1 4d QFT.

A "4d QFT with G symmetry" Q is a pair $Q = (\mathcal{V}, Z)$ where

- $\mathcal{V}$ is a $\mathbb{Z}/2$ filtered $\mathbb{C}$-vector space

$$\mathcal{V}_0 \subset \mathcal{V}_{1/2} \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V} \tag{1.1}$$

  such that each filtered piece $\mathcal{V}_i$ is a finite-dimensional representation of $\mathfrak{so}(4) \times G$.

- $\mathcal{V}$ has a lot of noncommutative, nonassociative products.

- For each principal G-bundle $W \to X$ with connection on a compact spin real 4-dimensional manifold X without boundary with finite-volume Riemannian metric with k marked points $x_1, \ldots, x_k$, there is a multi-linear map

$$Z(W; x_1, \ldots, x_k) : \mathcal{V}^{\otimes k} \to \mathbb{C} \tag{1.2}$$

  satisfying various conditions, such as continuity under the change of the metric, the marked points, the G-connection, etc. The object $Z(W; x_1, \ldots, x_k)(v_1 \otimes \cdots \otimes v_k)$ is often denoted as

$$\langle v_1(x_1) v_2(x_2) \cdots v_k(x_k) \rangle \tag{1.3}$$

  in the physics literature.

- When X has a boundary $\partial X \neq \varnothing$, one instead has

$$Z(W; x_1, \ldots, x_k) : \mathcal{V}^{\otimes k} \to \mathcal{H}(\partial X) \tag{1.4}$$

  where $\mathcal{H}(Y)$ is a vector space such that

$$\mathcal{H}(Y_1 \times Y_2) = \mathcal{H}(Y_1) \otimes \mathcal{H}(Y_2) \tag{1.5}$$

  and

$$\mathcal{H}(\bar{Y}) = \mathcal{H}(Y)^* \tag{1.6}$$

  where $\bar{M}$ is M with reversed orientation. Eq (1.2) is a special case where $\partial X = \varnothing$ and $\mathcal{H}(\varnothing) = \mathbb{C}$.

- These assignments $\mathcal{H}$ and Z are functorial. E.g., if there is a isometric equivariant map between $W_1 \to X_1$ and $W_2 \to X_2$, $Z(W_1) \in \mathcal{H}(\partial X_1)$ and $Z(W_2) \in \mathcal{H}(\partial X_2)$ can be naturally identified.

- Furthermore, if $W \to X$ is obtained by gluing $W_1 \to X_1$ and $W_2 \to X_2$ along a component M of the boundaries $M \subset \partial X_1$, $\bar{M} \subset \partial X_2$, we should have

$$Z(W; x_1, \ldots, x_k; y_1, \ldots, y_l) = Z(W_1; x_1, \ldots, x_k) Z(W_2; y_1, \ldots, y_l) \tag{1.7}$$

  where the product on the RHS uses the pairing (1.6).

1.1 QFT on infinite-volume spaces

So far the space X is assumed to be of finite volume. If X has infinite volume, or a boundary component $Y \subset \partial X$ has infinite volume, one needs to specify a vacuum $p \in M$, where M is called the moduli space of the vacuum of the QFT Q, to have Z and $\mathcal{H}$:

$$Z(W; p; x_1, \ldots, x_k) : \mathcal{V}^{\otimes k} \to \mathcal{H}(\partial X; p). \tag{1.8}$$

In other words, Z and $\mathcal{H}$ are defined for

- a finite volume X or a finite volume ∂X,
- a pair (X, p) when X has infinite volume, or $(\partial X, p)$ when ∂X has infinite volume.

One important property is that the algebra of $\mathbb{C}$-valued continuous functions on M is a subspace of $\mathcal{V}$.

1.2 Unitary QFT.

A unitary QFT has further properties that

- There is a real structure $* : \mathcal{V} \to \mathcal{V}$ (called the CPT conjugation map)
- and a Hilbert space structure on $\mathcal{H}(Y)$ such that

  $$\Psi = Z(W; x_1, \ldots, x_k)(v_1, \ldots, v_k) \in \mathcal{H}(Y) \tag{1.9}$$

  and

  $$\bar{\Psi} = Z(\bar{W}; x_1, \ldots, x_k)(v_1^*, \ldots, v_k^*) \in \mathcal{H}(\bar{Y}) \tag{1.10}$$

  where $\bar{W} \to \bar{X}$ is $W \to X$ with reversed orientation satisfies

  $$\langle \bar{\Psi} \Psi \rangle \geq 0. \tag{1.11}$$

  This is called the reflection positivity.

1.3 Renormalization group.

There is a natural action of the multiplicative group $\mathbb{R}_{>0}$ on a QFT:

$$Q \mapsto \mu_t Q, \quad t \in \mathbb{R}_{>0} \tag{1.12}$$

where we define $\mathcal{V}(\mu_t Q) = \mathcal{V}(Q)$ and

$$Z(\mu_t Q)(W; x_1, \ldots, x_k) = Z(Q)(\mu_t W; x_1, \ldots, x_k) \tag{1.13}$$

where $\mu_t W$ is defined so that if the base of $W \to X$ has is a Riemannian manifold (X, g) with metric g, $\mu_t W \to \mu_t X$ has the base with a metric given by (X, tg). This action of $\mathbb{R}_{>0}$ is called the renormalization group.

A QFT is called scale invariant if $\mu_t Q \simeq Q$.

If there's a limit $\mu_{+\infty} Q = \lim_{t \to +\infty} \mu_t Q$ in a suitable sense, it's called the infrared (IR) limit of Q.

2 4d $\mathcal{N} = 2$ supersymmetric QFT.

2.1 Basics

A "4d $\mathcal{N} = 2$ supersymmetric QFT with G symmetry" Q is a particular kind of 4d QFT with $G \times \mathrm{SU}(2)$ symmetry, with various extra axioms. A few important consequences are

- There's a linear map $\delta : \mathcal{V}_i \to \mathcal{V}_{i+1/2}$ with $\delta^2 = 0$.
- A distinguished subspace $M_{\text{susy}} \subset M$ is called the moduli space of supersymmetric vacuum, and has a complex structure. The algebra of holomorphic functions on M_{susy} is isomorphic to $H(\mathcal{V}, \delta)^{\mathfrak{so}(4)}$.
- M_{susy} contains two distinctive subspaces, called the Higgs and the Coulomb branches

$$M_{\text{susy}} \supset M_{\text{Coulomb}} \cup M_{\text{Higgs}} \tag{2.1}$$

- if the metric of Y is flat, there is $\delta : \mathcal{H}(Y) \to \mathcal{H}(Y)$ with $\delta^2 = 0$. Its cohomology group $H(\mathcal{H}(Y), \delta)$ is called the BPS states.
- For each simple factor $G_0 \subset G$, one has a positive real number k_{G_0}.

2.2 Topological twisting

Given a 4d $\mathcal{N} = 2$ supersymmetric QFT $Q = (\mathcal{V}(Q), Z(Q))$, one can construct a 4d QFT $T(Q) = (\mathcal{V}(T(Q)), Z(T(Q)))$ via the following procedure :

$$Z(T(Q))(X) = Z(Q)(K) \tag{2.2}$$

where $K \to X$ is a principal SU(2) bundle obtained from the tangent bundle TX by decomposing its affine connection in Spin(4) into one of the subgroups $\mathrm{SU}(2) \times \mathrm{SU}(2)$. This $T(Q)$ is known to be independent of the continuous defomation of the metric of X. This is used by Witten to give a QFT interpretation to the Donaldson invariant.

2.3 Basic ways of constructing 4d $\mathcal{N} = 2$ supersymmetric QFT.

2.3.1 Hypermultiplets

For a pseudo-real representation V of G, one has a 4d $\mathcal{N} = 2$ supersymmetric QFT with G-symmetry $H(V)$, called a half-hypermultiplet in the representation V. When $V = R \oplus \bar{R}$, $H(V)$ is called a hypermultiplet in the representation R.

$$M_{\text{Coulomb}}(H(V)) = \{0\}, \quad M_{\text{Higgs}}(H(V)) = V. \tag{2.3}$$

For a simple factor $G_0 \subset G$, $k_{G_0}(H(V))$ is given as follows. One first decomposes V (as a complex representation) to irreps of G_0:

$$V = \oplus_i R_i \tag{2.4}$$

then

$$k_{G_0}(H(V)) = \sum_i c_2(R_i) \tag{2.5}$$

where c_2 is the quadratic Casimir invariant of the representation, normalized so that $c_2(\mathfrak{g}_\mathbb{C}) = h^\vee(G_0)$.

When V is zero-dimensional, one has a trivial theory triv $= H(\{0\})$.

2.3.2 Gauging.

Given a 4d supersymmetric QFT Q with $G \times F$ symmetry where G is a simple Lie group and $k_G(Q) \leq 2h^\vee(G)$, one can construct a one-parameter family of 4d supersymmetric QFTs with F symmetry

$$(Q///G)_\tau \tag{2.6}$$

where τ is a complex number in the upper half plane. For a theory of this form, F is known as the flavor symmetry, and G is known as the gauge symmetry. τ is known as the complexified gauge coupling.

We have

$$M_{\text{Coulomb}}(Q///G) = M_{\text{Coulomb}}(Q) \times \operatorname{Spec} \mathbb{C}[\mathfrak{g}_\mathbb{C}]^{G_\mathbb{C}} \tag{2.7}$$

and

$$M_{\text{Higgs}}(Q///G) = M_{\text{Higgs}}(Q)///G. \tag{2.8}$$

Note that $\mathbb{C}[\mathfrak{g}_\mathbb{C}]^{G_\mathbb{C}} \subset \mathcal{V}(Q///G)$. Denote the degree-2 generator of $\mathbb{C}[\mathfrak{g}_\mathbb{C}]^{G_\mathbb{C}}$ by d_2. Consider the function

$$f_X(x,y) = Z((Q///G)_\tau)(X;x,y)(d_2, {d_2}^*) \in \mathbb{C}. \tag{2.9}$$

When $\ell = |x-y|$ is very small, it is known to behave as

$$f_X(x,y) \sim \left(\frac{1}{(\operatorname{Im}\tau) + (2h^\vee(G) - k_G(Q))\log(1/\ell)}\right)^2 \frac{1}{\ell^4} \tag{2.10}$$

Historically, the combination

$$b_1 = 2h^\vee(G) - k_G(Q) \tag{2.11}$$

is known as the one-loop beta function. This can't be negative, as it will violate the reflection positivity when ℓ is very very small. When $2h^\vee(G) = k_G(Q)$, the theory is called superconformal.

As an example, consider $G = \mathrm{SU}(N_c)$, $F = \mathrm{SU}(N_f)$, and $V = A \otimes B^* \oplus A^* \otimes B$ where $A \simeq \mathbb{C}^{N_c}$, $B \simeq \mathbb{C}^{N_f}$ are the defining representaitons of G and F. $k_G(H(V)) = 2N_f$. Therefore, when $N_f \leq 2N_c$, one can consider the gauge theory

$$(H(V)///G)_\tau. \tag{2.12}$$

3 Duality

. The theory

$$\mathcal{Q}(G)_\tau = (H(\mathfrak{g}_\mathbb{C} \oplus \mathfrak{g}_\mathbb{C})/\!/\!/G)_\tau \tag{3.1}$$

is called the $\mathcal{N} = 4$ super Yang-Mills theory with gauge group G. It is believed to satisfy

$$\mathcal{Q}(G)_\tau = \mathcal{Q}(G^\vee)_{-1/(n\tau)} \tag{3.2}$$

where $G^\vee$ is the Langlands dual group, and n is the ratio of the squared lengths of the long and short roots.

These nontrivial equality between QFTs are called dualities. Another famous example is

$$(H(V_2 \otimes_\mathbb{R} V_S)/\!/\!/\,\mathrm{SU}(2))_\tau = (H(V_2 \otimes_\mathbb{R} V_V)/\!/\!/\,\mathrm{SU}(2))_{-1/\tau} = (H(V_2 \otimes_\mathbb{R} V_C)/\!/\!/\,\mathrm{SU}(2))_{1/(1-\tau)} \tag{3.3}$$

where $V_2 \simeq \mathbb{C}^2$ is the defining representation of $\mathrm{SU}(2)$, and $V_{S,V,C}$ are three eight-dimensional irreducible representations of $\mathrm{Spin}(8)$.