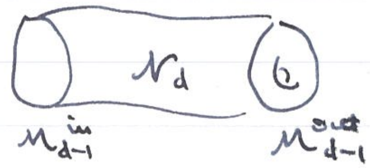
(Note that the references are given at the end, and are not explicitly given in the main handwritten part of the notes.)

A d-dim'l QFT Q with such and such structure

does...

① assign $\mathcal{H}_Q(M_{d-1})$: a Hilb. space of states
for a $(d-1)$-dim'l mfd with such and such structure

② for



one has

$$Z_Q(N_d) : \mathcal{H}_Q(M_{d-1}^{in}) \longrightarrow \mathcal{H}_Q(M_{d-1}^{out})$$

the "transition function"

③ s.t. they glue correctly, etc.

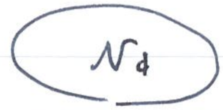
$$\mathcal{H}(M \sqcup M') = \mathcal{H}(M) \otimes \mathcal{H}(M'), \text{ etc.}$$

comments

1. I'm using HEP convention for the dimensions.
In cond-mat, M_d N_{d+1} instead.

2. "such and such structure" can include things such as
smooth str, spin str, metric,
G-bundle with or without connection,
a map to a given space X, etc ...

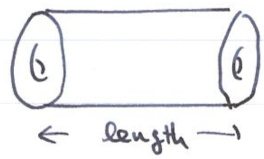
3. $\mathcal{H}(\phi) = \mathbb{C}$. then, for $\partial N_d = \phi$,



$$Z(N_d) : \underset{\mathbb{C}}{\underset{\shortparallel}{\mathcal{H}(\phi)}} \longrightarrow \underset{\mathbb{C}}{\underset{\shortparallel}{\mathcal{H}(\phi)}}$$

is a complex number, called the partition function.

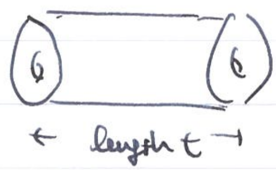
4. If the structure includes the metric,



$Z_Q(M_{d-1}\times I_t): \mathcal{H}(M_{d-1}) \xrightarrow{t} \mathcal{H}(M_{d-1})$, the Hamiltonian

is given by $e^{-tH(M_{d-1})}$

5. If the str. include a G-bundle with connection,



holonomy $g \in G$

gives $e^{-tH(M_{d-1})}\rho(g) : \mathcal{H}(M_{d-1}) \to \mathcal{H}(M_{d-1})$

G-action on $\mathcal{H}(M_{d-1})$.

this means:

A QFT has a symmetry G when the structure includes G-bundle with connection.

6. Q is called gapless if
lowest eigenvalue of $H(\mathbb{R}^{d-1})$ is not isolated.

is called gapped if
of $H(\mathbb{R}^{d-1})$ is isolated.

is called TQFT if
$\mathcal{H}(M_{d-1})$ is finite dim'l for any M_{d-1}.

is called invertible if
$\mathcal{H}(M_{d-1})$ is 1-dim'l for any M_{d-1}.

~~is called broken~~ if.

7. What was defined above should better be called a non-anomalous / absolute QFT.

We'll discuss <u>anomalous</u> QFT where in particular $Z(N_d)$ for $\partial N_d = \emptyset$ is defined only up to a multiplication by a phase $c \in U(1) = \{|z|=1\}$.

8. Given a G-symmetric QFT Q, the gauging of G, denoted by Q/G, assigns

$$Z_{Q/G}(N_d) = \int_{\substack{A:\ G\text{-bundles with} \\ \text{connection on } N_d}} Z_Q(N_d, A)\, \underset{\substack{\uparrow \\ \text{a nice measure.}}}{d\mu}$$

when G is not a finite group and $d>1$, very hard to make precise.

when G is finite, perfectly well defined.

<u>Examples</u>

1. 0+1 d G-symmetric TQFT.

$$\underset{\text{pt}}{M^{in}} \xrightarrow{\ N\ } \underset{\text{pt}}{M^{out}}$$

$$G \overset{\rho}{\curvearrowright} \mathcal{H}(M^{in}) = \mathcal{H}(\text{pt}) = \mathcal{H}(M^{out})$$

a (unitary) G-representation.

$Z($  $) = \mathrm{tr}_{\mathcal{H}(\text{pt})}\, \rho(g)$: character.

holonomy g

$\rightsquigarrow$ the study of QFT is a generalization of the rep. theory of groups.

2. 1+1d finite G gauge theory.

. consider a G-sym. QFT where

$$\mathcal{H}_{\text{triv}}(M_{d-1}) = \mathbb{C} \quad \text{for } {}^\forall M_{d-1}$$

$$\underset{\substack{\parallel \\ \text{id}}}{Z_{\text{triv}}} : \mathcal{H}(M_{d-1}) \to \mathcal{H}(M_{d-1}') \qquad \text{for any}$$

trivial QFT.

then $$Z_{\text{trivial}+G}(N_d) = \sum_{\substack{P:\ \text{isom. class of} \\ G\text{-bundle on } N_d}} \frac{1}{\#(\mathrm{Aut}\,P)} \underbrace{Z_{\text{triv}}\begin{pmatrix} P \\ \downarrow \\ N_d \end{pmatrix}}_{1.}$$

$$= \frac{1}{\# G} \sum_{\phi:\ \mathrm{Hom}(\pi_1(N_d),\, G)} 1$$

specialize to $d=2$.

$V := \mathcal{H}(S^1)$ has two nice sets of orthogonal bases.

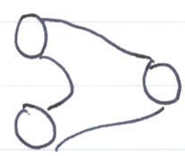
i) $|[g]\rangle \in V$ where $[g]$: conj. class

ii) $|R\rangle \in V$ where R : irrep

s.t. $\langle R|[g]\rangle = \chi_R(g)$

orthogonality relations of characters means ↺ is unitary.

it turns out



$Z : V \otimes V \to V$

is $$\langle R''|Z(|R\rangle \otimes |R'\rangle) = \begin{cases} \dfrac{|G|}{\dim R} & \text{if } R = R' = R'' \\ 0 & \end{cases}$$

A genus-γ surface Σ_γ is made of $2\gamma-2$ pairs of pants.

$$\Rightarrow \quad Z_{\text{triv}+G}(\Sigma_\gamma) = \sum_R \left(\frac{|G|}{\dim R}\right)^{2\gamma-2}$$

$$\Rightarrow \quad \#\mathrm{Hom}(\pi_1(\Sigma_\gamma), G) = |G|^{2\gamma-1} \sum_R (\dim R)^{2-2\gamma}.$$

$\gamma=0 \rightsquigarrow |G| = \sum_R (\dim R)^2$.

$\gamma=1 \rightsquigarrow \#\{(g,h) \text{ s.t. } gh=hg\} = |G| \times \#\{\text{conj. class}\}$

$\gamma=2 \rightsquigarrow \#\{(g_1,h_1,g_2,h_2) \text{ s.t. } g_1h_1g_1^{-1}h_1^{-1}g_2h_2g_2^{-1}h_2^{-1}=e\}$

$$= |G|^3 \sum_R (\dim R)^{-2}$$

$\vdots$

3. Cohomology-type invertible G-symmetric QFT.
"G-SPT phase"

Let's quickly recall group cohomology.

G : finite group. $\ni g_1 \cdots g_n$

A : abelian group.

$C^n(G,A)$: abelian gp. of functions $f(g_1,\cdots,g_n) \in A$

$$d: C^n(G,A) \longrightarrow C^{n+1}(G,A)$$
$$f \longmapsto df$$

is defined by the formula

$$\begin{aligned}(df)(g_1,\cdots,g_n,g_{n+1}) := {} & f(g_2,g_3,\cdots,g_{n+1}) \\ & - f(g_1g_2,g_3,\cdots,g_{n+1}) \\ & + f(g_1,g_2g_3,\cdots,g_{n+1}) \\ & \vdots \\ & + (-1)^n f(g_1,g_2,\cdots,g_ng_{n+1}) \\ & + (-1)^{n+1} f(g_1,\cdots,g_n)\end{aligned}$$

We can check that $d^2=0$.

kernel of d : cocycles $\in Z^n(G,A)$

image of d : coboundaries

$H^n(G,A)$: its cohomology.

given $\alpha \in H^d(G,U(1))$,

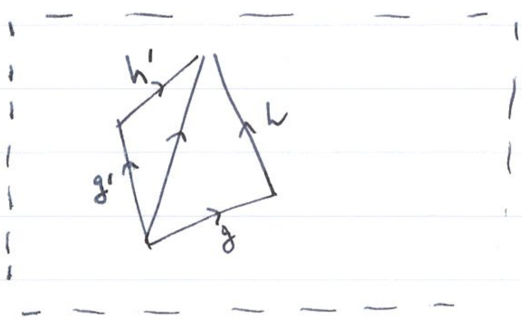
a G-symmetric invertible QFT Q_α is given as follows.

pick a specific cocycle $\alpha \in Z^d(G, U(1))$.

$$Z_{Q_\alpha}(N_d \text{ with } G\text{-bundle}) = \prod_{\triangle} \alpha(g_1, \cdots, g_d)^{\pm 1}$$


: simplex in a simplicial decomp. of N_d

say $d=2$.

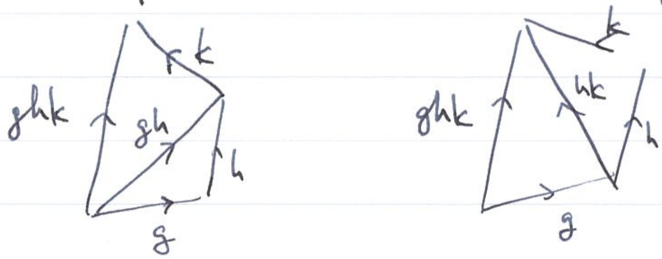


contributes by $\alpha(g,h)^{+1}$

contributes by $\alpha(g,h)^{-1}$

- This defintion would be familiar to people with stat. mech. bkg.

- Does not depend on the choice of triangulation.



$$\alpha(g,h)\,\alpha(gh,k) = \alpha(g,hk)\,\alpha(h,k)$$

↑ equivalent to $d\alpha = 0$

- Alg. topologist would prefer the following def instead:

$\alpha \in H^d(G, U(1)) = H^d(BG, U(1))$.

a G-bundle on N_d defines a map $f: N_d \to BG$.

the pull back $f^*(\alpha) \in H^d(N_d, U(1))$
can be integrated against the fund. class of N_d
(assuming it's oriented)

$$Z_{Q_\alpha}(f: N_d \to BG) = \int_{N_d} f^*(\alpha) \in U(1).$$

4. Dijkgraaf-Witten th.

in dimension d, pick $\alpha \in H^d(G, U(1))$.

Q_α : invertible phase just discussed.

$$Z_{Q_\alpha/G}(N_d) = \sum_{P:\ G\text{-bundle on } N_d} \frac{1}{\#(\mathrm{Aut}\,P)} Z_{Q_\alpha}\left(\begin{smallmatrix} P \\ \downarrow \\ N_d \end{smallmatrix}\right)$$

when $\alpha = 0$, reduces to the example 2.

5. more general invertible phase. (with such and such structure)

Let $\Omega_d^{\text{such and such str}}$ = bordism group

= N_d with such and such str,

identified if $\exists$ N_d (X_{d+1}) N_d'

sum given by disj. union.

in particular. ~~if~~

$$\Omega_d^{\text{such and such str + G-bundle}} = \Omega_d^{\text{str}}(BG).$$

Let $\alpha \in \mho^d_{\text{str}}(BG) = \mathrm{Hom}(\Omega_d^{\text{str}}(BG), U(1))$.

$$Z_{Q_\alpha}(f: N_d \to BG) = \alpha(\underbrace{[f: N_d \to BG]}_{\text{class in } \Omega_d^{\text{str}}(BG)}) \in U(1).$$

- Example 3 is a special case : $\exists$ natural transf.

$$\begin{array}{ccc} H^d(BG, U(1)) & \longrightarrow & \mho^d_{\text{str}}(BG) \\ \cup\!\!\!\!\!\shortmid & & \cup\!\!\!\!\!\shortmid \\ \alpha & \longmapsto & \underline{\alpha} \end{array}$$

s.t. $\underline{\alpha}([f]) = \int_{N_d} f^*(\alpha)$.

not necessarily either surjective or injective.

- all homotopical invertible phases are Q_α for some $\alpha \in \mho^d_{\text{str}}$.

6. generalized DW theory : Q_α / G.

END of 1.

- An anomalous d-dim'l QFT Q with such and such structure has $Z_Q(N_d)$ defined only up to a (controllable) phase.

For simplicity assume $\partial N_d = \phi$.

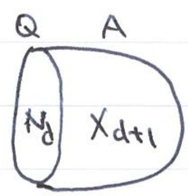
$Z_Q(N_d)$ is a cpx number def'd up to a phase.

$\simeq$ takes the values in a one-dim vector sp without a distinguished basis vector.

$Z_Q(N_d) \in \mathcal{H}_A(N_d)$: 1-dim ~~Hilb~~ sp.

A natural method to assign 1-dim Hilb spaces to N_d ...

Use an invertible QFT in $(d+1)$ dimension! Call it A.



$$\begin{cases} Z_Q(N_d) \in \mathcal{H}_A(N_d) \\ Z_A(X_{d+1}) : \mathcal{H}_A(N_d) \longrightarrow \mathcal{H}_A(\phi) = \mathbb{C} \end{cases}$$

$Z_Q(N_d)\, Z_A(X_{d+1}) \in \mathbb{C}$: combined system has a well-defined cpx number as the partition function.

A is the anomaly of Q.

Q lives on the boundary of A.

- Example. $(0+1)$-dim'l anomalous G-symmetric TQFT.

$$pt \xrightarrow{g} pt \qquad pt \xrightarrow{h} pt$$

$$pt \xrightarrow{gh} pt$$

$\rho(g), \rho(h), \rho(gh) : V \to V$

where $V = \mathcal{H}(pt)$.

$\rho(g)\,\rho(h) = \alpha(g,h)\,\rho(gh)$ where $\alpha(g,h) \in U(1)$.

a projective representation.

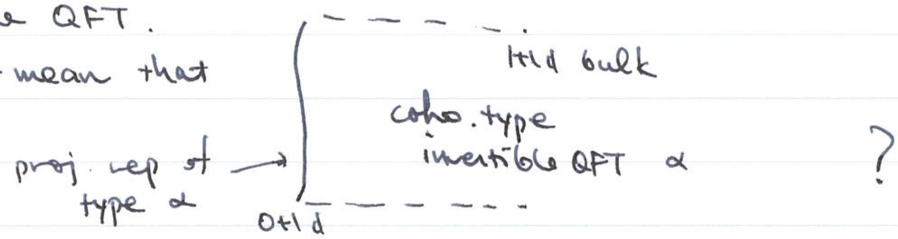
$$\rho(g)\rho(h)\rho(k) = \rho(gh)\rho(k)\,\alpha(g,h) = \rho(ghk)\,\alpha(g,h)\,\alpha(gh,k)$$
$$\qquad\qquad\qquad '' \ \rho(g)\rho(hk)\,\alpha(h,k) = \rho(ghk)\,\alpha(g,hk)\,\alpha(h,k)$$

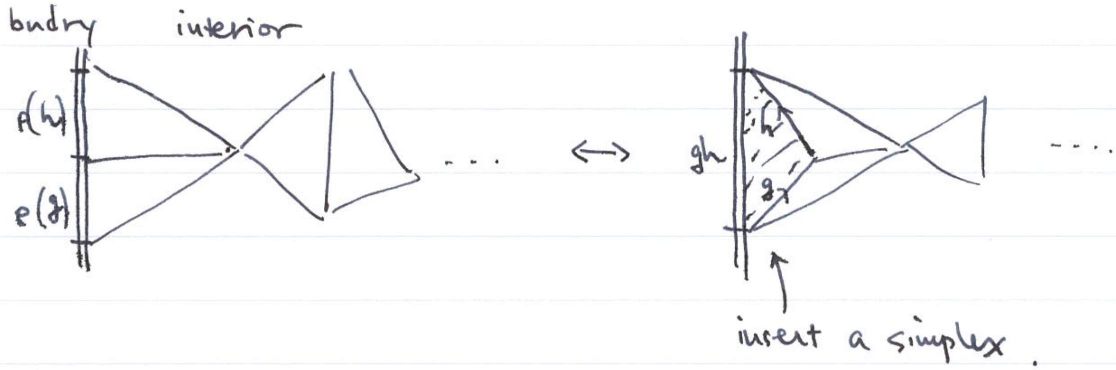
means $\alpha \in Z^2(G, U(1))$.

$$Z\left(\bigcirc^{g}\right) = \mathrm{tr}_V\,\rho(g) = \mathrm{tr}_V\,\rho(h^{-1}gh) \times \underline{\text{a phase}}.$$

We saw that the same α determines a $1+1$ d G-symmetric invertible QFT.

What does it mean that





thanks to $\rho(g)\rho(h) = \rho(g,h)\,\alpha(g,h)$,

the total partition function is invariant under the change of the triangulation.

if we <u>don't</u> have the proj. rep ρ on the bndry.

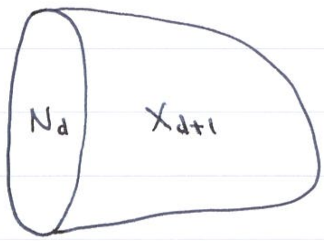
the bulk system is <u>not</u> invariant under triangulation change.

$\Rightarrow$ the top. inv. of the total system

+ the specific form of the bulk <u>forces</u> <u>something to appear</u> on the boundary.

"topological material" "bulk-boundary correspondence"

$\boxed{\text{Central Question}}$ pick d, structure, G.



Q $\quad A_\alpha$: specified by $\alpha : \Omega^{\text{str}}_{d+1}(BG) \to U(1)$

or, as a special case,

$\alpha \in H^{d+1}(BG, U(1))$.

$\boxed{\text{Question}}$ Is there a d-dimensional anomalous QFT Q whose anomaly is given by A_α ?

$\boxed{\text{gapped Question}}$ Is there ... $\begin{pmatrix}\text{gapped} \\ \text{TQFT}\end{pmatrix}$ Q ... A_α ?

$\boxed{\text{invertible Question}}$ invertible Q ... A_α ?

has full solution: only for trivial α, $\exists$ invertible Q.

in other words: an invertible Q is automatically non-anomalous.

partial answer:

$\exists$ gapped (TQFT) Q for any $\alpha \in H^{d+1}(BG, U(1))$

$\exists$ for any $\alpha : \Omega^{\text{unoriented}}_{d+1}(BG) \to U(1)$

But $\exists\ \alpha \in \Omega^{\text{spin}}_5(BSU(2)) \to U(1)$, of order 2,

s.t. $\exists$ physics argument saying no gapped Q

?

$SU(2)$ bundle on N_5 defines an element of $KSp(N_5) \simeq KO^4(N_5)$.

with spin str, we can integrate it $\in KO^{4-5}(pt) \simeq \mathbb{Z}_2$.

known as Witten's global anomaly. $\exists$ gapless Q.

For general α, even the existence of gapless Q is not clear.

NOTE From physical perspective, one wants to require that the gapped phase is without symmetry breaking.
I don't know how to formulate it mathematically yet.
For TQFT, one requirement is $\dim \mathcal{H}(S^{d-1}) = 1$.
The examples mentioned above satisfy this.
If one drops this cond, constructing anomalous TQFT for $\alpha \in H^{d+1}(G, U(1))$ is almost trivial.

The rest of the talks will be spent to describe how to construct them.
Along the way, we learn a few other things of independent interest.

—. —. —. —

Q : 1+1 d QFT with non-anomalous symmetry A : finite Abelian gp.

$$Z_Q(\Sigma, v) \in \mathbb{C}$$

Σ: 2d surface; v: A-bundle on it, i.e. $v \in H^1(\Sigma, A)$.

Q/A : gauged th.

$$Z_{Q/A}(\Sigma) = \underbrace{\frac{1}{|A|^g}}_{\text{normalization factor}} \sum_v Z_Q(\Sigma, v)$$

g: genus of Σ

in fact, Q/A naturally has $\hat{A}$-symmetry.
$\hat{A}$: group of homomorphisms $A \to U(1)$.

$$Z_{Q/A}(\Sigma, w) := \frac{1}{|A|^g} \sum_v e^{2\pi i \int w \wedge v} Z_Q(\Sigma, v)$$

$w \in H^1(\Sigma, \hat{A})$

$w \wedge v \in H^2(\Sigma, \underbrace{A \otimes \hat{A}}_{\downarrow \; U(1) \simeq \mathbb{Q}/\mathbb{Z}})$

dually,

$$Z_Q(\Sigma, v) := \frac{1}{|A|^g} \sum_w e^{2\pi i \int v \wedge w} Z_{Q/A}(\Sigma, w).$$

This is the discrete Fourier transformation!

$Q/A/\hat{A} = Q$.

Generalize this to d-dim'l A-symmetric Q:

$$Z_{Q/A}(\Sigma, w) := \frac{1}{|A|^g} \sum_v e^{2\pi i \int w \wedge v} Z_Q(\Sigma, v)$$

now $\in H^{d-1}(\Sigma, \hat{A})$

$$Z_Q(\Sigma, v) := \frac{1}{|A|^g} \sum_w e^{2\pi i \int w \wedge v} Z_{Q/A}(\Sigma, w)$$

$v \in H^1(\Sigma, A)$.

ordinary symmetry gp A ⇝ background } $\in H^1(M, A)$
" 0 - corresp. geom. obj

p-symmetry gp A $:\overset{\text{def}}{\Leftrightarrow}$ background $\in H^{p+1}(M, A)$.
or just $A_{[p]}$.

⌜ when Q has 0-symmetry $A_{[0]}$, Q/A has $(d-2)$-symmetry $\hat{A}_{[d-2]}$,
such that $Q/A_{[0]}/\hat{A}_{[d-2]} = Q$. ⌟

Easy to generalize further:

Q : having p-symmetry A, $v \in H^{p+1}(M, A)$
Q/A : q-symmetry $\hat{A}$, $w \in H^{q+1}(M, \hat{A})$
} $p+q+2 = d$.

<u>MOTTO</u> : Finite gauging is a reversible process.
The gauged theory should contain enough info to reconstruct the original theory.

Let's consider next a Q, having a symmetry Γ, s.t.

$$0 \to A \to \Gamma \to G \to 0$$

s.t. $A \subset \Gamma$ is normal and $\Gamma/A = G$.

as a set, $\Gamma \simeq A \times G$

$$(0, g) \circ (0, h) = (e(g,h),\ gh)$$

$e \in Z^2(G, A)$: two cocycle of the extension.

Consider

$Q + A$: has $(d-2)$-symmetry $\hat{A}_{[d-2]}$

0-symmetry $G = \Gamma / A$.

where did the data $e \in Z^2(G, A)$ go ?

Answer. it has the anomaly $e^{2\pi i \int_{d+1} w \wedge f^*(e)}$

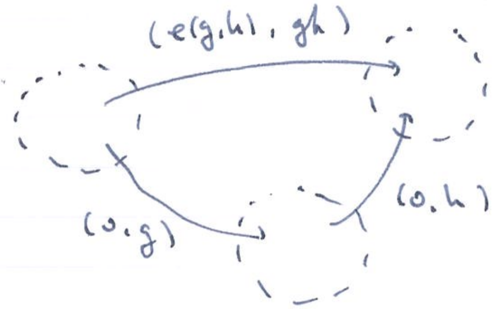
where $f : X_{d+1} \to BG$. $e \in H^2(G, A)$

$\rightsquigarrow$ $f^*(e) \in H^2(X_{d+1}, A)$

$w \in H^{d-1}(X_{d+1}, \hat{A})$

Why ? A Γ-bundle is A G-bundle $+$ $a \in C^1(M, A)$

s.t. $\partial a = f^*(e) \in Z^2(M, A)$.



then $Z_{Q+A}(M, \underset{\overset{\Uparrow}{H^{d-1}(M, \hat{A})}}{w}, f : M \to BG) \overset{?}{:=} \frac{1}{|A|^{\#}} \sum_a e^{2\pi i \int_M w \cup a} Z_Q(M, a)$

two cochains $w, w' \in Z^{d-1}(M, \hat{A})$

with $[w] = [w'] \in H^{d-1}(M, \hat{A})$

can give different values, since

$w' = dx + w \rightsquigarrow \int (w' \cup a - w \cup a)$

$= \int dx \cup a = \int x \cup \partial a = \int_M x \cup f^*(e) \neq 0$.

M X

introduce $\int_X w \cup f^*(e)$. $\int_X (w' - w) \cup f^*(e) = \int_X \partial x \cup f^*(e) = \int_M x \cup f^*(e)$

cancels the variation above.

$\rightsquigarrow$ $Q+A$ lives on the boundary of the anomaly theory.

suppose next

$Q + A_{[p]}$ has $(d-p-2)$-symmetry $\hat{A}_{[d-p-2]}$

0-symmetry G

s.t. the anomaly is $e^{2\pi i \int_{X_{d+1}} w \wedge f^*(e)}$

where $w \in H^{d-p-1}(X, \hat{A})$

$e \in H^{p+2}(G, A)$ s.t. $f^*(e) \in H^{p+2}(X, A)$.

what's the symmetry of

$Q = Q + A_{[p]} + \hat{A}_{[d-p-2]}$?

↑ can couple to the pair

$v \in C^{p+1}(M, A)$

$f : M \to BG$ s.t. $\partial v = f^*(e) \in Z^{p+2}(M, A)$.

when $p=0$, this corresponds to a Γ-bundle where

$0 \to A \to \Gamma \to G \to 0$. specified by $e \in H^2(G, A)$.

$p>0$, this corresponds to a 'Γ-bundle' where

'$0 \to A_{[p]} \to \Gamma \to G_{[0]} \to 0$' sp. by $e \in H^{p+2}(G, A)$.

alg. topologist would say

$$\begin{array}{ccc} K(A, p+1) & \to & B\Gamma \\ & & \downarrow \\ & & BG = K(G, 1) \end{array}$$

: fibration is specified by the postnikov class e.

a 'Γ-bundle' $\leftrightarrow$ $f : M \to B\Gamma$.

more generally, any space Π s.t. $\begin{cases} \pi_i(\Pi) \text{ all finite} \\ \pi_{j \gg i_0}(\Pi) = 0 \end{cases}$

is a repeated fibration

$$\left.\begin{array}{ccc} & & \vdots \\ K(A', 2) & \to & \Pi_2 \\ & & \downarrow \\ K(A, 2) & \to & \Pi_1 \\ & & \downarrow \\ & & K(G, 1) \end{array}\right\} \text{finite height.}$$

a map $M \to \Pi$ is a background for a symmetry

which is a 0-symmetry G extended by 1-symmetry A

extended by 2-symmetry A',

...

So far, we discussed that 'generalized symmetries' appear
e.g. if we gauge a subgroup A of a group Γ with certain anomaly.

We now need to study if $\exists$ a theory with a given anomaly.

So: take G : a finite group.
$\alpha \in H^{d+1}(G, U(1))$.
is there a (gapped) TQFT Q with this anomaly?

- starting point : when α is trivial,
  you can just take Q to be a trivial theory.
  $Z_{triv}(M, v) = 1$ identically.

- <u>symmetry extension method</u>

  suppose $\exists \Gamma$ s.t. $0 \to A \xrightarrow{\iota} \Gamma \xrightarrow{p} G \to 0$
  and $p^*\alpha \in H^{d+1}(\Gamma, U(1))$
  $\overset{\shortparallel}{0}$.

  take the trivial Γ-symmetric theory $triv_\Gamma$.

  then $triv_\Gamma / A$ is a G-symmetric theory
  with the required anomaly α.

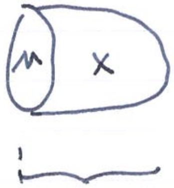
  <u>why?</u>

  pick a specific cocycle $\alpha \in Z^{d+1}(G, U(1))$.
  $p^*\alpha \in Z^{d+1}(\Gamma, U(1))$.
  this vanishes in cohomology, so
  $\exists \beta$ s.t. $\partial\beta = p^*\alpha$.
  $\beta \in C^d(\Gamma, U(1))$ $\quad = 0$ restricted to $C^{d+1}(A, U(1))$.
  $\rightsquigarrow \beta \in H^d(A, U(1))$.

  define
  $$Z_{triv_\Gamma / A}(M, \underset{\substack{\uparrow \\ G\text{-bundle on it}}}{v}) = \sum_{\substack{\Gamma\text{-bundle } \gamma \\ \text{lifting } v}} e^{2\pi i \int_M \gamma^*(\beta)}$$

  when v is trivial, this reduces to D-W theory for A
  with action $\beta \in H^d(A, U(1))$.

when v is nontrivial, ~~$\gamma^*(\beta)$~~ we do the following:



G-bundle v on X

v lifts to γ on the boundary M.

$\int_X v^*(\alpha)$ is not good, since the cochain $v^*(\alpha)$ does not vanish on $\partial X = M$.

But on M, $\partial \gamma^*(\beta) = v^*(\alpha)$

$\rightsquigarrow -\int_X v^*(\alpha) + \int_M \gamma^*(\beta)$ is well defined.

so, triv_Γ / A naturally lives on the anomaly theory with $Z = e^{-2\pi i \int_X \alpha}$.

· <u>a math question.</u>

G: finite, $\alpha \in H^{d+1}(G, U(1))$.

is there $0 \to A \to \Gamma \xrightarrow{p} G \to 0$ s.t. $p^*\alpha = 0$?

<u>answer</u>

$\exists$ a universal A s.t. $\exists$ $e \in H^2(G, A)$

s.t. $H^{d-1}(G, \hat{A}) \simeq H^{d+1}(G, U(1))$

via $w \longmapsto w \cup e = \alpha$.

Then, as a cochain, $\exists$ $a \in F^1(\Gamma, A)$ s.t. $\partial a = e$

so $\partial(w \cup a) = w \cup e = \alpha$

i.e. α pulled back to Γ is exact.

(known as the dimension shifting in group coh. literature.
when $d+1 = 2$, basically what Schur did.)

so, triv_Γ / A has the anomaly $\alpha \in H^{d+1}(G, U(1))$.

· we saw previously that for <u>any</u> Γ-symmetric theory Q,

Q / A has the symmetry $\hat{A}_{[d-2]} \times G$

with the anomaly $\int \underline{w} \cup e$

where $\underline{w} \in H^{d-1}(X, \hat{A})$ and $e \in H^2(G, A)$ describes the extension.

it's just that we are "embedding"

$$G_{[0]} \longrightarrow \hat{A}_{[d-2]}$$

using the coh. class

$$\underline{w} \in H^{d-1}(G, \hat{A}) \simeq [BG, K(\hat{A}, d-1)]$$

when $d=2$ this reduces to a homomorphism

$$G_{[0]} \to \hat{A}_{[0]}$$

Then, under $G_{[0]} \text{ diag} \to \hat{A}_{[d-2]} \times G_{[0]}$

the anomaly $\int \underline{w} \cup e$ for $\hat{A}_{[d-2]} \times G_{[0]}$ becomes the anomaly $\int w \cup e$ for $G_{[0]}$.

- more importantly, we see that G is not playing much role.

<u>what we use is that</u>

$\text{triv}_{A_{[0]}} + A_{[0]}$ has the sym. $\hat{A}_{[d-2]} \times A_{[1]}$

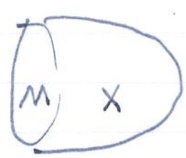
s.t. for the background $\underline{w} \in H^{d-1}(X, \hat{A})$, $\underline{e} \in H^2(X, A)$

the anomaly is $\int_X \underline{w} \cup \underline{e}$.

<u>Why?</u>

$$Z_{\text{triv}+A}[M, \underline{w}, \underline{e}] = \sum_{\substack{\text{cochain } a \in C^1(M,A) \\ \text{s.t. } \partial a = \underline{e} \text{ up to coboundaries}}} e^{2\pi i \int_M a \cup \underline{w}}$$

But as before, we need the combination



$$\int_X \underline{w} \cup \underline{e} - \int_M \underline{w} \cup a.$$

$\underline{e}, \underline{w}$

$\partial a = \underline{e}$ on the boundary

$\Rightarrow \text{triv}_{A_{[0]}} + A_{[0]}$

lives on the boundary of the anomaly theory $e^{2\pi i \int \underline{w} \cup \underline{e}}$.

"the topological Green-Schwarz effect"

so we were just doing

$$G_{[0]} \xrightarrow{w} \hat{A}_{[d-2]} \times A_{[1]} \quad (e: G_{[0]} \to A_{[1]})$$

Of course it is immediate to generalize to

triv $\uparrow A_{[p]}$ having $\hat{A}_{[d-2-p]} \times A_{[p+1]}$ symmetry

with the anomaly $\int_X \underline{w} \cup \underline{e}$

for $\underline{w} \in H^{d-1-p}(X, \hat{A})$, $\underline{e} \in H^{p+2}(X, A)$.

where $Z_{\text{triv} \uparrow A_{[p]}}[M, \underline{w}, e] = \sum_{\substack{a \in C^{p+1}(M,A) \\ \text{s.t. } \partial a = \underline{e}}} e^{2\pi i \int_M a \cup \underline{w}}$.

These observations allow us to solve the following question easily:

Q. Let the structure on the mfd to be just the topology ...

i.e. consider non-oriented mfds

with symmetry $G_{[0]}$: finite.

The anomaly is characterized by

$$\alpha \in \mathrm{Hom}(\Omega^{\text{unoriented}}_{d+1}(BG), U(1)).$$

Is there a gapped boundary?

A. Yes. Thom showed $\Omega^{\text{unoriented}}_{d+1}(X) = \Omega^{\text{uno.}}_{*} \otimes H_*(X, \mathbb{Z}_2)$

and therefore

$$\mathrm{Hom}(\Omega^{\text{uno}}_{*}(BG), U(1))$$

$$\simeq \underbrace{\mathrm{Hom}(\Omega^{\text{uno}}_{*}, U(1))}_{} \otimes H^*(X, \mathbb{Z}_2).$$

known to be given by polynomials of Striefel-Whitney classes w_n.

They are overcomplete but the relations are all known.

Now, the top. GS mechanism allows us to construct
gapped boundaries for any anomaly of the form

$$\underset{\uparrow}{A} \cup \underset{\uparrow}{B}$$
degree ≥ 2.

Question remains for indecomposable things i.e.g. $\int_{X_{d+1}} w_{d+1}$ itself.

so we need to be a bit more systematic.

any poly. of Stiefel Whitney classes is a poly. of
$w_1, w_2, w_4, w_8, \cdots$ and Sq^I acting on it.

we first kill $(w_1)^2$, w_2 by $\mathbb{Z}_2[0] \times \mathbb{Z}_2[0]$ gauging: $\partial a = (w_1)^2$, $\partial a' = w_2$

then w_4 by $\mathbb{Z}_2[2]$ gauging: $\partial b = w_4$

$\vdots$

It happens that, by Wu's formula, once w_{2^n} is killed,
up to $w_{2^{n+1}}$ is killed automatically
when evaluated on 2^{n+1}-dim mfd $M_{2^{n+1}}$.

so, it's safe enough.

At this point the remaining anomaly is $w_1 x + y$
for $x \in H^d(G, \mathbb{Z}_2)$
$y \in H^{d+1}(G, \mathbb{Z}_2)$

but they can be killed by extending $0 \to A \to \Gamma \to G \to 0$ DONE.

It's good that we are / I am finally using alg. top. developed in the late 1960s ... in something related to physics!

In the remaining time I'd like to mention that
what we've been discussing are the simplest possible cases
and usually it's more complicated!

<u>Recall.</u> the 2d theory Q with finite Abelian symmetry A

$$\begin{aligned} &\Downarrow \\ Q/A \quad &\text{has fin. Abel. sym } \hat{A} \\ &\Downarrow \\ Q/A/\hat{A} &= Q . \end{aligned}$$

How do we generalize this to non-Abelian finite G ?

Take a different view on the Abelian case :

$$Z_{Q/A}[\Sigma, w] = \sum_v e^{2\pi i \int w \wedge v} Z_Q[\Sigma, v]$$

$w \in H^1(\Sigma, \hat{A})$, $v \in H^1(\Sigma, A)$.

$\quad \cong$

$H_1(\Sigma, \hat{A})$: basically, loops labeled by elements of $\hat{A}$.

$$= \sum_v \underbrace{e^{2\pi i \int_w v}}_{\substack{\text{Holonomy of } A\text{-connection} \\ \text{along loops labeled by } \hat{A} = \text{f.irreps of } A}} Z_Q[\Sigma, v]$$

then $$Z_Q[\Sigma, v] = \underbrace{\sum_w e^{2\pi i \int v \wedge w}}_{\substack{\Uparrow \\ \text{insert all possible loop operators} \\ \text{with an appropriate weight.}}} Z_{Q/A}[\Sigma, w] .$$

In this language, can be generalized to non-abelian G :

$$Z_{Q/G}[\Sigma, (C_1, R_1) \cdots (C_n, R_n)] := \sum_{v: G\text{ conn.}} \Big(\prod_i \mathrm{Tr}_{R_i} \mathrm{Hol}_v(C_i)\Big) Z_Q[v] .$$

Then $$Z_Q[\Sigma, \text{triv. } G\text{-bundle}] = Z_{Q/G}[\Sigma, (A_1, \underset{\substack{\uparrow \\ \text{regular rep}}}{\mathrm{reg}}) (B_1, \mathrm{reg}) \cdots]$$

s.t. $\mathrm{Tr}_{\mathrm{reg}} \mathrm{Hol}_v = \begin{cases} 1 & \text{if trivial} \\ 0 & \text{or not} \end{cases}$

i.e. Q : has G-symmetry = has line op labeled by $g \in G$

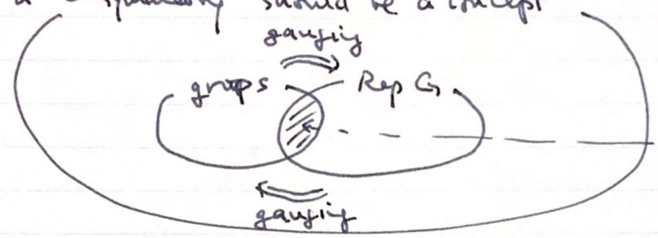
$$\Big\vert_g \ \Big\vert_h \to \Big\vert_{gh}$$

$\Longleftrightarrow$

Q/G : has "Rep G - symmetry" = has line op labeled by $R \in \mathrm{Rep}\, G$

$$\Big\vert_{R_1} \ \Big\vert_{R_2} \to \Big\vert_{R_1 \otimes R_2}$$

In 2d, a (finite) 0-symmetry should be a concept



Abelian groups where gauging acts by taking Pontrjagin duals.

closed under gauging, containing both G's and Rep G's

Fusion categories do the job.

$\supset$ a group with anomaly specified.

A general question: is there a (gapless / gapped) 2d system with a given fusion category as the sym?

Totally unsolved! cf. Haagerup fusion category.

In a higher dimension,

0-form symmetry G
$\updownarrow$
$(d-2)$-form 'symmetry Rep G'

1. not perfectly dual.
$\Leftarrow$ 2. already more general than any sym. of the form spacetime $\to$ some space with finite homotopy groups.

also: p-symmetry G $\Leftrightarrow$ a background $\nu \in H^{p+1}(M, G)$
$\Leftrightarrow$ dual operator inserted at $n \in H_{d-p-1}(M, G)$

But: there are cases where* a theory has a $(d-p-1)$-dim'l operator which requires spin structure on it, etc.

* a theory which can couple to a background $\nu \in KO^{p+1}(M)$ ...

generalization continues. WHAT IS A SYMMETRY?

An extremely subjective set of references is as follows:

- The pseudo-mathematical definition of QFTs is based on my article [Tac17a].
- The 2d finite-group gauge theory was discussed in detail in [Fre92] as a TQFT. This lecture note contains much more.
- The Dijkgraaf-Witten theory was introduced in [DW90]. Its relevance in the condensed-matter physics in the context of SPT phases was noted in [CGLW11]. The proposal that the bordism groups classifies SPT phases originates from [Kap14, KTTW14]. This put on a firmer mathematical ground in [FH16, Yon18].
- The symmetry-extension method to construct gapped boundaries was first pointed out in [Wit16]. It was then shown in [WWW17] that for any G and the anomaly $\alpha \in H^{d+1}(G, U(1))$ one can find a suitable extension. As I did not understand the argument there, I gave a different argument in [Tac17b]. The (easy) extension to $\Omega_{d+1}^{\text{unoriented}}(BG)$ was my ongoing work with K. Ohmori.
- In 1+1d, a finite symmetry (together with the anomaly) is described by a unitary fusion category. This is a known fact, see e.g. [FFRS09, CR12]; Bhardwaj and I wrote an exposition for high-energy physicists [BT17]. A physics application was described in [CLS+18].

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