D-PARTICLE BOUND STATES
MATRIX INTEGRALS AND KP

based on

G. Moore, N. Nekrasov and S.Shatashvili

- hep-th 97122241, hep-th/9803265

V.Kazakov, I. Kostov and N. Nekrasov

to appear

INTRODUCTION

M -theory — existence of a sequence of massive BPS states in IIA theory electrically charged with respect to RR 1-form which are interpreted as KK particles in 11D SUGRA compactified on S^1 (Witten, Townsend).

Later this particles were identified with "D0 Branes". In certain energy regime dynamics of N such particles — SUSY QM of $N \times N$ matrices; dimensional reduction of $D = 10$ SYM theory.

The existence of these M-theory KK tower of states is equivalent to statement that QM has exactly 1 bound state for each N. Existance of bound state was proven for $N = 2$ (Sethi-Stern).

The part of the proof of the existence of the bound state is the computation of the index. For $N = 2$ case this index has been computed (Yi, Sethi-Stern - 97). For $N = 3$ recently numerical computation has been performed (Krauth, Nicolai, Staudacher - March 98) and for general N is done analytically (MNS - March 98).

Here is presented the derivation for all N in $D = 10$, compute the integral also for reductions of $D = 4$ and $D = 6$ $\mathcal{N} = 1$ SYM theories and will demonstrate the validity of the conjecture by Green and Gutperle.

Witten Index

Bound states in SUSY QM are detected by Witten's Index:

$$\lim_{\beta\to\infty} \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta H} = N_B - N_F$$

This Tr is β-independent in theories with a discrete spectrum but if there is a continuous spectrum one can have non-trivial behavior (densities of Fermionic and Bosonic degrees of freedom may differ). SUSY allows to relate the expression above to the simpler one:

$$\lim_{\beta\to 0} \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta H}$$

In the case of N D0 branes we can rewrite this as:

$$I_{10}(N) = \frac{1}{\mathrm{Vol}(G)} \int d^{10}X d^{16}\Psi e^{-S}$$

S - dim. reduced action of $\mathcal{N} = 1$ $D = 10$ SYM with the gauge group $G = SU(N)/\mathbb{Z}_N$.

We wish to compute more general integral:

$$I_D(N) \equiv \left(\frac{\pi}{g}\right)^{\frac{(N^2-1)(D-3)}{2}} \frac{1}{\mathrm{Vol}(G)} \int d^D X d^{2^{D/2-1}} \Psi e^{-S}$$

for $D = 3+1, 5+1, 9+1$ with

$$S = \frac{1}{g}\left(\frac{1}{4}\sum_{\mu,\nu=1,\ldots,D} \mathrm{Tr}[X_\mu, X_\nu]^2 + \frac{i}{2}\sum_{\mu=1}^{D} \mathrm{Tr}(\bar{\Psi}\Gamma^\mu[X_\mu, \Psi])\right),$$

and Γ^μ are the Clifford matrices for $Spin(D)$.

The difference between Witten's Index and zero β limit is called boundary term and is non-zero (we will see that integral is not integer); it should be analyzed separately:

$$\lim_{\beta\to\infty} \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta H} - \lim_{\beta\to 0} \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta H} =$$

$$\int_0^\infty d\beta \frac{d}{d\beta} \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta H}$$

We start with change of variables:

$$\phi = X_{D-1} + iX_D$$
$$B_j = X_{2j-1} + iX_{2j}$$
$$j = 1, \ldots, D/2 - 1$$

Sometimes we use

$$\mathbf{X} = \{X_a, a = 1, \ldots, D-2\}$$

We also add bosonic auxillary field $\vec{H}$.

Same for fermions:

$$\Psi \to \Psi_a = (\psi_j, \psi_j^\dagger), \vec{\chi}, \eta$$

The origin of these fields for example in $D = 10$ is following: original fermions were 16 of $SO(1,9)$. We think of the fermions (we "break" in 10d language $SO(1,9)$) first as of a representation of $SO(1,1) \times SO(8)$ and then as that of $SO(1,1) \times SO(2) \times SO(6)$ or $SO(1,1) \times SO(7)$.

$16 \to 8_s + 8_c$, or using triality $8_c \to 8_v$. $8_s \to 7 + 1$ for $SO(7)$, thus one gets 8 (real) ψ's, 7 - χ's and 1 - η.

When 10d theory is considered on flat space (T^{10} of zero size in our case - dimensional reduction) this change of variables doesn't break original Lorentz group.

Action:

$$S = \frac{1}{16g}\mathrm{Tr}[\phi,\bar{\phi}]^2 - i\mathrm{Tr}\vec{\mathcal{E}}(\mathbf{X})\vec{H}+$$

$$g\mathrm{Tr}\vec{H}^2 - \frac{1}{4g}\sum_{a=1}^{D-2}\mathrm{Tr}|[X_a,\phi]|^2 + fermions$$

Here $\vec{\mathcal{E}}$ are "equations":

$$D=4: \qquad \vec{\mathcal{E}} = [B_1, B_1^\dagger]$$

$$D=6: \qquad \vec{\mathcal{E}} = \left([B_1,B_1^\dagger]+[B_2,B_2^\dagger],[B_1,B_2],[B_2^\dagger,B_1^\dagger]\right)$$

$$D=10: \qquad \vec{\mathcal{E}} = \left([B_i,B_j]+\frac{1}{2}\epsilon_{ijkl}[B_k^\dagger,B_l^\dagger], i<j, \quad \sum_i [B_i,B_i^\dagger]\right)$$

In $D=6$ case $\vec{\mathcal{E}}$ can be also written as a three-vector

$$\mathcal{E}_A = [X_A, X_4] + \frac{1}{2}\varepsilon_{ABC}[X_B,X_C], \qquad A=1,2,3$$

and for $D = 10$ using octonions structure constants as a seven-vector:

$$\mathcal{E}_{\mathbf{A}} = [X_{\mathbf{A}}, X_8] + \frac{1}{2}c_{\mathbf{ABC}}[X_{\mathbf{B}}, X_{\mathbf{C}}]$$

$\mathbf{A} = 1,\ldots,7$

BRST symmetry (one of the original SUSY generators):

$$QX_a = \Psi_a \quad Q\Psi_a = [\phi, X_a]$$

$$Q\vec{\chi} = \vec{H} \quad Q\vec{H} = [\phi, \vec{\chi}]$$

$$Q\bar{\phi} = \eta \quad Q\eta = [\phi, \bar{\phi}] \quad Q\phi = 0$$

Action: $S = Q(\mathcal{R})$,

$$\mathcal{R} = \mathrm{Tr}\frac{1}{16g}\eta[\phi, \bar{\phi}] - i\mathrm{Tr}\vec{\chi} \cdot \vec{\mathcal{E}} +$$

$$g\mathrm{Tr}\vec{\chi} \cdot \vec{H} + \frac{1}{4g}\sum_{a=1}^{D-2} \mathrm{Tr}\Psi_a[X_a, \bar{\phi}]$$

$$S = \frac{1}{16g}\mathrm{Tr}[\phi, \bar{\phi}]^2 + \frac{1}{4g}\sum_{a=1}^{D-2} \mathrm{Tr}[X_a, \phi][X_a, \bar{\phi}] - i\mathrm{Tr}\vec{H} \cdot \vec{\mathcal{E}} + g\mathrm{Tr}\vec{H} \cdot \vec{H}$$

$$-i\mathrm{Tr}\left(\vec{\chi} \cdot \frac{\partial \vec{\mathcal{E}}}{\partial X_a}\Psi_a + \frac{1}{4g}\Psi_a[X_a, \eta] - g\mathrm{Tr}\vec{\chi} \cdot [\phi, \vec{\chi}] + \frac{1}{16g}\mathrm{Tr}\eta[\eta, \phi]\right)$$

Ghost charges:

ϕ	$+2$
Ψ_α	$+1$
H, X_a	0
χ, η	-1
$\bar{\phi}$	-2

Now we note that all fields are paired with fermions except for ϕ. We conclude that above action gives a canonical measure for all fields except ϕ. Thus we need to fix the measure for ϕ on Lie Algebra of G.

G is a simple Lie group and we have unique Killing form up to a constant multiple. This form determines measure on Group and on Algebra, thus ratio is independent of above constant mutiple and we conclude that measure normalized against the Volume of Group is canonical:

$$\frac{D\phi}{\mathrm{Vol}(G)}$$

However, this measure depends whether the group has a center or not. Our group is $G = SU(N)/\mathbb{Z}_N$ because all fields are in adjoint representation. When we reduce the computation to an integral over the Lie algebra of the maximal torus $T \subset SU(N)$ the measure $\mathcal{D}\phi$ will be normalized in such a way that the measure on T obtained by the exponential map integrates to one. Therefore there is an extra factor $\#Z$ in front of the integral since in passing to the measure on $\underline{\mathfrak{t}}$ we get as a factor a volume of the generic adjoint orbit:

$$\frac{\mathrm{Vol}(G/T)}{\mathrm{Vol}(G/Z)} = \frac{\#Z}{\mathrm{Vol}(T)}$$

Gaussian integral over auxillary field H leads to the extra factor

$$\left(\frac{\pi}{g}\right)^{\frac{(D-3)(N^2-1)}{2}}$$

which people usualy forget to put when writing integral in on-shell variables.

Now we have the integral where we can apply the general method of "Integrating Over Higgs Branch" (G. Moore, N. Nekrasov, S. Sh., hep-th/9712241).

In order to apply this method we need 1. global symmetries of the problem and 2. deformed BRST transformation (our integral is ill-defined because it is over non-compact space and integrad looks to be order 1 at infinity). We have for symmetries:

$$\begin{aligned} 4 &: K = Spin(2),\ \vec{\mathcal{E}} \in \mathbf{1}; \\ 6 &: K = Spin(4),\ \vec{\mathcal{E}} \in \mathbf{3}_L; \\ 10 &: K = Spin(6),\ \vec{\mathcal{E}} \in (\mathbf{6} \oplus \bar{\mathbf{6}})_r \oplus \mathbf{1} \end{aligned}$$

or for $D = 10$ we can also use $K = Spin(7), \quad \vec{\mathcal{E}} \in \mathbf{7}$

We will deform Q using generic element ϵ in the Cartan subalgebra of global group K. Let

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then:

$$D = 4 \quad \epsilon = E \cdot R$$

$$D = 6 \quad \epsilon = \begin{pmatrix} E_1 \cdot R & 0 \\ 0 & E_2 \cdot R \end{pmatrix}$$

$$D = 10 \quad \epsilon = \begin{pmatrix} (E_1 + E_2) \cdot R & & \\ & (E_2 + E_3) \cdot R & \\ & & (E_1 + E_3) \cdot R \end{pmatrix}$$

for generic real E_i.

Now we write the deformed nilpotent charges:

$$Q_\epsilon X_a = \Psi_a \quad Q_\epsilon \Psi_a = [\phi, X_a] + X_b T_v(\epsilon)^b_a$$

$$Q_\epsilon \vec{\chi} = \vec{H} \quad Q_\epsilon \vec{H} = [\phi, \vec{\chi}] + T_s(\epsilon) \cdot \vec{\chi}$$

$$Q_\epsilon \bar{\phi} = \eta \quad Q_\epsilon \eta = [\phi, \bar{\phi}]$$

$$Q_\epsilon \phi = 0$$

Here T_v denotes action of $Lie(K)$ on X's and T_s action of $Lie(K)$ on the equations. Explicitly for the case of $D = 10$ we have:

$$X_b T_v(\epsilon)^b_a = (iE_1(B_1 - B_1^\dagger) + iE_2(B_2 - B_2^\dagger) + iE_3(B_3 - B_3^\dagger) + iE_4(B_4 - B_4^\dagger))_a$$

$$T_s(\epsilon)\chi = ((E_1 + E_2)(\chi_{12} - \chi_{34}) + (E_2 + E_3)(\chi_{23} - \chi_{14}) + (E_1 + E_3)(\chi_{13} - \chi_{24}))$$

Now we can write the deformed action simply by replacing Q by Q_ϵ in our action:

$$S_\epsilon = Q_\epsilon \mathcal{R}$$

for the old $\mathcal{R}$.

We get: $S =$

$$\frac{1}{16\tilde{g}}\mathrm{Tr}[\phi,\bar{\phi}]^2 + \frac{1}{4\hat{g}}\sum_{a=1}^{D-2}\mathrm{Tr}[X_a,\phi][X_a,\bar{\phi}] - i\mathrm{Tr}\vec{H}\cdot\vec{\mathcal{E}} + g\mathrm{Tr}\vec{H}\cdot\vec{H} -$$

$$i\sum_{a=1}^{D-2}\mathrm{Tr}\left(\vec{\chi}\cdot\frac{\partial\vec{\mathcal{E}}}{\partial X_a}\Psi_a + \frac{1}{4\hat{g}}\Psi_a[X_a,\eta]\right) - g\mathrm{Tr}\vec{\chi}\cdot[\phi,\vec{\chi}] +$$

$$\frac{1}{16\tilde{g}}\mathrm{Tr}\eta[\eta,\phi] + g\mathrm{Tr}\vec{\chi}\cdot T_s(\epsilon)\vec{\chi} + \frac{1}{4\hat{g}}T_v(\epsilon)^{ab}\mathrm{Tr}\bar{\phi}[X_a,X_b]$$

We want to treat coupling constants $g, \hat{g}, \tilde{g}$ separately although so far they were equal. Note that because each term is Q-exact separately we can take limits in any order. Later we will take the limit

$$\tilde{g} \to \infty$$

Added piece is equal

$$S_\epsilon - S_0 = g\mathrm{Tr}\left(\vec{\chi}\cdot T_s(\epsilon)\vec{\chi}\right) + \frac{1}{4\hat{g}}T_v(\epsilon)^{ab}\mathrm{Tr}\bar{\phi}[X_a,X_b]$$

and has ghost charge -2 (ϵ is assigned charge 0 temporarily). This means integral is not changed and is convergent if the original one was.

When we look on T_s we see that there is always one zero mode for the mass matrix of χ and that the rest is non-vanishing for generic ϵ. So we add a new Q_ϵ exact term:

$$sQ_\epsilon \mathrm{Tr}(\chi_0 \bar{\phi}) = -s\mathrm{Tr}\chi_0\eta + s\mathrm{Tr}H_0\bar{\phi}$$

with large s. Its ghost charge is -2 and again doesn't change the integral. Now together with gH^2 this term gives non-zero masses to all fermions of negative ghost charge.

We can integrate out all negative ghost charge fermions (taking limit $s \to \infty$, $g \to \infty$) and produce a very simple action but without "kinetic term" for Ψ_a. We add another term with positive ghost charge:

$$\frac{1}{2}tQ_\epsilon \left(\sum_{i=1}^{D/2-1} B_i \Psi_i^\dagger - B_i^\dagger \Psi_i \right) =$$

$$t\sum_{i=1}^{D/2-1} \mathrm{Tr}\Psi_i\Psi_i^\dagger + t\mathrm{Tr}B_i \left(\mathrm{ad}(\phi)\delta_i^j + T_v(\epsilon)_i^j \right) B_j^\dagger$$

The "standard" ghost charge for ϵ is 2, then the insertion of coupling t must be compensated by the insertion of coupling s.

We conclude that answer can depend only on combination st. But, one can repeat Witten's derivation ("2d Gauge theories revisited", 92) by first $s \to \infty$ with g much smaller than s. Result is the effective action:

$$S_{eff} \sim \frac{1}{s}\{Q_\epsilon, \mathrm{Tr}\Psi_a[X, \mathcal{E}]\}$$

We will see that for large s, t the dependence on either variable disappears thus the value we get is the SAME as the original integral.

For large s, t, g we can proceed by semiclassical (saddle point) approximation.

Usualy one takes small g limit and gets diagonal matricies. At the same time corresponding integral is ill-defined because we have zero from fermionic zero mode and infinity from bosonic zero mode. Here, we take large coupling constant limits and this makes an action trivial up to simple contribution from quadratic terms - gaussian integrals.

Large s limit sets χ_0, η, H_0 and $\bar{\phi}$ to zero. After this, only nonzero terms left in the action in large g, t limit will come from the terms proportional to g and from the terms proportional to t, but these fields come without zero modes.

First we take the gaussian integral over BRST quartet - $(\eta, \bar{\phi}, \vec{\chi}, \vec{H})$. Result is (factors of g cancel between H and χ integral because we have same amount of these fields):

$$\text{Det}\,(T_s(\epsilon) + ad(\phi))$$

with determinant in the representation of the space of equations.

Second gaussian integral is over pair B_i, Ψ_i. Result is (again, without any factors of t) :

$$\frac{1}{\text{Det}\,(T_v(\epsilon) + ad(\phi))}$$

in the space of complex matrices B.

Now we rewrite the integral over ϕ from $Lie(G)$ to $\underline{\mathrm{t}} = \mathrm{Lie}(T)$. Once we include corresponding Vandermonde factor the result is:

$$I_{D=10}(N) = \left(\frac{(E_1+E_2)(E_2+E_3)(E_3+E_1)}{E_1E_2E_3E_4}\right)^{N-1}$$

$$\frac{N}{N!}\int_{\underline{\mathfrak{t}}}\mathcal{D}\phi\prod_{i\neq j}\frac{P(\phi_{ij})}{Q(\phi_{ij})}$$

$$P(x) = x(x+E_1+E_2)(x+E_3+E_2)(x+E_1+E_3)$$

$$Q(x) = \prod_{\alpha=1}^{4}(x+E_\alpha+i0)$$

$$\sum_\alpha E_\alpha = 0$$

For $D=6$ and $D=4$ we get:

$$I_{D=6}(N) = \left(\frac{E_1+E_2}{E_1E_2}\right)^{N-1}\frac{N}{N!}\int_{\underline{\mathfrak{t}}}\mathcal{D}\phi\prod_{i\neq j}\frac{\phi_{ij}(\phi_{ij}+E_1+E_2)}{\prod_{\alpha=1}^{2}(\phi_{ij}+E_\alpha+i0)}$$

$$I_{D=4}(N) = \frac{N}{N!E_1^{N-1}}\int_{\underline{\mathfrak{t}}}\mathcal{D}\phi\prod_{i\neq j}\frac{\phi_{ij}}{(\phi_{ij}+E_1+i0)}$$

One can get $D=6$ case from $D=10$ by formal limit $E_3 \to \infty$ and $D=4$ from $D=6$ by formal limit $E_2 \to \infty$.

Let me remind the origin of the factor $\frac{N}{N!}$. The denominator is the order of the Weyl group of $SU(N)$ which enters in passing to the integral over the conjugacy classes of ϕ. We then rewrite this integral as an integral over $\underline{\mathrm{t}}$, divided by $|W(G)| = N!$. The numerator N is the order of the center of $\mathbb{Z}_N$ which appears in comparing the volumes of $SU(N)$ and G. The measure $\mathcal{D}\phi$ is defined as follows. The maximal Cartan subalgebra of $SU(N)$ can be identified with $\mathbb{R}^{N-1}$ by means of the imbedding:

$$(\phi_1, \ldots, \phi_{N-1}) \to \mathrm{diag}\,(\phi_1, \ldots, \phi_{N-1}, -\phi_1 - \ldots - \phi_{N-1})$$

into the space of traceless hermitian matrices. The measure $\mathcal{D}\phi$ is simply the normalized Euclidean measure on $\mathbb{R}^{N-1}$:

$$\mathcal{D}\phi = \prod_{k=1}^{N-1} \frac{d\phi_k}{2\pi i}$$

If we think about these integrals as of contour integrals then the fear that they are ill-defined because the measure approaches 1 at ∞ dissapears. The fact that they should be treated as a contour integrals follows from the integration which led to the them - for example we need the shift $E \to E + i0$ in order to make sense of gaussian integrals.

N=2

For $D = 10$ and $N = 2$ we have:

$$I = \frac{1}{2\pi i}\frac{P'(0)}{Q(0)}\int_{\mathbb{R}} d\phi \frac{P(2\phi)P(-2\phi)}{Q(2\phi)Q(-2\phi)}$$

$$P(x) = x(x + E_1 + E_2)(x + E_3 + E_2)(x + E_1 + E_3)$$

$$Q(x) = \prod_{\alpha=1}^{4}(x + E_\alpha + i0)$$

We close the contour in upper half-pane (prescription). We have four poles at

$$\phi = \frac{1}{2}E_\alpha + i0$$

With residue at each of them:

$$\mathrm{Res}_{\frac{1}{2}E_\alpha + i0} = \frac{1}{12}\frac{R(-2E_\alpha)}{E_\alpha R'(E_\alpha)}$$

$$R(x) = \prod_{\alpha=1}^{4}(x - E_\alpha)$$

Thus we conclude:

$$\sum_{\alpha=1}^{4}\mathrm{Res}_{\frac{1}{2}E_\alpha} = \frac{1}{12}\left(\oint \frac{R(-2x)}{xR(x)}dx - 1\right) = 5/4$$

For other values of d computation is same and we get:

$$\frac{1}{12}\left(2^{D/2-1} + (-1)^{D/2}\right)$$

$$5/4, 1/4, 1/4, \quad \text{for} \quad D = 10, 6, 4 \quad \text{respectively}$$

$SU(N)$ for $D = 4$

First we simplify the integrand by Bose-Cauchy identity:

$$\frac{1}{E_1^N} \prod_{i \neq j} \frac{\phi_{ij}}{(\phi_{ij} + E_1 + i0)} = \sum_{\sigma \in S_N} (-1)^\sigma \prod_{i=1}^{N} \frac{1}{\phi_i - \phi_{\sigma(i)} + E_1 + i0}$$

Only the cycles of maximal length contribute to the residue formula (there are $(N-1)!$ of those), since otherwise we get less then $N-1$ residues. We pick up a residue for all i except one, j when

$$\phi_{\sigma(i)} = \phi_i + E_1 + i0, \quad \text{for} \quad \text{all} \quad i \neq j.$$

Let us assume that $j = N$ and that σ is a long cycle $\sigma(i) = i+1$ (both can be achieved with the help of Weyl group); we get the pole at

$$\phi_i = \frac{1}{2}(2i - N - 1)E_1$$

and claim is that residue is

$$\frac{1}{N^2}$$

D=10, SU(N)

General $D = 6$ case can be treated in a similar way as $D = 4$ but is technicaly more complicated. We will turn directly to $D = 10$ case where we will take a little bit different route. The strategy is to reduce the number of matrices by enforcing deformed octonionic instanton equations. As opposed to section 2 where we were basically taking strong coupling limits here we are taking mixed weak and strong coupling limits, imposing the weak coupling limit to enforce some of the equations.

Introduce the formal variable m. Deform the "equation" to

$$\mathcal{E}_{ij} = \Phi_{ij} - \frac{1}{2}\epsilon_{ijkl}\Phi^{\dagger}_{kl}$$

where

$$\Phi_{ij} = [B_i, B_j] - m\epsilon_{ijk4}B_k \qquad\qquad 1 \leq i,j \leq 4$$

and

$$\frac{1}{2}\sum_{1\leq i,j\leq 4} \mathrm{Tr}\mathcal{E}_{ij}\mathcal{E}^{\dagger}_{ij} = \sum_{1\leq i,j\leq 4} \mathrm{Tr}\Phi_{ij}\Phi^{\dagger}_{ij}$$

Thus, imposing these equations we are led to the equations describing vacua of $\mathcal{N} = 4$ SYM broken down to $\mathcal{N} = 1$

(Vafa, Witten, 94) in 4d language, together with B_4 generating the gauge transformations in the complexified unbroken group:

$$[B_i, B_j] = m\epsilon_{ijk4}B_k,$$
$$[B_4, B_k] = 0$$

From first equation we coclude that B_i forms the N-dimensional representation of $SU(2)$ (reducible) Now we want to split the couplings to g' and g'' for the equations $\mathcal{E}_{ij}$ and for equation $\sum_{i=1}^{4}[B_i, B_i^\dagger]$ respectively (we can do this without spoiling Q-symmetry). Take the limit $g' \to 0$. This limit enforces above equations. We rewrite:

$$\frac{1}{\hat{g}}\sum_{a=1}^{4}\mathrm{Tr}|[X_a,\phi]|^2 \to \frac{1}{\hat{g}'}\sum_{i=1}^{3}\mathrm{Tr}|[B_i,\phi]|^2 + \frac{1}{\hat{g}''}\mathrm{Tr}|[B_4,\phi]|^2$$

Now we take the limit $\hat{g}' \to 0$ and enforce the equations:

$$[B_i, \phi] = 0, i = 1, 2, 3$$

Now we use the argument that extra $U(1)$'s kill the contributions to the partition function. Thus we need to count the vacua where the adjoint group is broken down to $SU(d)/\mathbb{Z}_d$, $N = ad$. This also means that all irreducible components of $SU(2)$ in B_i are the same, so we have d copies of a-dimensional representation of $SU(2)$:

$$B_\alpha = \|L_\alpha\|_{a\times a} \otimes I_{d\times d}$$

and $\alpha = 1, 2, 3$, L_α being $SU(2)$ generators in the a-dimensional irreducible representation of $SU(2)$. Now, because B_4 and ϕ commute with B_i we get for B_4, ϕ:

$$(B_4)_{N\times N} = I_{a\times a} \otimes (B_4)_{d\times d}\,, \quad (\phi)_{N\times N} = I_{a\times a} \otimes (\phi)_{d\times d}$$

But, in the limit $\hat{g}' \to 0$ we can integrate out ("ignore") $B_\alpha, \alpha = 1, 2, 3$ and stay with B_4, ϕ. We get back to $D = 4$ integral but for gauge group $SU(d)/\mathbb{Z}_d$. Moreover, due to supersymmetry, not only the degrees of freedom but also the measure is appropriate to interpret the integral as $I_{D=4}(d)$. Using our result for $D = 4$ we prove that:

$$I_{D=10}(N) = \sum_{d|N} \frac{1}{d^2}$$

Correlation functions and KP

Here we present the results of the computations of the correlation functions of $\mathrm{Tr}\phi^l$. In all these computations we keep ϵ as a regulator. Using the tricks described above the generating function of such correlators reduces to the integral over the eigenvalues of ϕ with the insertion of $\exp - \sum_n g_n \phi_i^n$: for example in $D = 4$ case we get:

$$\int \frac{d\phi}{\mathrm{Vol}(G)} \frac{e^{\sum_n g_n \mathrm{Tr}\phi^n}}{\mathrm{Det}\,(ad(\phi) + \epsilon)}$$

One can show that the grand partition function has the following properties:

$$Z_{\mu,V,4} \equiv \sum_N e^{\mu N} \langle e^{-V} \rangle = \mathrm{Det}(1 + e^{\mu} K)$$

where K is the integral operator:

$$K f(x) = e^{-V(x)} \int dy \frac{f(y)}{x - y + i\epsilon}$$

As a function of the chemical potential $\mu = t_1$ and the parameters t_n which are related to g_n via:

$$V(x) = i\,(W(x + i\epsilon) - W(x - i\epsilon))$$

for $W(x) = \sum_n t_n x^n$ it is equal to the τ-function of KP hierarchy. In particular

$$u = 2\partial_\mu^2 \mathrm{log} Z$$

obeys:

$$3\frac{\partial^2 u}{\partial t_2^2} + \frac{\partial}{\partial t_1}\Big[-4\frac{\partial u}{\partial t_3} + 6u\frac{\partial u}{\partial t_1} + \frac{\partial^3 u}{\partial t_1^3}\Big] = 0.$$

For the $D = 10$ case the grand partition function can be represented as:

$$Z_{\mu,V,10} = \prod_{l=1}^{\infty} \mathrm{Det}(1 + e^{l\mu} K)$$

which suggests certain speculations about GKM and Borcherds products.