Classification of invertible phases (and Anomalies)

Recommended Refs:

Freed "Lectures on Field Theory and Topology"
CBMS 133, AMS, 2019

Yonekura "On the cobordism classification of symmetry protected topological phases"
CMP 368 (2019) 1121, arXiv: 1803.10796

should've invited him instead of me.

Aim of this set of lectures:

Classify unitary invertible phases.

a particularly simple but important class of QFT.

Answer:

depends on the two variant formulations of the question above

$\begin{pmatrix}\text{Pontryagin}\\ \text{Anderson}\end{pmatrix}$ dual of the bordism group associated to the structure in question.

?

A (relativistic, Euclidean) QFT Q

in spacetime dimension d

with a structure S (such as

orientation, spin str, metric,

a map to another space X,

a principal G-bundle with connection, ...)

does the following:

assumed throughout this lecture

M_d : d-dim'l closed mfd $\boxed{\text{with structure } S}$

$\rightsquigarrow$

$Z_Q(M_d) \in \mathbb{C}$, partition function

M_{d-1} : $(d-1)$-dim'l closed mfd

$\rightsquigarrow$

$\mathcal{H}_Q(M_{d-1})$: a $\mathbb{C}$-vector space , space of states

↑ WE'LL ONLY USE THIS PART

M_{d-2} : $(d-2)$-dim'l closed mfd

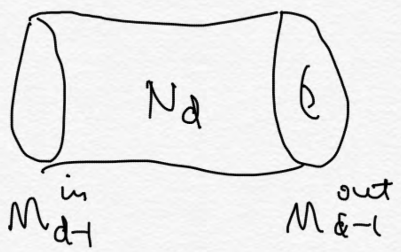
$\rightsquigarrow$

$\mathcal{C}_Q(M_{d-2})$: a category , data to 'factorize' space of states across entangling surface M_{d-2}

⋮

satisfying various consistency conditions:

For



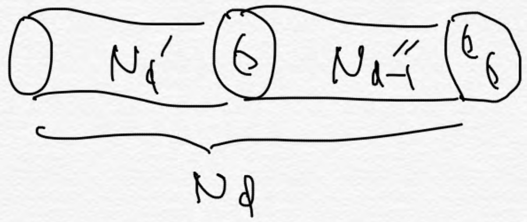
orientation reversal

$$\partial N_d = M^{in}_{d-1} \sqcup \overline{M^{out}_{d-1}}$$

$$Z_Q(N_d) : \mathcal{H}_Q(M^{in}_{d-1}) \longrightarrow \mathcal{H}_Q(M^{out}_{d-1})$$: linear map

transition amplitude

such that they compose correctly :



$$Z_Q(N_d) = Z_Q(N_{d-1}'') \circ Z_Q(N_{d-1}')$$

We also require

$$\begin{cases} \mathcal{H}_Q(M_{d-1} \sqcup M_{d-1}') = \mathcal{H}_Q(M_{d-1}) \otimes \mathcal{H}_Q(M_{d-1}') \\ \mathcal{H}_Q(\phi) = \mathbb{C} . \end{cases}$$

Then a closed N_d has $M^{in}_{d-1} = M^{out}_{d-1} = \phi$

and

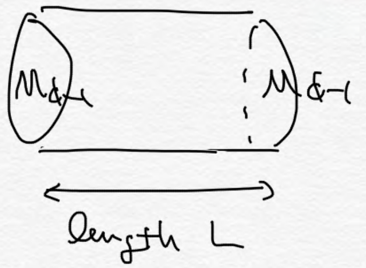
$$Z_Q(N_d) : \mathbb{C} \to \mathbb{C}$$: a linear map

$$\overset{\cap}{\mathbb{C}} .$$

partition function

A QFT is (contains as part of its data)
a functor from the bordism category
to the category of vector spaces.

When the structure S contains the metric,



$N_d = M_{d-1} \times [0,L]$

defines

$$Z_Q(N_d) : \mathcal{H}_Q(M_{d-1}) \longrightarrow \mathcal{H}_Q(M_{d-1})$$

$$\| $$

$$e^{-L\,\underline{H(M_{d-1})}}$$

↖ Hamiltonian on M_{d-1}.

Given Q, Q' with the same (d, S),

$Q \times Q'$ and $Q + Q'$ can be defined

st $\mathcal{H}_{Q\times Q'}(M_{d-1}) = \mathcal{H}_Q(M_{d-1}) \otimes \mathcal{H}_{Q'}(M_{d-1})$.

$\mathcal{H}_{Q+Q'}(M_{d-1}) = \mathcal{H}_Q(M_{d-1}) \oplus \mathcal{H}_{Q'}(M_{d-1})$.

Trivial QFT

$$\mathcal{H}_{triv}(M_{d-1}) = \mathbb{C}.$$

$$Z_{triv}(N_d) : \mathcal{H}_{triv}(M_{d-1}^{in}) = \mathbb{C} \xrightarrow{id} \mathcal{H}_{triv}(M_{d-1}^{out}) = \mathbb{C}$$

Clearly the identity under $Q \times Q'$.

Invertible QFT

$$\mathcal{H}_Q(M_{d-1}) \simeq \mathbb{C} \quad \text{(but not canonically)}$$

we can define $\mathcal{H}_{Q^{-1}}(M_{d-1}) := \mathcal{H}_Q(M_{d-1})^\vee$.

$Q \times Q^{-1} = $ triv.

Invertible QFTs for (d, S) form an Abelian group. under $Q \times Q'$.

Why interesting?

Math : well, QFTs form a monoidal category. one needs to understand invertibles.

Physics : believed / conjectured to be closely related / identical to the low energy limit of the symmetry protected topological phases.

& captures the anomaly of $(d-1)$-dimensional anomalous QFTs.

Unitarity = Reflection Positivity

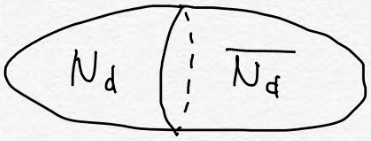
conjugate linear involution

$$Z_Q(N_d) : \mathcal{H}_Q(M_{d-1}) \longrightarrow \mathcal{H}_Q(M'_{d-1})$$

$$Z_Q(\overline{N_d}) : \mathcal{H}_Q(M'_{d-1}) \rightarrow \mathcal{H}_Q(M_{d-1})$$

such that $Z_Q(\overline{N_d}) \circ Z_Q(N_d)$ is positive definite.

In particular,

$$Z_Q\Big($$


$$\Big) > 0.$$

∃ many interesting non-unitary theories but far less understood.

not a standard terminology. introduced to simplify the discussion

Let Strongly unitary invertible theory Q be such that

$$Z_Q(N_d) : \mathcal{H}_Q(M_d) \to \mathcal{H}_Q(M_d')$$

is unitary. ← note that this is a natural assumption in Lorentzian QFT but not in Eucledianized QFT

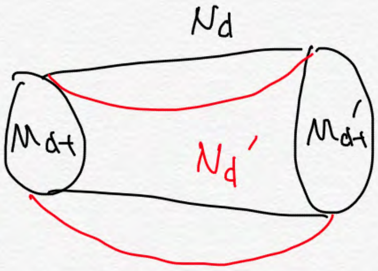
Strongly unitary inv. theories are characterized by

$$Z_Q(N_d) \in U(1) \text{ for closed } N_d.$$

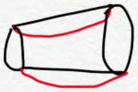
Rough argument:

set $\mathcal{H}_Q(M_{d-1}) = \mathbb{C}$

then



$$Z_Q(N_d), Z_Q(N_d') : \underset{\mathbb{C}}{\underset{\shortparallel}{\mathcal{H}_Q(M_{d-1})}} \longrightarrow \underset{\mathbb{C}}{\underset{\shortparallel}{\mathcal{H}_Q(M_{d-1}')}}$$

Choose s.t. $\dfrac{Z_Q(N_d)}{Z_Q(N_d')} = Z_Q\Big($

$\Big) \in U(1)$

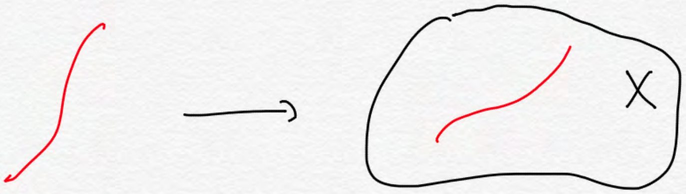
↑
closed mfd

Examples

$d = 1$

S : orientation + a map to X : a mfd

To construct a Q, pick a line bundle $L \to X$ together with a $U(1)$ connection.



$$\begin{cases} Z_{Q_L}(f : S^1 \to X) = \text{holonomy of the } U(1) \text{ connection around } f(S^1). \\ \mathcal{H}_{Q_L}(x : pt \to X) = \text{fiber over } x \in X. \end{cases}$$

with a suitable definition,

strongly unitary invertible QFTs ($d=1$, S: orientation + map to X)

$\updownarrow$

line bundle with $U(1)$ connections on X.

respects products:

$$Q_L \times Q_{L'} = Q_{L \otimes L'}$$

Classification is very detailed.

Two rougher classifications:

1. Take only flat line bundles + $U(1)$ connections.
   - $Z_Q(S^1)$ only depends on the homotopy class of the map $S^1 \to X$.
   - Classified by $\underline{H^1(X, U(1))}$.
2. Introduce equivalences $Q \sim Q'$ when they are homotopic (i.e. can be continuously deformable to each other.)
   - Any connection on the line bundle with a fixed $c_1(L)$ is mutually deformable
   - Classified by $\underline{H^2(X, \mathbb{Z})}$.

$\boxed{\text{THEOREM}}$

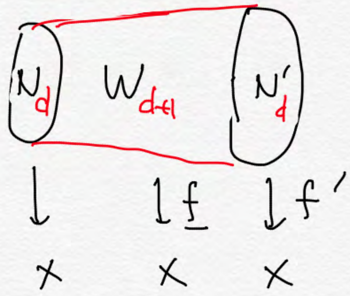
Fix d, $S = \begin{array}{l}\text{orientation} + \text{spin structure} \\ + \\ \text{map to } X.\end{array}$

1. strongly unitary homotopic invertible QFTs are classified by $\mathrm{Hom}(\Omega^{\overset{\text{spin}}{\text{oriented}}}_d(X), U(1))$
2. Homotopy classes of strongly unitary invertible QFTs are classified by $\mho^{d+1}_{\underset{\text{spin}}{\text{oriented}}}(X)$, the Anderson dual of $\Omega_d^{\underset{\text{spin}}{\text{oriented}}}(X)$.

Here $\Omega^{d}_{\text{oriented/spin}}(X)$: oriented/spin bordism group of X

$$\begin{array}{c}\rotatebox{90}{$\in$}\\ [N_d,\ f: N_d \to X]\end{array}$$

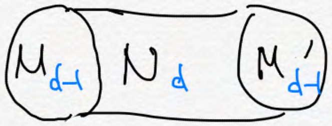
where $(N,f) \sim (N',f') \Leftrightarrow \exists$



and the sum is just the disjoint union ;

$\mathrm{Hom}(\Omega^{d}_{\text{oriented}}(X), U(1))$ is its Pontryagin dual.

[NOTE : def. of d-dim'l QFT used



instead.]

The Anderson dual is more elaborated.

Recall the universal coeff thm. of (co)homology :

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(H_d(X,\mathbb{Z}),\mathbb{Z}) \to H^{d+1}(X,\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_{d+1}(X,\mathbb{Z}),\mathbb{Z}) \to 0$$

← torsion part

free part ↙

For any generalized homology theory $F_d(-)$,

$\exists$ its Anderson dual $Ⅎ^{d}(-)$ s.t.

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(F_d(X,\mathbb{Z}),\mathbb{Z}) \to Ⅎ^{d+1}(X,\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(F_{d+1}(X,\mathbb{Z}),\mathbb{Z}) \to 0$$

← torsion part

free part ↙

- Generalized homology F_d
  comes with generalized cohomology F^d.
  $\text{Ⅎ}^d \neq F^d$ in general.
- $H^d = H^d$, $\text{Ʞ}^d = K^d$, $\text{Oƞ}^d = KO^{d \pm 4}$.
- no simple relation between $\Omega^d_{\text{structure}}$ & $\text{℧}^d_{\text{structure}}$.
- For $d<4$, $H_d(-,\mathbb{Z}) \simeq \Omega^{\text{oriented}}_d(-)$
  $\rightsquigarrow$ reproduces the $d=1$ case we saw.

$\boxed{\text{proof}}$ 1. [Yonekura] ← very concrete.

2. [Freed-Hopkins] [Freed]

↑ uses a lot of alg. top; I haven't been able to follow.

idea of proof of 1

any bordism

can be decomposed into elementary procedures of removing $S^p \times B^{q+1}$ from N_d and pasting $B^{p+1} \times S^q$ back to it where $p+q+1=d$ by using a Morse function.

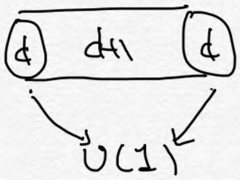
these steps are controllable with the Atiyah axioms of Q.

If you want to know more, invite Kazuya Yonekura to your seminar!

✱ NOTE : the theorem identifies

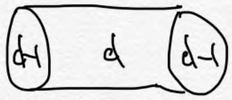
elems of $\{\mathrm{Hom}(\Omega_d(X), U(1))\}$

i.e. bordism invariants



$\updownarrow$

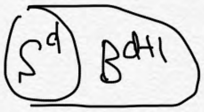
functors from



to $Z: \mathcal{H} \longrightarrow \mathcal{H}$ with positivity.

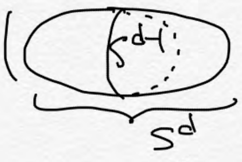
it's somewhat surprising that this condition automatically appears on this side.

But note :



$\rightsquigarrow$ $Z(S^d) = 1$ if bordism inv.

This is implied from unitarity $Z($



$) > 0$.

$\exists$ counterexample of a non-unitary invertible theory which is not a bordism invariant.

$d = 4n+1$ [Yonekura, private communication]

S : orientation.

$$\text{action} = \int b\,(*d + d*)\,c + m\, b \wedge * c \quad , m \to \pm\infty$$

where b, c : sections of $\bigoplus_{p:\text{even}} (p\text{-forms on } M)$ and fermionic.

$$Z_Q(M_d) = e^{2\pi i \eta(\underline{*d+d*})}$$

↖ acting on even forms

$$= (-1)^{\sum_{p:\text{even}} \dim H^p(M_d)}$$

In particular, $Z_\Theta(S^d) = -1 < 0$.

An example of spin strongly unitary invertible phase

$d=4$

S: spin structure + a map to $SU(N)$ (group mfd; not the classif. sp.)

known as the Wess-Zumino-Witten term. [Witten 1983]

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(\overset{0}{\Omega_4^{\mathrm{spin}}(SU(N))}, \mathbb{Z})$$

$$\to \mho^5_{\mathrm{spin}}(SU(N)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(\Omega_5^{\mathrm{spin}}(SU(N)), \mathbb{Z}) \to 0$$

$$\begin{tikzcd} \pi_5(SU(N)) \simeq \mathbb{Z} \arrow[d, "\times 2"'] & = & \Omega_5^{\mathrm{spin}}(SU(N)) \arrow[d] & \ni & {[M_5, f: M_5 \to SU(N)]} \arrow[d, mapsto] \\ \mathbb{Z} & = & H_5(SU(N), \mathbb{Z}) & \ni & {[f(M_5)]} \end{tikzcd}$$

Therefore

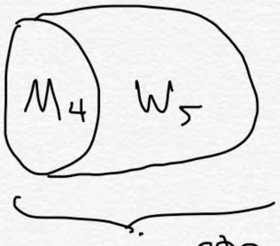
$$\begin{tikzcd} \mathbb{Z} \arrow[d, "\times 2"'] & = & H^5(SU(N), \mathbb{Z}) \arrow[d] \\ \mathbb{Z} & = & \mho^5_{\mathrm{spin}}(SU(N)) \end{tikzcd}$$

More concretely, take the de Rham representative ω of the generator of $H^5(SU(N), \mathbb{Z})$.

Then, for an integer k,

$$Z_{\theta_k}(f: M_4 \to SU(N)) = e^{2\pi i \, k/2 \int_{W_5} f^*(\omega)}$$

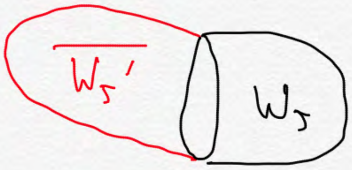
where f:



$\longrightarrow SU(N)$

This does not depend on the choice of W_5 since

given



.

$$e^{2\pi i\frac{k}{2}\int_{W_5}\omega} \Big/ e^{2\pi i\frac{k}{2}\int_{W_5'}\omega} = e^{2\pi i\frac{k}{2}\underbrace{\int_{W_5\sqcup\overline{W_5'}}\omega}_{\text{integrates to even integer on spin manifolds.}}} = 1.$$

(This subtle factor of 2 was already alluded to in [Witten 1983].)

So far I assumed $N\geqslant 3$. $N=2$ is somewhat special.

$SU(N=2)=S^3$.

$\pi_4(S^3)=\mathbb{Z}_2.\ \xrightarrow{\sim} \tilde{\Omega}_4^{\mathrm{spin}}(S^3)=\mathbb{Z}_2\ ,\quad \Omega_5^{\mathrm{spin}}(S^3)=0$

$\hookrightarrow \tilde{\Omega}_4^{\mathrm{oriented}}(S^3)=0\ ,\quad \Omega_5^{\mathrm{oriented}}(S^3)=0.$

$$0\to \overset{\mathbb{Z}_2}{\overset{\shortparallel}{\mathrm{Ext}(\Omega_4^{\mathrm{spin}}(S^3),\mathbb{Z})}}$$

$$\xrightarrow{\simeq} \underset{\mathbb{Z}_2}{\underset{\shortparallel}{\mho^5_{\mathrm{spin}}(S^3)}} \to \overset{0}{\overset{\shortparallel}{\mathrm{Hom}(\Omega_5^{\mathrm{spin}}(S^3),\mathbb{Z})}}\to 0$$

$\mathbb{Z}_2 \leftarrow$ given by $Z(f: M_4\to S^3)=\begin{cases}+1 & \text{if bordant}\\ -1 & \text{otherwise}\end{cases}$

Using $SU(2)\hookrightarrow SU(N)$ we can pull back :

$$\underset{\mathbb{Z}}{\underset{\shortparallel}{\mho^5_{\mathrm{spin}}(SU(N))}} \xrightarrow[\text{mod } 2]{} \underset{\mathbb{Z}_2}{\underset{\shortparallel}{\mho^5_{\mathrm{spin}}(SU(2))}}$$

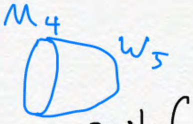
You can also consider spin-c structure.

$$\tilde{\Omega}_4^{\mathrm{spin-c}}(S^3)=0,\quad \Omega_5^{\mathrm{spin-c}}(S^3)=\mathbb{Z}$$

$$0 \to \overset{\overset{0}{\parallel}}{\mathrm{Ext}(\Omega_4^{\mathrm{spin}^c}(S^3), \mathbb{Z})}$$

$$\to \underset{\underset{\mathbb{Z}}{\parallel}}{\mho^5_{\mathrm{spin}^c}(S^3)} \xrightarrow{\simeq} \overset{\overset{\mathbb{Z}}{\parallel}}{\mathrm{Hom}(\Omega_5^{\mathrm{spin}^c}(S^3), \mathbb{Z})} \to 0$$

← given by



spin-c curvature ↓

$$Z(M_4) = e^{2\pi i k \int_{W_5} \frac{F}{2\pi} \wedge \omega}$$

↖ gen. of $H^3(S^3, \mathbb{Z})$

again, we have $\underset{\underset{\mathbb{Z}}{\parallel}}{\mho^5_{\text{spin-c}}(S^3)} \xrightarrow[\text{mod } 2]{} \underset{\underset{\mathbb{Z}_2}{\parallel}}{\mho^5_{\text{spin}}(S^3)}$

Whether an inv. QFT comes from the torsion part or the free part is not an invariant information when we $\begin{pmatrix}\text{extend}\\ \text{restrict}\end{pmatrix}$ the str S.

Another example of spin strongly unitary invertible phase

$d=3$ $\quad S =$ spin str + $U(1)$ bundle with connection.

known as spin U(1) Chern-Simons term.

Given k: integer,

$$Z_{Q_k}(M_3, \underset{\text{with curvature } F}{U(1) \text{ connection } A}) = e^{2\pi i \frac{k}{2} \int_{W_4} \frac{F}{2\pi} \wedge \frac{F}{2\pi}}$$

where

spin str, U(1) bundle with connection.

Then

$$\frac{e^{2\pi i \frac{k}{2} \int \frac{F}{2\pi} \wedge \frac{F}{2\pi}}}{e^{2\pi i \frac{k}{2} \int \frac{F}{2\pi} \wedge \frac{F}{2\pi}}} = e^{2\pi i \frac{k}{2} \underbrace{\int \frac{F}{2\pi} \wedge \frac{F}{2\pi}}_{\text{self intersection number on spin 4-manifold is even}}} = 1.$$

The invertible ~ SPT phases discussed so far are interacting phases in cond-mat parlance.

Free fermionic phases

Massive free fermions have gapped 1-dim vacua.

⇝ can extract invertible phases in the low-energy = long-distance = infinitely massive limit.

Lagrangian in Lorentzian signature

$$\mathcal{L} = \sum_i \psi_i \not{D} \psi_i + \sum_{i,j} \boxed{m_{ij} \psi_i \psi_j}$$

Dirac op.

section of some copies of SO(d-1, 1) real spinor bundle.

- mass term.
- a real antisymmetric SO(d-1, 1)-invariant bilinear form.

massive $\underset{\text{def}}{=}$ non-degenerate.

$\underline{d=4}$

$$\sum_i \bar{\psi}_i \not{D} \psi_i + m_{ij} \psi_i \psi_j + \overline{m_{ij}}\, \overline{\psi_i \psi_j}$$

cpx symmetric matrix.

parameter sp. of massive mass term

$$\simeq \{ \text{cpx non-deg. sym. matrix} \}$$

$$\simeq U(\infty)/O(\infty).$$

d=3 $\sum \psi_i \not{\partial} \psi_i + \sum m_{ij} \psi_i \psi_j$

$\uparrow$ real symmetric.

space of non-deg real sym. mat

$$\simeq \underset{\substack{\uparrow \\ \text{signature}}}{\mathbb{Z}} \times \frac{O(\infty)}{O(\infty)\times O(\infty)}$$

d=2 $\sum \left(\psi_i^+ \not{\partial} \psi_i^+ + \psi_i^- \not{\partial} \psi_i^-\right) + \sum m_{ij} \psi_i^+ \psi_j^-$

$\uparrow$ real matrix

$$\text{space of } m \simeq \frac{O(\infty)\times O(\infty)}{O(\infty)} \simeq O(\infty)$$

d=1 $\sum \psi_i \partial_t \psi_i + m_{ij} \psi_i \psi_j$

$\uparrow$ real asym.

$$\text{space of } m \simeq \frac{O(\infty)}{U(\infty)}$$

etc, etc.

summary

d	1	2	3	4	5	6	7	8	9
space of non-deg. mass	$\frac{O}{U}$	$\frac{O\times O}{O}$	$\mathbb{Z}\times\frac{O}{O\times O}$	$\frac{U}{O}$	$\frac{Sp}{U}$	$\frac{Sp\times Sp}{Sp}$	$\mathbb{Z}\times\frac{Sp}{Sp\times Sp}$	$\frac{U}{Sp}$	$\frac{O}{U}$
		" O	" $\mathbb{Z}\times BO$			" Sp	" $\mathbb{Z}\times BSp$		

"

KO^{d-3} : the classifying sp of $KO^\bullet$ shifted by -3.

π_0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	

We can consider position-dependent mass terms :

$$\mathcal{L} = \sum \psi_i \not{D} \psi_i + \sum m_{ij}(x) \psi_i(x) \psi_j(x) .$$

Let

S : spin str + a map to X.

A free fermionic invertible phase

$\overset{\text{def}}{\equiv}$ a theory with

$$\mathcal{L} = \sum \psi_i \not{D} \psi_i + \sum m_{ij}(f(x)) \psi_i(x) \psi_j(x) .$$

where f : spacetime $\longrightarrow X$

and m_{ij} : $X \longrightarrow$ space of non-deg mass-term

$$\simeq KO^{d-3}$$

Deformation class of such m_{ij}'s

$= [X, KO^{d-3}] = KO^{d-3}(X)$

known as the free classification (of symmetry class D.)

Kitaev, Ryu-Schnyder-Furusaki-Ludwig

These are examples of strongly unitary spin invertible phases.

$\leadsto$ there should be a natural transformation

$$KO^{d-3}(X) \longrightarrow \mho^{d+1}_{spin}(X) .$$

describing the relationship between free v.s. interacting.

It would be interesting to study it using η invariants etc. in a differential-geometric manner.

An answer in terms of algebraic topology is as follows :

there's a natural transformation

$$\Omega^{spin}_d(X) \to KO_d(X)$$

called Atiyah-Bott-Shapiro orientation.

It sends $(M_d, f: M_d \to X]$

to the KO homology class represented by $f(M_d)$.

(KO homology class is very roughly
a pair (submanifold, Dirac operator on it)
and requires the submfd to be spin.)

Taking the Anderson dual, we have

$$\begin{tikzcd} OY^d(X) \arrow[r] \arrow[d, equal] & \mho^d_{spin}(X) \\ KO^{d-4}(X) & \end{tikzcd}$$

or equivalently

$$KO^{d-3}(X) \to \mho^{d+1}_{spin}(X).$$

so : Free-to-interacting homomorphism is the Anderson dual of the A-B-S orientation.

//.