# Generalization of the $R^4$ Conjecture

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D=10 Type IIB low-energy effective action:

$$S_{10} = \frac{1}{\kappa^2}\int d^{10}x\sqrt{G}\left(R + \alpha'^3 R^4\left(\underbrace{2\zeta(3)e^{-\frac{3\phi}{2}}}_{\text{tree}} + \underbrace{\frac{2\pi^2}{3}e^{\phi/2}}_{\text{1-loop}} + \dots\right)\right)$$

$R^4$ conjecture of Green + Gutperle:

$$= \frac{1}{\kappa^2}\int d^{10}x\sqrt{G}\left(R + \alpha'^3 R^4 \sum_{m_1,m_2}{}' \frac{\tau_2^{3/2}}{|m_1+m_2\tau|^3}\right)$$

$$\tau = A + ie^{-\phi}, \quad \sum_{m_1,m_2}{}' \frac{\tau_2^{3/2}}{|m_1+m_2\tau|^3} = 2\zeta(3)\tau_2^{3/2} + \frac{2\pi^2}{3}\tau_2^{-\frac{1}{2}} + \mathcal{O}(e^{-\tau})$$

Conjecture $\Rightarrow$ $R^4$ term is $SL(2,\mathbb{Z})$-invariant and gets perturbative contributions only at tree-level and one-loop

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For compactification on $T_3$, there is a stronger version of $R^4$ conjecture by Kiritsis + Pioline:

$$S_8 = \frac{1}{\kappa^2}\int d^8x\sqrt{G}\left(R + \alpha'^3 R^4 \sum_{m_1,m_2,m_3}{}' \left(m_\alpha m_\beta M^{\alpha\beta}(\lambda_s,\sigma_1,\sigma_2,A,B)\right)^{-3/2}\right.$$

7 scalars split into $\frac{SL(2,R)}{SO(2)}$ and $\frac{SL(3,R)}{SO(3)}$ moduli

$\rho_1 + i\rho_2$ = complex moduli ; $\lambda_s$ = coupling constant, $\sigma_1 + i\sigma_2$ = Kahler moduli, $A + iB$ = R-R scalar

Conjecture is $SL(2,\mathbb{Z}) \times SL(3,\mathbb{Z})$ inv. and implies D=10 conjecture when $\sigma_2 \to \infty$

Can one generalize these conjectures for other terms in the Type IIB effective action?

Our $R^4 H^{4g-4}$ conjectures: $\mathcal{S}_{10} = \frac{1}{\kappa^2}\int d^{10}x \sqrt{G}\,\Big(R + H_+ H_- + \text{"}H_5^2\text{"} +$

$$\sum_{g=1}^{\infty} c_g \alpha'^{2g+1} \sum_{p,q=0}^{4g-4} R^4 \; H_5^{4g-4-p-q} H_+^p H_-^q \sum_{m_1,m_2}{}' \frac{\tau_2^{g+1/2}}{(m_1+m_2\tau)^{\frac{2g+1+p-q}{2}}(m_1+m_2\bar\tau)^{\frac{2g+1+q-p}{2}}}\Big)$$

$H_+ = \tau_2^{-1/2}(H^{\mu\nu\rho}_{RR} - \tau H^{\mu\nu\rho}_{NS-NS})$, $H_- = \tau_2^{-1/2}(H^{\mu\nu\rho}_{RR} - \bar\tau H^{\mu\nu\rho}_{NS-NS})$

$H_5 = H^{\mu\nu\rho\sigma\tau}_{RR}$, $c_g$ = overall constant

Conjecture $\Rightarrow$ $R^4 H^{4g-4}$ term is $SL(2,\mathbb{Z})$ invariant and gets perturbative corrections only at tree-level and g-loops.

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In $d=9$, $R^4 H^{4g-4}$ term cannot come from <u>local</u> $d=11$ term compactified on $T^2$.

May come from <u>non-local</u> $d=11$ term compactified on $T^2$.

For compactification on $T_2$, $H$'s combine into an $SO(3)$ triplet: $H^{(jk)}_{\mu\nu\rho}$

$$H^{(\pm\pm)}_{\mu\nu\rho} = \sigma_2^{1/2}\lambda_8^{1/3}\left(H^{RR}_{\mu\nu\rho} - A\,H^{NS\text{-}NS}_{\mu\nu\rho}\right) \mp i\sigma_2\lambda_8^{-2/3}H^{RR}_{\mu\nu\rho 89}$$

$$H^{(+-)}_{\mu\nu\rho} = i\lambda_8^{-2/3}H^{NS\text{-}NS}_{\mu\nu\rho}$$

Define $M_{\mu\nu\rho} = H^{(jk)}_{\mu\nu\rho}\,u^L_j\,u^R_j$

harmonic variables

Our D=8 conjecture:

$$S_8 = \frac{1}{\kappa^2}\int d^8x\sqrt{G}\,\Big(R + H^{(jk)}H_{(jk)} + \sum_{g=1}^{\infty} c_g\,\alpha'^{3g+1}\int du_L\,du_R\,R^4 M^{4g-4}\sum_{m_1,m_2,m_3}{}' \frac{\left(m^\alpha C^{(jk)}_\alpha \bar{u}^L_j \bar{u}^R_k\right)^{4g-4}}{\left(m_\alpha m_\beta M^{\alpha\beta}\right)^{3g-3/2}}\Big)$$

$$C^{(++)}_\alpha = \lambda_8^{-1/3}\sigma_2^{-1/2}(0,1,\sigma)\ ,\quad C^{(--)}_\alpha = \lambda_8^{-1/3}\sigma_2^{-1/2}(0,1,\bar{\sigma})$$

$$C^{(+-)}_\alpha = -i\lambda_8^{2/3}(1, A, B + A\sigma_1)$$

$$M^{\alpha\beta} = C^{\alpha(jk)}C^{\beta}{}_{(jk)}$$

D=8 Conjecture is $SL(2,\mathbb{Z})\times SL(3,\mathbb{Z})$ inv. and implies D=10 conjecture when $\sigma_2\to\infty$.

Evidence for D=8 conjecture comes from explicit g-loop amplitude computation.

# Topological Amplitudes

Certain g-loop superstring amplitudes can be explicitly calculated. Integrand is independent of location of PCO's (picture-changing operators).

Amplitude only depends on moduli of twisted N=2 sc̄ft representing the Calabi-Yau compactification.

Can be computed in RNS approach, but easier to compute using modified GS approach where spacetime SUSY is manifest and no sum over spin structures.

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Ex: A) Type IIB on Calabi-Yau 3-fold (N=2 D=4)

$$S = \int d^4x \sqrt{G}\left(R + F^2 + H^2 + \sum_{g=1}^{\infty}\left(f_g R^2 F^{2g-2} + f_g' R^2 H^{2g-2}\right)\right)$$

F and H are R-R 2-forms and 3-forms

$f_g$ and $f_g' \to 0$ as volume of CY $\to \infty$

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Ex: B) Type IIB on Calabi-Yau 2-fold (N=2 D=6)

$$S = \int d^6x \sqrt{G}\left(R + H^2 + \sum_{g=1}^{\infty} f_g R^4 H^{4g-4}\right)$$

$f_g$ survives as volume of CY $\to \infty$!

Ex. A) Type IIB on Calabi-Yau 3-fold ($N=2$ $D=4$)

Scatter 2 gravitons and $2g-2$ R-R fields

(chiral/chiral : F , chiral/anti-chiral : H )

Choose R-R vertex operators in $(-\frac{1}{2},-\frac{1}{2})$ picture $\Rightarrow$ Need $3g-3$ PCO's

CY described by $c=9$ $N=2$ sc'ft $[T, G^+, G^-, J]$

Chiral/Chiral (or chiral/anti-chiral) vertex op's carry $(\frac{3}{2},\frac{3}{2})$ (or $\frac{3}{2},-\frac{3}{2}$) charge w/respect to $(J_L, J_R)$ $\Rightarrow$ Only $e^{-\phi}G^-$ and $e^{-\bar\phi}\bar G^-$ (or $e^{-\bar\phi}\bar G^+$) terms in PCO's contribute

After integrating vertex operators over genus $g$ surface, one finds the amplitude

$$\mathcal{A} = \int d^4x \left( \int d^2\theta_L\, d^2\theta_R\, (W_{\alpha\beta} W^{\alpha\beta})^{2g} f_g + \int d^2\theta_L\, d^2\bar\theta_R\, (W_{\alpha\dot\beta} W^{\alpha\dot\beta})^{2g} f'_g \right)$$

$$= \int d^4x \left( R^2 F^{2g-2} f_g + R^2 H^{2g-2} f'_g \right) + \dots$$

$$W_{\alpha\beta} = F_{\alpha\beta} + \theta^\gamma_L \theta^\delta_R R_{\alpha\gamma\beta\delta} + \dots, \quad W_{\alpha\dot\beta} = H_{\alpha\dot\beta} + \theta^\gamma_L \bar\theta^{\dot\delta}_R R_{\alpha\gamma\dot\beta\dot\delta} + \dots$$

$$f_g = \prod_{j=1}^{3g-3} \int d^2m_j \langle \int \mu_j G^- \int \bar\mu_j \bar G^- \rangle_{CY}, \quad f'_g = \prod_{j=1}^{3g-3} \int d^2m_j \langle \int \mu_j G^- \int \bar\mu_j \bar G^- \rangle_{CY}$$

$\langle\ \rangle_{CY}$ : twisted $N=2$ correlation fncn

## Ex B) Type IIB on Calabi-Yau 2-fold (N=2 D=6)

Scatter 4 gravitons and 4g-4 R-R fields

R-R vertex op's in $(-\frac{1}{2},-\frac{1}{2})$ picture $\Rightarrow$ Need 4g-4 PCO's

CY described by $c=6$ $N=2$ SCFT $[T, G^+, G^-, J]$

$\Rightarrow$ can define $N=4$ generators $[T, G^+, \tilde{G}^+, G^-, \tilde{G}^-, J^{++}, J, J^{--}]$

$J^{++} = e^{3J}$, $J^{--} = e^{-3J}$, $\tilde{G}^+ = [J^{++}, G^-]$, $\tilde{G}^- = [J^{--}, G^+]$

D=6 spinor carries internal SU(2) index $\theta_\alpha^i$ $(i=\pm)$

R-R states in ++ direction carry (+1,+1) U(1) charge $\Rightarrow$ Only $e^{-\phi} G_L^-$ and $e^{-\phi} G_R^-$ terms in PCO's contribute

After integrating vertex operators over genus g surface, one finds the amplitude

$$\mathcal{A} = \int d^6x \int du_L \int du_R \int d^4\theta_L^+ \int d^4\theta_R^+ \left(\epsilon^{a_1 \dots a_4} \epsilon^{b_1 \dots b_4} W_{a_1 b_1} \dots W_{a_4 b_4}\right)^g f_g$$

$$= \int d^6x \int du_L \int du_R \, R^4 \left(\bar{u}_L^i \bar{u}_R^j H_{ij}\right)^{4g-4} f_g(u_L, u_R) + \dots$$

$$W_{ab} = \bar{u}_L^i \bar{u}_R^j H_{ab\,ij} + \theta_{cL}^+ \theta_{dR}^+ R_{ab}^{cd} + \dots$$

$$f_g = \prod_{j=1}^{3g-3} \int d^2m_j \prod_{i=1}^{g} \int d^2v_i \left\langle \left| \prod_{j=1}^{3g-3} \int \mu_j \hat{G}^- \prod_{i=1}^{g-1} \hat{G}^+(v_i) \, J(v_g) \right|^2 \right\rangle_{CY}$$

$$\hat{G}^- = u_+ G^- + u_- \tilde{G}^-, \quad \hat{G}^+ = u_+ \tilde{G}^+ + u_- G^+$$

Ooguri + Vafa explicitly computed $f_g(u_L, u_R)$ when the CY 2-fold is $T^2 \times R^2$ and found

$$f_g(u_L, u_R) = \sum_{m,n}{}' \left( \frac{u_+^L u_+^R}{m+n\sigma} + \frac{u_-^L u_-^R}{m+n\bar{\sigma}} \right)^{4g-4} |m+n\sigma|^{2g-4} c_g$$

Plugging $f_g$ into $\mathcal{A}$, rewriting in D=8 notation, and integrating over $u_L$ and $u_R$,

$$\mathcal{A} = \int d^8x \sqrt{G}\, c_g \lambda_8^{\frac{2g-2}{3}} \sum_{p=2-2g}^{2g-2} R^4 H_{(++)}^{2g-2+p} H_{(--)}^{2g-2-p} \sum_{m,n}{}' \frac{\sigma_2^8}{(m+n\sigma)^{3+p}(m+n\bar{\sigma})^{3-p}}$$

$H_{(++)}$ and $H_{(--)}$ are D=8 R-R three-forms.
Index structure comes from extending D=6 vector indices to D=8.

$\mathcal{A}$ is $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$ invariant and is precisely the g-loop contribution to our D=8 $SL(2,\mathbb{Z}) \times SL(3,\mathbb{Z})$ invariant conjecture.

Can one prove that $R^4 H^{4g-4}$ gets no perturbative contributions above g-loops?

Recall that $N=2$ $D=4$ $R^2 F^{2g-2}$ terms only gets g-loop contribution since dilaton is in $N=2$ $D=4$ tensor multiplet in Euclidean gauge.

$\Rightarrow$ $N=2$ $D=4$ $R^2 H^{2g-2}$ only gets <u>perturbative</u> contribution at g-loops because of <u>perturbative</u> "c-map"

For $N=2$ $D=8$, 7 scalar moduli split into

$\frac{SL(3)}{SO(3)}$ : lowest components of tensor superfield $L_{ijkl}$

and $\frac{SL(2)}{SO(2)}$ : lowest component of chiral superfield $W$

But (except for $g=1$ at linearized level), not known how to write $R^4 H^{4g-4}$ terms in $N=2$ $D=8$ superspace.

For $R^4$ term, tensor and vector decoupling implies no perturbative contributions above 1-loop (to linearized level)

# Further Evidence for our Conjecture

$R^4 H^{4g-4}$ is probably related by spacetime SUSY to $\partial^{4g-4} R^4$ terms and $R^{2g+2}$ terms.

Based on the Veneziano 4-point tree amplitude, Russo has conjectured that $\partial^{4g-4} R^4$ terms appear in the Type IIB effective action as

$$\mathcal{S}_{10} = \int d^{10}x \sqrt{G} \left( R + \sum_{g=1}^{\infty} \partial^{4g-4} R^4 \alpha'^{2g+1} \sum_{m,n}{}' \frac{\tau_2^{g+\frac{3}{2}}}{|m+n\tau|^{2g+1}} \right)$$

This is precisely our conjecture if SUSY implies the combination $\partial^{4g-4} R^4 + c_g R^4 H^{4g-4}$

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Also, M(atrix) theory calculations suggest that $F^{2g+2}$ does not get perturbative contributions above g-loops for <u>open</u> superstring.

(Ooguri + Douglas, private comm & Becker, Polchinski, Tseytlin)

This suggests $R^{2g+2}$ does not get perturbative contributions above g-loops for <u>Type IIB</u> superstring since graviton vertex operator = $|\text{gluon vertex operator}|^2$

# Further Speculations

A) $N=2 \quad D=4 : \mathcal{A} = \sum_{g=1}^{\infty} R^2 F^{2g-2} f_g$

If $F$ has a constant value $\lambda$, what is $\sum_{g=1}^{\infty} \lambda^{2g-2} f_g$ computing?

Near singular CY 3-folds, Gopakumar + Vafa have shown it computes partition function of large $N$ $SU(N)$ Chern-Simons theory for level $k = \frac{1}{\lambda}$.

B) $N=2 \quad D=10 : \quad \mathcal{A} = \sum_{g=1}^{\infty} c_g \alpha'^{2g+1} R^4 H_5^{4g-4} \sum_{m,n}' \frac{\tau_2^{g+\frac{1}{2}}}{|m+n\tau|^{2g+1}}$

If $H_5$ has a constant value $\lambda$, what is $\sum_{g=1}^{\infty} c_g \lambda^{4g-4} \alpha'^{2g+1} \sum_{m,n}' \frac{\tau_2^{g+1/2}}{|m+n\tau|^{2g+1}}$ computing?

Using arguments similar to Banks + Green, it seems to compute 4-pt correlation function of large $N$ $SU(N)$ super-Yang-Mills theory.