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HETEROTIC T-DUALITY AND THE RENORMALIZATION GROUP

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1 Introduction

T-duality symmetry first came about in the context of string theory [1], but a proof of its existence can be given directly from sigma model path integral considerations [2, 3].

A study was initiated concerning the possibility of having T-duality as a symmetry of the quantum sigma model away from the (conformal) RG fixed points, when the target manifold admits an Abelian isometry. It was observed that the interplay between T-duality and the RG translates to consistency conditions to be verified by the RG flows of the model; and that indeed they were verified by, and only by, the correct RG flows of the bosonic sigma model. Such a study was carried out up to two loops, order $\mathcal{O}(\alpha'^2)$, in references [4, 5, 6, 7].

What happens for supersymmetric extensions of such models? Let us study the heterotic sigma model [8]. One extra feature is that one now has a target gauge field. We shall work to one loop, order $\mathcal{O}(\alpha')$, and we will see that T-duality is again a good quantum symmetry of this sigma model. This shall be done by deriving consistency conditions for the RG flows of the model under T-duality and observing that they are satisfied by, and only by, the correct RG flows of the heterotic sigma model. Moreover, the measure of integration over the quantum fields involves chiral fermions. Such fermions produce potential anomalies, and we therefore have a first example where we can analyze the interplay of T-duality and the RG flow in the presence of anomalies.

Following [4, 5, 6, 7], let us begin with a theory with an arbitrary number of couplings, g^i, $i = 1, \ldots, n$, and consider a duality symmetry, T, acting as a map between equivalent points in the parameter space, such that,

$$Tg^i \equiv \tilde{g}^i = \tilde{g}^i(g). \tag{1.1}$$

Let us also assume that our system has a renormalization group flow, R, encoded by a set of beta functions, and acting on the parameter space by,

$$Rg^i \equiv \beta^i(g) = \mu \frac{dg^i}{d\mu}, \tag{1.2}$$

where μ is some appropriate subtraction scale. Given any function on the parameter space of the theory, $F(g)$, the previous operations act as follows:

$$TF(g) = F(\tilde{g}(g)) \qquad , \qquad RF(g) = \frac{\delta F}{\delta g^i}(g) \cdot \beta^i(g). \tag{1.3}$$

For a finite number of couplings the derivatives above should be understood as ordinary derivatives, whereas in the case of the sigma model these will be functional derivatives, and the dot will imply an integration over the target manifold.

The consistency requirements governing the interplay of the duality symmetry and the RG can now be stated simply as,

$$[T, R] = 0, \tag{1.4}$$

or in words: duality transformations and RG flows commute as motions in the parameter space of the theory. This amounts to a set of consistency conditions on the beta functions of our system:

$$\beta^i(\tilde{g}) = \frac{\delta \tilde{g}^i}{\delta g^j} \cdot \beta^j(g). \tag{1.5}$$

2 Duality in the Heterotic Sigma Model

We shall start by reviewing the construction of the heterotic sigma model, and the standard procedure of dualizing such model.

We consider the target manifold endowed with a metric $g_{\mu\nu}$, anti-symmetric tensor field $b_{\mu\nu}$ and a gauge connection $A_{\mu IJ}$ associated to the gauge group G. The Lagrangian density of such model is given by [8, 10]:

$$\mathcal{L} = (g_{\mu\nu} + b_{\mu\nu})\partial_+ X^\mu \partial_- X^\nu + i g_{\mu\nu}\lambda^\mu(\partial_- \lambda^\nu + (\Gamma^\nu_{\rho\sigma} + \frac{1}{2}H^\nu{}_{\rho\sigma})\partial_- X^\rho \lambda^\sigma) +$$

$$+ i\psi^I(\partial_+ \psi^I + A_\mu{}^I{}_J \partial_+ X^\mu \psi^J) + \frac{1}{2}F_{\mu\nu IJ}\lambda^\mu\lambda^\nu\psi^I\psi^J, \qquad (2.1)$$

where,

$$H_{\mu\nu\rho} = \partial_\mu b_{\nu\rho} + \partial_\nu b_{\rho\mu} + \partial_\rho b_{\mu\nu} \qquad \text{and} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \qquad (2.2)$$

We need to assume that the sigma model has an Abelian isometry in the target manifold, which will enable duality transformations [9, 10, 11]. Let ξ be the Killing vector that generates the Abelian isometry. The diffeomorphism generated by ξ transforms the scalar superfields, and the total action is invariant under the isometry only if we can compensate this transformation in the scalar superfields by a gauge transformation in the spinor superfields [9, 10]. This introduces a target gauge transformation parameter κ, such that $\delta_\xi A_\mu \equiv \mathcal{L}_\xi A_\mu = \mathcal{D}_\mu \kappa$.

Choose adapted coordinates to the Killing vector, $\xi^\mu \partial_\mu \equiv \partial_0$, and split the coordinates as $\mu, \nu = 0, 1, ..., d = 0, i$, so that the $\mu = 0$ component is singled out. In these adapted coordinates the isometry is manifest through independence of the background fields on the coordinate X^0.

Moreover, in these coordinates the target gauge transformation parameter will satisfy [9, 10],

$$\mathcal{D}_\mu \kappa \equiv \partial_\mu \kappa + [A_\mu, \kappa] = 0. \tag{2.3}$$

The duality transformations are then [10, 11]:

$$\begin{aligned} \tilde{g}_{00} = \frac{1}{g_{00}} \quad , \quad \tilde{g}_{0i} = \frac{b_{0i}}{g_{00}} \quad , \quad \tilde{b}_{0i} = \frac{g_{0i}}{g_{00}}, \\ \tilde{g}_{ij} = g_{ij} - \frac{g_{0i}g_{0j} - b_{0i}b_{0j}}{g_{00}} \quad , \quad \tilde{b}_{ij} = b_{ij} - \frac{g_{0i}b_{0j} - g_{0j}b_{0i}}{g_{00}}, \end{aligned} \tag{2.4}$$

$$\tilde{A}_{0IJ} = -\frac{1}{g_{00}} \mu_{IJ}, \tag{2.5}$$

$$\tilde{A}_{iIJ} = A_{iIJ} - \frac{g_{0i} + b_{0i}}{g_{00}} \mu_{IJ}. \tag{2.6}$$

where we have used $\mu_{IJ} \equiv (\kappa - \xi^a A_a)_{IJ}$ following [9, 10], and which in adapted coordinates becomes $\mu_{IJ} \equiv (\kappa - A_0)_{IJ}$.

Equations (2.4) are well known since [2, 3], and their interplay with the RG has been studied in [4, 5, 6, 7]. They shall not be dealt with in here, as to our one loop order there is nothing new to be found relative to the work in [4]. We shall rather concentrate on the new additions (2.5) and (2.6) yielding the duality transformations for the gauge connection.

As is well known, in a curved world-sheet we have to include one further coupling in our action,

$$\frac{1}{4\pi} \int d^2z \, \sqrt{h} R^{(2)} \phi(X), \tag{2.7}$$

$\phi(X)$ is the background dilaton field in $\mathcal{M}$. Taking into account the one loop Jacobian from integrating out auxiliary fields in the dualization procedure, one finds as usual the dilaton shift [3, 14]:

$$\tilde{\phi} = \phi - \frac{1}{2} \ln g_{00}. \tag{2.8}$$

3 Renormalization and Consistency Conditions

The one loop, order $\mathcal{O}(\alpha')$, beta functions can be computed to be [15, 16]:

$$\beta^g_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} H_\mu{}^{\lambda\rho} H_{\lambda\rho\nu} + \mathcal{O}(\alpha'), \tag{3.1}$$

$$\beta^b_{\mu\nu} = -\frac{1}{2} \nabla^\lambda H_{\lambda\mu\nu} + \mathcal{O}(\alpha'), \tag{3.2}$$

$$\beta^A_\mu = \frac{1}{2} (\mathbf{D}^\lambda F_{\lambda\mu} + \frac{1}{2} H_\mu{}^{\lambda\rho} F_{\lambda\rho}) + \mathcal{O}(\alpha'), \tag{3.3}$$

where $R_{\mu\nu}$ is the Ricci tensor of the target manifold, ∇_μ is the metric covariant derivative, and $\mathbf{D}_\mu$ is the covariant derivative involving both the gauge and the metric connections.

The advantage of using Weyl anomaly coefficients ($\bar{\beta}$) in our studies is due to the fact that while both $\bar{\beta}$ and β satisfy the consistency conditions (1.5), the $\bar{\beta}$-functions satisfy them exactly, while the β-functions satisfy them up to a target reparameterization [4, 5]. Since both encode essentially the same RG information, in the following we shall simply consider RG motions as generated by the $\bar{\beta}$-functions. For the heterotic sigma model [17, 13], and for the loop orders considered in this work:

$$\bar{\beta}^g_{\mu\nu} = \beta^g_{\mu\nu} + 2\nabla_\mu \partial_\nu \phi + \mathcal{O}(\alpha'), \tag{3.4}$$

$$\bar{\beta}^b_{\mu\nu} = \beta^b_{\mu\nu} + H_{\mu\nu}{}^\lambda \partial_\lambda \phi + \mathcal{O}(\alpha'), \tag{3.5}$$

$$\bar{\beta}^A_\mu = \beta^A_\mu + F_\mu{}^\lambda \partial_\lambda \phi + \mathcal{O}(\alpha'). \tag{3.6}$$

The consistency conditions (1.5) can now be derived. The couplings are $g^i \equiv \{g_{\mu\nu}, b_{\mu\nu}, A_\mu, \phi\}$, and the duality operation (1.1) is defined through (2.4-6) and (2.8). The RG flow operation is defined in (1.2), for our couplings, with the only difference that we shall consider $\bar{\beta}$-generated RG motions as previously explained. The consistency conditions associated to (2.4) and (2.8) have been studied before [4, 5, 6] and are known to be satisfied by, and only by, (3.1-2) or (3.4-5). The consistency conditions associated to the gauge field coupling are:

$$\bar{\beta}_0^{\tilde{A}} = \frac{1}{g_{00}}\bar{\beta}_0^A + \frac{1}{g_{00}^2}(\kappa - A_0)\bar{\beta}_{00}^g, \tag{3.7}$$

$$\bar{\beta}_i^{\tilde{A}} = \bar{\beta}_i^A - \frac{1}{g_{00}}((\kappa - A_0)(\bar{\beta}_{0i}^g + \bar{\beta}_{0i}^b) - (g_{0i} + b_{0i})\bar{\beta}_0^A) + \frac{1}{g_{00}^2}(g_{0i} + b_{0i})(\kappa - A_0)\bar{\beta}_{00}^g, \tag{3.8}$$

where we have used the notation $\bar{\beta}_\mu^{\tilde{A}} \equiv \bar{\beta}_\mu^A[\tilde{g}, \tilde{b}, \tilde{A}, \tilde{\phi}]$.

These are the main equations to be studied in this paper. The task now at hand is to see if these two conditions on the gauge field $\bar{\beta}$-functions are satisfied by – and only by – expressions (3.3), (3.6); and if so under what conditions are they satisfied.

4 Duality, the Gauge Beta Function and Heterotic Anomalies

As previously mentioned the model has chiral fermions that, when rotated, introduce potential anomalies into the theory. These anomalies need to be canceled if the dualization is to be consistent at the quantum level. However, our strategy in here is to see if we can get any information on this anomaly cancelation from the consistency conditions (3.7-8). So, we will set this question aside for a moment and directly ask: are the consistency conditions (3.7-8) verified by (3.3), (3.6)?

We choose to start with torsionless backgrounds. Such choice can be seen to extremely simplify equation (3.8), as the metric is parameterized by:

$$g_{\mu\nu} = \begin{pmatrix} a & 0 \\ 0 & \bar{g}_{ij} \end{pmatrix}, \tag{4.1}$$

and we take $b_{\mu\nu} = 0$. Therefore, there is also no torsion in the dual background [5, 6]. Equations (3.7-8) become,

$$\bar{\beta}_0^{\tilde{A}} = \frac{1}{a}\bar{\beta}_0^{A} + \frac{1}{a^2}(\kappa - A_0)\,\bar{\beta}_{00}^{g}, \tag{4.2}$$

$$\bar{\beta}_i^{\tilde{A}} = \bar{\beta}_i^{A}. \tag{4.3}$$

Now, use the Kaluza-Klein tensor decomposition of (3.3), (3.6), under (4.1), and compute $\bar{\beta}_0^{\tilde{A}}$ and $\bar{\beta}_i^{\tilde{A}}$. By this we mean the following. One should start with (3.3), (3.6), and decompose it according to the parameterization (4.1). We will obtain expressions for $\bar{\beta}_0^{A}$ and $\bar{\beta}_i^{A}$. Then, dualize these two expressions by dualizing the fields according to the rules (2.4-6) and (2.8). This yields expressions for $\bar{\beta}_0^{\tilde{A}}$ and $\bar{\beta}_i^{\tilde{A}}$. Finally, one should manipulate the obtained expressions so that the result looks as much as possible as a "covariant vector transformation" (1.5). Hopefully one would obtain (4.2-3), if the gauge beta functions are to satisfy the consistency conditions. However, the result obtained is:

$$\bar{\beta}_0^{\tilde{A}} = \frac{1}{a}\bar{\beta}_0^{A} + \frac{1}{a^2}(\kappa - A_0)(-\bar{\beta}_{00}^{g}), \tag{4.4}$$

$$\bar{\beta}_i^{\tilde{A}} = \bar{\beta}_i^{A}. \tag{4.5}$$

What (4.4) is saying is that, in order for duality to survive as a quantum symmetry of the heterotic sigma model, we need to have,

$$(\kappa - A_0)\,\bar{\beta}_{00}^{g} = 0. \tag{4.6}$$

We shall see that this is just the requirement of anomaly cancelation.

As was mentioned before, equations (2.5-6) were obtained using classical manipulations alone. In general, however, there will be anomalies and in this case the original theory and its dual will not be equivalent. If we want the two theories to be equivalent one must find the required conditions on the target fields that make these anomalies cancel. The simplest way to do so is to assume that the spin and gauge connections match in the original theory, *i.e.*, $\omega = A$ [8, 10, 11, 12]. Under this assumption, the duality transformation then guarantees that in the dual theory spin and gauge connections also match, $\tilde{\omega} = \tilde{A}$. In the following we choose to cancel the anomalies according to such prescription.

We are then required to have $\mu = \Omega$, where we define:

$$\Omega_{\mu\nu} \equiv \frac{1}{2}(\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu), \tag{4.7}$$

with ξ the Killing vector generating the Abelian isometry and ∇_μ the metric covariant derivative. In particular for our adapted coordinates $\xi_\mu = g_{\mu 0}$, and as the affine connection is metric compatible, $\Omega = 0$. But then,

$$\mu_{IJ} = (\kappa - A_0)_{IJ} = 0, \tag{4.8}$$

and we are back to (4.6). Then, the consistency conditions are satisfied as long as the anomalies are canceled.

Putting together the information in (4.6) and (4.8), let us address a few questions. The first thing we notice is that $\beta_0^A = 0$ as $\kappa = A_0$ (recall that in adapted coordinates κ satisfies (2.3), and so $F_{0i} = 0$), which is consistent with the fact that the target gauge transformation parameter is not renormalized. Then, the consistency conditions become,

$$\bar{\beta}_0^{\tilde{A}} = 0 \qquad , \qquad \bar{\beta}_i^{\tilde{A}} = \bar{\beta}_i^{A}, \tag{4.9}$$

stating that the gauge beta function is self-dual under (2.5-6). But so, by (4.4-5) with (4.6) satisfied, this proves that (3.6) explicitly satisfies the consistency conditions (4.9) – to the one loop, order $\mathcal{O}(\alpha')$, we are working to.

Given that the gauge field $\bar{\beta}$-function satisfies the consistency conditions, the question that follows is whether scaling arguments joined with the consistency conditions (4.9) are enough information to uniquely determine (3.3). This would mean that (4.9) is verified by, and only by, the correct gauge RG flows of the heterotic sigma model. Replacing (3.6) in (4.9) and using the duality transformations, we obtain the beta function constraint:

$$\beta_i^{\tilde{A}} = \beta_i^{A} + \frac{1}{2} F_i{}^k \partial_k \ln a. \tag{4.10}$$

On the other hand, according to scaling arguments the possible tensor structures appearing in the one loop, order $\mathcal{O}(\alpha')$, gauge beta function are:

$$\beta_\mu^A = c_1 \, \mathbf{D}^\lambda F_{\lambda\mu} + c_2 \, H_\mu{}^{\lambda\rho} F_{\lambda\rho}, \tag{4.11}$$

where the notation is as in (3.3). Dealing with torsionless backgrounds (4.1) we set $c_2 = 0$, and are left with c_1 alone. Inserting (4.11) in (4.10) then yields,

$$(c_1 - \frac{1}{2}) F_i{}^k \partial_k \ln a = 0, \tag{4.12}$$

and as the background is general (though torsionless), we obtain $c_1 = \frac{1}{2}$ which is the correct result (3.3). Therefore, our consistency conditions were able to uniquely determine the one loop gauge field beta function, in this particular case of vanishing torsion. We shall later see that the same situation happens when one deals with torsionfull backgrounds.

A final point to observe is that the proof of $\mu_{IJ} = 0$ through (4.6) (and so, also the proof of validity of the consistency conditions) is telling us that only if the sigma model is consistent at the quantum level (no anomalies) can the duality symmetry be consistent at the quantum level (by having the consistency conditions verified). Still, one could argue that strictly speaking (4.6) requires either $\mu_{IJ} = 0$ or $\bar{\beta}^g_{00} = 0$. But we also need to cancel all anomalies in order to have an RG flow. So, if one wants to flow away from the fixed point along all directions in the parameter space, one needs to cancel the anomalies in such a way that $\mu_{IJ} = 0$ in the adapted coordinates to the Abelian isometry. Otherwise, if we were to choose an anomaly cancelation procedure yielding non-vanishing μ_{IJ}, it would seem that in order to preserve T-duality at the quantum level away from criticality, expression (4.6) would require that one could only flow away from the fixed point along specific regions of the parameter space (*i.e.*, regions with $\bar{\beta}^g_{00} = 0$). As we shall see next when we deal with torsionfull backgrounds, this is actually not a good option: the only reasonable choice one can make is $\mu_{IJ} = 0$.

5 Torsionfull Backgrounds

To complete our analysis, we are left with the inclusion of torsion to the previous results. We shall see that even though the calculations are rather involved, the results are basically the same (see the paper).

Thus, the consistency conditions are verified by, and only by, the correct RG flows of the heterotic sigma model. In other words, classical target space duality symmetry survives as a valid quantum symmetry of the heterotic sigma model.

6 Conclusions

We have studied the consistency between RG flows and T-duality in the $d = 2$ heterotic sigma model. The basic statement $[T, R] = 0$ that had been previously studied in bosonic sigma models was shown to keep its full validity in this new situation, with the added bonus of giving us extra information on how one should cancel the anomalies (arising from chiral fermion rotations) of the heterotic sigma model. Moreover, contrary to previously considered cases [4, 5, 6], the requirement $[T, R] = 0$ enabled us to uniquely determine the (gauge field) beta function at one loop order, without any overall global constant left to be determined.

Such a basic statement $[T, R] = 0$ has now been shown to be alive and well in a wide variety of situations, possibly validating the claim in [5, 7] that it should be a more fundamental feature of the models in question than the invariance of the string background effective action.

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