

Making manifest the symmetry enhancement for coinciding BPS branes

Sergei V. Ketov¹

Institut für Theoretische Physik, Universität Hannover, Appelstraße 2, D-30167
Hannover, Germany

Abstract. We consider $g = N - 1$ coinciding M-5-branes on top of each other, in the multiple KK monopole background Q . The worldvolume of an M-5-brane is the local product of the four-dimensional spacetime $R^{1,3}$ and an elliptic curve. Taken together, these genus-one Riemann surfaces are supposed to give a single (Seiberg-Witten) hyperelliptic curve Σ_g in the coincidence limit, where the gauge symmetry is known to be enhanced to $SU(N)$. We make this gauge symmetry enhancement manifest by analyzing the corresponding *hypermultiplet* LEEA which is given by the $N=2$ non-linear sigma-model having Q as the target space. The hyper-Kähler manifold Q is known to be given by the multicentre Taub-NUT space, which in the coincidence limit amounts to the multiple Eguchi-Hansen (ALE) space Q_{mEH} . The latter is most naturally described by using the hyper-Kähler coset construction in harmonic superspace, in terms of the auxiliary $N=2$ vector superfields as Lagrange multipliers in the presence of FI terms. The Maldacena limit, in which the LEEA is given by the $N=4$ SYM with the gauge group $SU(N)$, corresponds to sending all the FI terms to zero at large N , when all the auxiliary $N=2$ vector superfields become dynamical.

1 Brane technology and KK monopoles

The (Seiberg-Witten-type) exact solution to the $N=2$ super-QCD can be identified with the LEEA of the effective (called $N=2$ MQCD) field theory defined in the worldvolume of the *single* M-5-brane, locally given by the product of the uncompactified four-dimensional spacetime $R^{1,3}$ and the hyperelliptic (Seiberg-Witten) curve Σ_g of genus $g = N_c - 1$: Witten (1997) (see, e.g. Karch et al. (1998), Ketov (1997), Ketov (1998) for a review or an introduction). The hyperelliptic curve Σ_g is supposed to be holomorphically embedded into the hyper-Kähler four-dimensional multiple Taub-NUT space Q_{mTN} associated with the multiple KK-monopole. The identification of the *low-energy effective actions* (LEEA) in the two apparently different field theories (the $N = 2$ super-QCD in the Coulomb branch on the one side, and the $N = 2$ MQCD defined in the M-5-brane world-volume, on the other side) is highly non-trivial, since the former is defined as the leading contribution to the quantum LEEA in quantum field theory, whereas the latter is determined by the *classical* M-5-brane dynamics or the $D = 11$ supergravity equations of motion, whose BPS solutions preserving some part of supersymmetry are the M-theory branes under consideration.

1.1 Multiple KK monopole

The multiple KK monopole is a non-singular BPS solution to the eleven-dimensional supergravity equations of motion, given by the product of the seven-dimensional (flat) Minkowski spacetime $R^{1,6}$ and the four-dimensional Euclidean multicentre Taub-NUT space Q_{mTN} (Townsend (1995)):

$$ds_{[11]}^2 = dx^\mu dx^\nu \eta_{\mu\nu} + H(dy)^2 + H^{-1}(d\rho + \mathbf{C} \cdot d\mathbf{y})^2, \\ \nabla \times \mathbf{C} = \nabla H, \quad F_{(4)} \equiv dA_{(3)} = 0, \quad (1)$$

where $\mu = 0, 1, 2, 3, 7, 8, 9$, $\mathbf{y} = \{y_i\}$, $i = 4, 5, 6$, the eleventh coordinate ρ is supposed to be periodic (with the period $2\pi k$ – this is just necessary to avoid conical singularities of the metric), while the harmonic function $H(\mathbf{y})$ is given by

$$H(\mathbf{y}) = 1 + \sum_{p=1}^N \frac{|k|}{2|\mathbf{y} - \mathbf{y}_p|}. \quad (2)$$

The moduli (k, \mathbf{y}_i) can be interpreted as (equal) charges and the locations of the KK monopoles, respectively. The multiple Taub-NUT space can be thought of as a non-trivial bundle (Hopf fibration) with the base R^3 and the fiber S^1 of magnetic charge k . There exist N linearly independent normalizable self-dual harmonic 2-forms ω_p in Q_{mTN} , which satisfy the orthogonality condition (see Gibbons et al. (1988), and references therein)

$$\frac{1}{(2\pi k)^2} \int_{Q_{\text{mTN}}} \omega_p \wedge \omega_q = \delta_{pq}. \quad (3)$$

As is clear from eq. (1), two adjacent KK monopoles are connected by a homology 2-sphere having poles at the positions of the two monopoles. Near a singularity of H , the KK circle S^1 contracts to a point. A holomorphic embedding of the SW curve Σ_g into the hyper-Kähler manifold Q_{mTN} is the consequence of the *BPS condition* (Mikhailov (1997))

$$\text{Area}(\Sigma) = \left| \int_{\Sigma} \Omega \right|, \quad (4)$$

where Ω is the Kähler form in Q_{mTN} .

1.2 $N=2$ QCD LEEA in Coulomb branch from brane dynamics

The geometrical origin and the physical interpretation of the hyperelliptic curve Σ_g , parameterizing the exact Seiberg-Witten solution to the LEEA of $N=2$ supersymmetric QCD in the Coulomb branch, becomes transparent when using brane technology after M-theory resolution of UV singularities (Witten (1997)), where Σ_g appears to be the part of the M-5-brane worldvolume in eleven dimensions. The Nambu-Goto (NG) term (proportional to the M-5-brane worldvolume) of the effective M-5-brane action in the low-energy

approximation gives rise to the scalar *non-linear sigma model* (NLSM) having the special geometry after the KK compactification of the six-dimensional NG action on the Seiberg-Witten curve Σ_g down to four spacetime dimensions. This is enough to unambiguously restore the full $N = 2$ supersymmetric Seiberg-Witten LEEA, by the use of $N = 2$ supersymmetrization of the special bosonic NLSM, when considering its complex scalars as the leading components of abelian $N = 2$ vector multiplets in four spacetime dimensions and then deducing the Seiberg-Witten holomorphic potential out of the already derived special Kähler NLSM potential (see Ketov (1998) for a review).

Being applied to a derivation of the *hypermultiplet* LEEA of $N = 2$ super-QCD in the Coulomb branch, brane technology suggests to dimensionally reduce the effective action of a D-6-brane (to be described in M-theory by a KK-monopole) down to four spacetime dimensions (Ketov (1998)). In a static gauge for the D-6-brane, the induced metric in the brane worldvolume is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + G_{mn} \partial_\mu y^m \partial_\nu y^n, \quad (5)$$

where $\mu, \nu = 0, 1, 2, 3, 7, 8, 9$, $m, n = 4, 5, 6, 10$, and G_{mn} is the multicentre ETN metric. After expanding the NG-part of the D-6-brane effective action

$$S_{NG} = \int d^7\xi \sqrt{-\det(g_{\mu\nu})} \quad (6)$$

up to the second-order in the spacetime derivatives, and performing plain dimensional reduction down to four dimensions, one arrives at the hyper-Kähler NLSM

$$S[y] = -\frac{1}{2} \int d^4x G_{mn}(y) \partial_\mu y^m \partial^\mu y^n, \quad \underline{\mu} = 0, 1, 2, 3, \quad (7)$$

whose $N = 2$ supersymmetrization yields the full hypermultiplet LEEA, in precise agreement with the $N = 2$ supersymmetric quantum field theory calculations in harmonic superspace (Ivanov et al. (1997)).

2 Symmetry enhancement for 2 coinciding D-6-branes

As is well known, the isolated singularities of the harmonic function (2) are just the coordinate singularities of the *eleven*-dimensional metric (1), though they are truly singular with respect to the (dimensionally reduced) *ten*-dimensional metric to be associated with the D-6-branes in the type-IIA picture. The physical significance of these ten-dimensional singularities can therefore be understood due to the illegitimate neglect of the KK modes related to the compactification circle S^1 in ten dimensions, since the KK particles (also called D-0-branes) become massless near the D-6-brane core (Townsend (1995)). Their inclusion is equivalent to accounting instanton corrections in the four-dimensional $N=2$ supersymmetric gauge field theory.

When some parallel and similarly oriented D-branes coincide, the symmetry enhancement happens (Hull and Townsend (1995), Witten (1996)). Since the brane singularities become non-isolated in the coincidence limit, they first have to be resolved by considering the branes separated by some distance r and then taking the limit $r \rightarrow 0$. In the simplest non-trivial case of two D-6-branes, one considers the harmonic function

$$H(\mathbf{y}) = \lambda + \frac{1}{2} \left\{ \frac{1}{|\mathbf{y} - \xi \mathbf{e}|} + \frac{1}{|\mathbf{y} + \xi \mathbf{e}|} \right\}, \quad r = 2\xi, \quad (8)$$

describing the double Taub-NUT metric in (1) with non-vanishing constant potential at infinity, and both centers on the line \mathbf{e} in 6-th direction, $\mathbf{e}^2 = 1$. After being substituted into eq. (1), eq. (8) describes two parallel and similarly oriented KK-monopoles in M-theory, which dimensionally reduce to the double D-6-brane configuration in ten dimensions. The homology 2-sphere connecting the KK monopoles contracts to a point in this limit, which gives rise to a curvature singularity of the dimensionally reduced metric in ten dimensions.

The BPS states in M-theory, whose zero modes appear in the effective $D = 4$ field theory defined in the M-5-brane worldvolume, correspond to the minimal area M-2-branes ending on the M-5-brane. The topology of an M-2-brane determines the type of the corresponding $N = 2$ supermultiplet in $D = 4$: a cylinder leads to an $N=2$ vector multiplet, whereas a disc gives rise to a hypermultiplet (Mikhailov (1997)). The M-2-branes can wrap about the 2-sphere connecting the KK monopoles, while the energy of the wrapped M-2-brane is proportional to the area of the sphere (Hull and Townsend (1995)). When the sphere collapses, its area vanishes and, hence, an additional massless vector state appears due to the zero mode of the wrapped M-2-brane. One thus expects the gauge symmetry enhancement from $U(1) \times U(1)$ to $U(2)$ associated with the A_1 -type singularity (Ooguri and Vafa (1996)). From the ten-dimensional perspective, the wrapped M-2-branes are just the (6-6) superstrings stretched between the D-6-branes, so that it is the massless zero modes of these 6-6 superstrings that become massless in the coincidence limit.

In order to make this symmetry enhancement manifest, let's start with the hypermultiplet low-energy effective action to be obtained by $N = 2$ supersymmetrization of the bosonic NLSM (7) in four spacetime dimensions, whose hyper-Kähler (double Taub-NUT) metric is determined by the harmonic function (8). In terms of this NLSM metric, the symmetry enhancement amounts to the appearance of gauged isometries in the NLSM target space, while the latter can be made manifest in the $N = 2$ harmonic superspace, as we are now going to demonstrate. First, let's note that the $N = 2$ supersymmetric double Taub-NUT NLSM is known to be equivalent to the one with the *mixed* (=Eguchi-Hanson-Taub-NUT) metric (Gibbons et al. (1988)). The mixed NLSM is described by the following $N = 2$ harmonic superspace action over the analytic subspace:

$$S_{\text{mixed}}[q, V] = \int_{\text{analytic}} \left\{ q_A^{a+} D^{++} q_{aA}^+ + V_L^{++} \left(\frac{1}{2} \varepsilon^{AB} q_A^{a+} q_{aB}^+ + \xi^{++} \right) + \frac{1}{4} \lambda (q_A^{a+} q_{aA}^+)^2 \right\} , \quad (9)$$

which is written down in terms of a gauged $O(2)$ analytic hypermultiplet superfield q_A^+ , $A = 1, 2$, and the auxiliary $O(2)$ vector gauge analytic superfield (Lagrange multiplier) V_L^{++} having no kinetic term. The parameters λ in eqs. (8) and (9) can be identified, whereas the parameter $\xi^{++} = \xi^{ij} u_i^+ u_j^+$ in eq. (9) is simply related to the constant ξ appearing in eq. (8) after choosing the coordinate frame in which $\xi^{12} = 2i\xi$ and $\xi^{11} = \xi^{22} = 0$. The hyper-Kähler NLSM metric, which is deduced out of the superspace action (9) after eliminating all the auxiliary fields in components, yields the double Taub-NUT metric, as can be verified by explicit calculation (Gibbons et al. (1988)). This is, in fact, ensured by the manifest $U(1)_A \times U(1)_{\text{PG}}$ symmetry of the superspace action (9), where the first $U(1)_A$ factor is the unbroken part of the $SU(2)_A$ automorphisms of the $N = 2$ supersymmetry algebra rotating two supercharges, whereas the second $U(1)_{\text{PG}}$ symmetry only acts on the pseudo-real indices $a = 1, 2$ in $q^{a+} = (\tilde{q}^+, q^+)$ and implies an abelian isometry of the NLSM metric. Any four-dimensional hyper-Kähler metric having the $U(1)_{\text{PG}}$ isometry is known to be a multicentre Taub-NUT metric (see Gibbons et al. (1988) and references therein).

The mixed four-dimensional hyper-Kähler metric of the $N = 2$ supersymmetric NLSM (9) interpolates between the *Eguchi-Hanson* (EH) metric ($\lambda = 0$) and the Taub-NUT ($\xi = 0$), both having the maximal isometry group $U(2)$. The action of the $U(2)$ isometry is linear in both limiting cases, while it is even holomorphic in the second case. In the harmonic superspace approach, this symmetry enhancement can be simply understood either as the restoration of the $SU(2)_A$ automorphism symmetry in the Taub-NUT limit, or as the restoration of the $SU(2)_{\text{PG}}$ symmetry in the Eguchi-Hanson limit.

3 Symmetry enhancement for N coinciding D-6-branes

The $D = 11$ supergravity approximation to M-theory is only valid for well-separated KK monopoles. When the KK monopoles coincide, their low-energy dynamics can be approximated by weakly coupled superstrings propagating in the multi-Eguchi-Hanson (ALE) background (Sen (1997)). This background naturally originates from the multi-Taub-NUT space. Indeed, when all the D-6-branes coincide, they can be described in M-theory by sending all the moduli \mathbf{y}_p in the harmonic function (2) to zero, so that the additive constant (asymptotic potential) 1 in eq. (2) can be ignored near the core of N D-6-branes on top of each other. The multi-Eguchi-Hanson space thus possesses A_{N-1} simple singularity which implies the enhanced non-abelian gauge symmetry $SU(N)$ in the effective supersymmetric field theory defined in the common worldvolume of the coinciding D-6-branes.

The effective gauge field theory is supposed to be defined in the limit where the gravity decouples. The $D = 11$ supergravity has a 3-form $A_{(3)}^{[11]}$ which is decomposed in the full spacetime given by the product of the D-6-brane worldvolume $R^{1,6}$ and the multi-Taub-NUT space Q_{mTN} as

$$A_{(3)}^{[11]} = \sum_{p=1}^N A_{p(1)}^{[7]} \wedge \omega_{p(2)}^{[4]} , \quad (10)$$

where the 2-forms ω_p in Q_{mTN} have been introduced in subsect. 1.1, whereas A_p are N massless vectors (1-forms) in $R^{1,6}$. In addition, there are $3N$ scalar fields associated with the translational zero modes (or moduli) \mathbf{y}_p . All these vectors and scalars are the bosonic components of N massless vector supermultiplets in $1 + 6$ dimensions, each having $8_{\text{B}} + 8_{\text{F}}$ on-shell components. The gauge group of the effective field theory (in the case of separated KK monopoles) in the Coulomb branch is therefore given by $U(1)^N$. Since the intersection matrix of 2-cycles in Q_{mTN} is just given by the Cartan matrix of A_{N-1} , the abelian gauge symmetry $U(1)^N$ is to be enhanced to $U(N)$ (the non-abelian Coulomb branch) in the coincidence limit (Sen (1997)). The area of the 2-cycles vanishes in this limit, so that the M-2-branes wrapped around these 2-cycles give rise to the additional massless vectors as their zero modes. In the type-IIA picture, the 6-6 strings stretched between separated D-6-branes do not contribute to the effective LEEA in the abelian Coulomb branch. However, since the zero modes of 6-6 strings become massless when the brane separation vanishes, they do contribute to the LEEA in the non-abelian Coulomb branch. After plain dimensional reduction from $R^{1,6}$ to $R^{1,3}$ the effective $N = 1$ super-Yang-Mills theory in $1 + 6$ dimensions yields the $N = 4$ super-Yang-Mills theory in $1 + 3$ dimensions, which has the same number of on-shell components, the same number of conserved supercharges and the same gauge group.

The hypermultiplet LEEA in the coincidence limit is described by the $N=2$ NLSM in the four-dimensional spacetime, with the NLSM target space being given by the ALE space Q_{mEH} . A proper generalization of eq. (9) reads

$$S_{\text{mEH}} = \text{tr} \int_{\text{analytic}} \left\{ \dot{\bar{q}}^+ \mathcal{D}^{++} q^+ + V^{++} \xi^{++} \right\} , \quad \mathcal{D}^{++} = D^{++} + iV^{++} , \quad (11)$$

in terms of the $N=2$ vector multiplet V^{++} valued in the Lie algebra of $SU(N)$, the hypermultiplet q^+ valued in the Lie algebra of $U(N)$, and the Fayet-Iliopoluols terms $\xi^{++} = \xi^{(ij)} u_i^+ u_j^+$ valued in the Cartan algebra of $SU(N)$. Note that the FI terms explicitly resolve the A_{N-1} singularity in this action.

If the $(1 + 6)$ -dimensional $N = 1$ supersymmetric effective field theory were compactified on the circle S^1 , this would yield the gauge field theory in $(1 + 5)$ dimensions, whose T-dual is an $(2, 0)$ supersymmetric gauge field theory with N tensor multiplets. Therefore, in the type-IIB picture, we do not get the enhanced gauge symmetry but N tensor multiplets instead.

Yet another gauge symmetry enhancement pattern is known when N of the D-6-branes come on top of an *orientifold* six-plane, which leads to the $SO(2N)$ gauge symmetry (Ooguri and Vafa (1996)). In M-theory, the orientifold six-plane is to be represented by the *Atiyah-Hitchin* space (Atiyah and Hitchin (1988)) instead of a KK monopole (Sen (1997)). Indeed, far from the origin the Atiyah-Hitchin space has the topology $R^3 \times S^3/T_4$, i.e. it looks like Q_{mTN} whose points are supposed to be identified under the action of the discrete symmetry T_4 reversing signs of all four coordinates of Q_{mTN} , which matches with the definition of the orientifold six-plane (Sen (1997)).

4 Large N limit

To reproduce the Seiberg-Witten-type solution to $N = 2$ super-QCD from M-theory, merely a *single* and smooth M-5-brane in a KK-monopole background is needed. The M-5-brane worldvolume should just be compactified on the SW curve Σ_g down to $(1+3)$ dimensions i.e. to the worldvolume of a D-3-brane. Taking N M-5-branes (whose worldvolume is now locally given by the product $R^{1,3} \times \Sigma_1$, with Σ_1 being an elliptic curve) and allowing them to coincide in the background of multi-KK monopole yields (at a generic point in the moduli space) an $N = 2$ supersymmetric gauge field theory with the non-abelian gauge group $U(N)$ as the LEEA in the common (macroscopically $(1+3)$ -dimensional) brane worldvolume. The KK monopoles are supposed to merge too, which also implies the non-abelian gauge group and extra massless supermultiplets in the LEEA. Indeed, the configuration of N parallel M-5-branes can support M-2-branes ending on different M-5-branes. When some or all of these M-5-branes coincide, the zero modes of the M-2-branes stretched between them give rise to the additional massless $N = 2$ multiplets in the effective field theory defined in the common worldvolume. The type of a supermultiplet depends upon the topology of M-2-brane: a cylinder yields an $N = 2$ vector multiplet, while a disc yields a hypermultiplet. The M-2-branes wrapped about S^1 and connecting different M-5-branes are strings in ten dimensions, which become *tensionless* in the M-5-brane coincidence limit.

In the coincidence limit for N KK monopoles, near the A_{N-1} singularity, all FI parameters ξ in the hypermultiplet LEEA (11) vanish, so that this action formally exhibits the genuine non-abelian gauge symmetry $SU(N)$. At large N , the auxiliary $N=2$ vector gauge multiplet V^{++} valued in the Lie algebra of $SU(N)$ becomes *propagating* via the well-known mechanism of the dynamical generation of massless vector bosons in non-compact NLSM (Polyakov (1987)). This $N=2$ vector multiplet and the hypermultiplet q^+ , both in the adjoint of the gauge group, constitute an $N = 4$ super-Yang-Mills multiplet, whose action is a straightforward $N = 4$ extension of eq. (11)!

Our result is closely related to the recent conjecture of Maldacena (1997). He discussed ‘simple’ M-5-branes, having no Riemann surface in their worldvolumes, in the particular limit (down the ‘throat’) given by the product

$AdS_7 \times S^4$ whose both radii are proportional to $N^{1/3}$. At large N , the Maldacena LEEA is given by a $(2, 0)$ superconformally invariant gauge field theory in six dimensions, which is supposed to be dual to M-theory compactified on $AdS_7 \times S^4$ (Maldacena 1997). Our results imply that the Maldacena limit can also be approached from the hypermultiplet part of the LEEA near the singularity, after taking into account the dynamical generation of an $SU(N)$ massless $N=2$ vector multiplet at large N . Our approach can be easily generalized to the other simply-laced gauge groups.

References

- Atiyah, M.F., Hitchin, N.J. (1988): *The Geometry and Dynamics of Magnetic Monopoles*. (Princeton University Press, Princeton) pp. 1-134
- Gibbons, G.W., Olivier, D., Ruback, P.J., Valent, G., (1988): Multicentre metrics and harmonic superspace. *Nucl. Phys.* **B296**, 679-696
- Hull, C.M., Townsend, P.K. (1995): Enhanced gauge symmetries in superstring theories. *Nucl. Phys.* **B451**, 525-546
- Ivanov, E.A., Ketov, S.V., Zupnik, B.M. (1997): Induced hypermultiplet selfinteractions in $N=2$ gauge theories. *Nucl. Phys.* **B509**, 53-82
- Karch, A., Lüst, D., Smith, D.J. (1998): Equivalence of geometric engineering and Hanany-Witten via fractional branes. Berlin preprint HUB-EP 98/22, 31 pages; hep-th/9803232
- Ketov, S.V., (1997): Solitons, Monopoles and Duality. *Fortschr. Phys.* **45**, 237-292
- Ketov, S.V., (1998): Analytic tools to brane technology in $N=2$ gauge theories with matter. DESY and Hannover preprint, DESY 98-059 and ITP-UH-12/98, 80 pages; hep-th/9806009
- Klemm, A., Lerche, P., Mayr, P., Vafa, C., Warner, N. (1996): Self-dual strings and $N=2$ supersymmetric field theory. *Nucl. Phys.* **B477**, 746-766
- Maldacena, J. (1997): The large N limit of superconformal field theories and supergravity. Harvard preprint HUTP-97/A097, 21 pages; hep-th/9711200
- Mikhailov, A., (1997): BPS states and minimal surfaces. Moscow and Princeton preprint, ITEP-TH-33/97 and PUTP-1714, 31 pages; hep-th/9708068
- Polyakov, A.M. (1987): *Gauge Fields and Strings* (Harwood Academic Publishers, Chur, Switzerland) pp. 1-301
- Ooguri, H., Vafa, C. (1996): Two-dimensional black holes and singularities of CY manifolds. *Nucl. Phys.* **B463**, 55-72
- Sen, A. (1997): Dynamics of multiple KK monopoles in M- and string theory. *Adv. Theor. Math. Phys.* **1**, 115-126;
A note on enhanced gauge symmetries in M- and string theory. Electronically published in JHEP **9** 001
- Townsend, P.K., (1995): The 11-dimensional supermembrane revisited. *Phys. Lett.* **350B**, 184-188
- Witten, E., (1996): Bound states of strings and p-branes. *Nucl. Phys.* **B460**, 335-350
- Witten, E., (1997): Solutions of four-dimensional field theory via M theory. *Nucl. Phys.* **B500**, 3-42.