Generalization of the $R^n$ Conjecture

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$D=10$ Type IIB low-energy effective action:

$$ S_{10} = \frac{1}{k^2} \int d^{10}x \sqrt{G} \left( R + \alpha'^3 R^n \left( 2\xi(3) e^{\frac{3z}{2}} + \frac{\alpha'^2}{3} e^{\frac{z}{2}} + \ldots \right) \right) $$

$R^n$ conjecture of Green+ Gutperle:

$$ S_{10} = \frac{1}{k^2} \int d^{10}x \sqrt{G} \left( R + \alpha'^3 R^n \sum' \frac{\xi}{m_i m_j (1 + m_i m_j \xi)^3} \right) $$

$$ \xi = A + i e^{-\eta} \quad , \quad \sum' \frac{\xi}{m_i m_j (1 + m_i m_j \xi)^3} = 2\xi(3) \xi^{\frac{3}{2}} + \frac{2\pi^2}{3} \xi^{\frac{3}{2}} + \Theta(e^{\eta}) $$

Conjecture $\Rightarrow$ $R^n$ term is $SL(2,\mathbb{Z})$-invariant and gets perturbative contributions only at tree-level and one-loop

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For compactification on $T_2$, there is a stronger version of $R^n$ conjecture by Kachru+ Pioline:

$$ S_{10} = \frac{1}{k^2} \int d^{10}x \sqrt{G} \left( R + \alpha'^3 R^n \sum' \frac{m_i m_j}{m_i m_j + 1} (\lambda, \sigma, \varphi, A, B) \right) $$

7 scalars split into $SL(2,\mathbb{R})/SO(2)$ and $SL(3,\mathbb{R})/SO(3)$ moduli:

$\phi, \psi, \phi_3$: complex moduli; $\lambda_s$ coupling constant; $\theta, \phi_2$: Kahler; $A + i B$: $R-R$ scalar

Conjecture is $SL(2,\mathbb{Z}) \times SL(3,\mathbb{Z})$ inv. and implies $D=10$ conjecture when $\sigma_2 \to \infty$
Can one generalize these conjectures for other terms in the Type IIB effective action?

Our $R^n H^{9-n}$ conjectures:

\[ \mathcal{L} = \frac{1}{k^4} \int d^8 x \sqrt{g} \left( R + H^+ H^- + \cdots + \right) \]

\[ \sum_{g=0}^5 \sum_{p=0}^{9-g-n} \left( R^g H^+ H^- \epsilon_{m_0 m_2} \left( m_0^2 m_2^2 \right)^{2g+2p} \left( m_0 e \right)^{2g+2p} \right) \]

\[ H_+ = \varepsilon^{\mu \nu \rho} \left( H_{+ \mu \nu} - \bar{\epsilon} H_{+ \mu \nu} \right), \quad H_- = \varepsilon^{\mu \nu \rho} \left( H_{- \mu \nu} - \bar{\epsilon} H_{- \mu \nu} \right) \]

\[ H_5 = H_{5 \mu \nu \rho \sigma \epsilon} \quad C_5 = \text{overall constant} \]

Conjecture: $R^n H^{9-n}$ term is $SL(2,\mathbb{C})$ invariant and gets perturbative corrections only at tree-level and g-loops.

In $d=9$, $R^n H^{9-n}$ term cannot come from local $d=11$ term compactified on $T^2$.

May come from non-local $d=11$ term compactified on $T^2$. 
For compactification on $T_2$, $H$'s combine into an $SO(3)$ triplet: $H_{AB}^{(s)}$

$$H_{AVP}^{(++)} = i \lambda_8 \lambda_8 \left( H_{AVP}^{AA} - A H_{AVP}^{NS-NS} \right) = i \sigma_2 \lambda_8 \lambda_8 H_{AVP}^{AA}$$

$$H_{AVP}^{(++)} = i \lambda_8 \lambda_8 \lambda_8 H_{AVP}^{NS-NS}$$

Define $M_{AVP} = H_{AVP}^{(s)} u_j^L u_j^R$

Our D=8 conjecture: $\delta_{g} = \frac{1}{k^2} \int d^4 \phi \sqrt{\gamma} \left( R + H^{(s)j} H_{(s)j} \right) + \sum_{g=1}^{39} \int d\nu \nu R M^{\nu g-y} \sum' \frac{(m^2 C_2^{(s)k} u_i^L u_k^R)}{(m_i m_j m_k (m_m m_p M^m p)^{3g-3/2})}$

$$C_2^{(s)k} = \lambda_8^{1/2} \sigma_2 \lambda_8^{1/2} (0,1,\sigma), \quad C_2^{(s)k} = \lambda_8^{1/2} \sigma_2 \lambda_8^{1/2} (0,1,\bar{\sigma})$$

$$C_2^{(s)k} = -i \lambda_8^{1/2} (1, A, B + A \sigma)$$

$$M^{\nu g} = C_2^{(s)k} C_2^{(s)k}$$

D=8 Conjecture is $SL(2,\mathbb{Z}) \times SL(3,\mathbb{Z})$ inv. and implies D=10 conjecture when $\sigma_2 \to \infty$.

Evidence for D=8 conjecture comes from explicit g-loop amplitude computation.
Topological Amplitudes

Certain g-loop superstring amplitudes can be explicitly calculated. Integrand is independent of location of PCO's (picture changing operators).

Amplitude only depends on moduli of twisted N=2 soft representing the Calabi-Yau compactification.

Can be computed in RNS approach, but easier to compute using modified GS approach where spacetime SUSY is manifest and no sum over spin structures.

Ex: A) Type IIB on Calabi-Yau 3-fold \((N=2 \ D=4)\)

\[
\mathcal{S} = \int d^6x \sqrt{G} \left( R + F^2 + H^2 + \sum_{g=1}^g \left( f_g R^2 F^{2g-2} + f'_g R^2 H^{2g-2} \right) \right)
\]

\(F\) and \(H\) are R-R 2-forms and 3-forms
\(f_g\) and \(f'_g \to 0\) as volume of CY \(\to \infty\)

Ex: B) Type IIB on Calabi-Yau 2-fold \((N=2 \ D=6)\)

\[
\mathcal{S} = \int d^4x \sqrt{G} \left( R + H^2 + \sum_{g=1}^g f_g R^4 H^{4g-4} \right)
\]

\(f_g\) survives as volume of CY \(\to \infty\)!
Ex. A) Type II B on Calabi-Yau 3-fold (N=2 D=4)

Scatter 2 gravitons and 2g-2 R-R fields
(chiral/chiral = F , chiral/anti-chiral = H )

Choose R-R vertex \( \Rightarrow \) Need 3g-3 PCO's
operators in \((-\frac{1}{2}, \frac{1}{2})\) picture

CY described by c=9 N=2 soft \([T, G^+, G^-, J]\)

Chiral/Chiral (or chiral/anti-chiral) vertex op's carry \((\frac{3}{2}, \frac{3}{2})\) (or \(3\frac{1}{2}, -\frac{3}{2}\)) \(\Rightarrow\) Only \(e^{-\frac{4}{3}}G^-\)
and \(e^{-\frac{4}{3}}\bar{G}^-(\text{or } e^\frac{4}{3}G^+)\)
terms in PCO's contribute

After integrating vertex operators over

\[ \mathcal{A} = \int d^6x \left( \int d^2\theta_x d^2\theta_R (W_{\alpha\beta} W^{\alpha\beta})^{2g} f_3 \right) \]
\[ + \int d^2\theta_x d^2\bar{\theta}_R (W_{\alpha\beta} W^{\alpha\beta})^{2g} f'_3 \]
\[ = \int d^6x \left( R^2 F^{2g-2} f_3 + R^2 H^{2g-2} f'_3 \right) + \ldots \]

\(W_{\alpha\beta} = F_{\alpha\beta} + \theta^x \bar{\theta}^y \alpha \beta_{xy} + \ldots\)

\(f_3 = \frac{3g^3}{2} \pi \sum_{m_j} \left< S_{m_j} G^- S_{\bar{m}_j} \bar{G}^- \right>_{cy} \), \(f'_3 = \frac{3g^3}{2} \pi \sum_{j} \left< \sum_{m_j} G^- S_{m_j} \bar{G}^- \right>_{cy} \)

\(< >_{cy} = \text{twisted N=2 correlation func.}\)
Ex B) Type IIB on Calabi-Yau 2-fold (N=2 D=6)

Scatter 4 gravitons and 4g-4 R-R fields

R-R vertex op's in ($\tilde{t}_1, \tilde{t}_2$) picture ⇒ Need 4g-4 PCO's

CY described by $c=6$ $N=2$ scft $(T, G^+, G^-, J)$

⇒ can define $N=4$ generators $(T, G^+, G^-, J, J^+, J^-, J^+)$

$J^{++} = e^{J_5}, J^+ = e^{J_5}, G^+ = [J^{++}, G^-], G^- = (J^-, G^+)$

D=6 spinor carries internal SU(2) index $\theta^i_d (i=±)$

R-R states in ++ direction ⇒ Only $e^{u^a} G^a$ and carry (±1, ±1) U(1) charge $e^{-q} G^a$ terms in PCO's contribute

After integrating vertex operators over genus g surface, one finds the amplitude

$A = \int d^6 x \int d\theta^a_+ \int d\theta^a_- \int d\theta^a_+ \int d\theta^a_-$

$\left( e^{\theta^a_+ a \theta^a_- b \ldots \theta^a_{n-1} a \theta^a_n} \right)^{\gamma_j} f_j (u_i, u_n) + ...$

$W_{ab} = \bar{u}^i_a \bar{u}^j_b H_{ij} + \theta^+_{cd} \theta^+_{de} R^{cd}_{ab} + ...$

$f_j = \prod_j \int d\eta^a_j \int d\eta^a_{-j} \left\langle \prod_{j=1}^g \int d\eta^a_j \hat{G}^- \int d\eta^a_{-j} \hat{G}^+ \right\rangle_{c_j}^2$

$\hat{G}^- = u_+ G^- + u_- G^-, \hat{G}^+ = u_+ G^+ + u_- G^+$
Ooguri + Vafa explicitly computed $f_g(u_c, u_a)$ when the CY 2-fold is $T^2 \times R^2$ and found

$$f_g(u_c, u_a) = \sum_{m,n} \left( \frac{u_c^4 u_a^2}{m+n \sigma} + \frac{u_c^2 u_a^4}{m+n \bar{\sigma}} \right)^{4g-4} \frac{1}{m+n \sigma} C_g^{2g-4}$$

Plugging $f_g$ into $A$, rewriting in D=8 notation, and integrating over $u_c$ and $u_a$,

$$A = \frac{1}{\sqrt{6} C_g^2} \frac{2g-2}{3} \sum_{p:2\to 2g} \sqrt{R^{4g-2g+2p}} H_{(++)}^{2g-2g} H_{(--)}^{2g-2g} \sum_{m,n} \frac{C_g^2}{(m+n \sigma)^{3g+2}(m+n \bar{\sigma})^{2g}}$$

$H_{(++)}$ and $H_{(--)}$ are D=8 R-R three-forms.

Index structure comes from extending D=6 vector indices to D=8.$A$ is SL(2,2)$= SL(2,2)$ invariant and is precisely the g-loop contribution to our D=8 SL(2,2)$\times$SL(3,2) invariant conjecture.
Can one prove that $R^4 H^{4g-4}$ gets no perturbative contributions above g-loops?

Recall that $N=2$ $D=4$ $R^2 F^{2g-2}$ terms only get g-loop contribution since dilaton is in $N=2$ $D=4$ tensor multiplet in Euclidean gauge.

$\Rightarrow$ $N=2$ $D=4$ $R^2 H^{2g-2}$ only gets perturbative contribution at g-loops because of perturbative "c-map".

For $N=2$ $D=8$, 7 scalar moduli split into

\[
\frac{SL(3)}{SO(3)}: \text{lowest components of tensor superfield } L_{ijk}^e \quad \text{and } \frac{SL(2)}{SO(2)}: \text{lowest component of chiral superfield } W
\]

But (except for $g=1$ at linearized level), not known how to write $R^4 H^{4g-4}$ terms in $N=2$ $D=8$ superspace.

For $R^4$ term, tensor and vector decoupling implies no perturbative contributions above 1-loop (to linearized level).
Further Evidence for our Conjecture

$R^4 H^{g-4}$ is probably related by spacetime SUSY
to $g^{-4} R^4$ terms and $R^{2g+2}$ terms.

Based on the Veneziano 4-point tree amplitude,
Russo has conjectured that $g^{-4} R^4$ terms
appear in the Type IIB effective action as

$$\delta_{10} = \int d^9 x \sqrt{G} \left( R + \sum \frac{g}{g''} g^{-4} R^4 a^{2g+1} \sum_{g>0} \frac{\frac{g+2}{1+\frac{2g+1}{g'}}}{g''} \right)$$

This is precisely our conjecture if SUSY
implies the combination $g^{-4} R^4 + c g R^4 H^{g-4}$

Also, Matrix theory calculations suggest
that $F^{2g+2}$ does not get perturbative
contributions above g-loops for open superstring.

This suggests $R^{2g+2}$ does not
get perturbative contributions
above g-loops for Type IIB superstring since
graviton vertex operator = Gluon vertex operator - $l^2$
Further Speculations

A) \( N=2 \quad D=4 \quad A = \sum_{g=1}^{n} R^2 F^{2g-2} f_g \)

If \( F \) has a constant value \( \lambda \), what is \( \sum_{g=1}^{n} \lambda^{2g-2} f_g \) computing?

Near singular CY 3-folds, Gopakumar + Vafa have shown it computes partition function of large \( N \) SU(\( N \)) Chern-Simons theory for level \( k = \frac{1}{\lambda} \).

B) \( N=2 \quad D=10 \quad A = \sum_{g=1}^{n} \epsilon_{\alpha^{2g-2} R^{2g-2} H_{5}^{\alpha}} \frac{E_\alpha}{1+\alpha} \)

If \( H_5 \) has a constant value \( \lambda \), what is \( \sum_{g=1}^{n} \lambda^{2g-2} \alpha^{2g-2} \sum_{\alpha} \frac{E_\alpha}{1+\alpha} \) computing?

Using arguments similar to Banks+Green, it seems to compute 4-pt correlation function of large \( N \) SU(\( N \)) super-Yang-Mills theory.