Large N and the Dine-Rajaraman problem

J. McCarthy, L. Susskind and **Andy Wilkins**

We compute the effective action for scattering of three well separated extremal brane solutions, in 11d supergravity, with zero p_- transfer and small transverse velocities. Using an interpretation of the conjecture of Maldacena, following Hyun, this can be viewed as the large N limit of the Matrix theory description of three supergraviton scattering at leading order. The result is consistent with the perturbative supergravity calculation.

Introduction

Matrix theory [1] proposes that M-theory is described by the maximally supersymmetric quantum mechanics of $U(N)$ matrices in the large N limit. Moreover, a prescription for computations is given [1, 2, 3] in which it has been argued [4] that the finite N corresponds to the so-called discrete light cone frame quantisation of the M-theory (for a review see [5]). One test of these conjectures (see, eg, [6, 7]) is to compare low energy scattering of supergravitons in the Matrix theory with the corresponding results in eleven-dimensional supergravity (a subtle limit, see [8, 9]). In particular, the two theories agree for the scattering of two well-separated supergravitons with small transverse velocities [10, 1, 11]. However, supergravity seems to predict different behaviour to that of the matrix model for processes such as the scattering of three supergravitons [12].

It has been suggested [13, 14] that this discrepancy may vanish on taking the large N limit. However, only through the recent work of Maldacena [15] has it been possible to deal with this limit. In [15, 16] brane configurations were studied in the limit where the field theory on the brane decouples from the bulk, and it was observed that when the number of branes N, becomes large, the curvature of spacetime around the brane becomes small (for earlier discussions in the conformal case, see [17] and references therein). However, for small curvatures branes are well described by extremal black-hole type solutions of the associated supergravity. Moreover, as discussed in [18] , this limit corresponds to the infinite boost limit in the DLCQ Matrix theory prescription mentioned above.

Thus we are naturally led to the following conjecture.

Conjecture: In the large N limit of DLCQ Matrix theory, supergravitons are described by D0-brane solutions of IIA supergravity.

This leads immediately to the trivial resolution of the problem in Dine and Rajaraman [12].

Resolution: Since D0-branes are BPS states which can be identified with Kaluza-Klein supergraviton modes of 11d supergravity [19, 20], their leading order scattering amplitudes will be proportional to those of point particles in 11d supergravity. Therefore, leading order supergraviton amplitudes calculated using the large N limit of DLCQ Matrix theory *are* those of 11d supergravity.

In the rest of this poster we describe an explicit calculation of the three supergraviton amplitude as "extremal black hole" solutions, since the details may be of interest.

Summary of the calculation

We calculate the effective action for large separation and low transverse velocity scattering of these particles (neglecting spin effects as usual), following a "post-Newtonian" calculation similar to those going back to [21] – with the slight twist that we work in the lightcone frame.
The essence of the calculation is as follows.

1. Lift the static BPS solution of IIA supergravity with n_c clusters of D0-branes to eleven dimensional supergravity compactified on a spacelike circle. Infinitely boost along the spacelike circle to obtain the DLCQ of supergravity.

2. In the leading order, where spin effects are neglected, the D0-brane action is that of a massless point particle with constant p_-.

3. Promote the centres in the static solution to dynamical variables and, by solving Einstein's equations, determine corrections to the metric order-by-order in time (x^+) derivatives and separation of the branes.

4. Substitute the solution found back into the action to obtain the effective action for the centres.

In the following V will stand for a typical (small) transverse speed and L a typical (large) transverse separation.

1. Uplifting static D0-branes to 11D

The static BPS solution of IIA supergravity that describes n_c clusters of D0-branes at positions $\boldsymbol{y}_i$, ([22], and references therein) is,

$$
\begin{aligned}
d\bar{s}^2 &= \bar{g}_{\mu\nu}(x)dx^\mu dx^\nu = -f_0(\boldsymbol{y})^{-1/2}dt^2 + f_0(\boldsymbol{y})^{1/2}d\boldsymbol{y}\cdot d\boldsymbol{y} \ , \\
e^{\overline{\Phi}(x)} &= g_{st}f_0^{3/4} \ , \qquad \overline{A}_0(x) = f_0^{-1} - 1 \ , \\
f_0(\boldsymbol{y}) &= 1 + \sum_{i=1}^{n_c} f_0^{(i)} \ , \qquad f_0^{(i)} = \frac{\mu_i}{g_{st}^2\ell_{st}^2|\boldsymbol{y}-\boldsymbol{y}_i|^7} \ ,
\end{aligned} \tag{1}
$$

The statement that these are D0-branes means that the "charges" in this solution, μ_i, are determined in terms of string parameters [20] by $\mu_i/g_{st}^2\ell_{st}^2 \sim N_i\ell_{st}^7 g_{st}$ where N_i is the number of D0-branes in the $i^{\rm th}$ cluster. As described clearly in [18, 23], this can be lifted to eleven dimensions (on R_s) and infinitely boosted to obtain the theory compactified on a null circle with radius R

$$
\begin{aligned}
ds^2 &= dx^+dx^- + f_0^B(\boldsymbol{y})dx^-dx^- + d\boldsymbol{y}\cdot d\boldsymbol{y} \ , \\
f_0^B(\boldsymbol{y}) &= f_0(\boldsymbol{y}) - 1 = \sum_{i=1}^{n_c} \frac{\mu_i}{R^2|\boldsymbol{y}-\boldsymbol{y}_i|^7} \ .
\end{aligned} \tag{2}
$$

2. The source action

The appropriate point-particle action has been discussed in [7]. The momentum of each cluster in the minus direction p_-^i, is cyclic $p_-^i \equiv Q_i$, so that the physics is contained in the "Routhian" constructed from the usual action for a massive particle. After the massless limit is taken, the action for particle i with transverse velocity $v_i^a = dy_i^a/dx^+$, reads

$$
\begin{aligned}
S^{(i)} = Q_i \int dx^+ \frac{1}{g_{--}} \Big[& g_{+-} + g_{-a}v_i^a - \\
& \sqrt{(g_{+-}+g_{-a}v_i^a)^2 - g_{--}(g_{++} + 2g_{+a}v_i^a + g_{ab}v_i^a v_i^b)} \Big] \ .
\end{aligned} \tag{3}
$$

3. Einstein's equations

In summary, then, the system of interest is described by

$$S = \kappa^{-2} \int d^{11}x\sqrt{-g}R(g) + \sum_{i=1}^{n_c} S^{(i)}, \tag{4}$$

where the source action S^i is given in Eq. (3). Now let the centres in the static solution Eq. (2) become dynamical variables, and determine corrections to the metric so that we have a solution to Einstein's equations

$$g_{MN} = \bar{g}_{MN} + g_{MN}^{(>)}, \qquad g_{MN}^{(>)} \equiv \sum_{n>0} g_{MN}^{(n)}, \tag{5}$$

order-by-order in an expansion in x^+ derivatives. The zeroth order $\bar{g}$ is given by Eq. (2) and we will say that $g^{(n)}$ "has $d_t = n$" in this expansion. This is a nontrivial solution since it corresponds to a nontrivial "tangent deformation" in the moduli space of static solutions. The corrections $g^{(n)}$ vanish on the spatial infinity.

• <u>Zeroth order</u>. This is just the static solution. Einstein's equations imply

$$Q_i = \frac{32\pi^4}{15}\frac{\mu_i}{R^2}\frac{2\pi R}{\kappa^2} \equiv \frac{N_i}{R}. \tag{6}$$

• <u>First order</u>. The first order solution is easily understood in terms of transverse boosts. A "Galilean" transversal boost with velocity $\boldsymbol{v}$ in light-cone time, as is appropriate for our discussion, gives

$$\begin{aligned} x'^+ &= x^+ , \\ x'^- &= x^- + 2\boldsymbol{v}\cdot\boldsymbol{y} - \boldsymbol{v}^2 x^+ , \\ \boldsymbol{y}' &= \boldsymbol{y} - x^+\boldsymbol{v} . \end{aligned} \tag{7}$$

Taking x' as the coordinates of the "static frame", we obtain the metric for a single center moving transversally with constant velocity by using the boost as a coordinate transformation ($r' = |\boldsymbol{y} - x^+\boldsymbol{v}|$),

$$\begin{aligned} ds^2 = &(1 - 2v^2 f_0^B(r'))dx^+dx^- + f_0^B(r')dx^-dx^- + v^4 f_0^B(r')dx^+dx^+ \\ &+(\delta_{ab} + 4f_0^B(r')v_a v_b)dy^a dy^b + 4f_0^B(r')v_a dx^- dy^a - 4f_0^B(r')v^2 v_a dx^+ dy^a \end{aligned} \tag{8}$$

This is extended to n_c centers moving independently to give the ansatz

$$\begin{aligned} ds^2 &= \left(1 - 2\sum_{i=1}^{n_c} v_i^2 f_0^{(i)}(r_i)\right) dx^+dx^- + \sum_{i=1}^{n_c} f_0^{(i)}(r_i) dx^-dx^- \\ &\quad + \sum_{i=1}^{n_c} v_i^4 f_0^{(i)}(r_i) dx^+dx^+ + \left(\delta_{ab} + 4\sum_{i=1}^{n_c} f_0^{(i)}(r_i) v_i^a v_i^b\right) dy^a dy^b \\ &\quad + 4\sum_{i=1}^{n_c} f_0^{(i)}(r_i) v_i^a dx^- dy^a - 4\sum_{i=1}^{n_c} f_0^{(i)}(r_i) v_i^2 v_i^a dx^+ dy^a , \end{aligned} \tag{9}$$

where $r_i = |\boldsymbol{y} - \boldsymbol{y}_i(x^+)|$. It is now straightforward to check that this ansatz is indeed a solution of the 11d system to first order in V

$$g_{-a}^{(1)} = \sum_i 2\frac{\mu_i}{R^2 r_i^7} v_i^a \quad \text{with all other } g^{(1)} = 0 . \tag{10}$$

• <u>Leading large-distance behaviour of the d_t expansion</u> The expression for $\dot{\boldsymbol{v}}_i$ must be iteratively determined in the d_t-expansion. Clearly this only receives corrections at even orders of d_t. It is well known that the $d_t = 2$ contribution vanishes (see, e.g., [24, 25]) — this is "flatness of the moduli space". To see this in the present calculation, we use the first-order corrections Eq. (10), to write down the equation of motion for the centres (the geodesic equation) to second order

$$\dot{v}_i^a = -\frac{1}{2}\partial_a g_{++}^{(2)} + O(V^4) . \tag{11}$$

But from the Einstein equations to second order (still only using Eq. (10)), we find

$$-\frac{\pi R}{\kappa^2}\Delta_\perp g_{++}^{(2)} = T_{++}^{(2)} = 0 , \tag{12}$$

and thus $\dot{\boldsymbol{v}}_i \sim O(V^4)$ as stated.

Further, a detailed calculation[1] [26] shows that the ansatz Eq. (9) is correct to second order, except for $g_{--}^{(2)}$ which is given by

$$g_{--}^{(2)} = \sum_{ij} \frac{\mu_i}{R^2}\frac{\mu_j}{R^2}\frac{|\boldsymbol{v}_i - \boldsymbol{v}_j|^2}{|r_i|^7 |r_j|^7} + f^{(2)} , \tag{13}$$

[1]To regularise the point particle we replace $r_i \to (r_i^2 + \epsilon^2)^{1/2}$.

where,

$$\Delta_\perp f^{(2)} = 2\partial_a\partial_b \sum_{ij} \frac{\mu_i}{R^2}\frac{\mu_j}{R^2}\frac{(v_i - v_j)^a(v_i - v_j)^b}{|r_i|^7|r_j|^7} . \tag{14}$$

The important result in the above is the observation that the solution at second order differs from the boosted metric Eq. (9) by $O(\frac{V^2}{L^{14}})$.

At higher order the leading behaviour is equally simple. Let us separate out the "boosted" corrections $\widehat{g}$ which are contained in Eq. (9); ie,

$$g^{(>)}_{MN} = \widehat{g}_{MN} + h_{MN} , \tag{15}$$

Then

$$h^{(n)}_{MN} = O\left(\frac{V^n}{L^{14}}\right) . \tag{16}$$

To see this one just calculates the leading terms in the Einstein equations at n^{th} order in the expansion.

We now show that the leading term in the effective action is determined by "independently boosted" metric $\widehat{g}_{MN}$. Expanding the action around $\widehat{g}_{MN}$, the above result implies that

$$S[g] = S[\widehat{g}] + \left.\frac{\delta S}{\delta g}\right|_{\widehat{g}}[h] + \text{ higher order} . \tag{17}$$

The second term can be further expanded around the static solution $\overline{g}_{MN}$, given by Eq. (2), and only the first term in this expansion is required for the leading order result,

$$\left.\frac{\delta S}{\delta g}\right|_{\widehat{g}=\overline{g}+\overline{h}}[h] = \left.\frac{\delta^2 S}{\delta g \delta g}\right|_{\overline{g}}[\overline{h}, h] + \text{ higher order} .$$

Now, the fact that the boosted single center metric is a solution of the Einstein equations for a constant transverse velocity source implies that, up to derivatives of $\boldsymbol{v}$,

$$\left.\frac{\delta S}{\delta g}\right|_{\widehat{g}_i}[h] = 0 ,$$

where $\hat{g}_i$ denotes the boosted single center solution (for the ith center) — ie, the limit of $\hat{g}_{MN}$ as $\mu_j \to 0$, $j \neq i$. Thus we must have, up to derivatives of v,

$$\left.\frac{\delta^2 S}{\delta g \delta g}\right|_{\bar{g}} [\bar{h}, \cdot] = O(\frac{\mu_i \mu_j}{L^{14}}),$$

meaning that the RHS is at least quadratic in the μ_i since it vanishes if all but one of them is sent to zero. The notation $[\cdot]$ means "true for arbitrary insertions into the variational slot". By inserting h it is seen that this term is higher order, and can be ignored. Thus we only have to worry about the contribution of $\dot{v}, \ddot{v}, \ldots$ terms.

Using the previous results, we have so far shown that, in the second term of Eq. (17),

$$\left.\frac{\delta S}{\delta g}\right|_{\hat{g}} = O(\mu V^4). \tag{18}$$

Thus, we only need the terms with derivatives on v in the LHS of Eq. (18). To this order, the $\ddot{v}$ need only be contracted with $h^{(1)}$ which is zero. The $\dot{v}$ terms only appear, at this order, in the $+-$ and ab components of the Einstein tensor. But, as summarized above, $h^{(2)+-}$ and $h^{(2)ab}$ vanish. Thus, finally, the result is proved — all terms but the first in Eq. (17) are higher order in $1/L$.

4. Computing the action

At this point we simply compute the leading contribution up to $O(V^6)$. The result for the leading $O(V^4)$ contribution to two particle scattering is (we always drop "polarisation" terms with numerators including $\boldsymbol{v}\cdot\boldsymbol{y}$)

$$S_{\rm eff}^{(4)} = -\frac{15}{2}\frac{N_1 N_2}{R^3 M^9}\frac{|\boldsymbol{v}_1-\boldsymbol{v}_2|^4}{|\boldsymbol{y}_1-\boldsymbol{y}_2|^7}. \tag{19}$$

This is precisely the result reported in [11]. In the present calculation it results from a cancellation between Einstein and source contributions.
The result for the leading $O(V^6)$ contribution to three particle scattering, in the limit considered by Dine and Rajaraman ($|\boldsymbol{y}_3| >> |\boldsymbol{y}_1-\boldsymbol{y}_2|$) is (for brevity we only write the "Dine-Rajaraman" term)

$$S_{\rm eff}^{(6)} = -15^2\frac{N_1 N_2 N_3}{R^5 M^{18}}\frac{|\boldsymbol{v}_1-\boldsymbol{v}_2|^2|\boldsymbol{v}_2-\boldsymbol{v}_3|^2|\boldsymbol{v}_1-\boldsymbol{v}_3|^2}{|\boldsymbol{y}_3|^7|\boldsymbol{y}_1-\boldsymbol{y}_2|^7}. \tag{20}$$

It is interesting to note that there is clearly no contribution from the source action of this form. We see that the term required for agreement with the perturbative supergravity calculation does appear.
After completion of this work, a number of papers appeared which discuss the Dine-Rajaraman problem [27, 28, 29, 30] from the finite N side. In [27] it was suggested that the supersymmetry cancellations proposed in the Matrix theory calculation of [12] would not occur, but this has been disputed in [28] and [29]. The technical calculation in this paper has significant overlap with [30], where, further, the Matrix theory result is recalculated and shown to be in agreement at finite N. The present paper supports the supergravity side of their calculation.

References

[1] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *Phys. Rev.* **D55** (1997) 5112, `hep-th/9610043`.

[2] N. Seiberg, *Phys. Rev. Lett.* **79** (1997) 3577, `hep-th 9710009`.

[3] A. Sen, "D0 branes on T^n and matrix theory", `hep-th/9709220`.

[4] L. Susskind, "Another conjecture about M(atrix) theory", `hep-th/9704080`.

[5] A. Sen, "An introduction to nonperturbative string theory", `hep-th/9802051`.

[6] K. Becker, M. Becker, Nucl. Phys. **B506** (1997) 48, `hep-th/9705091`.

[7] K. Becker, M. Becker, J. Polchinski and A. Tseytlin, it Phys. Rev. **D56** (1997) 3174, `hep-th/9706072`.

[8] S. Hellerman and J. Polchinski, "Compactification in the Lightlike Limit," `hep-th/9711037`.

[9] A. Bilal, "DLCQ of M-theory as the light-like limit," `hep-th/9805070`.

[10] M. Douglas, D. Kabat, P. Pouliot and S. Shenker, Nucl. Phys. **B485** (1997) 85, `hep-th/9608024`.

[11] D. Berenstein and R. Corrado, *Phys. Lett. B* **406** (1997) 37, `hep-th/9702108`.

[12] M. Dine and A. Rajaraman, "Multigraviton scattering in the matrix model", `hep-th/9710174`.

[13] T. Banks, Nucl. Phys. Proc. Suppl. **67** (1998) 180, `hep-th/9710231`.

[14] D. Bigatti and L. Susskind, "Review of Matrix theory", `hep-th/9712072`.

[15] J. Maldacena, "The large N limit of superconformal field theories and supergravity", `hep-th/9711200`.

[16] N. Itzhaki, J. Maldacena, J. Sonnenschein and S. Yankielowicz, "Supergravity and the large N limit of theories with sixteen supercharges", `hep-th/9802042`.

[17] S. Gubaer and I. Klebanov, Phys. Lett. **413B** (1997) 41, `hep-th/9708005`.

[18] S. Hyun, "The background geometry of DLCQ supergravity", `hep-th/9802026`.

[19] E. Witten, Nucl.Phys. B443 (1995) 85, `hep-th/9503124`.

[20] J. Polchinski, Phys. Rev. Lett. **75** (1995) 4724, `hep-th/9510017`.

[21] R. Ferrell and D. Eardley, Phys. Rev. Lett. **59** (1987) 1671.

[22] K.S. Stelle, "BPS branes in supergravity", `hep-th 9803116`.

[23] S. Hyun, Y. Kiem and H. Shin, *Phys. Rev.* **D57** (1998) 4856, `hep-th/9712021`.

[24] C. Bachas, Phys. Lett. **374B** (1996) 37, `hep-th/9511043`.

[25] R. Khuri and R. Myers, Nucl. Phys. **B466** (1996) 60, `hep-th/9512061`.

[26] J. McCarthy and A. Wilkins, to appear.

[27] M. Fabbrichesi, G. Ferretti and R. Iengo, "Supergravity and Matrix theory do not disagree on multi-graviton scattering", `hep-th/9806018`.

[28] W. Taylor IV and M. Raamsdonk, "Three-graviton scattering in Matrix theory revisited", `hep-th/9806066`.

[29] R. Echols and J. Gray, "Comment on multigraviton scattering in the Matrix model", `hep-th/9806109`.

[30] Y. Okawa and T. Yoneya, "Multi-body interactions of D-particles in supergravity and Matrix theory", `hep-th/9806108`.