Negative AdS energy and its implications for large $N$ gauge theories

(Myers, G.H.)
Claim: There are smooth solutions to $R_{\mu\nu} = \Lambda g_{\mu\nu}$ ($\Lambda < 0$) which are asymptotically $AdS$ but have negative total energy.
Positive energy theorems use spinors which are "asymptotically constant".

But non-susy gauge theories are described by compactifying on $S'$ with anti-periodic fermions. (Witten)
Positive energy theorems don't apply

In asymptotically flat case these bdy. cond. allow solns with arbitrarily negative energy! (Witten) (Brill, G.H.)

Is same true for $\Lambda < 0$?
"Anything that is possible in asymptotic flat spacetimes should be possible in asymptotic hyperbolic spacetimes."

-Yau
Example of negative energy solution \((D=5)\)

\[
\begin{align*}
\mathrm{d}s^2 &= \frac{U^2}{l^2} \left[ (1 - \frac{U_0^4}{U^4}) \mathrm{d}\tau^2 + \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}t^2 \right] \\
&\quad + \frac{l^2 \mathrm{d}U^2}{U^2 (1 - \frac{U_0^4}{U^4})}
\end{align*}
\]

\(U = U_0, \quad l^4 = 4\pi g N\)

globally static

\(\tau\) has period

\[
\beta = \frac{\pi l^2}{U_0}
\]
This has total energy

\[ E_{\text{sugra}} = - \frac{N^2 \pi^2 V_2}{8 \beta^3} < 0 \]

Yang-Mills theory on \( S^1 \times \mathbb{R}^2 \) has negative Casimir energy. At weak coupling:

\[ E_{\text{ym}} = - \frac{N^2 \pi^2 V_2}{6 \beta^3} \]

\[ \Rightarrow E_{\text{sugra}} = \frac{3}{4} E_{\text{ym}} \]

Look familiar?
Recall that for the near extremal black 3-brane

\[ S_{\text{BH}} = \frac{3}{4} S_{\text{ym}} \]

(Gubser, Klebanov, Peet, Strominger)

This is not a coincidence: The same partition function on \( S' \times \mathbb{R}^3 \) yields both \( S \) and \( E \) depending on whether Euclidean time is \( S' \) or \( \mathbb{R} \).
Key question:

Is this the lowest energy SUGRA soln for given periodicity $\beta$?

If so, there must be a new positive energy theorem saying $E = E_{\text{SUGRA}}$

If not, either large $N$ non-susy gauge theory is unstable, or conjecture fails
Evidence for new positive energy theorem:

Solution is extremum of energy (since static)

Solution is local minimum of energy

⇒ small fluctuations are stable

(This was not true in asymptotic flat case)
Key difference between asymp. flat and asymp. AdS

Asymp. flat: The S' has constant radius asymp., so 5D metric approaches flat space like 4D soln.

Asymp AdS: The radius of S' $\to \infty$ asymp., so space effectively de-compactifies.
Generalizations

- There are negative energy SUGRA solutions corresponding to gauge theory on $S' \times S^2$, $S' \times \Sigma_g$

E.g. take 5D Schwarzschild AdS and analytically continue in $t \to 0$ (get $S' \times S^2$ asymp)
Conclusions

1) There are negative energy solutions which reflect Casimir energy in gauge theory.

2) AdS/CFT correspondence predicts a new (and strange) positive energy theorem in general relativity with $\Lambda < 0$. 