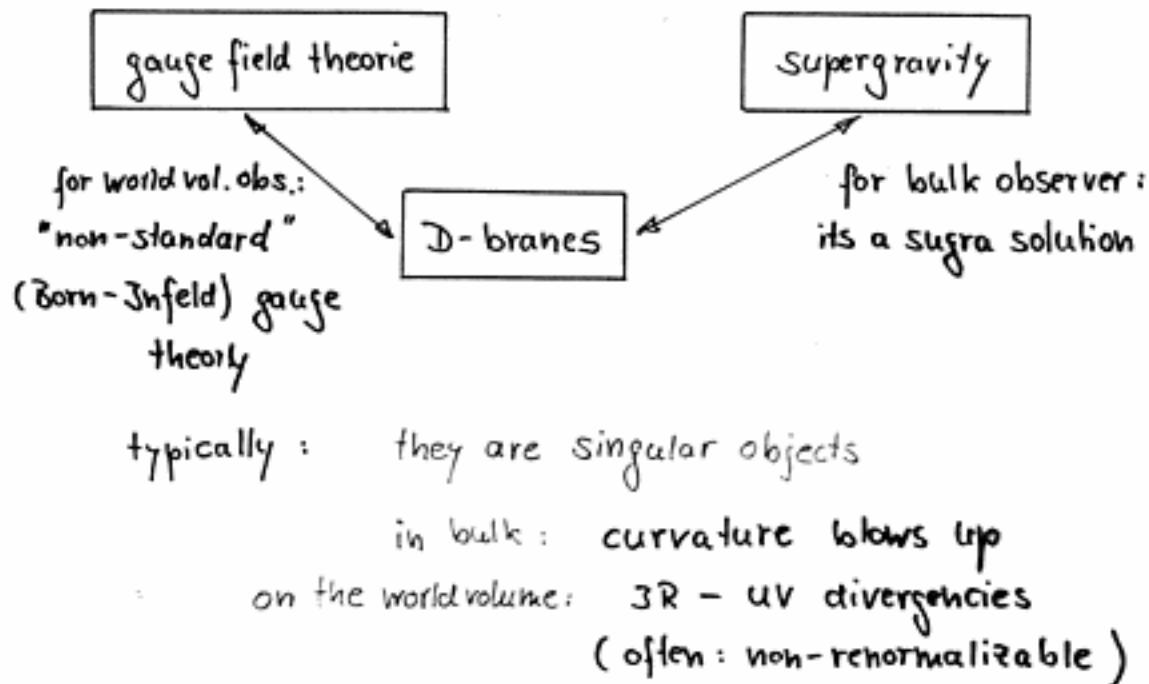


# CFT's on the boundaries of $AdS_3$

(Klaus Behrndt, Jilka Brunner, Jürg Gaida)

## Motivation

D-branes: provide link between  
two (completely different) theories



but still : both parts may be well-defined

→ singularities signals:

both parts do not decouple

i.e. starting at a certain energy scale  
one cannot neglect one or the other side

e.g. for worldvolume theory one has to  
take into account bulk gravity & vice versa

however : it is very difficult to consider both sides at the same time

better : to look for limits/cases in which both theories decouple  
→ one can expect to obtain two dual formulations

obvious candidates: branes with non-singular horizon

they have in common: 1) space time factorizes:

$$\text{AdS}_p \times S_q \times E_r$$

2) they scalar-free  
(solution of Einstein - Maxwell)

dimension	p	geometry
11	5	$\text{AdS}_7 \times S_4$
	2	$\text{AdS}_4 \times S_7$
10	3	$\text{AdS}_5 \times S_5$
(*) 8	2	$\text{AdS}_4 \times S_4$
6	1	$\text{AdS}_3 \times S_3$
5	1	$\text{AdS}_3 \times S_2$
	0	$\text{AdS}_2 \times S_3$
4	0	$\text{AdS}_2 \times S_2$

(\*) is special:  $ds^2 = H^{-\frac{2}{3}} dx_{11}^2 + H^{\frac{2}{3}} dx_{\perp}^2$ ,  $H = 1 + g_{11}$   
does not appear as (standard)  
compactifications of string- or M-theory

AdS group : is realized as conformal group  
on the boundary of the AdS space

conjecture : boundary CFT is dual to superconformal  
world volume field theory

Maldacena

special interest : odd AdS cases

they are symmetric on both sides  
of the horizon (no singularity beyond  
the horizon)

Gibbons, Horowitz, Townsend

$\text{AdS}_7$  :  $\rightarrow$  non-critical string theory in 6d

Claus, Kallosh, van Proeyen

$\text{AdS}_5$  :  $\rightarrow$  4d super Yang Mills

Maldacena, Witten, Rey, Yee,  
Gubser, Klebanov, Polyakov

$\text{AdS}_3$  :  $\rightarrow$  2d non-linear  $\sigma$ -model

Strominger, Vafa, Verlinde

Callan, Maldacena, Kaloper, Cicuta,  
Larsen, ...

$\text{AdS}_{5/7}$  : in order to trust these solutions

$\rightarrow$  large radius of AdS space (large  $N$ )

&  $\lambda' \rightarrow 0$

## AdS<sub>3</sub> is Special:

1) conformal algebra on boundary  
↓

enhanced to  $\infty$ -dimensional Virasoro algebra (2d CFT)  
Brown, Henneaux

⇒ we can make exact statements

we do not need to consider large  $N$  and/or  $\alpha' \rightarrow 0$

→ we avoid "trouble" with possible phase transitions  
that would spoil the extrapolation from  
 $\text{large } N \rightarrow \text{finite } N$

we will see:  $\alpha'$  corrections → qualitative different  
picture emerge

→ "sensible" to different  
topology

2) AdS<sub>3</sub> = near horizon geometry of strings

→ this model is important for our  
understanding of the BH horizon / entropy

## The way to $AdS_3$ gravity

(consistent) examples: near-horizon geometry of self-dual strings in 6d

metric:  $ds^2 = \left(1 + \frac{q}{r^2}\right)^{-1} du dv + \left(1 + \frac{q}{r^2}\right) [dr^2 + r^2 d\Omega_3]$

torsion:  $H = d\frac{1}{H} \wedge du \wedge dv + \star(dH \wedge du \wedge dv)$ ,

for  $\tau \rightarrow 0$ :

$$ds^2 = \frac{1}{q} du dv + \frac{q}{r^2} dr^2 + q d\Omega_3 \quad (*)$$

$\underbrace{\hspace{100px}}$   
 $AdS_3$ 
 $\underbrace{\hspace{100px}}$   
 $S_3$

reminder:  $AdS_3$  is defined as  $\neg \rightarrow$  neg. cosm. const.

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 - (x^3)^2 = -\ell^2$$

= hyperboloid in 4d space with signature  $(- + + -)$

also:  $AdS_3$  is group space of  $SL(2, \mathbb{R})$

$$g = \frac{1}{\ell} \begin{pmatrix} x^0 + x^1 & x^2 - x^3 \\ x^2 + x^3 & x^0 - x^1 \end{pmatrix} = \frac{1}{\ell} \begin{pmatrix} x_+ & u \\ v & x_- \end{pmatrix}$$

choosing:  $U = \frac{u r}{\ell}$ ,  $V = \frac{v r}{\ell}$ ,  $X_+ = r$

inserting in:  $ds^2 = -dx_+ dx_- + dU dV$

$\rightarrow$  above  $AdS_3$  metric in (\*)

String model ( $\times$ )  $\rightarrow$  is too simple

(also it has too much supersymmetry)

0) consider:  $I_p \times S_{NS}$  (fund. string inside a S-brane)

therefore : 1) add waves (along the string)

2)  $S_3 \rightarrow S_3/\mathbb{Z}_n$

by adding a Taub-NUT Soliton

3) make it non-extremal

4) make transformation:

$$(u, v) \rightarrow \sqrt{\frac{q}{\epsilon}}(u, v)$$

$$t \rightarrow \frac{r^2}{\epsilon} - q_0 \tanh \beta$$

wave parameter

non-extremality param.

$I_p \times S_{NS}$  + wave + Taub-NUT

$$\Rightarrow ds^2 = -e^{-2V} dt^2 + e^{2V} dr^2 + \left(\frac{r}{\epsilon}\right)^2 \left[ dy - \frac{r_- r_+}{r^2} dt \right]^2 + ds^2_{S_3/\mathbb{Z}_n}$$

$$\text{with: } e^{-2V} = \frac{(r_-^2 - r_+^2)(r^2 - r_\pm^2)}{r^2 \ell^2}, \quad r_\pm^2 = \frac{\epsilon q_0}{\tanh \beta^2}, \quad \ell = 2R_{S_3}$$

$$= \text{BTZ black hole} + S_3/\mathbb{Z}_n \\ (= \underbrace{\text{AdS}_3/\mathbb{Z}_n}_{})$$

AdS<sub>3</sub> with a discrete subgroup modded out.

$\rightarrow$  example for topological black hole

$\rightarrow$  solution of 3d Einstein-deSitter:

$$S_3 \sim \int (R - 2\Lambda)$$

negative cosmological const.

can be reformulated as 3d Chern-Simons model

Achucarro, Townsend, Witten

i.e.  $S \rightarrow S_{CS}[A] - S_{CS}[\bar{A}]$

AdS group:  $SO(2,2) \simeq SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$

( left- & right moving sector )

with:  $A/\bar{A} = (\omega^a \pm \frac{1}{\epsilon} e^a) T_a$

$$S_{CS}[A] = \frac{k}{4\pi} \int \text{Tr} (A dA + \frac{2}{3} A^3)$$

$$\omega^a = \frac{1}{2} \epsilon^{abc} \omega_{bc} \quad \text{Spin connection}$$

$$e^a \quad \text{3-bein}$$

$$T_a \quad SL(2, \mathbb{R}) \text{ generators}$$

inserting the metric:

$$A = \left[ e^{-v} T_0 + \frac{r}{\epsilon} \left( 1 - \frac{T_+ T_-}{r^2} \right) T_1 \right] \frac{dv}{\epsilon} + e^v \left( 1 + \frac{T_+ T_-}{r^2} \right) T_2 \frac{dr}{\epsilon}$$

$$\bar{A} = \left[ e^{-v} T_0 - \frac{r}{\epsilon} \left( 1 + \frac{T_+ T_-}{r^2} \right) T_1 \right] \frac{du}{\epsilon} - e^v \left( 1 - \frac{T_+ T_-}{r^2} \right) T_2 \frac{dr}{\epsilon}$$



both fields are pure gauge!

$$\text{if: } g = e^{\Theta_1 T_1} e^{\lambda T_2} e^{\Theta_2 T_1}$$

$$\text{then: } A = g^{-1} dg \quad \text{with: } \Theta_1 = 0, \Theta_2 = v, \sinh \lambda = \frac{e^{-v}}{T_+ - T_-}$$

$$\bar{A} = \bar{g}^{-1} d\bar{g} \quad \text{with: } \Theta_1 = 0, \Theta_2 = u, \sinh \lambda = -\frac{e^{-v}}{T_+ + T_-}$$

## Way to the CFTs

2 Steps:

1) Solutions for  $A$  &  $\bar{A}$

- do not follow not as solutions from the variation of  $S$   
(boundary values do not comply with variational principle)
- have to add boundary terms to the action

2) non-trivial boundary behavior



Chern Simons gauge symmetry is broken



gauge degrees of freedom become dynamical on boundaries



degrees of freedom of CFT

but: not all gauge degrees of freedom are broken

→ BTZ model has residual symmetry



has to be modded out



boundary CFT: gauged  $SL(2, \mathbb{R})$  modd

to 1):  $S_{CS}[A] - S_{CS}[\bar{A}] \rightarrow S_{CS}[A] - S_{CS}[\bar{A}] + \frac{k}{8\pi} \int_{\partial M} (A_u A_u + \bar{A}_v \bar{A}_u)$

then  $\delta S = 0 \Leftrightarrow$

$F = \bar{F} = 0 \text{ in } M$
$A_u = \bar{A}_v = 0 \text{ on } \partial M$

→ correct eq. of motion + boundary condition

→  $F = \bar{F} = 0 \hat{\wedge}$  only gauge degrees of freedom

for 2)  $\rightarrow$  have to consider both boundaries separately

a) asymptotic boundary ( $r \rightarrow \infty$ )

residual symmetry:  $g \rightarrow \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} g, \bar{g} \rightarrow \begin{pmatrix} 1 & \alpha_2 \\ 0 & 1 \end{pmatrix} \bar{g}$

$\rightarrow$  correspond to reparameterization of  $u \in U$   
 (recall:  $A = g^{-1} dg, \bar{A} = \bar{g}^{-1} d\bar{g}$ )

these are isometry  
directions

- next: inserting  $A$  &  $\bar{A}$  into the action
  - $\rightarrow S_{CS} \rightarrow S_{WZW}$  get a WZW model
- taking into account the residual symmetry
  - $\rightarrow$  the WZW model has to be gauged ( $T_\pm$  direction)
- $\Rightarrow S_{WZW} \rightarrow S_L$  WZW becomes Liouville model



$$S_L = \frac{k-2}{4\pi} \int ((\partial_u \lambda)^2 \nu \lambda + Q R'' \lambda + \mu e^{-2\lambda})$$

$\uparrow \quad \uparrow \quad \uparrow$   
 $Q = \frac{1-k}{k-2} \quad \sqrt{\mu} = \epsilon \quad e^{-\lambda} = \frac{r}{e}$

Liouville field = radial fluctuations

central charge:

$$C_L = \frac{3k}{k-2} - 2 + 6k$$

$(k \sim \frac{g^2}{x})$

classical value  
(for  $\alpha' \rightarrow 0$ )

### b) Horizon boundary

• residual symmetry:  $g \rightarrow g e^{\alpha T_1}$ ,  $\bar{g} \rightarrow \bar{g} e^{\alpha T_1}$

which again corresponds to reparameterizations of  $u$  &  $v$ , which are isometry directions for  $A$  &  $\bar{A}$  ( $A = g^{-1} dg$ ,  $\bar{A} = \bar{g}^{-1} d\bar{g}$ )

⇒ we have to gauge this group direction



$$S_{WZW} \rightarrow S_{WZW} + \frac{k}{2\pi} \int_{\partial\Omega} [a_v (\partial_u \theta_L + \cosh \lambda \partial_u \theta_R) + \\ + a_u (\partial_v \theta_R + \cosh \lambda \partial_v \theta_L) - a_u a_v (\cosh \lambda + 1)]$$

→ this CFT describes a 2-d black hole

In order to obtain target space metric → one integrates out  $a_v$ ,  $a_u$  or considers the  $L_0$  operator as target space Laplacian:

$$\Rightarrow ds^2 = Z(k-z) [d\lambda^2 - \tilde{B}^{-2} d\theta_i^2]$$

$$\tilde{e}^{-2\tilde{\phi}} = \tilde{B} \cosh \lambda \sinh \lambda, \quad \tilde{B}^2 = (\coth^2 \lambda - \frac{z}{k})$$

Verlinde, Verlinde, Dijkgraaf

however: BTZ black hole is asymptotically not flat  
and is dilaton-free

So, where is the 2-d black hole?

→ One can first do a T-duality ⇒ BTZ → black string

⇒ string direction = isometry direction

Horowitz, Welch, Keeler

→ reduction → yields 2-d black hole

central charge:

$$C_{AdS_3} = \frac{3k}{k-2} - 1$$

11

Putting things together

Starting point:

$$(I_p \times S_{NS} + \text{Ward + KK mon.})_{\text{non-extreme}}$$

$\tau \rightarrow 0$  (near the horizon)

$$AdS_3/\mathbb{Z}_n \otimes S_3/\mathbb{Z}_m \otimes E_4$$

CFT's:



$$\frac{SU(2)}{\mathbb{Z}_n} - WZW$$

BTZ horizon

- remarks:
- 1) BTZ black-hole  $\hat{=}$   $AdS_3$  space has spatial annulus geometry
  - 2) 2d-BH  $\hat{=}$   $\frac{SL(2, \mathbb{R})}{U(1)}$  - WZW model
  - 3) Liouville  $\hat{=}$   $\frac{SL(2, \mathbb{R})}{SO(1, 1)}$  - WZW model

- BTZ black hole interpolates between 2d-BH & Liouville
- it rotates the  $U(1)$  direction: spatial to lightcone  $U(1)$  is modded out

total central charge:

$$C_{\text{tot}} = C_{SU} + C_{2d-3H} + C_L = \left( \frac{3k}{k+2} - 1 \right) + \left( \frac{3k}{k+2} - 1 \right) + \left( \frac{3k}{k+2} - 2 + 6k \right)$$

classical limit:  $\alpha' \rightarrow 0$ ,  $k = \frac{e^2}{4\alpha'} \rightarrow \infty$

$$C_{\text{tot}} \rightarrow 6k$$

Susy case:  $k \rightarrow k-2$  in  $SU(2)$  part  
 $k \rightarrow k+2$  in  $SL(2)$  part

$$C_{\text{tot}} = 6(k + \frac{1}{k}) + \text{const.} \quad \text{sum is truncated !!}$$

→ invariant under:

$$k \rightarrow \frac{1}{k} \quad \text{or} \quad \ell \rightarrow \frac{4\alpha'}{\ell} \quad \text{or} \quad \ell \alpha' \rightarrow \frac{e^2}{\ell \alpha'}$$

"T-duality"  
 in cosmological constant

"S-duality"  
 in  $\alpha'$

- note:
- 1) this is a symmetry of  $AdS_3$  gravity  
but in our case  $k$  gives also  
 the level of the  $SU(2)$  part  
 $\Rightarrow$  it is a positive integer  
 $\Rightarrow$   $k=1$  lowest possible value
  - 2) at the self-dual point ( $k=1$  susy,  $k=3$  bos)  
 $\Rightarrow$  Liouville model  $\hat{=}$  "massless tachyon"  
 (puncture of 2d boundary)
  - 3)  $\frac{1}{k}$  term produces energy bound

## Entropy

Let us restrict on extreme case:

$$S = 2\pi \sqrt{\frac{1}{6} c N} + \text{logarithmic terms}$$

Cardy

$N = N_L$  : momentum number

→ can be determined from level matching

heterotic case:  $N = 1 + g_0 g_1$

$$\text{central charge: } c = 6 \left( \frac{p^2 p^3}{\alpha'} + \text{const.} + \underbrace{\frac{\alpha'}{p^2 p^3}}_{\alpha' \text{-corrections}} \right)$$

$$(k_c = \frac{p^2 p^3}{\alpha'})$$

$p^2, p^3$  - 5-brane &

Taub-NUT charge

$\alpha'$ -corrections

Correspond to  $R^2$  corrections on Sugra side

$$D=4, N=4 \text{ sugra: } S_4 \rightarrow S_4 + \frac{i}{16\pi} \int \text{Re} \left[ \frac{1}{2\pi i} \log \gamma^2(s) \text{tr}(Z - i \star R)^2 \right]$$

Harvey, Moore

Bek.-Haw. entropy:

$$S = \pi \left( r^2 e^{-2u} \right)_{r=0}$$

$$(ds^2 = -e^{2u} dt + e^{2u} d\vec{x}^2)$$

inclusion of  $R^2$  by:

$$F(x) \rightarrow F^0(x) + F^1(x) W^2 + \dots$$

prepotential  $\stackrel{\dagger}{F}{}^1$ -function

Weyl superfield

$$e^{-2u} = i(\bar{x}^I \bar{F}_I - \bar{F}_I x^I)$$

$$\text{with: } i(x^I - \bar{x}^I) = H^I$$

$$i(\bar{F}_I - \bar{F}_I) = H_I$$

$(\bar{x}^I, \bar{F}_I)$ : sympl. section

$(H^I, H_I)$ : harm. function

gives the same entropy as in CFT !