Anomaly of 0+1 d system with

T : time-reversal

and $(-1)^F$: fermion number parity.

Two versions :

$T^2 = (-1)^F$ Pin$^+$

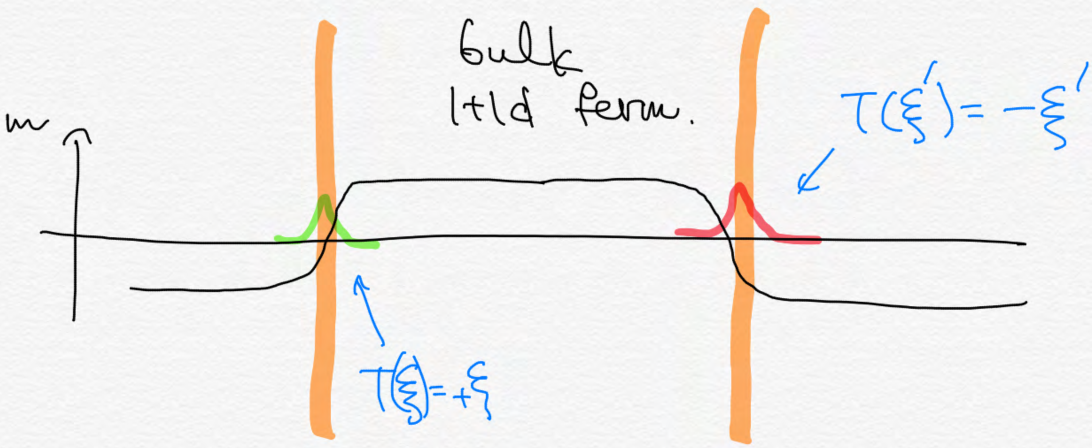
Today's focus

$\boxed{T^2 = 1 \quad \text{Pin}^-}$

simplest example:

Maj. fermion $\xi = (\xi)^\dagger$

$$T(\xi) = \pm\xi$$



ξ and ξ' realizable on a 2d Hilb. sp. $\mathcal{H}_{\xi,\xi'}$

$$\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \xi' = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

$$(-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \overline{a} \\ \overline{b} \end{pmatrix}$$

You would want to decompose

$$\mathcal{H}_{\xi,\xi'} \underset{?}{=} \mathcal{H}_{\xi} \otimes \mathcal{H}_{\xi'}$$

dim = $\sqrt{2}$? impossible!

Single $T(\xi) = \pm\xi$ $\longrightarrow$ anomalous.

$T(\xi) = \xi$ and $T(\xi') = -\xi'$ $\rightarrow$ non-anomalous.

$H = i\xi\xi' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$: time-reversal inv Hermitean.

ground state : 1-dim, T-inv, bosonic.

What if you have ξ_1, ξ_2 with both $T(\xi_i) = +\xi_i$?

$$\xi_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$(-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \bar{b} \\ \bar{a} \end{pmatrix}$$

$$\leadsto (-1)^F T = -T(-1)^F$$

anomaly!

How about $\xi'_{1,2}$ with $T(\xi'_i) = -\xi'_i$?

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\bar{b} \\ \bar{a} \end{pmatrix}$$

no T-inv Hermitean op except $\mathbb{1}$.

$$\leadsto (-1)^F T = -T(-1)^F$$

$$T^2 = -1.$$

How about $T(\xi_{i=1,2,3,4}) = +\xi_i$?

To construct it, take $T(\xi_{i=1,2}) = +\xi_i$

we already constructed, denote as $\hat{\xi}_i$.

Then

$$\xi_1 = \hat{\xi}_1 \otimes \mathbb{1} \qquad \xi_2 = \hat{\xi}_2 \otimes \mathbb{1}$$

$$\xi_3 = (-1)^{\hat{F}} \otimes \hat{\xi}_1 \qquad \xi_4 = (-1)^{\hat{F}} \otimes \hat{\xi}_2$$

$$(-1)^F = (-1)^{\hat{F}} \otimes (-1)^{\hat{F}} \qquad T = \hat{T} \otimes \left(\hat{T}(-1)^{\hat{F}}\right)$$

$\rightsquigarrow$ $T(-1)^F = (-1)^F T$ but $T^2 = -1$

Consider the Hamiltonian

$$\mathcal{H} = \xi_1 \xi_2 \xi_3 \xi_4$$

time-reversal invariant & Hermitean.

$\mathcal{H} = +1$ fermionic $T^2 = -1$

$\mathcal{H} = -1$ bosonic $T^2 = -1$

adding a large multiple of $\mathcal{H}$ to the system, equivalent to a bosonic anomalous state with $T^2 = -1$ we discussed before.

Consider bosonic $T^2=-1$ 2-state system.
'Kramers doublet'

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\bar{b} \\ \bar{a} \end{pmatrix}.$$

Hermitean op: $\mathbb{1}$, $\underbrace{\underset{\sigma_x}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}, \underset{\sigma_y}{\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}}, \underset{\sigma_z}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}}_{T\sigma_i T^{-1} = -\sigma_i}$

T-inv

T-inv Hamilt. can't make vacuum non-deg.
can't gap the system.

In the fermionic desc above,

$$\sigma_x \leftrightarrow i\xi_1\xi_2 = -i\xi_3\xi_4$$
$$\sigma_y \leftrightarrow i\xi_1\xi_3 = -i\xi_4\xi_2$$
$$\sigma_z \leftrightarrow i\xi_1\xi_4 = -i\xi_2\xi_3$$

Now take $T(\xi_i) = +\xi_i$,

$$i = 1,2,3,4,\ 5,6,7,8.$$

- can be constructed by taking two copies of 4-fermion system discussed right now.
- can check $(-1)^F T = T(-1)^F$ and $T^2 = +1$.
- another way is to introduce

$$\mathcal{H}_0 = \xi_1\xi_2\xi_3\xi_4 + \xi_5\xi_6\xi_7\xi_8$$

with a large coeff.

The ground state is now

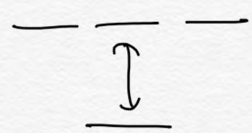
$$\left(\mathbb{C}^2 \circlearrowleft T^2=-1\right) \otimes \left(\mathbb{C}^2 \circlearrowleft T^2=-1\right)$$

as discussed before.

we then introduce

$$\mathcal{H}_1 = \sigma_x \otimes \hat{\sigma}_x + \sigma_y \otimes \hat{\sigma}_y + \sigma_z \otimes \hat{\sigma}_z$$

which is T-inv and Hermitian.



ground state is unique bosonic T-inv.

In the end we had

$$a\mathcal{H}_0 + b\mathcal{H}_1 \quad , \quad a \gg b$$

$$= a(\xi_1\xi_2\xi_3\xi_4 + \xi_5\xi_6\xi_7\xi_8)$$
$$- b(\xi_1\xi_2\xi_7\xi_8 + \xi_1\xi_3\xi_6\xi_8 + \xi_1\xi_4\xi_6\xi_7)$$

whose ground state is unique.
(There is a more symmetric choice, but this does the job for our purpose here.)

$\rightsquigarrow$ eight Maj. f. with $T(\xi_i) = +\xi_i$
is non-anomalous.

Fidkowski-Kitaev 0904.2197.

Summary

$$T(\xi_i) = +\xi_i \qquad i=1,\cdots,r$$

$$T(\xi_i') = -\xi_i' \qquad i=1,\cdots,s$$

Anomaly only depends on $\underline{r-s}$ mod $\underline{8}$.

because
$T(\xi) = \xi$
$T(\xi') = -\xi'$
can be gapped by
$H = i\xi\xi'$

because
$T(\xi_i) = \xi_i$
$i=1 \sim 8$
can be gapped
by $H = C_{ijkl}\xi_i\xi_j\xi_k\xi_l$.

Corresponding math ~~facts~~ known from long time ago.

① So far we used Hermitean ξ, ξ'.

Math prefers T-invariant

$$\gamma_a := \xi_a \quad \rightsquigarrow \quad \gamma_a^2 = +1 \qquad a=1,\cdots,r$$

$$\gamma_a' := i\xi_a' \quad \rightsquigarrow \quad \gamma_a'^2 = -1 \qquad a=1,\cdots,s$$

Known as real Clifford alg

$Cl(r,s)$.

known to show periodicity in $r-s$ mod 8.

Atiyah-Bott-Shapiro 1964

(2) More generally, consider an algebra $\mathcal{A}$ (of observables)

$$\underset{\substack{\uparrow \\ \text{bos}}}{\mathcal{A}} = \mathcal{A}_0 \oplus \mathcal{A}_1, \quad \text{with a } T\text{-action}$$

$\mathcal{A}_0$: bos, $\mathcal{A}_1$: ferm

"real graded algebra"

8 types of irreducible representations

- Whether or not $\exists\, (-1)^F$ s.t.
$$(-1)^F a_0 = a_0 (-1)^F \qquad a_i \in \mathcal{A}_i$$
$$(-1)^F a_1 = -a_1 (-1)^F$$
- $T(-1)^F = \pm (-1)^F T$
- $T^2 = \pm 1$

Tensor product of types makes the type $\in \mathbb{Z}_8$.

C.T.C. Wall, 1964

(3)

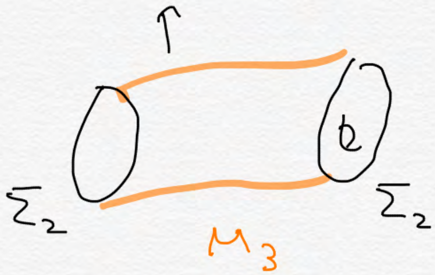
admits T-reversal
$\rightsquigarrow$ bulk can be unoriented

has fermion
$\rightsquigarrow$ bulk is $\mathrm{Pin}^{\pm}(n)$
$$\mathrm{Pin}^{\pm}(n) \xrightarrow{2:1} O(n)$$

Here we're dealing with Pin^-

anomaly classified by

$$\Omega^{pin-}_{d=2} = \mathbb{Z}_8 \leftarrow \text{generated by } \mathbb{RP}^2 = S^2/\mathbb{Z}_2 .$$



$\partial M_3 = \Sigma_2 \sqcup \overline{\Sigma_2'}$

$\Updownarrow$ def

$\Sigma_2 \sim \Sigma_2'$

Brown, "The Kervaire-invariant of a manifold" 1971

nice exposition in Kirby-Taylor 1990

④ The part. func. on $\mathbb{RP}^2$ of Massive maj. fermion = continuum limit of time-reversal-inv. Kitaev chain

$= \eta$-inv. of $\mathbb{RP}^2 = e^{2\pi i/8}$

i.e. massive fermion detects $\Omega^{pin-}_{d=2}$.

Gilkey, 1985