Term-end projects for "Algebraic topology for physicists"

for the autumn-inter semester, academic year 2024

Last update: Dec. 18, 2024

Please answer *at least one* of the questions below, write it as a term-end paper, and upload it to the UTokyo LMS page of this lecture series. I haven't decided the deadline yet, but it is probably around the last week of January.

1 Hopf fibration

We discussed the Hopf fibration $S^1 \to S^3 \to S^2$. S^3 can be projected to $\mathbb{R}^3$ using a stereographic projection. Objects in $\mathbb{R}^3$ can then be further projected to a computer screen and or drawn on a paper.

- Visualize the Hopf fibration itself, for example by drawing the inverse images of various points on S^2. If you create an animation or an interactive program, please upload it online and provide a link to it in the term-end paper.
- (Optional) We also discussed the construction of the connecting homomorphism $\partial : \pi_2(S^2) \to \pi_1(S^1)$ associated to the Hopf fibration. Visualize this in a way which satisfies you.

2 Topological solitons, 1

By now there are tons of papers about experimental observations of topological solitons. Pick (at least) one paper and give a summary, again to your heart's content.

3 Topological solitons, 2

The order parameter of superfluid Helium-3 is a homogeneous space G/H_A or G/H_B, where

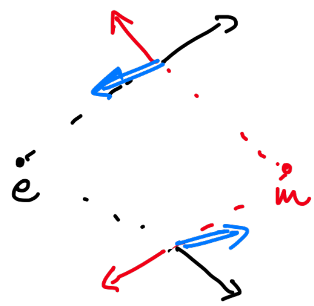
- $G = SO(3) \times SO(3) \times U(1)$, and
- H_A and H_B are subgroups depending on whether we are in the superfluid A-phase or the B-phase,

as described during the lectures and in the lecture note.

Compute their homotopy groups π_1 and π_2. It is of course OK to refer to various textbooks, or original papers!

4 Alternative "derivation" of the Dirac quantization law

We discussed the Dirac quantization law from a rather mathematical perspective. It is known that the same condition can be derived from the following consideration. Put an electrically charged particle of charge e at $p = (-1, 0, 0)$, and a magnetic monopole of charge m at $q = (+1, 0, 0)$. The pattern of the electric field and the magnetic field creates a nonzero Poynting vector, as seen below:



This creates an angular momentum around the axis connecting the points p and q. Demand that it equals the minimal allowed value $\hbar/2$ in quantum mechanics. What do you find as the magnetic charge m of the magnetic monopole?

(Optional) The derivation here is very different from the one given during the lectures. Are there any relation?

5 Kitaev's toric code and cohomology groups

During the lectures and in the lecture notes, we discussed how Kitaev's toric code gives a Hamiltonian whose ground state has basis states labeled by elements of $H^1(M; \mathbb{Z}_2)$ as realized by simplicial or cellular cohomology groups.

- Generalize this to $H^1(M; A)$ for an arbitrary finite Abelian group A.
- (Optional) generalize this to $H^p(M; \mathbb{Z}_2)$ for a higher p.
- (Optional) generalize this to $H^p(M; A)$ for a higher p and an arbitrary finite Abelian A.

6 Some computation of (co)homology groups, 1

Let Σ_g be an oriented compact two-dimensional surface of genus g. Give it a simplicial or cellular decomposition, and compute $H_p(\Sigma_g; \mathbb{Z})$ and $H^p(\Sigma_g; \mathbb{Z})$.

(Optional) How about the same question for non-orientable surfaces?

7 Some computation of (co)homology groups, 2

Realize S^{2n-1} as the unit sphere in $\mathbb{C}^n$, parameterized by $(z_1, \ldots, z_n)$. Let $\mathbb{Z}_k$ act on it via

$$(z_1, \ldots, z_n) \mapsto e^{2\pi i/k}(z_1, \ldots, z_n).$$

We have a quotient $M = S^{2n-1}/\mathbb{Z}_k$. Compute its homology groups $H_p(M; \mathbb{Z})$ and $H^p(M; \mathbb{Z})$.

Hint: the cell decomposition and much more is explained in Hatcher's textbook, Example 2.43.

8 $U(1)$ bundles with zero $[F]$ and nonzero c_1

In the exercise above, you find that $H^2(M;\mathbb{Z}) = \mathbb{Z}_k$ for $M = S^{2n-1}/\mathbb{Z}_k$. This means that it is possible to have a $U(1)$ bundle P over it such that its first Chern class $c_1(P)$ is the generator 1 of this $\mathbb{Z}_k$. As $k \cdot 1 = 0 \in \mathbb{Z}_k$, $kc_1(P) = 0$. Correspondingly, $k[F] = 0$ in the de Rham cohomology class, but this simply means that $[F] = 0$. This means that the curvature *does not* contain all the information of the first Chern class.

Let us study a systematic construction of such $U(1)$ bundles. We first consider a trivial $U(1)$ bundle over S^{2n-1}, parameterized by

$$(z_1, \ldots, z_n; w)$$

where $w \in \mathbb{C}$ parameterizes the fiber. We put a trivial connection, whose curvature is also trivial $F = 0$. We now let $\mathbb{Z}^k$ to act also on the fiber:

$$(z_1, \ldots, z_n; w) \mapsto e^{2\pi i/k}(z_1, \ldots, z_n; w).$$

Compute c_1 of this bundle following the steps described below.

- Argue that this line bundle comes from a $\mathbb{Z}_k$ bundle over M. As discussed in the lectures, it determines an element in $a \in H^1(M;\mathbb{Z}_k)$.
- Using a fine enough cover and the definition of $c_1(P)$ in the lecture notes, show that

  $$c_1(P) = \beta(a) \in H^2(M;\mathbb{Z})$$

  where β is the Bockstein associated to $0 \to \mathbb{Z} \xrightarrow{\times k} \mathbb{Z} \to \mathbb{Z}_k \to 0$. (Hint: in the lecture notes we used the sequence $0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0$ instead.)
- Now that we established the formula above, we can compute the Bockstein in the cellular cohomology, not in the Čech cohomology. Compute a and $\beta(a)$ using a cell decomposition of $M = S^{2n-1}/\mathbb{Z}_k$, and show that $\beta(a)$ is indeed $1 \in \mathbb{Z}_k$.

9 Density matrices vs. wavefunctions for n-level systems

In the lectures, we learned how to decide when a family of 2×2 density matrices over some parameter space M can be lifted to a family of qubits, i.e. a complex vector bundle $\mathbb{C}^2 \to E \to M$.

- Generalize the discussion there to n-level systems. That is, discuss when a family of $n \times n$ density matrices over M can be lifted to a family of n-dimensional Hilbert spaces $\mathbb{C}^n \to E \to M$. Describe the characteristic class obstructing this.
- Discuss an example of a family of $n \times n$ density matrices over M for $n = 3$, which cannot be lifted to a family of three-dimensional Hilbert spaces.