Anomalies and top. phases

Study of anomalies is an old subject.

(a nice summary of old hep-th perspective on anomalies can be found in Harvey's TASI 2003 lectures.)

Recent advances in cond-mat gave us a unifying view:

A d-dim'l anomaly is controlled by a special type of (d+1)-dim'l top. theory.

Aim.

Review various known old facts from this modern unifying point of view.

Motto.

Analects of Confucius 2:11

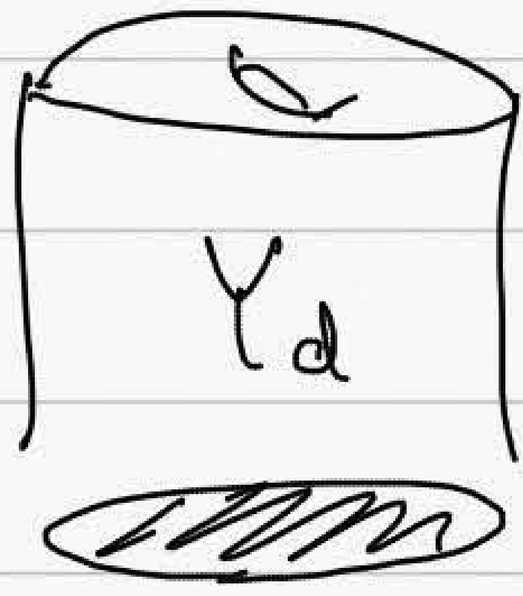
溫故而知新，可以爲師矣。

warm up old and know new, can become master indeed.

Content.

- Generalities.
- Example 1. Dirac quantization
- ——— 2. Fermions
- ——— 3. Maxwell and antisym. tensor fields
- ——— 4. Anomalies of finite sym. in 1d
- ——— 5. ——————— in 2d
- Fun with gauging.

What does a QFT Q in d dimensions do?



$$X'_{d-1} \rightsquigarrow \mathcal{H}(X'_{d-1})$$

$$\uparrow Z(Y_d) : \text{evol. op.}$$

$$X_{d-1} \rightsquigarrow \mathcal{H}(X_{d-1}) : \text{Hilb. space}$$

in particular,

$$\emptyset \rightsquigarrow \mathcal{H}(\emptyset) = \mathbb{C}$$

$$\uparrow Z(Y_d) : \text{part. func}$$

$$\emptyset \rightsquigarrow \mathcal{H}(\emptyset) = \mathbb{C}$$

Y_d can be equipped with

- metric
- orientation
- spin structure
- other background fields ...

(Quantum) Anomaly:

$Z(Y_d)$ has a phase ambiguity.
controlable

How do we characterize it?

$Z_Q(Y_d)$ is ~~a number with controlable phase ambig.~~

a vector in a one-dim vector space without a canonical basis.

$v \in V$: 1-dim

controlable phase change

$$\frac{v}{b} = \left(\frac{b'}{b}\right) \times \frac{v}{b'} \quad : \text{cpx numbers}$$

two bases

to describe an anomalous theory Q,

one needs to first specify in which vect. sp

$$Z_Q(Y_d) \in V(Y_d)$$

the part func. takes values in.

Q. What gives us a vect. sp for each Y_d ?

A. A $(d+1)$-dim'l QFT! Call it A. for anomaly.

$$Z_Q(Y_d) \in \mathcal{H}_A(Y_d)$$

A is rather special:

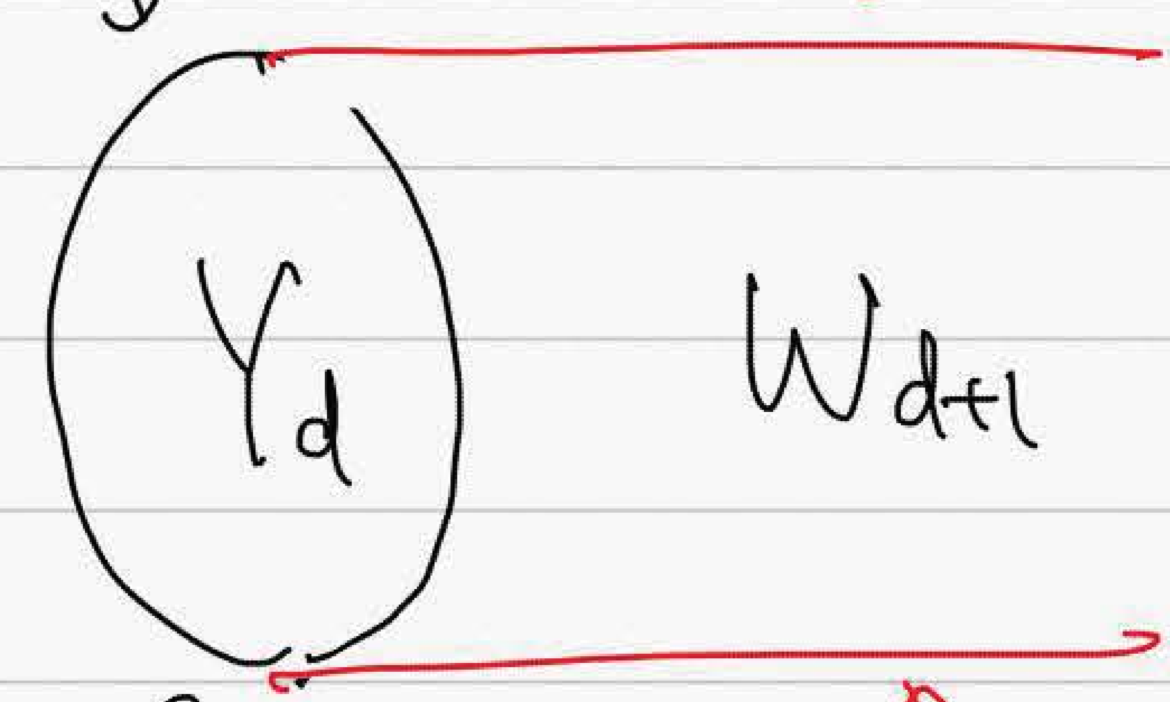
the Hilb. sp is one-dimensional

such QFTs are called <u>invertible</u>.

cond-mat:

boundary edge mode

bulk SPT phase



hep-th:

anomalous QFT Q

invertible QFT A

Q lives <u>at the boundary of</u> A.

$\boxed{\text{Example 1}}$ (Dirac 1931, Coleman 1976, Witten 1978 ...)

Consider a 0+1 d QFT coupled to background $U(1)$ field.

S^1, $\varphi = \int A\,dx$.

$g = e^{i\varphi}$: holonomy around S^1.

large gauge tr. sends $\varphi \to \varphi + 2\pi$

$$Z_Q(S^1, g) = e^{iq\int A\,dx} = g^q \ :$$

well-defined when $q \in \mathbb{Z}$.

There's a problem when $q \notin \mathbb{Z}$:

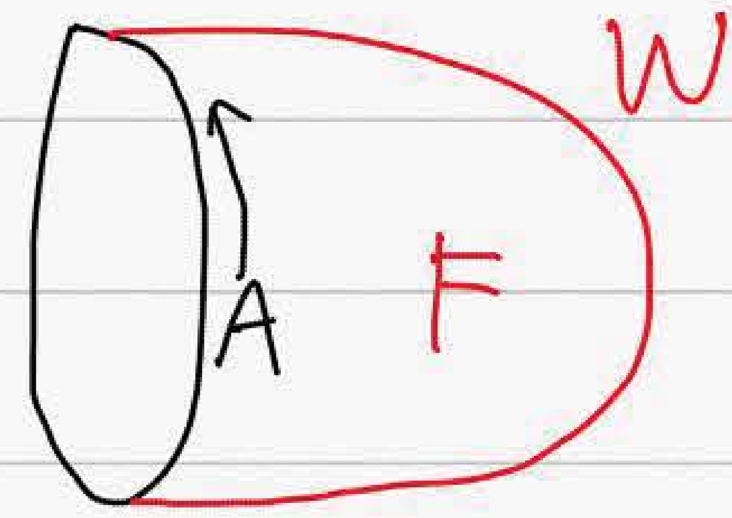
$$e^{iq\varphi} \longleftrightarrow e^{iq(\varphi+2\pi)} = \boxed{e^{2\pi i q}} \times e^{iq\varphi}$$

controlable phase ambig.

Let's pick $\theta/_{2\pi} = q \mod 1$.

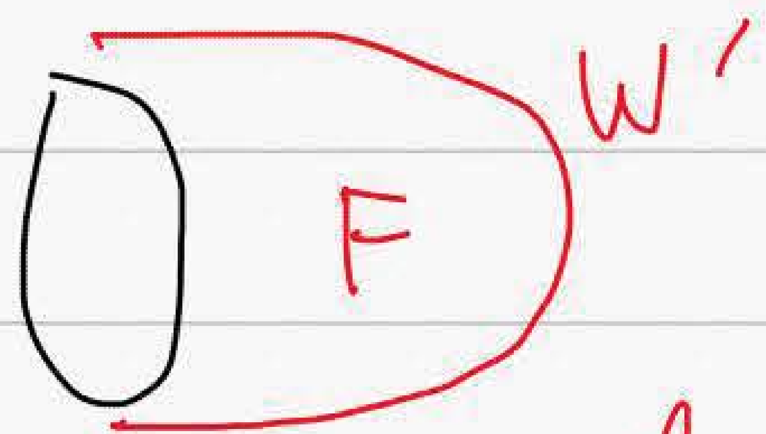
The bulk theory has the action

$$S = i \int \theta \frac{F}{2\pi}.$$ ← U(1) gauge field strength.



$e^{iq\int A}\, e^{i\theta \int_W F/2\pi}$: now has a definite value.

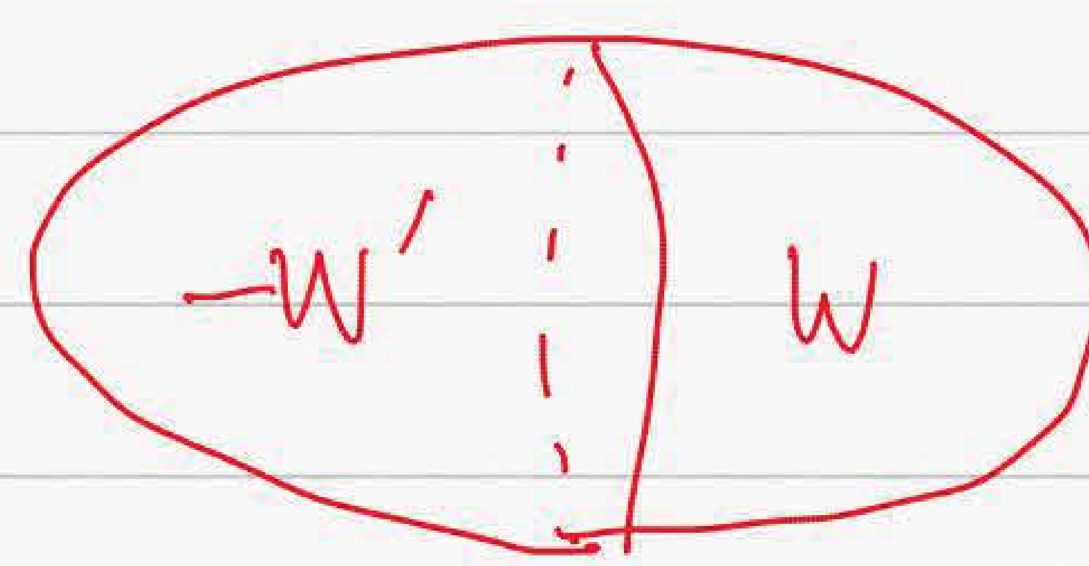
(large gauge tr. no longer possible)



$e^{iq\int A}\, e^{i\theta \int_{W'} F/2\pi}$

The ratio is

$$e^{i\theta \int_{W-W'} F/2\pi} = e^{i\theta n}.$$

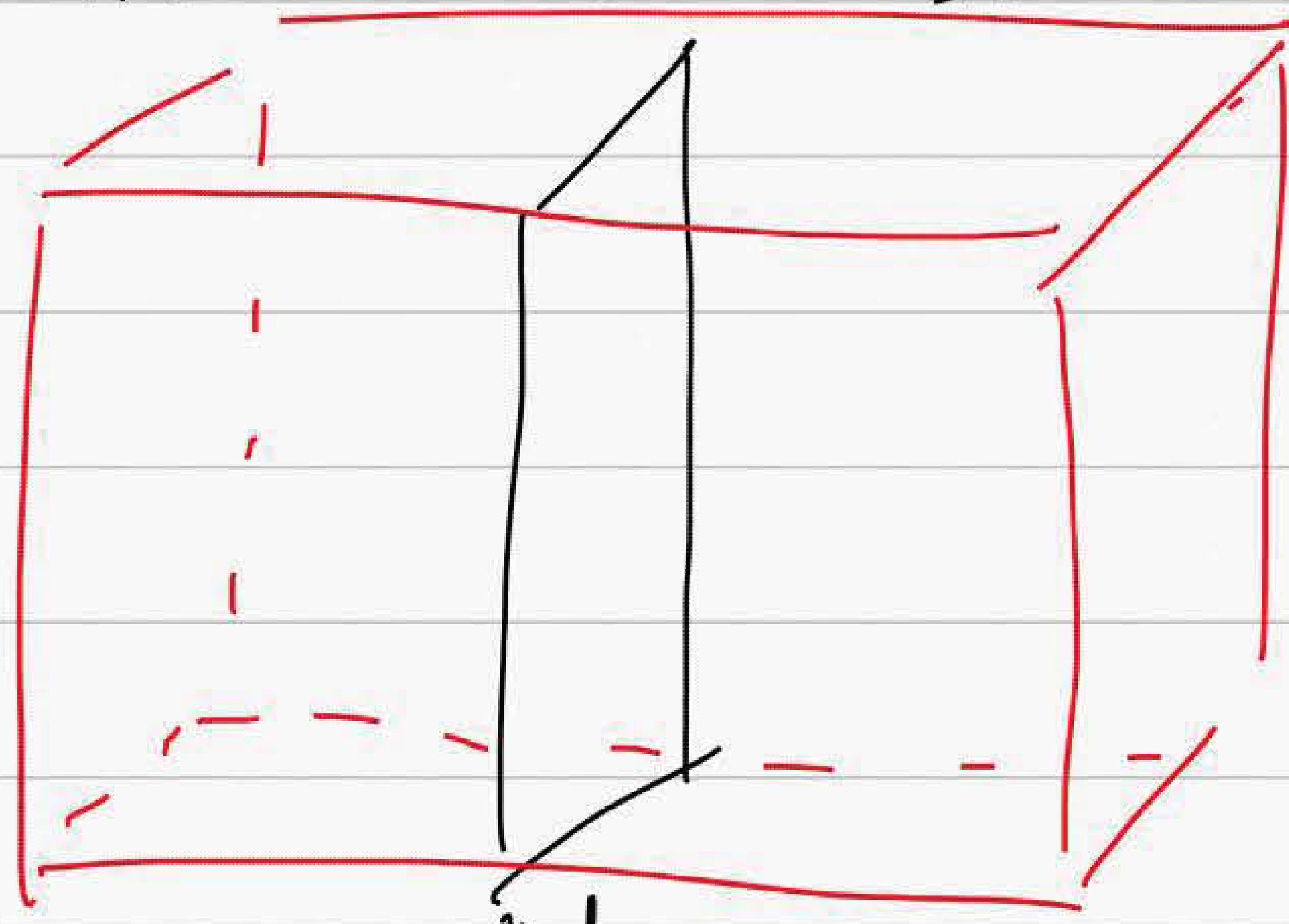


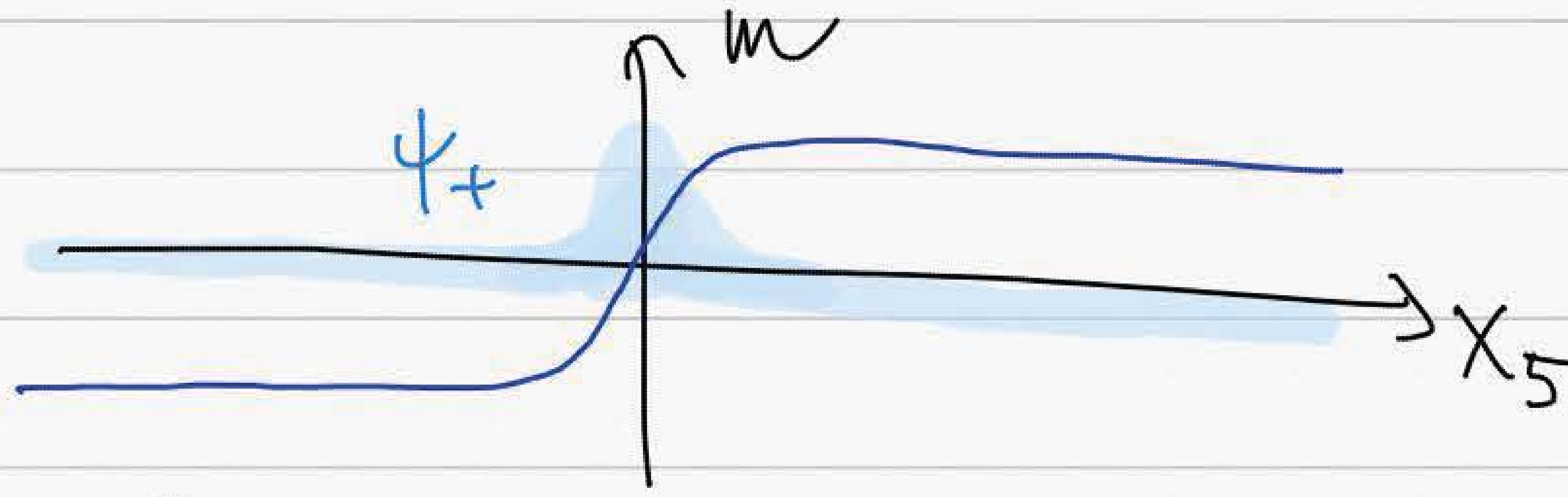
Phase change under gauge tr.

⇝ replaced by dependence on how to choose the bulk W and how to extend F.

$\boxed{\text{Example 2}}$ (Alvarez-Gaumé - Della Pietra - Moore 1985)

Massless fermions famously has anomalies.





Consider Dirac f. in 5d with varying mass.

$$\bar{\psi}\,(\partial_5 \gamma^5 + m)\,\psi \quad \text{in the Lagrangian}$$

$$\rightsquigarrow \begin{pmatrix} \partial_5 + m & \\ & -\partial_5 + m \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = 0$$

$\psi_+ \sim e^{-\int m\,dx}$: normalizable

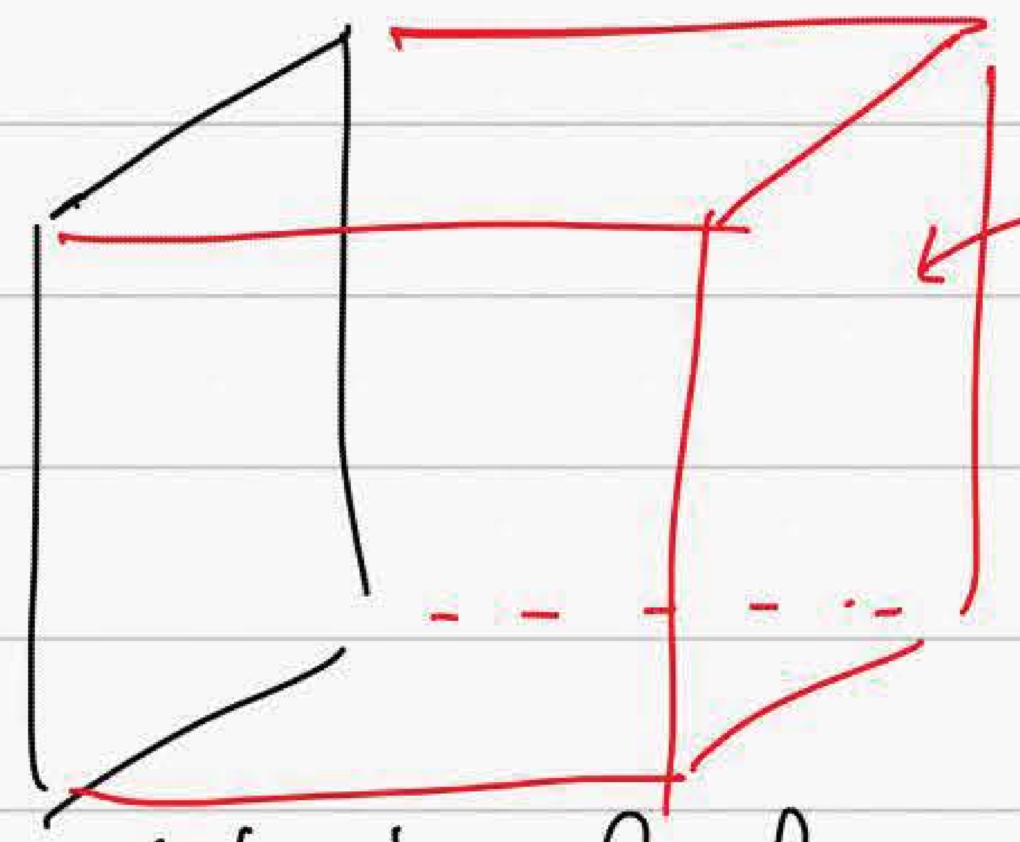
$\psi_- \sim e^{+\int m\,dx}$: not at all !

$\Longrightarrow$ 4d chiral fermion naturally arises on a domain wall of 5d fermion.

still ∞ d.o.f.

For the purpose of characterizing the 4d anom, taking the limit $m \to \pm\infty$ suffices.

$\rightsquigarrow$ isolates the vacuum.



the anomaly theory has the part. func. given by

$$\frac{Z(\text{5d fermion},\ m=+\infty)}{Z(\text{5d fermion},\ m=-\infty)}$$

4d chiral fermion.

Known as $e^{2\pi i \eta}$

$$\eta = \lim_{m\to\infty} \sum_{E:\ \text{eigenval. of } \not{D}} \frac{1}{2\pi} \operatorname{Arg}\left(\frac{iE+m}{iE-m}\right)$$

$$= \text{“}\ \frac{1}{2}\sum_{E} \operatorname{sign} E\ \text{”}$$

(• In math, ζ-func reg. is usually used but any phys. sensible reg. works.

• In the original APS paper, this was in fact the ξ invariant, slightly diff. from η.)

Let's do some exercise.

① $U(1)$-charged fermion on S^1 holonomy $e^{i\varphi}$.

Dirac op is just $\partial_x + \varphi$.

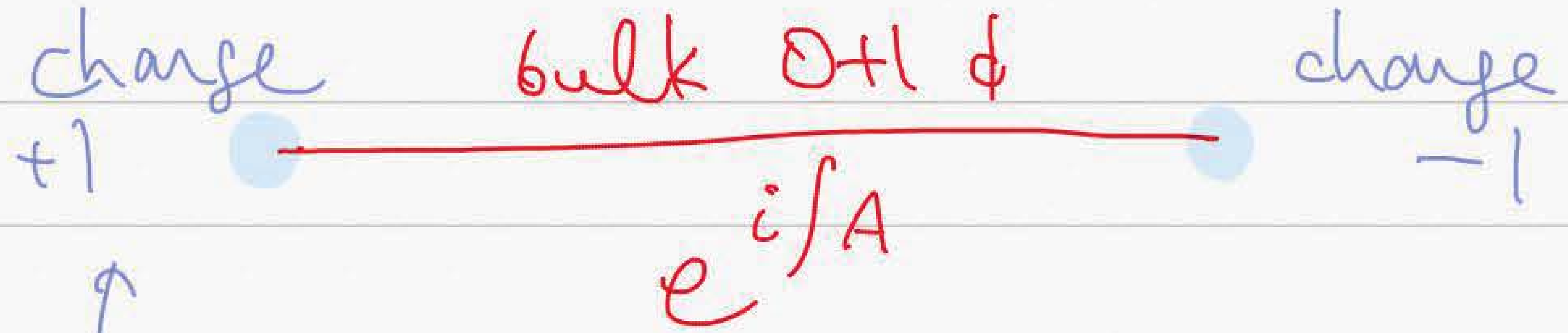
$\rightsquigarrow \quad E = n + \varphi/2\pi, \quad n \in \mathbb{Z}$.

Use $e^{-T|E|}$ for regularization:

$$\eta = \lim_{T\to\infty} \frac{1}{2} \sum \left[\mathrm{sign}\left(n+\frac{\varphi}{2\pi}\right)\right] e^{-T\left|n+\frac{\varphi}{2\pi}\right|}$$

$$= \frac{1}{2} - \frac{\varphi}{2\pi}.$$

$$e^{2\pi i \eta} = -e^{-i\varphi} = -e^{-i\int A}.$$



$\uparrow$

anomaly of a 0+0 d charged fermion

(a bit ~~too~~ degenerate and confusing, though).

② η of $S^3/\mathbb{Z}_n$

$\uparrow$

quotient of

$S^3 \leftarrow SU(2)_L \times SU(2)_R$

$\begin{pmatrix} x & \\ & x^{-1} \end{pmatrix}$

$x^n = 1$.

Dirac eigenmodes here is charged under $\mathbb{Z}_n$.

$\psi \to x^{2j} \psi$.

Let $\eta_{S^3}(x) := "\frac{1}{2}\sum_E x^{2j} \operatorname{sgn} E"$.

$\pi_1(S^3/\mathbb{Z}_n) = \mathbb{Z}_n$.

$\rightsquigarrow$ can consider $U(1)$ holonomy

$\omega^n = 1$.

modes on $S^3/\mathbb{Z}_n$ with this $U(1)$ holonomy

" modes on S^3 transforming with $\left(e^{\frac{2\pi i}{n}}\right)^{2j} = \omega$.

$\rightsquigarrow \eta_{S^3/\mathbb{Z}_n}(\omega) = \frac{1}{n}\sum_{k=1}^{n} \omega^{-k}\, \eta_{S^3}(e^{2\pi i k/n})$.

Still Need to compute $\eta_{S^3}(x)$.

$$S^3 = \frac{SU(2)_L \times SU(2)_R}{SU(2)_{diag}}.$$

spinor bundle on S^3: fiber is $\mathbf{2}$ of $SU(2)_d$.

in general, for $M = G/H$ consider a bundle s.t. the fiber on $p \in G/H$ is a rep R_H of H.

The section of this bundle is a G-rep induced from H

$\operatorname{Ind}_H^G R_H$.

We have: $\langle \rho_G, \operatorname{Ind}_H^G R_H \rangle = \langle \operatorname{Res}_H^G \rho_G, R_H \rangle$.

In our case, $G = SU(2)_L \times SU(2)_R$
$H = SU(2)_d$
$R_H = \mathbb{1}$.

$\leadsto$ # of times $V_L \otimes V_R$ appears in the decomp. of the sections of Dirac spinor on S^3 is equal to the # of times $\mathbb{1}$ appears in the irr. decomp. of $V_L \otimes V_R$ under $SU(2)_d$.

$\Rightarrow$ Section of Dirac sp. on S^3

$$= \bigoplus_{d=1}^{\infty} \Big(\underbrace{V_d \otimes V_{d+1}}_{\not{D} = d + 1/2} \oplus \underbrace{V_{d+1} \otimes V_d}_{\not{D} = -(d+1/2)} \Big).$$

$$\Rightarrow \eta_{S^3}(x) = \frac{1}{2} \lim_{s\to 0} \sum_d \Big(d\,\chi_{d+1}(x) - (d+1)\,\chi_d(x) \Big) e^{-s(d+\frac{1}{2})}$$

where $\chi_d(x) = x^{d-1} + x^{d-3} + \cdots + x^{3-d} + x^{1-d}$.

$$= \frac{x}{(1-x)^2}.$$

$$\leadsto \eta^{\text{Dirac}}_{S^3/\mathbb{Z}_2}(\omega = +1) = \frac{1}{2} \cdot \frac{-1}{(1-(-1))^2} = \frac{-1}{8}$$

$$\eta^{\text{Dirac}}_{S^3/\mathbb{Z}_2}(\omega = -1) = \frac{1}{2} \cdot \frac{1}{(1-(-1))^2} = +\frac{1}{8}.$$

In the Lorentzian sig, 3d ψ is in $\mathbf{2}$ of $SL(2,\mathbb{R})$
$\rightsquigarrow$ can consider Majorana fermion.

After Wick ~~rotation~~, 3d ψ is $\mathbf{2}$ of $SU(2)$
$\longrightarrow$ pseudoreal. This is exactly what's necessary to <u>divide η by 2</u>

$$\left.\begin{aligned} \not{D}\psi &= E\psi \\ \not{D}\psi^* &= E\psi^* \end{aligned}\right) \text{cpx. conj.}$$

pseudoreality guarantees ψ and ψ^* orthogonal

$\Rightarrow$ Dirac eigenvalues come in pairs.

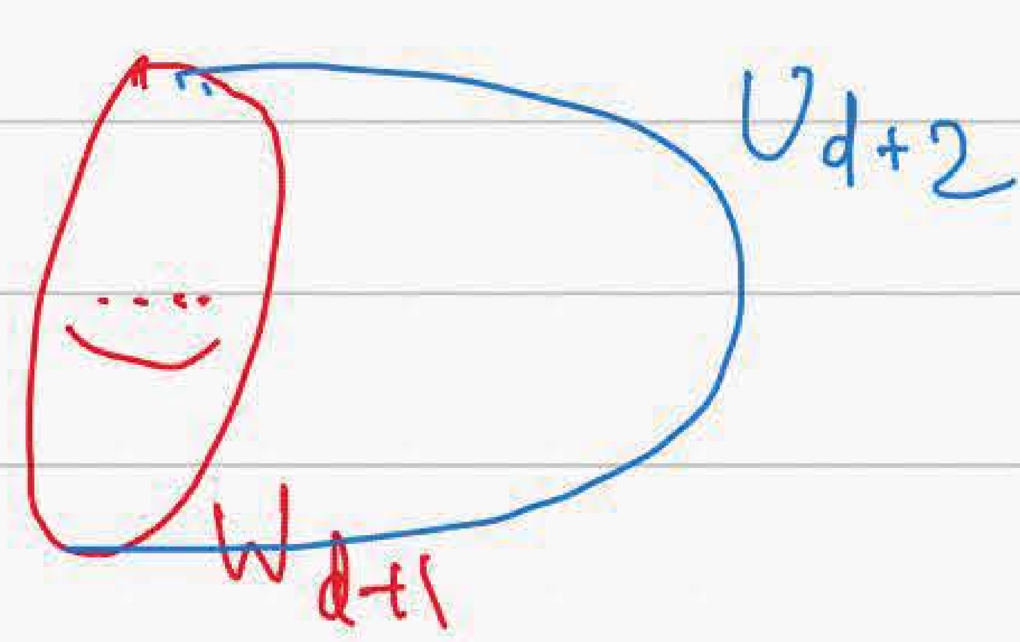
$$\eta^{\text{Maj.}}_{S^3/\mathbb{Z}_2}(\omega=+1) = -\frac{1}{16}$$

$$\eta^{\text{Maj.}}_{S^3/\mathbb{Z}_2}(\omega=-1) = +\frac{1}{16}.$$

End of a long exercise.

The Index

Such explicit computations are only possible when the manifold is very symmetric.
Another method uses the APS index theorem:



$$\eta = \boxed{(\#\text{ of zero modes on } U)} + \int_U \hat{A}\,\mathrm{tr}\left(e^{F/2\pi}\right).$$

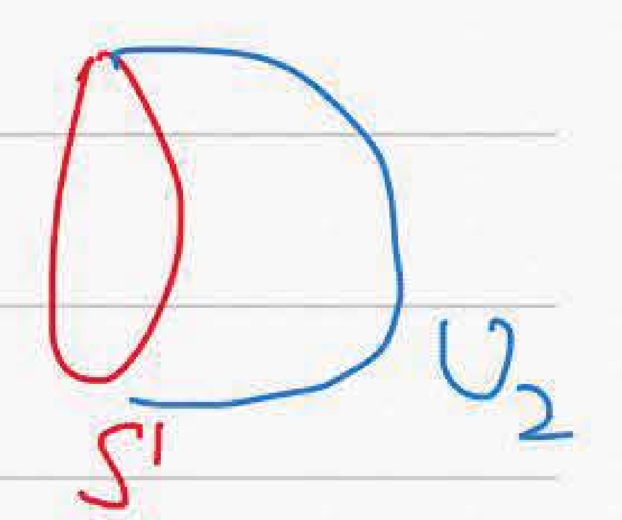
- When $W_{d+1} = \partial U_{d+2} = \phi$, this is the Atiyah-Singer index theorem.

- There, we typically care the index.
  But for this lecture we just need
  $$e^{2\pi i \eta} \longrightarrow \text{the index drops out.}$$

- $$\hat{A} = 1 - \frac{1}{24} p_1 + \cdots$$ where
  $$p_1 = -\frac{1}{2} \operatorname{tr} \left(\frac{R}{2\pi} \right)^2$$

- so, for $U(1)$ on U_4,
  $$\int_{U_4} \hat{A} \operatorname{tr} e^{F/2\pi} = -\frac{1}{24} \int_{U_4} p_1 + \frac{1}{2} \int_{U_4} \left(\frac{F}{2\pi} \right)^2 .$$

For $U(1)$ on U_2,
$$\int_{U_2} \hat{A} \operatorname{tr} e^{F/2\pi} = \int_{U_2} \frac{F}{2\pi}$$



NS spin str.

We saw $\eta = \frac{1}{2} + \int \frac{A}{2\pi}$

used the R spin str.

This illustrates the following:

$$\eta_{W_{d+1}} \sim (\text{index}) + \int_{U_{d+2}} (F^2 + R^2)$$

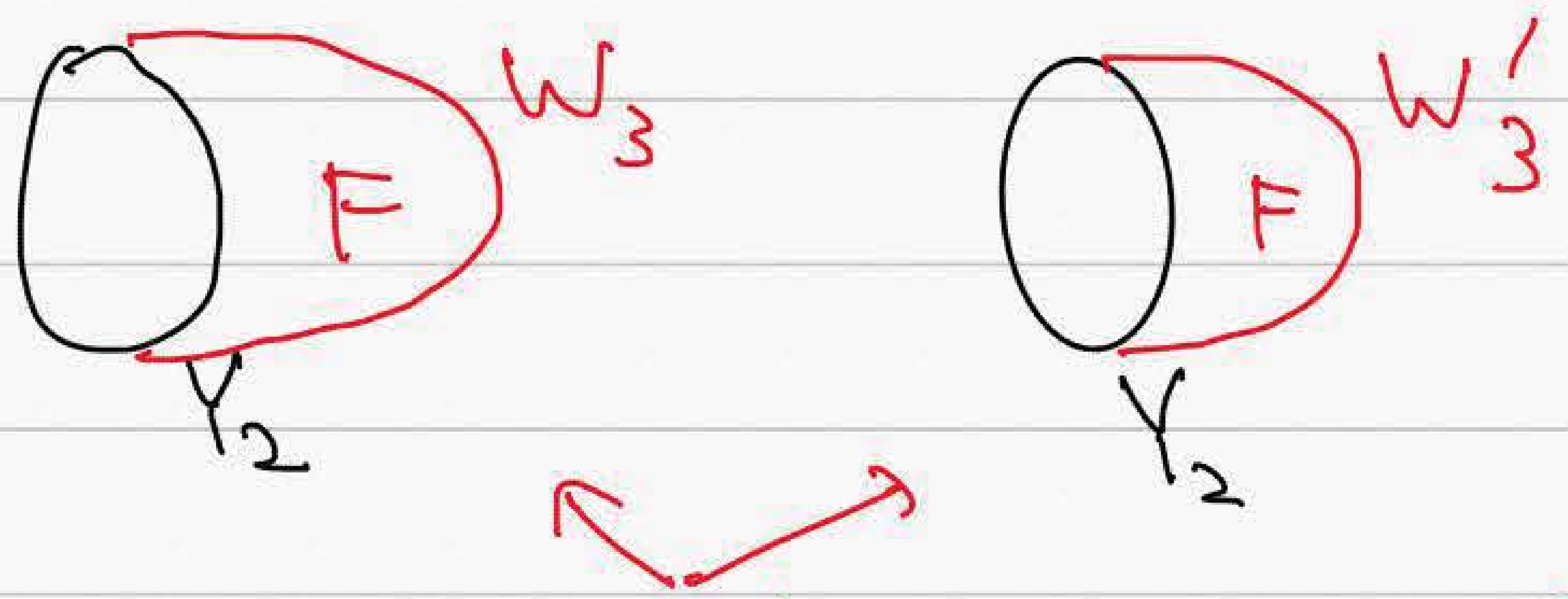
means that, <u>more or less</u>,

$$\eta_{W_{d+1}} \sim AdA + \omega d\omega$$

but only up to a subtle integration constant.

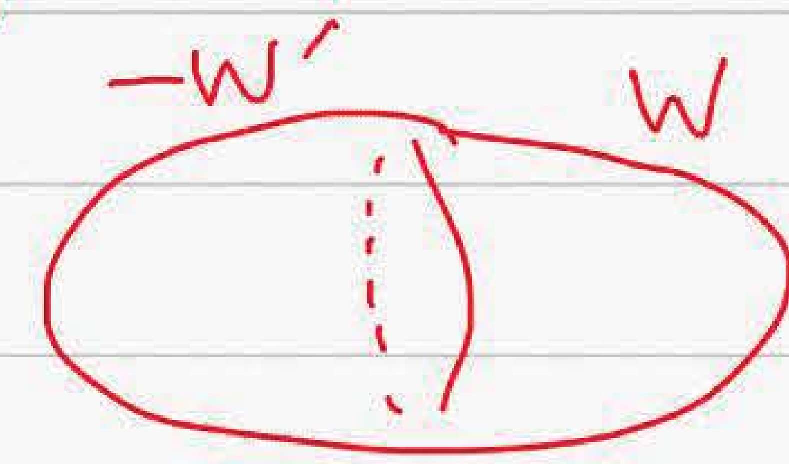
Let's finally come back to physics.

Consider 2d massless chiral complex fermion charged under $U(1)$.



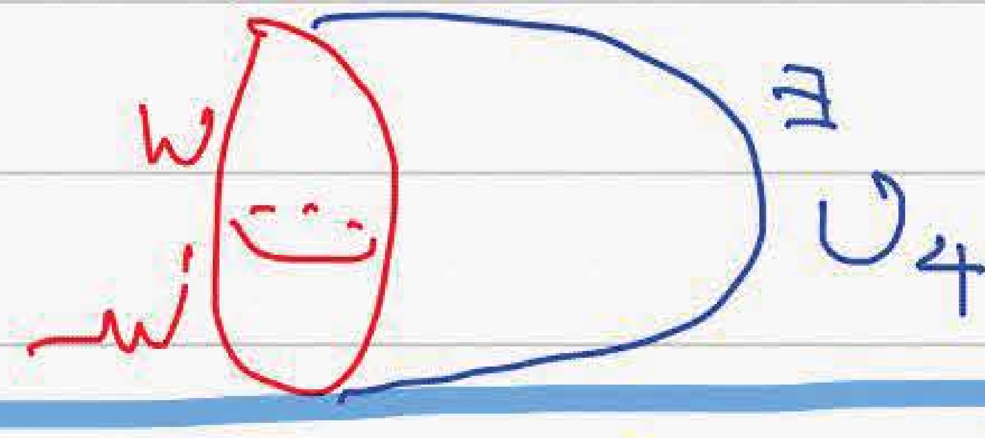
The ratio of the part. func

$= e^{2\pi i \eta}$ on



This is nontrivial as we saw for $S^3/\mathbb{Z}_n$.

Furthermore, when



Known as the anomaly poly.

$$= e^{2\pi i \int_{U_4} \boxed{-\frac{1}{24}\frac{1}{2}\mathrm{tr}\left(\frac{R}{2\pi}\right)^2 + \frac{1}{2}\left(\frac{F}{2\pi}\right)^2}}$$

But it's not always that $\exists\, U_4$.

Consider for e.g.

a left-moving real fermion unchanged under $\mathbb{Z}_2$.

+ a right-moving real fermion charged

The anomaly poly is

$$+\frac{p_1}{48} - \frac{p_1}{48} = 0.$$

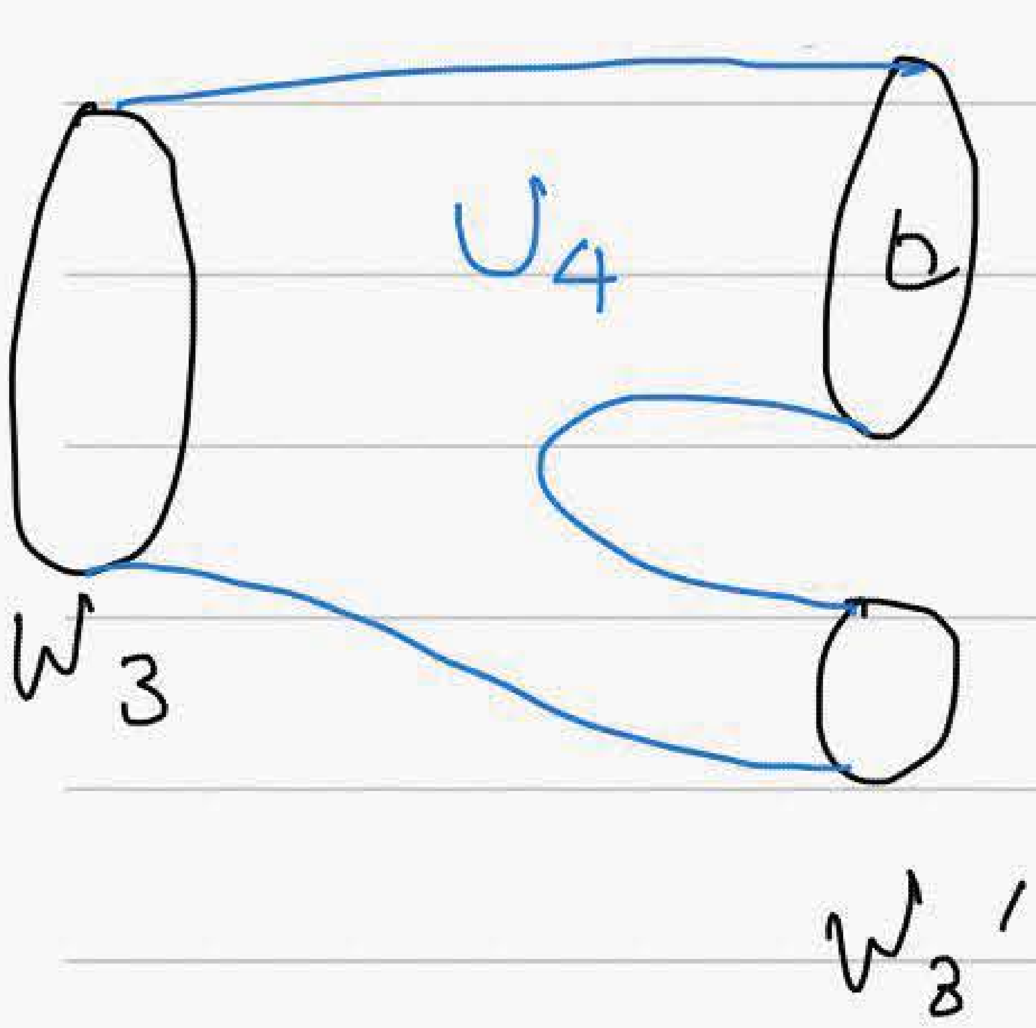
Still, $\eta_{S^3/\mathbb{Z}_2}(\omega = +1) = -\frac{1}{16}$

$\eta_{S^3/\mathbb{Z}_2}(\omega = -1) = +\frac{1}{16}$

$$\longrightarrow \underline{e^{2\pi i[\eta(\text{unchanged}) - \eta(\text{charged})]}} = e^{-2\pi i \frac{1}{8}}$$

$\longrightarrow$ Anomalous !

Now, consider **this part** as a function $Z_A(W)$.



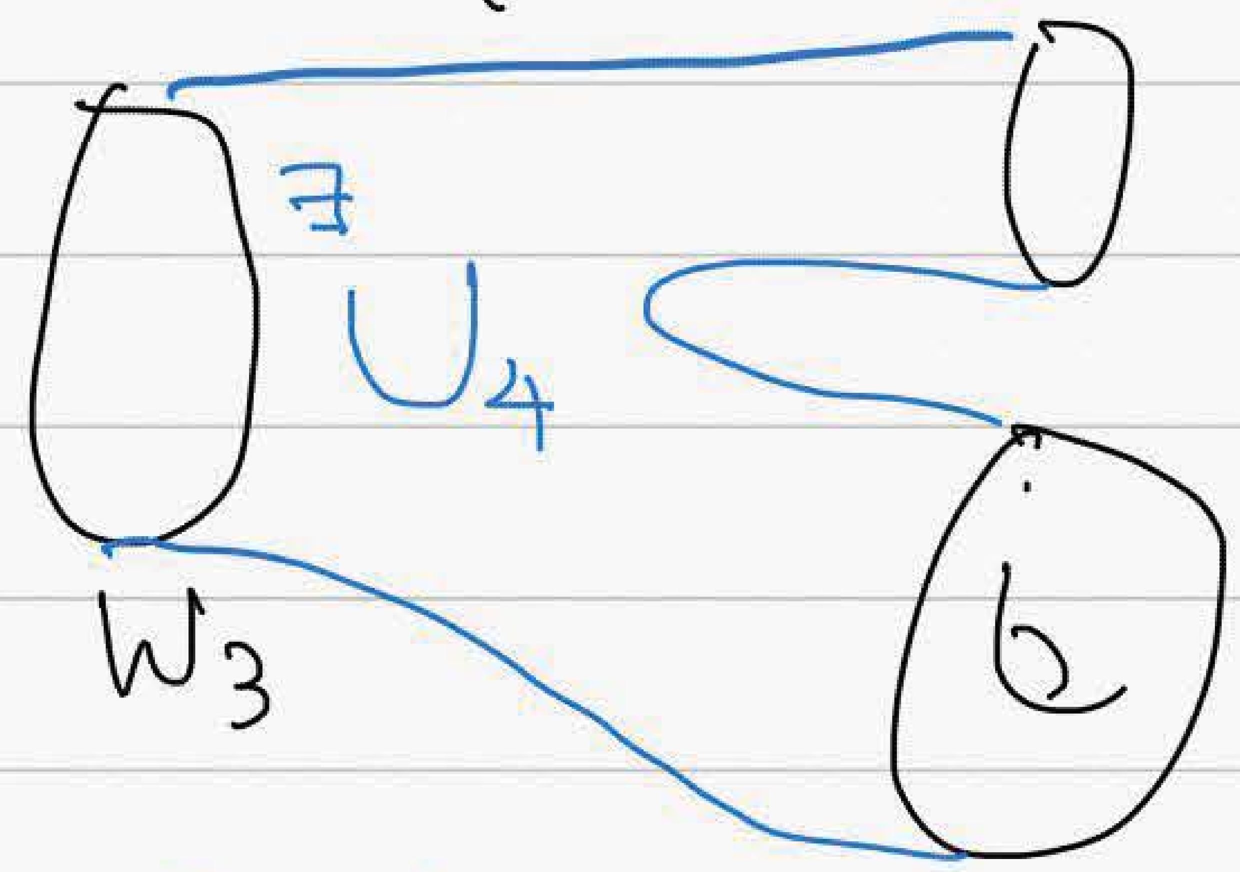
$$\frac{Z_A(W_3)}{Z_A(W_3')} = e^{2\pi i \int_{U_4} (\text{anom. poly})}$$

$$= 1.$$

In general, the set of

3d mfd + spin structure + $\mathbb{Z}_2$ bundle

under the equiv. relation



$$W_3 \overset{\exists U_4}{\sim} W_3'$$

Has spin str.

forms a bordism group

has $\mathbb{Z}_2$ bundle.

$$\Omega^{\mathrm{spin}}_{3}(B\mathbb{Z}_2)$$

under the group law

dim = 3

$$W_3 \sqcup W_3' \longleftarrow W_3 + W_3'.$$

An anomalous 2d theory with zero anom. poly determines a homomorphism

Kapustin-Thorngren-Turzillo-Wang 2014

$$\begin{array}{ccc} \Omega^{\mathrm{spin}}_{3}(B\mathbb{Z}_2) & \longrightarrow & U(1) \\ \Psi & & \cup \\ (M, \underset{\text{bundle}}{\mathbb{Z}_2}) & \longmapsto & Z_A(M, \underset{\text{bundle}}{\mathbb{Z}_2}) \end{array}$$

Our fermion system gave

$$Z_A(S^3/\mathbb{Z}_2, \underset{\text{bundle}}{\text{nontrivial}}) = e^{2\pi i/8}.$$

This shows

$\left(S^3/\mathbb{Z}_2, \begin{smallmatrix}\text{nontriv}\\ \text{bundle}\end{smallmatrix}\right)$ is a nontrivial

element in $\Omega_3^{spin}(B\mathbb{Z}_2)$

(called not null-bordant)

of order a multiple of 8.

In fact, $\Omega_3^{spin}(B\mathbb{Z}_2) = \mathbb{Z}_8$

and is generated by $S^3/\mathbb{Z}_2$.

$\longrightarrow$ Bordism class in this case is detected by the eta invariant.

Significance in String Theory

Gliozzi-Scherk-Olive 1977

Recall the GSO projection:

one needs to sum over the spin structures of the left-movers and the right-movers <u>seperately and independently</u>.

Left-moving spin st

= Right-moving spin st. + $\mathbb{Z}_2$ gauge field

A pair of left-moving Maj. spinor
+ right-moving Maj. spinor
has the anomaly $e^{2\pi i/8}$.

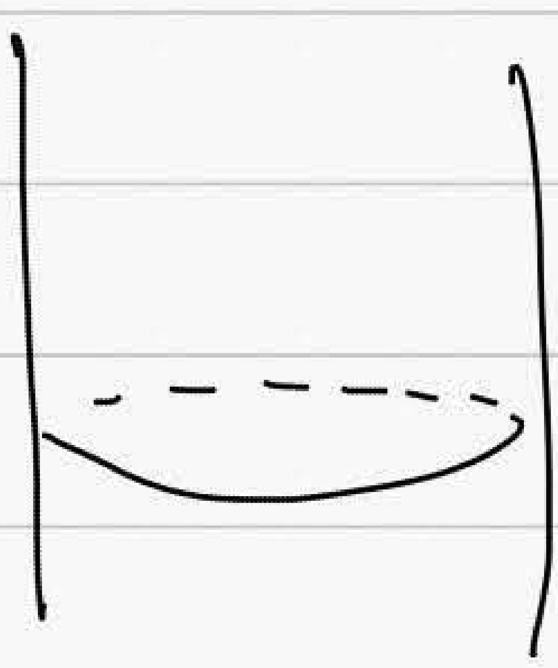
8 copies ⋯⋯> $(e^{2\pi i/8})^8 = 1$

Non - anomalous !

↑ # of fermion pairs in the lightcone gauge, $10-2=8$.

Another 'mysterious mathematical accident' underlying string theory.

<u>NOTE</u>. The same anomaly also visible in the following way:



R-NS sector:

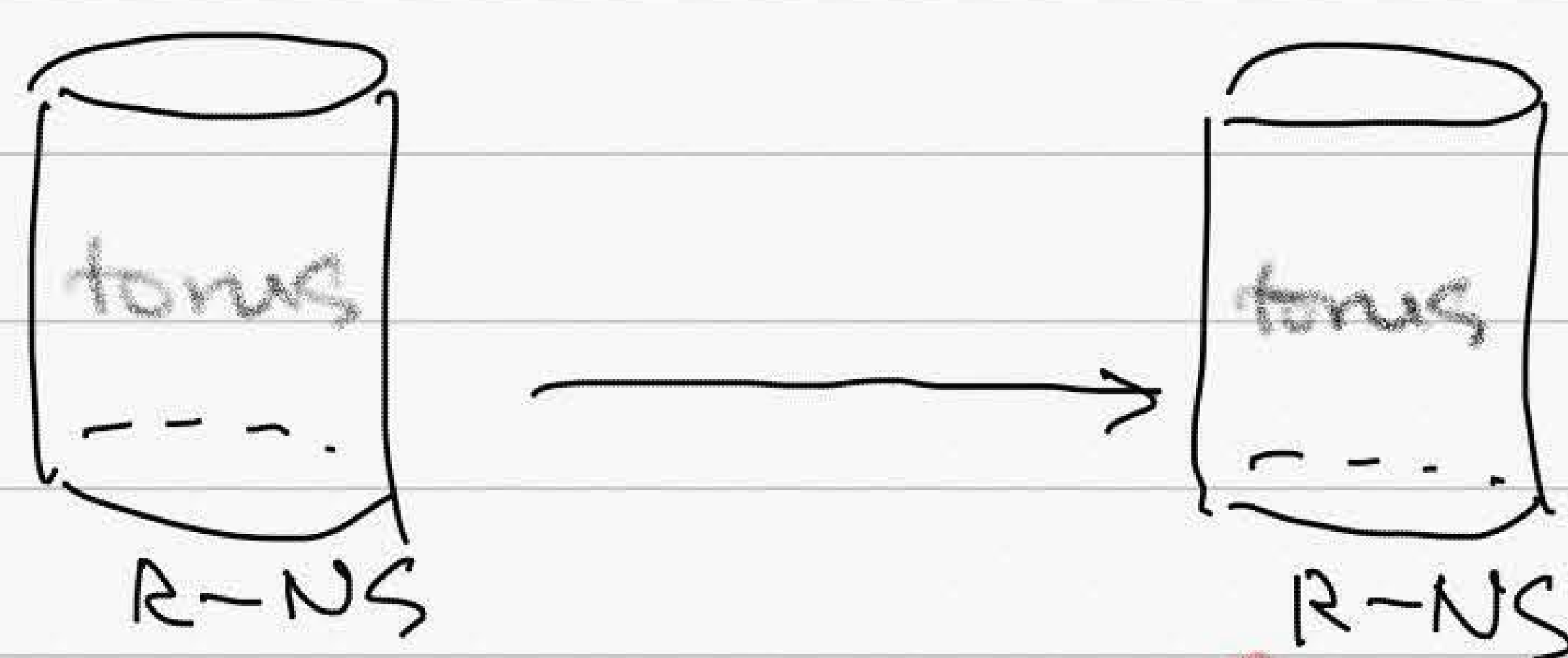
$L_0 = +1/24$

$\bar{L}_0 = -1/48$

} $+1/16$ difference.

⇝ 720° rotation gives the phase $e^{2\pi i/8}$.

⇝ needs 8 copies to be consistent.

$Z_A\Big($  $\Big) = e^{2\pi i/8}$

glue using '720° twist' $= T^2 \in SL(2,\mathbb{Z})$

$\boxed{\text{Example 3}}$ (Green-Schwarz 1984)

Consider a 2d periodic scalar $\theta \sim \theta + 2\pi$.

$$S = \int d^2x \, \frac{1}{2e^2} \partial_\mu \theta \partial^\mu \theta .$$

$\exists$ 2 U(1) symmetries $\begin{cases} \text{momentum} \\ \text{winding number} \end{cases}$

Introduce background fields

A_μ^{mom}, A_μ^{win}.

$S[A_\mu^{mom}, A_\mu^{win}] =$

$$\int_{Y_2} d^2x \left[\frac{1}{2e^2} |d\theta + A^{mom}|^2 + i \underline{(d\theta + A^{mom}) \wedge A^{win}/2\pi} \right]$$

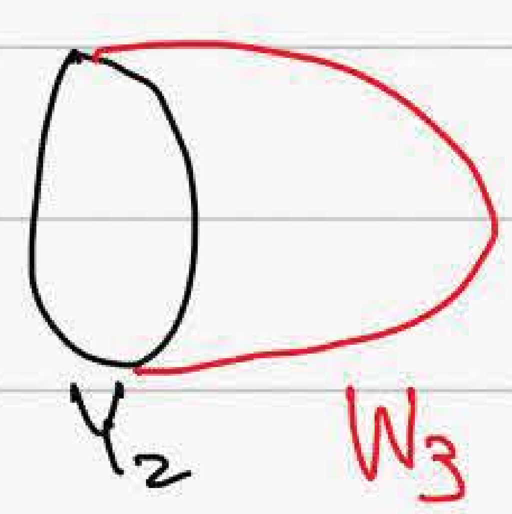
not gauge inv because

under $\delta A^{win} = d\chi$,

$$\delta \left[(d\theta + A^{mom}) \wedge A^{win}/2\pi \right] = (d\theta + A^{mom}) \wedge d\chi/2\pi$$

$\downarrow$ partial integral

$$\int F^{mom} \wedge \frac{\chi}{2\pi} \neq 0 .$$

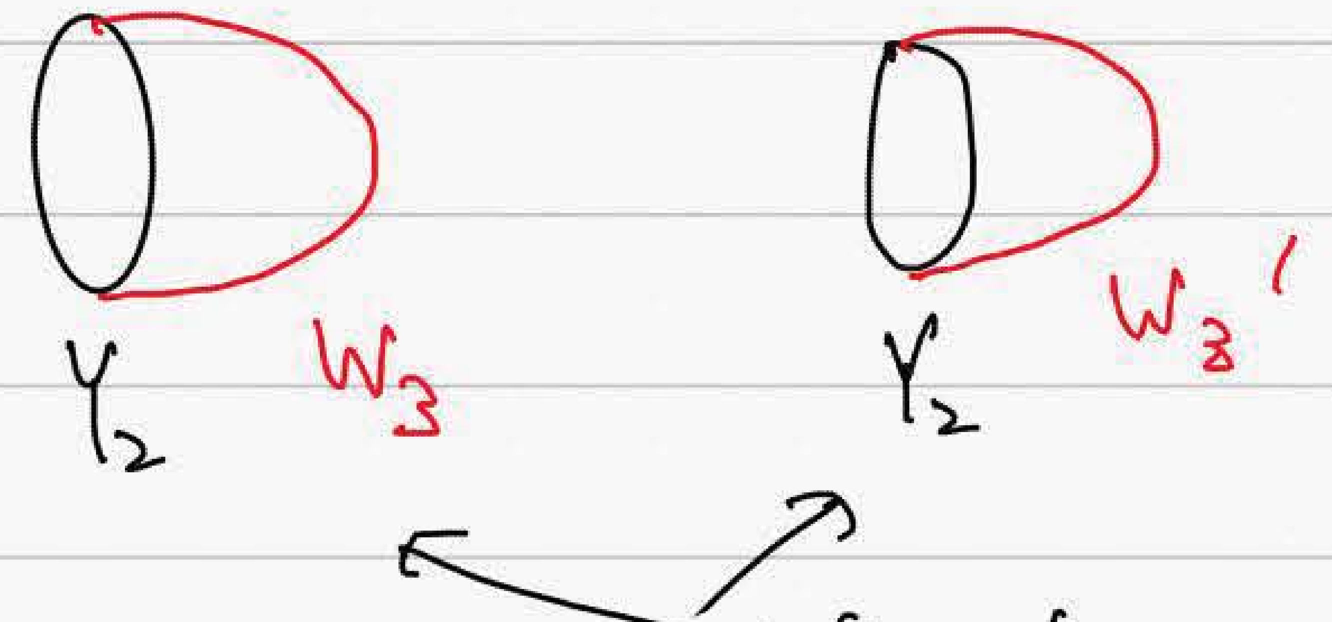


Add the bulk term

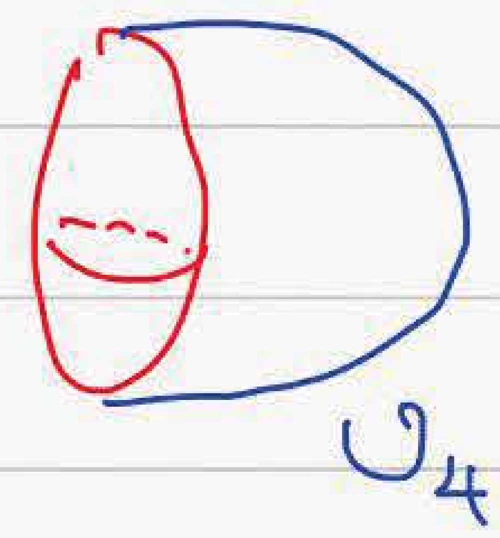
$$\int_{W_3} F^{mom} \wedge A^{win}/2\pi$$

cancels!

$$\delta \left[\int_{W_3} F^{mom} \wedge A^{win}/2\pi \right] = \int_{Y_2} F^{mom} \wedge \chi/2\pi$$



$$\text{diff. of action} = \int_{-W_3' \;\; W_3} i\, F^{mom} \wedge \frac{A^{win}}{2\pi}$$



$$= 2\pi i \int_{U_4} \boxed{\frac{F^{mom}}{2\pi} \wedge \frac{F^{win}}{2\pi}} .$$

The anomaly poly. of
U(1) momentum & U(1) winding number symmetry.
mixed anomaly.

Natural generalization

4d Maxwell

$$S = \int_{Y_4} \frac{1}{2e^2} |F|^2 , \quad F = dA .$$

introduce 2-form backgrounds B & C

$$S[B, C] = \int_{Y_4} \left[\frac{1}{2e^2} |dA + B|^2 + \frac{i}{2\pi} \underbrace{C \wedge (dA + B)} \right] .$$

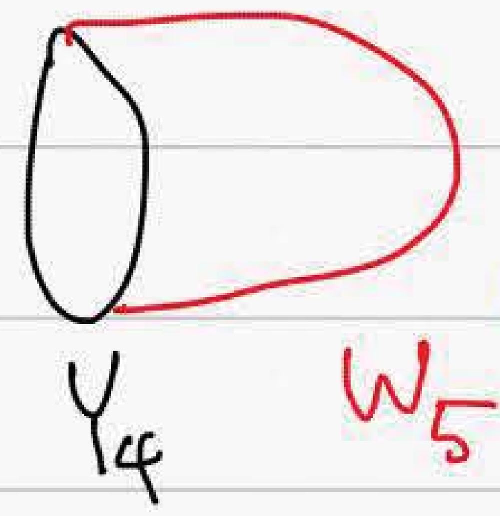
$$\downarrow \delta$$

background gauge tr.

$$\delta A = \eta , \quad \delta B = -d\eta$$
$$\delta C = d\chi .$$

$$\frac{i}{2\pi} \int d\chi \wedge (dA + B)$$
$$\shortparallel$$
$$\frac{i}{2\pi} \int \chi \wedge \underset{\neq 0}{dB} .$$

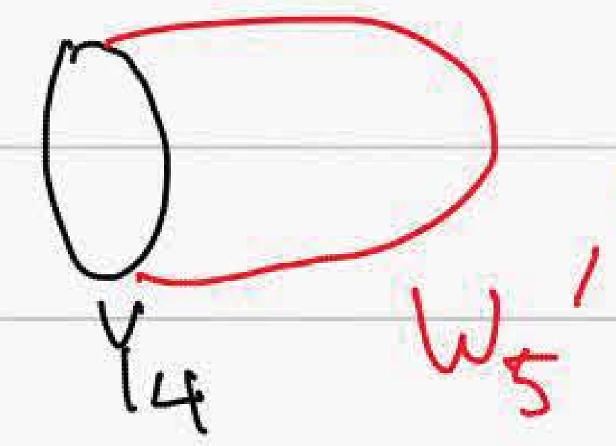
Introduce the bulk coupling



$$\frac{i}{2\pi}\int_{W_5} C\, dB \xrightarrow{\delta} \frac{i}{2\pi}\int_{W_5} d\chi\wedge dB$$

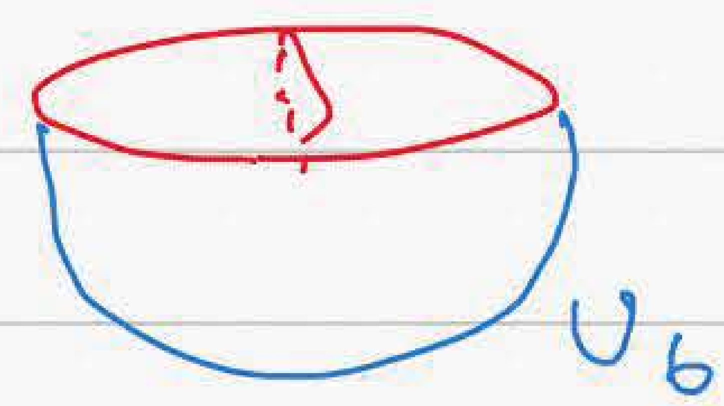
"

cancels the gauge variation of the boundary theory. $\Leftarrow$ $\frac{i}{2\pi}\int_{Y_4} \chi\wedge dB$



The diff. of the action

$$= \frac{i}{2\pi}\int_{-W_5' \,\cup\, W_5} C\wedge dB$$



$$= 2\pi i \int_{U_6} \text{“}\frac{dC}{2\pi}\text{”}\wedge\text{“}\frac{dB}{2\pi}\text{”} \quad \text{if } \exists U_6.$$

※ 2-forms B, C are not quite just diff. forms, just as a $U(1)$ gauge field A is not just a differential form.

periodic scalar $\theta \longrightarrow d\theta$, $\int \frac{d\theta}{2\pi} \in \mathbb{Z}$

$U(1)$ gauge f. $A_{(1)} \longrightarrow F_{(2)} = dA_{(1)}$, $\int \frac{F}{2\pi} \in \mathbb{Z}$

2-form $U(1)$ gauge f. $B_{(2)} \longrightarrow H_{(3)} = dB_{(2)}$, $\int \frac{H}{2\pi} \in \mathbb{Z}$

3-form " $C_{(3)} \longrightarrow G_{(4)} = dC_{(3)}$, $\int \frac{G}{2\pi} \in \mathbb{Z}$

$\vdots \qquad \vdots \qquad \vdots$

$\Theta_{(0)}$ is the background for (-1)-form $U(1)$ sym.

$A_{(1)}$ is the background for 0-form $U(1)$ sym.

$B_{(2)}$ is the background for 1-form $U(1)$ sym.

$C_{(3)}$ is the background for 2-form $U(1)$ sym.

$\vdots$

mathematically known as
Cheeger-Simons differential characters.

physically known as
generalized global symmetries.
Gaiotto-Kapustin-Seiberg-Willett 2014

Summarizing.

4d Maxwell th. has two $U(1)$ 1-form sym's

with the anom. poly $2\pi i \int_{U_6} \frac{dB_{(2)}}{2\pi} \wedge \frac{dC_{(2)}}{2\pi}$.

More generally, consider p-form gauge f. in D-dim

$$\int d^D x \left[\frac{1}{2e^2} \left| dB_{(p)} + X_{(p+1)} \right|^2 \right] + \frac{i}{2\pi} \int Y_{(q+1)} \wedge \left(dB_{(p)} + X_{(p+1)} \right)$$

where $p+q+2=D$.

$$\text{anomaly poly} = 2\pi i \int_{U_{D+2}} \frac{dX_{(p+1)}}{2\pi} \wedge \frac{dY_{(q+1)}}{2\pi}.$$

mixed anomaly between
p-form $U(1)$ symmetry
&
q-form $U(1)$ symmetry.

Significance in String theory

$E_8 \times E_8$ heterotic string has chiral fermions and B-field.

(anomaly poly)$_{\text{fermion}}$ =

$$2\pi i \int_{U_{12}} \left[\underbrace{\text{inst}(E_8^{(1)}) + \text{inst}(E_8^{(2)}) - \frac{p_1(\text{metric})}{2}}_{\text{4-form}} \right] \wedge Z_{(8)}$$

(anomaly poly)$_{\text{B-field}}$ =

$$2\pi i \int_{U_{12}} \text{“}\frac{dX_{(3)}}{2\pi}\text{”} \wedge \text{“}\frac{dY_{(7)}}{2\pi}\text{”} .$$

So, we see that when the background fields for 2-form U(1) symmetry is

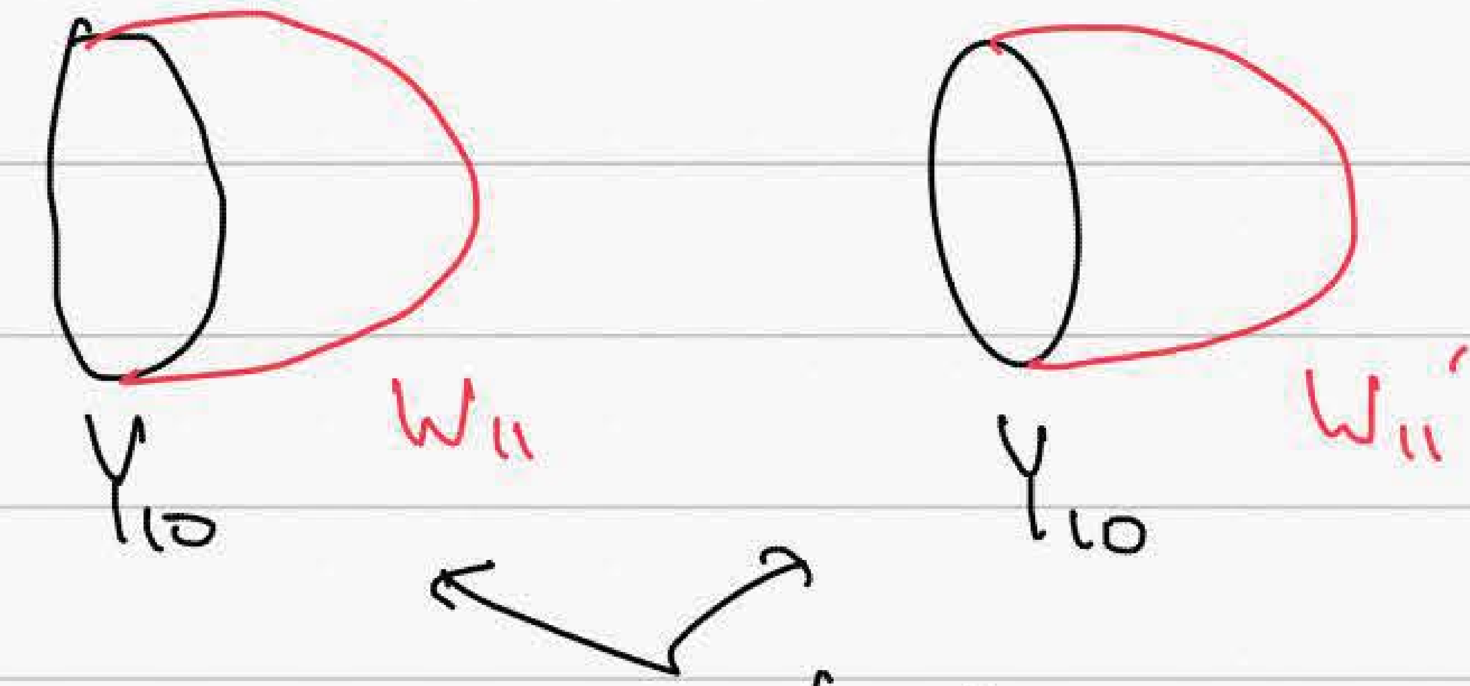
$$\text{“}\frac{dX_{(3)}}{2\pi}\text{”} = \text{inst}\left(E_8^{(1)}\right) + \text{inst}\left(E_8^{(2)}\right) - \frac{p_1(\text{metric})}{2}$$

and 6-form U(1) symmetry is

$$\text{“}\frac{dY_{(7)}}{2\pi}\text{”} = Z_{(8)}$$

the total anom poly vanishes.

There can still be global anomalies:



Let The ratio of the exponentiated action

$=:$ $Z_{\text{Anom.}}$[] .

$$\frac{Z_{\text{Anom}}[\text{here}]}{Z_{\text{Anom}}[\text{there}]} = e^{2\pi i \int_{U_{12}} (\text{anom. poly})} = 1 .$$

$\rightsquigarrow$ $Z_{\text{Anom}}[\ \cdots\ ]$ defines a homomorphism

$$\boxed{\begin{array}{l}\text{bordism classes of} \\ \text{spin 11-manifold} \\ \text{equipped with two } E_8 \text{ bundles}\end{array}} \longrightarrow U(1).$$

(But due to $dH = \underline{\text{inst}(E_8) + \text{inst}(E_8) - \frac{p_1}{2}(\text{metric})}$

This part is cohomologically trivial

$\rightsquigarrow$ essentially one E_8.)

$\Omega^{\text{spin}}_{11}(BE_8) = 0$ (Witten 'Top. Tools' 1986 with an appendix by Stong)

$\rightsquigarrow$ no possibility of global anom.

Another significance in string theory

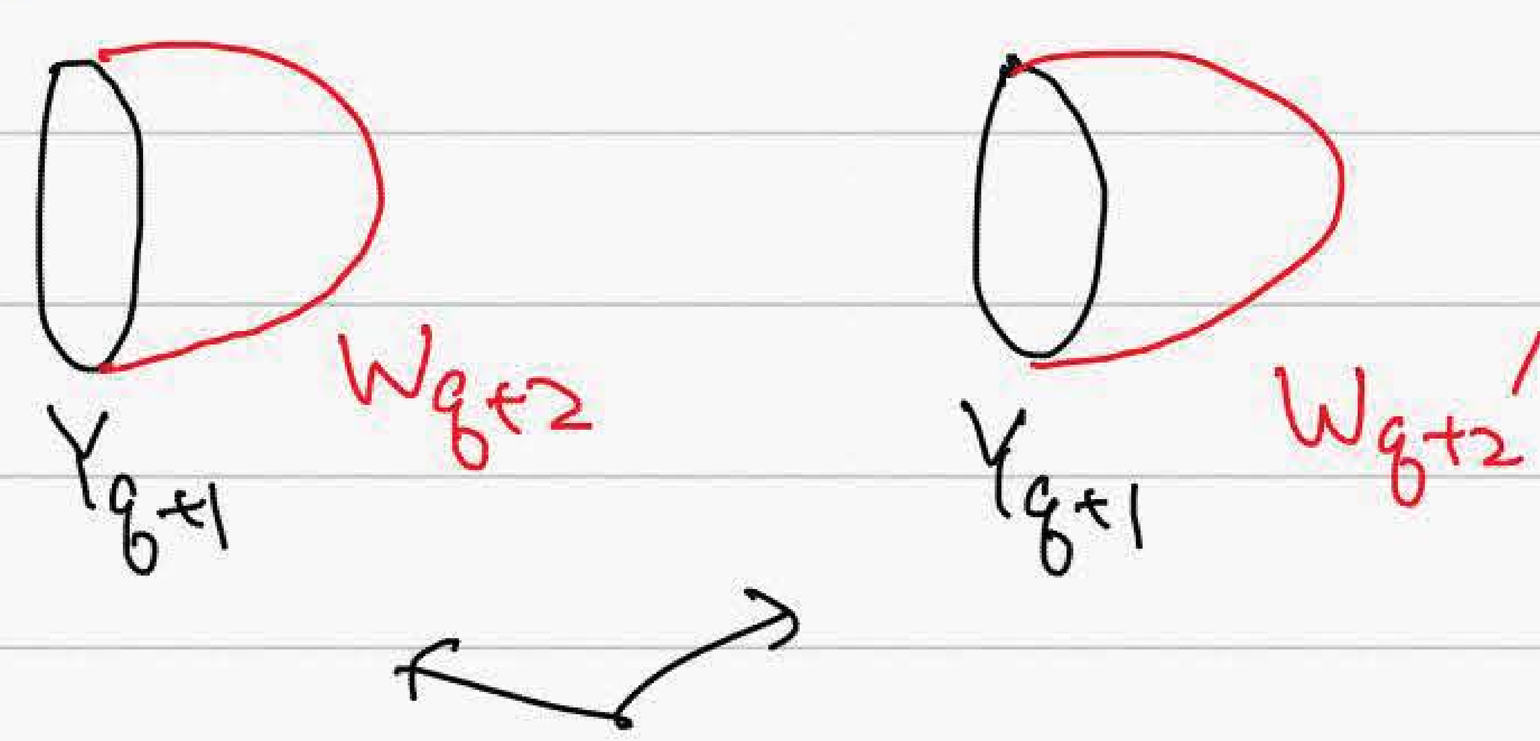
Type I $so(32)$ has $O9^-$-plane
+ 16 D9-branes.

$\longrightarrow$ $O9^-$ has D9-charge -16.

$\downarrow$ T-dual

$O8^-$	D8-charge	-8	
$O7^-$		-4	
$O6^-$		-2	
$O5^-$		-1	
$O4^-$	Are these compatible with Dirac quantization?	$-1/2$	?
$O3^-$		$-1/4$	???
$\vdots$		$\vdots$	

Recall: $\boxed{\int_{Y_{q+1}} A_{(q+1)}}$: worldvolume coupling on D_q-brane



$$\text{difference} = \int_{W_{q+2}} dA_{(q+1)} - \int_{W_{q+2}'} dA_{(q+1)} = \int F_{q+2}$$

$-W_{q+2}'$ W_{q+2}

$\rightsquigarrow$ anomalous unless $\int F_{q+2}/2\pi \in \mathbb{Z}$

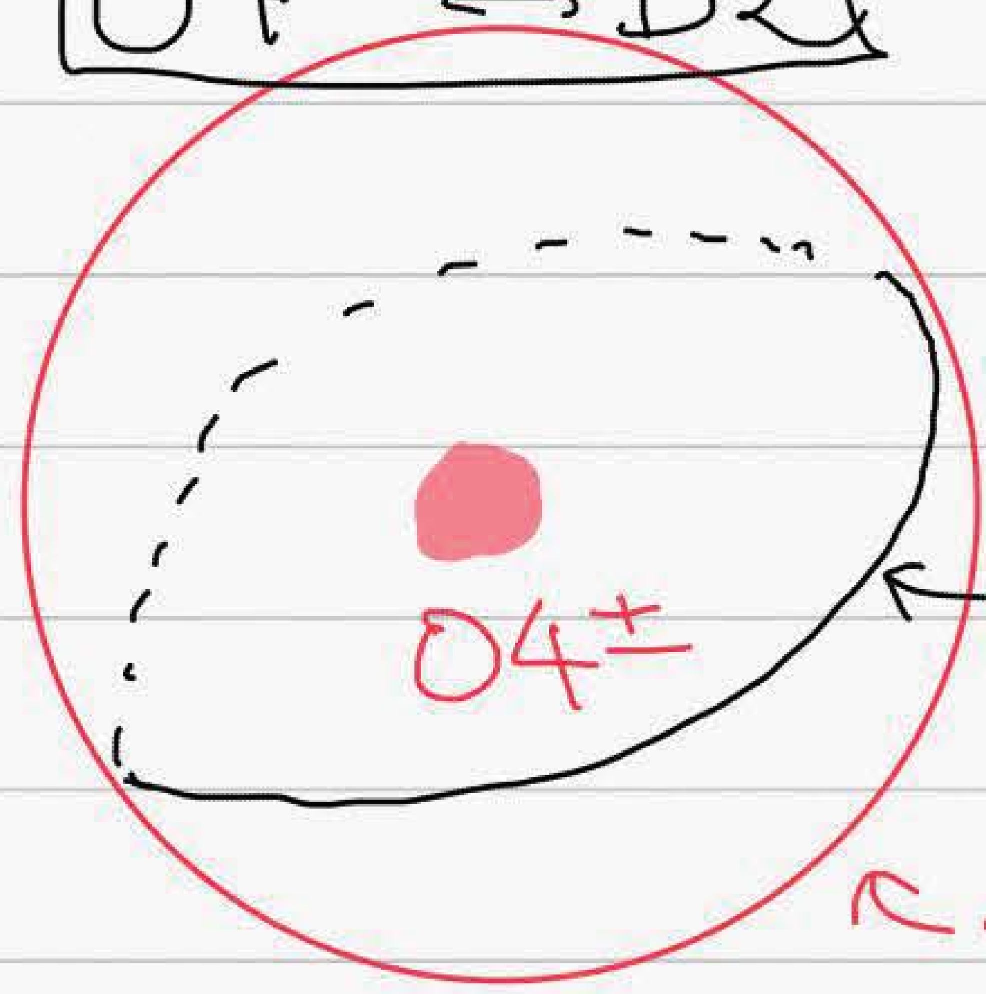
if $\int_{Y_{q+1}} A_{(q+1)}$ is the sole coupling.

But D-branes also have worldvolume fermions
+ U(1) gauge field.

We saw both can be anomalous.

$\rightsquigarrow$ need to consider the total anomaly.

$\boxed{O4^{\pm} \leftrightarrow D2}$



has $\mathcal{N}=8$ SUSY = has 8 fermions.

$\downarrow$

D2 worldvolume Y_3

$W_4 = S^4/\mathbb{Z}_2$ surrounding $O4^{\pm}$

fermion anomaly

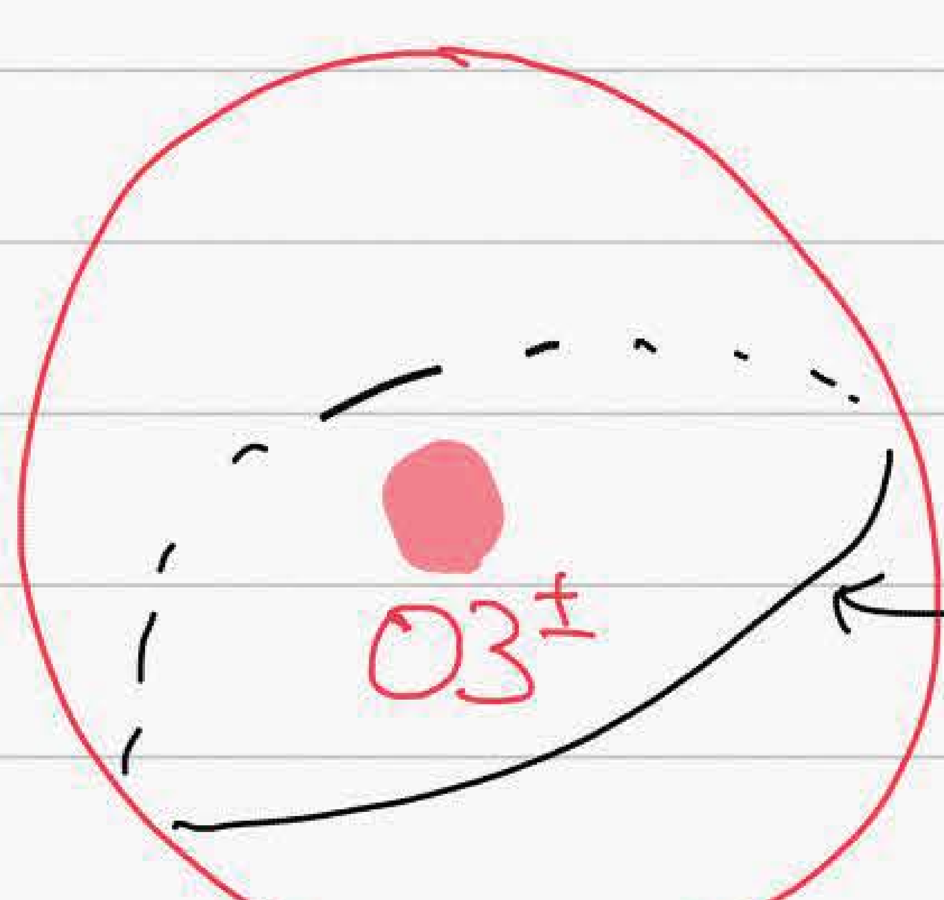
$$= \Big[e^{2\pi i \boxed{\overset{\pm 1/16}{\eta(\text{single fermion})}}} \Big]^8 = -1$$

$$e^{2\pi i \int_{S^4/\mathbb{Z}_2} F_{(4)}} = e^{2\pi i \cdot (\pm 1/2)} = -1$$

cancels!

Witten 1997

$\boxed{O3^{\pm} \longleftrightarrow D3}$



has $\mathcal{N}=4$ SUSY = has 4 fermions.

$\downarrow$

D3 worldvolume Y_4

$W_5 = S^5/\mathbb{Z}_2$ surrounding $O3^{\pm}$

$$e^{2\pi i \int_{S^5/\mathbb{Z}_2} F_{(5)}} = e^{2\pi i (\pm 1/4)}$$

$$(\text{fermion anomaly}) = \left[e^{2\pi i \boxed{\eta(\text{single fermion})}} \right]^4 = e^{2\pi i (-1/4)}$$

known to be $-1/4$

$$\text{Maxwell anomaly} = e^{2\pi i \int_{S^5/\mathbb{Z}_2} \frac{B}{2\pi} \frac{dC}{2\pi}} = e^{\pi i bc}$$

$$b := \text{`}\frac{dB}{2\pi}\text{'}, \quad c := \text{`}\frac{dC}{2\pi}\text{'} \in H^3(S^5/\mathbb{Z}_2, \tilde{\mathbb{Z}}) = \mathbb{Z}_2$$

		b	c	$\frac{1}{2}bc$	4η	$\int F_{(5)}$
permuted by $SL(2,\mathbb{Z})$	$\widetilde{O3}^+$	0	0	0	$-1/4$	$+1/4$
	$O3^+$	0	1	0	$-1/4$	$+1/4$
	$\widetilde{O3}^-$	1	0	0	$-1/4$	$+1/4$
$SL(2,\mathbb{Z})$ invariant	$O3^-$	1	1	$1/2$	$-1/4$	$-1/4$
				Maxwell	fermion	'scalar'

$SL(2,\mathbb{Z})$ acts as a spin structure on the torus

where 0 : NS, 1 : R

Hsieh-Yonekura-YT 2018 2019

O_p^+ with $p=2,1,0$ $\xleftrightarrow[\text{cancel}]{\text{nicely}}$ fermion anomaly

O_p^- with $p=2,1,0$: Maxwell contribution still mysterious.

$\boxed{\text{Example 4}}$ Anomalies of finite groups in 0+1 d.

0+1 d QFT = QM

G-symmetric $\qquad G \curvearrowright \mathcal{H}$

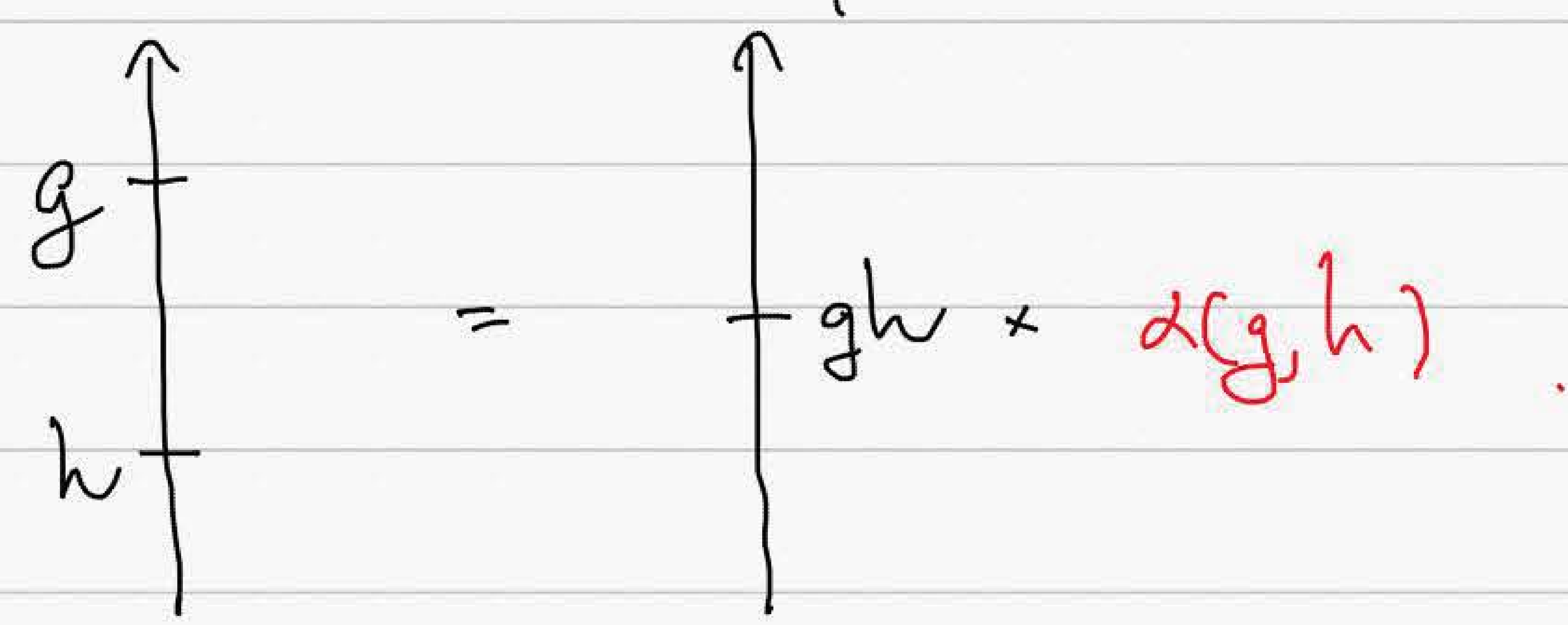
$$\rho(g)\rho(h) = \rho(gh)$$

In QM, only wavefunctions up to an overall phase matter

$\rightsquigarrow \quad \rho(g)\rho(h) = \rho(gh)\underbrace{\alpha(g,h)}_{\in U(1)}$ allowed.

(Wigner 1931)

called $\begin{cases} \text{projective reps.} \\ \text{anomalous } G\text{-sym.} \end{cases}$ equivalent!



$$\rho(g)\rho(h)\rho(k) = \rho(g)\rho(hk)\alpha(h,k)$$
$$= \rho(ghk)\alpha(g,hk)\alpha(h,k)$$

$\rho(gh)\rho(k)\alpha(g,h)$

$= \rho(ghk)\alpha(g,h)\alpha(gh,k)$

$$\therefore \quad \alpha(g,hk)\alpha(h,k) = \alpha(g,h)\alpha(gh,k) \quad \cdots \#$$

"α is a 2-cocycle valued in $U(1)$".

We can also redefine

$$\tilde{\rho}(g) = \rho(g)\underbrace{\beta(g)}_{\in U(1)}.$$

Then $\tilde\rho(g)\tilde\rho(h) = \tilde\rho(gh)\tilde\alpha(g,h)$ where

$$\tilde\alpha(g,h) = \alpha(g,h)\,\frac{\beta(g)\beta(h)}{\beta(g,h)}. \quad \cdots\cdots\ ♡$$

$\rightsquigarrow$ $\boxed{\text{proj. rep / anm in 0+1d is classified by } \alpha \text{ satisfying } ☆ \text{ modulo identification by } ♡}$.

For G: finite, we let

an abelian gp

$$C^n(G,A) = \{ f(g_1, g_2, \cdots, g_n) \in A \}$$

and

$$\delta : C^n(G,A) \to C^{n+1}(G,A)$$

via

$$\begin{aligned}(\delta f)(g_1,\cdots,g_n,g_{n+1}) = {} & f(g_2,g_3,\cdots,g_{n+1}) \\ & - f(g_1g_2,g_3,\cdots,g_{n+1}) \\ & + f(g_1,g_2g_3,\cdots,g_{n+1}) \\ & \vdots \\ & (-1)^n f(g_1,g_2,\cdots,g_ng_{n+1}) \\ & -(-1)^n f(g_1,g_2,\cdots,g_n).\end{aligned}$$

This satisfies $\delta^2 = 0$

Then

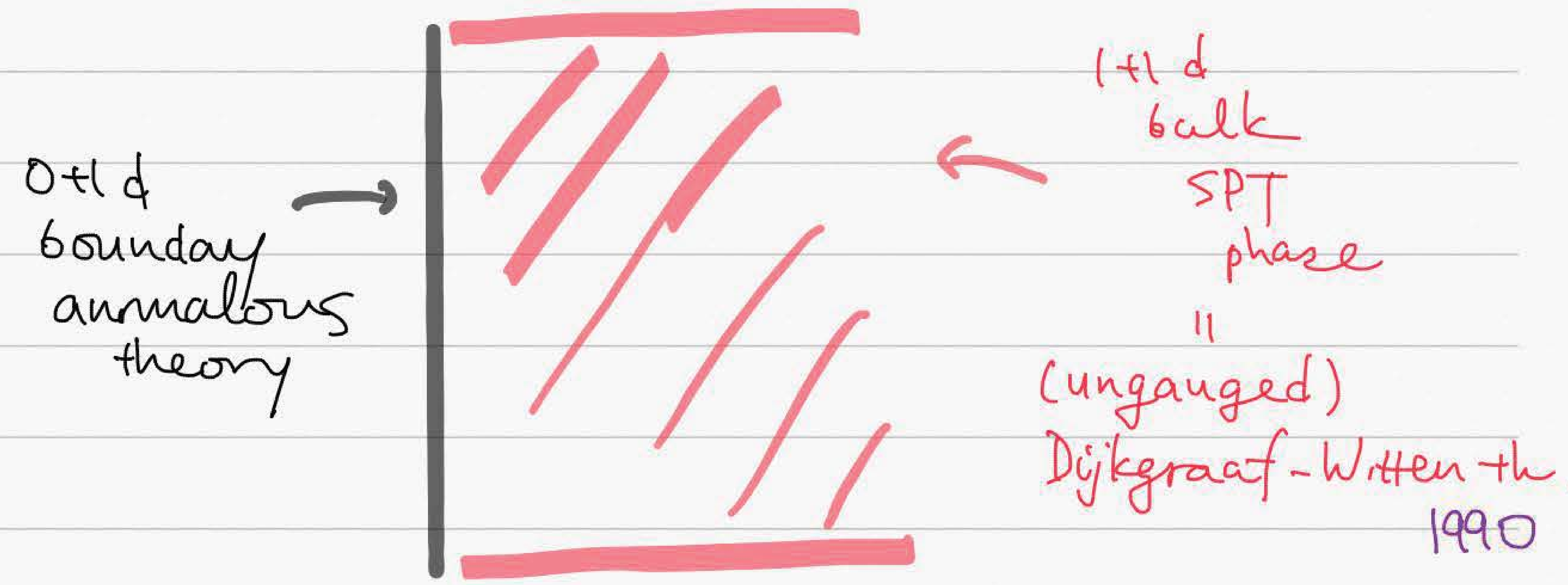
$$H^n(G,A) := \frac{\operatorname{Ker} \delta : C^n(G,A) \to C^{n+1}(G,A)}{\operatorname{Im} \delta : C^{n-1}(G,A) \to C^n(G,A)}.$$

The box $\boxed{}$ above is $H^2(G, U(1))$ written in the multiplicative notation.

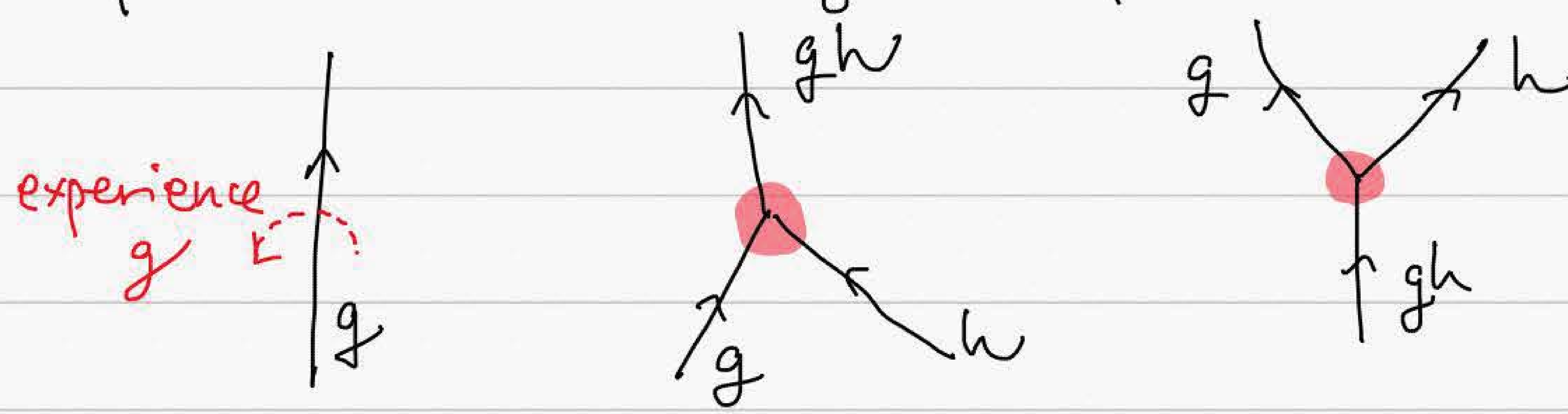
$\Big($ $\exists$ space BG s.t. $H^n(BG,A) = H^n(G,A)$ $\Big)$

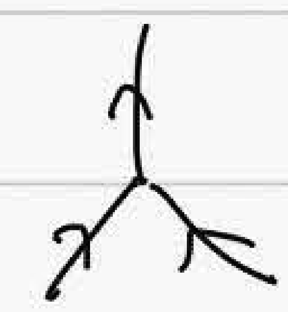
in the geometric sense — algebraically defined above.

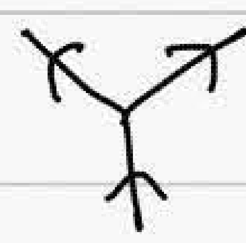
What's the bulk theory?



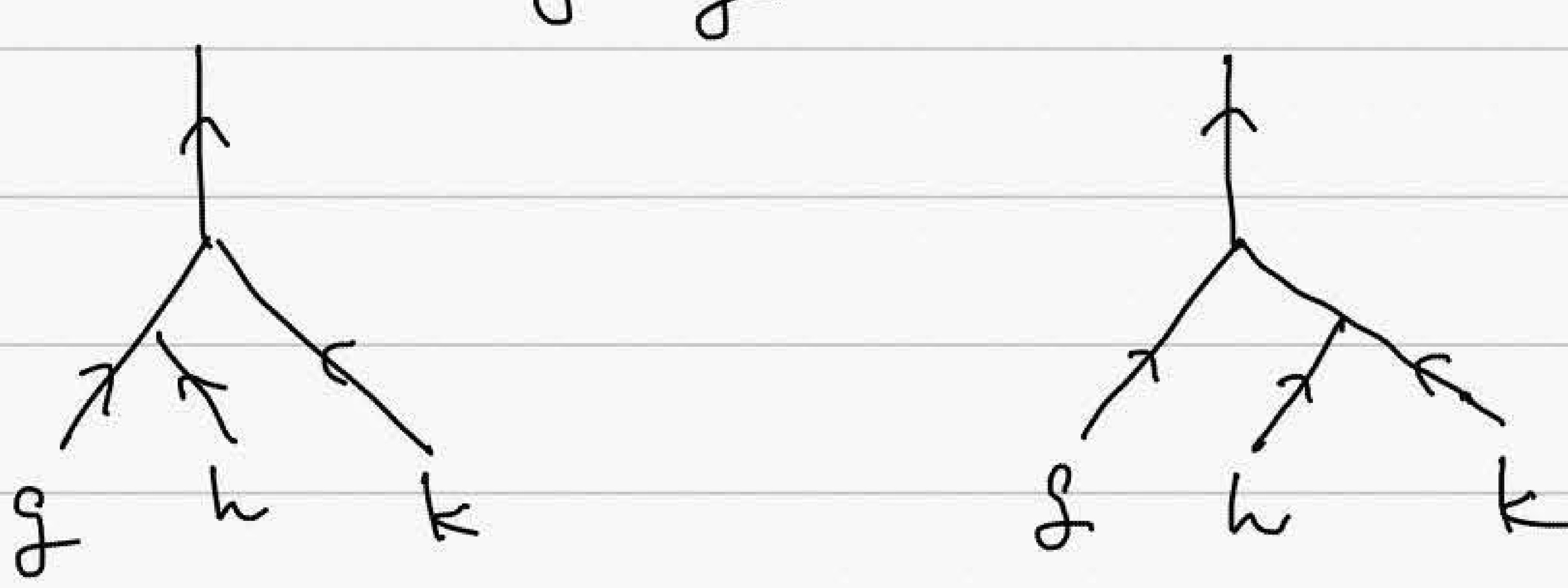
To describe ungauged DW theory,
represent the G-background by walls and their junctions



Then associate a factor of $\alpha(g,h)$ to 

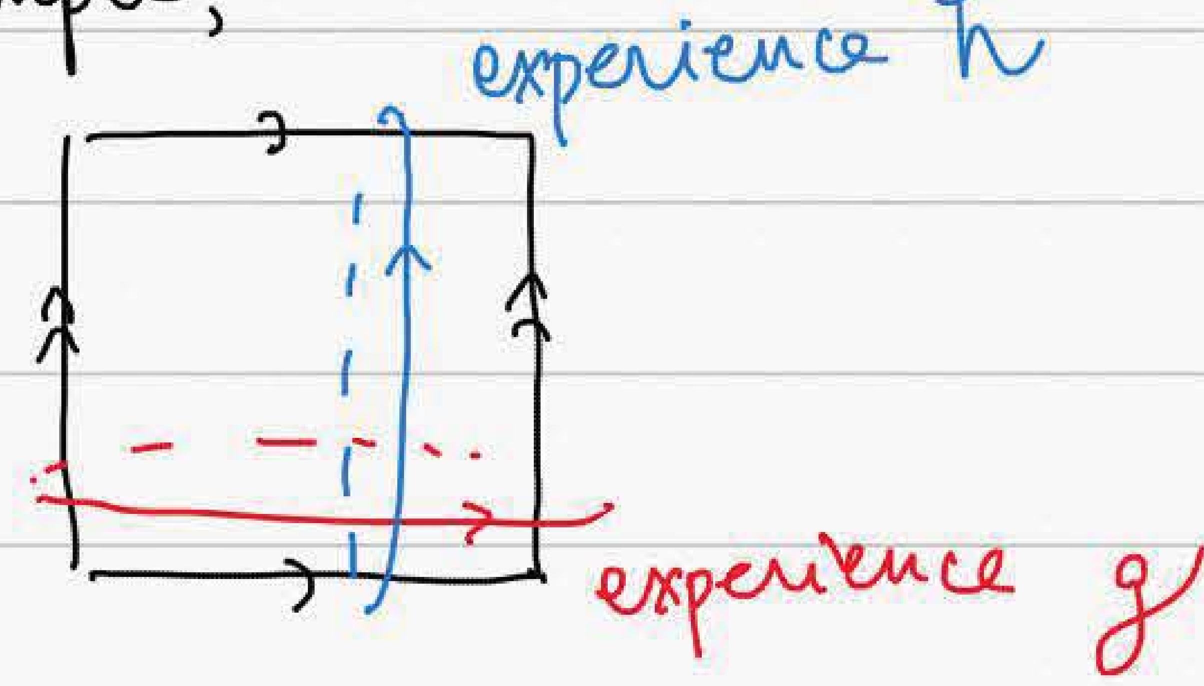
and $\alpha(g,h)^{-1}$ to  .

The combined factor independent of the way
one draws the junctions representing the same
G-backround gauge field :

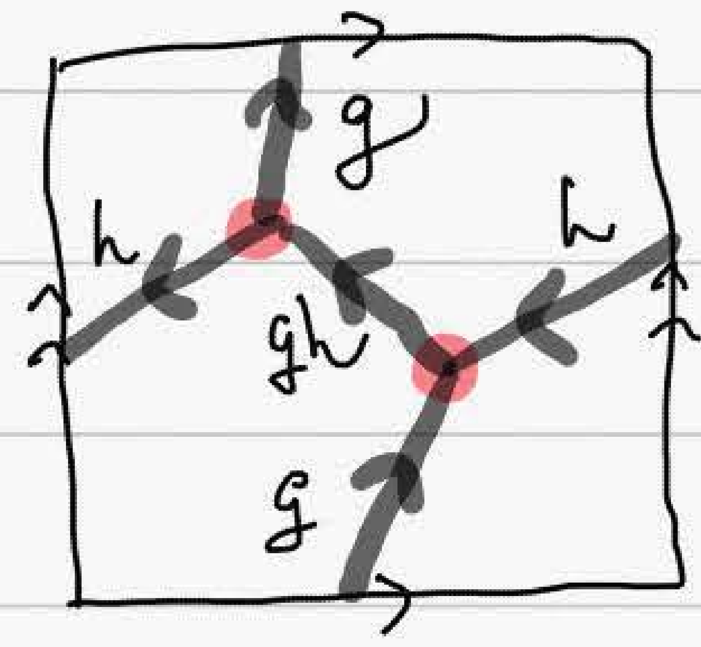


$$\alpha(g,h)\,\alpha(gh,k) = \alpha(g,hk)\,\alpha(h,k).$$

For example,



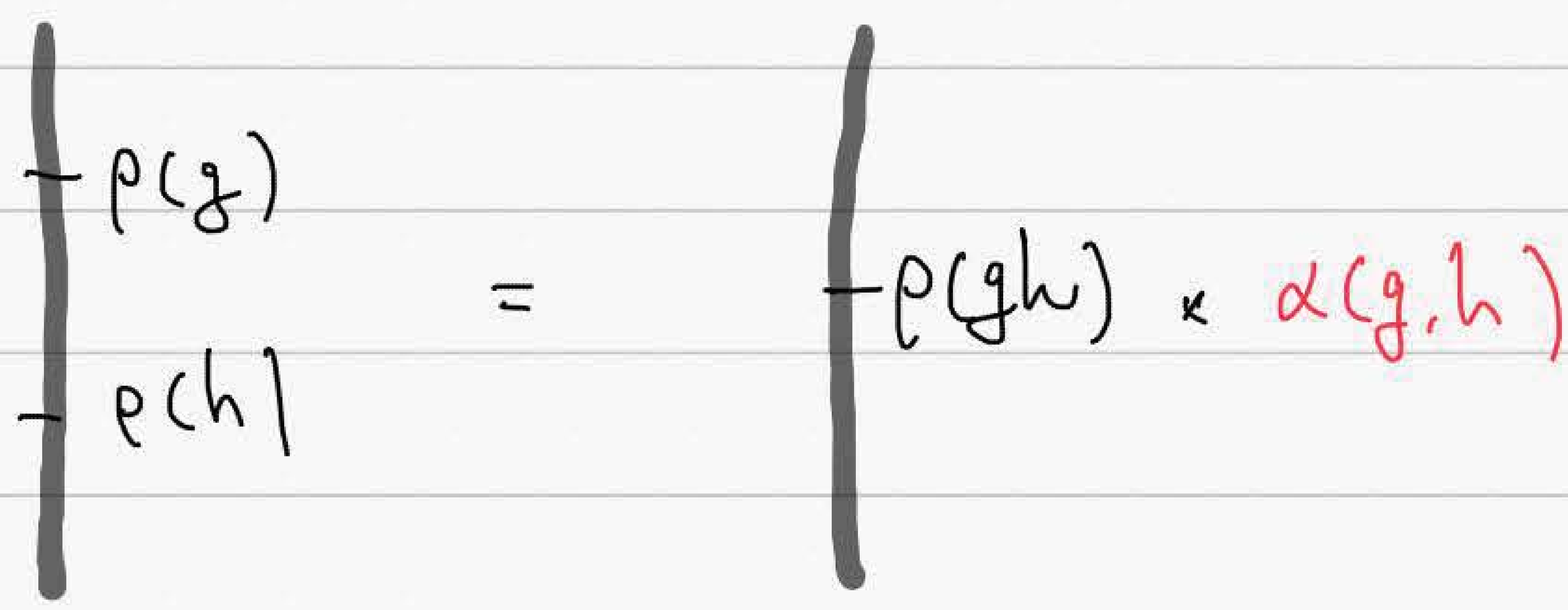
can be represented by the following junction



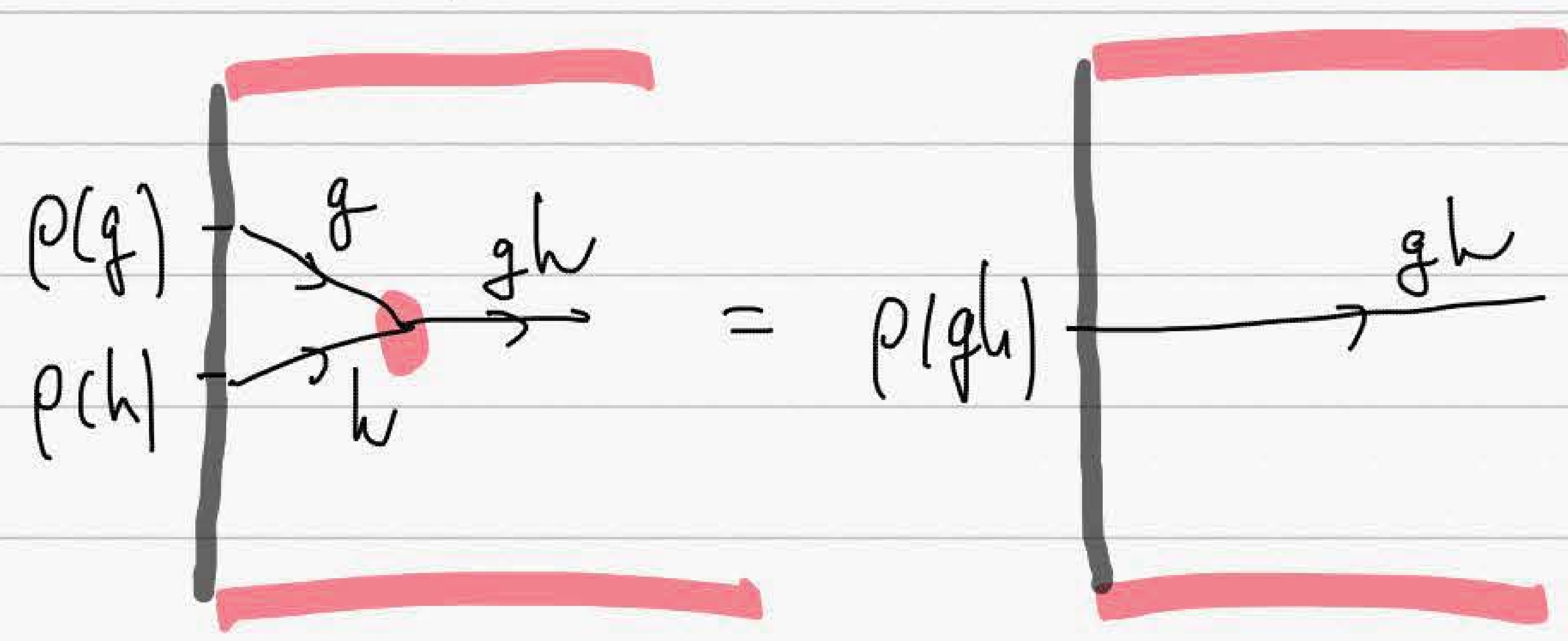
with $Z = \dfrac{\alpha(g,h)}{\alpha(h,g)}$.

How does this bulk theory cancel the boundary anomaly?

Recall



With the bulk,



Very simple.

Chen-Gu-Liu-Wen 2011 (general d)

Appearance in String Theory

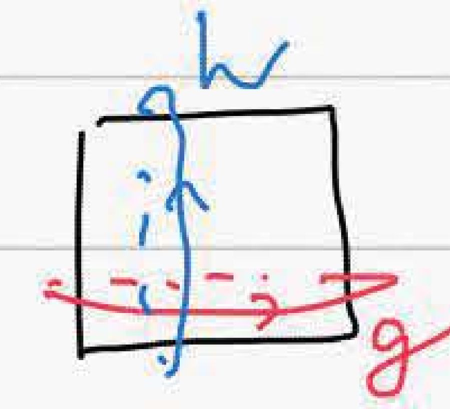
Given a 2d worldsheet theory Q with non-anomalous G-symmetry, we can consider gauging G, ("orbifolding")

forming a new theory Q/G.

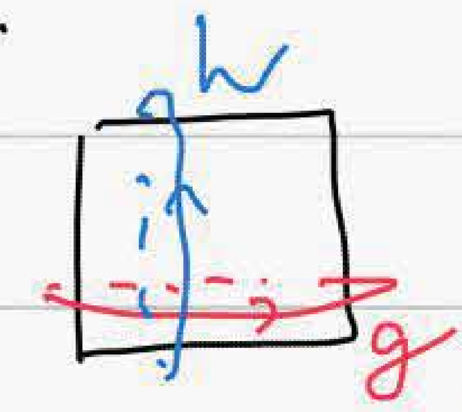
Call the bulk theory just introduced as SPT_α. This is also G-symmetric and non-anomalous. We can then also form

$$(Q \times SPT_\alpha)/G .$$

$$Z_{Q/G}(T^2) = \frac{1}{|G|} \sum_{gh=hg}$$



$$Z_{(Q\times SPT_\alpha)/G}(T^2) = \frac{1}{|G|} \sum_{gh=hg} \Bigg[$$



$$\frac{\alpha(g,h)}{\alpha(h,g)} \Bigg] .$$

Called Discrete Torsion. (Vafa 1986)

A Boundary condition = a brane.

Brane in an orbifold with discrete torsion carries a projective representation. (Douglas 1998)

Anomaly of a spacetime symmetry ... in 0+1 d.

The only possibility = time reversal.

T is anti-unitary.

$T^2 = c$: constant. unitary.

$$T^3 = T^2 T = cT$$
$$\parallel \quad TT^2 = Tc = \bar{c}T.$$

$\therefore \quad c = \bar{c}. \qquad \therefore \quad c = \pm 1$

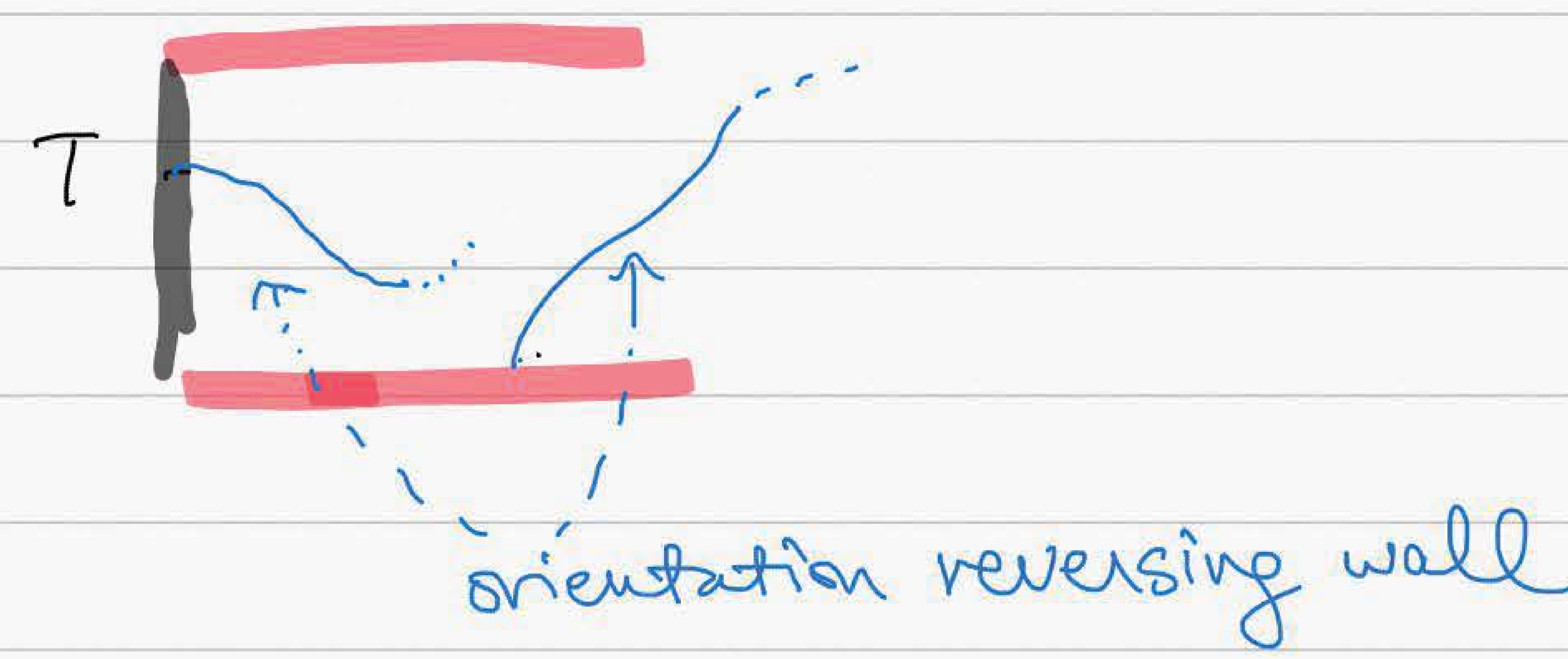
ordinary sym compatible with

$T^2 = +1$: (a subgroup of) $O(n)$

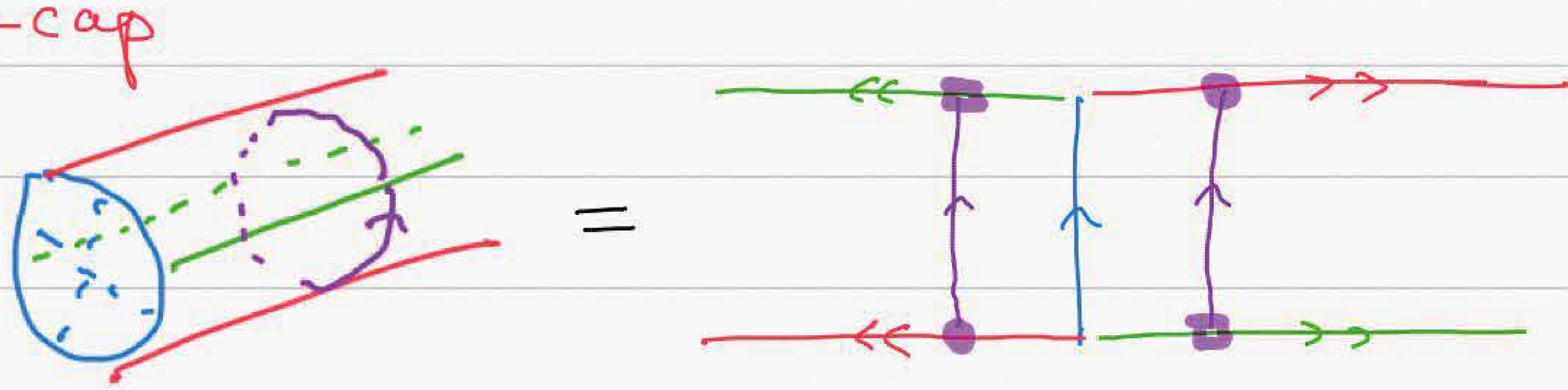
$T^2 = -1$: (a subgroup of) $Sp(n)$

The corresponding bulk th. allows

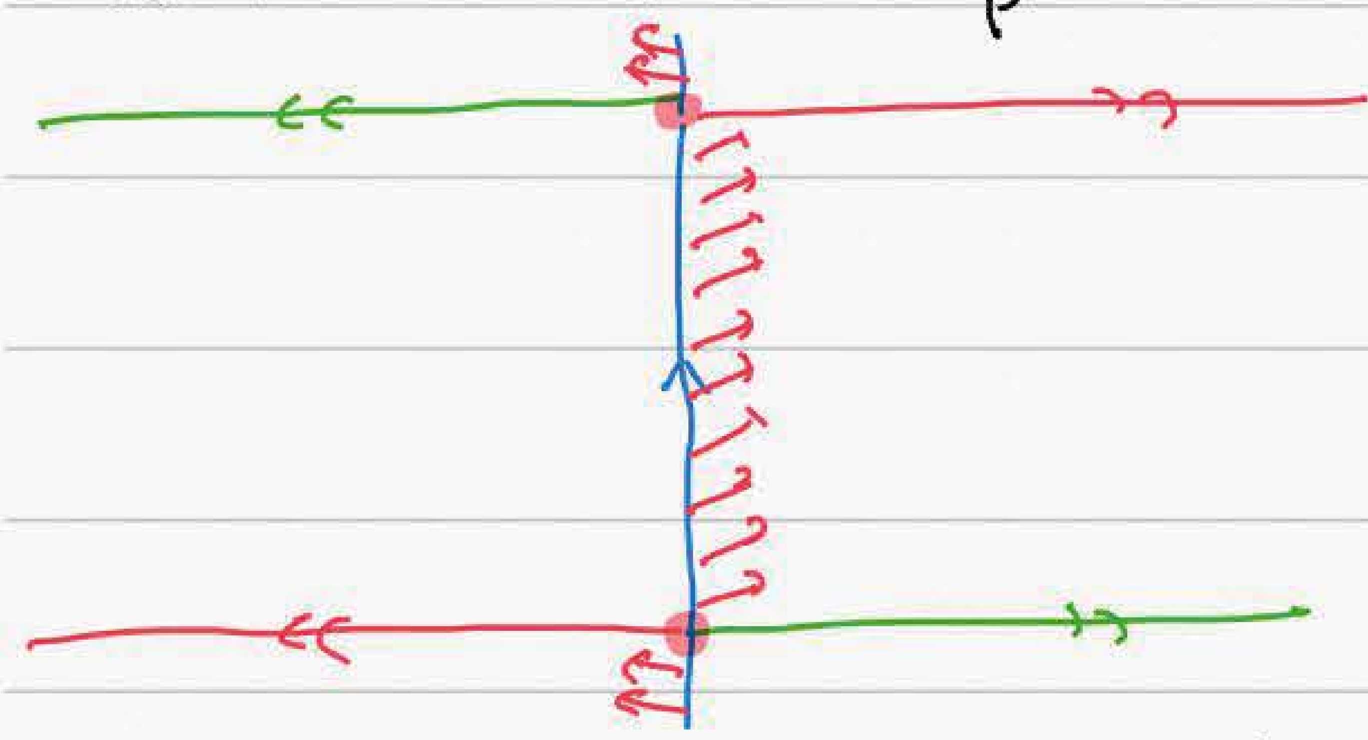
time reversal = parity = orientation reversal.



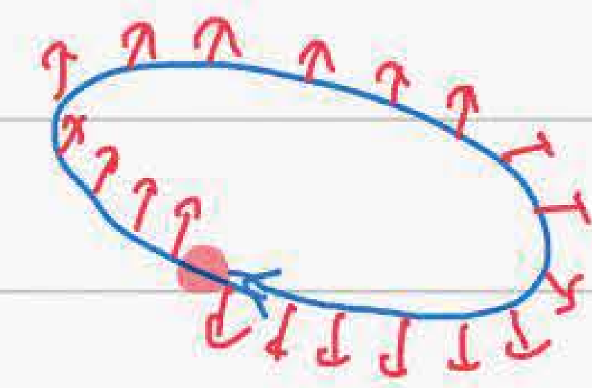
cross-cap



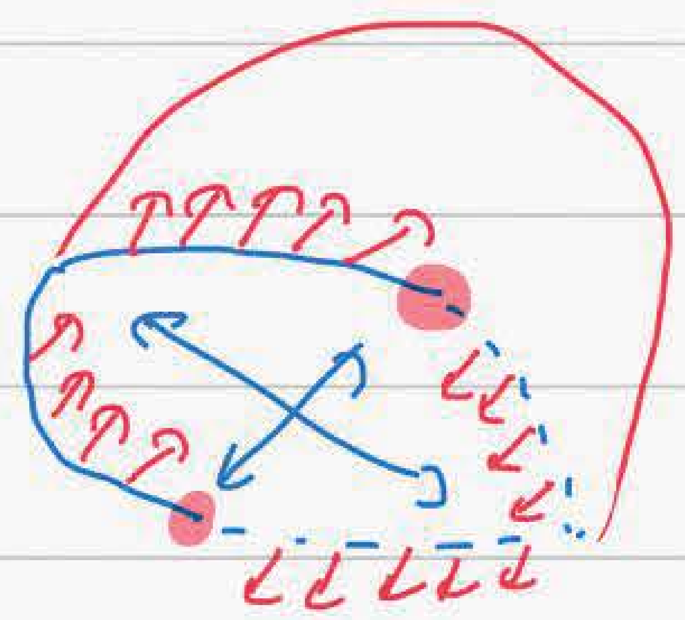
Orientation reversing wall comes with local orientation which can flip:



or

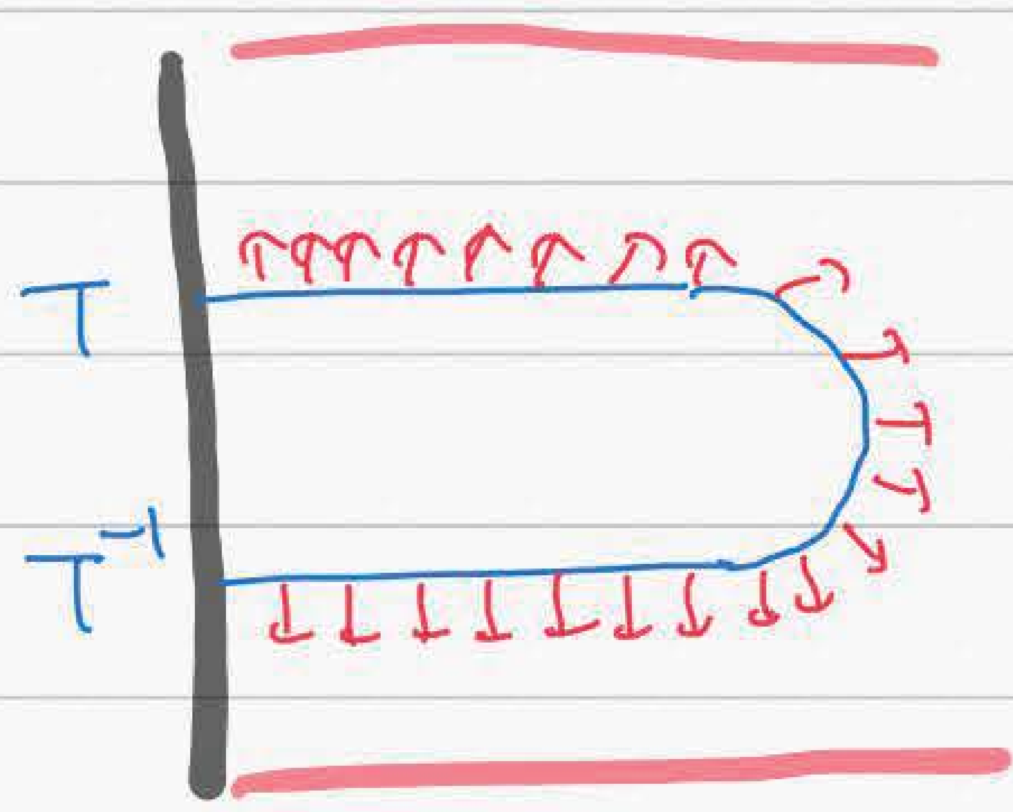


we can assign ± 1 per such flip.

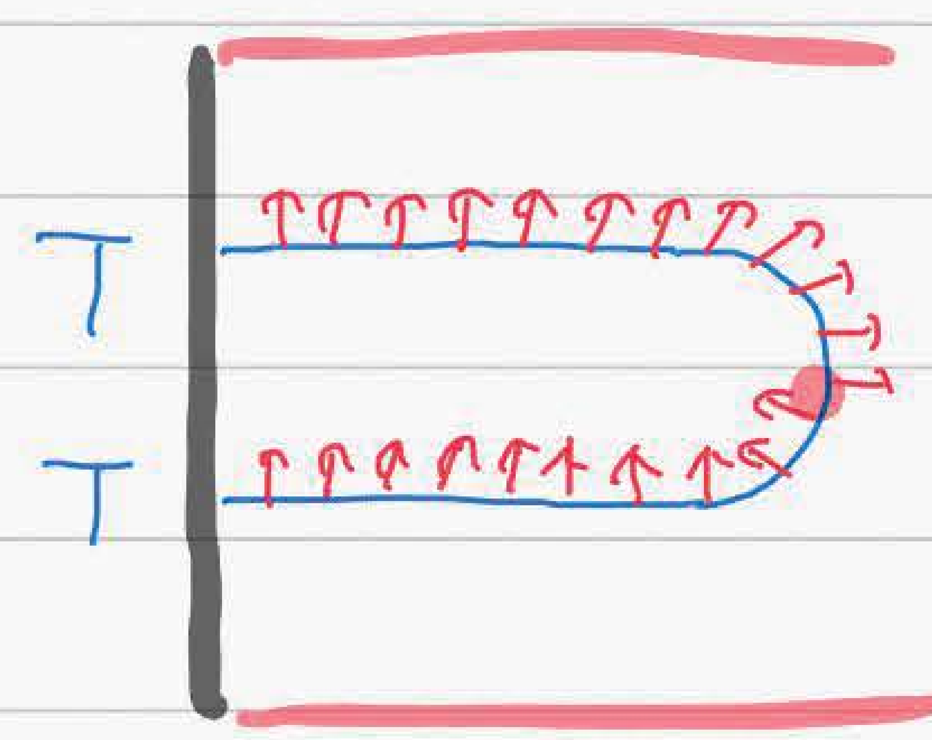


$$Z(\mathbb{RP}^2) = \pm 1.$$

equivalently, ± 1 per cross-cap.



$TT^{-1} = 1.$



$T^2 = \pm 1.$

$\rightsquigarrow$ crosscap $= \pm 1$

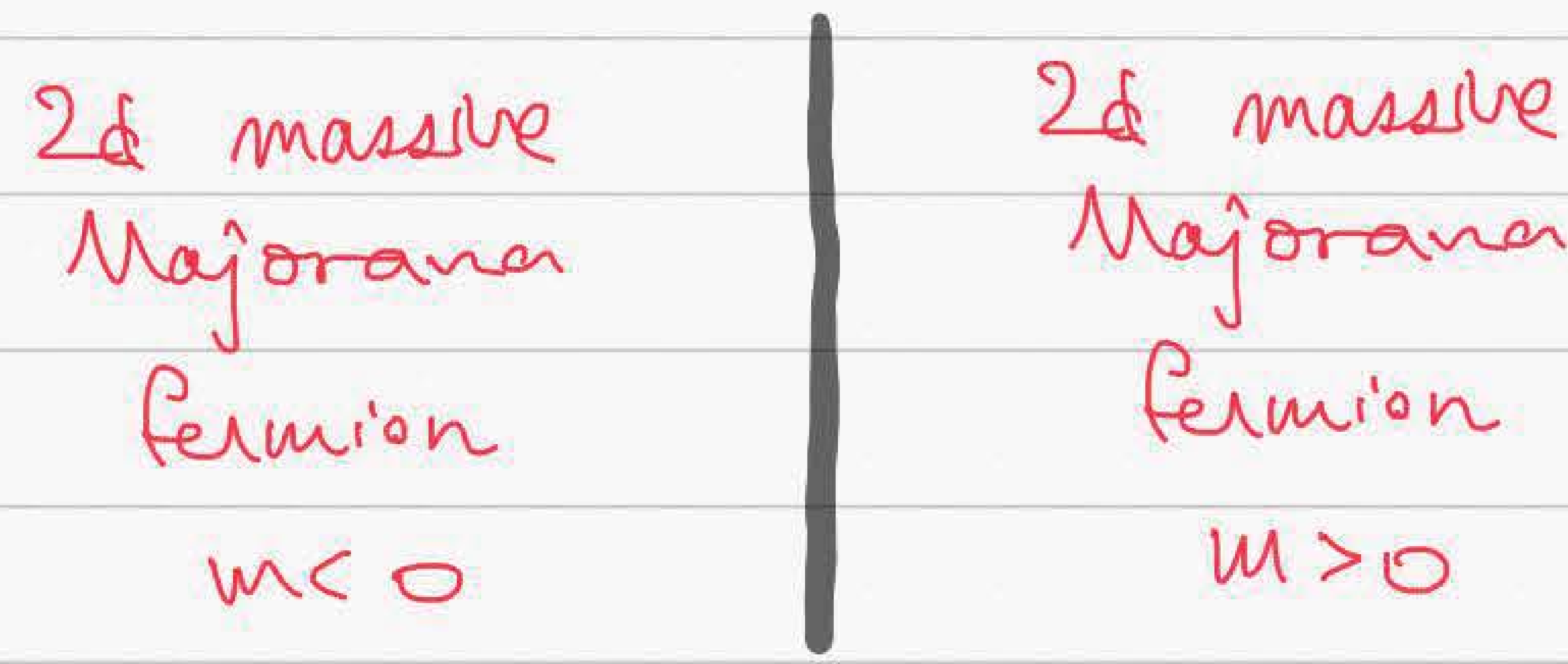
$\Longleftrightarrow$ $T^2 = \pm 1$

$\Longleftrightarrow$ sym. on the boundary is $\begin{cases} \text{orthogonal} \\ \text{symplectic} \end{cases}$.

In string theory, orientifolds have two types $O^{\pm}$ distinguished by ± 1 attached to a crosscap which also determines the gauge group to be $\begin{cases} \text{ortho. or} \\ \text{sympl.} \end{cases}$

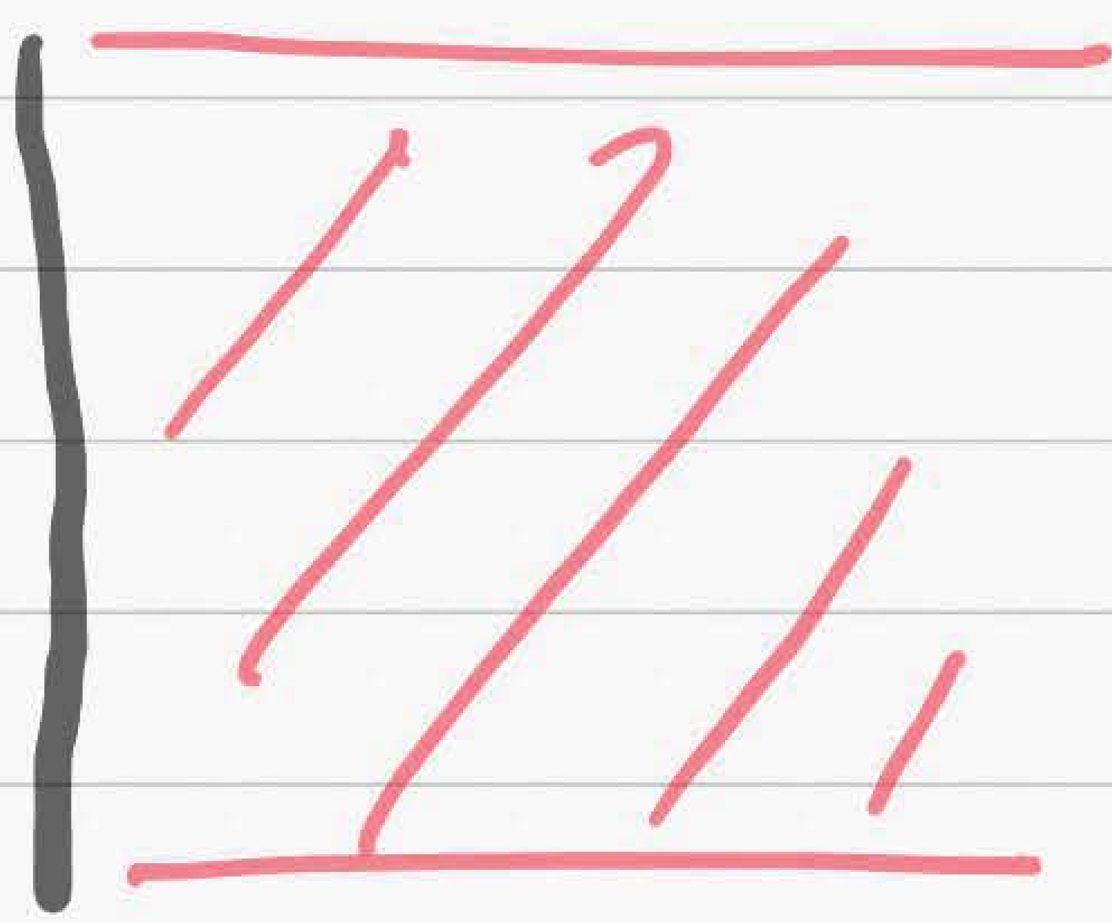
Marcus - Sagnotti, Schwarz 1982

Anomaly of a fermion ... in 0+1 d.



single Majorana fermion zero mode ψ.

What's the bulk theory?



$$Z_{\text{bulk}} = \frac{Z_{\text{fermion}}(m\to+\infty)}{Z_{\text{fermion}}(m\to-\infty)}.$$

$$= e^{2\pi i \eta}$$

where

$$\eta = \frac{1}{2} \sum_{E:\text{ eigenvalue of } \not{D}} \text{sgn}(E),$$

$\not{D}$ anticommutes with the chirality operator $\Gamma^3 := \Gamma^1\Gamma^2$.

$\rightsquigarrow$ E and $-E$ come in pairs.

declare to be $+1$.

$$\rightsquigarrow \eta = \frac{1}{2} \sum_{\substack{\text{zero eigenvalue}\\ E \text{ of } \not{D}}} \text{sgn}(E)$$

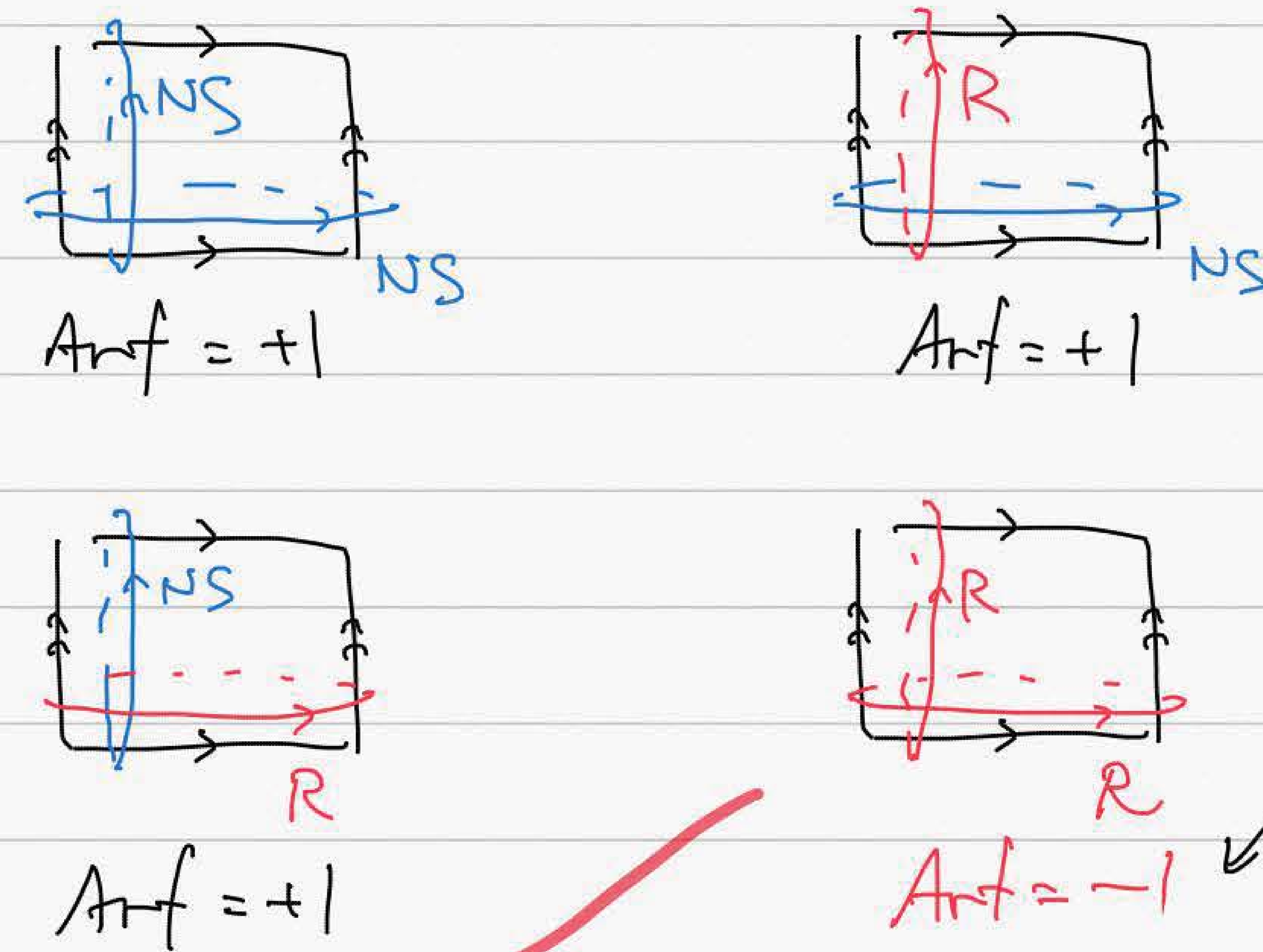
$$= \frac{1}{2} \text{ index } \not{D}$$

an example of mod-2 index.

$$\rightsquigarrow Z_{\text{bulk}}(M) = (-1)^{\text{index } \not{D}} =: \text{Arf}(M)$$

also known as the Kitaev chain..

e.g.



R-sector vac. has $(-1)^F = -1$.

Essentially,

$$Z = \int d\psi_0 d\bar{\psi}_0 \, e^{m\psi_0\bar{\psi}_0} = m . \rightsquigarrow \frac{Z(m)}{Z(-m)} = -1 .$$

In string theory,

GSO projection involves $\left\{ \begin{array}{l} \text{summation over} \\ \text{gauging} \end{array} \right\}$ spin structure.

$$\tilde{Q} = Q / \text{spin str} . \qquad \tilde{Z} = \frac{1}{2}\sum_{NS,R} \square$$

$$\longrightarrow$$

$$\tilde{Q}' = [Q \times \text{Arf}] / \text{spin str.} \qquad \tilde{Z}' = \frac{1}{2}\sum_{NS,R}\left[\square \times (\pm 1)\right]$$

This is the origin of the mysterious phase distinguishing Type IIA / Type IIB.

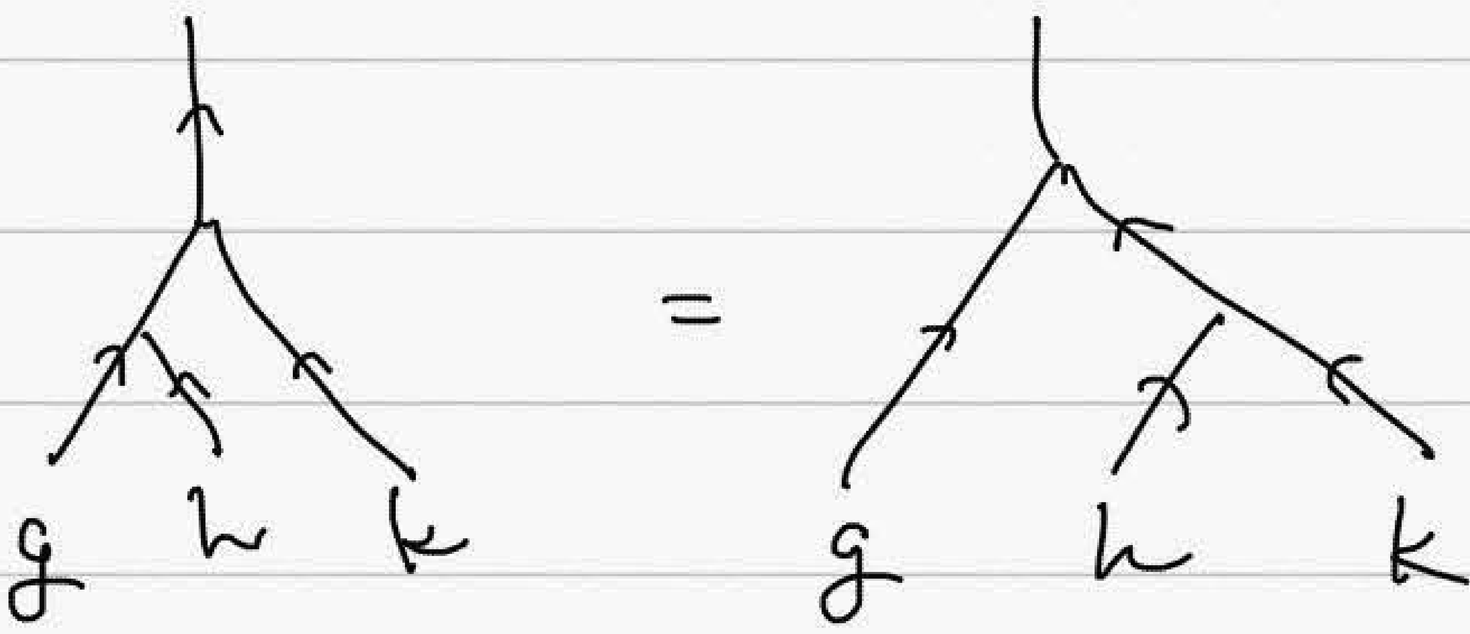
Seiberg - Witten 1986

Example 5 Anomalies of finite groups in 2d.

Consider bosonic systems first.

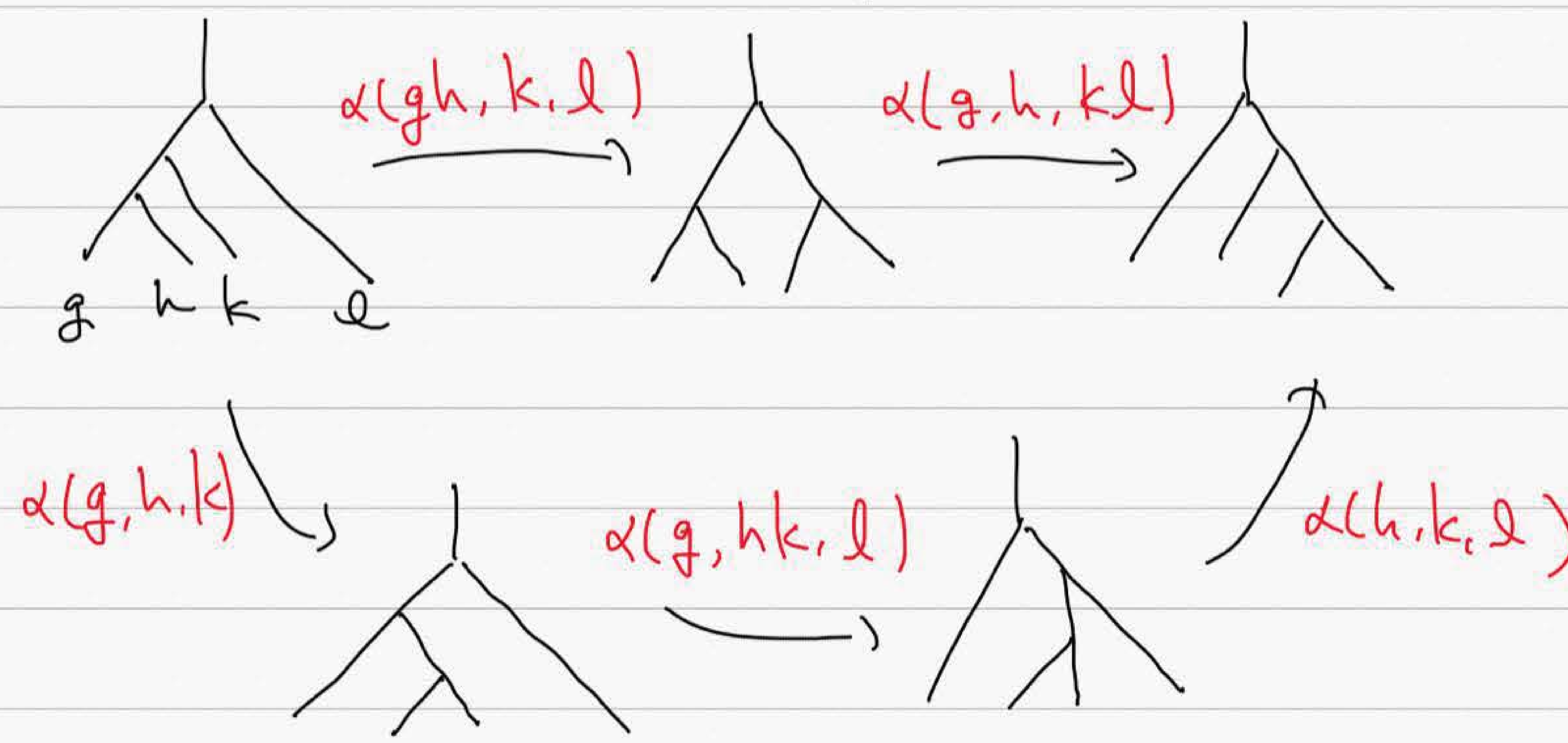
Represent G-backgrounds by walls and junctions.

Now two equivalent junctions can give a phase difference:



$\times\ \alpha(g,h,k)$

α needs to satisfy the following pentagon relation:



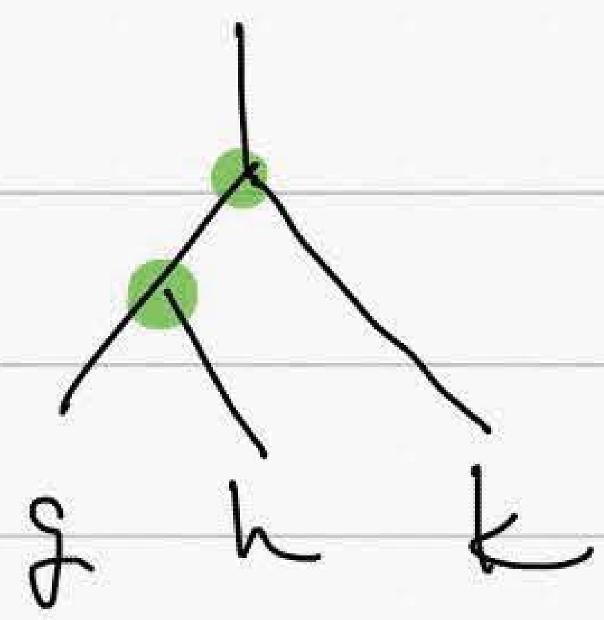
i.e.
$$\alpha(gh,k,l)\,\alpha(g,h,kl) = \alpha(g,h,k)\,\alpha(g,hk,l)\,\alpha(h,k,l)$$

i.e. $\delta\alpha = 1$.

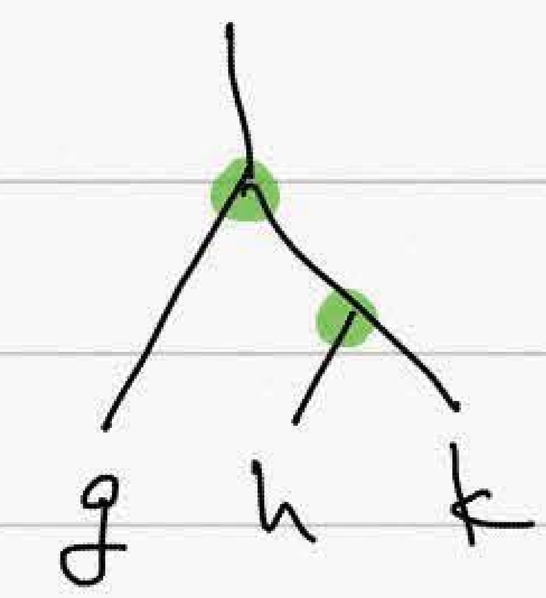
We can also declare that

carries an additional factor $\beta(g,h)$.

$\Longrightarrow$



$=$



$\times\ \tilde{\alpha}(g,h,k)$

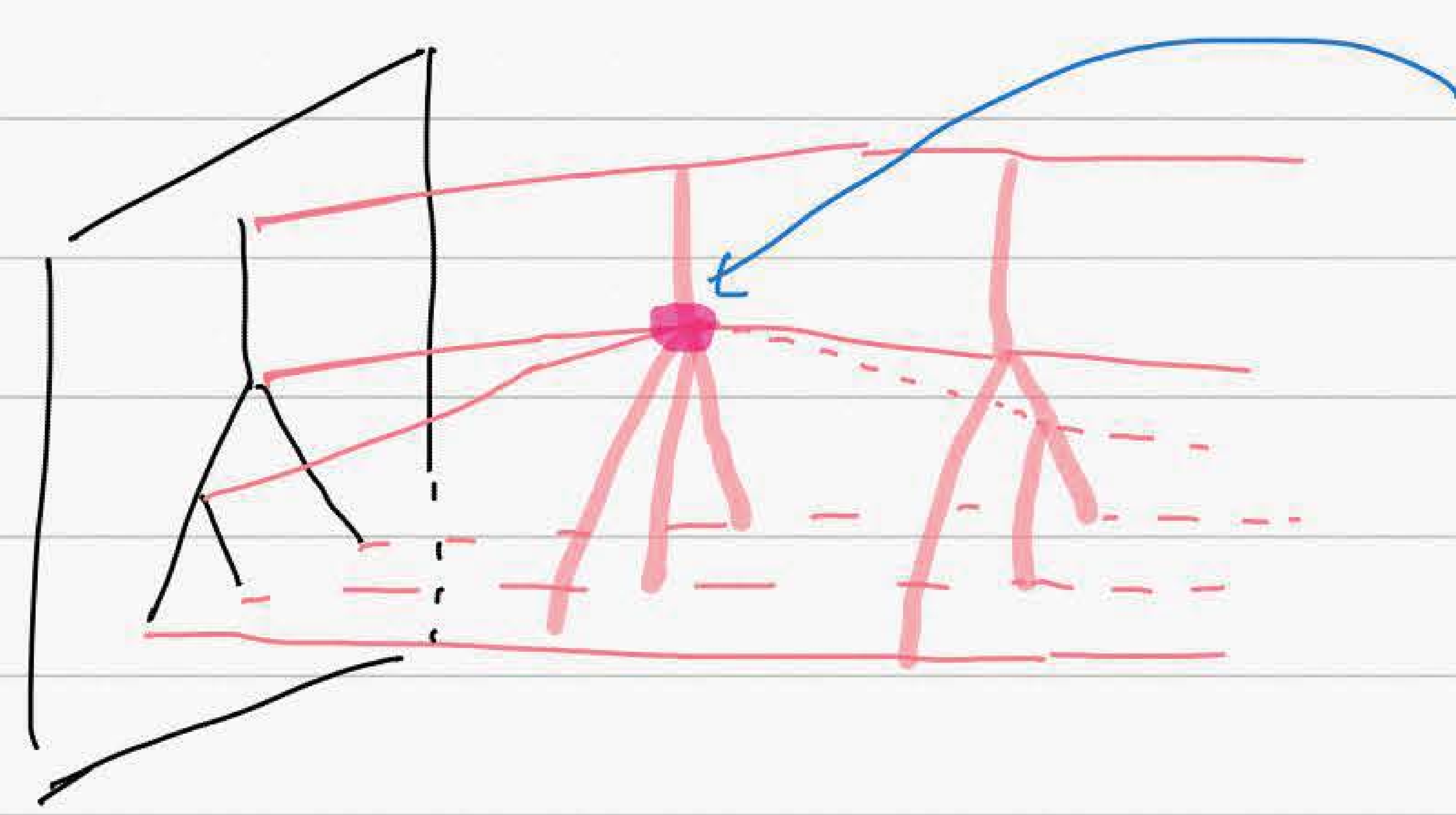
where $\tilde{\alpha}(g,h,k) = \alpha(g,h,k)\,\dfrac{\beta(g,h)\,\beta(gh,k)}{\beta(g,hk)\,\beta(h,k)}$

$= \alpha(g,h,k)\,\delta\beta(g,h,k)\,.$

$\rightsquigarrow$ anomaly of a (bosonic) G-symmetry is characterized by

$$H^3(G,U(1)) = \frac{\{\delta\alpha = 1\}}{\{\delta\beta = 1\}}\,.$$

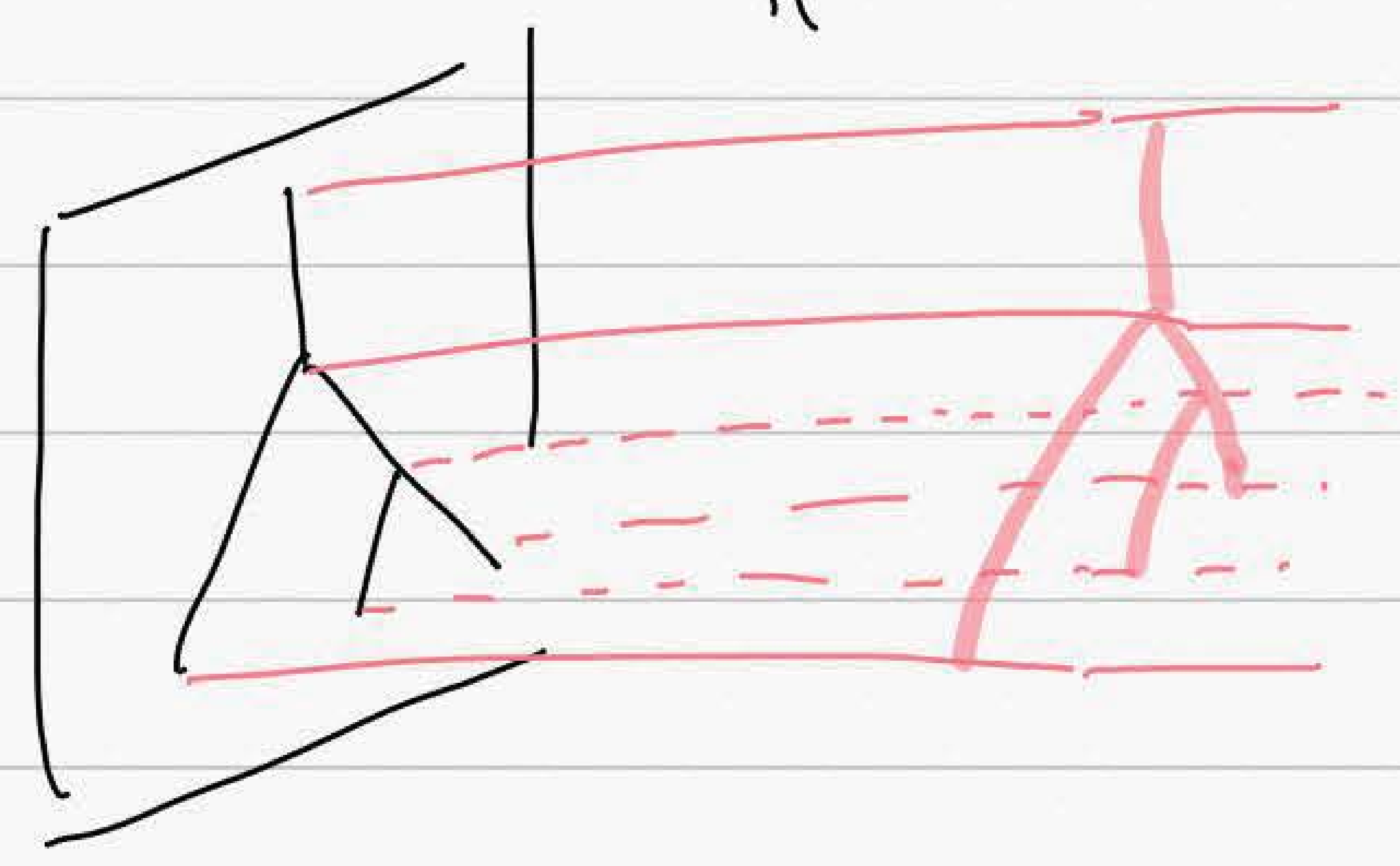
This anomalous phase is canceled by the 3d ungauged Dijkgraaf-Witten theory, exactly as before.



a factor of $\alpha(g,h,k)$

$\|$

$\alpha(g,h,k)\ \times$

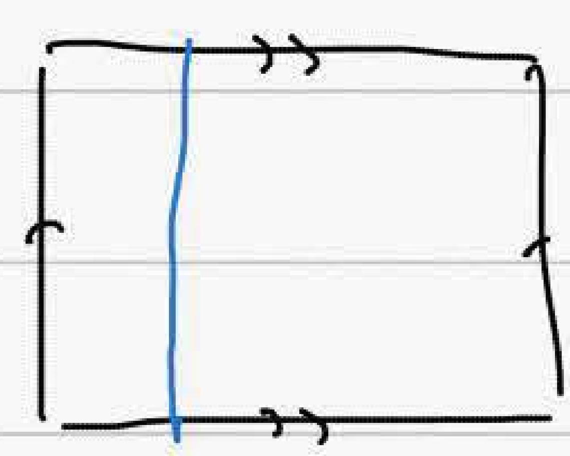


Take $G = \mathbb{Z}_2$ as an example.

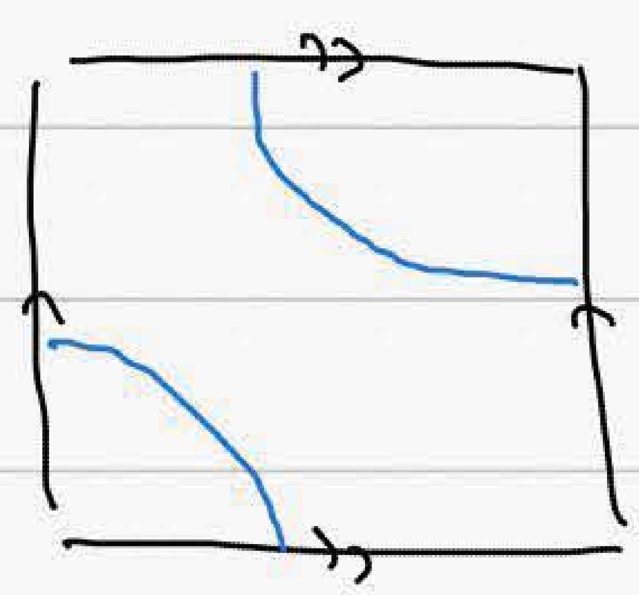
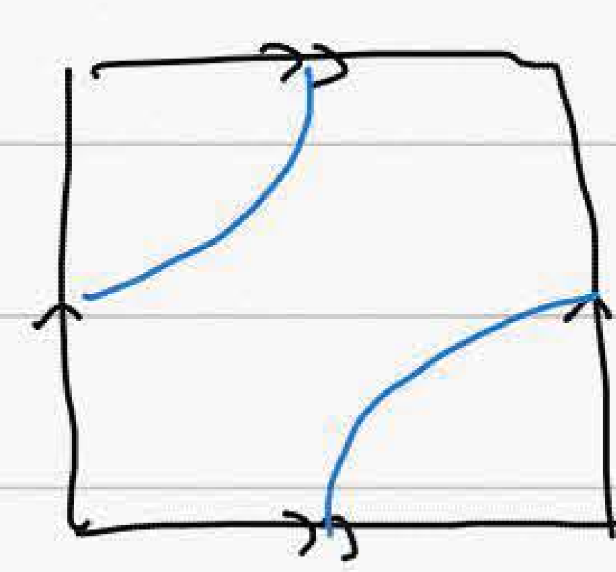
 $=$  $\times \pm 1$.

One manifestation:

Euclidean time

 $= \mathrm{tr}_{\mathcal{H}_{\text{twisted}}} e^{-\beta H}$

$T \in SL(2, \mathbb{Z})$ $\qquad$ $T \in SL(2, \mathbb{Z})$

 $=$  $\times \pm 1$

$$\mathrm{tr}_{\mathcal{H}_{\text{twisted}}} e^{-2\pi i P} e^{-\beta H} \qquad \qquad \mathrm{tr}_{\mathcal{H}_{\text{twisted}}} e^{+2\pi i P} e^{-\beta H}$$

meaning that:

$$P = L_0 - \overline{L_0} \in \tfrac{1}{2}\mathbb{Z} \quad \text{if } +1$$

$$\tfrac{1}{2}\mathbb{Z} + \tfrac{1}{4} \quad \text{if } -1.$$

The first case is well-known: a 2d boson on $S^1/\mathbb{Z}_2$ in the twisted sector is expanded in terms of

$$\alpha_{n\pm 1/2} \quad \text{and} \quad \tilde{\alpha}_{n\pm 1/2} \quad \rightsquigarrow \quad L_0 - \bar{L}_0 \in \tfrac{1}{2}\mathbb{Z}.$$

So the $\mathbb{Z}_2$ symmetry orbifolding a 2d boson is non-anomalous.

An example of the second case is the T-duality of S^1 at the self-dual radius. Recall that T-duality is

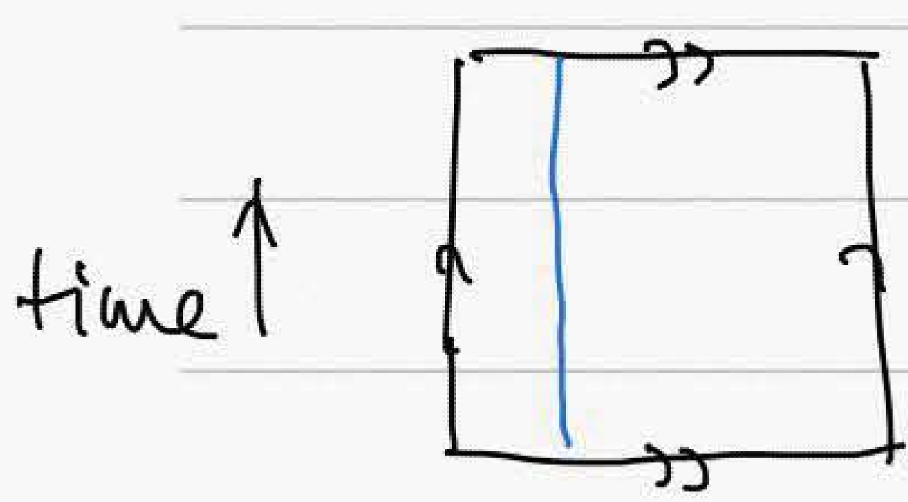
$$(X_L, X_R) \longrightarrow (-X_L, X_R)$$

in general. In the self-dual = SU(2) radius, it's equivalent to the half-shift of X_L.

Untwisted sector $\ni$ $j_\pm(z) = \,:e^{i\sqrt{2}X_L(z)}$

twisted sector $\ni$ $:e^{iX_L(z)/\sqrt{2}}:$

$L_0 = +1$ $\qquad\qquad$ $L_0 = +1/4$

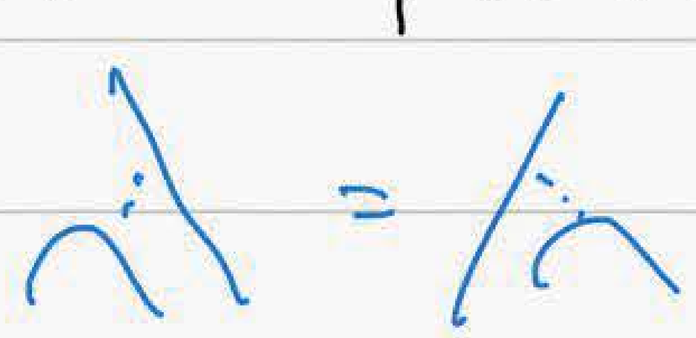


$X_L \qquad X_R$

$\leftarrow \mathcal{H} =$ (twisted, untwisted) sector

$$\rightsquigarrow \quad L_0 - \bar{L}_0 \in \tfrac{1}{2}\mathbb{Z} + 1/4.$$

(It's also possible to directly compute



$\times(-1)$ by computing fusion of Verlinde line operators.)

So far, so good. But we already saw that chiral $\mathbb{Z}_2$ on Majorana fermion has

$$L_0 - \bar{L}_0 \in \frac{\mathbb{Z}}{2} + \frac{1}{16}$$

in the twisted = NS-R sector.

Very abstractly, we saw that

bosonic G-anomaly $\qquad\qquad$ fermionic G-anomaly

$$H^3(BG, U(1)) \xrightarrow{\quad} \mathrm{Hom}(\Omega_3^{spin}(BG), U(1))$$

a homomorphism.

not necessarily an injection

or a surjection

When $G = \mathbb{Z}_2$,

$$n \in \mathbb{Z}_2 \longrightarrow \mathbb{Z}_8 \ni m$$

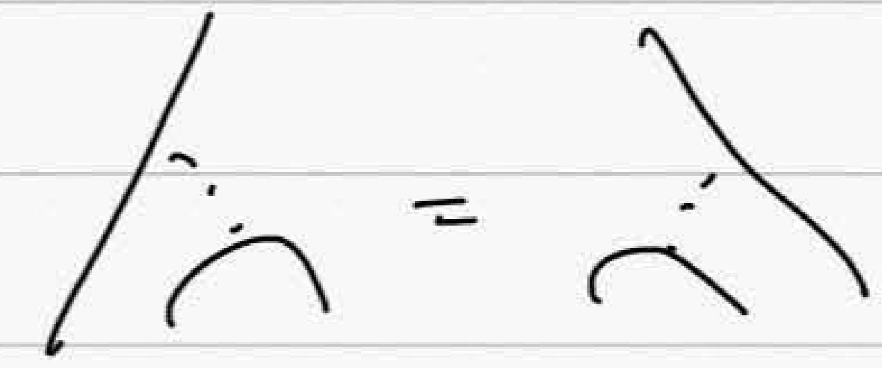
Characterized by the momentum

$$L_0 - \bar{L}_0 = \frac{\mathbb{Z}}{2} + \frac{n}{4} \longmapsto \frac{\mathbb{Z}}{2} + \frac{m}{16}$$

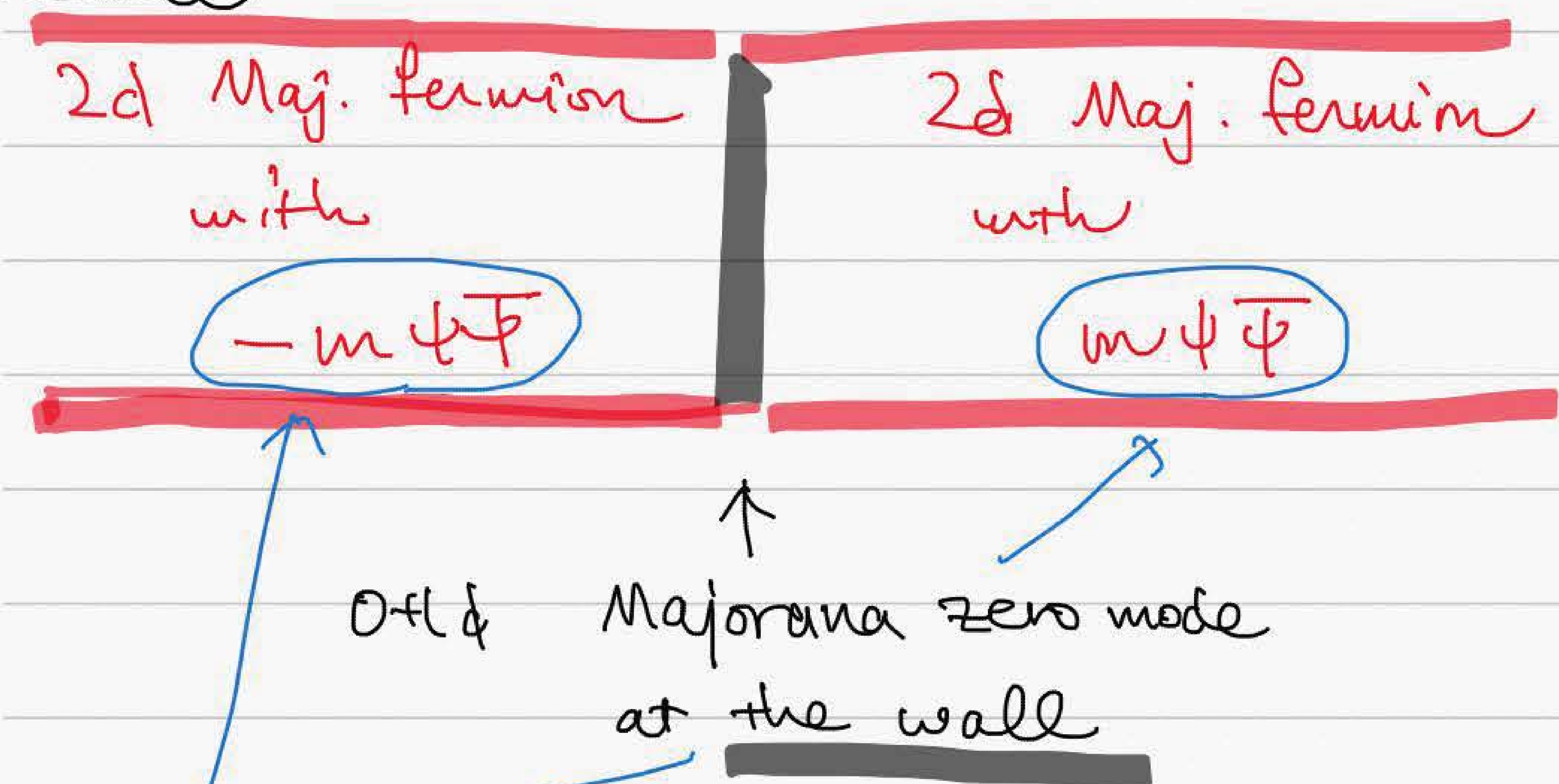
in the twisted sector, or the bulk partition function

$$Z_{bulk}(S^3/\mathbb{Z}_2) = e^{2\pi i \frac{n}{2}} \longmapsto e^{2\pi i \cdot \frac{m}{8}}.$$

But we need to ask:

Which part of  $\times \pm 1$ was wrong?

Recall:



Related by chiral $\mathbb{Z}_2$ $\begin{pmatrix}\psi\\ \bar{\psi}\end{pmatrix} \to \begin{pmatrix}\psi\\ -\bar{\psi}\end{pmatrix}$.

i.e. the wall is the $\mathbb{Z}_2$ wall. and

even after taking $m \to 0$, it has Majorana zero mode.

In general, in a fermionic system with G symmetry,
we need to specify

$$\mu : G \longrightarrow \mathbb{Z}_2$$

which tells which wall $g \in G$ has a Maj. zero mode

This is not all.

In addition, the junction point itself can be either fermionic or bosonic.

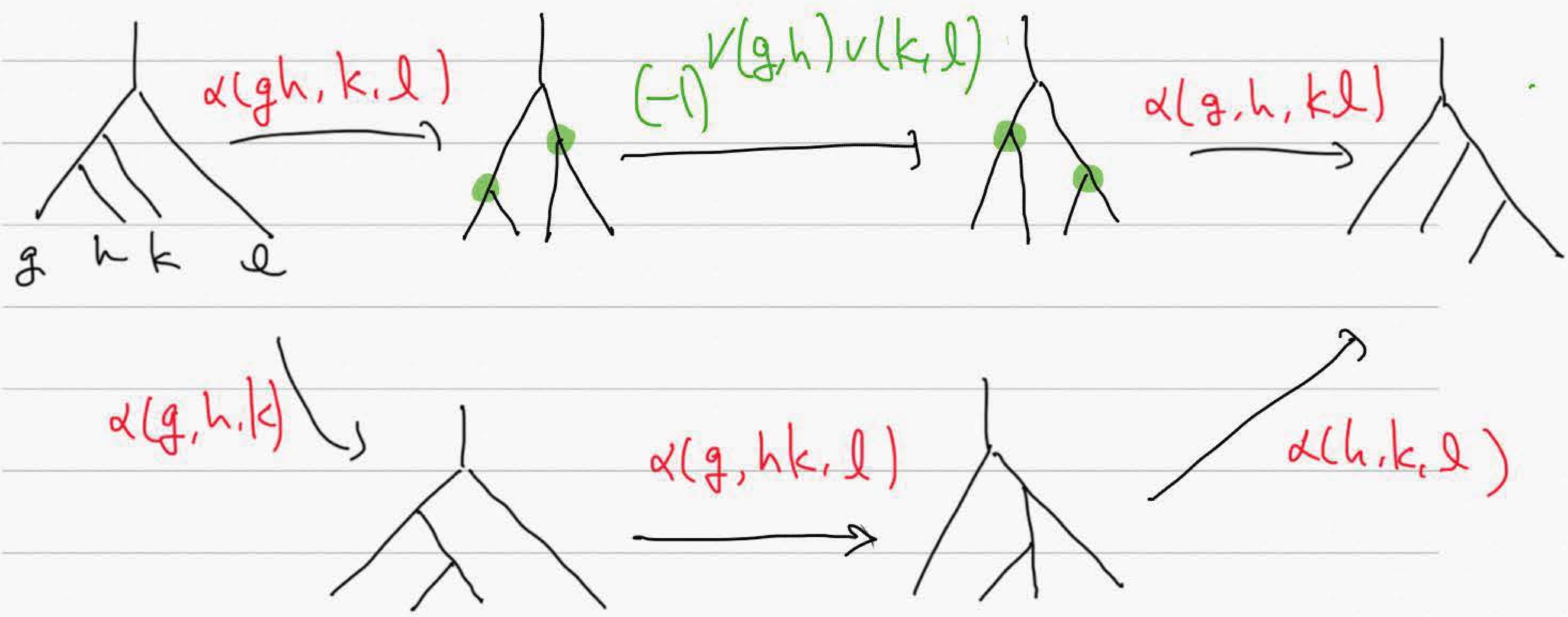
$\nu(g,h) \in \mathbb{Z}_2 = \{0,1\}.$

you want

$$\nu(g,h) + \nu(gh,k) = \nu(g,hk) + \nu(h,k)$$

i.e. $\delta\nu = 0$.

This modifies the pentagon eq. to



i.e. $$\frac{\alpha(g,h,k)\,\alpha(g,hk,l)\,\alpha(h,k,l)}{\alpha(gh,k,l)\,\alpha(g,h,kl)} = (-1)^{\nu(g,h)\nu(k,l)}.$$

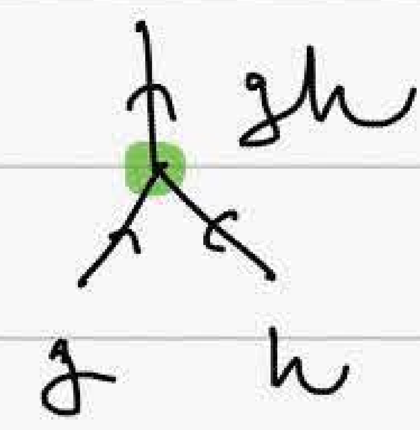
or $\delta\alpha = (-1)^{\nu^2}$ if you know some alg. top.

Gu–Wen 2012
Aasen–Lake–Walker 2017

Summarizing, the anomaly of fermionic G-sym. has

$\mu : G \to \mathbb{Z}_2$ telling which line has Maj. fer,

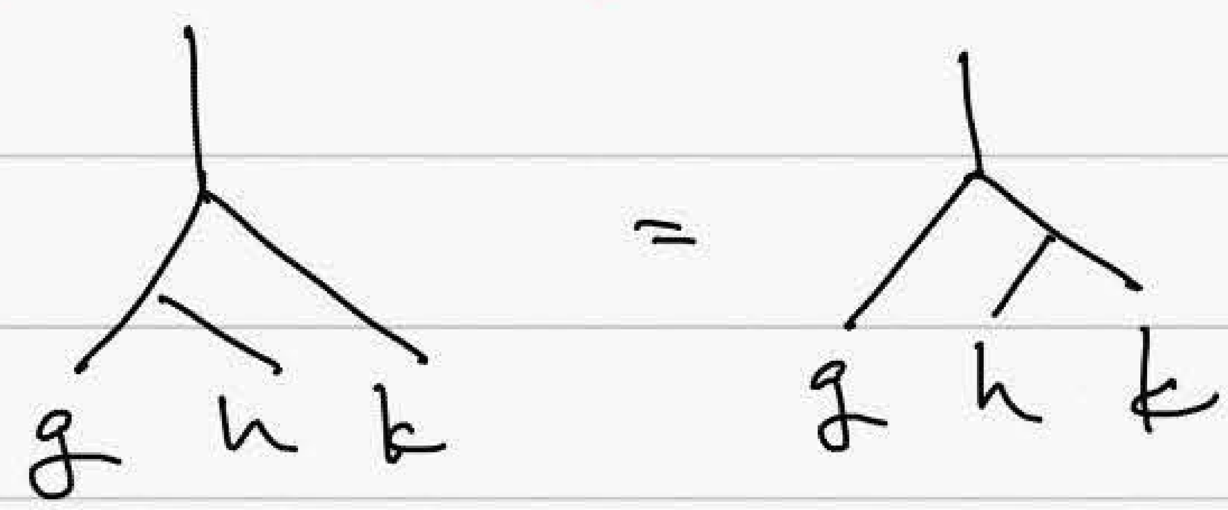
$\nu : G \times G \to \mathbb{Z}_2$ specifying whether



is bosonic or fermionic,

and

$\alpha : G \times G \times G \to U(1)$ specifying



$\times\ \alpha(g.h.k)$

such that

$$\begin{cases} \cdot\ \mu \text{ is homomorphism} \\ \cdot\ \delta\nu = 0, \\ \cdot\ \delta\alpha = (-1)^{\nu^2} \end{cases}$$

It's known that this fully describes $\mathrm{Hom}(\Omega_3^{spin}(BG), U(1))$.

Brumfiel-Morgan 2016

When $G = \mathbb{Z}_2$, very roughly,

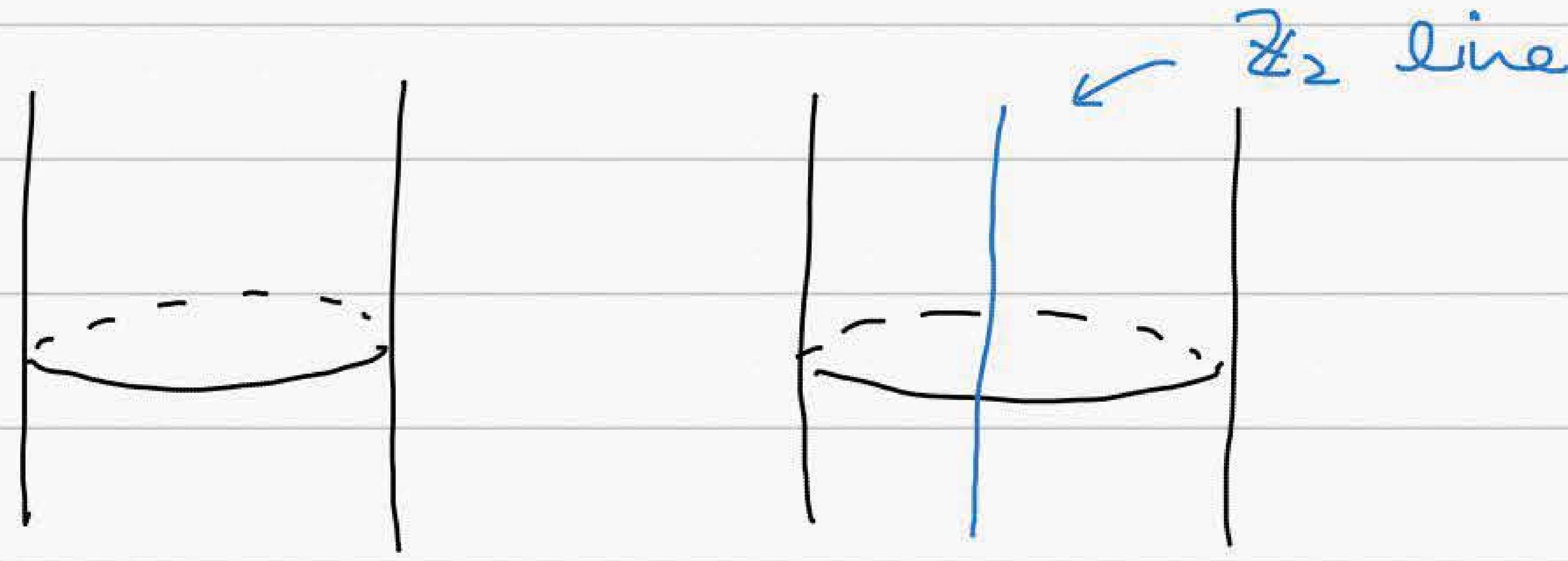
$$\left.\begin{array}{l} \mu : \text{two choices} \\ \nu : \text{two choices} \\ \alpha : \text{two choices} \end{array}\right\} 2^3 = 8 \text{ choices}$$

explaining

$$\underset{\text{bosonic}}{\mathbb{Z}_2} \longrightarrow \underset{\text{fermionic}}{\mathbb{Z}_8}.$$

Fun with gauging in 2d

Consider a 2d QFT Q with non-anomalous $\mathbb{Z}_2$ sym.



$\mathcal{H}^{+}_{Q,\,\text{untwisted}}$ $\qquad\qquad$ $\mathcal{H}^{+}_{Q,\,\text{twisted}}$

$\mathcal{H}^{-}_{Q,\,\text{untwisted}}$ $\qquad\qquad$ $\mathcal{H}^{-}_{Q,\,\text{twisted}}$

$\mathbb{Z}_2$ charge.

Let's orbifold = gauge $\mathbb{Z}_2$ to get $Q' = Q/\mathbb{Z}_2$.

$$\mathcal{H}_{Q',\,\text{untwisted}} = \mathcal{H}^{+}_{Q,\,\text{untwisted}} \oplus \mathcal{H}^{+}_{Q,\,\text{twisted}}$$

This theory has a dual $\mathbb{Z}_2$ under which is even & is odd.

We can also set

$$\mathcal{H}_{Q',\,\text{twisted}} = \mathcal{H}^{-}_{Q,\,\text{untwisted}} \oplus \mathcal{H}^{-}_{Q,\,\text{twisted}}$$

so that

$$Q'/\mathbb{Z}_2 = Q.$$

I.e.

$$Q/\mathbb{Z}_2/\mathbb{Z}_2 = Q.$$

At the level of part. func, let

$$Z_Q(M,v)$$

be its part. func. under $\mathbb{Z}_2$ background $v \in H^1(M,\mathbb{Z}_2)$.

$$Z_{Q/\mathbb{Z}_2} \propto \sum_v Z_Q(v).$$

we can refine it by considering

$$\boxed{Z_{Q/\mathbb{Z}_2}(w) \propto \sum_v e^{\pi i \int v \wedge w} \; Z_Q(v)}$$

where $w \in H^1(M,\mathbb{Z}_2)$ is the background for the dual $\mathbb{Z}_2$ symmetry.

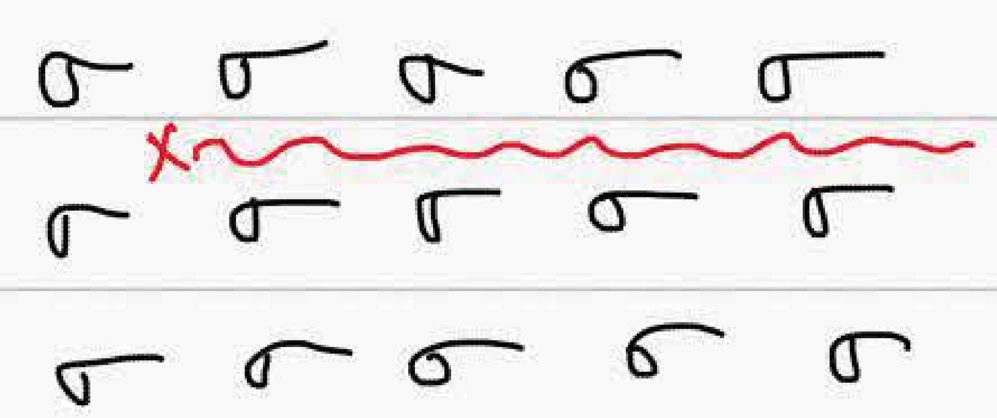
The inverse process is

$$\boxed{Z_Q(v) \propto \sum_w e^{\pi i \int w \wedge v} \; Z_{Q/\mathbb{Z}_2}(w)}.$$

note the symmetry!

In hep-th, goes back to Vafa 1989

In fact, goes back to Kramers-Wannier 1941



Ising is $\mathbb{Z}_2$ symmetric.
disorder field $\tilde{\sigma}$
is the endpoint of $\mathbb{Z}_2$ line.

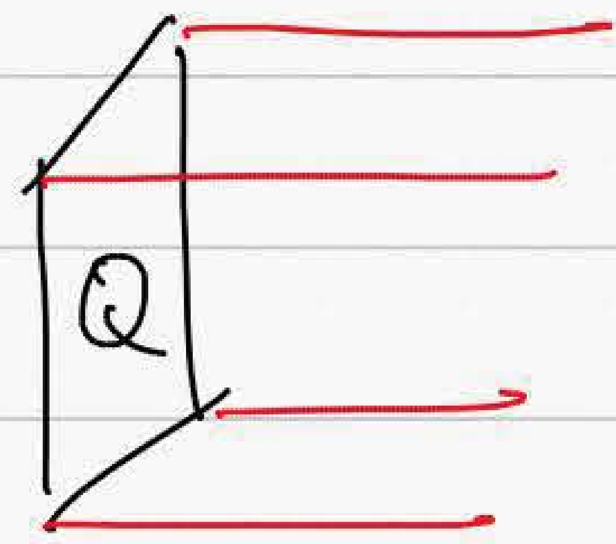
i.e. $\tilde{\sigma}$ is in the twisted sector.

$\rightsquigarrow$ $\text{Ising}_\beta / \mathbb{Z}_2 = \text{Ising}_{\tilde{\beta}}$.

(N.B. in general, Q and $Q/\mathbb{Z}_2$ are quite different.
Why is Ising self-dual? We'll come back to this.)

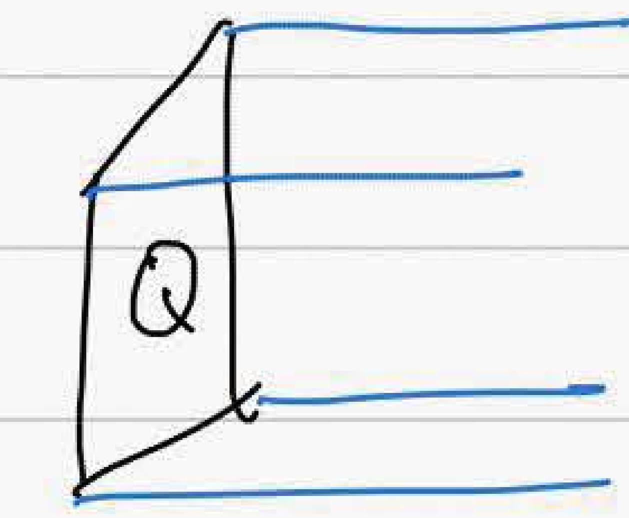
known also as toric code theory

Let's give it a 3d interpretation.



trivial $\mathbb{Z}_2$ SPT

$\xrightarrow{\text{gauge only the bulk}}$

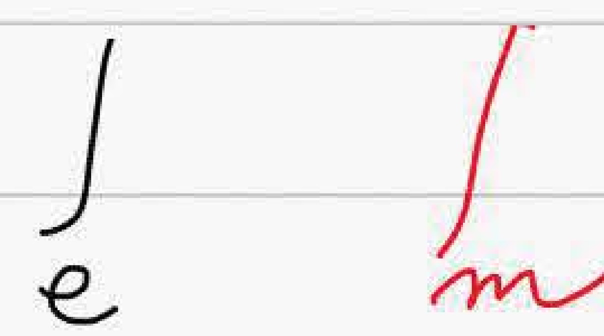


$\mathbb{Z}_2$ pure gauge th.

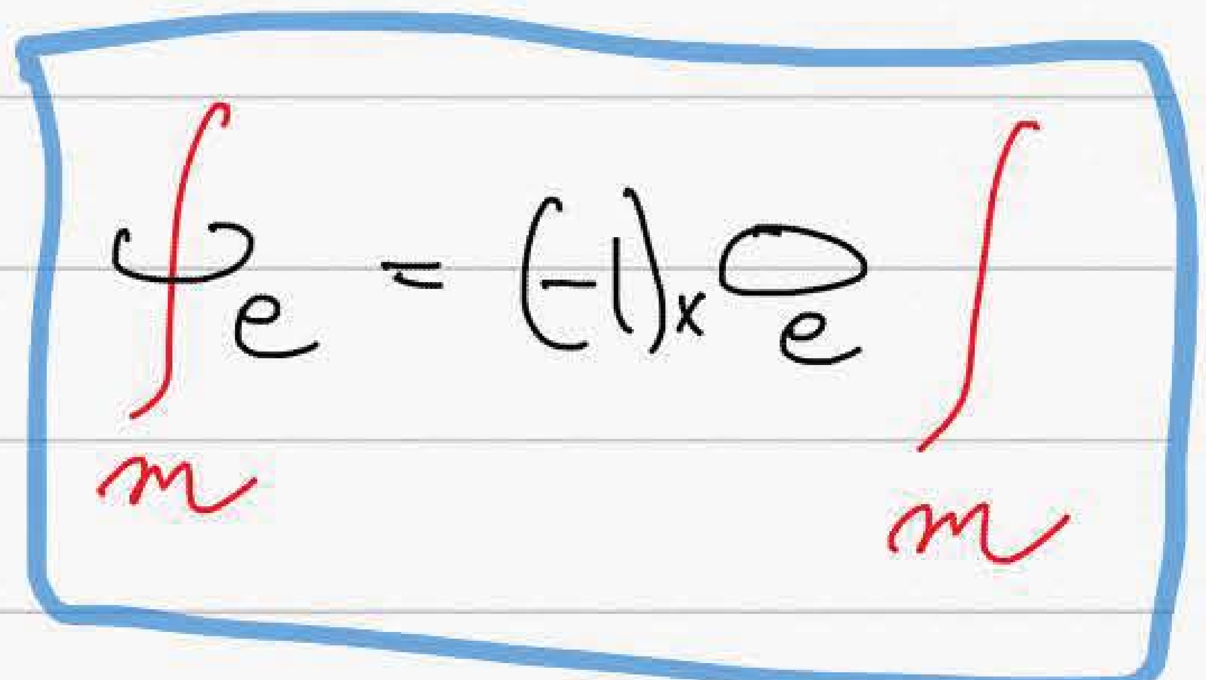
equivalent to $U(1)^2$ CS

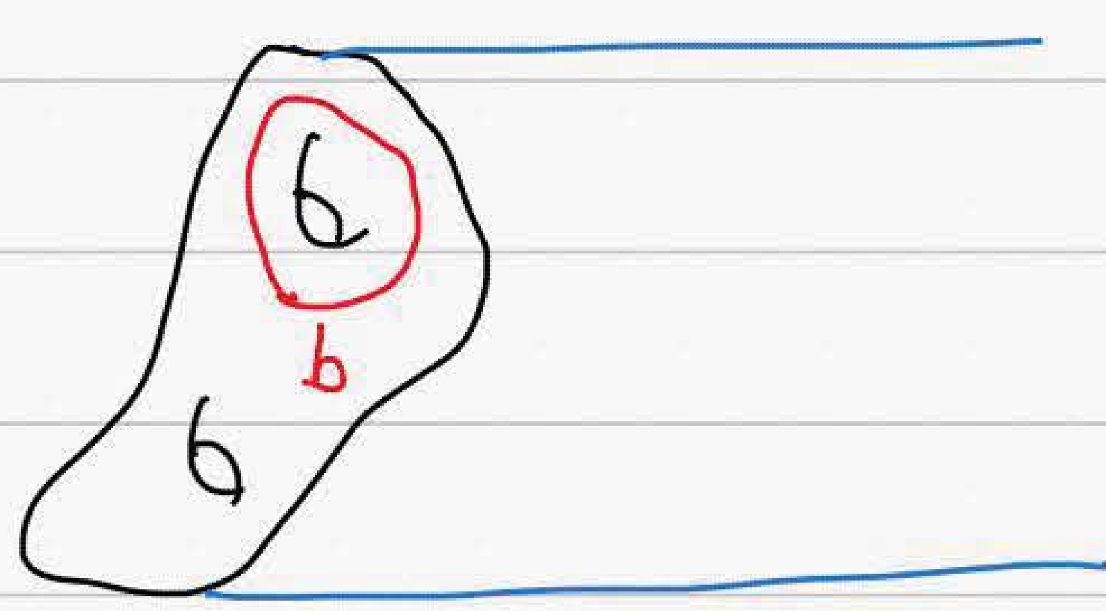
$$S = 2 \cdot 2\pi i \int \frac{A}{2\pi} d \frac{B}{2\pi}$$

$$K_{IJ} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$



Wilson line 't Hooft line





Σ

$\rightsquigarrow$

$\mathcal{H}^{3d}_{\Sigma} \leftarrow Le(a),\ Lm(b)$ where $a, b \in H_1(\Sigma, \mathbb{Z}_2) \simeq H^1(\Sigma, \mathbb{Z}_2)$.

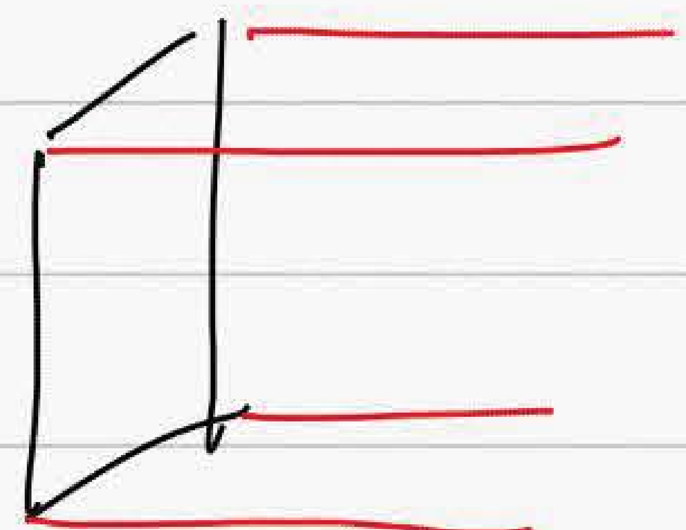
$$\boxed{Le(a)\, Lm(b) = Lm(b)\, Le(a)\, (-1)^{\int a \wedge b}}.$$

We can diagonalize $L_e(a)$ for all a

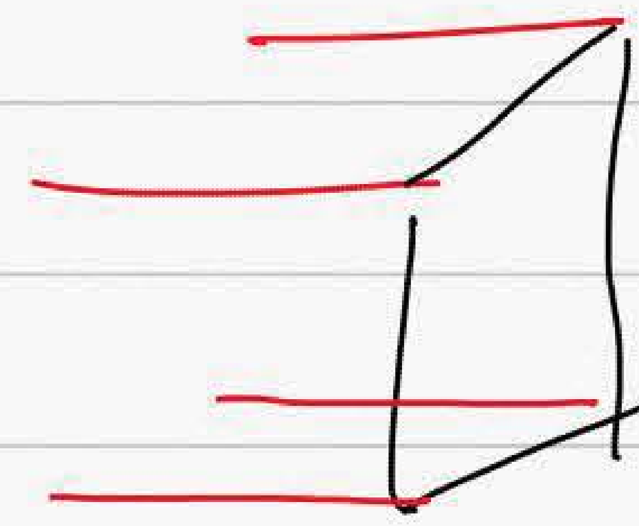
or $L_m(b)$ for all b.

leading to states

$$\begin{cases} L_e(a)\,|e,v\rangle = (-1)^{\int a v}\,|e,v\rangle \\ L_m(b)\,|m,w\rangle = (-1)^{\int b w}\,|m,w\rangle \end{cases}$$



$\mathbb{Z}_2^{(e)}$ ungauged



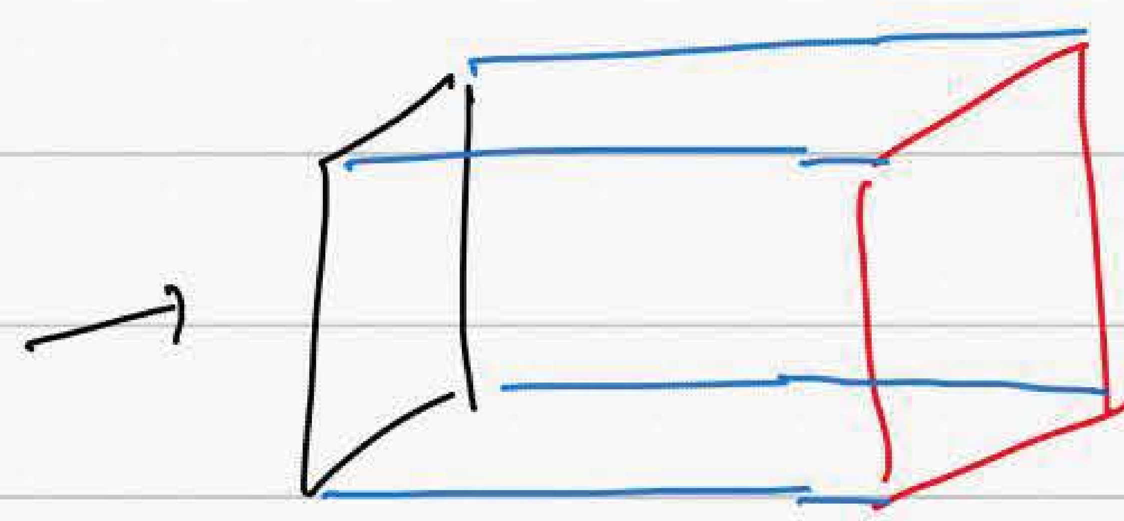
$\mathbb{Z}_2^{(m)}$ ungauged

gauge bulk $\mathbb{Z}_2^{(e)}$

gauge bulk $\mathbb{Z}_2^{(m)}$

Dirichlet for $\mathbb{Z}_2^{(e)}$

$\langle e,v|$



Dirichlet for $\mathbb{Z}_2^{(m)}$

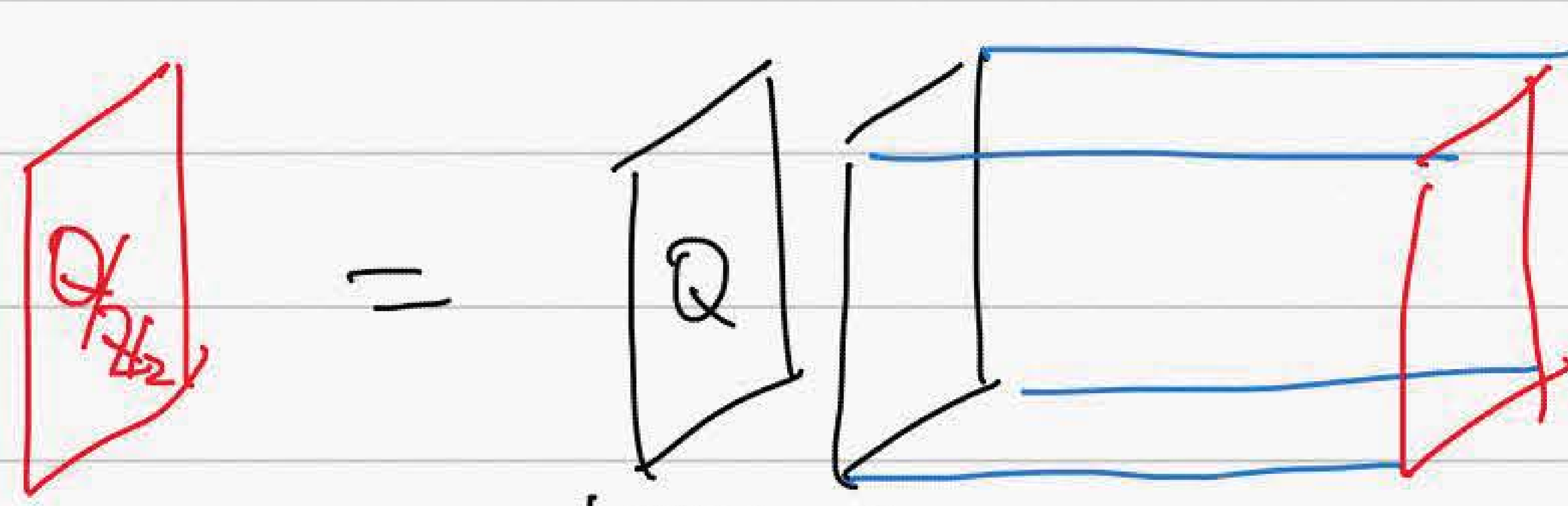
$|m,w\rangle$

pure $\mathbb{Z}_2^{(e)}$ gauge th.

pure "$\mathbb{Z}_2^{(m)}$" gauge th.

$$\langle e,v|m,w\rangle = (-1)^{\int v \cup w}.$$

Then



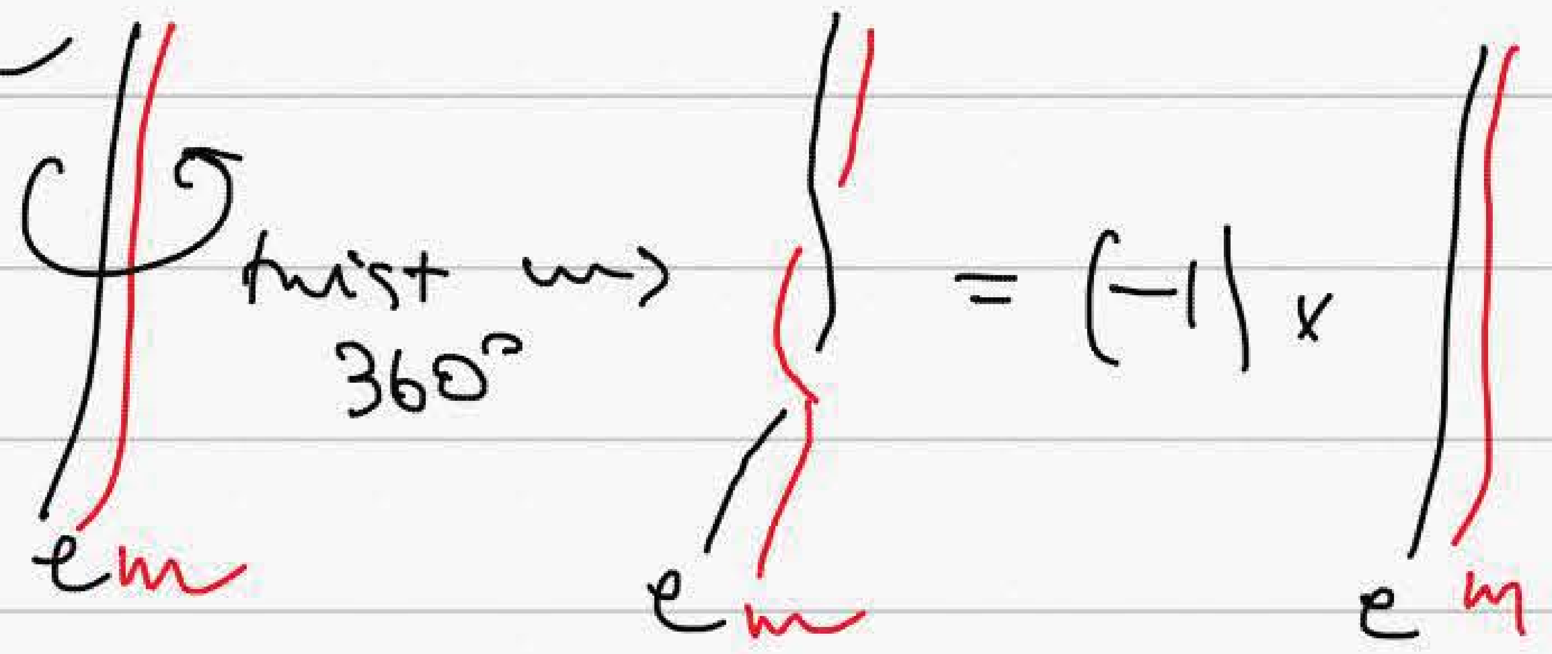
"The duality wall"

$\mathbb{Z}_2^{(m)}$ sym.

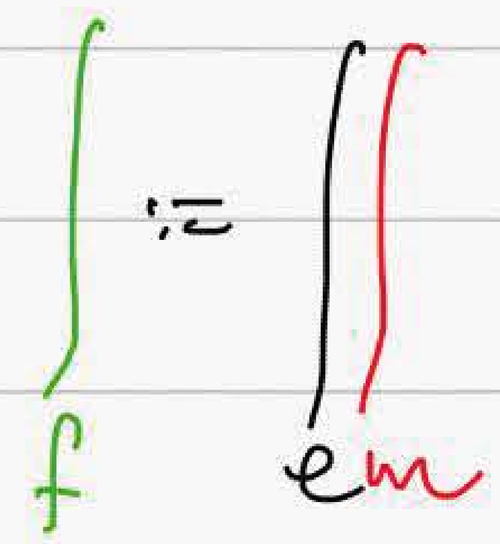
gauge $\mathbb{Z}_2$

i.e. $Z_{Q/\mathbb{Z}_2}(w) = \sum_v Z_Q(v)\,(-1)^{\int v\cup w}$.

Consider



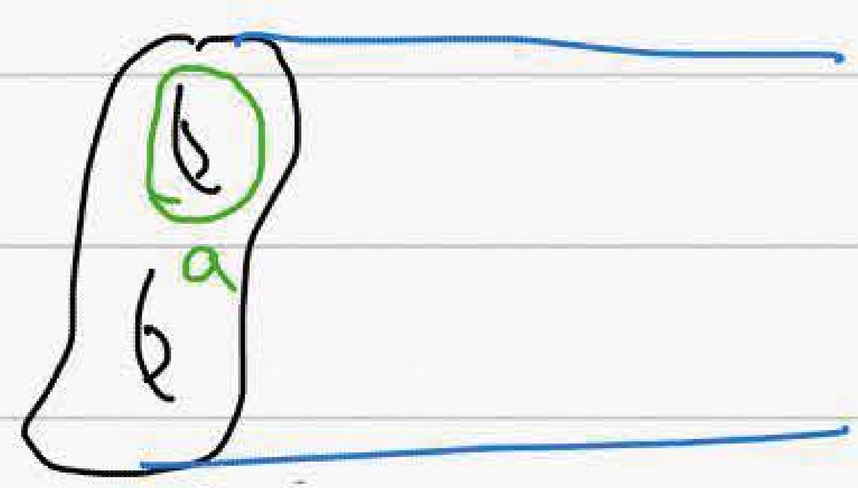
i.e.



is a fermion.

$$L_f(a) := L_e(a) L_m(b).$$

Then $\quad L_f(a) L_f(b) = L_f(b) L_f(a) (-1)^{\int ab}$.



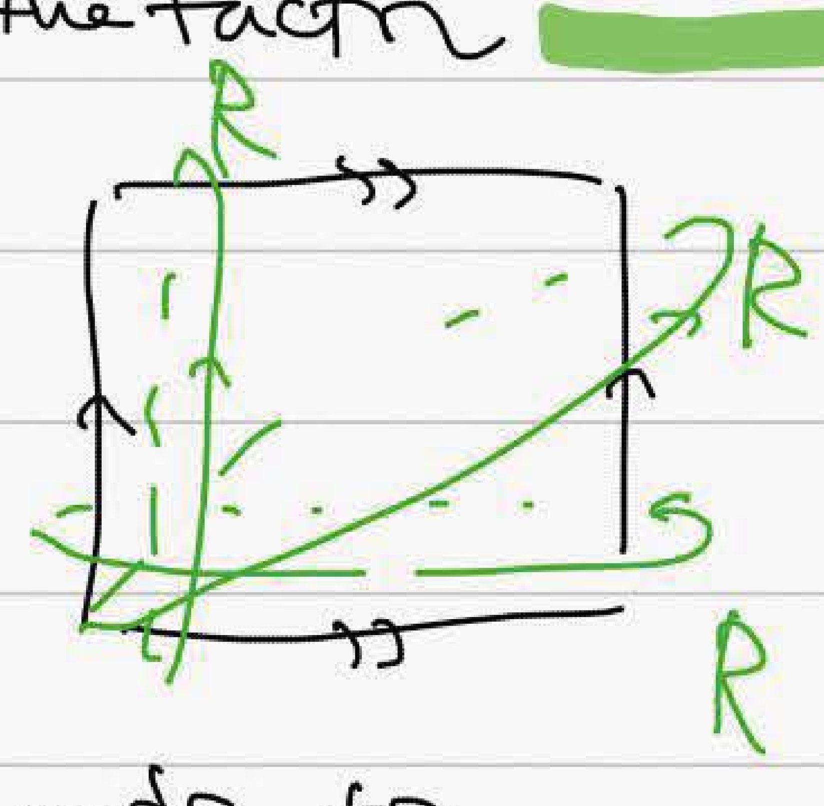
$\mathcal{H}^{3d}_{\Sigma} \curvearrowleft L_f(a) \Leftarrow$ simultaneously 'diagonalizable' given a spin structure σ on Σ

$$L_f(a)\, |f, \sigma\rangle = (-1)^{\sigma(a)}\, |f, \sigma\rangle$$

0 for NS, 1 for R

where $\quad \sigma(a+b) = \sigma(a) + \sigma(b) + \int_\Sigma ab$. Atiyah 1971

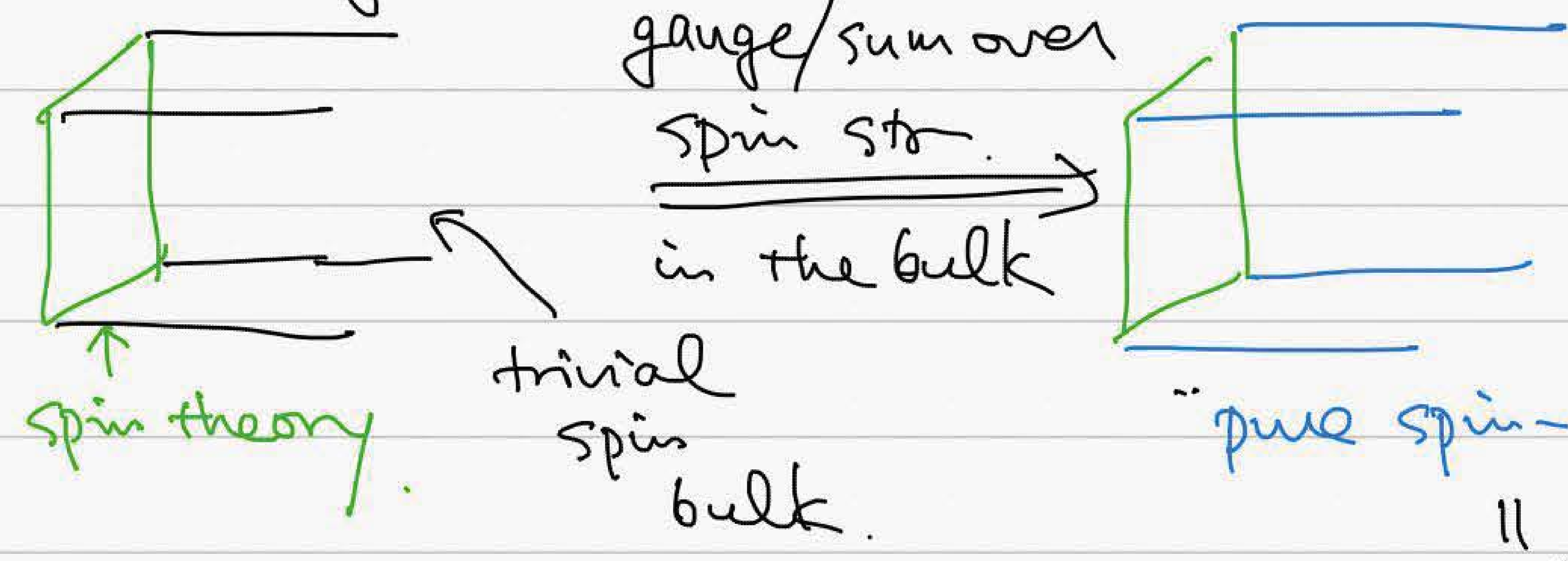
NOTE: the factor ______ above is necessary since



corresponds to

$$L_f(a) = L_f(b) = L_f(a+b) = -1.$$

The point being:



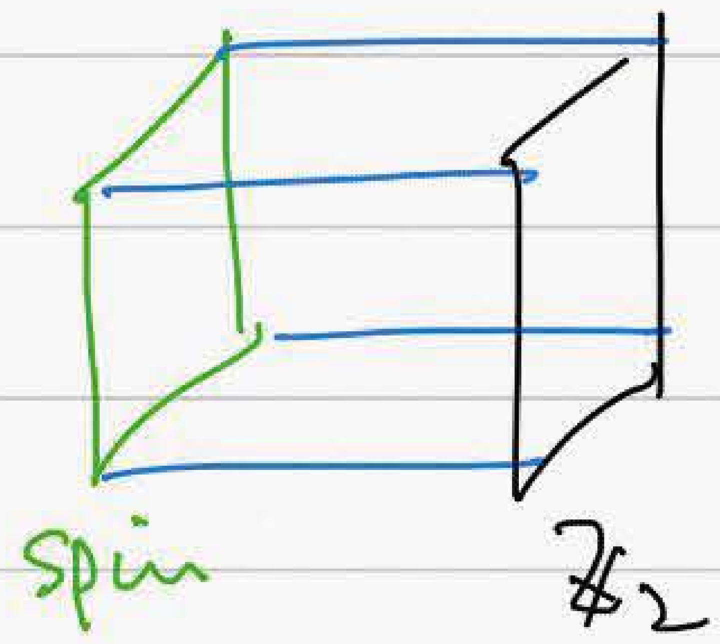
"pure spin-str gauge th."

$\parallel$

pure $\mathbb{Z}_2^{(e)}$ gauge th.

$\parallel$

pure $\mathbb{Z}_2^{(m)}$ gauge th.



$$\langle f,\sigma | e, v\rangle = (-1)^{\sigma(v)}$$

This allows us to convert

a spin theory $\longleftrightarrow$ a $\mathbb{Z}_2$-symmetric theory.

$Z_{\tilde{Q}}(\sigma)$ $\qquad$ $Z_Q(v)$

$$Z_{\tilde{Q}}(\sigma) \propto \sum_v (-1)^{\sigma(v)} Z_Q(v)$$

$$Z_Q(v) \propto \sum_\sigma (-1)^{\sigma(v)} Z_{\tilde{Q}}(\sigma).$$

Jordan-Wigner transformation. $\qquad$ GSO projection.

1928 $\qquad$ 1977

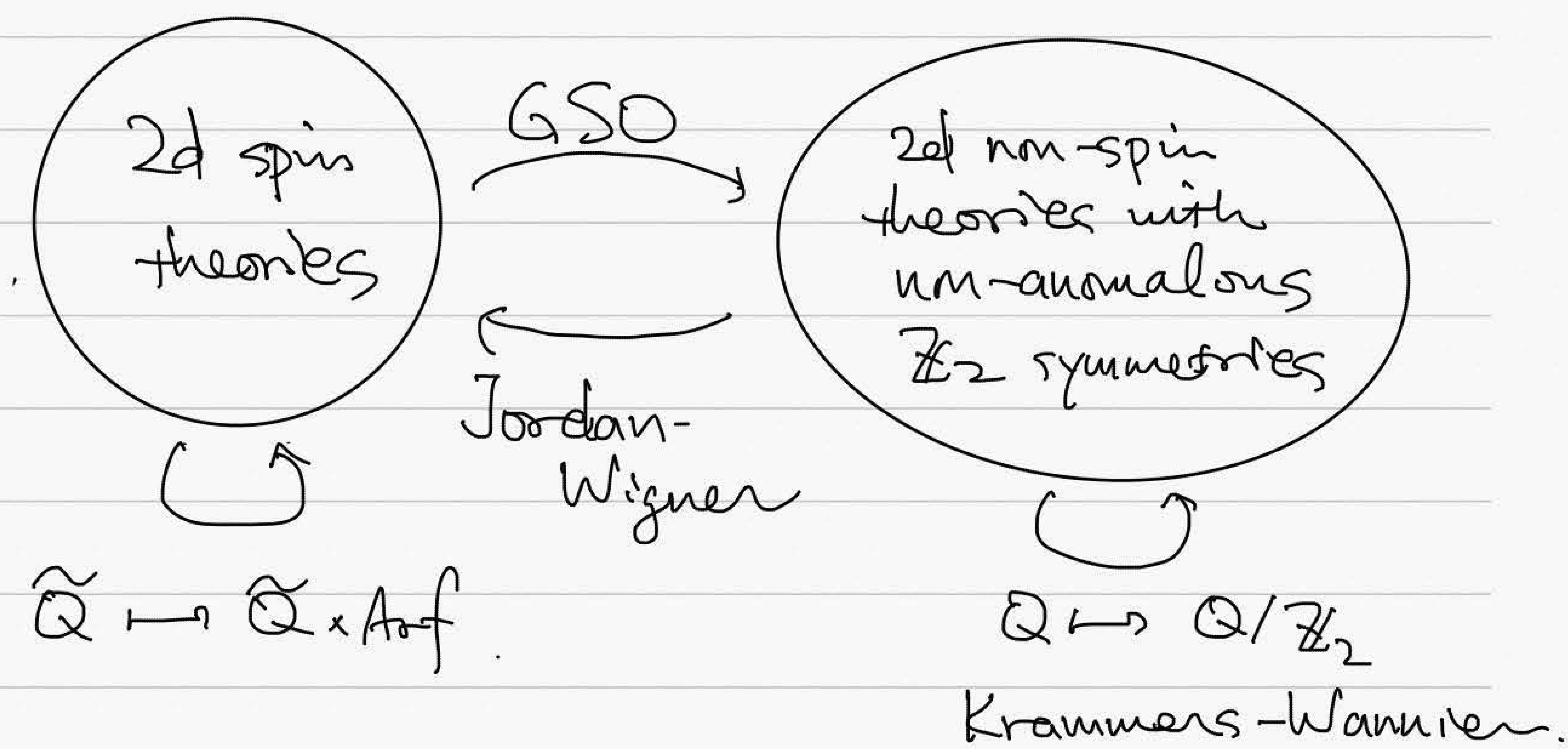
Inverses.

A very simple 2d spin theory "Arf"

$$Z_{\mathrm{Arf}}(M,\sigma) = \mathrm{Arf}(\sigma) = (-1)^{\#\text{Dirac zero mode}}.$$

$\tilde{Q}$: a spin theory

$\tilde{Q} \times \mathrm{Arf}$: another spin theory.



In fact, we have

$$\begin{tikzcd} \tilde{Q} \arrow[rr, leftrightarrow, "GSO", "JW"'] \arrow[d, mapsto] & & Q \arrow[d, mapsto] \\ \tilde{Q} \times \mathrm{Arf} \arrow[rr, leftrightarrow, "GSO", "JW"'] & & Q/\mathbb{Z}_2 \end{tikzcd}$$

To see this, compute

$$\begin{tikzcd} \cdot \arrow[d] & Q \arrow[l] \\ \cdot \arrow[r] & Q'. \end{tikzcd}$$

which is

$$Z_{Q'}(w) = \sum_{\sigma, v} (-1)^{\sigma(w)} \mathrm{Arf}(\sigma) (-1)^{\sigma(v)} Z_Q(v) .$$

We need to compute

$$\sum_\sigma (-1)^{\sigma(v)} \operatorname{Arf}(\sigma) (-1)^{\sigma(w)} \quad \cdots\cdots (\ast)$$

We use

$$(-)^{\sigma(v)+\sigma(w)} = (-1)^{\sigma(v+w)} (-1)^{\int vw}$$

and then

$$\operatorname{Arf}(\sigma)(-1)^{\sigma(x)} = \operatorname{Arf}(\sigma+x)$$

so that

$$(\ast) = \sum_\sigma \operatorname{Arf}(\sigma+v+w) (-1)^{\int vw}$$

$$= \Big(\underbrace{\sum_\sigma \operatorname{Arf}(\sigma)}\Big)(-1)^{\int vw}$$

When genus $=g$, always $\left\{\begin{array}{ll} 2^{g-1}(2^g-1) & \text{odd} \\ 2^{g-1}(2^g+1) & \text{even} \end{array}\right\}$ spin str.

$$= 2^g \, (-1)^{\int vw} .$$

so indeed, $Z_{Q'}(w) \propto \sum_v (-1)^{\int vw} Z_Q(v)$

i.e.

$$Q' = Q/\mathbb{Z}_2 .$$

It is also instructive to see this in the Hilbert space.

$$\boxed{Q} \xrightarrow{\ /\mathbb{Z}_2\ } \boxed{Q/\mathbb{Z}_2}$$

$\boxed{Q}$	$\mathbb{Z}_2$ even	$\mathbb{Z}_2$ odd
untwisted	$\boxed{\mathcal{H}_u^+}$	$\boxed{\mathcal{H}_u^-}$
twisted	$\boxed{\mathcal{H}_t^+}$	$\boxed{\mathcal{H}_t^-}$

$\boxed{Q/\mathbb{Z}_2}$	$\mathbb{Z}_2$ even	$\mathbb{Z}_2$ odd
untwisted	$\boxed{\mathcal{H}_u^+}$	$\boxed{\mathcal{H}_t^-}$
twisted	$\boxed{\mathcal{H}_u^-}$	$\boxed{\mathcal{H}_t^+}$

JW $\downarrow$ $\uparrow$ GSO　　　　JW $\downarrow$ $\uparrow$ GSO

$\boxed{\tilde{Q}}$	$(-1)^F$ even	$(-1)^F$ odd
NS-sector	$\boxed{\mathcal{H}_u^+}$	$\boxed{\mathcal{H}_t^-}$
R-sector	$\boxed{\mathcal{H}_u^-}$	$\boxed{\mathcal{H}_t^+}$

$\boxed{\tilde{Q}\times \mathrm{Arf}}$	$(-1)^F$ even	$(-1)^F$ odd
NS-sector	$\boxed{\mathcal{H}_u^+}$	$\boxed{\mathcal{H}_t^-}$
R-sector	$\boxed{\mathcal{H}_t^+}$	$\boxed{\mathcal{H}_u^-}$

$$\boxed{\tilde{Q}} \xleftrightarrow{\ \times \mathrm{Arf}\ } \boxed{\tilde{Q}\times \mathrm{Arf}}$$

$(-1)^F$ of the R-sector is exchanged.

This is due to the unique R-sector state of Arf theory having $(-1)^F = -1$.

So what? This generalizes

$$\begin{array}{ccc} \text{maj. fermion}_{m} & \longleftrightarrow & \text{Ising}_{\beta} \\ \times\text{Arf} \downarrow & & \downarrow /\mathbb{Z}_2 \\ \text{maj. fermion}_{-m} & \longleftrightarrow & \text{Ising}_{\tilde\beta} \end{array}$$

In particular

$$\times\text{Arf} \circlearrowleft \text{Maj. fermion}_{m=0} \longleftrightarrow \text{Ising}_{\beta=\tilde\beta} \circlearrowright /\mathbb{Z}_2$$

fermion number symmetry $\longleftrightarrow$ $\mathbb{Z}_2$ on spin variables of Ising model.

also has chiral $\mathbb{Z}_2$

$$(\psi_L, \psi_R) \to (\psi_L, -\psi_R)$$

under which

$$m\psi_L\psi_R \to -m\psi_L\psi_R.$$

Corresponding wall: $\mathbb{Z}_2$, g

This does

$$\frac{Z_{\text{fermion}}(m)}{Z_{\text{fermion}}(-m)} = Z_{\text{Arf}}$$

i.e. $\times$Arf above is chiral $\mathbb{Z}_2$.

duality is a $\mathbb{Z}_2$ symmetry here.

corresponding wall $\mathbb{Z}_2$, h $\longleftrightarrow$ corresponding wall: D

has Maj. fermion zeromode on it.

(In passing, $(\psi_L, \psi_R) \to (\psi_L, -\psi_R)$ does $\times$Arf means T-duality exchanges Type IIA and IIB.)

Summarizing, under

$$\hat{Q} \underset{JW}{\overset{GSO}{\rightleftarrows}} Q \quad ,$$

we have

$$\text{spin structure} \longleftrightarrow \quad \overset{\dashrightarrow \mathbb{Z}_2}{\underset{g}{|}} \quad .$$

In the case of Maj. fermion$_{m=0}$ $\leftrightarrow$ Ising$_{\beta=\tilde{\beta}}$,

we also have

$$\text{chiral } \mathbb{Z}_2 \quad \overset{\dashrightarrow \mathbb{Z}_2}{\underset{h}{|}} \longleftrightarrow \underset{D}{|} \text{ duality wall}$$

with Majorana zero mode.

Combining,

- spin structure
- $h^2 = id$
- h has Maj. zeromode

$\longleftrightarrow$

- $g^2 = id$
- $D^2 = id \oplus g$

This is in fact a general formal result,
not just between Maj. fermion $\leftrightarrow$ Ising.