

DLCQ of M-theory as the light-like limit

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M-theory on S^1 of radius R_{11}

= IIA with $g_s = R_{11} / \sqrt{\alpha'} = R_{11} / l_s$

- BFSS conjecture: M-theory boosted along x^n to the infinite momentum frame:
only relevant dof are DO branes : $(p_{11} = \frac{N}{R_{11}})$
described by a quantum mechanics of $U(N)$ matrices as obtained by dim. reduction of ten-dim susy $U(N)$ YM. IMF : $N \rightarrow \infty$
- Susskind's DLCQ conjecture:

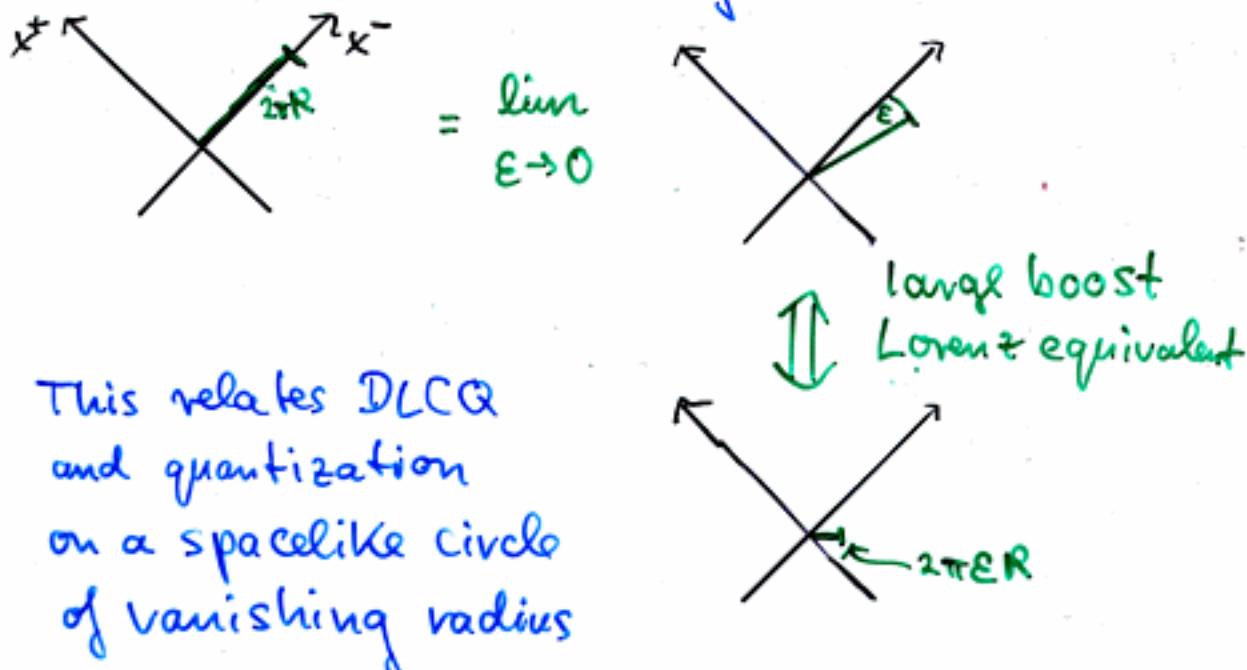
M-theory compactified on a light-like circle $(x^- \equiv \frac{1}{\sqrt{2}}(x^0 - x^n) \simeq x^- + 2\pi R)$

in a sector of total momentum $p_- = \frac{N}{R}$
is described by the same $U(N)$ susy QM
but at finite N. (dualities manifest
at finite N, \dots)

②

Seiberg's and Sen's proofs of DLCQ conjecture

DLCQ : compact x^- viewed as a limit
of an almost light-like circle



This relates DLCQ
and quantization
on a spacelike circle
of vanishing radius

+ Scaling arguments :

M-theory on light-like
circle R , $p_- = \frac{N}{R}$ \Leftrightarrow M-theory on
space-like circle
 $R_s = \epsilon R$, $p_- = \frac{N}{R_s}$
same N

$$RM_p^2 = R_s \tilde{M}_p^2 = \text{fixed}$$

$$\text{as } \epsilon \rightarrow 0 : R_s \rightarrow 0 \text{ and } \begin{cases} \tilde{q}_s \rightarrow 0 \\ \tilde{M}_s \rightarrow \infty \end{cases}$$

\Rightarrow weak coupling II A string with $\tilde{M}_S \rightarrow \infty$:
Only D0-branes survive:

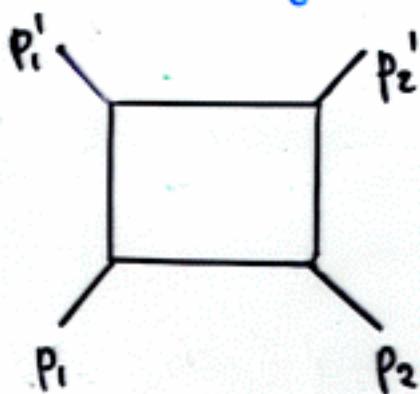
This is the matrix model.

Perfect, but:

- * The equivalence relies on an infinite Lorenz boost.
- * Can one really view a light-like circle as a limit of a space-like one?
- ** Does the light-like limit of M-theory make sense? is it = DLCQ?

Hellerman & Polchinski:

In a generic QFT this limit is divergent:



$$G_4((p_1 - p_1')_- = 0) \sim \frac{1}{\epsilon}$$

casts doubt on the limit for M-theory!

But M-theory is not a QFT:
 it contains extended objects: M2, M5
 can wrap compact dimensions,
 similar to string theory winding states

\Rightarrow study light-like limit of
 type II superstrings

2 motivations:



analogy of extended
 objects between
 string & M-theory

more important:

$$\mathcal{M}_{R_{\parallel}} = \text{IIA} \Big|_{q_S \sim R_{\parallel}}$$

If we can prove some
 property in IIA for
any $q_S \Rightarrow$ proven
 for M on any R_{\parallel}
 \Rightarrow proven for M-theory

Kinematics

space-like circle of radius $R_s \equiv R_g = \epsilon R$

$$x^3 \simeq x^3 + 2\pi R_s \quad , \quad p_3 = \frac{n}{R_s} = \frac{n}{\epsilon R}$$

Lorenz boost \downarrow with $\beta = \frac{1 - \epsilon^2/2}{1 + \epsilon^2/2}$

$$x'^3, x'^0 \rightarrow x^\mp$$

$$x^- \simeq x^- + 2\pi R \quad , \quad x^+ \simeq x^+ - \epsilon^2 \pi R$$

$$ds^2 = -2 dx^+ dx^- + (dx^i)^2$$

or with $t = x^+ + \frac{\epsilon^2}{2} x^-$

$$x^- \simeq x^- + 2\pi R \quad , \quad t \simeq t$$

$$ds^2 = -2 dt dx^- + \epsilon^2 (dx^-)^2 + (dx^i)^2$$

\Rightarrow almost light-like circle (x^-)

Truly light-like as $\epsilon \rightarrow 0$.

momenta and energies :

$$p_3 = \frac{1}{\epsilon} p_- \quad , \quad p_0 = \frac{1}{\epsilon} p_- + \epsilon p_t$$

↑

$$p_3 = \frac{n}{\epsilon R} \Leftrightarrow p_- = \frac{n}{R} \quad \text{same } n!$$

↑

also the energies
for the space-like
circle go as $\frac{1}{\epsilon}$.

⑥

The light-like limit of genus-one type II superstring amplitudes

a well-known result:

string theory compactified on S^1 radius $R\varepsilon$
at fixed α' has diverging one-loop amplitudes
as $(E \rightarrow 0)$.

but: this is for external states with
no momenta in the compact direction!

$$n_r = 0$$

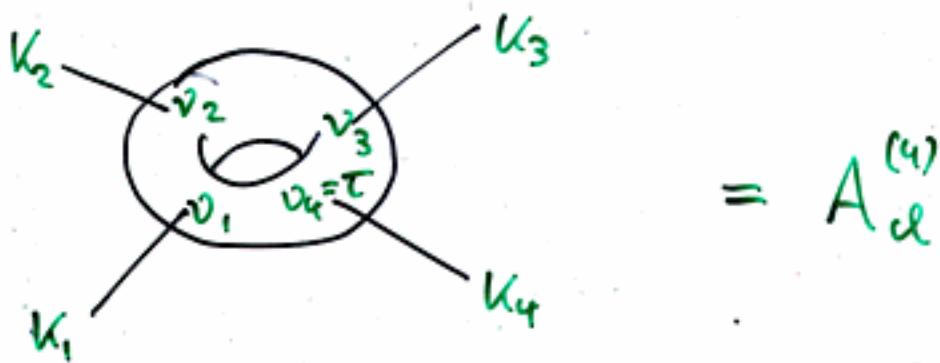
Here: interested in $n_r \neq 0$.

Dramatic change!

$$\text{amplitude} \sim \frac{1}{\varepsilon} \frac{1}{E} \times (\text{new factor for } n_r \neq 0)$$

from momentum integral as in QFT from condensation of light winding modes

→ finite as $\varepsilon \rightarrow 0$.



$$= A_{el}^{(4)}$$

$$= (\pi K)^4 K_{cl} \int d^2\tau d^2v_1 d^2v_2 d^2v_3 I$$

↑
kinematic factor as in tree amplitude

$$I = (\alpha' \Im m \tau)^{-9/2} \prod_{s>r=1}^4 \chi(v_{sr}, \tau) \prod_{s>r} \frac{\alpha' k_r \cdot k_s}{v_s - v_r} \gamma_{\text{ten-dim products}}$$

$$\gamma = \exp \left\{ i\alpha' \sum_{s>r=1}^4 \frac{n_s n_r}{(\epsilon R)^2} \left[\frac{(\Im m v_{sr})^2}{\Im m \tau} - \frac{v_{sr}^2}{2i\tau} + \frac{\bar{v}_{sr}^2}{2i\bar{\tau}} \right] \right\} S$$

$$S = \frac{1}{\epsilon R} \sum_{n,m} \exp \left\{ i\frac{\pi\tau}{2} \alpha' \left(\frac{n}{\epsilon R} - \frac{\epsilon R m}{\alpha'} + \sum_s \frac{n_s v_s}{\epsilon R \tau} \right)^2 - i\frac{\pi\bar{\tau}}{2} \alpha' \left(\frac{n}{\epsilon R} + \frac{\epsilon R m}{\alpha'} + \sum_s \frac{\bar{n}_s \bar{v}_s}{\epsilon R \bar{\tau}} \right)^2 \right\}$$

partial Poisson resummation :

$$A_{el}^{(4)} = \frac{(\pi K)^4}{\alpha'^4} K_{cl} \int \frac{d^2\tau}{(\Im m \tau)^2} \prod_{r=1}^3 \frac{d^2v_r}{\Im m \tau} \prod_{s>r} \chi(v_{sr}, \tau) \times \sum_{n,m} \frac{\alpha'}{\epsilon^2 R^2} \exp \left\{ -i\frac{\alpha'}{\epsilon^2 R^2} \frac{1}{\Im m \tau} \left[m + n\tau + \sum_{s=1}^4 n_s v_s \right]^2 \right\}$$

The $\varepsilon \rightarrow 0$ limit:

$\frac{\alpha'}{R^2}$ is dimensionless \Rightarrow rescale $\frac{1}{\varepsilon^2} \frac{\alpha'}{R^2} \rightarrow \frac{1}{\varepsilon^2}$

study $\varepsilon \rightarrow 0$ limit of

$$\frac{1}{\varepsilon^2} \exp \left\{ -\frac{\pi}{\varepsilon^2} \frac{1}{3m\tau} \left(m + n\tau + \sum_s u_s v_s \right)^2 \right\}$$

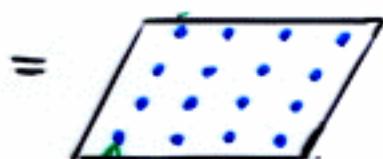
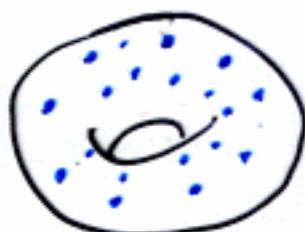
$$\downarrow \quad \varepsilon \rightarrow 0$$

$$(3m\tau) \delta^{(2)}(m + n\tau + \sum_s u_s v_s)$$

use to eliminate $\int \frac{d^2 v_3}{3m\tau}$

But also $\sum_{m,n} \dots$

$\int \frac{d^2 v_3}{3m\tau} \sum_{m,n} (3m\tau) \delta^{(2)}(\dots) \rightarrow$ sum over n_3^2
values of v_3 on a lattice



$v_3^{(0)}$ depends on v_1, v_2, T

\Rightarrow finite result as $\varepsilon \rightarrow 0$!

Possible singularities of the amplitude

divergences as $v_r \rightarrow v_s$



1) usual poles of closed string amplitudes

e.g. at $\alpha' t = 4$

2) new singularities :

$$\text{since } v_3 = v_3^{(0)}(v_1, v_2, \tau) + \frac{n+m\tau}{n_3}$$

as $v_1 \rightarrow v_2$, v_3 may be driven $v_4 \equiv \tau \simeq 0$
at $\alpha' t = 2$

just o.k. since for compact dimension
level matching is $N - \bar{N} = nm \in \mathbb{Z}$

$\Rightarrow N + \bar{N}$ is even or odd \Rightarrow new states at
 $\alpha' M^2 = 2$

Direct DLCQ computation

$\varepsilon = 0$ from the outset

$$A_{\text{DLCQ}}^{(n)} = \frac{(2\pi)^4}{\alpha'^3} K_{\text{d}} \frac{\int d^2 \tau d^2 v_1 d^2 v_2}{(2\pi)^4} \prod_{s>r} \chi(v_{sr\tau})^{\alpha' k_s \cdot k_r} S_{\text{DLCQ}}$$

$$S_{\text{DLCQ}} = \sum_{n,m} \exp \left[2\pi p_+ \sum_n \text{Im}(n\tau + \sum_s n_s v_s) - 2\pi i m \text{Re}(n\tau + \sum_s n_s v_s) \right]$$

$p \sim p_+$ can fully divergent because no
Wick rotation $p_0 \rightarrow ip_0$

How to do a Wick rotation in light cone coordinates?

just try $p_+ \rightarrow ip_+$:

$$\begin{aligned} S_{\text{DLCQ}} &\rightarrow \sum_n \delta(\text{Im}(n\tau + \sum_s n_s v_s)) \sum_m \delta(\tilde{m} + \text{Re}(n\tau + \sum_s n_s v_s)) \\ &= \sum_{n,m} \delta^{(2)}(m + n\tau + \sum_s n_s v_s) \end{aligned}$$

just as derived before from $\varepsilon \rightarrow 0$ limit!

\Rightarrow The light-like limit of the superstring 4-point one-loop amplitude ($\varepsilon \rightarrow 0$) exists and is finite and coincides with the naive ("Wick"-rotated) DLCQ computation!

What does this tell us about the light-like limit of M-theory?

* By analogy: it's encouraging

* But better:

Suppose we prove:

A) IIA superstring theory with q_s on $S^1_{R_g}$
has a well-defined limit as $R_g \rightarrow 0$

This is equivalent to and \equiv DLCQ of IIA

B) M-theory on (space-like) $S^1_{R_{11} = \sqrt{g_s} q_s} \times S^1_{R_g}$

has a well-defined limit as $R_g \rightarrow 0$

and \equiv DLCQ of $M/S^1_{R_g}$

If we can prove A) as uniform convergence
in q_s , then we have shown

\tilde{B}) M-theory on a space-like circle $S^1_{R_g}$

has a well-defined limit as $R_g \rightarrow 0$

and \equiv DLCQ of M-theory

\rightarrow requires perturbative and
non-perturbative evidence.

Perturbative evidence

- so far:



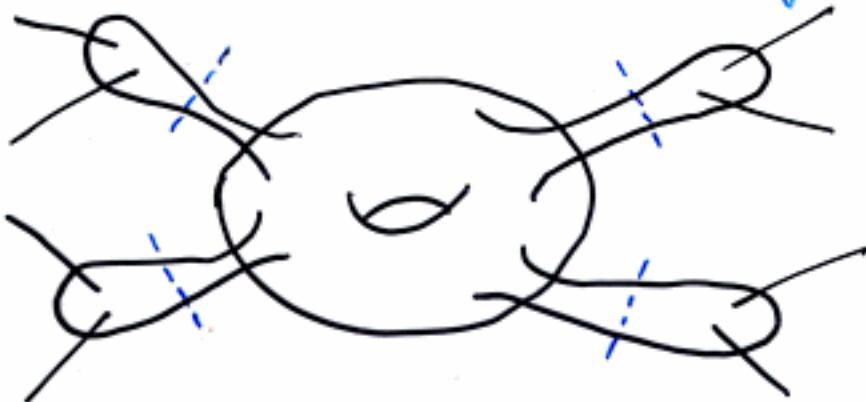
genus one,
4 massless (in 10 dim)
external states.

- obvious generalisation to genus-one N-point
(massless ext. states):

$$\frac{1}{\epsilon^2} \exp \left(-\frac{\pi}{\epsilon^2} \frac{1}{3m\tau} \left(m + N\tau + \sum_{s=1}^4 n_s v_s \right)^2 \right)$$

$$\rightarrow \frac{1}{\epsilon^2} \exp \left(-\frac{\pi}{\epsilon^2} \frac{1}{3m\tau} \left(m + N\tau + \sum_{s=1}^N n_s v_s \right)^2 \right)$$

- Then massive external states from factorisation



Higher genus:

genus g : a factor $\frac{1}{R_s^2} = \frac{1}{\epsilon^2 R^2}$ for each handle

from momentum measure and

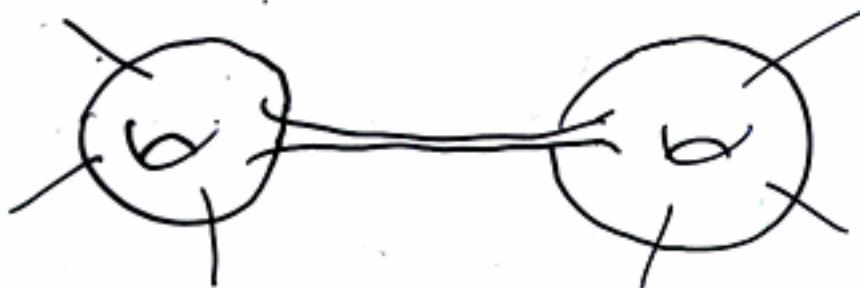
Poisson resumming of winding modes

\Rightarrow total $\frac{1}{(\epsilon R)^{2g}}$ as expected from T-duality

This $\frac{1}{\epsilon^{2g}}$ must combine with some

$$\prod_{i=1}^g \exp\left(-\frac{\pi i}{\epsilon^2} [\dots]?\right) \text{ to yield a product of } g \text{ } \delta\text{-fcts.}$$

simple example:



$$\underbrace{\text{S}(\text{modular parameter})}_{\text{(one-loop)}} \times \underbrace{\text{S}(\text{modular parameter})}_{\text{(one-loop)}}$$

finite limit as
 $\epsilon \rightarrow 0$:
 discretises
 αV_r

finite limit
 as $\epsilon \rightarrow 0$:
 discretises e.g.
 insertion point
 of the long tube

\Rightarrow finite as $\epsilon \rightarrow 0$ as well !

Non-perturbative evidence

Non-perturbative states : D-branes

finite DLCQ energies $p_t \approx p_+ = p^-$ correspond to energies for space-like compactification on $\mathbb{E}R$ that scale as $\frac{1}{\epsilon}$.

- D0 branes : before compactification of X^9 : finite mass T_0 . Compactification of X^9 :
 $D0 \rightarrow D1$ wrapped on dual circle $\hat{R}_9 = \frac{\alpha'}{R_9} = \frac{\alpha'}{\mathbb{E}R}$
 \Rightarrow energy $T_1 2\pi \hat{R}_9 = \frac{2\pi T_0 \alpha'}{\mathbb{E}R} \sim \frac{1}{\epsilon}$

- D2 branes : before compactif. of X^9 , to get finite energy, compactify e.g. X^7, X^8 and wrap D2 around this T^2 :
finite mass $T_2 (2\pi)^2 R_7 R_8$

Compactification of X^9 : $D2 \rightarrow$ wrapped D3
with energy $T_3 (2\pi)^3 R_7 R_8 \frac{\alpha'}{\mathbb{E}R} \sim \frac{1}{\epsilon}$

instead start with D2 in 8-g direction.

infinite energy since X^9 non compact.

compactify X^9 : $D2 \rightarrow D1$ in the 8-direction
 \Rightarrow energy indep of ϵ , no $\frac{1}{\epsilon}$ scaling.

Exactly those D-branes that had a finite energy before compactifying X^3 on $R_g = \epsilon R$, have an energy scaling as $\frac{1}{\epsilon}$ as required once X^3 is compactified.
 \Rightarrow finite DLCQ energy!

Non-perturbative amplitudes

Difficult, but one can extract some information from known results on low-energy effective actions,
e.g. R^4 -couplings

Note: we are interested in high energies $\sim \frac{1}{\epsilon}$
but still: the low-energy limit may give some hint on the amplitudes at arbitrary energy.

example: genus-one 4-point amplitude $A_{cl}^{(4)}$
before taking $\epsilon \rightarrow 0$:

$$A_{cl}^{(4)} \sim \dots \frac{1}{R_g^2} \exp \left(-\frac{\pi i}{R_g^2} \frac{1}{3m^2} (m + n + \sum_s n_s v_s)^2 \right)$$

low-energy limit: $n_s = 0$ and then:

$$A_{cl}^{(4)} \underset{R_g \rightarrow 0}{\sim} \frac{1}{R_g^2}$$

* Turn the argument around:

The $\frac{1}{R_g^2} \sim \frac{1}{\epsilon^2}$ divergence of the low-energy amplitude is just what is needed to get a finite result for $u_s \neq 0$.

R^4 -coupling: BPS protected \rightarrow only tree, one-loop and non-perturbative D-instanton contributions:

$$\begin{aligned} \frac{1}{R_g} f_{D=3}^{IIA} &= 2 \zeta(3) e^{-2\phi} + \frac{2\pi^2 \alpha'}{3 R_g^2} + \\ &+ \frac{4\pi \sqrt{\alpha'}}{R_g} e^{-\phi} \sum_{m,n \neq 0} \left| \frac{m}{n} \right| K_1 \left(2\pi |mn| R_g e^{-\phi} \right) e^{2\pi i mn \phi} \end{aligned}$$

$$\xrightarrow{R_g \rightarrow 0} \text{tree} + \frac{\alpha'}{R_g^2} \left(\frac{2\pi^2}{3} + 2 \sum_N e^{2\pi i N \phi} - \sum_{n \in N} \frac{1}{n^2} \right)$$

\uparrow
not uniform
for all N

\uparrow
same overall R_g -dependence!

\rightarrow encouraging

Conclusions

- * 4-pt one-loop amplitude of IIA
on a circle $R_g = \epsilon R$ with $p_g^{(r)} = \frac{n^r}{\epsilon R} \neq 0$
has a finite limit as $\epsilon \rightarrow 0$
which coincides with the "naive" DLCQ result.
- * argued: generalises to all N-point genus g amplitudes for IIA.
- * non-perturbative evidence:
D-branes of finite energy before compactif.
survive the light-like limit
- * some evidence the non-perturbative amplitudes also have a finite limit

\Rightarrow IIA string theory has a well-defined
 $\epsilon \rightarrow 0$ limit \equiv DLCQ for all g_s

If this limit is uniform

\Rightarrow M-theory has a well-defined light-like
($\epsilon \rightarrow 0$) limit \equiv DLCQ of M-theory.