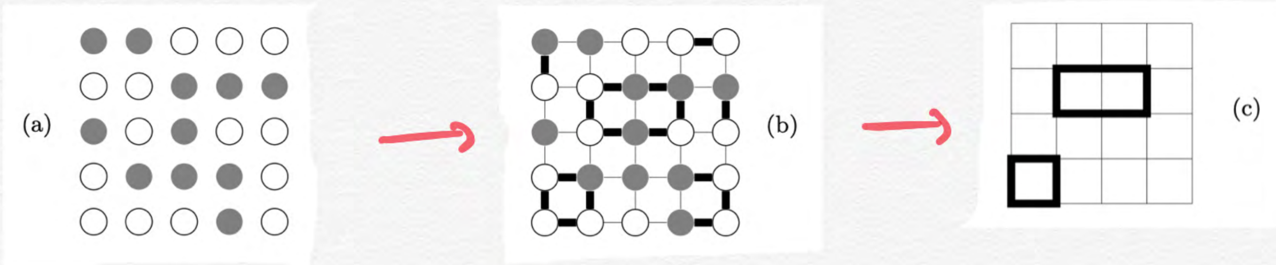
Ising Model



Start from (a)

$$Z = \sum_{\{\sigma\}} \prod_{\sigma - \sigma'} e^{K\sigma\sigma'} \quad \text{where} \quad e^{K\sigma\sigma'} = (\cosh K) \sum_{t=0,1} (\sigma\sigma' \tanh K)^t$$

introduce $\{t\}$ because $\sigma\sigma' = \pm 1$.

Therefore

$$Z = (\cosh K)^N \sum_{\{\sigma\}} \sum_{\{t\}} \prod (\sigma\sigma' \tanh K)^t$$ (b)

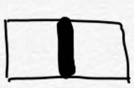
$$= (2\cosh K)^N \sum_{\partial t = 0} (\tanh K)^{L(t)}$$ (c)

sum $\{\sigma\}$ first. t forms closed path.

where $L(t)$ is the length of the closed path.

Regard t as $\mathbb{Z}_2$-walls. Introduce $\tilde{\sigma} \equiv 1$ on every plaquette.

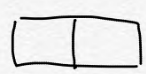
$$Z = (2\cosh K)^N \sum_{\substack{\mathbb{Z}_2\text{-wall} \\ \tilde{a}}} \prod (\tanh K)^{1/2} (\tanh K)^{-\frac{1}{2}\tilde{\sigma}\tilde{a}\tilde{\sigma}'}$$



$\tilde{a} = -1$

$\tilde{\sigma} = 1 \qquad \tilde{\sigma}' = 1$

Sum over gauge equiv classes



$\tilde{a} = +1$

$\tilde{\sigma} = 1 \qquad \tilde{\sigma}' = 1$

$$= (\sinh 2K)^{\frac{N}{2}} \boxed{\sum_{\tilde{a}} \sum_{\tilde{\sigma}}} e^{\tilde{K}\tilde{\sigma}\tilde{a}\tilde{\sigma}'} \quad \text{where} \quad e^{\tilde{K}} = (\tanh K)^{-\frac{1}{2}}.$$

$$\sinh 2K \sinh 2\tilde{K} = 1.$$

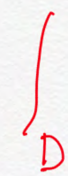
Abstractly.

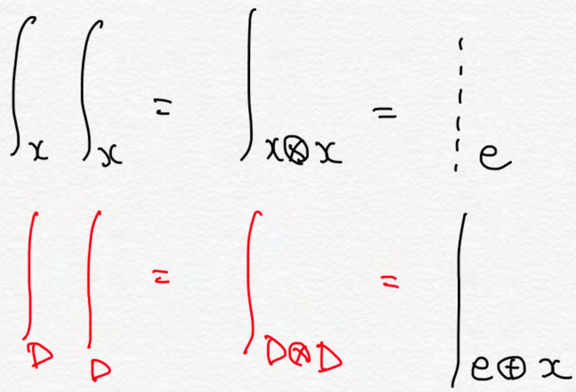
Two basic operations:

$\boxed{\text{Spin flip}}$ $\sigma_I \mapsto -\sigma_I$ denote by x.

$\boxed{\text{Kramers-Wannier duality}}$ $\sigma_I \mapsto \tilde{\sigma}_I$

denote by D.





$$\boxed{\begin{aligned} x\otimes x &= e \\ D\otimes x &= D \\ D\otimes D &= e\oplus x \end{aligned}}$$

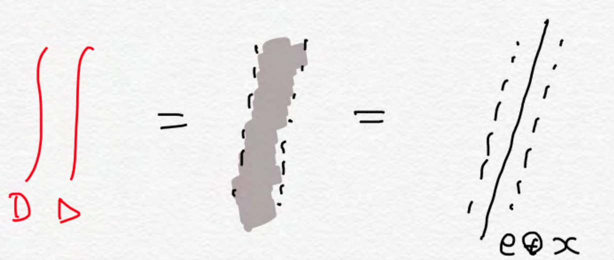
Why?

KW duality is in fact gauging $\mathbb{Z}_2$.

Then

fine mesh of $\mathbb{Z}_2$-walls, summed over.

Then



$\mathbb{Z}_2$ gauged only between two D lines.

Another way to derive it:

the IR limit is the $c=\frac{1}{2}$ minimal model.

Three primaries $\quad 1, \quad \sigma, \quad \varepsilon,$

$$L_0 = \quad 0, \quad \frac{1}{16}, \quad \frac{1}{2}$$

with the fusion rule:

$$\varepsilon \otimes \varepsilon = 1$$

$$\sigma \otimes \varepsilon = \sigma$$

$$\sigma \otimes \sigma = 1 \oplus \varepsilon$$

The corresponding Verlinde line operators are

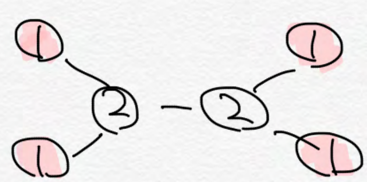
$$\varepsilon \leftrightarrow x \qquad \sigma \leftrightarrow D$$

A natural generalization:

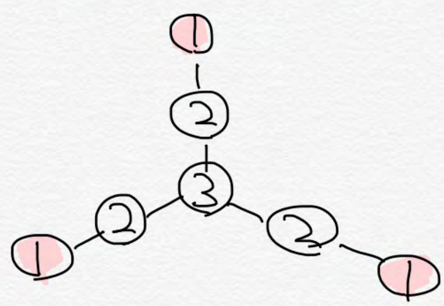
2d $\mathrm{Spin}(n)_1$ WZW model, $\quad c = \frac{n}{2}$

G_1 WZW model: primaries correspond to nodes of extended Dynkin diagram with label 1.

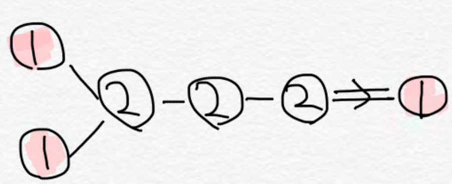
e.g. $\quad D_5$



E_6

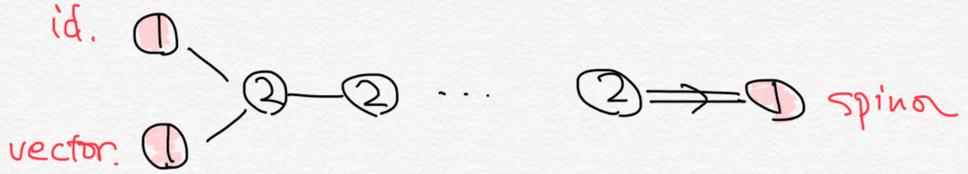


B_4



Fusion rule ~ tensor prod. of corresponding rep. of fin. dim. Lie group G.

$Spin(n)_1$, n: odd



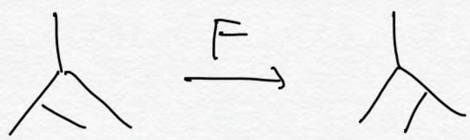
Fusion rule:

vec $\otimes$ vec $=$ id

vec $\otimes$ spinor $=$ spinor

spinor $\otimes$ spinor $=$ vec $\otimes$ vec.

the same as the Ising fusion rule.

F-symbols  are known to

depend on n mod 8.

$\exists$ exactly two choices of compatible F symbols with Ising fusion rule.

$Spin(n)_1$ with $n \equiv 1, 3, 5, 7$ (In give the same F)

provides standard realizations. Ising: $n=1$ limit.

———— . ———— . ———— . ————

$\mathbb{Z}_2$ symmetry generated by vec

$=$ center $-1 \in Spin(n)$ $\qquad c = \frac{n}{2}$.

can be 'fermionized' as we saw before.

It's just a theory of n Maj. fermions $\psi_I, \tilde\psi_I$

$I = 1, 2, \cdots, n$.

Clearly $SO(n)$ acts on it ;
the level is easily computed to be 1.

It's in fact a version of $SO(n)_1$

where $g : M_2 \to SO(n)$

$\rightsquigarrow$

$w(g) \in H^1(M_2, \mathbb{Z}_2)$: obs. to lifting to $M_2 \to Spin(n)$

$\rightsquigarrow$

$(-1)^{z(w(g))}$: spin str. around the Poincaré dual of $w(g)$

added to the exponentiated action

take $\boxed{n : \text{odd}.}$

Spin$(n)_1$ WZW — n Maj. fermions

bosonic theory. — fermionic theory.

$\int_x$ $\mathbb{Z}_2$ line $\longleftrightarrow$ $\int$ spin structure

$\int_D$ duality line $\longleftrightarrow$ $\int$ chiral $\mathbb{Z}_2$ sending $(\psi, \tilde\psi) \mapsto (\psi, -\tilde\psi)$

why ?

For the Ising $\longleftrightarrow$ Majorana case,

the deformation away from criticality

is the mass term $m\psi\tilde{\psi}$.

gets inverted by KW duality D

and also by chiral $\mathbb{Z}_2$.

Note: $D \otimes D = e \oplus x$, $\quad x \otimes x = e$

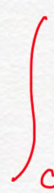
qdim $\sqrt{2} \times \sqrt{2} = 1 + 1$ $\quad 1 \times 1 = 1$

while in the fermionic description

chiral $\mathbb{Z}_2$ is anomalous: Kitaev chain

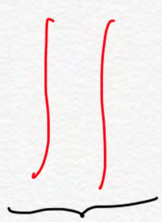
$$\frac{Z_{\text{fermion}}(m > 0)}{Z_{\text{fermion}}(m < 0)} = (-1)^{\overset{\text{Kitaev chain}}{\overset{\shortparallel}{\mathrm{Arf}(\sigma)}}}$$

so $\exists$ Maj. zero mode on



chiral $\mathbb{Z}_2$.

then



supports two Maj. zero modes ψ, ψ'

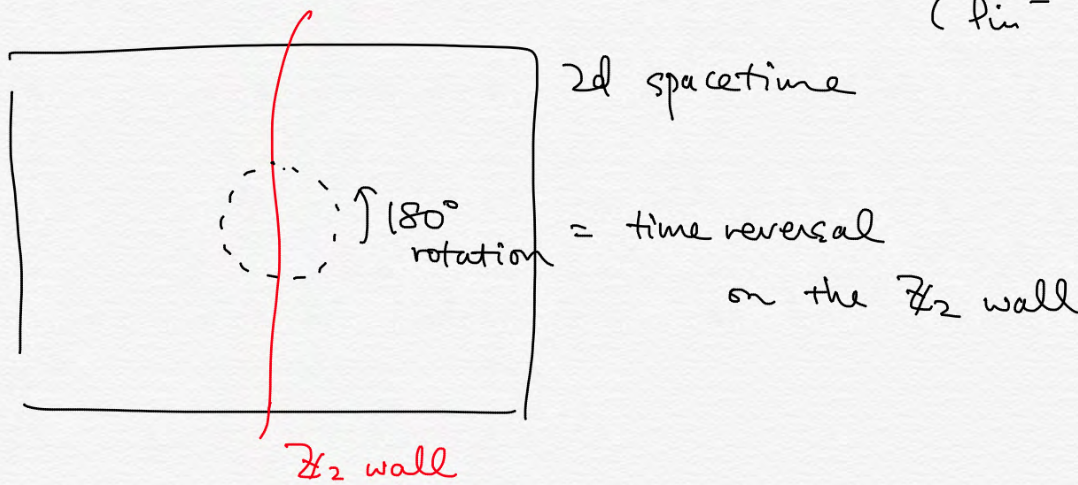
$\Rightarrow$ 2d Hilb. sp.

therefore qdim  $= \sqrt{2}$.

chiral $\mathbb{Z}_2$

Recall anomaly of $\mathbb{Z}_2$ in 2d fermionic theory is a mod 8 integer n.

This is the same mod 8 characterizing the anomaly of 1d fermionic system with $T^2 = 1$. (Pin^-)



$\rightsquigarrow$ Majorana zero modes on the $\mathbb{Z}_2$ wall has a natural T action

$\rightsquigarrow$ mod 8.

(can be generalized :

anomaly of $\mathbb{Z}_2$ in D-d fermionic theory

$\overset{1:1}{\Longleftrightarrow}$ anomaly of time reversal with $T^2 = 1$ (Pin^-) in (D-1)d fermionic theory)

cf. Hason - Komargodski - Thorngren 1910.14039

Cordova - Ohmori - Shao - Yan 1910.14046

Both on Oct 31 !

Consider turning spin str. to $\mathbb{Z}_2$ sym by gauging :

chiral $\mathbb{Z}_2$ with anom $n \in \mathbb{Z}_8$

spin str.

$\xrightarrow[\text{spin str.}]{\text{gauge}}$

D line

x $\mathbb{Z}_2$ line

form a fusion cat. depending on n.

This is a formal process ;
the result can be found by studying an example,

n Maj fermion $\leftrightarrow$ $\mathrm{Spin}(n)_1$ WZW .

$n=0$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$D^2 = e$
1	Ising	$D^2 = e + x$
2	$\mathbb{Z}_4$	$D^2 = x$
3	Ising$'$	$D^2 = e + x$
4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$D^2 = e$
5	Ising$'$	$D^2 = e + x$
6	$\mathbb{Z}_4$	$D^2 = x$
7	Ising	$D^2 = e + x$
$8 \equiv 0$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$D^2 = e$

More generally,

anomaly of G of 2d fermionic system is characterized by

$\mu : G \to \mathbb{Z}_2$ homomorphism

$\nu : G \times G \to \mathbb{Z}_2$ 2-cocycle $\delta\nu = 0$

$\alpha : G \times G \times G \to U(1)$ s.t. $\delta\alpha = (-1)^{\nu^2}$.

gauge $\mathbb{Z}_2$ $\Uparrow$ $\Downarrow$ gauge spin str.

2d bosonic system with fusion cat. sym.

s.t. x : $\mathbb{Z}_2$ sym coming from spin str.

A_g : $\mu(g) = 0$

D_g : $\mu(g) = 1$

with the fusion rule

$$A_g A_{g'} = A_{gg'} x^{\nu(g,g')}$$

$$A_g D_{g'} = D_{gg'}$$

$$D_g D_{g'} = A_{gg'} \oplus A_{gg'} x$$

and the F-symbols determined by α.

see Bhardwaj - Gaiotto - Kapustin 1605.01640

in particular Sec 3.6 and Appendix B.