Branes, Strings and Noncommutative Geometry

Strings '98

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Based on work with A. Connes, A. Schwarz, C. Hull, B. Morowitz and the papers

M. Kontsevich, q-alg/9709040
M. Rieffel, quant-ph/9712009
N. Nekrasov and A. Schwarz, hep-th/9802068
P.-M. Ho and Y.-S. Wu, hep-th/9801147
H. Garcia-Compean, hep-th/9804188

We all know and use differential geometry and basic topology.

In recent developments in duality, algebraic geometry has played an important role. Why?

- Supersymmetry $\Rightarrow$ complex geometry
  special geometry
  period integrals, Kähler theory...
- Classification and resolution of singularities
- GAGA

...

Can always treat questions about a large class of complex manifolds in an algebraic framework, and translate topological (eg cohomology, intersection theory) and analytic (eg moduli spaces of metrics) questions into algebraic analogs.

$$CY \leftrightarrow \{z^i\} / W'(z) = 0$$

The defining characteristic of **noncommutative geometry** - coordinate rings are replaced by general operator algebras - is too broad. It is hard to make any general picture of a "noncommutative space."

Nevertheless many similar examples exist, and tools with a strong analogy to the familiar tools of topology and differential geometry.

topological manifold	c	C^* algebra
cohomology		?
K-theory		K theory
characteristic classes		cyclic cohomology
vector bundle		module
metric		(various constructions)

In particular, **gauge theory** with "fields taking values in the algebra" can be defined.

Why is this relevant to string/M/... - theory?

Main application discovered so far is to brane theories on torus with B field.

As we will argue here this generalises to branes on orbifolds with discrete torsion, and likely to general spaces with B field, asymmetric orbifolds.

Another potential application is gauge theories on singular quotients.

Finally, we are seeing new classes of preferred non-local theories in which short distance singularities are resolved in new ways.

manifold $M \longleftrightarrow$ algebra functions on M

$f(x)$ $\qquad$ $f(x)\,\delta(x-y)$

$$\begin{pmatrix} f(x_1) & & & 0 \\ & f(x_2) & & \\ & & f(x_3) & \\ 0 & & & \ddots \end{pmatrix}$$

Quotienting M by an equivalence relation $x_0 \sim x_0'$.

Usual: $\quad f(x_0) = f(x_0')$.

New: enlarge the algebra.

allow $\quad f(x)\,\delta(x-x_0)\,\delta(y-x_0')$

$\qquad\qquad f(x)\,\delta(x-x_0')\,\delta(y-x_0)$

$$\begin{array}{c} x_0 \\ x_0' \\ x_1 \\ \vdots \end{array} \begin{pmatrix} f & f & 0 \\ f & f & 0 \\ 0 & 0 & f \end{pmatrix}$$

algebra $\bigoplus_i M_{k_i}(\mathbb{C})$.

For <u>some</u> questions, these two algebras are interchangeable.

Example: What is large N limit of $N \times N$ matrices with elements from $\mathcal{A}$ (and with suitable limit, i.e. $\mathcal{A} \otimes K$ compact operators).

$$\bigoplus M_{k_i}(\mathbb{C}) \otimes M_N(\mathbb{C}) \cong \bigoplus M_{Nk_i}(\mathbb{C})$$

are $M_{Nk_i} \to K$.

"Morita equivalence"

When usual quotient is well defined, the two definitions agree in this sense. But new definition is more general.

Example: $S^1 / x \to x + B$.

Enlarge functions

$$f(x) = \sum_n f_n e^{2\pi i n x}$$

to

$$\sum_{mn} f_{mn} e^{2\pi i n x} \delta(x - y - mB)$$

let $U = e^{2\pi i x}$, $V = \delta(x - y - B)$,

$$UV = e^{2\pi i B} VU$$

"noncommutative torus" - well defined for any B.

For $B = p/q$ rational, this reduces to $M_q \otimes$ fns on S^1 rad. R/q.

$V^q = 1$

let $U^q = e^{2\pi i \sigma}$ ...

By writing

$$S^1 = \mathbb{R} / x \to x + 1$$

one can see action of $SL(2, \mathbb{Z})$.

$$\mathbb{R} / \{x \to x + aB + b, x + cB + d\}$$

Note that this representation (module) of the algebra $UV = e^{2\pi i \beta} VU$ is NOT unique - the relation $V^q = 1$ for rational case was special to this construction, we could have had

- $U^q = V^q = 1$ finite dim matrix algebra M_q. "zero dimensional"
- $V^q = 1$, U^n distinct "one dimensional"
- $U^m V^n$ All distinct "two dimensional".

These are distinct representations which lead to different field theories. Further distinctions (also in β irrational case) correspond to distinct vector bundles over torus.

Gauge theory on NC torus is described by the standard action

$$S = \int \mathrm{tr}\, F^2 + i\bar{\chi} \not{D} \chi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + iA_\mu A_\nu - iA_\nu A_\mu$$

where the fields take values in the algebra. We must also define derivative + integral.

$$\int U^m V^n = \delta_{m,0}\, \delta_{n,0}$$

$$\partial_\nu (U^m V^n) = i_m U^m V^n \quad \text{for "}d=2\text{"}$$

$$= i \sin\frac{\pi m}{q} U^m V^n \quad \text{for } d=0,1$$

$$= [V, U^m V^n]$$

We do <u>not</u> take $\theta \to 0$ or $q \to \infty$.

Result is

- matrix integral $\quad d=0$
- matrix QM on circle $\quad d=1$
- (matrix) 2D YM in given topological sector. $\quad d=2$

$\vdots \qquad \vdots$

$SL_2(\mathbb{Z})$

$$S^1/x \to x+B \approx \mathbb{R}/\{x \to x+1, x \to x+B\}$$

$$\approx \mathbb{R}/\{x \to ax + aB + b,\ x \to x + cB + d\}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

This $SL_2(\mathbb{Z})$ acts as

$$B \to \frac{aB+b}{cB+d}$$

and is in addition to the $SL_2(\mathbb{Z})$ acting on the metric as

$$ds^2 = V|dx^1 + \tau dx^2|$$

$$\tau \to \frac{a'\tau + b}{c'\tau + d}$$

It should be thought of as the zero volume limit of the second $SL_2(\mathbb{Z})$ of T-duality, acting on

$$B + iV/\alpha'.$$

Choice of torus was not essential:

- can take volume $\to \infty$.

$$[x^i, x^j] = \varepsilon \omega^{ij}$$

- can take other base spaces in principle (more later)

- can take $d > 2$ (commuting or noncommuting).

A large class of gauge theories with 16 supersymmetries?

- Are they really new?
- Are they consistent + unitary?
- Do they play a role in string theory?

We can analyze them with more familiar physical tools by writing the NC multiplication as an operation on conventional functions (symbols):

$$U^m V^n \longrightarrow e^{im\sigma_1 + in\sigma_2}$$

$$fg \to (f * g)(\sigma)$$

$$= \exp\, i\pi B \varepsilon^{ij} \frac{\partial}{\partial \sigma^i} \frac{\partial}{\partial \sigma'^j} f(\sigma) g(\sigma') \Big|_{\sigma' = \sigma}$$

$$f * g - g * f = \{f, g\}_{\text{Moyal}}$$

$$= 2\pi i B \{f, g\}_{\text{Poisson}} + \mathcal{O}(B^3).$$

The YM action becomes

$$S = \int \operatorname{tr} F^2$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \{A_\mu, A_\nu\}_{\text{Moyal}}$$

higher derivative (even non-local)

but if $\varepsilon^{0i} = 0$ clearly sensible.

Gauge fixing procedure and Feynman rules as are for conventional YM with replacement

$$f_{abc}(\zeta_1 \cdot k_2 \zeta_2 \cdot \zeta_3 + \dots)$$

$$f_{abc} \to \sin 2\pi i B \, \varepsilon^{\mu\nu} k_{1\mu} k_{2\nu}$$

Tree level scattering is modified:

$$A_4 = \frac{\sin k_1 \times k_4 \, \sin k_2 \times k_3}{st} + \frac{\sin k_1 \times k_3 \, \sin k_2 \times k_4}{su}$$

so barring non-local field redefinitions these theories are new. Clear modification at small k in noncompact space.

Loop amplitudes are similar to usual YM/SYM. (Even 'U(1)' theory is asymptotically free).

(SYM)

$$= K \int \frac{d^D p}{p^2 (p+k_1)^2 (p+k_1+k_2)^2 (p-k_4)^2} \sin p \times k_1 \, \sin(p+(k_1+k_2)) \times k_2)$$
$$\times \sin(p+k_1+k_2) \times k_3$$
$$\times \sin p \times k_4$$

These factors preserve or improve convergence.

On the other hand, for rational B on torus, these theories are exactly equivalent to conventional YM/SYM.

$$B = P/q \text{ on radius } R \cong U(q) \text{ on } R/q.$$

The spectrum at radius R can be precisely reproduced by postulating Wilson lines $e^{iA_1} = U$, $e^{iA_2} = V$.
(eg see Ko 9803166, Bigatti 9804120 ...)
But nothing forces the Wilson lines to take these values — we must integrate over zero modes of A.

This fits with $\delta(z(x))$.

Is theory new/sensible for θ irrational? Still open.

Relation to branes.

Dp-brane winding modes in presence of B are naturally thought of as noncommuting variables.

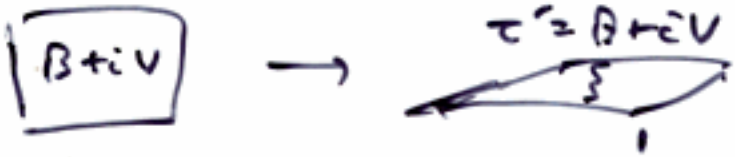
2 gets $e^{2\pi i B}$

so $A(w_1, w_2)\, A(w_1', w_2')$

$$= e^{2\pi i B(w_1 w_2' - w_2 w_1')}\, A(w_2')\, A(w').$$

These modes give pure gauge theory in zero volume limit.

Can also use T-duality to see relation to quotient construction:



$x \sim x+1$

$\sim x + B + iV$

$\to x + B.$

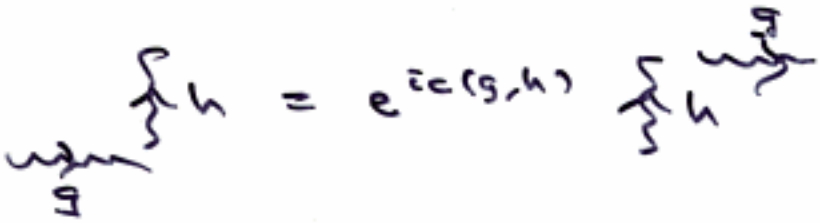
This argument is more general than torus. Consider discrete torsion on orbifold:

genus 1 $\sum_{g,h}$ 

$e^{ic(g,h)}$

$e^{ic(g,h)}$ is (multiplicative) 2-cocycle.

We must use the same cocycle in describing interactions of open strings twisted by g with twisted by h. (e.g. Ho+Wu 9801149)



This is reflected in standard D-brane on orbifold construction by taking a projective representation of the point group.

e.g. $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$

	z_1	z_2	z_3
g	$-$	$-$	$+$
h	$-$	$+$	$-$
gh	$+$	$-$	$-$

can take $gh = +hg$
$gh = -hg$

$$\gamma(g)^{-1} z^i \gamma(g) = g z^i$$

has sol'n $z^i \sim \gamma(g)$ representing string twisted by g.

These models were studied in Vafa + Witten (9409188).

48 fixed planes $z^i \sim -z^i$ for one i

$gh = hg \Rightarrow$ blow up to $\mathbb{P}^1$

$h_{1,1} = 3 + 48$

$h_{2,1} = 3$

$gh = -hg \qquad h_{2,1} = 3 + 48 \qquad h_{1,1} = 3$

Consider local behavior near one fixed point, $z^i = 0$.

Invariants under g and h are

$u_i = (z_i)^2 \qquad y = z_1 z_2 z_3$

$$y^2 = u_1 u_2 u_3$$

$\exists$ 48 complex structure perturbations which remove codim 2 singularities, leaving fixed point at origin:

$$y^2 = u_1 u_2 u_3 + \varepsilon(u_1^2 + u_2^2 + u_3^2).$$

D-branes near $\mathbb{C}^3/\mathbb{Z}_2\times\mathbb{Z}_2$ singularity w. discrete torsion.

Regular representation w. $gh^2=hg$ is

$$g\begin{pmatrix}1\\g\\h\\gh\end{pmatrix}=\begin{pmatrix}g\\1\\gh\\h\end{pmatrix}\qquad h\begin{pmatrix}\ \\ \ \\ \ \\ \ \end{pmatrix}=\begin{pmatrix}h\\hg\\1\\-g\end{pmatrix}$$

equivalent to

$$\gamma(g)=1\otimes\sigma_3\qquad \gamma(h)=\tau_1\otimes\sigma_1$$

$\gamma^{-1}A\gamma=A$ is $U(2)$ generated by $1,\ \tau_3\otimes\sigma_3,\ \tau_1\otimes 1,\ i\tau_2\otimes\sigma_2$.

D-brane at this singularity has enhanced $U(2)$ gauge sym.

$$\gamma^{-1}(g)\,z^i\,\gamma(g)=g^i{}_j\,z^j$$

solution = 1 singlet + 1 triplet for each z^i.

call these z^i and $\tilde{z}^i$.

Substitute sol'ns into $N=4$ L.

Resulting theory has Higgs branch $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ and should admit superpotential deformations (work in progress)

$$W + \xi_i z^i$$

blowing up fixed planes. No FI terms for SU(2) gauge group. The result is a conifold singularity which cannot be resolved due to discrete torsion.

Discrete torsion has no conventional geometric interpretation, but it fits naturally in the framework of NC geometry.

A visit to AdS.

(w. G. Horowitz hep-th/0001001)

Many interesting curved backgrounds can be described as quotients of AdS by discrete groups.

$$\Gamma \subset SO(d,2).$$

It is natural to conjecture (Horowitz + Marolf, 9805207) that string theory on these spaces corresponds to gauge theory on the quotient of the boundary B/Γ.

However these quotients are typically NOT smooth manifolds.

e.g. $g>1$ Riemann surface Σ as

$$\Sigma = H/\Gamma$$

H = 'upper half plane' AdS_2

$\partial H \cong S^1$

Γ is generated by $2g$ boosts γ_i

$$z \to \frac{az+b}{b^* z + a^*} \quad \in U(1,1)$$

has two fixed points, one attractive.

$\Rightarrow$ the set of attractive fixed points (for some $\gamma \in \Gamma$) is dense in S^1.

This is exactly the type of quotient which is well-defined in the NC sense. We can define a quotient NC gauge theory by choosing a representation $r(\gamma)$ of Γ (eg regular) and projecting

$$r(\gamma)^{-1} A_\mu(z)\, r(\gamma) = \left(\frac{\partial z^\nu}{\partial(\gamma(z))^\mu}\right) A_\nu(\gamma z).$$

a classical symmetry of 3+1 g.t.

This algebra can also be defined using $H \cong \mathbb{R} \backslash SL(2,\mathbb{R})$

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \backslash \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

i.e. foliation of 3-manifold.

Open problems:

- q.t. on more general curved spaces. Implicit def'n exists in terms of 'spectral triple' but need more explicit.

Deformation quantization:

$$f * g = fg + i\hbar \{f, g\}_{PB} + \dots$$

(is in fact only way to deform function algebras). Leads to (gauge equiv class of) algebra for any Poisson str. Need to also develop appropriate ideas of metric + connection.

Given NC g.t. on general curved space,

- what are conditions on B for sensible g.t.? for susy? $B \in H^{1,1}$?
- does it come from branes?
- what is duality group?

  e.g. can we realize $O(4,20;\mathbb{Z})$ on K3 in explicit way?

- explicit Nahm duality
- instanton moduli spaces (Nekrasov + Schwarz 9802068).