Gaiotto–Johnson-Freyd 1712.07950
Yonekura 1803.10796
Brumfiel–Morgan 1612.02860 (3d) 1803.08147 (4d)

①

Kitaev's homotopy theoretic approach to the classification of symmetry-protected topological phases

AIM ⓪ Classify ① SPTs / invertible phases ② under various conditions.

from my distorted point of view

① SPT / invertible phase := gapped phase (no zero energy excit.)
no topological order (1-dim Hilb sp. on closed spatial slice)

- (phase = QFT)
- what's the name "SPT" or "invertible"? → comes back soon.

② · condition fix: { free / interacting — bosonic / fermionic; relativistic / non-relativistic; unitary / possibly non-unitary; with flavor symmetry G } vary

- set of QFTs satisf. fixed cond. with G sym. forms a commutative monoid
  - $Q_1 \times Q_2$: two QFTs considered as one
    partition func. multiplied
    Hilb. sp tensored
  - "trivial" : Hilb. sp 1 dim, $Z = 1$ } really trivial

- invertible QFTs have inverses.

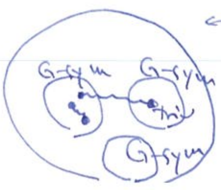
$$\mathcal{H}_{Q^{-1}} = \overline{\mathcal{H}_Q} \quad \text{s.t.} \quad \mathcal{H}_{Q^{-1}} \otimes \mathcal{H}_Q \simeq \mathbb{C}$$

$$Z_{Q^{-1}} = Z_Q^{-1}$$

QFT comes in continuous families, param. by $E_d^{\mathcal{E},G}$

⓪ classify : identify two QFTs (with fixed cond.) with sym G if they can be connected by a continuous change.

← inv. QFT with no add'l sym



"sym protected"

what's $\pi_0(E_d^{\mathcal{E},G})$?

various people have various proposals/results for different set of fixed conditions.

Wen et al. "bosonic, sym G: finite" $\rightsquigarrow$ $H^d(BG, U(1))$ d: spacetime dimension

cf. if G is gauged, this is Dijkgraaf-Witten).

Ryu et al. fermionic, free, no sym $\rightsquigarrow$ $KO^{d+2}(pt)$

G: finite $\rightsquigarrow$ $KO^{d+2}(BG)$

pontrjagin dual

Kapustin et al fermionic, interacting, G: finite $\rightsquigarrow$ $\Omega^{spin}_d(BG)^\wedge$

Freed-Hopkins ask Kazuya about the difference.

always a generalized coh. th.

Kitaev's approach 'answers' this. $\longrightarrow$ why?

short answer: $E_d^{e,G} = \mathrm{Map}(BG, E_d^{e,\phi})$.

So let's consider general gapped/massive free fermionic system without any symmetry (except CPT coming from unitarity).

reflection positivity in Euclidean spacetime

$d=4$ $\quad \bar\psi^i \not\partial \psi_i + m_{ij}\psi_i\psi_j + c.c.$ (symmetric)

$SO(4) \simeq SU(2)\times SU(2)$

massive $\rightsquigarrow$ $\det m \neq 0$.

what's the space of such m? $\curvearrowleft$ $U(N)$ acts.

a single such m breaks it to $O(N)$.

essentially $U(N)/O(N)$ ($N\to\infty$ in an appropriate sense).

$d=3$ $\quad \psi^i \partial \psi_i + m_{ij}\psi_i\psi_j$

has $O(N)$ rotation $\qquad$ real symmetric, breaks it to $O(N_+)\times O(N_-)$

space of such $m \simeq \mathbb{Z} \times \dfrac{O(\infty)}{O(\infty)\times O(\infty)}$

$d=2$ $\quad \psi_+^i \not\partial \psi_+^i + \psi_-^j \not\partial \psi_-^j + m_{ij}\psi_+^i\psi_-^j$ $\qquad \dfrac{O(\infty)\times O(\infty)}{O(\infty)}$

$d=1$ $\quad \psi^i \partial_t \psi^i + \overset{\text{asym.}}{\widehat{m_{ij}}}\psi^i\psi^j$ $\qquad \dfrac{O(\infty)}{U(\infty)}$

$d=9$ $\qquad$ determines $J^2=-1$

$d=8$ $\qquad U(\infty)/Sp(\infty)$

$d=7$ $\qquad \mathbb{Z}\times Sp(\infty)/Sp(\infty)\times Sp(\infty)$

$d=6$ $\qquad Sp(\infty)\times Sp(\infty)/Sp(\infty)$

$d=5$ $\qquad Sp(\infty)/U(\infty)$

rule of thumb for SUSY persons:

$$\frac{\text{R-sym of } \mathcal{N}=\infty \text{ in } d\text{-dim}}{\text{R-sym of } \mathcal{N}=\infty \text{ in } (d-1)\text{-dim}}$$

Summary

d	1	2	3	4	5	6	7	8	9
$M_d :=$	O/U	$\frac{O\times O}{O} = O$	$\mathbb{Z}\times\frac{O}{O\times O} = \mathbb{Z}\times BO$	$\frac{U}{O}$	$\frac{Sp}{U}$	$\frac{Sp\times Sp}{Sp} = Sp$	$\mathbb{Z}\times\frac{Sp}{Sp\times Sp} = \mathbb{Z}\times BSp$	$\frac{U}{Sp}$	$\frac{O}{U}$
π_0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$		

$KO^{d-3}(pt)$.

Not only that, $KO^{d-3}(X) = [X, M_d]$ $\leftarrow$ gp of homotopy classes of maps from X to M_d.

$M_d \cong \Omega M_{d+1}$: Ω-spectrum

Ω: space of loops

NOTE. of course Dirac ops are related to $K(O)$. But the way it appears is <u>different</u>.

generalized coh. th $E^*(-)$: satisfies the same set of axioms as $H^*(-,\mathbb{Z})$

except $H^i(pt,\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i\neq 0 \end{cases}$

$\updownarrow$ Brown representability

$\exists$ Ω-spectrum $E_d \simeq \Omega E_{d+1}$ e.g. $H^*(X,\mathbb{Z})$

s.t. $E^d(X) = [X, E_d]$. $= [X, K(n,\mathbb{Z})]$

QFT with $\boxed{\text{a fixed condition } \begin{pmatrix}\text{free}\\ \text{interacting}\end{pmatrix} \begin{pmatrix}\text{bos.}\\ \text{fer}\end{pmatrix} \text{(non)rel (non)unit}}$ ... $\mathcal{C}$

with a fin. sym. G

$\rightsquigarrow$ $\exists$ $E_d^{\mathcal{C}} \simeq \Omega E_{d+1}^{\mathcal{C}}$ s.t. gp of conn. comp. of Inv. phase. with condition $\mathcal{C}$ with sym G

$= E^d_{\mathcal{C}}(BG) = [BG, E_d^{\mathcal{C}}]$.

$E_d^{\mathcal{C}}$: space of local gapped "Lagrangian" with the condition set $\mathcal{C}$.

e.g. when $d=4$, $\mathcal{C}$ = free relativ. fermionic unitary,

$$E_4^{\mathcal{C}} = \frac{U}{O}$$

$m: X_4 \to E_4^{\mathcal{C}}$: position-dependent mass term.

$$\int d^4x \ \bar{\psi}^i \partial\!\!\!/ \psi_j + m_{ij}(x)\psi^i\psi^j + \text{c.c.}$$

still everywhere gapped.

X_4 with a bkg G gauge field.

$\updownarrow$ equiv

$A: X_4 \to BG$ up to homotopy. an inv. th. with G-sym transform this to

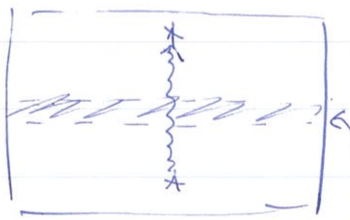
$m: X_4 \to E_4^{\mathcal{C}}$. to do this, we use $\alpha : BG \to E_4^{\mathcal{C}}$

and just define $m = \alpha \circ A$.

$\rightsquigarrow$ inv. phase with G-sym $\Leftrightarrow$ $[BG, E_4^{\mathcal{C}}] = E^4_{\mathcal{C}}(BG)$.

why should we expect $E_d^e \simeq \Omega E_{d+1}^e$ in general ?

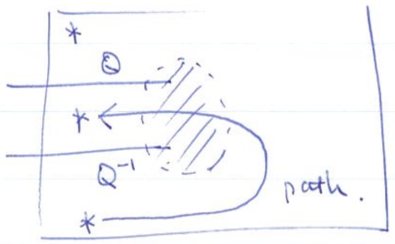
$\leftarrow$:



$* \in E_{d+1}^e$

produces an inv. phase in one lower dim.

$\rightarrow$:



use the fact that $Q \times Q^{-1} \simeq$ trivial

$\boxed{\text{Kapustin et al.}}$: interacting fermionic : $\Omega_d^{spin}(BG)^\wedge = [BG, X_d]$.

their idea : QFT_d should associate , a la Atiyah,

functor Q :

$$M_{d-1} \rightsquigarrow \mathcal{H}_Q(M_{d-1})$$

$$M_d : M_{d-1}^{in} \to M_{d-1}^{out} \rightsquigarrow Z_Q(M_d) : \mathcal{H}_Q(M_{d-1}^{in}) \to \mathcal{H}_Q(M_{d-1}^{out})$$

invertible : $\mathcal{H}_Q(M_{d-1})$ one dimensional $\simeq \mathbb{C}$ (not cannonical)

for $\partial M_d = \phi$, $Z_Q(M_d) \in U(1)$: unitary on $\mathbb{C} = \mathcal{H}(\phi)$.

demand Z_Q is a func. on bordism class

$\rightsquigarrow$ determines an element of $\mathrm{Hom}(\Omega_d^{spin}(BG), U(1))$.

$\boxed{\text{Yonekura}}$: $\leftarrow$ March this year 1803

shows 1:1 correspondence between

refl. positive functor Q s.t. $\mathcal{H}_Q \simeq \mathbb{C}$

$$\longleftrightarrow \quad \Omega_d^{spin}(BG)^\wedge$$

constructively.

$\boxed{\text{Gaiotto - JF}}$: $\begin{cases} \text{reviews} \\ \text{describes } X_d \text{ combinatorially (explicitly for small } d. \end{cases}$

$\uparrow$ 1712.

$\boxed{\text{Brumfiel-Morgan}}$ describes $[M, X_3]$, $[M, X_4]$ and determine the pairing with elements of $\Omega_d^{spin}(M)$ explicitly.

$\uparrow$ 1803, 1612