T, T' \neq 0$, so in fact in this vacuum (n_m being non-zero) a magnetically charged object has obtained a vacuum expectation value. Thus as expected a magnetically charged field obtained a vacuum expectation value just in the confining case. (To pursue this with greater precision, it is important that the magnetically charged field in question has n_m odd. Indeed, as the center of $SU(2)$ is $\mathbf{Z}/2\mathbf{Z}$, the explanation of confinement via a dual of the usual Higgs mechanism requires a nonzero expectation value of a field of odd n_m. We will see that n_m actually is odd, in fact ± 1, for T and T'.)

Notice something surprising has happened. The magnetic monopoles have mass at infinity on the order of $\sqrt{|u|}\ln(|u|)$ whereas there are other BPS states of masses on the order of $\sqrt{|u|}$. Thus, at infinity the monopoles are not the least massive of the massive particles. Nevertheless, they are the ones becoming massless at the singular points.

[2]This is a somewhat imprecise description as we have seen in previous lectures, but is valid in weak coupling.

1.6 Explicit nature of the family of elliptic curves

We are now ready to describe the macroscopic theory associated to the quantum theory labeled by mass parameter Λ by describing explicitly the family of elliptic curves and the two-form η associated with the space of quantum vacua of this theory. As we have seen, for this theory the moduli space of quantum vacua is $\mathbf{C}$ with parameter $u = \mathrm{Tr}\phi^2$. The $SL_2(\mathbf{Z})$-monodromy at infinity for the family is

$$\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}.$$

There are in addition two other points $\pm\Lambda^2$ in the u-plane where there are singularities. The $SL_2(\mathbf{Z})$-monodromy, $m_{\pm\Lambda^2}$ at each of these points has trace two. We write

$$m_{\Lambda^2}^{-1} = \begin{pmatrix} 1+b & -b^2/a \\ a & 1-b \end{pmatrix},$$

for integers a, b. (This is the general form of a matrix of trace two and determinant one.) Since

$$m_{-\Lambda^2} = m_{\Lambda^2}^{-1} m_\infty$$

has trace two, we conclude that

$$2 = \mathrm{Tr}\begin{pmatrix} 1+b & -b^2/a \\ a & 1-b \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}.$$

It follows by a direct computation that $a = 2$. Since $-b^2/a$ is an integer, we see that b is even. Now let us conjugate the matrix for $m_{\Lambda^2}^{-1}$ by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which commutes with the monodromy at infinity. The result is to change b by 2. Thus, after a number of these conjugations we arrive at $b = 0$. This means that in an appropriate basis the monodromy at infinity is

$$\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

and the monodromy at Λ^2 is of the form

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

Thus, we have shown that the monodromy representation is uniquely determined up to conjugation in $SL_2(\mathbf{Z})$. Notice also that according to Equation 1.6 the magnetic charge n_m of the hyper-multiplet that is becoming masses at Λ^2 is ± 1. The same of course holds at $-\Lambda^2$. In fact, because the $SL_2(\mathbf{Z})$ is uniquely determined up to conjugation it is invariant, up to conjugation, under the symmetry $u \mapsto -u$.

It is easy to identify the family of elliptic curves over the twice punctured plane with this monodromy. The base of the family can be identified with the modular curve $\mathbf{H}/\Gamma(2)$ given by dividing out the upper half-plane by the (free) action of the subgroup of $PSL_2(\mathbf{Z})$ consisting of all matrices congruent to the identity modulo two. The family is the universal family of elliptic curves with 'level two structure' over this base. Since the singularities are at $u = \pm\Lambda^2$, the Weierstrass equation for the family of elliptic curves associated to this quantum theory is

$$y^2 = (x-u)(x-\Lambda^2)(x+\Lambda^2).$$

Now let us consider the two-form η as described in the last lecture. We know that in the Weierstrass model given above η has the general form

$$\eta = f(u)du\frac{dx}{y}.$$

Implicitly in the discussions before we assumed that η was a complex symplectic form without zeros or poles at least away from the exceptional values of u. Such zeros or poles would produce non-physical singularities. If we take f to be a non-zero constant, then the Kähler form $\pi_*(\eta \wedge \overline{\eta})$ is smooth except at $\infty, \pm\Lambda^2$ where it has the correct singular behavior. Any other f would produce either poles or zeros at other points or the wrong singular behavior at one of $\infty, \pm\Lambda^2$. It follows that $\eta = Cdu\frac{dx}{y}$ for an appropriate constant which can be computed by comparing to the formula $a^2 = u$ near $u = \infty$.

Clearly,

$$\eta = d\left(C\frac{udx}{y}\right) = d\left(C\frac{2ydx}{x^2 - \Lambda^4}\right).$$

The form $\lambda = 2Cydx/(x^2 - \Lambda^4)$ is a differential of the second kind on the total space $\mathcal{E}$ of the family of elliptic curves. This means, as we have been asserting all along, that $[\eta] \in H^2(\mathcal{E}; \mathbf{C})$ is trivial and hence that the monodromy of our family of representations of the theory is contained in $SL_2(\mathbf{Z})$, inside the bigger group $ISL_2(\mathbf{Z})$. This was expected since the theory has no conserved charges except n_e, n_m.

1.7 Description of the BPS spectrum

The line in the u-plane where a/a_D is real is a simple closed curve that passes through the points $u = \pm\Lambda^2$. The spectrum of BPS states is continuous off this circle. Approaching from $u = \infty$, we see that at $u = \Lambda^2$ we have $a = 0$ and $a_D \neq 0$, whereas at $u = -\Lambda^2$ we have $-a + a_D = 0$. (Of course, the exact values depend on the path chosen from ∞ to $\pm\Lambda^2$, but if we come in along rays from infinity toward the origin then given equations hold.) Since we can approach the two singularities from infinity without crossing the jumping locus for the BPS states, we see that the BPS states that are becoming massless at $u = \pm\Lambda^2$ are indeed among the original BPS states at infinity. That is to say, these particles are analytic continuations of the original magnetic monopoles at infinity.

Now let us examine whether, at other points on the jumping locus besides $\pm\Lambda^2$, there are particles forced to become massless. As we move along the jumping line on the open upper arc from Λ^2 to $-\Lambda^2$, we find that for every $s, 0 < s < 1$ there is a point where $sa - a_D = 0$. So if there is a BPS state at infinity has charges (n_e, n_m) with $(n_e, n_m) = t(s, -1)$ for some $t \in \mathbf{R}$, then at some point along this open arc we find that this state becomes massless. This, and the symmetric argument for the open lower arc of the jumping line, show that all the BPS states at infinity that satisfy

$$(n_e, n_m) = (p, \pm q)$$

with $0 < p < q$ become massless somewhere on the jumping curve minus $\pm\Lambda^2$. But we know by our analysis of the singularities that there are no points on these open arcs where BPS states (or any other charged states) become massless. The conclusion is that there are no BPS states at infinity satisfying the above inequality. Said another way the only BPS states at infinity are those of charges

$$(n_e, n_m) = (\pm 1, 0)$$

$$(n_e, n_m) = (p, \pm 1).$$

Of course, these are exactly the BPS states that do exist at infinity. The first are the vector multiplets and the second are the magnetic monopoles. This gives us a check of the internal consistency of the theory.

This analysis of the BPS spectrum in the quantum theory can be compared in an interesting way to the following classical computation. Let $\mathcal{M}_k$ be the moduli space of k-monopole solutions of this theory, reduced by dividing by translations. It is a hyperkähler manifold of dimension $4k-4$ that has been much studied mathematically. It can be shown that L^2-holomorphic forms (possibly valued in a flat line bundle) on $\mathcal{M}_k$ would give k-monopole bound states, for $k > 1$. From what we have seen, there should be no such bound states, so there should be no L^2-holomorphic forms on $\mathcal{M}_k$. More generally, an argument by Sen using duality of $N = 4$ super Yang-Mills (rather than the $N = 2$ theory that we have considered in this lecture) determines the full L^2-cohomology of the $\mathcal{M}_k$.

Lecture II-19: $N = 2$ Super-symmetric Yang-Mills theories in dimension four: Part III, Topological Applications

Edward Witten[1]

1 A survey of $N = 2$ super-symmetric gauge theories in dimension four

Let us begin with a brief overview of the range of $N = 2$ super-symmetric gauge theories. We must specify a gauge group G which will be a compact group. We shall be doing gauge theory, which means our fields include a vector multiplet consisting of the gauge fields and their super partner fields with values in the adjoint bundle of the principal G-bundle. We can also have hypermultiplets. $N = 2$ super-symmetry forces these hypermultiplets to lie in vector bundles associated to the principal G-bundle by a representation $\rho\colon G \to Sp(n)$. This means that the associated bundles are quaternion bundles. The most important invariant for the qualitative nature of the theory is the β function, and in particular, the one-loop β-function. The formula for the one-loop β-function is

$$\beta_1(g) = \frac{-g^3}{8\pi^2}(4h - c_2(\rho)),$$

where h is the dual coxeter number of the group G and $c_2(\rho)$ is the trace of the quadratic Casimir for ρ, normalized so that the defining representation of $SU(2)$ has c_2 equal to one. Of course, $2h$ is simply c_2(adjoint), the quadratic Casimir of the adjoint representation of G on its Lie algebra. There are three possibilites:

- The one-loop β-function vanishes.
- The one-loop β-function is negative.
- The one-loop β-function is positive.

Since

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi^2 i}{g^2}$$

we have the expansion

$$\mu\frac{d}{d\mu}(\tau) = \text{constant} + \sum_{\text{ell}\geq 1} C \cdot g^{2\text{ell}} + \sum c_{n,\text{ell}} g^{2\text{ell}} \exp(2\pi i n\tau),$$

where the constant term is the computed from the one-loop β-function, the higher powers of g arise from higher loop contributions and the exponential terms come from instanton corrections (which are of course not seen in perturbation theory). By super-symmetry $\mu\frac{d}{d\mu}(\tau)$ is a holomorphic function, and hence the positive powers of g in this expansion vanish identically. Thus, if the one-loop β function vanishes then β vanishes in perturbation theory. In this case it is believed that $\beta \equiv 0$. This result is not completely clear – it is not always the case in $N = 2$ super-symmetric theories that the exponentially small terms vanish. (Recall that in the pure $SU(2)$-theory that we studied last time we found instanton corrections to τ which gave exponentially small, but non-zero, corrections to the one-loop β-function.) On the other hand, one case when the one-loop β-function does vanish is when the hypermultiplet representation ρ is the tensor product of the adjoint representation of G with the quaternions. This

[1] Notes by John Morgan

theory in fact has $N = 4$ super-symmetry. Using this enhanced super-symmetry one can show that the β-function vanishes identically. The general belief is that when the one-loop β-function vanishes then the β-function is identically zero. When $\beta \equiv 0$, the theory has a well-defined τ just like the classical theory. It is believed that in this case there is a subgroup Γ of finite index in $SL_2(\mathbf{Z})$ with the property that the theory is invariant under a faithful action of Γ. Of course, when $\beta \neq 0$ the theory has a mass scale and τ is no longer constant. For these theories one would expect, as we saw for the $U(1)$-theory in Lecture II-17, that there is not an exact duality of the theory, but rather that there is an action of the duality group on the representations of the low energy behaviour of the theory by Lagrangians.

When the one-loop β-function is negative, then the β-function is negative, at least for sufficiently high energy scales. (The pure $SU(2)$-theory we studied last time is of this type.) In this case, we have an asymptotically free theory which is a good fundamental theory. That is to say we have a well-defined *problem* (to describe the low energy effective version of the theory). We solved this problem in the last lecture for $G = SU(2)$ or $SO(3)$ with ρ being the trivial representation.

When the one-loop β-function is positive, we do not have a well-defined problem, since the theory is not asymptotically free. Rather these theories are candidate *solutions* to the problems posed by the theories with one-loop β-function negative. A theory T_{mac} with positive β-functioin theory is a candidate solution in the sense that its low energy effective theory is well-defined and infrared free. It is a solution to the problem posed by a theory, T_{mic}, with negative β, if T_{mic} flows in the infrared to T_{mac} in the sense that at long distances (or equivalently low energies) the correlation functions of the two theories converge to each other.

The gauge theories we will study today are Lagrangian models that come by quantizing classical physics. But there are more exotic $N = 2$ theories in four-dimensions, some of which come from string theory. Not all of these have Lagrangian formulations. Whether or not they have Lagrangians, these $N = 2$ supersymmetric theories should also lead to four-manifold invariants. Those without Lagrangians should lead to four-manifold invariants which do not have a classical description in terms of the moduli space of solutions to some differential equation.

2 From Minkowski space to a compact riemannian four-manifold

Let us fix one of the $N = 2$ super-symmetric asymptotically free gauge theories (one with $\beta < 0$). For example, if we wish to study Donaldson theory we will take pure $N = 2$ super-symmetric $SU(2)$-gauge theory. But for a while we can be more general. The theory, as we have discussed it so far, has been on Minkowski four-space. That is to say, we have written down an action which involves the integration over Minkowski space of a Lagrangian function of various fields on this space. We wish to pass from this to a theory defined on a compact riemannian four-manifold. This passage is carried in two steps – i) rotating from Minkowski four-space to Euclidean four space, and ii) globalizing to a compact riemannian four-manifold.

Let us describe how to Wick rotate the theory to Euclidean four-space. Our theory consists of a family of local operators with an operator product expansion (O.P.E). A realization of the theory gives values to the correlation functions of these operators at various points of Minkowski four-space. But there is an open subset of the complexification of Minkowski four-space where the O.P.E. and the correlation functions are defined by analytic continuation. This open subset contains Euclidean four-space. In this way we Wick rotate to define the theory on Euclidean four-space (see, for example, Kazhdan's lectures from last fall). In our case we are dealing with (supersymmetric) actions

$$S = \int_{M^4} \mathcal{L}$$

which are integrals of Lagrangians $\mathcal{L}$ which are analytic expressions involving fields ϕ on the underlying Minkowski space. These expressions extend to holomorphic expressions on complexified Minkowski space and then can be restricted to the Euclidean subspace to produce a Euclidean Lagrangain $\mathcal{L}_E$. The Euclidean action is defined by

$$S_E = \int_{\mathbf{R}^4} \mathcal{L}_E.$$

The path integral for a correlation function in Minkowski space is

$$\int \mathcal{D}\phi O_1(x_1) \cdots O_t(x_t) e^{iS},$$

and under this process it Wick rotates to the path integral

$$\int \mathcal{D}\phi O_1(x_1) \cdots O_t(x_t) e^{-S_E}$$

computing the correlation function in Euclidean space. (Of course, Feynmann diagram computations of path integrals are easier to do in the Eulcidean framework and this process is reversed to give answers in Minkowski space.) For more details on this see the article 'Actions and Reality' by D. Freed.

2.1 Globalizing by twisting – the case of the super-symmetric algebra

Now let us examine the question of putting the quantum field theory on an oriented riemannian four-manifold X. Our supersymmetric theory is defined over four-space by an action S which is an integral of a Lagrangian function of local fields which are spinors and vector representations of certain types under $Spin(4)$. Of course, there are bundles on Minkowski or Euclidean four-space associated to these representations. In order to write the 'same' action on a four-manifold X we need to be able to write down the supersymmetric Lagrangian on that manifold. This means that all fields that appear in the Lagrangian must be globally defined (as sections of bundles) on X. This will not in general be the case for the theory as it has been presented so far because the relevant spin bundles may not exist globally on X. In fact one sees that the 'physically untwisted' theory on a riemannian four-manifold requires that X be spin and generically breaks all supersymmetry. We have seen before (see Lecture II-??) how to deal with this problem – we twist the theory by an appropriate homomorphism of the spin group $Spin(4)$ to the R-symmetry group. As we shall see an appropriate twist will both allow us to write down the Lagrangian on any oriented riemannian four-manifold and also allow us to preserve a crucial piece of the supersymmetry.

The decomposition of $\Lambda^2 T^*\mathbf{R}^4$ into self-dual and anti-self-dual components is covered by a decomposition of the group $Spin(4)$ as $SU(2)_+ \times SU(2)_-$. Fortunately, the $N = 2$ the R-symmetry group is $U(2)_R = SU(2)_R \times U(1)_R$. (The $U(1)_R$ is often absent, e.g., if the hypermultiplet has nonzero bare mass. It is also anomalous if $\beta \neq 0$. But even so, we still have $SU(2)_R$ as an R-symmetry group.) With this R-symmetry group twists are possible. (For $N = 1$ the R-symmetry is $U(1)_R$ and there is no possibility for twisting.) Recall from the super-homework that the super-symmetric generators are right-invariant vector fields on $\mathbf{R}^{4|8}$ $\{Q^i_\alpha\}$ and $\{\overline{Q}_{j\dot{\alpha}}\}$ where $1 \leq i, j, \alpha \leq 2$. Both of these sets are acted on by $SU(2)_R$ in the standard fashion on the i, j indices. (The α and $\dot{\alpha}$ are the spinor indices for $SU(2)_+ \times SU(2)_-$.) The actions of the spin group and the $SU(2)_R$ are summarized in the following

table.

Charges	Transformation under		
	$SU(2)_+$	$SU(2)_-$	$SU(2)_R$
$\{Q^i_\alpha\}$	$\frac{1}{2}$	0	$\frac{1}{2}$
$\{\overline{Q}_{j\dot{\alpha}}\}$	0	$\frac{1}{2}$	$\frac{1}{2}$

We define $SU(2)'$ as the diagonal subgroup of $SU(2)_+ \times SU(2)_R$ under the obvious identification of these two groups. We shall be interested in the group $SU(2)' \times SU(2)_-$. If follows from the information in the table that under $SU(2)' \times SU(2)_-$ the $\{Q^i_\alpha\}$ transform in the representation

$$(\frac{1}{2} \otimes \frac{1}{2}, 0) = (0,0) \oplus (1,0).$$

This means that the twisted form of the vector fields $\{Q^i_\alpha\}$ decompose as a complex-valued vector field and a vector field with values in complex self-dual two-forms. Similarly, the $\{\overline{Q}_{j\dot{\alpha}}\}$ transform in representation

$$(\frac{1}{2}, \frac{1}{2})$$

of this same group and thus are tangent vectors, so that the $\{\overline{Q}_{j\dot{\alpha}}\}$ become vector fields with values in the complexified tangent bundle. Thus, both these representations are vector representations of $Spin(4)$ and hence exist globally on any oriented riemannian 4-manifold. Also, notice that these representations are naturally real representations (even though the original ones were only complex). This means that on the riemannian manifold we can take real fields.

In the superspace language, we are working on the split supermanifold whose even part is X and whose odd part is the parity reversed vector bundle $\wedge^0 TX \oplus TX \oplus \wedge^2_+(TX)$. On this supermanifold we have a globally defined real vector field Q defined as follows. Suppose that we have local coordinates $(x^1, \ldots, x^4)$ on an open subset of X. We let $\theta^{\mathbf{R}}$ be the natural coordinate in the odd $\wedge^0 TX$-direction and θ^i be the coordinates in the odd TX-direction determined by the x^i. We denote by $\partial_{\mathbf{R}^{\text{odd}}}$ and $\partial_{i,\text{odd}}$ be the corresponding vector fields in these odd directions. Lastly, $\partial_{i,\text{even}}$ is the usual partial derivative in the x^i-direction. We write

$$Q = \partial_{\mathbf{R}^{\text{odd}}} + \theta^i \partial_{i,\text{even}}.$$

Clearly, as we change local coordinates, this expression is invariant. This gives us a globally defined real vector field Q on the super manifold which is the globalization of the $(0,0)$ component of the twisted form of the local supersymmetry generators $\{Q^i_\alpha\}$. It is because of the presence of this one global, everywhere nonzero, super-symmetric generator satisfying $Q^2 = 0$ that we can do global topological quantum field theory on X.

We also have a globally defined one-form K on X with values in vector fields on the supermanifold derived from the twisted form of $\{\overline{Q}_{j,\dot{\alpha}}\}$. A formula for it in the same local coordinates is:

$$K = \left(\partial_{i,\text{odd}} + \theta^{\mathbf{R}} \partial_{i,\text{even}}\right) dx^i.$$

One sees directly from the formula that K is invariant under change of coordinates, and hence is a well-defined global object on X.

It is easy to deduce from the super-Poincaré algebra structure, or from the above explicit formulas, that $Q^2 = 0 = \{K, K\}$. The direct computation, or the general super-symmetric algebra structure, also gives

$$\{Q, K\} = d, \tag{2.1}$$

where d is the exterior derivative on X.

In fact, by Noether's theorem there is a local (or unintegrated) form of these equations. Working in Minkowski space and choosing a time-slice Γ, the charge $\tilde{K}$ associated to the operator K can be written as

$$\tilde{K} = \int_\Gamma S$$

for a local operator S, which is the tensor product of three-form on Minkowski space with a one-form on Minkowski space. Letting T be the stress-energy tensor (which will exist once we have managed to formula the theory on a general riemannian manifold) , the unintegrated form of Equation 2.1 is

$$\{Q, S\} = T.$$

This equation can either be viewed as an equation for Poisson bracket of charges or as an equation for the bracket of the associated vector fields (the symplectic gradients of the charges). Physicists often do not distinguish between the charge and the vector field. In Euclidean space there is the usual relation between the stress-energy tensor and the translations

$$\begin{aligned} P_\mu &= \int_\Gamma d^3x T_{\mu 0} \\ \overline{Q}_{\dot\alpha} &= \int_\Gamma d^3x S_{\dot\alpha 0} \\ \{Q^i_\alpha, \overline{Q}_{j\dot\alpha}\} &= P_{\alpha\dot\alpha}\delta^i{}_j \\ \{Q_\alpha, S_{\mu\dot\alpha}\} &= T_{\mu\alpha\dot\alpha}. \end{aligned}$$

When we pass from Minkowski space to a riemannian manifold, we can no longer view K and Q as operators on Hilbert space. Nevertheless, in this more general context the charges associated to K and Q act by Poisson bracket on the local fields to produce new local fields (or equivalently the associated vector fields act by differentiation to produce new local fields).

2.2 Topological Quantum Field Theory on X

We shall be considering expectation values of products of Q-closed operators. Since the expectation value of any Q-exact operator is trivial (see Lecture II-??), it follows that the expectation value of any product of Q-closed operators is zero provided that at least one of them is Q-exact.

The fact that $\{Q, T\} = 0$ means that as we vary the metric, the correlation functions change by the insertion of an operator which is in the image of Q. It then follows that the correlation functions we are considering are independent of the metric. The operators that we have in mind are the analogues in this more general gauge theory setting of the Donaldson classes. The independence of the metric of the correlation functions of Q-closed operators is the physics analogue of the independence of the Donaldson invariants under change of metric.

Let us recall the general scheme of generating operators with values in higher dimensional differential forms out of local operators with values in functions. Here we are generalizing from two-dimensions to four-dimensions the discussion and results of Lecture II-10.

Begin with a polynomial function P of degree d on Lie algebra $\mathfrak{g}$ of the gauge group G, invariant under the adjoint representation. Classically, this produces a Chern form $P(F_A)$ of degree $2d$ when applied to any connection A on a principal G-bundle. Applying this to the universal connection on the universal bundle over the product of the manifold X with the space $\mathcal{B}$ of gauge equivalence classes of configurations yields a closed form of degree $2d$ on $X \times \mathcal{B}$. Recall from the last two lectures that part of the $N = 2$ supersymmetric gauge multiplet is an $N = 1$ chiral multiplet which includes a scalar

field ϕ with values in the adjoint bundle. Quantum mechanically, the polynomial function P applied to ϕ produces a Q-closed local operator $\mathcal{O}_P^{(0)}$ in the theory. Applying K repeatedly yields operators $\mathcal{O}_P^{(n)} = K^n \mathcal{O}_P^{(0)}$. These are local operators with values in n-forms on X. For an n-cycle $\Sigma^n \subset X$, the operator

$$\int_{\Sigma^n} \mathcal{O}_P^{(n)}$$

is the analogue of taking the slant product of the Chern form $P(F_A)$ of the universal connection on $X \times \mathcal{B}$ with the cycle Σ^n to produce a closed form $\mu_P(\Sigma)$ of degree $2d - n$ on $\mathcal{B}$. As we shall see, the expectation value

$$\langle \prod_{i=1}^{s} \int_{\Sigma_i} \mathcal{O}_P^{(n_i)} \rangle$$

computes

$$\sum_E \int_{\mathcal{M}_E} \mu_P(\Sigma_1) \cup \cdots \cup \mu_P(\Sigma_s)$$

where E ranges over the topological types of principal G-bundles over X and where $\mathcal{M}_E \subset \mathcal{B}_E$ is the moduli space of classical solutions. This integral of course is only taken when the virtual dimension of $\mathcal{M}_E$ is equal to $\sum_{i=1}^{s}(2d - n_i)$.

From the fact that $\{Q, K\} = d$ and the fact that $\left\{Q, \mathcal{O}^{(0)}\right\} = 0$, we have

$$\left\{Q, \mathcal{O}^{(n)}\right\} = d\mathcal{O}^{(n-1)}.$$

Thus, we see that the expectation value of products these operators are topological in nature in the sense that these expectation values depend only on the homology classes of the Σ_i in the four-manifold.

The $N = 2$ supersymmetric theory that we have been discussing on $\mathbf{R}^4$ or Minkowski four-space is induced by dimensionally reducing from a theory on $\mathbf{R}^{6|8}$, i.e., an $N = 1$ supersymmetric theory in six dimensions. For more details on this, see the superhomework.

2.3 Twisting and globalizing the $N = 2$ vector multiplet

We still have work to do before we have installed the supersymmetric theories on a riemannian manifold. So far we have studied the effect of twisting on the super Euclidean group, and seen how to implement that on an oriented riemannian manifold. We must still examine the fields in the Lagrangian. In this section we study the vector multiplet.

Let us recall the basic configurations of the (untwisted) fields from an $N = 2$ vector multiplet in the $N = 1$ language. First the $N = 2$ vector multiplet decomposes as an $N = 1$ vector multiplet

$$\mathcal{A} = (A, \lambda, D)$$

and an $N = 1$ chiral multiplet

$$\Phi = (\phi, \zeta, F).$$

In the first (vector) multiplet A is a gauge field, λ is a spinor with values in the adjoint bundle and D is a real auxiliary field with values in the adjoint bundle. In the chiral multiplet ϕ is section of the complexification of the adjoint bundle, ζ is a spinor with values in the adjoint bundle, and F is a complex-valued auxiliary field with values in the adjoint bundle. Under the $SU(2)_R$-symmetry the pair (λ, ζ) transform in a two-dimensional representation denoted ψ_α^i and $\vec{D} = (D, \operatorname{Re} F, \operatorname{Im} F)$ transforms under the adjoint representation of $SU(2)_R$.

The following table summarizes the $U(1)_R$-charge of the various fields

The $N = 2$ Vector Multiplet

$U(1)_R$ charge	-2	-1	0	1	2
Field	$\overline{\phi}$	$\overline{\psi}_{\dot{\alpha}j}$	A $\vec{D}$	ψ^i_α	ϕ

Now we twist as described above using $SU(2)_R$. The $SU(2)_R$ acts trivially except on $\vec{D}$ and on the fermions, thuys these are the only fields which change character when we twist. After twisting the $\psi^i_{\dot{\alpha}}$ transform in the $(1/2, 1/2)$ representation. This means that after twisting the $\psi^i_{\dot{\alpha}}$ become a one-form with values in the adjoint representation, denoted $\psi^{(1)}$ on X. Similarly, after twisting the $\overline{\psi}_{\alpha j}$ transform in $(\frac{1}{2} \otimes \frac{1}{2}, 0)$ and hence decompose as a direct sum of a function $\psi^{(0)}$ and a self-dual two-form $\psi^{(2)}$. The auxiliary field $\vec{D}$ also transforms in $(1, 0)$ and hence becomes a self-dual two-form on X.

We see that, after twisting, all these fields exist globally on X since they have become differential forms. This means that as far as pure gauge theories are concerned, we can write the Lagrangian on any closed oriented 4-manifold perserving the supersymmetric charge Q.

Let us examine the action of Q on these fields. The fields $(A, \psi^{(1)}, \phi)$ with Q, acting as a differential, form a model of the equivariant cohomology of the gauge group action on the space of connections. To make the notation more suggestive of a differential we let δ denote the action of Q. Then we have

$$\begin{aligned} \delta A &= \psi^{(1)} \\ \delta\psi^{(1)} &= -\partial_A \phi \\ \delta\phi &= 0. \end{aligned}$$

The Q-closed operators have representatives modulo Q-exact operators using only these fields. We begin with any operator $\mathcal{O}^{(0)}_P = P(\phi)$ given by a gauge-invariant polynomial in ϕ. For example, in the $SU(2)$-case we take $P(\phi) = \mathrm{Tr}\,\phi^2$. This polynomial has $U(1)_R$-charge four and corresponds to the Donaldson class $\mu(pt)$ in the fourth cohomology. From these local operators we build operators with values in forms $\mathcal{O}^{(j)}_P = K^j \mathcal{O}^{(0)}_P$.

The action of δ on the other fields is:

$$\begin{aligned} \delta\overline{\phi} &= \psi^{(0)} \\ \delta\psi^{(0)} &= [\phi, \overline{\phi}] \\ \delta\psi^{(2)} &= F^+_A - \vec{D} \\ \delta(F^+_A - \vec{D}) &= [\phi, \psi^{(2)}]. \end{aligned}$$

Here we are following the notation that the curvature F_A is decomposed into its self-dual and anti-self-dual components $F^+_A + F^-_A$.

2.4 Twisting and Globalizing the $N = 2$ hypermultiplet

Let us turn now to the $N = 2$ hypermultiplet. We have the following table describing the nature of the fields in this $N = 2$ multiplet before twisting.

The $N = 2$ Hypermultiplet

Helicity	$-1/2$	0	$1/2$
$U(1)_R$ − charge	-1	0	1
$SU(2)_R$ − action	trivial	$1/2$	trivial
Field	$\overline{\lambda}_{\dot{\alpha}}$	M	λ_α
Type	−chirality spinor	Boson	+chirality spinor

The boson and the spinors of both chiralities in the above table all take values in the quaternionic bundle over X associated to the representation ρ.

After twisting the boson M becomes a spinor of type $(1/2, 0)$, that is to say M becomes a plus chirality spinor (a section of the plus spin bundle). The fermions λ and $\overline{\lambda}$ are unchanged under twisting since the $SU(2)_R$ acts trivially on them. Thus, λ remains a spinor of plus chirality and $\overline{\lambda}$ remains a spinor of minus chirality. Thus, in order to write the part of the Lagrangian involving the hypermultiplet we must be able to make sense of the tensor product of the spin bundles $S^{\pm}(X)$ of each chirality with the quaternionic representation ρ. (Of course, it is not necessary that the spin bundles actually exist globally, only that these tensor products exist.)

Assuming that we are in this situation, let us describe the action of the differential Q on these fields. Again denoting it by δ we get

$$\begin{aligned} \delta M &= \lambda \\ \delta\lambda &= [a, M] \\ \delta\overline{\lambda} &= \not{\partial}(M) + \text{fermions}. \end{aligned}$$

3 The general form of the high energy computations

At this point we have succeeded in defining a twisted version of our $N = 2$ super-symmetric gauge theory on any oriented riemannian four-manifold X for which the bundles $S^{\pm}(X) \otimes \rho$ are defined. In particular, we have implemented all pure gauge theories on any oriented riemannian four-manifold. Since the ultraviolet behavior of the theory is independent of the metric on the manifold and independent of any global topology, we see that if we began with a fundamental theory (one which has a negative β-function and hence is asymptotically free), then the implementation of the theory on a complete riemannian four-manifold will also be asymptotically free. Thus, it is a well-defined problem to compute the correlation functions in this theory.

We are interested in the correlation functions of the Q-closed local operators in the theory. There is a localization result to the effect that to compute these in the UV limit one does not need to integrate over the entire infinite dimensional space of fields, but rather only over the fixed points of the action of Q on the space of fields (see Lecture II-??). Another way to think about this is that the correlation functions we are computing are independent of the metric and we can compute in the limit when as the metric shrinks to zero, or equivalently by asymptotic freedom, as the coupling constant goes to zero. In the limit we are doing a classical computation over the minima of the Lagrangian – this space of minima is exactly the fixed points of Q.

Let us examine the bosonic part of the fixed point set of Q. From the equations above, ignoring the fermion components and working modulo the action of the gauge group, we see that the fixed points

of Q are given by:

$$\begin{aligned} d_A\phi &= 0 \\ 0 &= F_A^+ - \vec{D} \\ 0 &= \not{\partial}_A(M) \end{aligned}$$

Now using the equations of motion allows us to integrate out the auxiliary field $\vec{D}$ giving the equation

$$\vec{D} = \mu(M)$$

where μ is the hyperkähler moment map. Thus, the above system becomes

$$d_A\phi = 0 \tag{3.1}$$
$$F_A^+ = \mu(M) \tag{3.2}$$
$$0 = \not{\partial}_A(M) \tag{3.3}$$

Away from solutions where A is reducible, the first equation implies that $\phi = 0$, and we can ignore ϕ and this equation. At the reducible solutions however this equation is important. Notice that Equations 3.2 and 3.3 are the Seiberg-Witten equations in the more general setting where the gauge group is not required to be $U(1)$. Also, notice that if we are considering a pure $N = 2$ supersymmetric gauge theory (with no hypermultiplets), then $M = 0$ and these equations become the usual Yang-Mills equations.

What we see is that our correlation functions will localize to be integrals over supermanifolds whose underlying geometric manifold is a disjoint unoin of the moduli spaces

$$\mathcal{M}_E = \left\{ \begin{matrix} d_A\phi = 0 \\ F_A^+ = \mu(M) \\ 0 = \not{\partial}(M) \end{matrix} \right\} / (\text{Modulo gauge}),$$

where the union is taken over all the topological types E for the principal G-bundle. Recall that, assuming that X is compact, there is an index theorem which computes the virtual dimension of $\mathcal{M}_E$. It is

$$\text{virt. dim}(\mathcal{M}_E) = k\,(4h - c_2(\rho)) - \frac{1}{2}\dim(G)\,(\chi(X) + \sigma(X)) - \frac{\sigma(X)}{4}\text{quaternion dim}(\rho),$$

where k is the instanton number of the bundle E, $\chi(X)$ is the Euler characteristic of X and $\sigma(X)$ is its signature. If G is simply connected and simple then

$$k = \int_X \frac{\text{Tr}(F \wedge F)}{8\pi^2}.$$

Notice that if $\beta = 0$ then the virtual dimension of $\mathcal{M}_E$ is independent of E. But when $\beta < 0$, the virtual dimension of the $\mathcal{M}_E$ go to infinity as the instanton number of E goes to infinity, so that our localization process becomes an integral over an infinite number of finite dimensional moduli spaces.

As we have said before, we are interested in computing the expectation values of products of operators of the form

$$\int_{\Sigma_i} \mathcal{O}_P^{(k_i)}$$

for an invariant polynomial P on the Lie algebra $\mathfrak{g}$ and for surfaces Σ_i in X. In the case of pure $SU(2)$-gauge theory these operators correspond to the usual operators in Donaldson theory in the following way. Since the gauge group is $SU(2)$ the space of irreducible polynomials is generated by $\mathrm{Tr}(\phi^2) = u$. If we use this polynomial invariant to define $\mathcal{O}^{(0)}$, then this operator corresponds to what is usually denoted $\mu(\mathrm{pt})$ in Donaldson theory. That is to say, the operator corresponds to the cohomology class on the moduli space which is minus one quarter the first Pontrjagin class of the $SO(3)$-bundle over the moduli space given by the based moduli space. Furthermore, the derived operators

$$\int_\Sigma \mathcal{O}^{(k)}$$

correspond to the cohomology classes $\mu([\Sigma]) \in H^{4-k}(\mathcal{M}_E)$.

One can begin with any polynomial function P of degree d on the Lie algebra invariant under the adjoint representation to produce $\mathcal{O}_P^{(0)}$ and derived operators of $\mathcal{O}_P^{(n)}$. In general, we know from topological considerations (the Künneth formula) how to express the products of integrals of operators derived out of a decomposable invariant polynomial function in terms of the integrals of products of operators constructed from indecomposable invariant functions. Thus, in general it suffices to work exclusively with products of higher dimensional operators derived from indecomposable invariant polynomial functions on the Lie algebra. In the case of $su(2)$ there is only indecomposable invariant polynomial $\mathrm{Tr}\,\phi^2$. Hence, it suffices to compute operators derived from this one local zero dimensional operator. These are the computations we shall do when we come to Donaldson theory.

In general, to compute a correlation function of the type where $\mathcal{O}_P^{(0)}$ comes from an invariant polynomial P of degree d

$$\langle \prod_{i=1}^{s} \int_{\Sigma_i} \mathcal{O}_P^{(b_i)} \rangle$$

we must do the integration over all components of the supermanifold of solutions whose underlying geometric manifold, $\mathcal{M}_E$, satisfies

$$\text{virt. dim}(\mathcal{M}_E) = \sum_{i=1}^{s} (2d - b_i).$$

Actually, we can do slightly better in the case that G is not simply connected. We can sum over all principal G-bundles E which have a fixed isomorphism class over the two-skeleton of X. (Of course, if G is simply connected, all principal G-bundles on X have isomorphic restriction to the two-skeleton, so in that case we are summing over all E.)

As we have seen before, integration of functions over the odd directions in the supermanifold simply become integration of differential forms over the underlying geometric manifold. The dimension equation ensures that we are integrating a top dimensional form over the geometric manifold $\mathcal{M}_E$ underlying the supermanifold of solutions.

3.1 A case when $\beta = 0$

Let us continue with the case when $\beta = 0$. For simplicity, we take the gauge group G to be a connected (compact) semi-simple group. Then, since the $\mathcal{M}_E$ all have the same formal dimension, our integration is over all $\mathcal{M}_E$ at once. We have not written down the Lagrangian explicitly on the curved four-manifold, only on flat four-space. In putting the theory on a curved space we can add terms to the

action which involve the metric. Since $\beta = 0$, there is a well-defined coupling constant τ for these theories. There are two such terms that preserve the topological invariance, namely:

$$\int_X f(\tau)\mathrm{Pf}(R) + g(\tau)L(R)$$

where R is the riemann curvature tensor of the metric on the four-manifold, Pf denotes the Pfaffian, and L denotes the L-polynomial, and f, g are holomorphic functions of τ. (Since $\overline{\partial}/\partial\overline{\tau} = \{Q, \cdot\}$, any topological invariant must be a holomorphic function of τ.) These give topological terms computing universal multiples of $\chi(X)$ and $\sigma(X)$, respectively. Adding in such terms inserts a multiplicative factor in the correlation function. Even if we begin with a classical Lagrangian without these extra terms, they would be added in the renormalization process, and the coefficients with which they appear will depend on the explicit nature of that process. Thus, in the end what we are computing is of the form

$$q^{a(\tau)\chi(X)+b(\tau)\sigma(X)} \sum_k \langle \prod_{i=1}^{s} \int_{\Sigma^{b_i}} \mathcal{O}^{(b_i)} \rangle_k q^k$$

for functions a, b derived from f, g. For example, if f, g are linear functions of τ, then a, b are constants. Here, we have fixed the isomorphism class of E over the two skeleton and indicated by k the instanton number. (This action may not be an integer but its value modulo $\mathbf{Z}$ is determined by the isomorphism class of the bundles over the two-skeleton.) Also, $q = e^{2\pi i\tau}$, and the factor $q^k \cdot q^{a(\tau)\chi(X)+b(\tau)\sigma(X)}$ is the value of the $\exp(-S)$ on the component of the moduli space of classical solutions with instanton number k. Because of the proposed invariance of such a theory under a subgroup $\Gamma \subset SL_2(\mathbf{Z})$ of finite index, this result should be a modular form under Γ possibly with poles at the cusps of the quotient of the upper half-plane by Γ. We can compute this expression by showing that there is k_0 such that if $\mathcal{M}_E$ is non-empty then $k \geq k_0$. Also, if $k >> 0$ then the index of $\not{\partial}$ is negative and hence generically at least we expect no solutions to $\not{\partial}(M) = 0$ except the trivial solution $M = 0$. Thus, in this range we expect to have to do a computation over the moduli space $\mathcal{M}_E^{\text{instantons}}$ of solutions to the anti-self-dual equations. In this case we are computing

$$\int_{\mathcal{M}_E^{\text{instanton}}} \prod_i \int_{\Sigma_i} \mathcal{O}^{(b_i)} \chi(\mathcal{V})$$

where $\mathcal{V}$ is the obstruction bundle, which in this case is identified with the cokernel of $\not{\partial}\colon S^+ \otimes \rho \to S_- \otimes \rho$. One case where we can show that there is a vanishing theorem for $\not{\partial}$ for all instanton number is the case when the gauge group is $SU(2)$ and $\rho = su(2) \otimes$ (quaternions) and the manifold is a hyper-Kähler manifold. In this case we have $N = 4$ supersymmetry, and $\mathcal{V}$ is identified with the tangent bundle of the instanton moduli space. In particular, the virtual dimension of $\mathcal{M}_E$ is zero. Thus, in this case, for dimension reaons there can be no positive dimensional surfaces in the integrals. That is to say, we are computing

$$q^C \sum_k q^k \chi\left(\mathcal{M}_{E_k}\right),$$

for an appropriate constant C. Our conclusion is that this series in q is a modular form (possibly with poles at infinity), for a subgroup $\Gamma \subset SL_2(\mathbf{Z})$ of finite index. (In fact, Γ is the 2-congruent subgroup Γ_0^2.) This prediction has been compared successfully in some detail to mathematical computations [Vafa-Witten, *A strong coupling test of S-duality*, Nuclear Physics B, 431 (1994) 3-77].

3.2 Donaldson Theory

Now we are ready to specialize to Donaldson theory which is the case of pure $SU(2)$-theory, pure in the sense that the representaion ρ is trivial so that there are no hypermultiplets. As we remarked above the basic operator is $\mathcal{O}^{(0)} = u = \operatorname{Tr}\phi^2$, coming from the quadratic invariant polynomial on $su(2)$, and the derived operators $\mathcal{O}^{(i)}, i = 1, \ldots, 4$ which are to be integrated over i-cycles in X. We denote the various components of the geometric manifold underlying the supermanifold of classical solutions by $\mathcal{M}_k$ where $k \in \mathbf{Z}$ is the instanton number, i.e., the value of the second Chern class of the principal $SU(2)$-bundle. The short distance arguments above show that

$$\langle \prod_{i=1}^{s} \int_{\Sigma_i} \mathcal{O}^{(b_i)} \rangle_k = \int_{\mathcal{M}_k} \mu(\Sigma_1) \cup \cdots \cup \mu(\Sigma_s)$$

where $\mathcal{M}_k$ is the moduli space of instantons on X with second Chern class k, provided that

$$\text{virt. dim}(\mathcal{M}_k) = \sum_{i=1}^{s} (4 - b_i),$$

or equivalently that

$$8k - \frac{3}{2}(\chi(M) - \sigma(M)) = \sum_{i=1}^{s} (4 - b_i).$$

The integral on the right-hand-side is the value of the usual Donaldson polynomial invariant

$$D([\Sigma_1], \ldots, [\Sigma_s]) = \int_{\mathcal{M}_k} \mu(\Sigma_1^{b_1}) \cup \cdots \cup \mu(\Sigma_s^{b_s}).$$

4 Low Energy Computations for Donaldson thoery

From now on we concentrate exclusively on Donaldson theory. To get something interesting we must compare this high energy computation, which we have just showed limits to the usual Donaldson polynomial invariants, with a computation using the low energy effective limit of the theory. In terms of metrics, we imagine a one-parameter family of metrics $t^2 g_0$ for some fixed metric g_0. As we have already remarked, since we are commuting correlation functions of Q-cohomology classes of operators, the answers are independent of the metric. As $t \mapsto \infty$ we are recovering the short distance (or high energy) computations we sketched above and we find that the results are the Donaldson polynomial invariants. As $t \mapsto 0$ we can recover the correlation functions by computing in terms of the space of vacua of the theory.

Thus, we obtain a result of the form

$$\langle \prod_{i} \int_{\Sigma_i} \mathcal{O}^{(b_i)} \rangle = \int_{u \text{ plane}} E(\Sigma_1, \ldots, \Sigma_s).d\mu$$

for an appropriate expression E that depends on the dimensions of the operators and a measure $d\mu$ on the u-plane that depends on the four-manifold and the path of metrics going to infinity. (Recall that all our operators $\mathcal{O}^{(b_i)}$ are higher dimensional operators derived from the invariant polynomial $\operatorname{Tr}\phi^2$.) It will turn out that as long as $b_2^+(X) > 1$, that this integral over the u-plane becomes a sum of two computations, one at each of $u = \pm 1$. When $b_2^+(X) = 1$, then the answer has three contributions – discrete contributions at $u = \pm 1$ as before and an actual integral of a function over the u-plane. In

this case the terms are not topological invariants. As we vary the metric in a generic one-parameter family (which of course is all that is necessary to check independence of the metric) there will be a finite set of points where things degenerate and on the two sides of each of these points the terms will have changed. The changes at $u = \pm 1$ are the wall-crossing formulas for the Seiberg-Witten invariants whereas the sum of the changes of the three terms is the wall-crossing formula for the Donaldson polynomial invariants.

When we do the integral over the u-plane, everywhere except at $u = \pm 1$, we are computing in a pure $U(1)$-theory, but at $u = \pm 1$ we cannot ignore the hypermultiplets which become massless at these points. At $u = \pm 1$ the classical equations are

$$\begin{aligned} F_A^+ &= (M\overline{M})^+ \\ \not{\partial}(M) &= 0. \end{aligned}$$

Here A is a $U(1)$-connection on a line bundle $\det S^+ \otimes \mathcal{L}$ and M is a section of $S^+ \otimes \mathcal{L}$. Of course, we must sum over all complex line bundles. Actually, as we remarked earlier, it is not necessary that S^+ exist, only that $S^+ \otimes \mathcal{L}$ exist. Thus, we are really summing over $Spin_c$-structures on X (which form a principal homogeneous set for the group of isomorphism classes of complex line bundles on X).

The appearance of M only at $u = \pm 1$ is of course the reason that there are delta-function contributions to the measure at these points.

4.1 Operators in the effective low energy theory

In order to do the low energy effective computations we need to know the image of our operators (defined in the fundamental or high energy theory) in the low energy effective theory. In defining the low energy effective version of an operator we must integrate out all the high energy modes. This operation does not commute with taking products. Thus, while we shall see that several of the elementary operators will correspond in the obvious way, funny things will happen to products. In the high energy theory we have the operator $u = \mathrm{Tr}\,\phi^2$. We have seen that this operator flows in the low energy limit to an operator u which we know explicitly. Its expectation values are used to parameterize the vacua of the low energy theories. Of course, local operators at the point x in the high energy theory map to local operators at the same point x in the low energy theory, and hence the derivative of a local operator $\mathcal{O}$ in the high energy theory maps to derivative of the image $\overline{\mathcal{O}}$ of $\mathcal{O}$ in the low energy theory and similarly $\int_\Sigma \mathcal{O}$ in the high energy theory flows to $\int_\Sigma \overline{\mathcal{O}}$. Since the low energy theory is still $N = 2$ supersymmetric, it has the operators $\overline{Q}$ and $\overline{K}$, which are the images of the operators Q and K in the high energy theory. Thus, the image in the low energy theory of $K^n u$ is $\overline{K}^n u$. Thus, as long as Σ_1, Σ_2 are disjoint cycles in X, the operation of taking products commutes with integrating over the high energy modes. This means that the high energy operator

$$\int_{\Sigma_1} K^{n_1} u \cdot \int_{\Sigma_2} K^{n_2} u$$

goes in the low energy theory to the product of the images of the individual operator:

$$\int_{\Sigma_1} \overline{K}^{n_1} u \cdot \int_{\Sigma_2} \overline{K}^{n_2} u$$

as long as

$$\Sigma_1 \cap \Sigma_2 = \emptyset.$$

Where products will not be preserved in passing to the low energy effective theory is when the cycles intersect. In this case the general form of the answer is that the product

$$\int_{\Sigma_1} K^{n_1} u \cdot \int_{\Sigma_2} K^{n_2} u$$

of high energy operators flows in the low energy limit to an operator of the form

$$\int_{\Sigma_1} \overline{K}^{n_1} u \cdot \int_{\Sigma_2} \overline{K}^{n_2} u + \int_{\Sigma_1 \cap \Sigma_2} \overline{K}^{n_1+n_2} f(u)$$

for some function $f(u)$. If we take products of more than two operators of this form then there will be higher corrections over triple and higher intersections of the cycles. The reason for all this is that if $O_i(x) \mapsto \overline{O}_i(x)$ then

$$O_1(x)O_2(y) \mapsto \overline{O}_1(x)\overline{O}_2(y) + \delta(x-y)\tilde{O}_{12}(x)$$

for an appropriate operator $\tilde{O}(x)$ in the low energy theory. In our case if $O_i = K^{n_i}u$, then it must be the case that $\tilde{O}_{12} = \overline{K}^{n_1+n_2} f(u)$. Integrating this result for local operators gives us the formula above for the image in the low eneryg theory of the product.

Let us simplify the dicussion by supposing that $b_1 = 0$. Then the only operators that we need to consider are u, K^2u, K^4u. But there is never any problem with K^4u since it is a numerical invariant which is simply the second Chern class, so that inserting it in a correlation function simply multiples the expectation value on the bundle by the second Chern class. From now on we drop this operator from consideration. Among the remaining operators there are no triple intersections assuming, as we always can by topological invariance, that the cycles over which we integrate are in general position. Furthremore, the only double intersections are finite sets of points where a pair of two-cycles meet transversely. In this case we have that the image of product of the high energy operators

$$\int_{\Sigma_1} K^{n_1} u \cdot \int_{\Sigma_2} K^{n_2} u$$

in the low energy effective theory will be of the form

$$\int_{\Sigma_1} \overline{K}^{n_1} u \cdot \int_{\Sigma_2} \overline{K}^{n_2} u + (\Sigma_1 \cap \Sigma_2) T(u),$$

where T is a holomorphic function of u and where $\Sigma_1 \cap \Sigma_2$ denotes the algebraic intersection of Σ_1 and Σ_2. The reason that $T(u)$ is holomorphic is that since Q-closed operators map to Q-closed operators, we must have that the correction term in formula is Q-closed. On the other hand, up to Q-exact terms any Q-closed function of u is holomorphic.

4.2 Warm-up with $U(1)$-theory with a hypermultiplet

We shall now do a practice computation that will help in learning to relate the $SU(2)$ and the $U(1)$-theories. But we shall do this practice computation in a simpler theory – the $U(1)$-theory with a hypermultiplet and a minimal Lagrangain, minimal in the sense that we will omit certain couplings. We simply take the Lagrangian for the vector multiplet and hypermultiplet as we wrote them down in flat space with τ constant. We set $q = \exp(2\pi i\tau)$. The only invariant polynomials for the Lie algebra $u(1)$ are polynomials in a. We are assuming that $H^1(X) = 0$ so that the operators under discussion are the zero-dimensional operator $O^{(0)} = a$ and the two-dimensional operator $O^{(2)} = c_1(\mathcal{L})$. Once again

insertion of this latter operator simply multiplies the correlation function by the first Chern class of the line bundle, and hence can be ignored. That is to say the only operators whose expectation values we need to compute are holomorphic (or polynomial) functions $P(a)$.

Recall that in this theory the bosonic part of the hypermultiplet field M becomes after twisting a section of $S^+ \otimes \mathcal{L}$. This means that $\mathcal{L}$ should be though of as a $Spin_c$-structure on X rather than as a complex line bundle. (We shall come back to this point in the next lecture.)

Let us suppose that we wish to compute $\langle P(a)\rangle_\lambda$ for some $Spin_c$-structure λ. The high energy arguments outlined above show that this expectation value concentrates along the geometric manifold $\mathcal{M}_\lambda$ underlying the moduli space of solutions to the classical equations for the $Spin_c$-structure λ. Since the value of $\exp(-S)$ on $\mathcal{M}_\lambda$ is q^{λ^2}, we have:

$$\langle P(a)\rangle_\lambda = q^{\lambda^2} \int_{\mathcal{M}_\lambda} P(a).$$

Since a is a two-form on $\mathcal{M}_\lambda$, this integral is equal to

$$q^{\lambda^2} c_{n_\lambda} \int_{\mathcal{M}_\lambda} a^{n_\lambda},$$

where n_λ is one-half the virtual dimension of $\mathcal{M}_\lambda$ and c_{n_λ} is the coefficient of a^{n_λ} in the Taylor series for $P(a)$ at $a = 0$. Of course,

$$SW_X(\lambda) = \int_{\mathcal{M}_\lambda} a^{n_\lambda}$$

is by definition the Seiberg-Witten invariant of the manifold X evaluated on the $Spin_c$-structure λ. (By convention, all of these integrals are interpreted to be zero if n_λ is not an integer.) Thus, we have evaluated the expectation value $\langle P(a)\rangle_\lambda$ to be the coefficient of a^{n_λ} in the Taylor series for $P(a)$ times $q^{\lambda^2} SW_X(\lambda)$. It is convenient for some of the later computations to rewrite this as

$$\langle P(a)\rangle_\lambda = q^{\lambda^2} SW_X(\lambda) \cdot \mathrm{Res}_{a=0}\left(\frac{da}{a^{1+n_\lambda}} P(a)\right).$$

Summing over all $Spin_c$-structures yields

$$\langle P(a)\rangle = \sum_\lambda \langle P(a)\rangle = \sum_\lambda q^{\lambda^2} SW_X(\lambda) \cdot \mathrm{Res}_{a=0}\left(\frac{da}{a^{1+n_\lambda}} P(a)\right).$$

There is a similar formula when we allow a two-dimensional operator as well.

$$\begin{aligned}\langle P(a) \cdot \exp\left(\alpha \int_\Sigma O^{(2)}\right)\rangle &= \sum_\lambda q^{\lambda^2} \cdot \alpha^{\langle c_1(\lambda),\Sigma\rangle} \langle P(a)\rangle_\lambda \\ &= \sum_\lambda q^{\lambda^2} \cdot \alpha^{\langle c_1(\lambda),\Sigma\rangle} SW_X(\lambda) \cdot \mathrm{Res}_{a=0}\left(\frac{da}{a^{1+n_\lambda}} P(a)\right).\end{aligned}$$

Now let us generalize this $U(1)$-gauge theory computation by allowing more general terms that preserve topological invariance. The most general form for the Lagrangian on a curved manifold X, preserving topoloogical invariance, is

$$\mathcal{L}(a, A, M) = \{Q, \cdot\} + \int K^4 e(a) + \int g(a)\mathrm{Pf}(R_X) + \int h(a) L(R_X)$$

where R_X is the riemannian curvature tensor of X. By supersymmetry, topological invariance implies that the functions e, g, h must be holomorphic functions. Of course $\int \mathrm{Pf}(R_X)$ gives a multiple of the Euler characteristic whereas $\int L(R_X)$ gives a multiple of the signature of X.

The expectation value is

$$\sum_{\lambda} SW(\lambda)\mathrm{Res}_{a=0}\left\{\frac{da}{a^{1+n_\lambda}}\left(E(a)^{\lambda^2}G(a)^{\chi(X)}H(a)^{\sigma(X)}\right)\right\},$$

where $G = e^{-g}, H = e^{-h}$, and $E = \exp(-\partial^2 e/\partial a^2)$. The reason for the last equation is that since $K(a) = \psi$ and $K(\psi) = F$, we have $K(e(a)) = e'(a)\psi$ and

$$K^2(e(a)) = e''\psi \wedge \psi + e'(a)K(\psi) = e''\psi \wedge \psi + e'(a)F.$$

Continuing the differentiation, we find that

$$K^4(e(a) = e''(a)F \wedge F + \psi \wedge (\cdots).$$

In our case all the terms involving ψ vanish since $b_1(X) = 0$.

4.3 The case of $SU(2)$

Now we are ready to return to the pure $SU(2)$-theory and understand the low energy effective computation in terms of the $U(1)$-computations we just did. Suppose that we fix $\alpha \in \mathbf{C}$ and a class $\Sigma \in H^2(X; \mathbf{C})$. We wish to compute

$$\langle \exp\left(\alpha u + \beta \int_\Sigma \mathcal{O}^{(2)}\right)\rangle_{SU(2)}.$$

The general form of the answer will be

$$\langle \exp(\alpha u + \beta \int_\Sigma \mathcal{O}^{(2)}\rangle_{u=1} + \langle \exp(\alpha u + \beta \int_\Sigma \mathcal{O}^{(2)}\rangle_{u=-1} + \int_{u-\mathrm{plane}} \cdots .$$

Let us see what operators we obtain in the low energy effective theory. By our above computations of $K^2(e(a))$ we have

$$\int_\Sigma \mathcal{O}^{(2)} \mapsto \int_\Sigma K^2 u = \int_\Sigma \frac{\partial u}{\partial a}F + \int_\Sigma \frac{\partial^2 u}{\partial a^2}\psi \wedge \psi.$$

Here, this expression is valid in some region of the u-plane and we have picked one of the possibilities of the $U(1)$-variables a (remeber there is an $ISL_2(\mathbf{Z})$-family of such a) which is a local parameter in this region. Near $u = \infty$ we use the usual a but near $u = 1$ we have to use the a different a. If $b_1(X) = 0$ then we can drop the second term so that we have

$$\int_\Sigma \mathcal{O}^{(2)} \mapsto \int_\Sigma \frac{\partial u}{\partial a}F.$$

Now using the rules for multiplication of the two-dimensional operators worked out above we see

$$\exp\left(\alpha u + \beta \int_\Sigma \mathcal{O}^{(2)}\right) \mapsto \exp\left(\alpha u + \beta \frac{\partial u}{\partial a}\int_\Sigma F + \beta^2\Sigma^2 T(u)\right).$$

Now let us compute the $u = 1$ contribution to the Donaldson invariant. The a we use here is the one produced by the monopole field that is massless at $u = 1$. We translate this coordinate so that

$u = 1$ corresponds to $a = 0$. The effective low energy theory is a $U(1)$-theory and is a special case of the one we discussed in the last section with soem unkonwn universal functions e, g, h (or equivalently unknown functions E, G, H).

So, we have

$$\begin{aligned}&\langle \exp\left(\alpha u + \beta\frac{\partial u}{\partial a}(\Sigma,\lambda) + \beta^2\Sigma^2 T(u)\right)\rangle \\ &= \quad 2\sum_\lambda SW(\lambda)\mathrm{Res}_{a=0}\left\{\frac{da}{a^{1+n_\lambda}}\exp\left(\alpha u + \beta\frac{\partial u}{\partial a}(\Sigma,\lambda) + \beta^2\Sigma^2 T(u)\right)E^{\lambda^2}H^{\sigma(X)}K^{\chi(X)}\right\}.\end{aligned}$$

The factor of 2 is introduced because of the following reason. The group of gauge transformations is $SU(2)$-bundle automorphisms, but its action on the space of connections is not faithful. There is a central group of order 2 which acts trivially. In performing the path integrals in gauge theory one has to normalize by dividing out by the volume of the group of gauge transformations. The fact that this group is twice as big as the group that acts faithfully introduces a factor of two in the comparison of the path integral with the topological formulas obtained by integrating over the moduli space.

Of course, by the index theorem, the one-half the dimension of the moduli space, n_λ, is a linear function of $\lambda^2, \chi(X), \sigma(X)$.

Let us suppose that the manifold X is of simple type (Seiberg-Witten simple type). By definition this means that all the invariants $SW_X(\lambda)$ vanish except those for which $n_\lambda = 0$. Our expression then simplifies to

$$2\sum_\lambda SW_X(\lambda)\mathrm{Res}_{a=0}\frac{da}{a}\exp(\cdots)$$

where the unwritten expression is a regular expression in a. We can evaluate at $u = 1$ to obtain

$$2\sum_\lambda SW_X(\lambda)\exp\left(\alpha u|_{u=1} + (\Sigma,\lambda)\frac{du}{da}|_{u=1} + \Sigma^2 T(1)\right)\tilde{G}^{\chi(X)}\tilde{H}^{\sigma(X)}.$$

Here $\tilde{G}$ and $\tilde{H}$ are complex numbers derived from E, G, H by setting $u = 1$ and using the index theorem (recall $n_\lambda = 0$) to express λ^2 as a linear compbination of $\chi(X), \sigma(X)$. To complete determine the answer we must evaluate

$$u(1), \left.\frac{\partial u}{\partial a}\right|_{u=1}, \tilde{G}, \tilde{H}, T(1).$$

Of course, we also have the contribution at $u = -1$ which differs from this one by the involution $u \mapsto -u$.

In any event, assuming that, as we shall show in the next lecture, the integral over the u-plane does not contribute to the answer when $b_2^+(X) > 1$, in this case the Donaldson polynomial invariant of X is completely determined by the function $SW_X(\lambda)$ on $Spin_c$ structures and the explicit determination involves five universal constants that must be computed. Fortunately,there is a list of four-manifolds whose Donaldson invariants are explicitly known, and whose Seiberg-Witten function is easy to compute, elliptic surfaces. From these known answers it is an easy computation to determine the constants. In fact, there is more than enough information available to determine these constant since there are infinitely many manifolds for which both SW and the Donaldson invariants are known. This gives infinitely redundant determinations of the constants and gives then infinitely many checks on the method. All these computations produce the same values for the five unknown constants in the above discussion.

The final result is that for manifolds X with $b_2^+(X) > 1$ which are of Seiberg-Witten simple type we have:

$$\left\langle \exp\left(\sum_a \beta_a \int_{\Sigma_a} \mathcal{O}^{(2)} + \alpha \mathcal{O}^{(0)}\right)\right\rangle = 2^{1+\frac{1}{4}(7\chi+11\sigma)}\left(\exp\left(\frac{v^2}{2}+2\alpha\right)\sum_\lambda SW_X(\lambda)e^{v\cdot c_1(\lambda)} + i^\Delta \exp\left(-\frac{v^2}{2}-2\alpha\right)\sum_\lambda SW_X(\lambda)e^{-iv\cdot c_1(\lambda)}\right),$$

where $v = \sum_a \beta_a[\Sigma_a]$ and where $\Delta = (\chi(X)+\sigma(X))/4$. The first of the two terms on the right-hand-side is the contribution at $u = 1$ and the second is the contribution at $u = -1$. It can be derived from the first by using the involution.

The factor of 2 in front of the α seems to appear mysteriously, and is not consistent with the formulas that we developed because in those formulas the coefficient was always one. The reason for this descrepancy is that we made an implicit assumption throughout this discussion which now has to be corrected in order to compare with Donaldson theory. Recall that in the low energy theory there is a mass scale Λ^2 and that the monopoles become massless at $u = \pm\Lambda^2$. We have implicitly assumed that the monopoles become massless at $u = \pm 1$. But there is no reason that this value of Λ^2 should be the one that is compatible with topology. So we should do the computations with an extra unknown variable $\pm\Lambda^2$ and then determine this variable by comparison with topological computations in examples. Since $\pm\Lambda^2$ is the expectation value of u at the two special points, the way that $\pm\Lambda^2$ enters the formulas derived from the low energy computations is as a multiplicative factor in front of α (the two different signs appearing in the two different terms). It also changes the points at which we evaluate the unknown functions, but these values were not determined directly in terms of $\pm\Lambda^2$, but rather by comparing with known answers. The fact that $\pm\Lambda^2 = \pm 2$ in order for the field theory computations to give the usual Donaldson polynomial answers is a reflection of the fact that the simple type condition implies that $u^2 = 4$ in Donaldson theory.