Branes, Strings
and
Noncommutative Geometry

Strings '98
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Based on work with A. Connes, A. Schwarz, C. Hull, C. Norvitt and the papers

M. Kontsevich, q-alg/9709040
M. Rieffel, quant-ph/9712009
N. Nekrasov and A. Schwarz,
hep-th/9802068
P.-M. Ho and Y.-S. Wu,
hep-th/9801147
H. Garcia-Compean, hep-th/9804188
We all know and use differential geometry and basic topology.

In recent developments in duality, algebraic geometry has played an important role. Why?

- Supersymmetry 2) complex geometry
  special geometry
  period integrals, heterotic theory...

- Classification and resolution of singularities

GA GA

... Can always treat questions about large class of complex manifolds in an algebraic framework, and translate topological (e.g. cohomology, intersection theory) and analytic (e.g. moduli spaces of matrices) questions into algebraic analogs.

\( C(8) \rightarrow \mathbb{R}^3 / W(\mathbb{R}) = 0 \)
The defining characteristic of noncommutative geometry - coordinate rings are replaced by general operator algebras - is too broad. It is hard to make any general picture of a "noncommutative space."

Nevertheless, many similar examples exist, and tools with a strong analogy to the familiar tools of topology and differential geometry.

<table>
<thead>
<tr>
<th>topological manifold</th>
<th>C* algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>cohomology</td>
<td>?</td>
</tr>
<tr>
<td>K-theory</td>
<td></td>
</tr>
<tr>
<td>characteristic classes</td>
<td></td>
</tr>
<tr>
<td>vector bundle</td>
<td></td>
</tr>
<tr>
<td>metric</td>
<td></td>
</tr>
<tr>
<td>cyclic cohomology module (various constructions)</td>
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</tr>
</tbody>
</table>

In particular, gauge theory with "fields taking values in the algebra" can be defined.
Why is this relevant to string/M-theory?

Main application discovered so far is to brane theories on torus with B field.

As we will argue here this generalizes to branes on orbifolds with discrete torsion, and likely to general spaces with B field, asymmetric orbifolds.

Another potential application is gauge theories on singular quotients.

Finally, we are seeing new classes of preferred non-local theories in which short distance singularities are resolved in new ways.
manifold $M \rightarrow$ algebra of functions on $M$

\[
f(x) \rightarrow f(x_1, f(x-x')_1, \ldots, f(x_2), 0, f(x_2), 0, f(x_2), 0, \ldots)
\]

Quotienting $M$ by an equivalence relation $x_0 \sim x'_0$.

**Usual:** $f(x_0) = f(x'_0)$.

**New:** enlarge the algebra.

allow $f(x)f(x-x_0)f(y-x_0')f(x-x_0')f(y-x_0')$

$\begin{pmatrix}
  f & f & 0 \\
  f & f & 0 \\
  0 & 0 & f
\end{pmatrix}$

algebra $\oplus M_{bc}(\mathbb{C})$. 

For some questions, these two algebras are interchangeable.

Example: What is large $N$ limit of $N \times N$ matrices with elements from $A$ (and with suitable limit, i.e. $A \otimes K$ compact operators).

\[ \bigoplus M_{k_i}(C) \otimes H_N(C) \cong \bigoplus M_{Nk_i}(C) \]

are $M_{Nk_i} \to K$.

"Morita equivalence"

When usual quotient is well defined, the two definitions agree in this sense. But new definition is more general.
Example: $S^1 / x \rightarrow x + B$.

Enlarged functions

$$f(x) = \sum_n f_n e^{2\pi i n x}$$

to

$$\sum_{mn} f_{mn} e^{2\pi i n x} \delta(x - y - mB)$$

let $U = e^{2\pi i x}$, $V = \delta(x - y - B)$,

$$UV = e^{2\pi i cB} VU$$

"noncommutative torus" - well defined for any $B$.

For $B = p/q$ rational, this reduces to $M_q \otimes \text{fns on } S^1, \text{rad.}$

$V^q = 1$

let $U = e^{2\pi i x}$

By writing

$$S^1 = \mathbb{R} / x \rightarrow x + 1$$

one can see action of $SL(2, \mathbb{R})$.

$\mathbb{R} / \{ x + x + aB + b, x + cB + d \}$
Note that this representation (module) of the algebra \( UV = e^{2i\pi} VU \) is NOT unique - the relation \( V^2 = 1 \) for rational case was special to this construction, we could have had

- \( U^2 = V^2 = 1 \)
- Finite \( 2 \times 2 \) matrix algebra \( M_2 \)
- "zero dimensional"
- \( V^2 = 1, \ U^m \) distinct "one dimensional"
- \( U^n V^n \) all distinct "two dimensional"

These are distinct representations which lead to different field theories. Further distinctions (also \( \pi \) irrational case) correspond to distinct vector bundles over torus.
Gauge theory on NC torus is described by the standard action
\[ S = S + tr \, F^2 + i \bar{X} \partial X \]
\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + c A_\mu A_\nu - c A_\nu A_\mu \]
where the fields take values in the algebra. We must also define derivative and integral.
\[ \int_0^\infty U^m V^n = \delta_{m,0} \delta_{n,0} \]
\[ \tau_0 \left( U^m V^n \right) = i \pi \frac{U^m V^n}{\pi m} \text{ for } \delta = 2 \]
\[ = i \sin \frac{\pi m}{\delta} U^m V^n \text{ for } \delta = 0, 1 \]
\[ = \left[ V, U^m V^n \right] \]
We do not take \( \theta \to 0 \) or \( q \to \infty \).

Result is
- matrix integral
- matrix DQH on circle
- (matrix) 2D YM in quenched topological sector.
$SL_2(\mathbb{R})$

$S^1/\mathbb{Z} \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \cong \mathbb{R}/(x \rightarrow ax+\delta a \beta + b, x \rightarrow x+c \beta + d \beta)$

$(a \ b \ c \ d) \in SL_2(\mathbb{R})$.

This $SL_2(\mathbb{R})$ acts as

$\beta \rightarrow \frac{a \beta + b}{c \beta + d}$

and is in addition to the $SL_2(\mathbb{R})$ acting on the metric as

$ds^2 = \sqrt{dx^1 + \tau^2 dx^2}$

$\tau \rightarrow \frac{a' \tau + b}{c' \tau + d}$

It should be thought of as the zero volume limit of the second $SL_2(\mathbb{R})$ of $T$-duality acting on $B+\sqrt{\kappa} \tau$. 
Choice of terms was not essential:

- can take volume $\to \infty$.
  \[ \sum \mathcal{L} \rightarrow \infty \]

- can take other base spaces in principle (more later).

- can take $d > 2$ (commuting or noncommuting).

A large class of gauge theories with 16 supersymmetries?

- Are they really new?

- Are they consistent + unitary?

- Do they play a role in string theory?
We can analyze them with more familiar physical tools by writing the NC multiplication as an operation on conventional functions (symbols):

\[ u^m v^n \rightarrow e^{i m \sigma_1 + i n \sigma_2} \]

\[ f g \rightarrow (f \ast g)(\sigma) \]

\[ = \exp \left( \pi i B E \int_{\sigma-\delta}^{\sigma+\delta} \frac{1}{2} \cdot \frac{f(\sigma-\delta)g(\sigma+\delta)}{\delta^2} \right) \]

\[ f \ast g - g \ast f = \{ f, g \}_{\text{Moyal}} \]

\[ = 2 \pi i B E f, g \mathbb{P}_{\text{Poisson}} + O(\delta^3) \].

The YM action becomes

\[ S = S + \text{tr} \ F^2 \]

\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \varepsilon_{\mu \nu} A_\rho, A_\rho \mathbb{P}_{\text{Moyal}} \]

higher derivative (even non-local) but if \( \varepsilon^{0 \hat{v}} = 0 \) clearly sensible.
Gauge fixing procedure and Feynman rules are as are for conventional YM with replacement

\[ f_{abc} \rightarrow \sin 2\pi i B \sum_{k_1 k_2} \delta_{k_1 k_2} \]

Tree level scattering is modified:

\[ A_4 = \sin k_1 \times k_2 \frac{\sin k_1 \times k_3 + \sin k_1 \times k_3}{\sin k_1 \times k_4} \]

so barring non-local field redefinition, these theories are now. Clear
modification at small k is noncompact space.

Loop amplitudes are similar to usual YM/SYM. (Even 't Hooft theory is
asymptotically free.)

\[ \mu_1 \int \frac{d^{4}p}{(2\pi)^{4}} \frac{1}{p^{2}(p + k_{1})^{2}(p + k_{2})^{2}(p - k_{1})^{2}} \]

\[ \times \frac{\sin(p \cdot k_{1}, k_{2}) \times k_{3}}{k_{1} \cdot k_{4}} \times \frac{\sin(p \cdot k_{1}, k_{2}) \times k_{3}}{k_{1} \cdot k_{4}} \]

These factors present or improve convergence.
On the other hand, for rational $\mathbb{R}$ on torus, these theories are exactly equivalent to conventional YM/SYM.

$B = P_f$ on radius $R \cong U(2)$ on $R/2$.

The spectrum at radius $R$ can be precisely reproduced by postulating Wilson lines $e^{iA_1} = U$, $e^{iA_2} = V$.

(e.g. see Ko 9903106, Asami 9804120...)

But nothing forces the Wilson lines to take these values — we must integrate over zero modes of $A$.

This fits with $5(2|\mathbb{R})$.

Is theory new/known for $\theta$ irrational? Still open.
Relation to branes.

Dp-brane winding modes in presence of $B$ are naturally thought of as noncommuting variables.

\[
A(w_1, w_2) A(w'_1, w'_2) = e^{2 \pi i B(w, w'_2 - w_2 w'_1)} A(w'_1) A(w').
\]

These modes give pure gauge theory in zero volume limit.

Can also do T-duality to see relation to quotient construction:

\[
\begin{array}{c}
\beta + i \nu \\
\nu \rightarrow 0
\end{array}
\rightarrow
\begin{array}{c}
\tau' = \beta + e \nu \\
\sim x + 1 \\
\sim x + B + e \nu \\
\rightarrow x + B.
\end{array}
\]
This argument is more general than torus. Consider discrete torsion on orbifold:

\[ \gamma \in \mathbb{Z} \]

\[ e^{i\pi \phi / g} = e^{i\pi \phi / g} \]

\[ e^{i\pi \phi / g} \] is (multiplicative) Z-cycle.

We must use the same cocycle in describing interactions of open strings twisted by g with twisted by h. (e.g., Hotwo 9801149)
This is reflected in standard 0-brane or orbifold construction by taking a projective representation of the point group.

e.g. $C^3 \times \mathbb{R}_2 \times \mathbb{R}_2$

\[
\begin{array}{ccc}
\varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\
g & - & + \\
h & - & + \\
gh & + & - \\
\end{array}
\]

can take $gh = +_g h$

$gh = -_g h$

$Y(g)\varepsilon_1^{-1}\varepsilon_1 Y(g) = g \varepsilon_1$

has some $\mathbb{Z}^3 \sim \mathbb{R}(g)$ representation space twisted by $g$. 
These models were studied in Vafa and Witten (9409188).

For fixed planes $z_i^n - z_i^i$ for one $i$,

$$gh = hg \Rightarrow \text{blow up to } \mathbb{P}^1$$

$$h_{1,1} = 3 + 4g$$

$$h_{2,1} = 3$$

$$gh = -hg \quad h_{2,1} = 3 + 4g \quad h_{1,1} = 3$$

Consider local behavior near one fixed point, $z_i = 0$.

Invariants under $g$ and $h$ are

$$u_i = (2z_i)^2$$

$$y = 2z_1 z_2 z_3$$

$$y^2 = u_1 u_2 u_3$$

A $g$ complex structure perturbation which remove codim 2 singularities, leaving fixed point at origin:

$$y^2 = u_1 u_2 u_3 + \varepsilon (u_1^2 + u_2^2 + u_3^2)$$.
D-branes near $C^3/Z_2 \times Z_2$

singularity w. discrete torsion.

Regular representation w. $gh^2 = hg$
is
\[
\begin{pmatrix}
1 \\
gh \\
gh
\end{pmatrix} =
\begin{pmatrix}
gh \\
h \\
gh
\end{pmatrix} =
\begin{pmatrix}
h \\
gh \\
1
\end{pmatrix}
\]
equivalent to
\[
\gamma(g) = 1 \otimes 0^2, \quad \gamma(h) = 2, \otimes 0^1.
\]

\[
\gamma^{-1}(A) \gamma = A \quad \text{is} \quad U(2) \quad \text{generated by}
\]
\[
1, \quad 2_g \otimes 0^2, \quad 2_h \otimes 1, \quad 2_h \otimes 0^2.
\]

D-brane at this singularity has enhanced $U(2)$ gauge w.r.

\[
\gamma^{-1}(g) 2^c \gamma(g) = g \otimes 2^c
\]
solution = 1 singlet + 1 triplet for each $2^c$.

call these $2^c$ and $2^c$.

Substitute solns w.r. $N^2 = 2$. 

Resulting theory has Higgs branch $C^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ and should admit superpotential deformations (work in progress $W + \gamma e^{2i}$)

blowing up fixed planes. No FI terms for SU(3) gauge group. The result is a conifold singularity which cannot be resolved due to discrete torsion.

Discrete torsion has no conventional geometric interpretation, but it fits naturally in the framework of NC geometry.
A visit to AdS.

Many interesting curved backgrounds can be described as quotients of AdS by discrete groups.

\[ \Gamma = \text{SO}(d, 2). \]

It is natural to conjecture (Horowitz and Marolf, 2005207) that string theory on these spaces corresponds to gauge theory on the quotient of the boundary \( \partial \Gamma \).

However, these quotients are typically not smooth manifolds. e.g. \( g \geq 1 \) Riemann surface \( \Sigma \) as

\[ \Sigma = \mathbb{H} / \Gamma \]

\( \mathbb{H} \) = 'upper half plane' AdS\(_2\)

\( \partial \Sigma \equiv S \)

\( \Gamma \) is generated by \( 2g \) boosts \( \sigma \).
This is exactly the type of quotient which is well-defined in the NC sense. We can define a quotient NC gauge theory by choosing a representation $r(x)$ of $\Gamma$ (e.g., regular) and projecting

$$r(x)^{-1} A_{\mu}(x) r(x) = \left( \frac{\partial x^*}{\partial (x(x^*))^\mu} \right) A_{\nu}(x^*) .$$

a classical symmetry of 3+1 g.t.

This algebra can also be defined using

$$H = \mathbb{R} \backslash \mathcal{C}(\mathbb{C} \times \mathbb{R})$$

$$(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}) \setminus (\begin{smallmatrix} i & 1 \\ 0 & 0 \end{smallmatrix})$$

i.e., foliation of 3-manifold.
Open problems:
- q.t. on more general curved spaces. Implicit def'n exists in terms of 'spectral triple', but need more explicit.
Deformation quantization:

\[ f \star g = fg + it \varepsilon f, g \phi_0 + \ldots \]

(is in fact only way to deform function algebras). Leads to (gauge equiv. class of) algebra for any Poisson str. Need to also develop appropriate ideas of metric + connection.
Given $\mathcal{N}=\frac{4}{5}$ on general curved space,

- what are conditions on $B$ for sensible g.s.t.? for susy? $B \in H^{1,1}$?

- does it come from branes?

- what is duality group?

e.g. can we realize $O(4,20,\mathbb{R})$ on $K3$ in explicit way?

- explicit Nahm duality

- instanton moduli spaces

(Nahmsov + Schwarz 9802068).