Arithmetic & Attractors

Santa Barbara, June 26, 1998

Gregory Moore

work in progress

This is a *highly speculative* talk about some possible connections between number theory and black holes in supergravity/superstring theory.

1. Review of the "attractor equations"
2. Solution for $K3 \times T^2, T^6, FHSV$: A product of three (isogenous) elliptic curves.
3. Three connections to arithmetic
4. Attractors and RCFT's
5. CY 3-folds
6. Grandiose speculation

Introduction

Modular forms, congruence subgroups, elliptic curves, are all mathematical objects of central concern both to string theorists and to number theorists. *Is there a deeper connection?*

PROBLEM: the detailed questions of the number theorists and the string theorists seem generally orthogonal... For example: In string perturbation theory we encounter the elliptic curve

$$\boxed{E_\tau \equiv \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})}$$

but there never was any compelling reason to restrict attention to elliptic curves defined over $\mathbb{Q}$ (or any other number field).

The point of this talk is that the "attractor mechanism" for susy black strings and black holes provides a framework which naturally isolates certain arithmetic varieties.

We *hope* that the arithmetic of these varieties will be intimately connected with questions about

- BPS states, their spectrum and existence
- Properties of the CFT at the Dbrane horizon.

THE ATTRACTOR EQUATIONS

X : CALABI-YAU 3-FOLD

$\tilde{\mathcal{M}}$: MODULI OF COMPLEX STRUCTURES

$\hat{\gamma} \in H^3(X:\mathbb{Z})$ "charge vector"

$z \in \tilde{\mathcal{M}} \Rightarrow$ Hodge decomposition:

$$\hat{\gamma} = \hat{\gamma}^{3,0} + \hat{\gamma}^{2,1} + \hat{\gamma}^{1,2} + \hat{\gamma}^{0,3}$$

$$\boxed{\text{ATTRACTOR EQ'S:} \quad \hat{\gamma} = \hat{\gamma}^{3,0} + \hat{\gamma}^{0,3}}$$

Ferrara
Kallosh
Strominger

- LOCAL MINIMA FOR BPS MASS [F.G.K]
- CONSTRUCT $\mathcal{N}=2$ BLACK HOLES [F.K.S.]

EQUIVALENT FORMULATION:

$$\bar{c} X^I - c \bar{X}^I = i p^I$$
$$\bar{c} F_I - c \bar{F}_I = i q_I \qquad p^I, q_I \in \mathbb{Z}$$

The minimization principle

$d = 4, \mathcal{N} = 2$ compactification: IIB$/M_4 \times X_3$ has abelian gauge fieldstrengths (with duals):

$$\mathcal{F} \in \Omega^2(M_4; \mathbb{R}) \otimes H^3(X; \mathbb{R})$$

Dyonic charges:

$$\int_{S^2_\infty} \mathcal{F} = \hat{\gamma} \in H^3(X; \mathbb{Z})$$

Definition: For $\gamma \in H_3(X; \mathbb{Z})$, $\Omega \in H^{3,0}(X)$ in complex structure $z \in \widetilde{\mathcal{M}}$:

$$|Z(z;\gamma)|^2 \equiv \frac{|\int_\gamma \Omega|^2}{i \int \Omega \wedge \bar{\Omega}}$$

- A well defined nonnegative function of $z \in \widetilde{\mathcal{M}}$.
- If γ supports a BPS state then $M^2(z;\gamma)/M^2_{\text{Planck}} = |Z(z;\gamma)|^2$

Theorem [Ferrara, Gibbons, Kallosh].

a.) $|Z(z;\gamma)|^2$ has a stationary point at $z = z_*(\gamma) \in \widetilde{\mathcal{M}}$, with $Z(z_*;\gamma) \neq 0$ iff $\hat{\gamma}$ has Hodge decomposition $\hat{\gamma} = \hat{\gamma}^{3,0} + \hat{\gamma}^{0,3}$.

b.) If such a stationary point exists it is a local minimum.

Black holes and dynamical systems

Static, spherically symmetric, dyonic, extremal black holes:

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}(dr^2 + r^2 d\Omega^2)$$
$$\mathcal{F}^- = E^- \otimes (\hat{\gamma}^{3,0} \oplus \hat{\gamma}^{1,2})$$

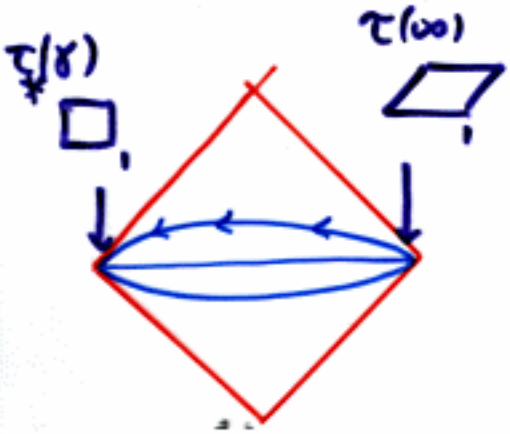
$\delta\psi = \delta\lambda = 0 \Rightarrow$dynamical system for $z(r)$ on $\mathcal{M}_{VM}$:

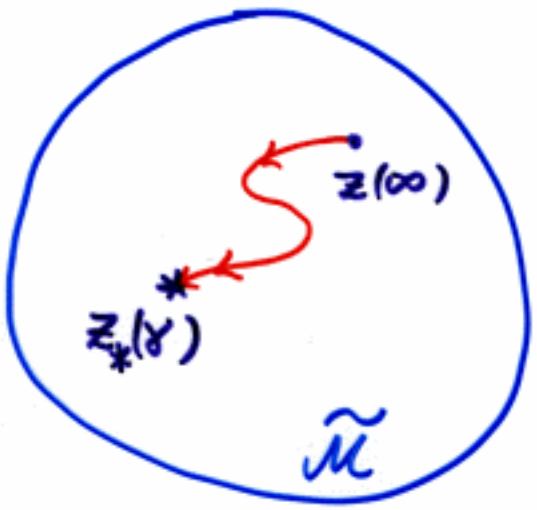
$$r^2 \frac{d}{dr}(e^{-U}) = -|Z(z(r);\gamma)|$$
$$\Pi^{2,1}\left[e^{K/2}\frac{d\Omega(r)}{dr}\right] = i\hat{\gamma}^{2,1}e^U \frac{Z}{|Z|}\frac{1}{r^2}$$

Attractor equations are the fixed point equations $\hat{\gamma}^{2,1} = 0$

Near horizon geometry:

$$ds^2 = -\frac{r^2}{Z_*^2}dt^2 + Z_*^2\frac{dr^2}{r^2} + Z_*^2 d\Omega^2$$





SOLUTION FOR $X = K3 \times T^2$

CHARGE: $\hat{\gamma} \in H^3(X;\mathbb{Z}) = H^2(K3;\mathbb{Z}) \otimes H^1(T^2;\mathbb{Z})$

$$= H^2(K3;\mathbb{Z}) \oplus H^2(K3;\mathbb{Z})$$

$$\hat{\gamma} = p \oplus q$$

$$X_{p,q} = Y_{2Q_{p,q}} \times E_{\tau(p,q)} = \left(\frac{E_{\tau(p,q)} \times E_{\tau'(p,q)}}{\mathbb{Z}_2} \right) \times E_{\tau(p,q)}$$

$$\tau(p,q) = \frac{p\cdot q + i\sqrt{-D_{p,q}}}{p^2} \qquad \tau'(p,q) = \frac{-p\cdot q + i\sqrt{-D_{p,q}}}{2}$$

$$D_{p,q} = (p\cdot q)^2 - p^2 q^2 < 0$$

$$\tau, \tau' \in \mathbb{Q}(\sqrt{D}) \equiv \mathbb{Q} + i\sqrt{|D|}\,\mathbb{Q}$$

$Y_{2Q_{p,q}}$: "Exceptional K3 surface"

- $\rho = \text{rank}(NS) = 20$, dense, isolated in $\tilde{\mathcal{M}}$
- Classified by <u>binary</u> <u>quadratic</u> <u>forms</u>

$$Q_{p,q} \equiv \frac{1}{2}\begin{pmatrix} p^2 & -p\cdot q \\ -p\cdot q & q^2 \end{pmatrix} \quad ; \quad -4\det Q_{p,q} = D_{p,q}$$

Solution for $X = K3 \times T^2$

Charge: $\hat{\gamma} \in II^{22,6} \oplus II^{22,6}$.

Use U-duality and the product structure:

$$\begin{aligned}\hat{\gamma} \in H^3(X;\mathbb{Z}) &\cong H^2(K3;\mathbb{Z}) \otimes H^1(T^2;\mathbb{Z}) \\ &\cong H^2(K3;\mathbb{Z}) \oplus H^2(K3;\mathbb{Z})\end{aligned}$$

$$\hat{\gamma} \cong p \oplus q \qquad\qquad \Omega^{3,0} = \Omega^{2,0} \wedge dz$$

Attractor equations:

$$2\mathrm{Im}\bar{C}\Omega^{2,0} = p \qquad\qquad 2\mathrm{Im}\bar{C}\tau\Omega^{2,0} = q$$

Solution:

$$\Omega^{2,0} \sim C(q - \bar{\tau}p)$$

$$\int_{K3} \Omega^{0,2} \wedge \Omega^{0,2} = 0 \Rightarrow p^2\tau^2 - 2p \cdot q\tau + q^2 = 0$$

$$\tau = \tau(p,q) \equiv \frac{p \cdot q + \sqrt{D_{p,q}}}{p^2}$$

$$D_{p,q} \equiv (p \cdot q)^2 - p^2q^2 < 0$$

By the Torelli theorem, the complex structure of the $K3$ surface is uniquely determined by $\Omega^{2,0} = C(q - \bar{\tau}p)$.

Exceptional $K3$ Surfaces

We now show that these attractor varieties are closely related to products of three special (arithmetic) elliptic curves

Neron-Severi lattice: $NS(S) \equiv \ker\{\gamma \to \int_\gamma \Omega\}$.

Transcendental lattice $T_S \equiv (NS(S))^\perp$.

- Generic $K3$: $NS(S) = \{0\}$
- Generic algebraic $K3$: $NS(S) = H\mathbb{Z}, \qquad \rho(S) = 1$
- Generic elliptic $K3$: $NS(S) = B\mathbb{Z} \oplus F\mathbb{Z}, \qquad \rho(S) = 2$
- Attractor K3:

$$H^{2,0} \oplus H^{0,2} = T_S \otimes \mathbb{C} \qquad T_S = \langle p, q \rangle_{\mathbb{Z}}$$

$NS(S) = \langle p, q \rangle^\perp \subset H^2(K3; \mathbb{Z})$ has rank $\rho(S) = 20$!

These define *exceptional K3 surfaces*:

They form an isolated dense set of the moduli of algebraic $K3$'s and have been completely classified.

Exceptional $K3$ surfaces & Quadratic Forms

Definition: An integral binary quadratic form:

$$Q = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \qquad a, b, c \in \mathbb{Z},$$

Equivalence: $\exists m \in SL(2, \mathbb{Z})$:

$$m \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} m^{tr} = \begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix}$$

Theorem[Shioda-Inose] There is a 1-1 correspondence between exceptional K3 surfaces, S, and $SL(2, \mathbb{Z})$ equivalence classes of positive even binary quadratic forms.

$S \Rightarrow$Quadratic forms:

$$T_S = \langle p, q \rangle_{\mathbb{Z}} \Rightarrow 2Q_{p,q} \equiv \begin{pmatrix} p^2 & -p \cdot q \\ -p \cdot q & q^2 \end{pmatrix}$$

Form $Q \Rightarrow S$: Consider $A_Q = E_{\tau_1} \times E_{\tau_2}$:

$$\tau_1 = \frac{-b + \sqrt{D}}{2a} \qquad \tau_2 = \frac{b + \sqrt{D}}{2}$$

$$\begin{array}{ccc} & A_Q & \\ & \downarrow & \\ Y_Q & \overset{2:1}{\to} & \mathrm{Km}(A_Q) \end{array}$$

The Attractor Varieties: $IIB/K3 \times T^2$

Corollary. Suppose $p, q \in II^{22,6}$ span a rank two primitive sublattice $L_{p,q} = \langle p, q \rangle \subset H^2(K3; \mathbb{Z})$.

Then the attractor variety $X_{p,q}$ determined by $\hat{\gamma} = p \oplus q$ is $Y_{2Q_{p,q}} \times E_{\tau(p,q)}$.

$$\tau(p,q) = \frac{p \cdot q + i\sqrt{-D_{p,q}}}{p^2}$$

and $Y_{2Q_{p,q}}$ is the Shioda-Inose K3 surface associated to the even quadratic form:

$$2Q_{p,q} \equiv \begin{pmatrix} p^2 & -p \cdot q \\ -p \cdot q & q^2 \end{pmatrix}$$

The variety is a double-cover of a Kummer surface constructed from

$$Y_{2Q_{p,q}} \times E_\tau \to Km\left(E_{\tau(p,q)} \times E_{\tau'(p,q)}\right) \times E_{\tau(p,q)}$$

$$\tau'(p,q) = \frac{-p \cdot q + i\sqrt{-D}}{2}$$

Similar results hold for $X = T^6, (K3 \times T^2)/\mathbb{Z}_2$

Three Relations to arithmetic

- Enumeration of U-duality classes with the same near-horizon metric is a class number problem.

- The BPS mass-spectrum at an attractor point is given in terms of norms of ideals in a (quadratic imaginary) number field.

- By the theory of "complex multiplication" the attractor varieties are arithmetic varieties, i.e., defined by polynomial equations over certain number fields (classfields).

Class Numbers

How many equivalence classes of quadratic forms are there?

$$m\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} m^{tr} = \begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix} \qquad m \in SL(2,\mathbb{Z})$$

[Fermat, Euler, Lagrange, Legendre, Gauss]. There are a **finite number** at fixed discriminant $D = b^2 - 4ac$. If g.c.d.$(a,b,c) = 1$ ("primitivity") this number $h(D)$ is called the *class number*. $h(D) > 1$ except for 13 values of D.

The distinct classes may be labelled by points $\tau_i \in \mathcal{F}$:

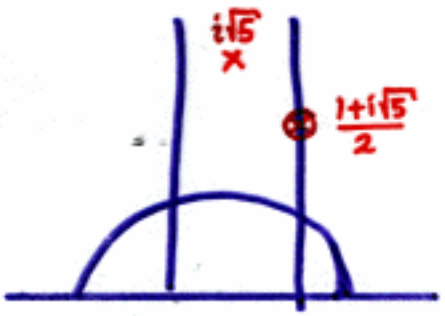
$$ax^2 + bxy + cy^2 \equiv a|x - \tau y|^2 \to \tau = \frac{-b + \sqrt{D}}{2a}$$

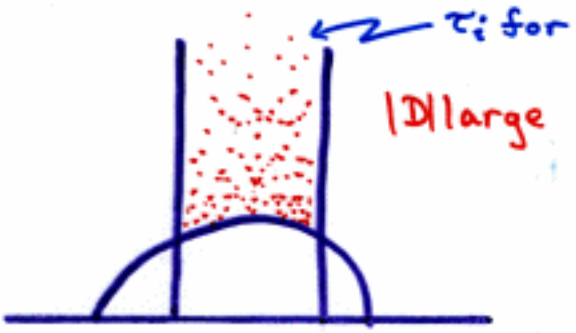
$SL(2,\mathbb{Z})$ acts on τ in the standard way.

Example: $D = -20$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \qquad x^2 + 5y^2 \qquad \tau_1 = i\sqrt{5}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \qquad 2x^2 + 2xy + 3y^2 \qquad \tau_2 = \frac{\pm 1 + i\sqrt{5}}{2}$$





HORIZON AREA & U-DUALITY

* NEAR-HORIZON METRIC ONLY DEPENDS ON $|Z_*(\gamma)|$

* $|Z_*(\gamma)|$ IS U-DUALITY INVARIANT.

$$\boxed{\text{? HOW MANY U-DUALITY INEQUIVALENT } \gamma\text{'s LEAD TO THE SAME NEAR-HORIZON METRIC?}}$$

A: $(p,q) \underset{U(\mathbb{Z})}{\sim} (p',q') \Longleftrightarrow \exists\, m \in SL(2,\mathbb{Z})$

$$Q_{p,q} \equiv \frac{1}{2}\begin{pmatrix} p^2 & -p\cdot q \\ -p\cdot q & q^2 \end{pmatrix} = m\, Q_{p',q'}\, m^{tr}$$

Cvetic + Youm
F. + K.

B: $|Z_*(\gamma)|^2 = \sqrt{-4\det Q_{p,q}} = \sqrt{-D_{p,q}} = \sqrt{p^2q^2-(p\cdot q)^2}$

RESULT: $\#[\gamma]_U = h(D_{p,q}) \gtrsim |D_{p,q}|^{1/2} \sim A$

$$\boxed{\text{FOR LARGE } A\text{, the \# of U-inequivalent charges with horizon area } A \text{ grows like } A.}$$

Near-horizon metric vs. U-duality

$|Z(\Omega_*(\gamma);\gamma)|^2$ is defined for *all* charges, not necessarily large.
$\Rightarrow$extend notion of "horizon area" to all BPS states.

Example: $II/K3 \times T^2$

$$\hat{\gamma} = p \oplus q \in \Lambda = II_e^{19,3} \oplus II_m^{19,3} \cong H^3(X_3;\mathbb{Z})$$

$$|Z_*|^2 = \sqrt{-D_{p,q}} = \sqrt{p^2q^2 - (p\cdot q)^2} \qquad D_{p,q} = -4\det Q_{p,q}$$

Define *the discriminant of a BPS state* to be $|Z_*|^2 \equiv \sqrt{-D(\gamma)}$.
In the SUGRA approximation the near-horizon metric only depends on the discriminant $D(\gamma)$:

$$\frac{A(\gamma)}{4\pi} = M_*^2(\gamma) = \sqrt{-D(\gamma)}$$

While $D(\gamma)$ is invariant under $U(\mathbb{Z})$, it might be that U-inequivalent γ's have the *same* $D(\gamma)$. (i.e., the same near-horizon metric).

$$\mathcal{N}(D) \equiv \#\{[\gamma]_U : D(\gamma) = D\}$$

Example: $K3 \times T^2$: $U(\mathbb{Z}) = SL(2,\mathbb{Z}) \times O(22,6;\mathbb{Z})$.

$(p',q') \sim (p,q)$ *iff* $\exists m \in SL(2,\mathbb{Z})$ with $mQ_{p,q}m^{tr} = Q_{p',q'}$

The growth of U-duality inequivalent classes

Conclusion: $\mathcal{N}(D) = h(D)$, $\quad Q_{p,q}$ primitive

The number of classes *grows* with $|D|$!

$$\log \mathcal{N}(D) > \log |D|^{1/2}$$

Therefore, at large entropy the number of U-duality inequivalent black holes with fixed area A grows like A.

Is there some larger "symmetry" which unifies these ?

Results for other models:

1. *FHSV*:

$$\log \mathcal{N}(D) \geq \log |D|^5$$

2. $6D$ Strings: For n_T tensor multiplets:

$$\log \mathcal{N}(P) \geq \log |P^2|^{(n_T-1)/2}$$

BPS Masses and Norms of Ideals

FHSV: A charge $p, q \in II^{10,2}(2)$ defines a unique attractor point $z_*(p,q) \in \widetilde{\mathcal{M}}$.

What is the BPS mass spectrum at the attractor point?

The answer turns out to involve the arithmetic of the *quadratic imaginary number field* $K_{D_{p,q}}$, where

$$K_D \equiv \mathbb{Q}[i\sqrt{|D|}] \equiv \{a + ib\sqrt{|D|} : a, b \in \mathbb{Q}\}$$

- To each attractor point $z_*(p,q)$ we associate a class of ideals in K_D: $z_*(p,q) \leftrightarrow [\underline{\mathbf{a}}_{p,q}]$

$$\underline{\mathbf{a}}_{p,q} \equiv \frac{1}{2}p^2\mathbb{Z} + \frac{p \cdot q + \sqrt{D_{p,q}}}{2}\mathbb{Z}$$

To each BPS charge (r,s) we associate an ideal in $[\underline{\mathbf{a}}_{p,q}]$.

The BPS mass spectrum $M^2(r,s)$ at $z_*(p,q)$ is given by the norm of the ideal:

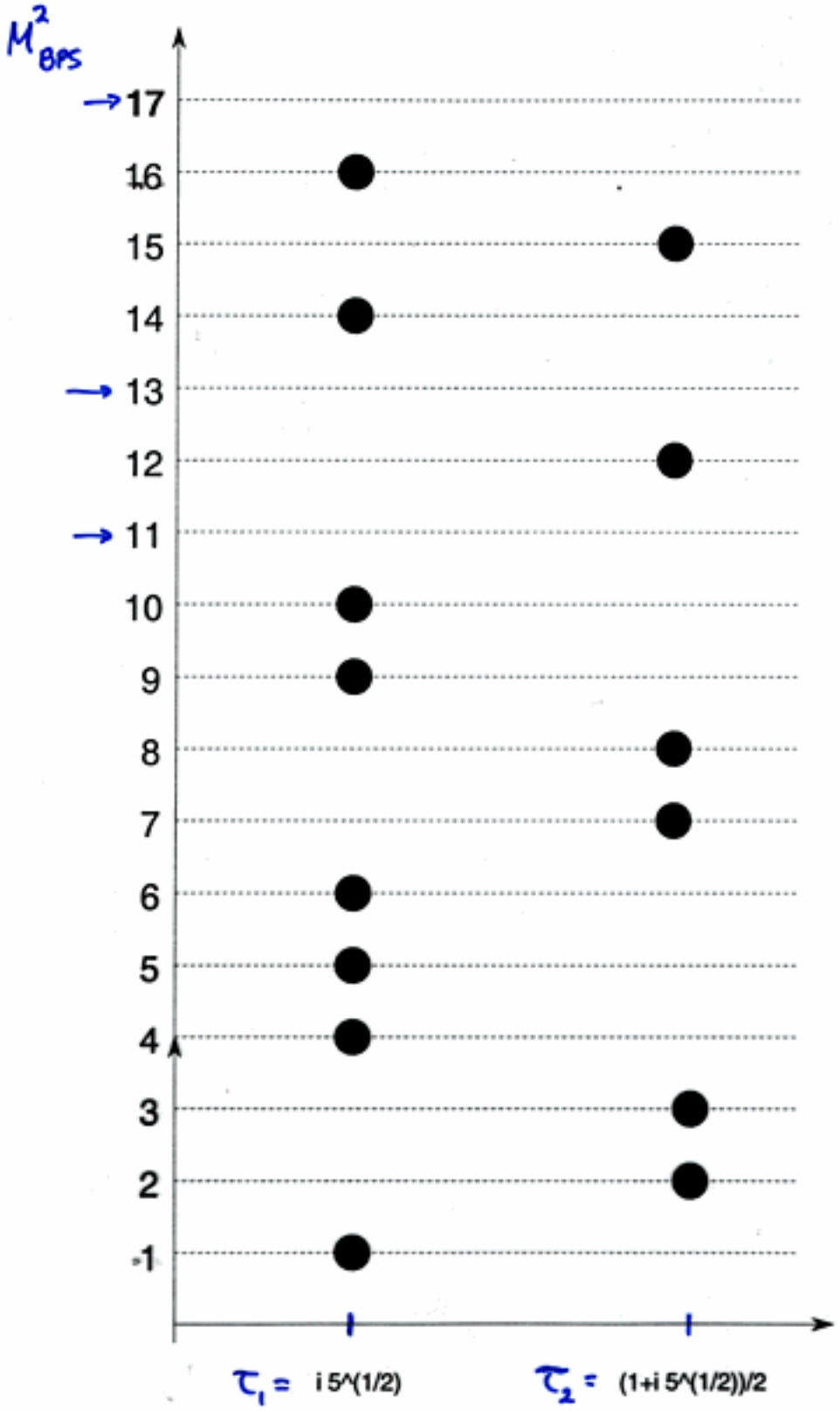
$$\kappa_{p,q}|Z(z_*(p,q);r,s)|^2 = \frac{1}{2}p^2|A - \tau_{p,q}B|^2 = N[\underline{\mathbf{a}}_{p,q}(A - \tau_{p,q}B)]$$

A, B are (rational) integers depending on p, q, r, s.

Corollary:

- The spectrum is integral
- At inequivalent τ_i these integers are disjoint.

BPS MASSES @ ATTRACTOR PTS. FOR D = −20



COMPLEX MULTIPLICATION

Abel, Gauss, Eisenstein, Kronecker, Weber:

$\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$: $a\tau^2 + b\tau + c = 0 \Rightarrow$

$j(\tau)$ is an algebraic integer of degree $h(D)$

$\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$: $j(i\sqrt{5}) = (50 + 26\sqrt{5})^3$

$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$: $j\left(\frac{1+i\sqrt{5}}{2}\right) = (50 - 26\sqrt{5})^3$

$$E_\tau \approx \{ y^2 = 4x^3 - c(\tau)(x+1) \}$$

$$c(\tau) = \frac{27 j(\tau)}{j(\tau) - (12)^3} \in \hat{K}_D \equiv K_D(j(\tau_i)) = \mathbb{Q}(\sqrt{D}, j(\tau_i))$$

Corr: THE $\mathcal{N} = 4, 8$ ATTRACTORS ARE ARITHMETIC VARIETIES, DEFINED/ $\hat{K}_D$

Remark: C.F.T. $\Rightarrow$ $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ permutes the $h(D_{p,q})$ U-inequivalent attractor points

Complex Multiplication

If $\tau \in K_D$ then $j(\tau)$ and the elliptic curve E_τ have very special properties...

Theorem [Abel, Gauss, Eisenstein, Kronecker, Weber]

Suppose $a\tau^2 + b\tau + c = 0$. Let D be the discriminant of the associated primitive quadratic form. Then,

i.) $j(\tau)$ is an algebraic integer of degree $h(D)$.

ii.) If τ_i correspond to the distinct ideal classes in $\mathcal{O}(D)$, the minimal polynomial of $j(\tau_i)$ is

$$p(x) = \prod_{k=1}^{h(D)} (x - j(\tau_k)) \in \mathbb{Z}[x]$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \qquad j(i\sqrt{5}) = (50 + 26\sqrt{5})^3$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \qquad j(\frac{1 + i\sqrt{5}}{2}) = (50 - 26\sqrt{5})^3$$

$$p(x) = x^2 - 1264000\, x - 681472000$$

The curve $E_\tau \cong \{(x, y) : y^2 = 4x^3 - c(x + 1)\}$ has special arithmetic properties:

$$c = \frac{27j(\tau)}{j(\tau) - (12)^3}$$

Consequence: There is an arithmetic Weierstrass model for E_τ defined over $\widehat{K_D} = K_D(j(\tau_i))$.

Attractors are arithmetic

Corollary: The $\mathcal{N} = 4, 8$ attractor varieties are arithmetic.

In fact: $\widehat{K_D} = K_D(j(\tau_i))$ is *independent of* τ_i; it is a "class-field of K_D."

The attractor varieties are arithmetic varieties, defined over classfield extensions of the field of definition, K_D, of the periods.

A result of classfield theory: $\widehat{K_D}$ is Galois over K_D, and $Gal(\widehat{K_D}/K_D)$ is in fact isomorphic to the class group $C(D)$

$$[\tau] \to \sigma_{[\tau]} \in Gal(\widehat{K_D}/K_D)$$
$$j([\bar{\tau}_i] * [\tau_j]) = \sigma_{[\tau_i]}(j[\tau_j])$$

Example:

$$D = -20 \quad \hat{K}_{D=-20} = K_{-20}(\sqrt{5}) = \mathbb{Q}(\sqrt{-1}, \sqrt{-5})$$
$$\langle\sigma\rangle = \mathrm{Gal}(\widehat{K_D}/K_D) \cong \mathbb{Z}/(2\mathbb{Z})$$

$$(50 - 26\sqrt{5})^3 = j(\frac{1 + i\sqrt{5}}{2}) = \sigma(j(i\sqrt{5})) = \sigma((50 + 26\sqrt{5})^3)$$

Thus, $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the (complex) attractor VM moduli, and permutes the attractor moduli at fixed discriminant: the Galois group extends the U-duality group and "unifies" the different attractor points at discriminant D.

RCFT's are attractive

F – Theory duality : $HET/T^2 \overset{\cong}{\Leftrightarrow} IIB/S \to \mathbb{P}^1$

Moduli $=\mathcal{B}^{18,2} = Gr_+(2, II^{18,2} \otimes \mathbb{R})$

Rightmover projection of $p \in II^{18,2}$

$$p_R = e^{K/2} \int_p \Omega^{2,0}$$

Compare "attractor points" for $p, q \in II^{18,2}$:

F-theory $\Rightarrow$Exceptional K3 surface:

- Transcendental lattice: $T_S = \langle p, q \rangle_{\mathbb{Z}}$
- (Generic) Mordell-Weil group $MW = \langle p, q \rangle_{\mathbb{Z}}^{\perp}$

Heterotic theory: $\mathcal{B}^{18,2} =$ a space of projectors:

$p = p^L + p^R = p^{18,0} + p^{0,2}$

"Attractor equations": $p^L = q^L = 0 \Rightarrow$ RCFT!

The heterotic RCFT's are F-theory dual to the exceptional K3 surfaces. The Mordell-Weil group generates the enhanced chiral algebra of the heterotic RCFT.

- Realizes part of an old dream of Friedan & Shenker.

- ♮ conjecture: II/attractor varieties is dual to Heterotic/RCFT in general!

ARITHMETIC & MIRROR MAPS

"complex multiplication"
$$\begin{cases} \tau \longmapsto j(\tau) \\ K_D \longrightarrow \hat{K}_D : \text{ classfield} \end{cases}$$

K3 MIRROR MRP = F-THEORY MAP

$$\text{HET}/T^2 \longleftrightarrow F/K3$$

$$(T, U, \vec{A}) \longmapsto y^2 = 4x^3 - f_8(z)x - f_{12}(z)$$

Expect: $K_D \longrightarrow$ K3/ Classfield of K_D

WE CHECKED IT FOR SEVERAL FAMILIES

ARITHMETIC ATTRACTOR CONJECTURE:

Suppose: $\gamma \longrightarrow z_*(\gamma) \in \widetilde{\mathcal{M}}$

(a.) $X^I, F_I \in K_\gamma$ # field

(b.) $\bar{X}_{z_*(\gamma)}$ is defined over $\hat{K}_\gamma$

$\hat{K}_\gamma =$ finite Galois extension of K_γ

Application to K3 mirror maps

Algebraic coordinates on $\mathcal{B}^{18,2}$

$$\begin{aligned} ZY^2 =& 4X^3 - f_8(s,t)XZ^2 - f_{12}(s,t)Z^3 \\ f_8(s,t) =& \alpha_{-4}s^8 + \cdots + \alpha_{+4}t^8 \\ f_{12}(s,t) =& \beta_{-6}s^{12} + \cdots + \beta_{+6}t^{12} \end{aligned}$$

$$\mathcal{B}^{18,2} = \Big[\{(\vec{\alpha},\vec{\beta})\} - \mathcal{D}\Big]/GL(2,\mathbb{C})$$

Flat coordinates on $\mathcal{B}^{18,2}$

$$Gr_+(2, II^{2,18} \otimes \mathbb{R}) \cong \mathbb{R}^{1,17} + iC_+ = \{y = (T, U, \vec{A})\}$$

T: Kahler, U: complex, $\vec{A}$: Wilson

Definition: F-map, or $K3$ mirror map: $\Phi_F : y \to (\vec{\alpha}, \vec{\beta})$. It is a vector-valued modular form for $O(18, 2; \mathbb{Z})$.

Example: [Morrison & Vafa, Cardoso-Curio-Lust,Lerche & Stieberger]:

$$y^2 = x^3 + \alpha z^4 x + (z^5 + \beta z^6 + z^7)$$

$$J(T)J(U) = -1728^2\left(\frac{\alpha}{3}\right)^3$$

$$(J(T) - 1728)(J(U) - 1728) = +1728^2\left(\frac{\beta}{2}\right)^2$$

Arithmetic Mirror Maps

Shioda-Inose + Complex multiplication motivates the

> Conjecture: the $K3$ mirror map should behave analogously to the elliptic functions in the theory of complex multiplication: $y^i \in K_D \to \alpha_i, \beta_i \in \hat{K}$

We have checked this for many families of K3's

1. Lian & Yau commensurability: 1-parameter families of $K3$-surfaces:

$$P(j, x) = 0$$

$x = x(q)$ is the mirror map. Therefore algebraic over $\hat{K}$.

2. Morrison-Vafa family: follows from CM theory.

3. Friedman-Morgan-Witten: $T \to \infty$ and stable degenerations:

$$(\vec{y}, U) \quad \overset{\text{ellipticfunctions}}{\to} \quad \{(x_i, y_i)\}_{i=1,\ldots 8}$$

$$\overset{\text{algebraic}}{\to} \quad \{(\vec{\alpha}, \vec{\beta})\}$$

Attractors for Calabi-Yau 3-folds

Arithmetic Attractor Conjecture: Suppose $\gamma \in \Lambda$ determines an attractor point $z_*(\gamma) \in \mathcal{M}$. Then the special coordinates X^I, F_I are valued in a number field K_γ, and X_γ is an arithmetic variety over some finite Galois extension of K_γ.

Important* distinction: Attractors of rank 1 & rank 2

Suppose

$$\mathrm{Im}(2\bar{C}_1\Omega) = \hat{\gamma}_1$$
$$\mathrm{Im}(2\bar{C}_2\Omega) = \hat{\gamma}_2$$

$$\langle \hat{\gamma}_1, \hat{\gamma}_2 \rangle = \mathrm{Im}(\bar{C}_1 C_2)\Big[2i\langle \Omega, \bar{\Omega}\rangle\Big] = 2e^{-K}\mathrm{Im}(\bar{C}_1 C_2)$$

If a complex structure satisfies the attractor equation for two different nonzero vectors $\gamma_1, \gamma_2 \in H_3(X;\mathbb{Z})$. Then either

a.) $\langle \hat{\gamma}_1, \hat{\gamma}_2 \rangle \neq 0$ "Attractor of rank 2" $\Rightarrow$

$$\Omega = \frac{1}{2\mathrm{Im}(\bar{C}_1 C_2)}\Big(C_1\hat{\gamma}_2 - C_2\hat{\gamma}_1\Big)$$
$$H^{3,0} \oplus H^{0,3} = T_X \otimes \mathbb{C} \qquad T_X \subset H^3(X;\mathbb{Z})$$

b.) $\hat{\gamma}_1 = r\hat{\gamma}_2$ for $r \in \mathbb{Q}$ "Attractor of rank 1"

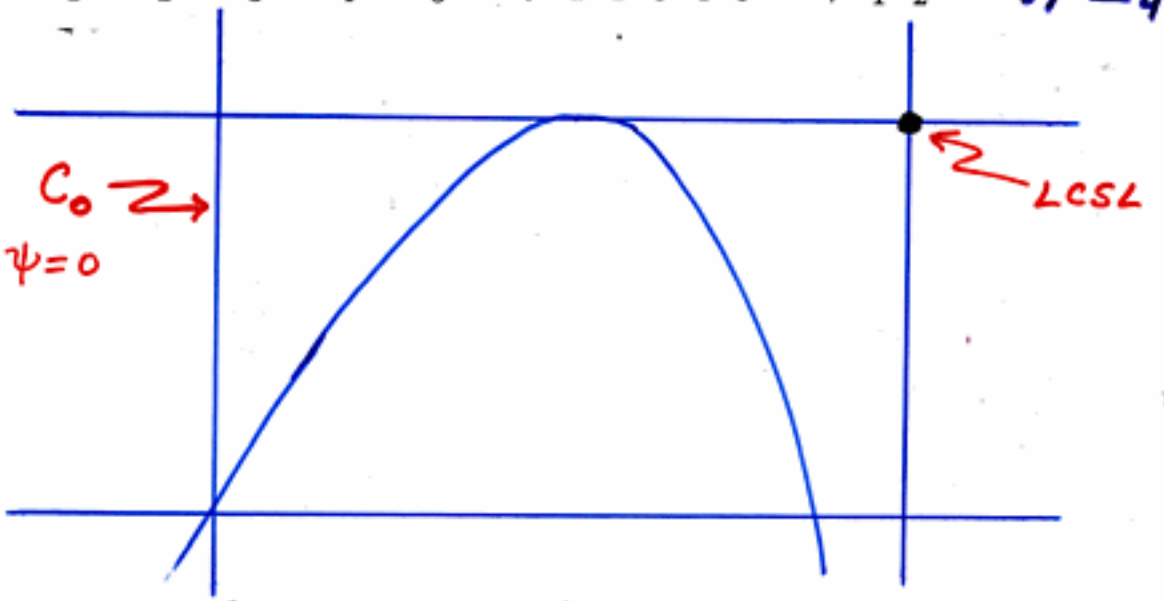
Weak Attractor Conjecture: Only asserts this for rank 2: much more likely*

* Thanks to P. Deligne for emphasizing this.

Nontrivial Exact CY Attractors

C, d10, F
K, M

$$\left\{ x_1^8 + x_2^8 + x_3^4 + x_4^4 + x_5^4 - 8\psi x_1 x_2 x_3 x_4 x_5 - 2\phi x_1^4 x_2^4 = 0 \right\} / \mathbb{Z}_4^3$$



$$\begin{aligned} \psi^{-1}\varpi_j(\psi,\phi)|_{\psi=0} &= \alpha^j u_\nu(\phi) \qquad j \text{ even} \\ &= \alpha^j u_\nu^D(\phi) \qquad j \text{ odd} \end{aligned}$$

where $\alpha = e^{2\pi i/8}$, $\nu = -1/4$.

Define

$$\tau(\phi) = i\sqrt{2}u_\nu^D(\phi)/u_\nu(\phi)$$

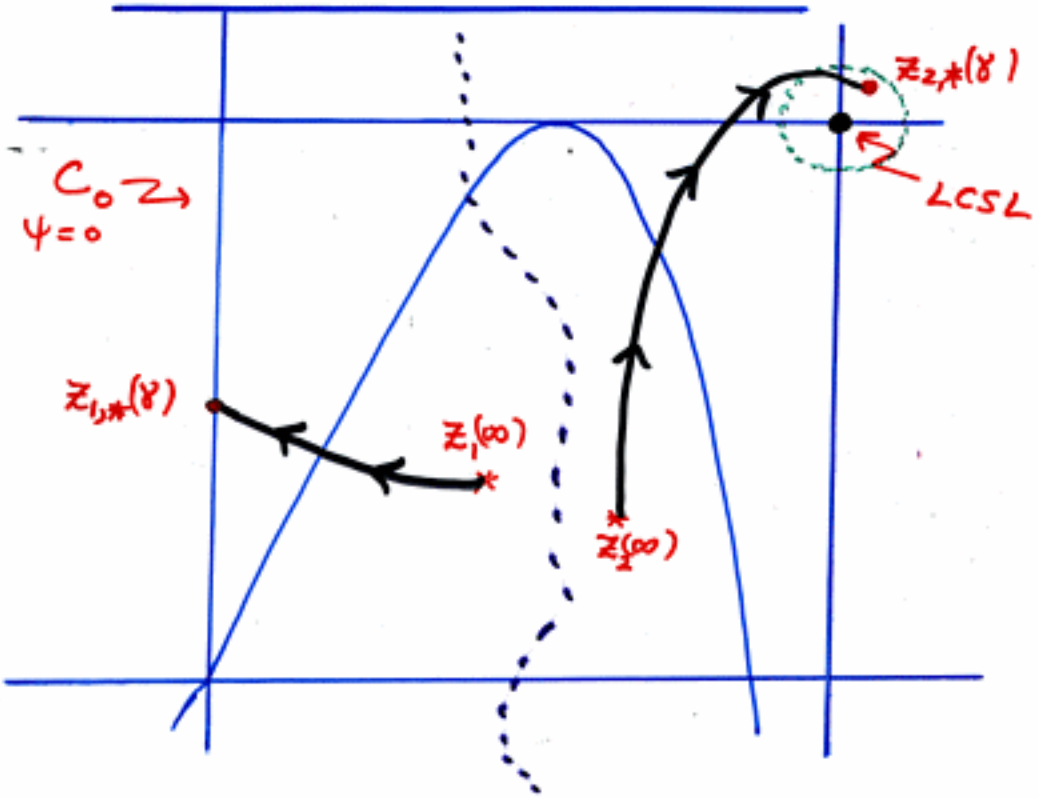
Attractor points: $\tau = a + bi \in \mathbb{Q}[i]$!

All these attractor points are of rank 2.

By CM theory, ϕ^2 is in a class field of K_D.

This confirms the weak attractor conjecture on the divisor C_0.

Multiple minima & attractive hair



The charges γ leading to attractor points on C_0 *also* have solutions near the LCSL (mirror to the large radius limit in IIA).

$\Rightarrow d=4, \mathcal{N}=2$ black holes have extra *attractive hair*: the label of the basin of attraction of the attractor dynamical system

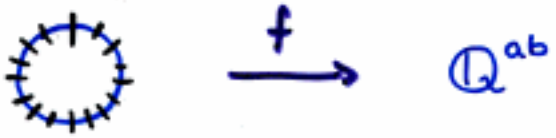
N.B. The extremal mass (=horizon area =entropy) is *different* at the two points: The entropy depends on attractive hair.

Kronecker's Jugendtraum ...

Kronecker considered the problem of finding all *abelian* extensions of the number field $\mathbb{Q}$.

Answer (the Kronecker-Weber theorem):

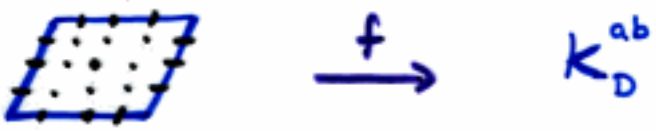
There is a "magical" transcendental function, $\theta \mapsto f(\theta) = \exp[2\pi i\theta]$ whose values $f(\theta)$ on torsion points in the circle $\theta \in \mathbb{Q}$ generate all abelian extensions of $\mathbb{Q}$.



Formally: all abelian extensions of $K = \mathbb{Q}$ are subfields of some cyclotomic extension $\mathbb{Q}[e^{2\pi i/n}]$.

Kronecker's Jugendtraum: An analogous situation holds with $K = \mathbb{Q}$ replaced by imaginary quadratic fields $K = K_D$, and $f(x)$ replaced by the elliptic functions: $j(\tau), \wp(z,\tau), \wp'(z,\tau)$ evaluated on torsion points of elliptic curves E_τ of CM type.

Kronecker's Jugendtraum is true: This is the theory of complex multiplication, mentioned above.



... and Hilbert's 12th problem

In his famous address to the ICM in 1900 Hilbert posed his 12th problem in which he encouraged mathematicians to

" ... succeed in finding and discussing those functions which play the part for any algebraic number field corresponding to that of the exponential function in the field of rational numbers and of the elliptic modular functions in the imaginary quadratic number field."

This has been partially solved by the Shimura-Taniyama theory of abelian varieties of CM type.

In view of the above remarks on complex multiplication, K3 mirror symmetry, and the attractor points, one cannot help speculating that the transcendental functions provided by the mirror map are just the functions which Hilbert was seeking.

The attractor conjecture proposes the generalization:

$$\begin{array}{rcl}
\text{elliptic curve} & \rightarrow & \text{Calabi} - \text{Yau d} - \text{fold} \\
\tau \in \mathcal{H}_1 & \rightarrow & t^a \in \widetilde{\mathcal{M}} \\
\text{discriminant } D & \rightarrow & \gamma \in H^d(X;\mathbb{Z}) \\
a\tau^2 + b\tau + c = 0 & \rightarrow & 2\mathrm{Im}\bar{C}\Omega = \hat{\gamma} \\
\mathbb{Q}[\sqrt{D}] & \rightarrow & K(\gamma) \\
\tau \mapsto j(\tau) & \rightarrow & \text{The mirror map}
\end{array}$$

Conclusions

• Black holes/strings in compactifications with 16,32 supercharges are connected with *lots* of interesting arithmetical phenomenon, thanks to the attractor mechanism.

• There is an action of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on attractor points, whose physical role is unclear...

• We believe these arithmetical phenomena carry over to theories with 8 supercharges, but we have only verified that in a few examples.

• The known examples suggest that the transcendental mirror map functions have arithmetical properties generalizing those of the j-function ($\Rightarrow$a generalization of Kronecker's Jugendtraum).