A brief review on anomaly cancellation in string theory

– Yuji , September 2025

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0 Introduction

This is a write-up of a review talk I gave in a mini-workshop in September 25, 2025, organized by young postdocs in Japan and held in Hongo. After all, superstring theory began to be accepted by the larger community only after Green and Schwarz found the anomaly cancellation in [GS84, GS85], so it would not be a complete waste of time to review these issues.

In the part where the local anomaly is discussed, I prepared an accompanying Mathematica notebook which does the computation of the anomaly polynomial and check the factorization. The reader is encouraged to play with it if s/he has an access to Mathematica.

I tried to refer to useful references I know, but they are by no means exhaustive. I also know I referred to too many of my own papers. Please do write to me if there are more papers you think should be cited.

The organization is as follows. Sec. 1 is a very brief overview of the modern general theory of anomalies. Then in Sec. 2, we discuss fermion anomalies, and see that the anomaly polynomial factorizes in heterotic $E_8 \times E_8$ and $so(32)$ theories. In Sec. 3, we discuss local anomalies of form fields. In particular, we show that the total Type IIB anomaly polynomial is zero, and explain the Green-Schwarz mechanism at the differential form level. In Sec. 4, we discuss a precise, global formulation of self-dual form fields. We first try to use differential cohomology, but will see that it does not suffice. We introduce differential K theory, and outline how it works. Finally in Sec. 5, we discuss two precise global formulations of the Green-Schwarz mechanism. One uses (twisted) string structure, which is more suitable for heterotic theories. The other uses differential KO theory, which is more suitable for Type I.

Going through all these, you will understand the outline of local and global anomaly cancellation of Type IIB theory, Type I theory, and two supersymmetric heterotic theories, in ten-dimensional spacetime. We do not treat the anomalies in the presence of branes or orientifolds (except for the space-filling O9 plane in the Type I). We do not treat nontrivial $SL(2,\mathbb{Z})$ background in Type IIB either. In fact these two cases just mentioned have not been settled yet.

1 General theory of anomalies

When I was a graduate student there was no general theory of global anomalies. We now have one, thanks to [FH16].[1]

Consider a d-dimensional theory T. Let $Z_T[N_d]$ be its partition function of T on N_d. That T anomalous means that the phase part of $Z_T[N_d]$ is not quite well defined, but in a very controlled way.

A modern take is to regard $Z_T[N_d] \in V$, where V is a 1-dimensional $\mathbb{C}$-vector space. It is a complex number, after a basis of V is chosen. Two different choices are related in a controlled manner, leading to the anomaly. For this to make sense, one needs to have a rule assigning

$$V : N_d \mapsto V[N_d]. \tag{1.1}$$

We need to have a way to compare $V[N_d]$ and $V[N'_d]$. N_d and N'_d can share the same underlying manifolds and/or principal bundles equipped with two different metrics and/or gauge connections This corresponds to traditional anomalies. In quantum gravity, you will eventually need to sum over topologies. This requires us to compare $V[N_d]$ and $V[N'_d]$ even in cases when the underlying manifolds and/or the principal bundles have different topologies.

[1] We will be brief here. Some other nice detailed discussions on the modern take are [Yon16, Yon18, WY19, YY21]. For reviews, see e.g. [Mon19, ÁGVM22]. If you read Japanese, the master's thesis [川24] by one of the organizer, Kawahira-san, is also a nice review and highly recommended. The old paper [Wit85b] by Witten on global gravitational anomalies is still an instructive read.

On the traditional side of anomalies, the review [Har05] by Harvey can be considered as a classic. The original papers on the general theory of anomalies in the 80s are also quite helpful. See e.g. [ÁGW84, ÁGG84, ÁGG85].

All this is conveniently done by a $(d+1)$-dimensional *invertible* QFT $A = A(T)$. It has a Hilbert space $V_A[N_d]$ associated to a closed 'spatial' slice, and the evolution operator

$$U_A[M_{d+1}] : V_A[N_d] \to V_A[N'_d] \tag{1.2}$$

for a bordism $M_{d+1} : N_d \to N'_d$ allows us to compare the Hilbert spaces. One important condition for us is that $V_A[N_d]$ are all one dimensional. Such a QFT is called invertible.

An invertible theory A in dimension $d+1$ is characterized by the partition function on closed $(d+1)$-manifolds:

$$Z_A : M_{d+1} \mapsto Z_A[M_{d+1}] \in U(1). \tag{1.3}$$

When $M_{d+1} = \partial L_{d+2}$, it can be written as

$$Z_A[M_{d+1}] = \exp(2\pi i \int_{L_{d+2}} P), \tag{1.4}$$

where P is a degree-$(d+2)$ differential form constructed from the gauge and spacetime curvature. There can be other choices of such L. Suppose $M_{d+1} = \partial L'_{d+2}$. We need

$$\exp(2\pi i \int_{L_{d+2}} P) = \exp(2\pi i \int_{L'_{d+2}} P) \tag{1.5}$$

which is

$$\int_{\mathsf{L}_{d+2}} P \in \mathbb{Z}, \tag{1.6}$$

where $\mathsf{L}_{d+2} = \overline{L_{d+2}} \sqcup L'_{d+2}$ is a closed $(d+2)$-manifold.

For example, for $d=2$ i.e. $d+2=4$ with $U(1)$ symmetry, P can be something like

$$P = \frac{k}{2}(\frac{F}{2\pi})^2. \tag{1.7}$$

For (1.6) to hold, k needs to be an even integer (on oriented manifolds) or an integer (on spin manifolds). Then

$$Z_A[M_3] \quad \text{“}{=} \exp(2\pi i \frac{k}{2} \int_{M_3} \frac{A}{2\pi} \frac{F}{2\pi})\text{”} \tag{1.8}$$

is more or less the (classical) Chern-Simons invariant, but the proper definition of the right hand side is tricky when the $U(1)$ connection is topologically nontrivial.

A is the anomaly theory of T and P is the anomaly polynomial of T. If P is zero, T is said to have a perturbative or local anomaly. Even when P is nonzero, the theory A can still be nontrivial. This is the global anomaly. In this case A defines a bordism invariant

$$A : \Omega_{d+1}^{\text{structure}} \to U(1) \tag{1.9}$$

where the structure needs to be appropriately chosen.

As the structure, we will encounter

- spin structure plus G gauge field, for which we use $\Omega_*^{\text{spin}}(BG)$,
- twisted string structure, with G gauge field, for which we use $\Omega_*^{\text{string},\,\tau}(BG)$, where τ specifies the twisting. Here the twisting at the differential form level is specified by a degree-4 class $\tau \in H^4(BG;\mathbb{Z})$, so that we have $dH_3 = p_1(R)/2 - \tau(F)$.

When the gauge bundle is assumed to be trivial, or equivalently if we are interested only in pure gravitational anomaly (in an generalized sense of being pure gravitational, of including the B-field if necessary), we can simply use $\Omega_*^{\text{string}}(\text{pt})$ or $\Omega_*^{\text{spin}}(\text{pt})$. It so happens that $\Omega_{11}^{\text{string}}(\text{pt}) = 0$ and $\Omega_{11}^{\text{spin}}(\text{pt}) = 0$, therefore there is no global anomaly if there is no local anomaly, as far as the pure gravitational anomaly is concerned. The main issue, then, is to formulate the theories under investigation carefully enough so that they are well-defined even in a topologically non-trivial situation. As there is no problem in doing so for fermions, the actual issue is to formulate the theory of p-form fields and their coupling to background fields carefully enough.

2 Fermions

Let us start with the anomaly contributions of fermions. We note that the local anomaly from fermions, particularly in the Type I case, was very nicely reviewed by Schwarz [Sch01].

2.1 Spin-1/2 fermions

For a Weyl fermion in d dimensions coupled to a complex vector bundle V,

$$Z_A[M_{d+1}] = \exp(2\pi i \eta(\not{D}_V)) \tag{2.1}$$

where $\eta(\not{D}_V) \in \mathbb{R}$ is the eta invariant of the corresponding Dirac operator in $(d+1)$ dimensions.[2] This has integer jumps but is continuous in $\mathbb{R}/\mathbb{Z}$. The anomaly polynomial is the degree-$(d+2)$ part of

$$\hat{\mathsf{A}}(R)\,\mathrm{ch}_V(F) \tag{2.2}$$

where $\hat{\mathsf{A}}(R)$ is the A-roof genus and

$$\mathrm{ch}_V(F) = \mathrm{tr}_V\, e^{iF/(2\pi)} \tag{2.3}$$

is the Chern character. Here we take the convention that F is anti-Hermitean, and tr is in the N-dimensional representation. For other gauge groups G and other representation R, simply regard it as a complex vector bundle and use the same formula above.

When the Majorana condition can be imposed on d-dimensional spacetime in the physical, Lorentzian signature, the integer jumps of $\eta(\not{D}_V)$ is actually always an even number, because

[2] In the original APS papers, this η was slightly different and the η here was written as ξ. Here I follow a convention more common in hep-th.

the spinor bundle tensored with V in $(d+2)$ dimensions is pseudoreal in Wick-rotated, Euclidean signature. This makes

$$\exp(2\pi i\frac{1}{2}\eta(\not{D}_V)) \tag{2.4}$$

smooth, which is the anomaly theory of the fermion with Majorana condition.

To actually compute the anomaly polynomial, the trick known as the *splitting principle* is useful. This allows us to pretend that the G curvature and the spacetime curvature are both "diagonalized", in the sense that the associated vector bundles can be thought of as a direct sum of complex line bundles.

In the case of an $SO(2N)$ bundle (which can either be tangent or gauge bundle), assume that the complexification of the $\mathbb{R}^{2N}$-bundle is the sum of $2N$ complex line bundles with first Chern classes $\pm x_{i=1,\ldots,N}$. We define the Pontryagin classes via

$$p_i = i\text{-th symmetric polynomial of } x_i^2. \tag{2.5}$$

Then

$$\hat{\mathsf{A}} = \prod_i \frac{x_i}{e^{x_i/2}-e^{-x_i/2}} \tag{2.6}$$

$$= 1 - \frac{p_1}{24} + (\frac{7p_1^2}{5760} - \frac{p_2}{1440}) + (-\frac{31p_1^3}{967680} + \frac{11p_1p_2}{241920} - \frac{p_3}{60480}) + \cdots . \tag{2.7}$$

Similarly,

$$\mathrm{ch}_{SO(2N)\ \mathrm{vec}}(F) = \mathrm{tr}_{SO(2N)\ \mathrm{vec}}\, e^{iF/(2\pi)} \tag{2.8}$$

$$= \sum_i (e^{x_i} + e^{-x_i}) \tag{2.9}$$

$$= 2N + p_1 + (\frac{p_1^2}{12} - \frac{p_2}{6}) + (\frac{p_1^3}{360} - \frac{p_1p_2}{120} + \frac{p_3}{120}) + \cdots . \tag{2.10}$$

while

$$\mathrm{ch}_{SO(2N)\ \mathrm{adj}}(F) = N + \sum_{i>j} e^{\pm x_i \pm x_j}. \tag{2.11}$$

For E_8, note that the adjoint of E_8 is the sum of the adjoint and a chiral spinor of $so(16)$, so

$$\mathrm{ch}_{E_8\ \mathrm{adj}}(F) = 8 + \sum_{i>j} e^{\pm x_i \pm x_j} + \frac{1}{2}(\prod_i (e^{x_i/2} + e^{-x_i/2}) - \prod_i (e^{x_i/2} - e^{-x_i/2})). \tag{2.12}$$

Then, for example, the anomaly polynomial for the 10d Majorana-Weyl fermion in the adjoint of $SO(32)$ is

$$\begin{aligned} \frac{1}{2}\hat{\mathsf{A}}(R)\,\mathrm{ch}_{SO(32)\ \mathrm{adj}}(F)\,\Big|_{12} = &-\frac{961p_1^3}{120960} + \frac{341p_2p_1}{30240} - \frac{31p_3}{7560} \\ &- \frac{5}{96}p_1q_1^2 + \frac{7}{384}p_1^2q_1 - \frac{p_2q_1}{96} + \frac{p_1q_2}{12} + \frac{q_1^3}{24} - \frac{q_2q_1}{12} \end{aligned} \tag{2.13}$$

where I used q_i for Pontryagin classes of the gauge bundle. Note that q_3 did not appear. This is special to $SO(32)$.

Similarly, the same for $E_8 \times E_8$ is

$$\begin{aligned}\frac{1}{2}\hat{\mathsf{A}}(R)\,\mathrm{ch}_{E_8\times E_8\,\mathrm{adj}}(F)\Big|_{12} = &-\frac{961p_1^3}{120960}+\frac{341p_2p_1}{30240}-\frac{31p_3}{7560}\\ &-\frac{1}{32}p_1(q_1^2+r_1^2)+\frac{7}{384}p_1^2(q_1+r_1)-\frac{p_2(q_1+r_1)}{96}+\frac{q_1^3}{48}+\frac{r_1^3}{48}\end{aligned} \tag{2.14}$$

where I used q_1 and r_1 for the first Pontryagin classes of $so(16) \subset E_8$ of two factors. Note that $q_{2,3}$ and $r_{2,3}$ did not appear. This is because the next independent characteristic class of E_8 has degree 16.

In older papers these characteristic classes were usually written in terms of differential forms such as $\mathrm{tr}\, R^2$ with a lot of 2π's in the denominator. The overall normalization of the anomaly polynomial was also often different and depended on the papers. If you are interested not just in local anomalies but global anomalies, it is useful to use characteristic classes coming from integral cohomology groups of classifying spaces, and use the standard normalization which comes from the theory of invertible phases. Then all the coefficients are rational, making the analysis of global anomalies somewhat easier.

2.2 Gravitinos

A gravitino is like a spinor with an additional vector index. So its anomaly theory is

$$Z_A[M_{d+1}] = \exp(2\pi i\eta(\not{D}_{TM-2\mathbb{R}})) \tag{2.15}$$

where $\not{D}_{TM-2\mathbb{R}}$ is the Dirac operator for the spinor bundle tensored with the tangent bundle, *minus* twice the contribution of the ordinary Dirac operator, of which one -1 comes from the fact that the gravitino lives on $N_d = \partial M_{d+1}$ and another -1 comes from removing the gauge degree of freedom (of local supersymmetry). The anomaly polynomial of the 10d Majorana-Weyl gravitino is then

$$\frac{1}{2}\hat{\mathsf{A}}(R)\left(-3+\mathrm{ch}_{SO(12)\,\mathrm{vec}}(R)\right) \tag{2.16}$$

where another -1 (in addition to -2 we already saw) came from the fact that L_{d+2} is one dimensional higher than M_{d+1}.

In string theory, a gravitino always appears together with dilatino, which has the opposite chirality. So the anomaly polynomial for the gravitino + dilatino for Type I theory is

$$\frac{1}{2}\hat{\mathsf{A}}(R)\left(-4+\mathrm{ch}_{SO(12)\,\mathrm{vec}}(R)\right)\Big|_{12} = \frac{p_1^3}{7560}-\frac{13p_2p_1}{15120}+\frac{31p_3}{7560}. \tag{2.17}$$

Then the total fermion anomaly of Type I $SO(32)$ is

$$-\underbrace{(\frac{p_1}{2}-\frac{q_1}{2})}_{X_4}\underbrace{\frac{1}{192}(3p_1^2-4p_2-4p_1q_1+16q_1^2-32q_2)}_{Y_8}, \tag{2.18}$$

while that of Type I $E_8 \times E_8$ is

$$-\underbrace{(\frac{p_1}{2} - \frac{q_1}{2} - \frac{r_1}{2})}_{X_4} \underbrace{\frac{1}{192}(3p_1^2 - 4p_2 - 4p_1(q_1 + r_1) + 8(q_1^2 - q_1 r_1 + r_1^2))}_{Y_8} . \tag{2.19}$$

We see they are factorized.[3] This will be canceled by the Green-Schwarz mechanism, as we review below.

It was mentioned alreaedy in the very first papers of anomaly cancellation that this factorization works only for $E_8 \times E_8$, $SO(32)$, $E_8 \times U(1)^{248}$ and $U(1)^{596}$. As I did not see it explicit described anywhere, I suggested it to one of my undergraduate summer intern as a project, the result of which is available as [Ant15] on the arXiv. It is also known that $E_8 \times U(1)^{248}$ and $U(1)^{596}$ are not compatible with supersymmetry [ADT10].

3 Local analysis of form fields

Let us move on to the discussion of form fields $F = dC$. The phrase 'p-form fields' can both mean that C is of degree p or F is of degree p. I try to stick with the convention that p-form fields have degree-p gauge potentials C_p, but I admit that I often misspeak.

3.1 Self-dual form fields and their anomaly

In $8k+2$ dimensions, we can consider self-dual form field F_{4k+1} satisfying $F = \pm{*F}$. It is quite hard to analyze. For example, a naive action $\int |F|^2 \sim \int F \wedge *F$ vanishes, since $F \wedge *F \propto F \wedge F = 0$, since the degree of F is odd. Another funny fact is that it has local anomaly.

Note that the tensor product of the spinor bundle with itself is the direct sum of all the p-form bundles, and the chirality operator of this $S \otimes S$ equals the Hodge star $*$. Then, the signature index theorem in $d = 2n+2$ dimensions is

$$\sigma(L_{2n+2}) = \int \hat{\mathsf{A}} \prod_i (e^{x_i/2} + e^{-x_i/2}) = \int \prod_{i=1}^{n+1} x_i \frac{e^{x_i/2} + e^{-x_i/2}}{e^{x_i/2} - e^{-x_i/2}} = \int \prod_{i=1}^{n+1} x_i \frac{e^{x_i} + e^{-x_i}}{e^{x_i} - e^{-x_i}} \tag{3.1}$$

where in the last step we used the fact that only the terms with exactly $n+1$ factors of x_i's contribute to the integral.[4] In terms of Pontryagin classes, the integrand is

$$\mathsf{L} = 1 + \frac{p_1}{3} + \frac{-p_1^2 + 7p_2}{45} + \frac{2p_1^3 - 13p_1 p_2 + 62p_3}{945} + \cdots . \tag{3.2}$$

[3] The details of this computations were famously quite wrong in the first edition of Polchinski [Pol07]. These were corrected in the newer editions. But I know papers which quoted the wrong equations in the old edition.

[4] To be slightly more explicit, we need the coefficient of t^{n+1} of $\prod_{i=1}^{n+1} tx_i \frac{e^{tx_i/2} + e^{-tx_i/2}}{e^{tx_i/2} - e^{-tx_i/2}}$. Letting $t = 2s$, we can take the coefficient of s^{n+1} and divide by 2^{n+1}. This is the right-most expression of (3.1).

This is L-genus of Hirzebruch. There is an invertible theory in $8k + 3$ dimensions, whose partition function is $\exp(2\pi i\eta(\not{D}_S))$ and with the anomaly polynomial given by the degree-$(8k + 4)$ part of the L genus.

The self-dual form field and its anomaly is closely related to this, but not quite equal to it. Actually, the local anomaly of a d-dimensional self-dual form field is known to be $\pm 1/8$ of this quantity. But you can not simply divide the anomaly polynomial by 8, because it needs to satisfy the quantization condition (1.6). Similarly, you can not simply take the 8-th root of the invertible theory $\exp(2\pi i\eta(\not{D}_S))$, since it is not guaranteed that we can take a smooth $1/8$-th root of a $U(1)$-valued function, due to possible branch cuts.

That said, one finds

$$\hat{\mathsf{A}}(R)\left(-4 + \mathrm{ch}_{SO(12)\ \mathrm{vec}}(R)\right)\Big|_{12} - \frac{1}{8}\mathsf{L}\Big|_{12} = 0, \tag{3.3}$$

since the degree-12 part of (3.2) is exactly 16 times (2.17). This means that the local anomaly cancels in Type IIB theory.

3.2 Green-Schwarz mechanism

Above, we saw that the fermion anomaly in the Type I theory showed the factorization,

$$P = X_4 Y_8. \tag{3.4}$$

In such a case we introduce a B-field whose gauge-invariant field strength H satisfies

$$dH_3 = X_4, \qquad d * H_3 = Y_8. \tag{3.5}$$

Such a B-field is known to have an anomaly, whose anomaly polynomial is $-X_4 Y_8$, so that the total anomaly cancels.

The traditional explanation at the local level is as follows. Note that at this point, X_4 and Y_8 can be field strengths of arbitrary form fields, not necessarily constructed from gauge and spacetime curvatures. The relation $dH_3 = X_4$ can be solved by writing

$$H_3 = dB_2 + C_3, \tag{3.6}$$

where C_3 is the gauge potential for X_4, satisfying $dC_3 = X_4$. Note that the relation $dH_3 = X_4$ is imposed at the kinematical level. To make H_3 is gauge-invariant under the gauge transformation

$$C_3 \mapsto C_3 + d\chi_2, \tag{3.7}$$

B_2 needs to have a nontrivial gauge transformation

$$B_2 \mapsto B_2 - \chi_2. \tag{3.8}$$

Let us now consider the coupling

$$-\int_{N_{10}} B_2 Y_8 \tag{3.9}$$

in the 10d action. Then the relation $d * H_3 = -Y_8$ is part of the EOM, i.e. is imposed dynamically. Furthermore, the coupling (3.9) has a nontrivial gauge variation under (3.8), which corresponds to an inflow from

$$-\int_{M_{11}} C_3 Y_8 \tag{3.10}$$

on a manifold with boundary. As $dC_3 Y_8 = X_4 Y_8$, this is exactly what we need to cancel the fermion anomaly.

Let us reformulate it in a somewhat more modern way. There are actually two related but slightly different ways to proceed.

1. One is to introduce a B-field solving $dH_3 = X_4$ not just on N_d but on M_{d+1} and L_{d+2}. Then we can simply consider the anomaly theory whose action is

$$\exp(-2\pi i \int_{M_{d+1}} H_3 Y_8) \tag{3.11}$$

which clearly satisfies

$$\exp(-2\pi i \int_{M_{d+1}} H_3 Y_8) = \exp(-2\pi i \int_{L_{d+2}} X_4 Y_8) \tag{3.12}$$

when $M_{d+1} = \partial L_{d+2}$. So this anomaly theory can cancel the fermion anomaly theory. And we can consider the coupling

$$-\int_{M_{d+1}} H_3 Y_8 \tag{3.13}$$

for $N_d = \partial M_{d+1}$ as the Wess-Zumino-Witten-type better rewriting of (3.9).

2. Another is to consider a bulk "BF-type" theory

$$\int_{M_{d+1}} C_3 dC_7 - C_3 Y_8 - X_4 C_7 \tag{3.14}$$

where C_3 and C_7 are two dynamical fields to be path-integrated, and X_4 and Y_8 are background fields (which can be composite of spacetime and gauge curvatures). As part of the EOM, we have $dC_3 = X_4$ and $dC_7 = Y_8$. Plugging it back, the bulk action is $-\int_L X_4 Y_8$.

Note that $C_{3,7}$ are topological in the bulk, but have massless modes localized at the boundary $N_d = \partial M_{d+1}$. Denoting the boundary values of C_3 and C_7 by H_3 and H_7, we have $dH_3 = X_4$ and $dH_7 = -Y_8$. We also have $H_3 = *H_7$ as part of the boundary equation of motion.

Global anomaly cancellation will be shown by following either of these two approaches more carefully, paying attention to global topological issues. There are trade-offs in two approaches. The first approach picks H_3 as more fundamental than H_7. In addition, H_3 needs to be added to the set of background fields. This changes the spacetime structure used to define bordism groups. Instead, the bulk coupling (3.11) is well-defined as written, and there is no need for the bulk path integral.

The second approach is more democratic with respect to the exchange $H_3 \leftrightarrow H_7$. We have not introduced new bulk background fields either. The fields C_3 and C_7 are path-integrated and do not change the set of background fields, so we produce a bulk theory which is defined on the original spacetime structure. The price we need to pay in this approach is that the action (3.14) needs to be made well-defined, and that we have to perform a dynamical path integral in the bulk.

In the heterotic theory we take the first approach; in the Type I we take the second approach. In particular, for the heterotic SO(32) and Type I, two rather different approaches will be used to show global anomaly cancellation of the theories S-dual to each other, with the same massless spectrum.

Before proceeding, let us discuss the case of self-dual 4-form fields. The discussion above can be directly generalized to the case of a non-chiral 5-form field strength F_5 satisfying $dF_5 = X_6$ and $d * F_5 = Y_6$, which has the anomaly polynomial $-X_6 Y_6$. For this both of the two approaches above can be used. (Note that in the second approach, we need *two* 5-form fields C_5 and C_5' in the bulk.)

When the self-duality is imposed, $F_5 = *F_5$, $Y_6 = X_6$, the anomaly polynomial has a Green-Schwarz contribution $(X_6)^2/2$ in addition to the one-loop effect discussed above. In this case only the second approach can be used, where the bulk action is

$$\int_{M_{11}} \frac{1}{2} C_5 dC_5 - C_5 X_6. \tag{3.15}$$

Now, the Type IIB equations of motion actually contains the piece $dF_5 = H_3 F_3$. Luckily, the Green-Schwarz contribution to the anomaly is $(H_3 F_3)^2/2 = 0$ thanks to the fact that the degrees of H_3 and F_3 are odd. Therefore this does not ruin the local anomaly cancellation in Type IIB we already saw.

4 Self-dual form fields

4.1 Self-dual form fields via differential cohomology

To discuss global anomaly cancellation, we need to discuss what is a p-form field even to get started. For $F_2 = dA_1$ it is well-established. It is a $U(1)$ connection. For $H_3 = dB_2$ there is also a geometric model, called a gerbe. $J_1 = d\phi_0$ for a circle-valued field ϕ_0 behaves in the same way. They cover the cases $p = 0, 1, 2$.

But this does not generalize nicely to higher p-form fields. Instead we can use something called differential cohomology. It turns out that it does not quite suffice for our purposes, but it is useful nonetheless as a preparation and an easier exercise.

On a manifold M, the topological information of a $U(1)$ connection is captured by its first Chern class in $H^2(M;\mathbb{Z})$. The topologically trivial part is instead given by a 1-form A_1 up to $d\chi_0$, where χ_0 is allowed to have nontrivial winding number. So, the space of 1-form gauge field A on M, or equivalently $U(1)$-connections on M, sits in a short exact sequence

$$0 \to \Omega^1(M)/\Omega^1_{\text{closed},\mathbb{Z}}(M) \to \{\text{1-form gauge fields}\} \to H^2(M;\mathbb{Z}) \to 0, \tag{4.1}$$

where $\Omega^1_{\text{closed},\mathbb{Z}}(M)$ is the space of closed 1-forms with integral periods, which is a more precise way to say "$d\chi_0$ for χ_0 allowed to have nontrivial winding number". Similarly, the space of 2-form gauge fields B is expected to sit in a short exact sequence

$$0 \to \Omega^2(M)/\Omega^2_{\text{closed},\mathbb{Z}}(M) \to \{\text{2-form gauge fields}\} \to H^3(M;\mathbb{Z}) \to 0. \tag{4.2}$$

Mathematicians have a nice generalization for all p, known as differential cohomology groups:

$$0 \to \Omega^{p-1}(M)/\Omega^{p-1}_{\text{closed},\mathbb{Z}}(M) \to \breve{H}^p(M) \to H^p(M;\mathbb{Z}) \to 0. \tag{4.3}$$

Note that the degree of a differential cohomology class refers to the degree of the gauge field *strength*. There are various realizations of the same Abelian group $\breve{H}^p(M)$, known as Cheeger-Simons differential character, Deligne cohomology, Hopkins-Singer differential cocycle, etc. The situation is analogous to the case of the ordinary cohomology groups $H^p(M;\mathbb{Z})$, which has various realizations which end up giving the same Abelian group.

It has a graded-commutative product

$$\breve{H}^p(M) \times \breve{H}^q(M) \to \breve{H}^{p+q}(M) \tag{4.4}$$

which lifts the product of ordinary cohomology

$$H^p(M;\mathbb{Z}) \times H^q(M;\mathbb{Z}) \to H^{p+q}(M;\mathbb{Z}) \tag{4.5}$$

and restricts to the map

$$(C_{p-1}, C_{q-1}) \mapsto C_{p-1} dC_{q-1} \tag{4.6}$$

in the topologically trivial case.

It also has an integration. Suppose we are given a fiber bundle $F \to E \to B$ where the fiber is oriented. Then, we can integrate a given element $\breve{\omega}_p \in \breve{H}^p(E)$ along the fiber:

$$\int_F \breve{\omega}_p \in \breve{H}^{p-\dim_{\mathbb{R}} F}(B). \tag{4.7}$$

In particular, applying this to $M \to M \to \text{pt}$, we have two basic types of integrations:

$$\int_M \breve{\omega}_{\dim M} \in \breve{H}^0(\text{pt}) \simeq \mathbb{Z}, \tag{4.8}$$

$$\int_M \breve{\omega}_{\dim M+1} \in \breve{H}^1(\text{pt}) \simeq \mathbb{R}/\mathbb{Z}. \tag{4.9}$$

The former measures the integer-quantized flux, while the latter measures the holonomy, which is $U(1)$-valued.

For $\breve{\omega}_p \in \breve{H}^p(M)$ and $\breve{\omega}_q \in \breve{H}^q(M)$ with $p+q = \dim M + 1$, we have a pairing

$$(\breve{\omega}_p, \breve{\omega}_q) := \int_M \breve{\omega}_p \breve{\omega}_q \in \mathbb{R}/\mathbb{Z}, \tag{4.10}$$

which generalizes

$$\int_M C_{p-1} dC_{q-1} \tag{4.11}$$

to the topologically non-trivial case. This pairing is known to be perfect, i.e. $\breve{H}^p(M)$ and $\breve{H}^q(M)$ are Pontryagin dual to each other. For physicists, this means that

$$\int D\breve{\omega}_p \, e^{2\pi i(\breve{\omega}_p, \breve{\omega}_q)} = \delta(\breve{\omega}_q). \tag{4.12}$$

4.2 Invertible field theory from a quadratic refinement

Let us try to use differential cohomology to discuss the self-dual five-form field in Type IIB theory. For the bulk action, let us try as a candidate

$$S = 2\pi i(\breve{\omega}_6, \breve{\omega}_6), \tag{4.13}$$

where $\breve{\omega}_6 \in \breve{H}^6(M_{11})$ is the dynamical field. This is a perfectly well-defined action, and has a chiral 5-form field at the boundary. But this is not the correct theory to be used in Type IIB theory, because $V(N_{10})$, the Hilbert space of the bulk theory in which the partition function of the boundary theory takes value, is not one-dimensional in general.

One way to see this is as follows. The equation of motion says that $2\breve{\omega}_6 = 0$. Therefore, the path integral is reduced to a sum over two-torsion elements of $\breve{H}^6(M_{11})$, which is quite nontrivial in general, and there is no reason to expect that it leads to a one-dimensional Hilbert space. To see it more explicitly, consider the operator

$$U(X) := \exp(2\pi i \int_X \breve{\omega}_6) \sim \exp(2\pi i \int_X C_5) \tag{4.14}$$

where $X \subset M_{11}$ is a 5-dimensional submanifold. (This is an analogue of the Wilson loop operator in the case of 3d Chern-Simons theory.) As $2\breve{\omega}_6 = 0$, we have $U(X)^2 = 1$. Now, take the spatial slice to be N_{10} and choose X, Y within it, It is not difficult to show the commutation relation

$$U(X)U(Y) = (-1)^{X \cdot Y} U(Y)U(X) \tag{4.15}$$

where $X \cdot Y \in \mathbb{Z}$ is the intersection number of X and Y in N_{10}. Take N_{10} to be $S^5 \times S^5$ and take $X = S^5 \times \{\text{pt}\}$ and $Y = \{\text{pt}\} \times S^5$. Then $X \cdot Y = 1$, so $U(X)$ and $U(Y)$ anti-commute. This means that $V(N_{10})$ has at least two dimensions. So the boundary theory does not have a partition function; it only have a partition vector taking value in $V(N_{10})$.

Roughly speaking, we want something like

$$S = 2\pi i \frac{1}{2}(\breve{\omega}_6, \breve{\omega}_6) \tag{4.16}$$

as the action. But this is not well-defined, since $(\breve{\omega}_6, \breve{\omega}_6)$ takes values in $\mathbb{R}/\mathbb{Z}$, which cannot be divided by 2 in a well-defined manner. We need a more sophisticated construction, known as the quadratic refinement.

Note that if $(\breve{a}, \breve{a})/2$ could be taken, it would satisfy

$$(\breve{a} + \breve{b}, \breve{a} + \breve{b})/2 = (\breve{a}, \breve{a})/2 + (\breve{b}, \breve{b})/2 + (\breve{a}, \breve{b}). \tag{4.17}$$

So let us suppose that we have a function $q : \breve{H}^{p+1}(M) \to \mathbb{R}/\mathbb{Z}$ satisfying

$$q(\breve{a} + \breve{b}) = q(\breve{a}) + q(\breve{b}) + (\breve{a}, \breve{b}). \tag{4.18}$$

This is known as a quadratic refinement of the pairing $(-,-)$.

We can then consider the bulk theory with the action

$$S = 2\pi i q(\breve{\omega}_{p+1}) \tag{4.19}$$

where $\breve{\omega}_{p+1} \in \breve{H}^{p+1}(M)$. We can show that it leads to a bulk theory whose partition function is always of unit norm, and therefore has one-dimensional $V(N_d)$ for any N_d. To see this, consider the absolute value of the partition function:

$$\left| \int D\breve{a} \exp(2\pi i q(\breve{a})) \right|^2 = \int D\breve{a} D\breve{b} \exp(2\pi i (q(\breve{a}) - q(\breve{b}))). \tag{4.20}$$

Redefining $\breve{a} = \breve{c} + \breve{b}$, this is

$$\int D\breve{c} D\breve{b} \exp(2\pi i (q(\breve{c} + \breve{b}) - q(\breve{b}))) = \int D\breve{c} D\breve{b} \exp(2\pi i (q(\breve{c}) + (\breve{c}, \breve{b}))) \tag{4.21}$$

where we used the property of the quadratic refinement. The integral over $\breve{b}$ gives a delta function setting $\breve{c} = 0$, so the whole integral is 1.

The issue is whether such a quadratic refinement exists. It is known to exist for self-dual 0-form fields in 2d (where $p = 0$ and M is 3-dimensional), and for self-dual 2-form fields in 6d (where $p = 2$ and M is 7-dimensional), if we assume that the spacetime is equipped with spin structure. But for self-dual 4-form fields in 10d (where $p = 4$ and M is 11-dimensional), it is not known if we insist on using differential cohomology.

Importance of quadratic refinements and the bulk Chern-Simons theories based on them in this context were pointed out and emphasized in a series of papers by Moore, e.g. [BM06a, BM06b] by Monnier, e.g. [Mon11a,Mon11b,Mon13], and by Monnier and Moore, e.g. [MM18]. They use something called Wu structure to define quadratic refinements in general dimensions, but for the particular case of string theory, it suffices to have spin structure on the spacetime. Therefore I am not going to discuss Wu structure here.

4.3 Type IIB anomaly cancellation via differential K theory

It is known that a quadratic refinement in this particular dimension exists if we use differential K theory. This is also natural from a string theory point of view, since the charges of D-branes, which are the sources of the RR fields, are not governed by ordinary integral cohomology but by K-theory, as pointed out by [MM97, Wit98, MW99]. This approach to RR fields using the K-theoretic quadratic refinement was pioneered in [FH00], and some more discussions were given in [HTY20].

Instead of (4.3), we have a sequence

$$\frac{\bigoplus_n \Omega^{p-1+2n}(M)}{\bigoplus_n d\Omega^{p-2+2n}(M)} \to \breve{K}^p(M) \to K^p(M) \to 0, \tag{4.22}$$

where $K^p(M)$ is the topological K-theory group. There is a graded-commutative product

$$\breve{K}^p(M) \times \breve{K}^q(M) \to \breve{K}^{p+q}(M) \tag{4.23}$$

and a perfect pairing

$$(-,-)_M : \breve{K}^p(M) \times \breve{K}^q(M) \to \mathbb{R}/\mathbb{Z} \tag{4.24}$$

for $p+q \equiv \dim M + 1$ mod 2 when M is spin. Finally, there is a quadratic refinement

$$q_M : \breve{K}^0(M) \to \mathbb{R}/\mathbb{Z} \tag{4.25}$$

of the pairing $(-,-)_M$ when $\dim M$ is odd.

In our case, we take $\breve{\omega}_0 \in \breve{K}^0(M_{11})$ as the bulk dynamical field and take

$$S = 2\pi i q_{M_{11}}(\breve{\omega}_0) \tag{4.26}$$

as the bulk action. This is exactly what we need. An important point is that the differential form content of $\breve{\omega}_0$ is a formal sum of $C_{1,3,5,7,9,11}$ with the kinetic term

$$\sim \int_{M_{11}} C_1 dC_9 + C_3 dC_7 + C_5 dC_5/2. \tag{4.27}$$

Their boundary values are $F_{1,3,5,7,9}$ satisfying the self-duality constraint

$$*F_1 = F_9, \quad *F_3 = F_7, \quad *F_5 = F_5. \tag{4.28}$$

This is exactly the set of RR fields in Type IIB theory. That we need to use K-theory to describe RR fields also matches the classification of D-brane charges by K-theory.

We do not go into the details of how $\breve{K}(M)$ is defined, or how the quadratic refinement q_M is constructed, as we will soon discuss the Type I case in more detail where we use KO theory instead.

Now that we have a well-defined bulk action, the global anomaly cancellation of Type IIB theory is easy. As already mentioned, $\Omega_{11}^{\text{spin}}(\text{pt}) = 0$, so there is no global anomaly if there is no local anomaly. We already checked that the local anomaly cancels. So we are done.

The extension to the case with nontrivial NSNS B-fields should be straightforward, by using twisted K-theory, although I am not sure if the necessary details of twisted differential K-theory has been worked out by mathematicians. Dealing with nontrivial Type IIB $SL(2,\mathbb{Z})$ background is much more subtle. For this, see e.g. [DDHM23, Yon24].

5 Green-Schwarz mechanism

5.1 Green-Schwarz via differential cohomology

Let us also try to use differential cohomology to make the bulk action (3.14) to describe the Green-Schwarz mechanism more precise. Let us say that

$$\breve{\omega}_4 \in \breve{H}^4(M_{11}), \qquad \breve{\omega}_8 \in \breve{H}^8(M_{11}) \tag{5.1}$$

are dynamical fields, and

$$\breve{X}_4 \in \breve{H}^4(M_{11}), \qquad \breve{Y}_8 \in \breve{H}^8(M_{11}) \tag{5.2}$$

are background fields, and the bulk action is

$$S = 2\pi i \left((\breve{\omega}_4, \breve{\omega}_8) - (\breve{\omega}_4, \breve{Y}_8) - (\breve{X}_4, \breve{\omega}_8) \right). \tag{5.3}$$

The path-integral over $\breve{\omega}_4$ and $\breve{\omega}_8$ can be easily performed, and gives

$$= -(\breve{X}_4, \breve{Y}_8). \tag{5.4}$$

The issue is whether the background fields can actually be taken to be differential cohomology elements. Setting the gauge fields to be zero, $X_4 Y_8$ as a differential form was

$$-\frac{p_1}{2}\frac{3p_1^2 - 4p_2}{192}. \tag{5.5}$$

On a spin manifold, it is known that there is a natural class $\lambda \in H^4(BSpin; \mathbb{Z})$ which satisfies $2\lambda = p_1$. This means that there is a differential cohomology element

$$\breve{\lambda} \in \breve{H}^4(M) \tag{5.6}$$

which lifts $p_1/2$. The problem is $\frac{1}{192}(3p_1^2 - 4p_2)$, which does not integrate to an integer even on a spin manifold. For example, it evaluates to $-1/12$ on $\mathbb{HP}^2$. This means that ordinary differential cohomology is not enough to formulate the Green-Schwarz mechanism for ten-dimensional string theory.

5.2 Heterotic solution

One solution to the issues above, at least on the heterotic side, is to take the first approach mentioned above, by introducing a B-field whose field strength H_3 satisfies

$$dH_3 = X_4 \tag{5.7}$$

as part of the background field. (More precise topological conditions for the B-field was discussed first in [Wit85a] and in more detail in [Yon22].)

Then the local part of the anomaly is already canceled by the well-defined bulk term (3.11). The issue is how the sum of the fermionic anomaly and the B-field anomaly,

$$\frac{1}{2}(\eta(\not{D}_{TM-3\mathbb{R}}) + \eta(\not{D}_{\text{adj}})) - \int_{M_{11}} H_3 Y_8 \tag{5.8}$$

behaves as the invertible field theory. By construction, this is a bordism invariant, where the relevant spacetime structure is the spin structure, the G bundle, and the B-field satisfying $dH_3 = X_4$. Once this bordism invariant is shown to be trivial, all what is left is to path-integrate over B.

(This is an example of a very general way of constructing a boundary theory with a specified anomaly α for some spacetime structure $\mathcal{S}$. First, we try to come up with a enlarged structure $\mathcal{S}'$ fitting in the sequence $X \to \mathcal{S}' \to \mathcal{S}$, so that the anomaly α trivializes as a theory with structure $\mathcal{S}'$. We then path-integrate over X on the boundary, and we are done. This method was discussed in [WWW17, Tac17] when α is given by an ordinary cohomology and X is finite. A more general version was then described in [KOT19]. Here we use this technique when $\mathcal{S}$ is the spin structure, $\mathcal{S}'$ is the string structure, and X is the space of B-fields.)

Let us discuss how one shows the vanishing of the bordism invariant, i.e. the global anomaly. When the G-bundle is assumed to be trivial, the resulting structure is known as string structure, which sits as a member of a natural sequence of spacetime structures: unoriented, oriented, spin, string $\ldots$, obtained by killing the homotopy groups of O one by one. Now, $\Omega_{11}^{\text{string}}$ is known to vanish, so there cannot be any nontrivial bordism invariant, and therefore the global anomaly cancels.

When the G-bundle is nontrivial, the relevant structure is known as twisted string structure. When $G = E_8 \times E_8$, the constraint is that

$$dH_3 = \frac{p_1}{2} - \frac{q_1}{2} - \frac{r_1}{2}, \tag{5.9}$$

where $q_1/2$ and $r_1/2$ are the generators of $H^4(BE_8;\mathbb{Z})$ of two E_8 factors. As BE_8 is 16-equivalent to $K(\mathbb{Z},4)$, for our purposes we can just consider it as $K(\mathbb{Z},4)$. Therefore, at the topological level, the constraint (5.9) simply says that $r_1/2 \in H^4(M;\mathbb{Z})$ is unconstrained while $q_1/2$ is determined by $p_1/2$ and $r_1/2$. Therefore the relevant bordism group is the same as $\Omega_{11}^{\text{spin}}(K(\mathbb{Z},4))$. This was computed to vanish, so the global anomaly cancels also in this case. It is shocking to me that this exact argument was already given by Witten [Wit86], just a year after the discovery of the anomaly-free superstring theory in 1984, and that Witten asked an algebraic topologist, Stong, to provide an appendix [Sto86] to providing the computation of $\Omega_{11}^{\text{spin}}(K(\mathbb{Z},4))$.

Unfortunately the same argument does not work for $G = Spin(32)$, as the relevant twisted string bordism group $\Omega_{11}^{\text{string},\tau}(BSpin(32))$ has not been computed as far as I know. Taking $G = Spin(32)/\mathbb{Z}_2$ or $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$ will make things even more complicated, but might be worth doing. (For $G = Spin(32)/\mathbb{Z}_2$, even the ordinary cohomology was only computed in 2024 [Kne24].)

There is a general vanishing argument for the heterotic anomaly given in [TY21, Yon22], but it uses a still largely conjectural relation between TMF and the 2d worldsheet superconformal field theory. The point which is lacking in the ten-dimensional cases is that the $c = 16$ current algebra theories have not been rigorously mathematically shown to give appropriate G-equivariant TMF classes.

5.3 Type I anomaly cancellation via differential KO theory

We can also try the second approach of Sec. 3.2, which is more suitable for Type I. In this case, we need to use differential KO theory. Let us first summarize the relevant mathematical facts, and then discuss physics afterwards. This approach was first explained in [Fre00]. Unfortunately, it was written before the modern understanding of anomalies using invertible field theories was developed, and therefore very hard to read, at least to me. Together with two postdocs at IPMU, we spent about a year to decipher it, resulting in the paper [HTZ25], which would hopefully be more easily understandable than the original one. This section is based on this reference.

5.3.1 Math

Differential KO theory groups sit in the sequence

$$\frac{\bigoplus_n \Omega^{p-1+4n}(M)}{\bigoplus_n d\Omega^{p-2+4n}(M)} \to \widecheck{KO}^p(M) \to KO^p(M) \to 0. \tag{5.10}$$

There is a graded-commutative product as always, and a perfect pairing

$$(-,-)_M : \widecheck{KO}^p(M) \times \widecheck{KO}^q(M) \to \mathbb{R}/\mathbb{Z} \tag{5.11}$$

for $p + q + 4 \equiv \dim M + 1$ mod 8 when M is spin. Note the appearance of the shift by $+4$ here, which was absent in the case of differential cohomology or differential K theory.[5]

In any case, the pairing for $p = q = 0$ is defined when $\dim M \equiv 3$ mod 8, and then there is a quadratic refinement

$$q_M : \widecheck{KO}^0(M) \to \mathbb{R}/\mathbb{Z} \tag{5.12}$$

in this case. We use this for M_{11}.

Note that the differential form content of $\widecheck{KO}^0(M_{11})$ is a formal sum of C_3 and C_7, with the kinetic term

$$\sim \int_{M_{11}} C_3 dC_7 + \cdots . \tag{5.13}$$

Their boundary values are $F_{3,7}$ satisfying the self-duality constraint $*F_3 = F_7$. This is exactly the RR fields in Type I theory.

[5] This shift is related to the shift in the self Anderson duality of KO theory.

Note that in the failed approach to describe the bulk theory in terms of ordinary differential cohomology, we used a pair of fields $\breve{\omega}_4 \in \breve{H}^4(M)$ and $\breve{\omega}_8 \in \breve{H}^8(M)$, containing the differential form fields C_3 and C_7, respectively. Here, a single field $\breve{\omega}^0 \in \widetilde{KO}^0(M)$ contains both C_3 and C_7. The bulk Lagrangian is therefore more like a Chern-Simons action, rather than the BF-action.

Let us give a concrete description of $\widetilde{KO}^0(M)$. It is a combination of $C = \sum_n C_{4n+3}$ and a (virtual) real vector bundle V over M. We further equip a connection A on V. These are too much, so we need to impose an equivalence relation. One is the usual one which makes topological KO theory out of real vector bundles. Another is the equivalence relation on a fixed V, between two data (C_0, A_0) and (C_1, A_1). We declare that they are equivalent if there is a homotopy (C_t, A_t), $t \in [0, 1]$ such that

$$C_1 - C_0 = -\int_0^1 dt \sqrt{\hat{\mathsf{A}}}\,\mathrm{ch}(F), \tag{5.14}$$

where $\hat{\mathsf{A}}$ is the $\hat{\mathsf{A}}$-genus (2.7) and F is the curvature of A_t regarded as a connection of $[0,1]\times M$. A class $\breve{a} \in \widetilde{KO}^0(M)$ is represented by (C, V, A). This gauge equivalence is designed so that the integral

$$\int_X \breve{a} := \kappa \left(\eta(\not{D}_{X,V,A}) + \int_X \frac{\sqrt{\hat{\mathsf{A}}(TX)}}{\sqrt{\hat{\mathsf{A}}(NX)}} C \right) \tag{5.15}$$

for a submanifold[6] $X \subset M$ is well-defined under the equivalence relation. Here, κ is $1/2$ or 1 depending on $\dim X$ mod 8, and $\not{D}_{X,V,A}$ is the Dirac operator on X coupled to the bundle V with connection A, and NX is the normal bundle of X in M.

Let us give an explicit expression for the pairing and the quadratic refinement. For $\breve{a}_i = (C_i, V_i, A_i)$, we have

$$\begin{aligned}(\breve{a}_1, \breve{a}_2)_M = \frac{1}{2}\eta(\not{D}_{M,V_1\otimes V_2, A_1\otimes 1+1\otimes A_2}) \\ + \frac{1}{2}\int_M \Big(C_1 dC_2 - (\sqrt{\hat{\mathsf{A}}}\,\mathrm{ch}(F_1) C_2 - C_1 (\sqrt{\hat{\mathsf{A}}}\,\mathrm{ch}(F_2) \Big),\end{aligned} \tag{5.16}$$

and for $\breve{a} = (C, V, A)$, we have

$$q_M(\breve{a}) = \frac{1}{2}\eta(\not{D}_{M,\wedge^2 V, A\otimes 1+1\otimes A}) + \frac{1}{4}\int_M C dC - \frac{1}{2}\int_M C \underbrace{(\sqrt{\hat{\mathsf{A}}}\,\mathrm{ch}(F) + \mathsf{X})}_{\mathsf{U}} \tag{5.17}$$

where X is a differential form constructed out of the spacetime curvature which is determined by the condition

$$\hat{\mathsf{A}}\,\mathrm{ch}_{\mathsf{adj}}(F)\Big|_{\dim M+1} + \frac{1}{2}\mathsf{X}^2\Big|_{\dim M+1} = \frac{1}{2}(\sqrt{\hat{\mathsf{A}}}\,\mathrm{ch}(F) + \mathsf{X})^2\Big|_{\dim M+1} \tag{5.18}$$

[6] This also applies to $M \subset M$.

where $\Big|_{\dim M+1}$ means that we only take the degree $\dim M+1$ part.[7] It is a good exercise to show that q_M as defined above is invariant under the equivalence relation (5.14).

The relation (5.18) allows us to recursively determine X from the lower degree terms to higher degree terms. For $\dim M = 11$, we find

$$\mathsf{X}_0 = 32, \qquad \mathsf{X}_4 = \frac{p_1}{3}, \qquad \mathsf{X}_8 = -\frac{p_1^2}{320} + \frac{7}{720}p_2 \tag{5.19}$$

and

$$\mathsf{X}_{12} = \frac{79p_1^3}{1935360} - \frac{5p_1p_2}{32256} + \frac{31p_3}{120960}. \tag{5.20}$$

Defining $\mathsf{U} := \sqrt{\hat{\mathsf{A}}}\,\mathrm{tr}\, e^{iF/2\pi} + \mathsf{X}$, we have

$$\mathsf{U}_0 = \dim V - 32, \tag{5.21}$$

$$\mathsf{U}_4 = q_1 - p_1, \tag{5.22}$$

$$\mathsf{U}_8 = \frac{1}{64}p_1^2 - \frac{1}{48}p_2 - \frac{1}{48}p_1q_1 + \frac{1}{12}q_1^2 - \frac{1}{6}q_2. \tag{5.23}$$

Surprisingly, one finds that $\mathsf{U}_4 = 2X_4$ and $\mathsf{U}_8 = Y_8$, where X_4 and Y_8 are exactly the forms appearing in the Green-Schwarz mechanism (2.18) for Type I theory. We also find that $\mathsf{U}_0 = \dim V - 32$ is strongly reminiscent of the required condition $\dim V = 32$ for Type I theory.

5.3.2 Physics

One useful fact from this explicit construction is that an O bundle V with connection A gives an element $\widetilde{KO}^0(M)$, which we can denote by $\check{A} := [C = 0, V, A]$. This makes it easy to describe the coupling of the RR fields to the gauge field. In Type I theory, we have a gauge bundle, which is known to be a $Spin(32)/\mathbb{Z}_2$ bundle in general. In the formalism used here, we can use an arbitrary $O(n)$ gauge field at this point. So we take the bulk action to be

$$S = 2\pi i \left(q_{M_{11}}(\check{\omega}_0) - (\check{\omega}_0, \check{A})_{M_{11}} \right). \tag{5.24}$$

The conditions that the $O(n)$ bundle V has $\dim V = 32$, orientable $w_1(V) = 0$, and spin $w_2(V) = 0$ all arise as a consequence of the bulk equation of motion. This sets the gauge group to be $Spin(32)$. A further quotient to make it into $Spin(32)/\mathbb{Z}_2$ will require the use of twisted KO theory, and is beyond the scope of this review.

To study the easiest case, take the variation with respect to $\check{\omega}_0$. This leads to $\check{\omega}_0 = \check{A}$. So we can identify V in $\check{A}$ and V in $\check{\omega}_0$. Now, taking the variation by C_{11} in (5.17), we have $\dim V - 32 = 0$. Obtaining $w_1(V) = w_2(V) = 0$ is harder; it involves taking the variation of the action with respect to discrete degrees of freedom within $\check{\omega}_0$, but it can be done[8]

[7] The two terms on the left hand side are the anomalies of the gaugino and of the gravitino+dilatino, and the right hand side is the Green-Schwarz contribution which is the square of the sum of the D9- and O9-contributions. See the review article [Sch01], which was in turn based on [SW01]

[8] There is still a gap in the argument in [HTZ25] in that it has not been shown that $\int_{X_2} w_2(V) = 0$ when X_2 is unoriented, but I believe this is due to a lack of ability on the authors side, and not on the formalism.

Next, let us vary C_3 and C_7 in (5.17). The kinetic term is $\frac{1}{2}\int_M C_3 dC_7$ there, which is different by a factor of 2 in the more conventional $\int_M C_3 dC_7$. It turns out that this is due to a difference in the normalization of C_3, so $C_3^{\text{KO}} = 2C_3^{\text{usual}}$. Using the usual normalization, we have the boundary equation of motion

$$dH_3^{\text{usual}} = \frac{1}{2}\mathsf{U}_4 = X_4, \qquad dH_7 = \mathsf{U}_8 = Y_8, H_7 = *H_3^{\text{usual}}, \tag{5.25}$$

which is exactly as we need in Type I. We emphasize that here the correct X_4 and Y_8 somehow emerge without even talking anything about anomaly cancellation; they just arise from the fact that we used a quadratic refinement of differential KO theory as the bulk action.

Finally let us discuss the anomaly cancellation. For this we rewrite the bulk action by shifting the integration variable to $\breve{\omega}_0' := \breve{\omega}_0 - \breve{A}$. We find

$$S = 2\pi i q_{M_{11}}(\breve{\omega}_0') - 2\pi i q_{M_{11}}(\breve{A}). \tag{5.26}$$

So, when M_{11} is closed, the two terms are completely independent and can be discussed separately.

Let us first discuss the second term. For $\breve{A} = [C = 0, V, A]$, it is simply $\frac{1}{2}\eta(\not{D}_{\wedge^2 V, A\otimes 1 + 1\otimes A})$ according to (5.16). This is the eta invariant for a Majorana-Weyl fermion in the adjoint of the gauge bundle. So, it exactly cancels the anomaly from the gaugino, both the local part and the global part. Note that there was no need to determine $\Omega_{11}^{\text{spin}}(BSpin(32))$ using difficult spectral sequence computations. We abstractly know that the gauge field dependence of the anomaly theory of the RR field is exactly the same eta invariant as that of the gaugino, and that was enough.

What remains is then the theory with the action

$$S = 2\pi i q_{M_{11}}(\breve{\omega}_0'). \tag{5.27}$$

We already know that it is an invertible theory from our general discussion in Sec. 4.2. Furthermore, it only depends on the spacetime itself. As $\Omega_{11}^{\text{spin}}(\text{pt}) = 0$, such an invertible theory is determined purely by the local part, i.e. the anomaly polynomial. Determining it requires dealing with only the infinitesimal variation of the action (5.17).

Let us temporalily work in arbitrary $8k+3$ dimensions. The variation of the eta invariant part is (5.18), which is

$$\left(\frac{1}{2}\mathsf{U}^2 - \frac{1}{2}\mathsf{X}^2\right)\Big|_{\dim M+1}. \tag{5.28}$$

To compute the variation of the differential form part, we first solve the equation of motion for C. Note that the bilinear term CdC in the action only contains C_3 to $C_{\dim M-4}$. The EOM of these components of C then set

$$dC = \mathsf{U} - (\mathsf{U}_0 + \mathsf{U}_{\dim M+1}) \tag{5.29}$$

where U_d is the degree-d piece of U. Plugging this back in, the differential form part of the action is effectively

$$-\frac{1}{2}(\mathsf{U} - (\mathsf{U}_0 + \mathsf{U}_{\dim M+1}))^2\Big|_{\dim M+1}. \tag{5.30}$$

We also note that there is a coupling $\int \mathsf{U}_0 C_{\dim M}$ in the action. The EOM of $C_{\dim M}$ then sets $\mathsf{U}_0 = 0$. Then the differential form part of the action (5.30) further simplifies to

$$= -\frac{1}{2}\mathsf{U}^2. \tag{5.31}$$

Summing (5.28) and (5.31), we find that the total variation is simply

$$-\frac{1}{2}\mathsf{X}^2\Big|_{\dim M+1}. \tag{5.32}$$

In our case $\dim M + 1 = 12$, we have

$$-\frac{1}{2}(\mathsf{X}_0\mathsf{X}_{12} + \mathsf{X}_4\mathsf{X}_8) = -\left(\frac{p_1^3}{3780} - \frac{13p_1p_2}{7560} + \frac{31p_3}{3780}\right), \tag{5.33}$$

which exactly cancels (2.17), the anomaly polynomial for the gravitino and the dilatino. This allows us to conlcude that the gravitational part of the anomaly is also completely cancelled, including the global part. This was what we wanted to show.

5.3.3 Comments

One comment is that the global anomaly cancellation was argued in two distinct ways on the heterotic side and on the Type I side, in the case of the 10d $so(32)$ superstring theory. There was a general difference in the approaches taken, as already explained in Sec. 3.2: on the heterotic side, we included the B-field as part of the spacetime structure, and on the Type I side, we wrote down the bulk action which treated H_3 and H_7 in a democratic fashion. Not only that, the analysis on the Type I side required the use of differential KO theory. This not only added the continuous B-field, but also the discrete KO-theoretic degrees of freedom. It would be nice to work out the relation of these two approaches in more detail.

Another comment is that Type I is just one of the many examples of NSR superstrings with orientifolds. How the K-theoretic anomaly cancellation should work in this generality was outlined in [DFM09,DFM10], but the details are still to be worked out. As the authors of these two references told me that they will probably not do that by themselves, any interested reader is free to pursue this direction.

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