Term-end projects for "Algebraic topology for physicists"

for the autumn-inter semester, academic year 2025

Jan. 8, 2026

Please answer *at least one* of the questions below, or come up with an equally interesting(?) question relevant to the topic of the lectures, and write it as a term-end paper, and upload it to the UTokyo LMS page of this lecture series.

(It's OK to use LLM for help, but please do verify by yourself if what LLM tells you is reasonable. These days, LLM is more or less as knowledgeable in any descipline as a master student in that descipline, but their replies are not yet perfectly realiable, so the replies need to be vetted carefully.)

The deadline is January 30th, 23:59, JST.

The set of suggested questions is exactly the same as last year's. I'm sorry for being lazy. I should also say that I might not be able to cover the required background content for Questions 8 and 9 in the lecture this year, as the pace is slower than last year.

1 Hopf fibration

We discussed the Hopf fibration $S^1 \to S^3 \to S^2$. S^3 can be projected to $\mathbb{R}^3$ using a stereographic projection. Objects in $\mathbb{R}^3$ can then be further projected to a computer screen and or drawn on a paper.

- Visualize the Hopf fibration itself, for example by drawing the inverse images of various points on S^2. If you create an animation or an interactive program, please upload it online and provide a link to it in the term-end paper.
- (Optional) We also discussed the construction of the connecting homomorphism $\partial : \pi_2(S^2) \to \pi_1(S^1)$ associated to the Hopf fibration. Visualize this in a way which satisfies you.

2 Topological solitons, 1

By now there are tons of papers about experimental observations of topological solitons. Pick (at least) one paper and give a summary, again to your heart's content.

3 Topological solitons, 2

The order parameter of superfluid Helium-3 is a homogeneous space G/H_A or G/H_B, where

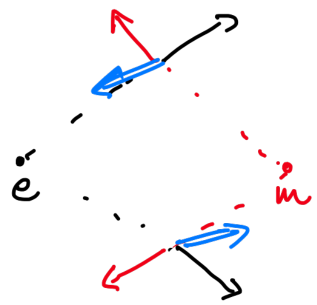
- $G = SO(3) \times SO(3) \times U(1)$, and
- H_A and H_B are subgroups depending on whether we are in the superfluid A-phase or the B-phase,

as described during the lectures and in the lecture note.

Compute their homotopy groups π_1 and π_2. It is of course OK to refer to various textbooks, or original papers!

4 Alternative "derivation" of the Dirac quantization law

We discussed the Dirac quantization law from a rather mathematical perspective. It is known that the same condition can be derived from the following consideration. Put an electrically charged particle of charge e at $p = (-1, 0, 0)$, and a magnetic monopole of charge m at $q = (+1, 0, 0)$. The pattern of the electric field and the magnetic field creates a nonzero Poynting vector, as seen below:



This creates an angular momentum around the axis connecting the points p and q. Demand that it equals the minimal allowed value $\hbar/2$ in quantum mechanics. What do you find as the magnetic charge m of the magnetic monopole?

(Optional) The derivation here is very different from the one given during the lectures. Are there any relation?

5 Kitaev's toric code and cohomology groups

During the lectures and in the lecture notes, we discussed how Kitaev's toric code gives a Hamiltonian whose ground state has basis states labeled by elements of $H^1(M; \mathbb{Z}_2)$ as realized by simplicial or cellular cohomology groups.

- Generalize this to $H^1(M; A)$ for an arbitrary finite Abelian group A.
- (Optional) generalize this to $H^p(M; \mathbb{Z}_2)$ for a higher p.
- (Optional) generalize this to $H^p(M; A)$ for a higher p and an arbitrary finite Abelian A.

6 Some computation of (co)homology groups, 1

Let Σ_g be an oriented compact two-dimensional surface of genus g. Give it a simplicial or cellular decomposition, and compute $H_p(\Sigma_g; \mathbb{Z})$ and $H^p(\Sigma_g; \mathbb{Z})$.

(Optional) How about the same question for non-orientable surfaces?

7 Some computation of (co)homology groups, 2

Realize S^{2n-1} as the unit sphere in $\mathbb{C}^n$, parameterized by $(z_1, \ldots, z_n)$. Let $\mathbb{Z}_k$ act on it via

$$(z_1, \ldots, z_n) \mapsto e^{2\pi i/k}(z_1, \ldots, z_n).$$

We have a quotient $M = S^{2n-1}/\mathbb{Z}_k$. Compute its homology groups $H_p(M; \mathbb{Z})$ and $H^p(M; \mathbb{Z})$.

Hint: the cell decomposition and much more is explained in Hatcher's textbook, Example 2.43.

8 $U(1)$ bundles with zero $[F]$ and nonzero c_1

In the exercise above, you find that $H^2(M;\mathbb{Z}) = \mathbb{Z}_k$ for $M = S^{2n-1}/\mathbb{Z}_k$. This means that it is possible to have a $U(1)$ bundle P over it such that its first Chern class $c_1(P)$ is the generator 1 of this $\mathbb{Z}_k$. As $k \cdot 1 = 0 \in \mathbb{Z}_k$, $kc_1(P) = 0$. Correspondingly, $k[F] = 0$ in the de Rham cohomology class, but this simply means that $[F] = 0$. This means that the curvature *does not* contain all the information of the first Chern class.

Let us study a systematic construction of such $U(1)$ bundles. We first consider a trivial $U(1)$ bundle over S^{2n-1}, parameterized by

$$(z_1, \ldots, z_n; w)$$

where $w \in \mathbb{C}$ parameterizes the fiber. We put a trivial connection, whose curvature is also trivial $F = 0$. We now let $\mathbb{Z}^k$ to act also on the fiber:

$$(z_1, \ldots, z_n; w) \mapsto e^{2\pi i/k}(z_1, \ldots, z_n; w).$$

Compute c_1 of this bundle following the steps described below.

- Argue that this line bundle comes from a $\mathbb{Z}_k$ bundle over M. As discussed in the lectures, it determines an element in $a \in H^1(M;\mathbb{Z}_k)$.

- Using a fine enough cover and the definition of $c_1(P)$ in the lecture notes, show that

$$c_1(P) = \beta(a) \in H^2(M;\mathbb{Z})$$

  where β is the Bockstein associated to $0 \to \mathbb{Z} \xrightarrow{\times k} \mathbb{Z} \to \mathbb{Z}_k \to 0$. (Hint: in the lecture notes we used the sequence $0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0$ instead.)

- Now that we established the formula above, we can compute the Bockstein in the cellular cohomology, not in the Čech cohomology. Compute a and $\beta(a)$ using a cell decomposition of $M = S^{2n-1}/\mathbb{Z}_k$, and show that $\beta(a)$ is indeed $1 \in \mathbb{Z}_k$.

9 Density matrices vs. wavefunctions for n-level systems

In the lectures, we learned how to decide when a family of 2×2 density matrices over some parameter space M can be lifted to a family of qubits, i.e. a complex vector bundle $\mathbb{C}^2 \to E \to M$.

- Generalize the discussion there to n-level systems. That is, discuss when a family of $n \times n$ density matrices over M can be lifted to a family of n-dimensional Hilbert spaces $\mathbb{C}^n \to E \to M$. Describe the characteristic class obstructing this.

- Discuss an example of a family of $n \times n$ density matrices over M for $n = 3$, which cannot be lifted to a family of three-dimensional Hilbert spaces.