Anomaly of Duality

Consider Maxwell theory.

'symmetric' under $\vec{E} \leftrightarrow \vec{B}$.

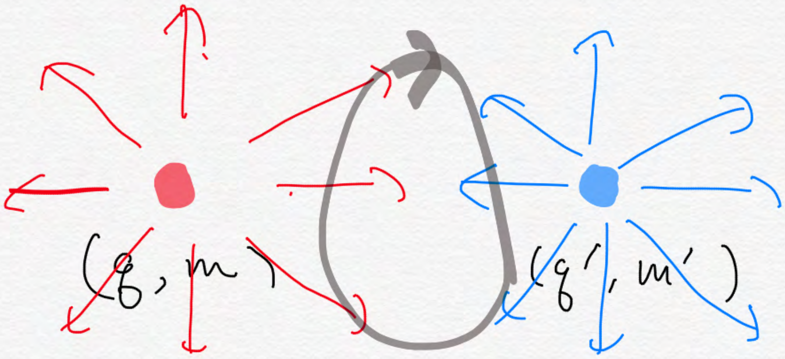
$$\mathrm{div}\,\vec{B} = 0 , \quad \mathrm{rot}\,\vec{E} + \partial_t \vec{B} = \vec{0} ,$$

$$\mathrm{div}\,\vec{E} = 0 , \quad \mathrm{rot}\,\vec{B} - \partial_t \vec{E} = \vec{0} .$$

$$S : \begin{pmatrix} E \\ B \end{pmatrix} \longmapsto \begin{pmatrix} -B \\ E \end{pmatrix}$$

$$\underset{\substack{\| \\ C}}{S^2} : \begin{pmatrix} E \\ B \end{pmatrix} \longmapsto -\begin{pmatrix} E \\ B \end{pmatrix}$$

$S^4 = \mathrm{id}$. part of $\underline{SL(2,\mathbb{Z})}$.



Poynting vec. give angular momentum

$$\frac{\hbar}{2}(qm' - q'm) = \frac{\hbar}{2} \det\begin{pmatrix} q & q' \\ m & m' \end{pmatrix} .$$

$(q, m) \in \mathbb{Z}^2$: quantized.

Transf.

$$\begin{pmatrix} q \\ m \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q \\ m \end{pmatrix}$$

should fix $\frac{\hbar}{2} \det \begin{pmatrix} q & q' \\ m & m' \end{pmatrix}$

$$\rightsquigarrow \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \quad \text{i.e.} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let's next study it using Lagrangian.

$$S = \frac{1}{g^2} \int F \wedge *F + \frac{i\theta}{2} \int \boxed{\frac{F}{2\pi} \wedge \frac{F}{2\pi}} \quad \leftarrow \text{even on spin mfd.}$$

$$= \frac{1}{2g^2} \int F_{\mu\nu} F_{\mu\nu} + \frac{i\theta}{16\pi^2} \int F_{\mu\nu} \tilde{F}_{\mu\nu}$$

$\theta \sim \theta + 2\pi$ equivalent.

T: $\theta \longmapsto \theta + 2\pi$ is the T transf.

To see the full $SL(2, \mathbb{Z})$ transf,

write down the EOM, assuming g^2 and θ position-dependent.

$$\partial_{[\mu} F_{\nu\rho]} = 0 \longrightarrow \mathrm{div}\,\vec{B} = 0.$$

$$\partial_\mu \left[\frac{4\pi}{g^2} F_{\mu\nu} + \frac{\theta}{2\pi} \tilde{F}_{\mu\nu} \right] = 0$$

$$\hookrightarrow \vec{D} := \frac{4\pi}{g^2}\vec{E} + \frac{\theta}{2\pi}\vec{B}.$$ satisfies $\mathrm{div}\,\vec{D} = 0$.

$$\int_{S^2} \vec{D}\cdot d\vec{n} = 2\pi q \qquad \int_{S^2} \vec{B}\cdot d\vec{n} = 2\pi m.$$

integers.

adiabatic change of θ keeps q

$\rightsquigarrow$ shifts $\frac{4\pi}{g^2}\vec{E}$ by $\frac{\theta}{2\pi}\vec{B}$.

$\rightsquigarrow$ shifting θ by 2π shifts $q \mapsto q+m$.

'Witten effect'

Let

$$F_D := \frac{4\pi}{g^2} F + \frac{\theta}{2\pi} *F.$$

one finds

$$\frac{1}{g^2} F\wedge *F + \frac{\theta}{8\pi^2} F\wedge F = \frac{1}{g_D^2} F_D \wedge *F_D + \frac{\theta_D}{8\pi^2} F_D \wedge F_D$$

where $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \longleftrightarrow \tau_D = \frac{4\pi i}{g_D^2} + \frac{\theta_D}{2\pi}$

satisfy $\tau_D = -\frac{1}{\tau}$.

summarizing,

$$S: \tau \longmapsto -\frac{1}{\tau}$$

$$T: \tau \longmapsto \tau + 1.$$

More generally

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ does $\tau \mapsto \dfrac{a\tau+b}{cd+d}$.

the linear fractional transformation.

Now, compute the part. func. of this free Maxwell theory on closed spin 4-mfd M_4.

Is $Z(M_4, \tau)$ invariant under S, T ?

NO! [Witten, 9505186].

$$T: \ Z(\tau+1) = Z(\tau)$$

$$S: \ Z\left(-\frac{1}{\tau}\right) = \tau^{(\chi+\sigma)/4}\,\bar{\tau}^{(\chi-\sigma)/4}\, Z(\tau)$$

where χ : Euler number $= \dfrac{1}{32\pi^2}\displaystyle\int R_{ab}\wedge R_{cd}\,\epsilon^{abcd}$

σ : signature $= \underbrace{\dfrac{1}{24\pi^2}\displaystyle\int R^a{}_b \wedge R^b{}_a}_{p_1/3}$.

$\uparrow$

multiple of 16 on spin mfd.

e.g. at $\tau = i$, fixed by S $\leftarrow$ $\mathbb{Z}_4$ symmetry.

$Z\left(-\frac{1}{\tau}\right) = Z(\tau)$, non-anomalous.

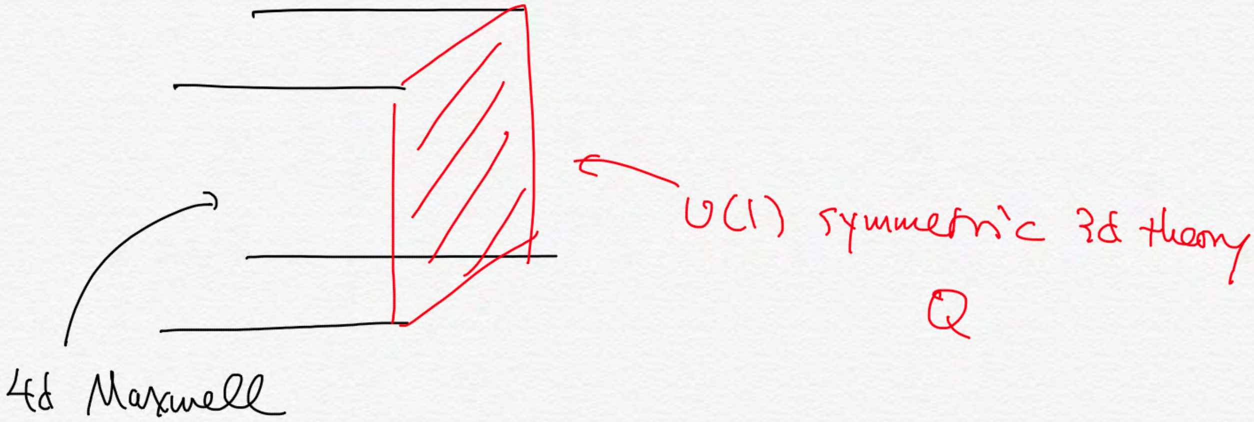
at $\tau = e^{2\pi i/3}$, fixed by ST : $\tau \to \dfrac{-1}{\tau+1}$.

$(ST)^3 = 1$ in $SL(2,\mathbb{Z})$.

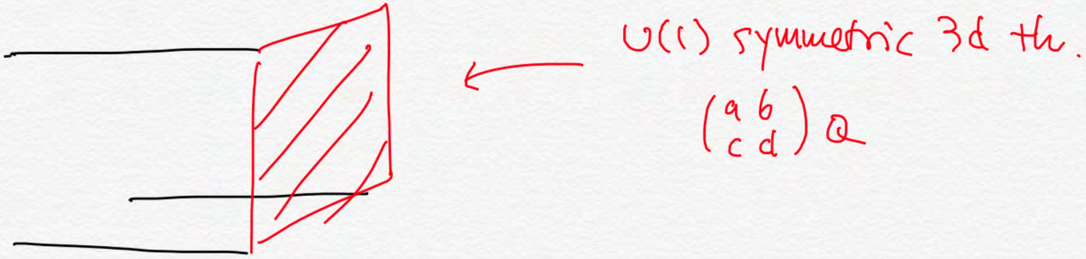
$$Z\left(\frac{-1}{\tau+1}\right) = e^{\frac{2\pi i \sigma}{12}}\, Z(\tau).$$

mixed $\mathbb{Z}_3$-gravitational anomaly!

A related phenomenon:



$$\Downarrow \text{ duality} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

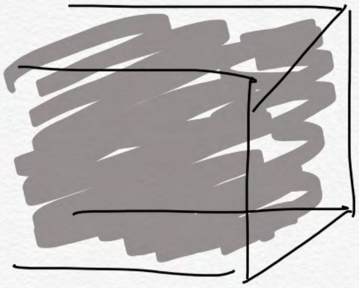


$$Z_{TQ}(A) = e^{i\pi \int \frac{A}{2\pi} \wedge \frac{F}{2\pi}} Z_Q(A)$$

partial integral.

because

performing T is to add $\frac{\theta = 2\pi}{2} \int \frac{F}{2\pi} \wedge \frac{F}{2\pi}$ here



$$Z_{SQ}(A) := \int \mathcal{D}a \; e^{2\pi i \int \frac{a}{2\pi} \wedge \frac{dA}{2\pi}} Z_Q(a).$$

$$Z_{S^2Q}(A) = \int \underline{\mathcal{D}a}\,\mathcal{D}b\; e^{2\pi i\int\left(\frac{A}{2\pi}d\frac{a}{2\pi} + \underline{\frac{a}{2\pi}d}\frac{b}{2\pi}\right.}\, Z_Q(b)\big)$$

$\uparrow$ enforces δ-functional

$A+b=0$

$$= Z_Q(-A).$$

i.e. $S^2 = \mathcal{C}$, $S^4 = 1$.

How about $(ST)^3 = 1 \in SL(2,\mathbb{Z})$?

A short computation shows

$$Z_{(ST)^3Q}(A) = e^{2\pi i\cdot\frac{1}{192\pi^2}\int \mathrm{Tr}\,\omega d\omega}\, Z_Q(A).$$

$\downarrow d$

$$\frac{1}{192\pi^2}\mathrm{tr}\,R\wedge R = \frac{p_1}{24}$$

thrice the minimal allowed grav. CS term. corresponding to $c=1$.

[Witten 0307041]

E.g. when Q is the trivial theory,

$(ST)^3Q$ is the $U(1)^3$ CS theory with the level matrix

$$K := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

this has $\det K = -1$ and therefore has no anyon.

signature is $(++-)$

$\rightsquigarrow$ equivalent to $U(1)_1$ invertible theory

$\simeq$ grav. CS for $c=1$.

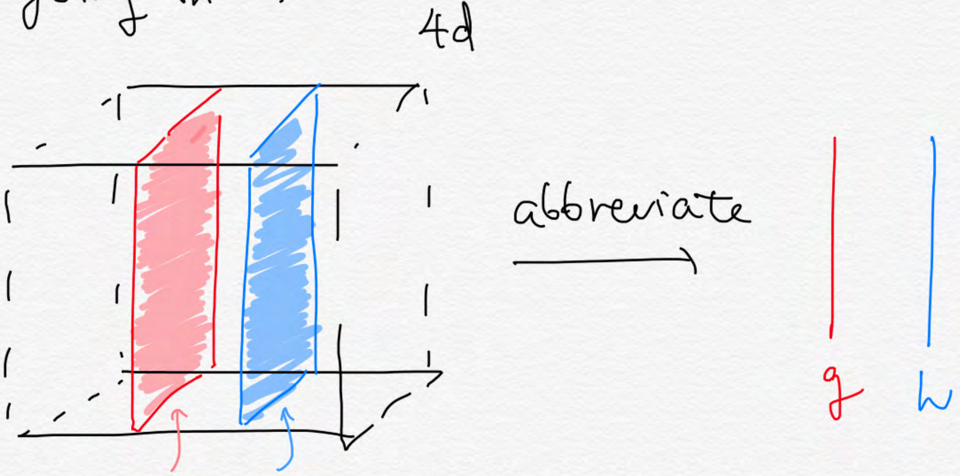
Summarizing:

the 4d Maxwell theory has
mixed $SL(2,\mathbb{Z})$ – gravitational anomaly
for $p_1 \sim \mathrm{tr} R \wedge R$.

the corresponding $SL(2,\mathbb{Z})$ action on the
U(1)-symmetric 3d theories on the boundary
is 'projective' in the sense that
the group law is satisfied up to the
multiplication by the invertible phase

$$e^{2\pi i \frac{k}{384\pi^2}\int \mathrm{tr}\,\omega d\omega} \xrightarrow{d} 2\pi i \int \frac{p_1}{48}$$

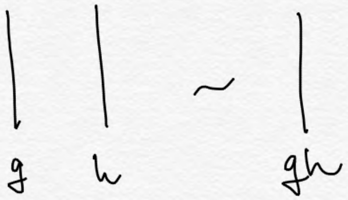
What's going on?



g h $\in G$ (= $SL(2,\mathbb{Z})$ in the discussion above)

up to 3d invertible phase.

in d dimensions. H^2(G, group of (d-1) dim'l invertible phase).



up to (d-1) -dim'l invertible phase.

generalization.

when d=1,

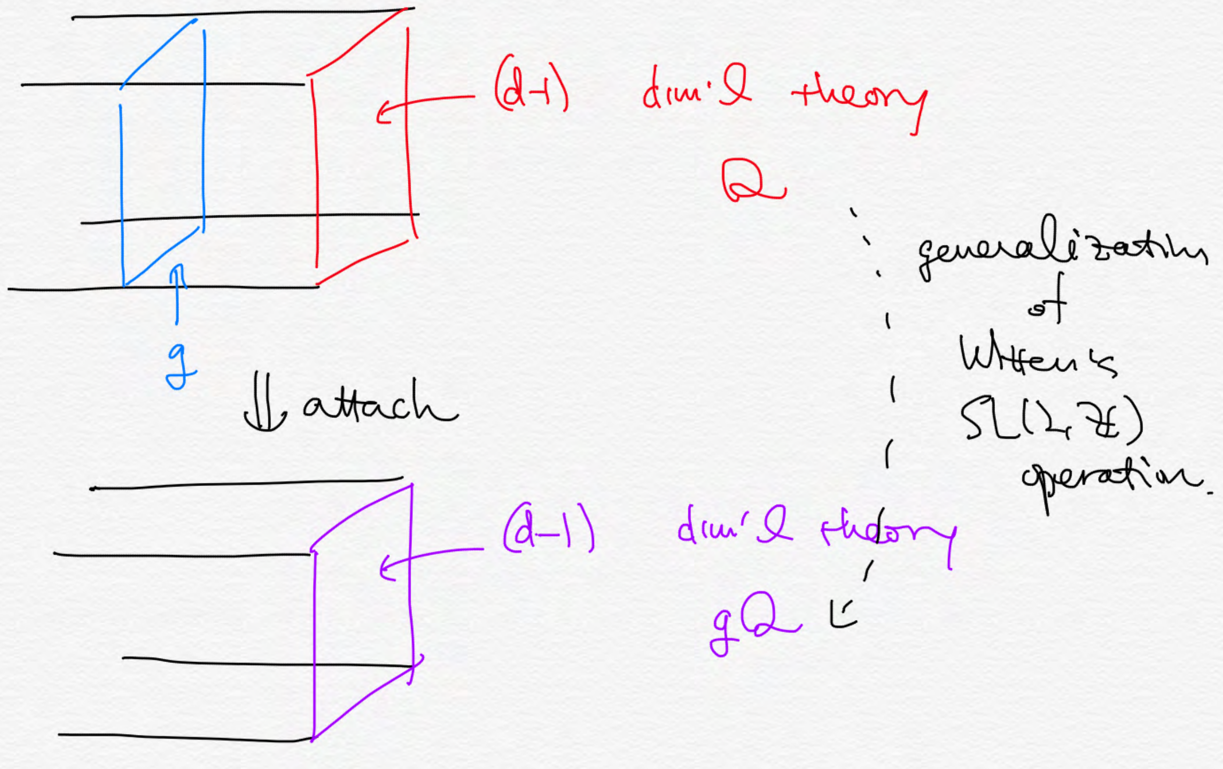
$$\overset{\bullet}{\rho}(g)\ \overset{\bullet}{\rho}(h) \sim \overset{\bullet}{\rho}(gh) \times \text{0-dim'l inv. phase}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

a complex phase.

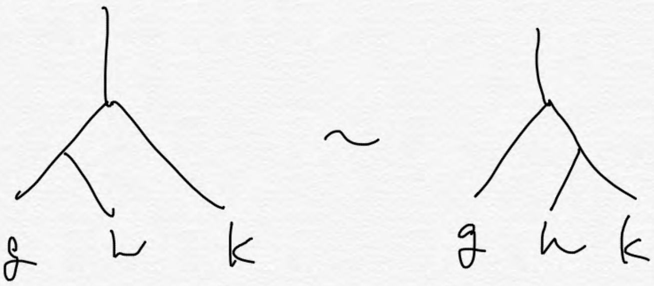
$H^2(G, U(1))$

Furthermore, this is also reflected on



the group law satisfied up to mult. by (d-1) dim'l inv. phase.

one can also have



up to (d-2)-dim'l inv. phase

characterized by $H^3(G, \text{group of } (d\text{-}2)\text{-dim'l inv. ph.}))$

In general,

$$H^p(G, \text{group of } {\color{red}q}\text{-dim'l inv ph})$$

without any sym.

with ${\color{red}p+q=d+1}$

contributes to the anomaly of d-dim'l G-sym theory

~ (d+1)-dim'l G-sym. inv. phases.

In particular, consider

$\mho^{d+1}_{spin}(BG)$: group of (d+1)-dim'l G-sym inv. phases with spin structure.

When G=pt (= point = trivial)

$d+1$	0	1	2	3	4	5
$\mho^{d+1}_{spin}$	$U(1)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	↑ phase	↑ NS/R	↑ Arf	↑ grav CS		

To the experts: it's more appropriate to set $\mho^{-1}=\mathbb{Z}$, $\mho^{0}=0$.
See Gaiotto-Johnson-Freyd 1712.07590 for details.
$\mho^0=U(1)$ works for G discrete.

$\mho^{d+1}_{spin}(BG)$ computable starting from

$H^p(G, \mho^q)$ where $p+q = d+1$

using the tool called

Atiyah-Hirzebruch Spectral Sequence
(AHSS).

In general, a SS computing groups X^d

is a sequence of tables of groups $E_r^{p,q}$

with diffeentials

$$d_r : E_r^{p,q} \longrightarrow E_r^{p-r+1, q+r}$$

s.t. $(d_r)^2 = 0$

and $$E_{r+1}^{p,q} = \frac{\mathrm{Ker}\, d_r}{\mathrm{Im}\, d_r}$$

Then $E_\infty^{p,q}$ is related to $X^{d=p+q}$ via

$$X^d =: F^d \supset F^{d-1} \supset F^{d-2} \cdots \supset F^1 \supset F^0$$

s.t $$\underset{E_\infty^{d,0}}{\overset{\shortparallel}{F^d/F^{d-1}}}, \quad \underset{E_\infty^{d-1,1}}{\overset{\shortparallel}{F^{d-1}/F^{d-2}}}, \quad , \quad \underset{E_\infty^{1,d-1}}{\overset{\shortparallel}{F^1/F^0}} \quad \underset{E_\infty^{0,d}}{\overset{\shortparallel}{F^0}}$$

AHSS for $\mho^d_{spin}(BG)$ has

$$E_2^{p,q} = H^p(G, \mho^q_{spin}).$$

non-chiral

E.G. $\mho^3_{spin}(B\mathbb{Z}_2) = \mathbb{Z} \oplus \mathbb{Z}_8$ ← # of Maj. fermions

↑ ← $c_L - c_R$

E_2 page

d_2

$$\begin{tikzcd}
q & & & & & & & \\
3 & \mathbb{Z} \arrow[rrd, "0"] & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & & \leftarrow H^p(\mathbb{Z}_2, \mathbb{Z}) \\
2 & \mathbb{Z}_2 \arrow[rrd, "0"] & \mathbb{Z}_2 \arrow[rrd, "0"] & \mathbb{Z}_2 \arrow[rrd, "id"] & \mathbb{Z}_2 & \mathbb{Z}_2 & & \leftarrow H^p(\mathbb{Z}_2, \mathbb{Z}_2) \\
1 & \mathbb{Z}_2 & \mathbb{Z}_2 \arrow[rrd, "0"] & \mathbb{Z}_2 \arrow[rrd] & \mathbb{Z}_2 \arrow[rrd, "id"] & \mathbb{Z}_2 & & \leftarrow H^p(\mathbb{Z}_2, \mathbb{Z}_2) \\
0 & U(1) & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 & & \leftarrow H^p(\mathbb{Z}_2, U(1)) \\
 & 0 & 1 & 2 & 3 & 4 & p &
\end{tikzcd}$$

invertible phases

- without additional symmetry

- d_r coming out of the $p=0$ column is known to be always zero. ($\simeq$ purely gravitational anomaly is unaffected by the addition of an independent symmetry group G.)

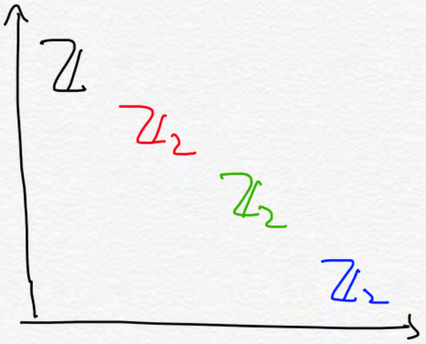
E_3 page
=
E_4
=
E_5
=
⋮

$$\begin{tikzcd}
0 & & & & \\
\mathbb{Z} & 0 & ? & & \\
\mathbb{Z}_2 \arrow[rrrdd, "0"] & \mathbb{Z}_2 \arrow[rrrdd, "0"] & 0 & 0 & \\
\mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & \\
U(1) & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0
\end{tikzcd}$$

$$\mho^3_{spin}(B\mathbb{Z}_2) = \mho^3_{spin} \oplus \tilde{\mho}^3_{spin}(B\mathbb{Z}_2)$$

↑ pure gravitational anomaly

↑ anomaly involving $\mathbb{Z}_2$



← $H^1(\mathbb{Z}_2, \mathbb{Z}_2)$

← $H^2(\mathbb{Z}_2, \mathbb{Z}_2)$

← $H^3(\mathbb{Z}_2, U(1))$

$$\mathbb{Z}_8 = \tilde{\mho}^3_{spin} = F^2 \supset F^1 \supset F^0$$

Ψ

$n \bmod 8$

$$\begin{array}{ccc} F^2/F^1 & F^1/F^0 & F^0 \\ \| & \| & \| \\ \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 = \{0 \bmod 8,\ 4 \bmod 8\} \\ \uparrow & & \uparrow \\ n \bmod 2 & & \end{array}$$

mixed $\mathbb{Z}_2$-grav anomaly

cohomological pure $\mathbb{Z}_2$ anomaly

↔ Gu-Wen phases

↔ beyond cohomology phases

Anomaly of $SL(2,\mathbb{Z})$ of Maxwell th.

$\cup$

let's concentrate on $\mathbb{Z}_3$, for simplicity.

$\boxed{E_2\text{-page}}$

$$E_2^{p,q} = H^p(\mathbb{Z}_3, \mho^q_{spin})$$

d_r entering and leaving $p+q=5$ are all zero, since entries with $p+q=4$ or 6 are all zero.

q							
5	0	0	0	0	0	0	
4	0	0	0	0	0	0	
3	$\mathbb{Z}$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$	0	
2	$\mathbb{Z}_2$	0	0	0	0	0	
1	$\mathbb{Z}_2$	0	0	0	0	0	
0	$U(1)$	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_3$	0
	0	1	2	3	4	5	p

so $\mho^5_{spin}(B\mathbb{Z}_3) = F^5 \supset F^4 \supset F^3 \supset F^2 \supset F^1 \supset F^0$

$$\begin{array}{cccccc} F^5/F^4 & F^4/F^3 & F^3/F^2 & F^2/F^1 & F^1/F^0 & F^0 \\ \parallel & \parallel & \parallel & \parallel & \parallel & \parallel \\ 0 & 0 & \mathbb{Z}_3 & 0 & 0 & \mathbb{Z}_3 \end{array}$$

i.e. $F^5 = F^4 = F^3$, $F^2 = F^1 = F^0$.

therefore

$$0 \to \mathbb{Z}_3 \to \mho^5_{spin}(B\mathbb{Z}_3) \to \mathbb{Z}_3 \to 0.$$

$\uparrow$

either $\mathbb{Z}_9$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$.

$\eta(S^5/\mathbb{Z}_3)$ can be computed as discussed before, and is $\frac{1}{9}$. i.e. $\exists$ 5d $\mathbb{Z}_3$-symmetric spin inv. phase where $Z(S^5/\mathbb{Z}_3) = e^{2\pi i/9}$.

$\rightsquigarrow$ $\Omega^5_{spin}(B\mathbb{Z}_3)$ is generated by this fermion and is $\mathbb{Z}_9$.

Typically: AHSS $\longrightarrow$ upper bound on $|\Omega(G)|$

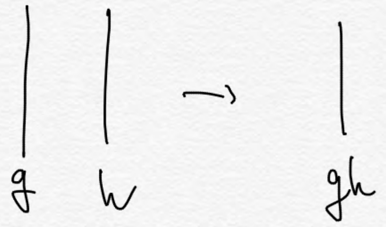
η $\longrightarrow$ lower bound on $|\Omega(G)|$

$\Uparrow$ agree for favorable cases.

$$\begin{array}{ccccccccc} \text{So: } 0 \to & \mathbb{Z}_3 & \to & \Omega^5_{spin}(B\mathbb{Z}_3) & \to & \mathbb{Z}_3 & \to 0 \\ & \| & & \| & & & \\ & \left\{\begin{array}{l} 0 \bmod 9 \\ 3 \bmod 9 \\ 6 \bmod 9 \end{array}\right\} & & \mathbb{Z}_9 & & & \\ & & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\ & & & n \bmod 9 & & n \bmod 3 & \end{array}$$

pure $\mathbb{Z}_3$

mixed $\mathbb{Z}_3$-gravitational



+ 3d inv. phase

tells us Maxwell has $2 \bmod 3 \in \mathbb{Z}_3$.

⇝ the full anomaly is

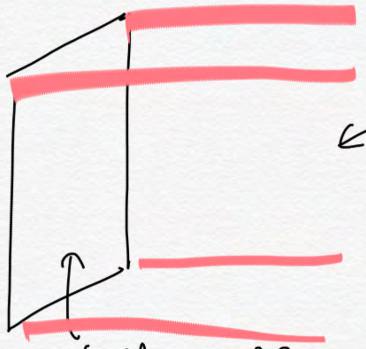
2 or 5 or 8 mod 9.

in the convention that

$\mathbb{Z}_3$-charged fermion has 1 mod 9.

It's in fact 2 mod 9 [Hsieh-YT-Yonekura]

Derivation 1



4d Maxwell

$$S = \frac{1}{g^2}\int F\wedge *F + \frac{1}{8\pi^2}\int F\wedge F$$

↻

SL(2, $\mathbb{Z}$)

2-forms

← 5d bulk : $S = \pi i \int \left[\frac{B}{2\pi} d\frac{C}{2\pi} - \frac{C}{2\pi} d\frac{B}{2\pi}\right]$

↻

SL(2, $\mathbb{Z}$) on (B, C)

↑

a bit subtle but it's a free theory.

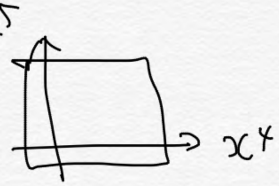
A careful evaluation on $S^5/\mathbb{Z}_3$ gives

$$Z = e^{2\pi i \times \frac{2}{9}}.$$

Derivation 2

4d Maxwell th. ⟵ 6d self-dual tensor theory on T^2.

$F_{\mu\nu}$ $H_{\mu\nu\rho} = *_{6d} H_{\mu\nu\rho}$.

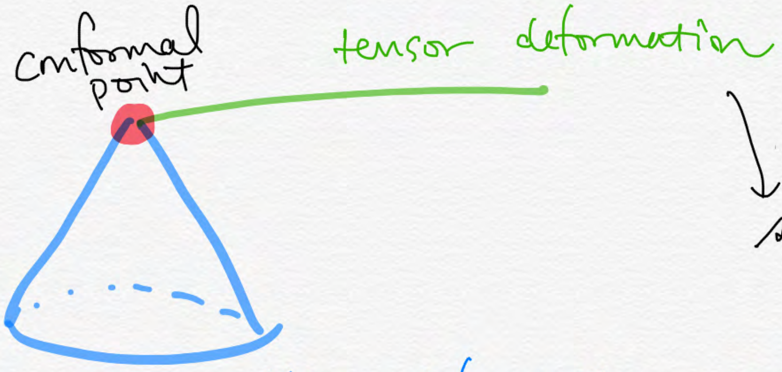
$T^2 =$ 

$$\begin{cases} F_{\mu\nu} = H_{4\mu\nu} \\ \tilde{F}_{\mu\nu} = H_{5\mu\nu} \end{cases} \longleftarrow \quad H_{4\mu\nu} = *_{4d} H_{5\mu\nu}.$$

↺ $SL(2,\mathbb{Z})$ duality

↺ coordinate change of T^2

∃ 6d superconformal theory called E-string theory : E_8 symmetry.



↓ low energy
self-dual tensor

Higgs deformation : $E_7 \subset E_8$ kept.

↘ low energy

smallest nontrivial irrep of E_7 → 56 fermions

So:

$$\begin{tikzcd} 6d & 56 \text{ fermions} \arrow[d] & \text{E-str} \arrow[l] \arrow[r] & 6d \text{ self-dual tensor} \arrow[d] \\ 4d & 56 \text{ fermions} \arrow[rr, leftrightarrow, "\text{continuously connected}", "\text{with the same anomaly.}"'] & & 4d \text{ Maxwell} \end{tikzcd}$$

56 mod 9 = 2 mod 9 .

An easier analogue :

2d chiral boson (at the free-fermion radius) $\overset{\text{equivalent}}{\longleftrightarrow}$ 2d Dirac fermion

(not just with the same anomaly)

↑ lives on the boundary of 3d $U(1)_1$ Chern-Simons

↑ lives on the boundary of 3d massive fermion

$\longleftrightarrow$ the same invertible phase.

Both are free, but the agreement is un-trivial.

e.g. they should give the same $Z(S^3/\mathbb{Z}_n)$.

We discussed long time ago how to compute

$$Z_{\text{fermion}}(S^3/\mathbb{Z}_n) = e^{2\pi i \eta(S^3/\mathbb{Z}_n)} .$$

$$\eta(S^3/\mathbb{Z}_n) = \frac{1}{2} \text{``} \sum_{E:\text{ eigenvalue of } \not{D}} \operatorname{sign}(E) \text{''}$$

$$= \frac{1}{n} \sum_{k=1}^{n-1} \frac{\alpha^k}{(\omega^{k/2} - \omega^{-k/2})^2}$$

where $\omega = e^{2\pi i/n}$,

$\alpha = \pm 1$ s.t. $\alpha^n = 1$ ← spin structure.

$\rightsquigarrow$ $\eta(S^3/\mathbb{Z}_2) = \pm\frac{1}{8}$, $\eta(S^3/\mathbb{Z}_3) = -\frac{2}{9}$ ...

How about the $U(1)_1$ Chern-Simons ?

$$Z_{U(1)CS}(M_3) = \int \mathcal{D}A \; e^{\pi i \int \frac{A}{2\pi} d \frac{A}{2\pi}}$$

$\uparrow$

very formally, analogous to $\psi \not\partial \psi$.

$$A = \underbrace{A_{\text{zero mode}}}_{\substack{\text{flat but} \\ \text{topologically nontrivial}}} + \underbrace{A_{\text{nonzero mode}}}_{\substack{\text{non-flat but} \\ \text{topologically trivial}}}$$

then

$$Z_{U(1)CS}(M_3) = \left(\int \mathcal{D}A_{\text{flat}} \; e^{\pi i \int \frac{A}{2\pi} d \frac{A}{2\pi}} \right) \swarrow ①$$

$$\times \left(\int \mathcal{D}A_{\text{nonzero mode}} \; e^{\pi i \int \frac{A}{2\pi} d \frac{A}{2\pi}} \right) \swarrow ②$$

② : computable if one knows the eigenvalues of $*d$.
essentially the eta inv. of $*d$,
but some additional factors such as

$$\int_{-\infty}^{\infty} dx \; e^{\pi i E x^2} \propto \pm\sqrt{i}$$

$$\{$$

$$② = e^{-\frac{2\pi i}{8} \eta_{sig}} , \quad \eta_{sig} = \frac{1}{n} \sum_{k=1}^{\infty} \left(\frac{1+\omega^k}{1-\omega^k} \right)^2 .$$

on $S^3/\mathbb{Z}_n$.

① : Flat bundles on $S^3/\mathbb{Z}_n$ are such that
going around $L \in \pi_1(S^3/\mathbb{Z}_n) = \mathbb{Z}_n$ gives
the phase $e^{2\pi i k/n}$, $k = 0, \cdots, n-1$.

Let A_k … the gauge connection for $k \in \mathbb{Z}_n$.

$$e^{2\pi i \int \frac{A_k}{2\pi} d \frac{A_\ell}{2\pi}} = e^{2\pi i \frac{k\ell}{n}}$$ are well defined.

But we need

$$e^{\pi i \int \frac{A_k}{2\pi} d \frac{A_k}{2\pi}} \quad : \quad \text{ill-defined !}$$

Well, there should've been some problems, otherwise $Z_{U(1)CS}$ won't depend on the spin structure.

The general mathematical problem is:

$$\boxed{\begin{array}{l} \text{given a pairing} \quad \langle a, b \rangle \quad \text{mod } 1 \\ \text{define} \quad \text{"}\frac{1}{2} a^2\text{"} . =: q(a) \end{array}}$$

Take

$$\begin{cases} q(a+b) - q(a) - q(b) = \langle a, b \rangle & \text{mod } 1 \\ q(na) = n^2 q(a) & \text{mod } 1 \end{cases}$$

be the defining equations for q.

Called the quadratic refinement of the pairing.

Then

$$① \propto \boxed{\sum_a e^{2\pi i q(a)}}$$

← the phase is known as the Arf invariant of q.

↑ up to real rescaling
i.e. LHS and RHS have the same phase

$S^3/\mathbb{Z}_2$: two flat bundles

$0, 1 \in \mathbb{Z}_2$.

$$\langle k, l \rangle = \frac{kl}{2} \mod 1.$$

$$\underbrace{q(2)}_{\overset{\shortparallel}{4q(1)}} - 2q(1) = \frac{1}{2} \mod 1$$

$$\rightsquigarrow \quad q(1) = \pm\frac{1}{4}.$$

Two sol'n

$\updownarrow$

Two spin str.

$$\rightsquigarrow \quad \sum_k e^{2\pi i q(k)} \propto e^{\pm 2\pi i \frac{1}{8}}.$$

$S^3/\mathbb{Z}_3$: three flat bundles

$0, 1, 2 \in \mathbb{Z}_3$

$$\langle k, l \rangle = \frac{kl}{3} \mod 1.$$

Uniq sol : $q(0) = 0, \quad q(1) = q(2) = -\frac{1}{3}$

$$\rightsquigarrow \quad \sum_k e^{2\pi i q(k)} \propto e^{-2\pi i \frac{1}{4}}.$$

$\updownarrow$

uniq. spin. str

In general, spin str in 3d $\Longleftrightarrow$ quadratic refinement of $\int \frac{A}{2\pi} d \frac{A'}{2\pi} \mod 1$.

Summary

	$\frac{\text{Arg}}{2\pi} Z_{\text{fermion}}$		$\frac{\text{Arg}}{2\pi} Z_{U(1) CS}$	=	non zero modes ①	+	flat ②
$S^3/\mathbb{Z}_2$	$\pm\frac{1}{8}$	=			0		$\pm\frac{1}{8}$
$S^3/\mathbb{Z}_3$	$-\frac{2}{9}$	=			$\frac{1}{8} \cdot \frac{2}{9}$		$-\frac{1}{4}$

Note that the infinitesimal variation of

$$Z_{\text{fermion}} = e^{2\pi i \eta_{\text{fermi}}}$$

and

$$Z_{U(1)CS} = \underbrace{e^{-\frac{2\pi i}{8}\eta_{sig}}}_{\text{②}} \; \underbrace{e^{2\pi i \,\mathrm{Arf}(q)}}_{\text{①}}$$

are carried solely by η invariants.

The APS index theorem says that

$$\eta_{Dirac}(M_3) - \eta_{Dirac}(M_3') = \int_{W_4} \hat{A} = -\int \frac{p_1}{24} \mod 1$$

$$\eta_{sig}(M_3) - \eta_{sig}(M_3') = \int_{W_4} L = \int \frac{p_1}{3} \mod 1$$

which give the standard perturbative anomaly of a Dirac fermion and/or a chiral boson (×8.)

To fix the constant parts requires more care, and that was what we discussed in the last few pages.

In particular. $e^{2\pi i \eta_{Dirac}}$ and $e^{2\pi i \eta_{signature}}$ give well-defined invertible phases, but $e^{2\pi i \eta_{signature}/8}$ alone does not, unless corrected by the Arf-invariant part.