

Open-closed String Mirror Symmetry

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- study D-brane interactions: branes usually close



- However, want to compute exact, non-trivial functions (couplings) to test string dualities

→ global structure of interactions (presence of all other branes) is important

- preferred couplings: semi-topological, holomorphic

eg. $\text{Re}[\tau(\phi)] \underbrace{F_n F_n \dots}_n F$ in $d=2n=2,4,8$ $N=2$ SUSY

$$\tau(\phi) = i\beta(\phi) + \frac{1}{2\pi} \theta(\phi) = \frac{1}{2\pi i} \log(\phi) + \dots$$

↑ 1-loop

- typically: exact computations \leftrightarrow period integrals
 \leftrightarrow Mirror Symmetry

In string theory, such couplings arise e.g. in T_2 compactifications of $d^{*2} = 2n+2 = 4, 6, 10$ dimensional heterotic strings at one loop order:



The effective coupling $\tau(T, U)$ depends on the torus moduli T, U .

"new index" (BPS saturated):

$$\text{Re } \tau(T, U) F_n F_n \dots F = \int \frac{d^2 \tau}{\tau_2^2} \left[\sum_{\substack{P_L, P_R \in \\ \Gamma_{2,2}}} q^{\frac{1}{2} P_L^2} \bar{q}^{\frac{1}{2} P_R^2} \right] \cdot \hat{\mathcal{A}}(F, q)$$

\nearrow T_2 partition fct \nearrow elliptic genus gives $F_n \dots F$

- Integral complicated but can be done

Many tests \checkmark in $d=4$ ($II_{1,1}$ on CY's)

Now do this in $d=8$ (new):

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het. string on T_2 at 1-loop

Re $\tau_{ijkl}(T,U)$ Fix $F_2 \wedge F_2 \wedge F_2$

Looks naively similar, but non-trivial that

$$\begin{aligned}\tau_{ijkl}(T,U) &= \sum_{n_T, n_U} g(n_T, n_U) n_i n_j n_k n_l \log[1 - q_T^{n_T} q_U^{n_U}] \\ &= \partial_i \partial_j \partial_k \partial_l g(T,U) \quad (i,j,\dots = \{T,U\})\end{aligned}$$

where the "prepotential" (?) is

$$g(T,U) = \sum_{n_T, n_U} g(n_T, n_U) \text{Li}_5(q_T^{n_T} q_U^{n_U}) \quad \boxed{\text{EXACT!}}$$

typical behavior:

$$\tau_{TTUU} = \frac{1}{2\pi i} \log[J(T) - J(U)] + \text{finite}$$

What is the dual theory in $d=8$ that potentially can reproduce this from geometry?

→ "F-theory" on K3

$$\begin{array}{c} \uparrow \\ \dim_{\mathbb{C}}(K3) = 2 \\ c_1 = 0 \end{array}$$

Generic form:

$$\frac{\partial}{\partial T_m} \tau_{\text{isnc}}(T) = \sum c(u_s, u_j) n_i n_j n_k n_l n_m \frac{\pi q_s^{u_s}}{1 - \pi q_s^{u_s}}$$

• Puzzle

How to get such non-trivial holomorphic functions from K3 surfaces?

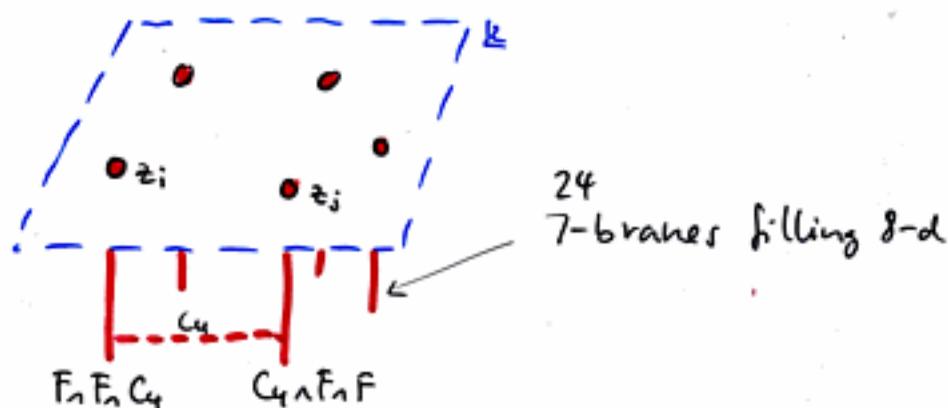
Usual mirror symmetry computations

→ 2pt fct = const.

⇒ Novel manifestation of mirror map!

How are $F_n F_n F_n F$ terms generated in F-theory?

type IIB strings
on \mathbb{P}^1 w/ 7-branes:



WV couplings:

$$I = \int C_n e^F \sim \int d^8x F^4 + C_2 F_n F_n F + C_4 F_n F + \dots$$

→ integrate out p-form RR fields C_p

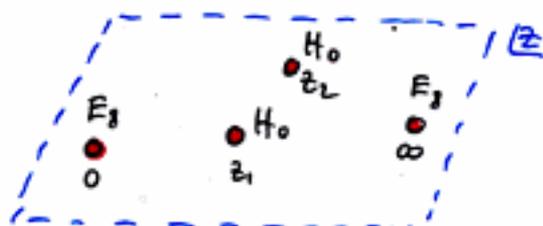
On the z -plane, C_p behave like scalars

→ Green's fun's:

$$\langle C(z_i) C(z_j) \rangle \sim \text{Log}[z_i - z_j] + \text{finite}$$

Example:

- for simplicity, group branes together:
"E₈-planes" at $z=0, \infty$ and "H₀-planes" at $z=z_1, z=z_2$



(10) (2) (2) (10) $\Sigma = 24$ 7-branes

corresponds to het string on T_2 , with $U \equiv e^{2\pi i/3}$

Relevant K3:

$$y^2 + x^3 + z^5(z - z_1(\tau))(z - z_2(\tau)) = 0$$

- Mirror map: $z_{1,2}(\tau) = \sqrt{J(\tau) - 1} \pm J(\tau)$

- Relevant 7-plane coupling: $C_4 \wedge F_7 \wedge F_0$

induces space-time coupling

$$\begin{aligned} \tau_{TTUU} &= \langle C_4(z_1) C_4(z_2) \rangle = \log [z_1 - z_2] + \text{finite} \\ &= \log [J(\tau)] + \text{finite} \end{aligned}$$

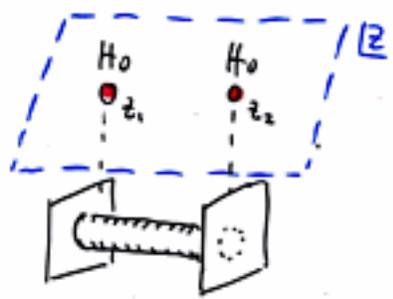
cf. 1-loop het. string:

$$\log [J(\tau) - J(\omega)] + \dots$$

" 0 for $U = e^{2\pi i/3}$ ✓

Important insight:

consider int. of two 7-branes:



Has two dual interpretations:

- Tree level closed string exchange

$$\tau = \langle C_\mu(z_1) C_\mu(z_2) \rangle = \log(z_1 - z_2) + \dots$$

- Open strings at one-loop order

$$\tau = \sum_{\substack{\text{open st.} \\ \text{states } i}} \log(\mu_i) + \dots$$

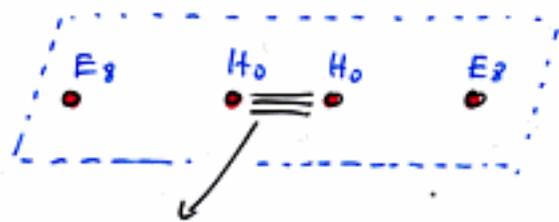
Mass of open string stretched between z_1, z_2 :
 distance in "w-plane"

$$\mu = |W(z_1) - W(z_2)|$$

$$W(z) = \int^z dw$$

string metric: $dw \cong \pi(z - z_i^*)^{-1/2} dz \cdot (p + q\tau_s)$

• Focus in our example on collision of H_0 - H_0 -planes:



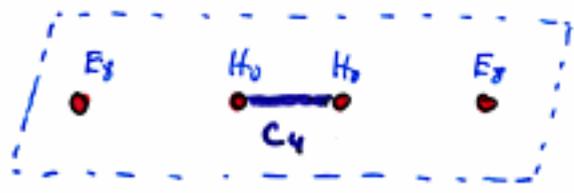
open strings with $(p, q) = (1, 0), (0, 1), (1, 1)$ become massless ($\rightarrow SU(3)$)

$$m = |a| \equiv \left| \frac{T-s}{T+s} \right| \rightarrow 0 \quad (s = e^{2\pi i/\beta})$$

so leading contribution to coupling is:

$$\tau \sim \sum \log(m_i) \sim \underline{3 \log a} + \dots$$

• Now closed string sector:



single C_4 -exchange:

mirror map
↓

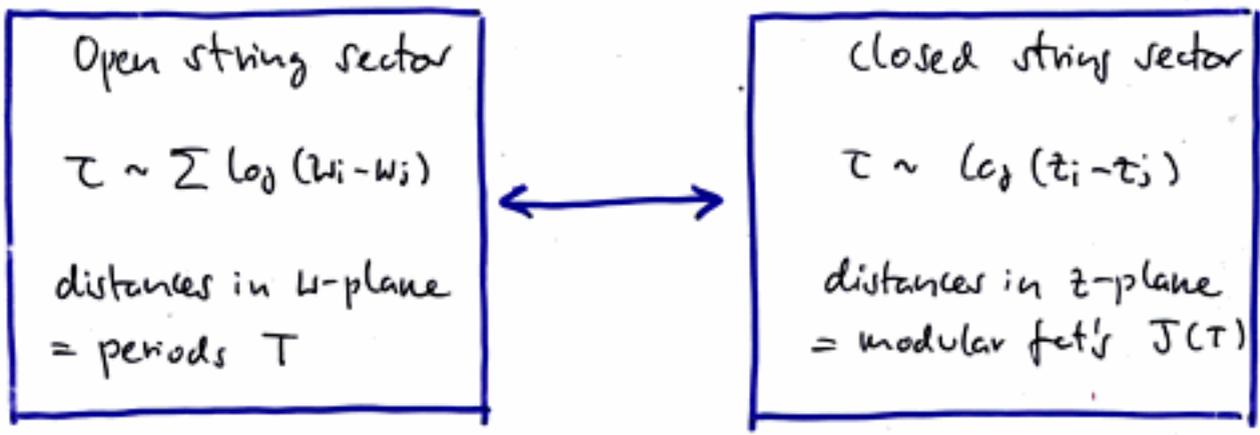
$$\tau \sim \cancel{\log(z_1 - z_2)} \sim \log[J(\tau)] \sim \underline{3 \log a} + \dots$$

$$(J(a) \sim a^3 + O(a^6))$$

\Rightarrow Tree level closed string exchange sums up
 ∞ many open string one-loop contributions!

(cf modular function)

→ Novel manifestation of mirror symmetry



How to explicitly compute full result

$$\tau(\tau, u) = \frac{1}{2\pi i} \log [J(\tau) - J(u)] + \log \gamma$$

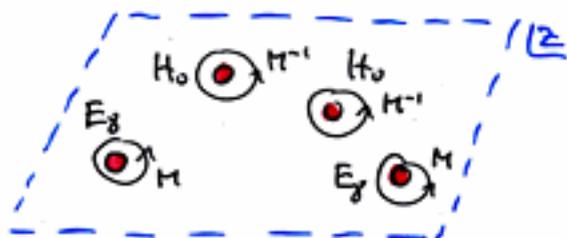
need to find finite pieces of Green's fat

$$\langle C(z_1) C(z_2) \rangle \sim \log [z_1 - z_2] + \log \gamma(z_1, z_2)$$

→ global structure of z -plane,
monodromies

Note:

In the subspace where $U = e^{2\pi i/\alpha} = \tau_s = \text{const}$,
all the monodromies have finite order (z_6):



$$M = (ST) \in SL(2, \mathbb{Z}), \quad M^3 = -\mathbb{1}$$

→ The 7-planes behave like z_6 twist fields $\sigma, \bar{\sigma}$
in a tree level z_6 orbifold computation

RR and NSNS p-forms C, B behave like
twisted scalar fields

→ Green's fct

$$\langle C, C \rangle \sim \langle X, \bar{X} \rangle \Big|_{\substack{\text{in presence of} \\ \sigma, \bar{\sigma}}}$$

Can be canonically generalized to Z_N -twists!
 ($N=2,3,4,6$)

Relevant K3's:

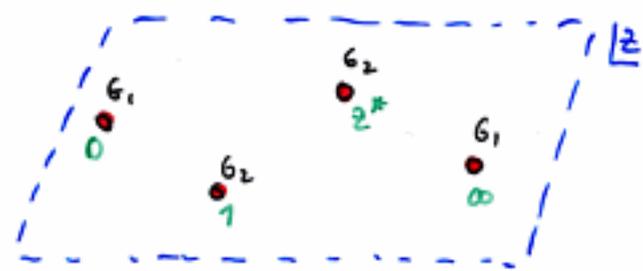
$Z_2: Y^2 + X^3 + z^2(z-1)^2(z-z^*)^2 = 0$

$Z_3: Y^2 + X^3 + z^3(z-1)^2(z-z^*)^2 = 0$

$Z_4: Y^2 + X^3 + Xz^2(z-1)(z-z^*) = 0$

$Z_6: Y^2 + X^3 + z^5(z-1)(z-z^*) = 0$

← considered so far



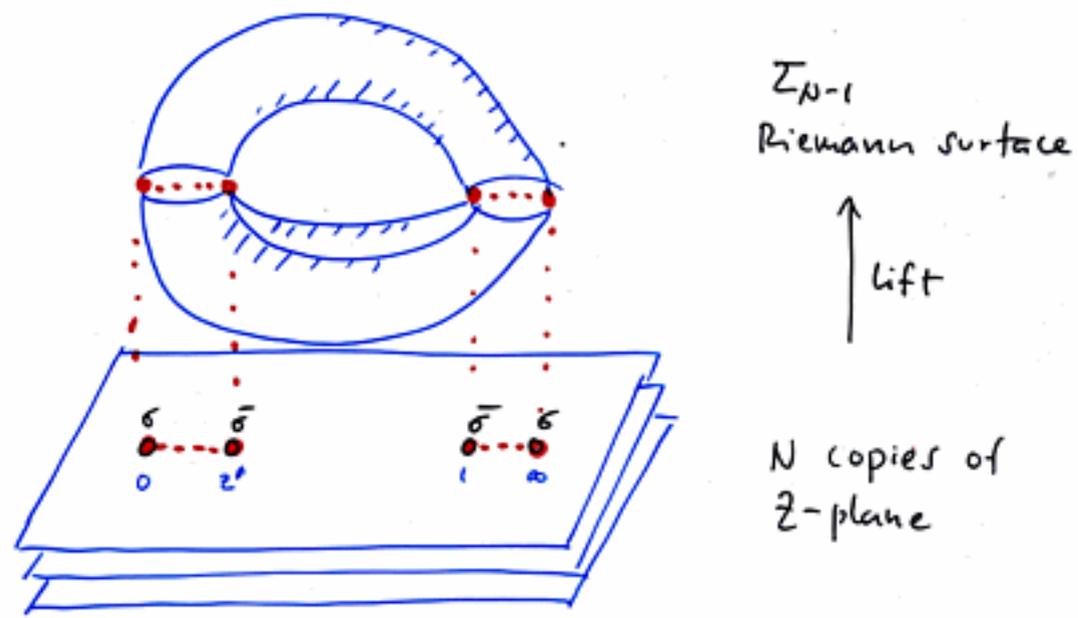
$G_{total} = (G_1 \times G_2)^2$

N	G_1	G_2	τ_S
2	$SO(3)$	$SO(3)$	any
3	E_6	$H_2 \sim SU(3)$	\mathcal{F}
4	E_7	$H_1 \sim SU(2)$	i
6	E_8	H_0	\mathcal{F}

(← Sen)

(+ other cases)

Multi-valuedness: suitable single-valued Green's fct can be obtained by going to a \mathbb{Z}_N -cover:



The covering spaces are none other than the \mathbb{Z}_N -symmetric Seiberg-Witten curves for $SU(N)$!

$$\Sigma_{N-1} : x^N = z^{-1}(z-1)(z-z^*)$$

NB:

The periods $\int \frac{dz}{z} \frac{1}{x} = \int dz \frac{1}{z^{-1/N}(z-1)^{1/N}(z-z^*)^{1/N}}$
 coincide with the relevant K3 periods!
 (capture relev geometry)

→

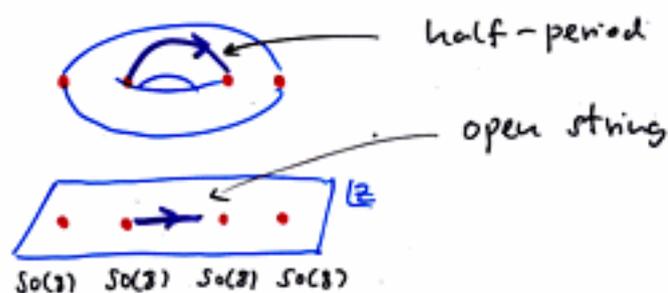
The Green's fct we seek are nothing but the Green's fct on Σ_{g-1} , given by the "prime form":

$$G(z_1, z_2) = \log \left[\frac{\theta_s \left[\int_{z_1}^{z_2} \vec{v} \mid \Omega \right]}{\sqrt{\zeta(z_1)} \sqrt{\zeta(z_2)}} \right] + \text{non-holom.}$$

$$\sim \log [z_1 - z_2] + \text{finite}$$

$$(\zeta(z) \equiv \sum \frac{\partial}{\partial v_i} \theta_s[\vec{v} \mid \Omega] \cdot u^i)$$

The simplest example is for $N=2$ (is, $SO(8)^4$ gauge sym.)
 Σ_1 has genus 1 (torus)



Consider couplings $\tau_{ijij} F_{SO(8)}^2$; $\tau_{ij} F_{SO(8)}^2$; arising from $C_4 + F_{SO(8)}^2$

Green's fct on Torus :

$$\langle C_n(z_1) C_n(z_2) \rangle = \log \left[\frac{\theta_1 \left[\frac{z_1 - z_2}{\tau} \middle| \tau \right]}{\theta_1'(z)} \right]$$

We that $z_i - z_j =$ half-periods $\frac{1}{2}, \frac{1}{2}\tau, \frac{1}{2}(\tau+1)$, and $\theta_1 \left[\frac{1}{2}\tau \middle| \tau \right] = \theta_2 \left[\tau \right]$ etc.

$$\Rightarrow \begin{cases} \tau_{1122} = \log \left[\theta_2 / \theta_1' \right] \\ \tau_{1133} = \log \left[\theta_3 / \theta_1' \right] \\ \tau_{1144} = \log \left[\theta_4 / \theta_1' \right] \end{cases}$$

|| Match precisely the 1-loop couplings on the heterotic side!

.. have thus indeed computed prepotential \mathcal{G} from K3!

Resumé

- A novel manifestation of mirror symmetry allows to compute holom. 4-pt couplings from K3's
- These coincide with the 1-loop couplings of the heterotic string on $T_2 \rightarrow$
quantitative test of heterotic/F-theory duality
- The mirror map essentially acts between open and closed string sectors (\rightarrow applications -)
- Intriguing math. structure

$$\frac{\partial}{\partial T_m} \tau_{\text{het}}(T) = \sum \tilde{c}(g_s, h_g) N_{\text{het}} N_{\text{het}} N_{\text{het}} \frac{\pi g_s^{h_g}}{1 - \pi g_s^{h_g}}$$

↑
integers count nodal
curves on K3

$\sim \frac{1}{2}$ BPS states