Gaiotto - Kapustin - Komargodski - Seiberg

1703.00501

· used (relatively) new understanding of anomalies to get new facts in pure Yang-Mills.

② — formal / kinematical aspect

$\boxed{\text{① — dynamical aspect}}$

Consider 4d $SU(2)$ pure YM

two parameters: temperature $t$

theta angle $\theta$

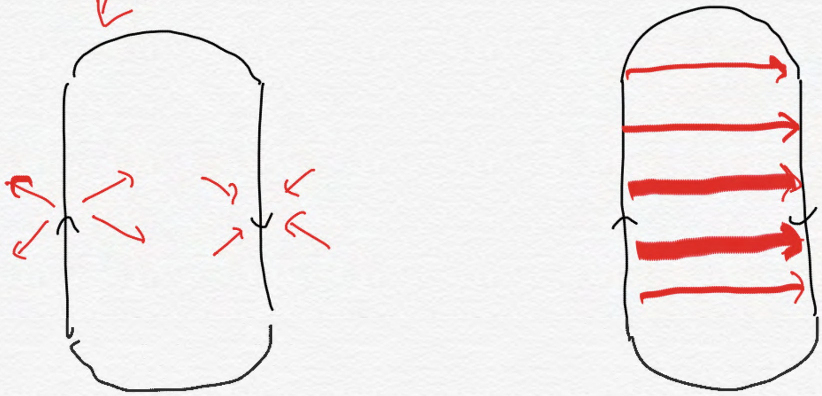
order parameters: Wilson lines

't Hooft lines

dyon lines

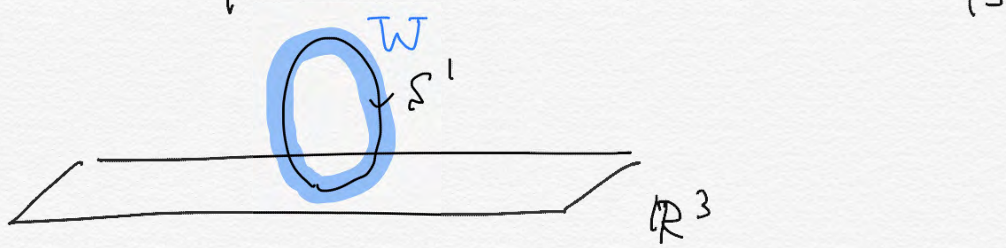
At $\theta = 0$.

| $t$ | | | |
|---|---|---|---|
| | deconfined | $\langle W \rangle$ | : perimeter law |
| | confined | $\langle W \rangle$ | : area law |
| 0 | | | |



A useful restatement:

temperature $\beta^{-1}$ $\longleftrightarrow$ thermal circle $S^1_\beta$




Wilson loop wrapped around the thermal circle

=: (Wilson) Polyakov loop.

$\uparrow$ $\beta^{-1}$

deconfined $\langle W \rangle \neq 0$ $\cancel{\mathbb{Z}_2}$

confined $\langle W \rangle = 0$ $\mathbb{Z}_2$

2nd order phase tr.

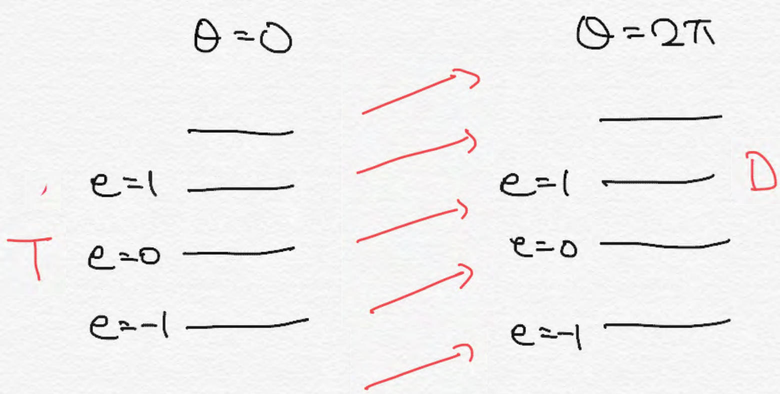
take $\beta^{-1}=0$ and vary $\theta$ instead.

Note $\theta \sim \theta + 2\pi$

spacetime parity $\theta \to -\theta$

$\rightsquigarrow$ $\theta = 0, \pi$ parity symmetric.

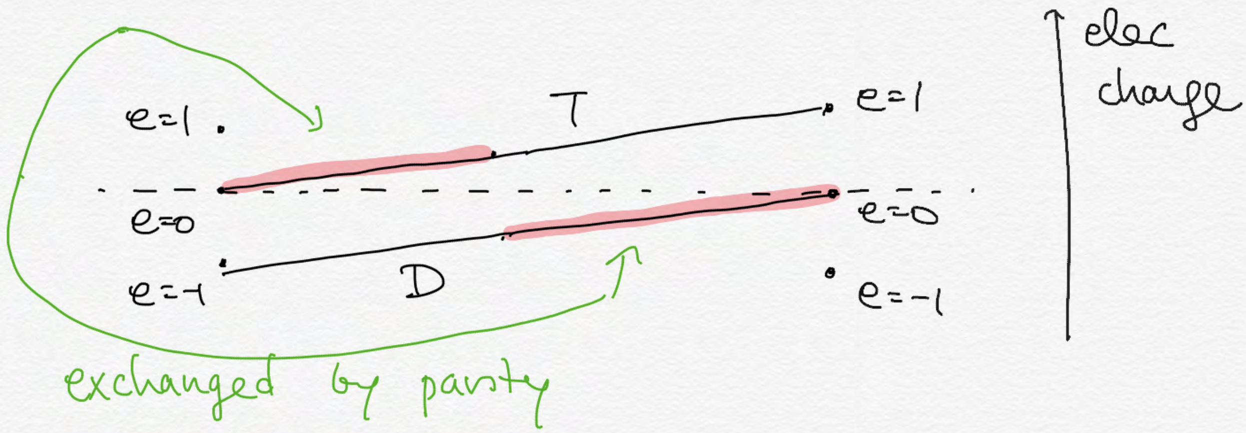
Witten effect: given a $m=1$ 't Hooft loop,



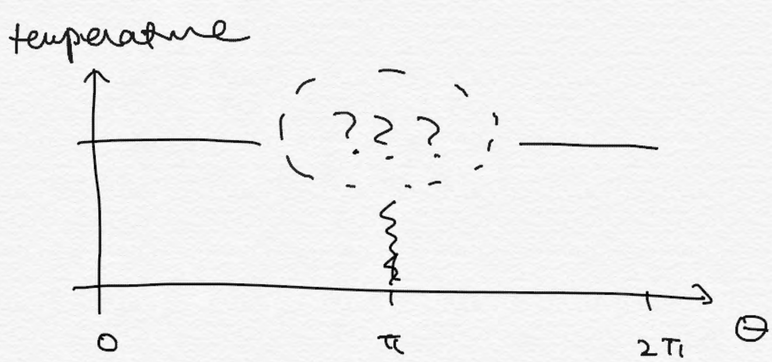

$\langle W \rangle = 0$ $\qquad$ $\langle W \rangle = 0$

$\langle T \rangle \neq 0$ $\qquad$ $\langle T \rangle = 0$

$\langle D \rangle = 0$ $\qquad$ $\langle D \rangle \neq 0$



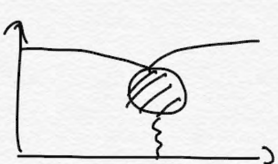

Then:




GKKS showed that

is impossible.

or



is possible.

How? Not very hard.

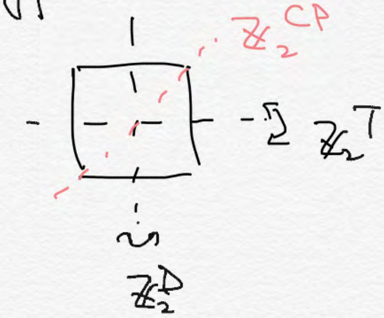
$\mathbb{Z}_2^{CP}$

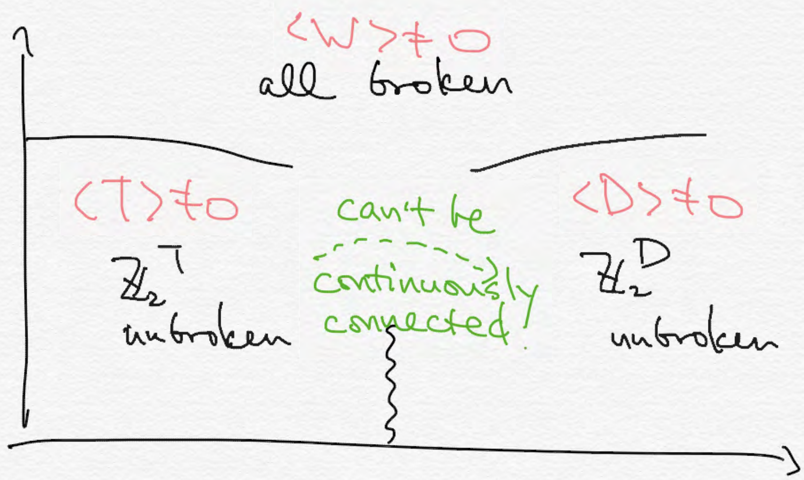
| | $\mathbb{Z}_2^T$ | $\mathbb{Z}_2^D$ | $\mathbb{Z}_2^W$ |
|---|---|---|---|
| $T$ | $+1$ | $-1$ | $-1$ |
| $D$ | $-1$ | $+1$ | $-1$ |
| $W$ | $-1$ | $-1$ | $+1$ |

$\rightsquigarrow$ symmetry group $\underbrace{\mathbb{Z}_2^{CP} \ltimes (\mathbb{Z}_2^T \times \mathbb{Z}_2^D)}$

dihedral gp of order 8

sym. of




That's the 'dynamical' part. The kinematical part answers:

- What are $\mathbb{Z}_2^T$ and $\mathbb{Z}_2^D$?
- Why dihedral?

## ② Formal aspects.

We'll ask a different-sounding question.

(We'll eventually come back to GKKS, so don't worry.)

Take a 2d theory $Q$ with $\mathbb{Z}_2$ symmetry.

We learned before that $Q/\mathbb{Z}_2$ has a new $\hat{\mathbb{Z}}_2$ sym

s.t. $Q/\mathbb{Z}_2/\hat{\mathbb{Z}}_2 = Q$.

More generally, say $Q$ has $G$ sym.

$$\begin{matrix} G \\ \cup \\ \mathbb{Z}_2 \end{matrix}$$

$Q/\mathbb{Z}_2$ has $\boxed{\hat{\mathbb{Z}}_2 \times G/\mathbb{Z}_2}$ sym.

(note: $G/\mathbb{Z}_2$ is a group only when $\mathbb{Z}_2$ is a normal subgroup.

When not normal, the sym. of $Q/\mathbb{Z}_2$ is <u>not</u> a group any more.)

$Q/\mathbb{Z}_2/\hat{\mathbb{Z}}_2$ should be $Q$ with $G$ sym.

$G = \mathbb{Z}_4$

$\hat{\mathbb{Z}}_2 \times G/\mathbb{Z}_2 = \hat{\mathbb{Z}}_2 \times \mathbb{Z}_2$

$G = \mathbb{Z}_4$

same sym group

different outcome

$G' = \mathbb{Z}_2 \times \mathbb{Z}_2$

$\hat{\mathbb{Z}}_2 \times G/\mathbb{Z}_2 = \hat{\mathbb{Z}}_2 \times \mathbb{Z}_2$

$G' = \mathbb{Z}_2 \times \mathbb{Z}_2$

The point is that the two theories in the middle has different anomalies.

Let's see why.

$$0 \to \mathbb{Z}_k \to G \to H \to 0$$

$$H = G/\mathbb{Z}_2$$

Then $G \simeq \mathbb{Z}_k \times H$ <u>as a set</u>.

write $g = (a, h)$

then $(a, h)(a', h') = (aa' e(h, h'), hh')$

associativity of $(a,h)(a',h')(a'',h'')$ demands

$$e(h,h')\,e(hh',h'') = e(h,h'h'')\,e(h',h'')$$

i.e. $e(h,h') \in \mathbb{Z}_k$ determines a cocycle

$$e \in Z^2(H, \mathbb{Z}_k) \rightsquigarrow [e] \in H^{\cdot} \quad \mathbb{Z}_k)$$

BTW you learned this first in elementary school!

$$0 \to \mathbb{Z}_{10} \to \mathbb{Z}_{100} \to \mathbb{Z}_{10} \to 0$$

$$e(h,h') = \begin{cases} 0 & \text{if } h+h' < 10 \\ 1 & \text{if } h+h' \geq 10 \end{cases}$$

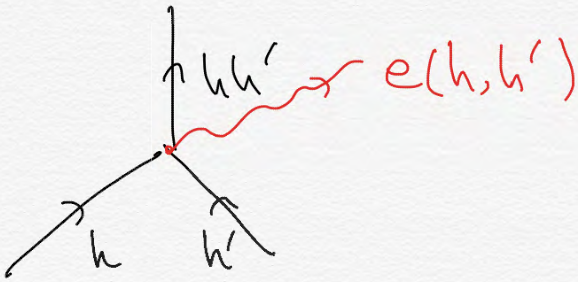
known as carry over

繰りあがり

Have you ever checked that it satisfies the cocycle condition?

$$(a,h)(a',h') = (aa'\,e(h,h'),\, hh')$$

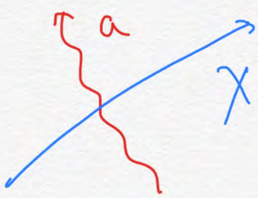
means that the $G$ walls can be expressed as $H$ walls + $\mathbb{Z}_k$ walls with the rule




Let's now gauge $\mathbb{Z}_k \rightsquigarrow \hat{\mathbb{Z}}_k$

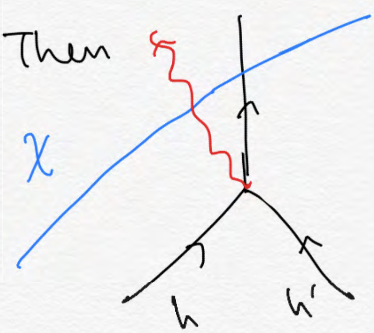
$$\begin{array}{ccc} \mathbb{Z}_k & & \hat{\mathbb{Z}}_k \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ a & & \chi_\ell \end{array} \qquad \chi_\ell(a) = e^{2\pi i a\ell/k}.$$

basic feature is

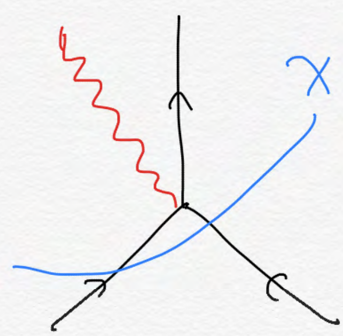



produces the phase $\chi(a)$.

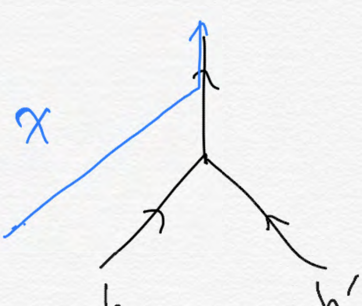
Then



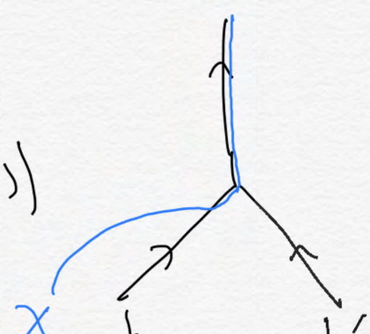

$= \chi(e(h,h'))$




i.e.




$= \chi(e(h,h'))$




in gauged theory.

D-dim'l

<u>summary</u> for 2d theory,

$Q$ : $\quad 0 \to \mathbb{Z}_k \to G \to H \to 0$

specified by $e(h,h')$ : 2-cocycle.

gauge $\hat{\mathbb{Z}}_k^{[D-2]}$ $\quad\quad$ gauge $\mathbb{Z}_k^{[0]}$

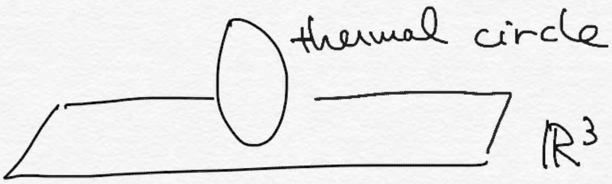
$Q/\mathbb{Z}_k$ : $\quad \hat{\mathbb{Z}}_k^{[D-2]} \times H$ with the anomaly

$$\chi(e(h,h')) \in H^{\overset{D+1}{3}}(\hat{\mathbb{Z}}_k \times H, U(1))$$

deg $D-1$ $\quad$ deg 2

( $\mathbb{Z}_k^{[p]}$ : $p$-form symmetry
acting on $p$ dim'l obj.
background field : deg $(p+1)$ cochain )

<u>Back to GKKS</u>




: effectively 3 dim'l.

$$G = \mathbb{Z}_2^{CP} \ltimes (\mathbb{Z}_2^{T} \times \mathbb{Z}_2^{D})$$
$$\cup$$
$$\mathbb{Z}_2^{\text{diag}}$$ : normal subgroup

( recall

| | $\mathbb{Z}_2^T$ | $\mathbb{Z}_2^D$ | $\mathbb{Z}_2^W$ |
|---|---|---|---|
| T | +1 | −1 | −1 |
| D | −1 | +1 | −1 |
| W | −1 | −1 | +1 |

)

a theory with $\mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$

gauge $\widehat{\mathbb{Z}_2}^{\mathrm{diag}}[1]$ ( ) gauge $\mathbb{Z}_2^{\mathrm{diag}}$

a theory with

$$\boxed{\widehat{\mathbb{Z}_2}^{\mathrm{diag}}[1]} \times \boxed{\mathbb{Z}_2^{T=D}} \times \mathbb{Z}_2^{CP}$$

with the anomaly $\int_{3d} w \cup v \cup a$

Wilson loop along uncompactified $S^1$ is charged

Wilson loop around thermal $S^1$ is charged

single obj in 4d.

Let $\int_{S^1} t = 1.$

then $w_{4d} = w + v \cdot t \longrightarrow w_{4d}^2 = w^2 + 2 \cdot w \cdot v \cdot t + (v \cdot t)^2$

$$\int_{3d} w \cup v \cup a = \int_{4d} w \cup v t \cup a$$

$$= \int \text{“}\frac{1}{2} w_{4d}^2\text{”} \cup a$$

should be the anomaly in 4d!

SU(2) gauge th.

Wilson line in the doublet rep. is

charged under 1-form $\mathbb{Z}_2$ sym.

$w \in H^2(X, \mathbb{Z}_2)$ : bkg field

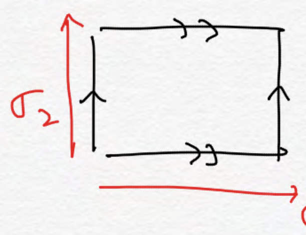
↑ measures the failure of SO(3) gauge f.

to be an SU(2) gauge f.

commute up to $-1$

in the doublet rep

e.g.




$$\sigma_1\sigma_2 = -\sigma_2\sigma_1$$

$$\begin{pmatrix} +1 & & \\ & -1 & \\ & & -1 \end{pmatrix}\begin{pmatrix} -1 & & \\ & +1 & \\ & & -1 \end{pmatrix} = \begin{pmatrix} -1 & & \\ & +1 & \\ & & -1 \end{pmatrix}\begin{pmatrix} +1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

↑ commute in the triplet rep

⇝ SO(3) gauge f. but not SU(2) gauge f.

$$\int_{T^2} w = 1 \mod 2.$$

also known as the Stiefel Whitney class.

<u>Math fact</u>

inst. number $\displaystyle\int \frac{1}{2} \operatorname{tr} \frac{F}{2\pi} \wedge \frac{F}{2\pi}$

is an integer on $\mathbb{R}^4$, but not necessarily

on more general manifold.

in particular,

on a spin mfd,

$$\#(\text{inst}) = \frac{1}{2} \underbrace{\int \frac{1}{2} w^2}_{\text{mod-2 integer}} \quad \text{mod } 1.$$

Now, the $\theta$ term contributes

$$e^{i\theta \#(\text{inst})}$$

in the path integral. At $\theta = \pm\pi$ this is

$$e^{i\pi \#(\text{inst})} = (\pm i)^{\int \frac{1}{2} w^2}$$

therefore, under CP which sends $\theta = \pi \to \theta = -\pi$,

This produces the phase ambiguity by

$$(-1)^{\int \frac{1}{2} w^2}.$$

i.e. $\exists$ mixed anomaly

$\mathbb{Z}_2$ bkg field for CP

$$\int_{5d} \frac{1}{2} w^2 \cup a$$

at $\theta = \pi$, as anticipated two pages ago.