

STRINGS and D3-branes in $AdS_5 \times S^5$

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hep-th/9805028,
9806095

- superstring in flat superspace
- superstring action in $AdS_5 \times S^5$
from $SU(2,2|4)$ superalgebra
- some properties: conformal invariance
- D3-brane action in $AdS_5 \times S^5$

Type II B Supergravity:

Maximally supersymmetric vacua

- Flat $D=10$ space

- $AdS_5 \times S^5 + F_{\mu_1 \dots \mu_5} \neq 0$

Schwarz 83

Symmetries:

- Super Poincare group

- $SU(2,2|4)$ supergroup

Superstring actions:

- σ -model on flat superspace
(super Poincare) / $SO(1,9)$

- σ -model on symmetric superspace
 $SU(2,2|4) / SO(1,4) \times SO(5)$

Green-Schwarz Superstring Action

as 2d σ -model on flat target superspace

Henneaux, Mezincescu 85

$$\text{Superspace} = (\text{Super Poincare}) / SO(1,9)$$

$$(P_m, J_{mn}, Q^I) \quad I = 1, 2$$

$$\{Q_\alpha^I, Q_\beta^J\} = 2 \delta^{IJ} \Gamma_{\alpha\beta}^m P_m$$

$$[P_m, P_n] = 0, \quad [P_m, J_{nk}] = \eta_{m[n} P_{k]}$$

Coset: $G(x, \theta) = e^{-ix^m P_m + \theta \cdot Q}$

$$X^M = (x^m, \theta_I^\alpha)$$

$$G^{-1} dG = L^m P_m + L^{\alpha I} Q_{\alpha I} + L^{mn} J_{mn}$$

M-C: $dL^m = \bar{L} \Gamma^m L, \quad dL = 0$

$$L^m = dx^m - i \bar{\theta}^I \Gamma^m \theta^I$$

$$L^{\alpha I} = d\theta^{\alpha I}$$

String Action:

- global super-invariant
- α -invariant (add WZ term)
Green, Schwarz

$$S = \int_{M_2} L^m L^n \eta_{mn} + \int_{M_3} H_3$$

$$H_3 = L^m \wedge \bar{L}^I \Gamma_m \wedge L^J \eta_{IJ} \quad dH_3 = 0$$

$$\eta_{IJ} = \text{diag}(1, -1) \quad U(1) \text{ broken}$$

$$H_3 = dB_2, \quad B_2 \sim dx^m \wedge \bar{\theta} \Gamma_m d\theta + \bar{\theta} \Gamma^m d\theta \wedge \bar{\theta} \Gamma_m d\theta$$

$$S_{GS} = \int d^2\sigma \left[(\partial_i x^m - i \bar{\theta}^I \Gamma^m \partial_i \theta^I)^2 - 2i \varepsilon^{ij} \eta_{IJ} \bar{\theta}^I \Gamma_m \partial_i \theta^J \left(\partial_j x^m - \frac{1}{2} i \bar{\theta}^N \Gamma^m \partial_j \theta^N \right) \right]$$

Generalization to the case when

$$\partial x^m \partial x^n \eta_{mn} \rightarrow \partial x^m \partial x^n g_{mn}(x) \quad ?$$

$$g_{mn} = \text{metric of } AdS_5 \times S^5$$

Superstring Action in $AdS_5 \times S^5$

General form of action in IIB supergravity background

$$I = \int (E_i^m E_i^m + \epsilon^{ij} E_i^A E_j^B B_{AB})$$

Grisaru, Howe
Mazur, Nilsson
Townsend (85)

Background superfields subject to IIB sugra constraints

κ -invariance if background fields are on-shell

How to find background superfields explicitly for a given bosonic solution ?

Order-by-order expansion in θ :

$$\begin{aligned} \mathcal{L} = & G_{mn}(x) \partial x^m \partial x^n + B_{mn}(x) \partial x^m \partial x^n \\ & + \bar{\theta} \mathcal{D} \theta \partial x + \bar{\theta} \mathcal{D} \theta \bar{\theta} \mathcal{D} \theta \\ & + R_{mnpq} \bar{\theta} \gamma^{mnp} \theta \bar{\theta} \gamma^{q} \theta \partial x^p \partial x^q \\ & + F_{m_1 \dots m_5} \bar{\theta} \gamma^p \gamma^{m_1 \dots m_5} \gamma^q \theta \partial x^p \partial x^q \\ & + \dots + O(\theta^{32}) \end{aligned}$$

$$R_{mnpq} \sim \gamma_{mk} \gamma_{ne} - \gamma_{me} \gamma_{nk}$$

$$F_{m_1 \dots m_5} \sim E_{m_1 \dots m_5}$$

in each factor of $AdS_5 \times S^5$

Alternative Approach:

- start directly with symmetry algebra
- consider coset superspace
- determine super vielbein:
left-invariant 1-form L^A
(solve Maurer-Cartan equations)
- find closed invariant 3-form
(WZ-term $\sim L \wedge L \wedge L$)
from condition of \mathcal{Z} -invariance

Manifest global supersymmetry

New "symmetric" superspace [no $U(1)$]

(comparison to action of GHMNT determines "standard" supergeometry, cf. Kallosh et al)

Advantage of using L^A over particular coordinates (x, θ)

cf. $G_{mn}(x) \partial x^m \partial x^n$ and $(g^{-1} \partial g)^2$
manifest \rightarrow global symmetry

$$\text{Ad } S_5 \times S^5 = \frac{SO(4,2)}{SO(4,1)} \times \frac{SO(6)}{SO(5)}$$

Superalgebra of $SU(2,2|4)$

$$SO(4,2) \oplus SO(6) \oplus 32 \text{ supercharges}$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow \\ (P_a, J_{ab}) & (P_{a'}, J_{a'b'}) & (Q^{\alpha\alpha'I}) \end{matrix}$$

Translations: $(P_a, P_{a'})$ $a, b = 0, 1, 2, 3, 4$

Rotations: $(J_{ab}, J_{a'b'})$ $a', b' = 5, 6, 7, 8, 9$

$$\Gamma^m \rightarrow (\Gamma^a, \Gamma^{a'})$$

$$\begin{aligned} \Gamma^a &= \gamma^a \times \mathbb{1} \times \epsilon_1 \\ \Gamma^{a'} &= \mathbb{1} \times \gamma^{a'} \times \epsilon_2 \end{aligned}$$

$Q^{\alpha\alpha'I}$: two MW spinors of same chirality

$$[P_a, P_b] = J_{ab}, \quad [P_{a'}, P_{b'}] = -J_{a'b'}$$

$$[Q^I, P_a] = -\frac{i}{2} \epsilon^{IJ} Q^J \gamma_a, \quad [Q^I, P_{a'}] = \frac{1}{2} \epsilon^{IJ} Q^J \gamma_{a'}$$

$$[Q^I, J_{ab}] = -\frac{1}{2} Q^I \gamma_{ab}, \quad [Q^I, J_{a'b'}] = -\frac{1}{2} Q^I \gamma_{a'b'}$$

$$\begin{aligned} \{Q^I, Q^J\} &= 2\delta^{IJ} (-iC' \gamma^a P_a + C \gamma^{a'} P_{a'}) \\ &+ \epsilon^{IJ} (C' \gamma^{ab} J_{ab} - C \gamma^{a'b'} J_{a'b'}) \end{aligned}$$

Superspace:

$$\frac{SU(2,2|4)}{SO(4,1) \times SO(5)}$$

Coset representative:

$$G(x, \theta)$$

$$G^{-1} dG = L^{\hat{a}} P_{\hat{a}} + L^{\hat{a}\hat{b}} J_{\hat{a}\hat{b}} + L^I Q_I$$

$$\hat{a} = (a, a')$$

$$\hat{a}\hat{b} = (ab, a'b')$$

↑
S-beins

↑
Cartan
connections

↑
32-spinor
vielbein

Maurer-Cartan [implied by $SU(2,2|4)$]

$$dL^{\hat{a}} + L^{\hat{a}\hat{b}} \wedge L^{\hat{b}} + \bar{L}^I \gamma^{\hat{a}} \wedge L^I = 0$$

$$dL^{\hat{a}\hat{b}} + L^{\hat{a}\hat{c}} \wedge L^{\hat{c}\hat{b}} + L^{\hat{a}} \wedge L^{\hat{b}} + \varepsilon^{IJ} \bar{L}^I \gamma^{\hat{a}\hat{b}} \wedge L^J = 0$$

$$dL^I + \frac{1}{2} \varepsilon^{IJ} \gamma^{\hat{a}} L^J \wedge L^{\hat{a}} + \frac{1}{4} \gamma^{\hat{a}\hat{b}} L^I \wedge L^{\hat{a}\hat{b}} = 0$$

String action:

- bosonic part is σ -model on $AdS_5 \times S^5$
- has global $SU(2,2|4)$ invariance
- invariant under α -symmetry
- reduces to flat-space GS action for radius $R \rightarrow \infty$

Exists and Unique

(contains expected "RR coupling" to F_5 , etc.)

Action has same form as
in flat superspace (!)

$$S = \int_{M_2} d^2\sigma \sqrt{g} g^{ij} L_i^{\hat{a}} L_j^{\hat{a}} + \int_{M_3} H_3$$

$$\partial M_3 = M_2$$

$$H_3 = L^{\hat{a}} \wedge \bar{L}^I \gamma^{\hat{a}} L^J \zeta_{IJ}$$

Unique closed super-invariant 3-form
(required by α -symmetry)

$$L^A = dX^M L_M^A, \quad L_i^A \equiv \partial_i X^M L_M^A$$

$$\delta_x X^{\hat{a}} = \delta X^M L_M^{\hat{a}}, \quad \delta_x \theta^I = \delta X^M L_M^I$$

$$\delta_x X^{\hat{a}} = 0, \quad \delta_x \theta^I = L_i^{\hat{a}} \gamma^{\hat{a}} \alpha_i^I$$

$$\delta_x (\sqrt{g} g^{ij}) = \sqrt{g} (P_-^{jk} \bar{L}_k^1 \alpha_i^1 + P_+^{jk} L_k^2 \alpha_i^2)$$

Eqs. of motion: manifestly invariant, in terms
of L's only

Explicit 2d form of WZ term:

choose θ -coordinates:

$$G(x, \theta) = g(x) e^{\theta \cdot Q}$$

Define:

$$L(x, t\theta) \equiv L_t(x, \theta)$$

$$S = \int d^2\sigma (\sqrt{g} g^{ij} L_i^{\hat{a}} L_j^{\hat{a}} + \epsilon^{ij} \int dt L_{ti}^{\hat{a}} \bar{\theta}^I \gamma^{\hat{a}} L_{tj}^J \eta_{IJ})$$

M-C eqs. imply basic equations:

$$\partial_t L_t^{\hat{a}} = \bar{\theta}^I \gamma^{\hat{a}} L_t^I$$

$$\partial_t L_t^I = d\theta^I + \gamma^{\hat{a}} \theta L_t^{\hat{a}} + \frac{1}{4} \gamma^{\hat{a}\hat{b}} \theta L_t^{\hat{a}\hat{b}}$$

$$\partial_t L_t^{\hat{a}\hat{b}} = \bar{\theta}^I \gamma^{\hat{a}\hat{b}} L_t^I$$

$t=0$: $L_t^{\hat{a}} = e^{\hat{a}(x)}$, $L_t^{\hat{a}\hat{b}} = \omega^{\hat{a}\hat{b}(x)}$, $L_t^I = 0$
(or $\theta=0$)

Determine complete dependence on θ : ($t=1$)

$$L^{\hat{a}} = e^{\hat{a}} - i \bar{\theta}^I \gamma^{\hat{a}} D\theta^I + \dots$$

$$L^I = D\theta^I + \frac{1}{6} \gamma^{\hat{a}} \bar{\theta} \gamma^{\hat{a}} D\theta^I + \dots$$

$$D\theta^I = \left(d + \frac{1}{4} \omega^{\hat{a}\hat{b}} \gamma^{\hat{a}\hat{b}} \right) \theta^I - \frac{1}{2} i \epsilon^{IJ} \gamma^{\hat{a}} e^{\hat{a}} \theta^J$$

$\text{Ad}_{S^5}^S$ covariant derivative

$$d\omega^{\hat{a}\hat{b}} + \omega^{\hat{a}\hat{c}} \wedge \omega^{\hat{c}\hat{b}} = \pm e^{\hat{a}} \wedge e^{\hat{b}} \quad (R=1)$$

Symmetric space : closed solution of M-C

$$e^a_b = \left(\frac{\sinh \sqrt{m}}{\sqrt{m}} \right)_b^a \quad m_{ab} \sim R_{abcd} X^c X^d$$
$$\sim X^2 \delta_{ab} - X_a X_b$$

$$e^a_b = \left(\delta_{ab} - \frac{X_a X_b}{X^2} \right) \frac{\sinh X}{X} + \frac{X_a X_b}{X^2}$$

Similar solution in θ -sector:

$$L^{\hat{a}} = e^{\hat{a}} - i \bar{\theta} \gamma^{\hat{a}} W(\theta) D\theta$$

$$L^I = V(\theta) D\theta^I$$

Kallosh
Rajaraman
Rahmfeld

$$V = \frac{\sinh \sqrt{m}}{\sqrt{m}} = 1 + \frac{1}{3!} m + \frac{1}{5!} m^2 + \dots$$

$$W = \frac{\cosh \sqrt{m} - 1}{m} = \frac{1}{2} + \frac{1}{4!} m + \frac{1}{6!} m^2 + \dots$$

$$m = \gamma^{\hat{a}} \theta \bar{\theta} \gamma^{\hat{a}} + \gamma^{\hat{a}\hat{b}} \theta \bar{\theta} \gamma^{\hat{a}\hat{b}}$$

String Action:

$$S = \int d^2\sigma \left\{ \left(e^{\hat{a}}_i - i \bar{\theta} \gamma^{\hat{a}} W(\theta) D_i \theta \right)^2 \right.$$

$$\left. + \varepsilon^{ij} \int dt \left[e^{\hat{a}}_i - \bar{\theta} \gamma^{\hat{a}} W(t\theta) D_i \theta \right] \bar{\theta} \gamma^{\hat{a}} V(t\theta) D\theta \right\}$$

Action to θ^4 -order: " $\partial \rightarrow D$ "

$$I = \int d^2\sigma \left[\left(e_{\hat{\mu}}^{\hat{a}} \partial_i x^{\hat{\mu}} - i \bar{\theta}^I \gamma^{\hat{a}} D_i \theta^I \right)^2 - 2i \varepsilon^{ij} e_{\hat{\mu}}^{\hat{a}} \partial_i x^{\hat{\mu}} \bar{\theta}^I \gamma^{\hat{a}} D_j \theta^J \eta_{IJ} + \varepsilon^{ij} \bar{\theta}^I \gamma^{\hat{a}} D_i \theta^J \bar{\theta}^N \gamma^{\hat{a}} D_j \theta^N \eta_{IJ} + \dots \right]$$

$$= \int d^2\sigma \left[g_{\hat{\mu}\hat{\nu}}(x) \partial_i x^{\hat{\mu}} \partial_i x^{\hat{\nu}} + \partial_i x^{\hat{\mu}} e_{\hat{\mu}}^{\hat{a}}(x) \bar{\theta}^I \gamma^{\hat{a}} D_i \theta^I + \dots \right]$$

$AdS_5 \times S^5$

$\partial + \frac{1}{4} \gamma^{ab} \omega^{ab} + \frac{i}{2} \gamma^a e^a$

↪ $\partial x^{\hat{\mu}} \partial x^{\hat{\nu}} \bar{\theta} \gamma_{\hat{\mu}} \gamma^{\hat{\mu}_1 \dots \hat{\mu}_5} \gamma_{\hat{\nu}} \theta F_{\hat{\mu}_1 \dots \hat{\mu}_5}$

Coupling to RR background field

$$F_{a_1 \dots a_5} = \varepsilon_{a_1 \dots a_5} \quad , \quad F_{a'_1 \dots a'_5} = \varepsilon_{a'_1 \dots a'_5}$$

Consistent with α -symmetry for on-shell background:

$AdS_5 \times S^5$ is solution of IIB if $F_5 \neq 0$

The presence of F_5 is automatically "encoded" in $SU(2,2|4)$ algebra and α -invariant action

Some properties of string theory

Conformal invariance at quantum level:

unique action fixed by $SU(2,2|4)$
and α -symmetry (implying WZ term)

\mapsto no renormalization at
quantum level: $AdS_5 \times S^5$ is
exact solution of string theory (CFT)

(i) Global symmetry \rightarrow only coeff. of LL but
not of $L \wedge L \wedge L$ term
can be renormalized
(as in WZW)

(ii) α -symmetry \rightarrow relates the two
coefficients
(cf. affine symmetry of WZW)

(2) Exact superstring vacuum: $C = C_{free}$

Central charge? $\beta^\phi = \text{const}$ in $AdS_5 \times S^5$

Leading correction $\sim R + F_s^2 = 0$

$R_{AdS_5 \times S^5} = 0$, $F_{\mu_1 \dots \mu_5}^2 = 0 \mapsto$ Free value
unchanged
($\phi = \text{const}$)

Higher corrections should cancel in special
 α -symmetric scheme (cf. WZW)

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Proof that WZ term
is not renormalized:

$$S_{WZ} = \int_{M_3} H_3, \quad \partial M_3 = M_2$$

$$H_3 \sim L \wedge L \wedge L, \quad dH_3 = 0$$

expansion: $X \rightarrow \bar{X} + \delta X$, $X = (x, \theta)$

$$\delta H_3 = dT_2, \quad T_2 = \delta X^M f_M(\bar{L})$$

Thus expansion of WZ-term has 2-d form:

$$S_{WZ} = \int_{M_2} \sum_n \underbrace{\xi^n}_{\text{quantum}} g_n(\underbrace{\bar{L}}_{\text{background}})$$

Quantum contributions to divergent part are

local, $\Delta S_{WZ}^{(\infty)} = \int d^2\sigma F(\bar{L})$
local function

Since $dF(\bar{L}) \neq H_3 \sim L \wedge L \wedge L$

$$S_{WZ} = \int_{M_3} H_3 \quad \underline{\text{is not renormalized}}$$

[Exactly the same argument applies to WZW model or flat GS action]

G/H sigma model

$$g \in G, \quad g^{-1}dg = (g^{-1}dg)_{G/H} + (g^{-1}dg)_H$$

$$L = \left[(g^{-1} \partial_i g)_{G/H} \right]^2 = L_i^a L_i^a$$

Non-conformal: (generation of mass, etc)

$$\beta = a_1 R + a_2 R^2 + \dots$$

Add WZ-term which is not renormalized and α -symmetry make whole model conformal

WZW model: Alternative argument

$$(G_{\mu\nu} + B_{\mu\nu}) \partial x^\mu \bar{\partial} x^\nu$$

$$B'_{\mu\nu} \sim \partial_\lambda H_{\lambda\mu\nu} \log \Lambda + \partial H H H \log \Lambda + \dots$$

$$H_{\mu\nu\lambda} \sim \epsilon_{\mu\nu\lambda}, \quad \partial H = 0$$

no corrections

[All corrections are proportional to ∂H as action is covariant: $H \dots H \dots R$]

Important: 3-form H_3 in the action
is covariantly constant \rightarrow

no corrections

related argument by Kallosh and Rajaraman

Direct check of conformal invariance:

1-loop:

α -symmetry gauge $\gamma^+ \theta = 0$ (quantum)

RR coupling $\sim \partial x \partial x \bar{\theta} \gamma \dots \gamma \theta e^\phi F_5$

produces $(\partial x)^2 \log \Lambda$:

$$R_{\mu\nu} \sim e^\phi (F_5)_{\mu\nu}^2$$

[cf. α -invariant action is conformal at 1-loop in general on-shell background but central charge is shifted in general] Grisaru
Zanon
88

Action defines a novel type of 2d CFT:

$$\mathcal{L} = g_{mn}(x) \partial x^m \partial x^n \leftarrow \begin{array}{l} \text{non-conformal} \\ \text{bosonic coset} \\ \text{space model} \end{array}$$

$$+ (\partial x \bar{\theta} \partial \theta + \dots) \leftarrow \begin{array}{l} \text{fermionic} \\ \text{terms} \end{array}$$

Divergences in bosonic sector are cancelled by fermionic contributions (cf. susy WZW: fermions do not contribute to 1-loop β)

Equivalent to some WZW model? No, $\frac{1}{2}$ susy?
Bosonization? Free-field representation?

Coset space model integrable \rightarrow use currents as basic objects?

$SU(2,2|4)$ symmetry of action \rightarrow

string spectrum should be in reps. of $SU(2,2|4)$

Marginal vertex operators ("massless modes") \leftrightarrow unitary irreps of $SU(2,2|4)$ representing IIB sugra modes on $AdS_5 \times S^5$
Günaydin, Marcus 85

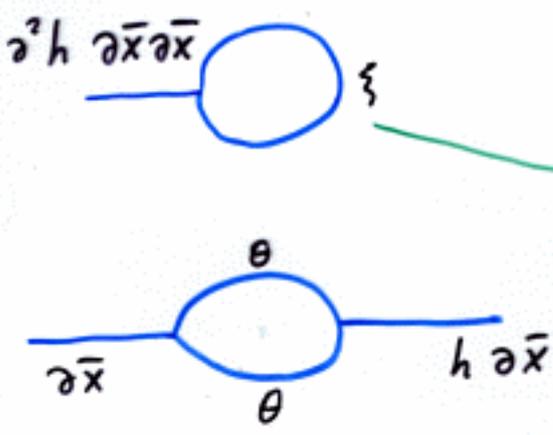
Can be seen directly in $\gamma^+ \theta = 0$ gauge:

IIB sugra: $R_{\mu\nu} = (F_5)_{\mu\nu}^2$
 $\partial H_3 = F_5 \tilde{H}_3$
 $\partial \tilde{H}_3 = -F_5 H_3$

Perturbations near $AdS_5 \times S^5$:

$(\partial^2 + 2) h_{\mu\nu} = 0$, $\partial_\rho H_{\mu\nu\alpha\beta} = \epsilon_{\mu\nu\alpha\beta\gamma} \tilde{H}_{\rho\gamma}$

Follow from $\mathcal{L} = (g_{\mu\nu} + h_{\mu\nu}) \left[\partial x^\mu \partial x^\nu + \bar{\theta} \gamma^\mu D \theta \partial x^\nu \right]$
 \downarrow
 $\partial \theta + \gamma \theta$



$(\partial^2 + 2) h_{\mu\nu} = 0$

$x = \bar{x} + \xi$

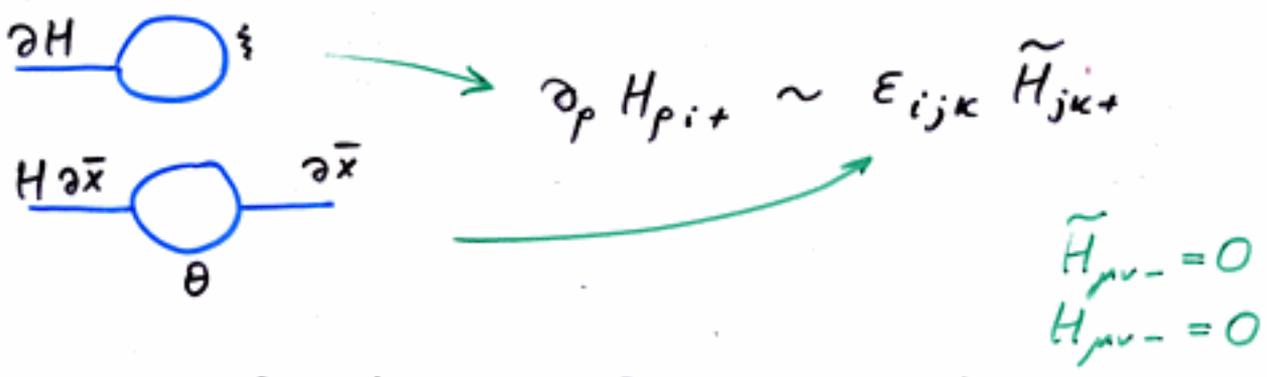
$$\mathcal{L} = \partial x \partial x + \bar{\theta} \gamma D \theta \partial x$$

$$+ B_{\mu\nu} \partial x^\mu \partial x^\nu + \tilde{H}_{\rho\sigma\lambda} \partial x^\rho \partial x^\sigma \partial x^\lambda \bar{\theta} \gamma^\rho \gamma^\sigma \gamma^\lambda \theta$$

\uparrow NS-NS
 \uparrow RR
 $\gamma^+ \theta = 0, x^+ = \tau$

$$\rightarrow \partial \xi \partial \xi + \xi \xi \partial H \partial \bar{x} \partial \bar{x} + \bar{\theta} \partial \theta$$

$$+ \partial x \bar{\theta} \gamma \gamma \theta + \bar{\theta} \gamma \gamma^{ij} \theta \tilde{H}_{ij}$$



Chirality of action in fermionic sector \rightarrow
 only one marginal $SO(6)$ vector vertex operator
 (marginality depends on non-zero RR 5-form term)
 in agreement with sugra compactified on S^5

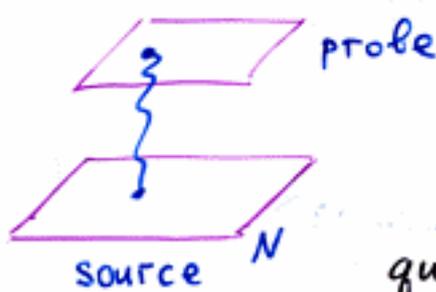
Open issues:

- put action in a simpler form
- (α -symmetry gauge ?)
- (x, θ) - are not adequate variables
- use conserved (non-chiral) currents?
- twistor-like variables? "Well, $N=4$ SYM defines the theory"

Supersymmetric D3-brane action in $AdS_5 \times S^5$

D3-brane probe in $AdS_5 \times S^5$ vacuum:

D3-brane near the core of D3-brane source



Integrating out massive modes
large N limit:

(A_i, X^S) $U(1)$ background \rightarrow
quantum effective action of
 $\mathcal{N}=4$ $U(N+1)$ SYM theory

Bosonic action (in sugra):

$$S = \int_{M_4} d^4 \sigma \sqrt{-\det (G_{ij} + F_{ij})} + \int_{M_5} F_5$$

$$\rightarrow \int d^4 \sigma \left[X^4 \left(\sqrt{-\det \left(\eta_{ij} + \frac{\partial_i X^S \partial_j X^S}{X^4} + \frac{1}{X^2} F_{ij} \right)} - 1 \right) + (\partial X)^4 \right]$$

$$\rightarrow \int d^4 \sigma \left(\frac{1}{2} \partial_i X^S \partial_i X^S + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{X^4} F^4 + \dots \right)$$

\leftarrow 1-loop SYM

Supersymmetric generalization:

- global $SU(2,2|4)$ invariance
- local α -symmetry

Unique action

- reduces to "SYM effective action" form in static gauge and α -gauge
- has expected bosonic limit (no static potential, etc)
- 16 linear and 16 non-linear susy's $SO(4,2) \times SO(6)$ by construction (before static gauge fixing)

Starting point: $SU(2,2|4)$ symmetry of $AdS_5 \times S^5$

Action in terms of invariant 1-forms on coset superspace

$$\frac{SU(2,2|4)}{SO(4,1) \times SO(5)}$$

$$S = S_{BI} + S_{WZ} = \int_{M_4} \sqrt{\det(G_{ij} + \mathcal{F}_{ij})} + \int_{M_5} H_5$$

$$\mathcal{F} = dA + B_2, \quad H_5 = d\Omega_4$$

Flat space :

D=10 Born-Infeld action

$$\mathcal{L} = \sqrt{-\det(\eta_{ij} + \alpha' F_{ij})} + \text{fermionic terms}$$

susy modified
order by order in α'

↑ string amplitudes
on the disc

Exact form :

(use flat superspace, etc)

$$\mathcal{L} = \sqrt{-\det(\eta_{ij} + F_{ij} + \bar{\lambda} \Gamma_i \partial_j \lambda + \bar{\lambda} \Gamma_i \partial_j \lambda \bar{\lambda} \Gamma_k \partial_l \lambda)}$$

Aganagic
Popescu
Schwarz 96

Non-trivial : $N=4$ susy

cf. $N=1$ superfield BI in $d=4$

Ferrara
Cecotti 86

D3-brane in flat space:

$$B_2 \sim \text{fermionic} \quad \alpha\text{-gauge}^* \rightarrow \bar{\lambda} \partial \lambda + (\bar{\lambda} \partial \lambda)^2$$

$$\Omega_4 \sim \text{fermionic} \rightarrow 0 \text{ (or } \bar{\lambda} \partial \lambda F)$$

Cederwall
Gussich, Nilsson
Westenberg

* Aganagic, Popescu
Schwarz
Bergshoeff, Townsend

AdS₅ × S⁵: super-invariant objects:

$$G_{ij} = L_i^{\hat{a}} L_j^{\hat{a}} \sim g_{MN}(x) \partial X^M \partial X^N$$

$$X^M = (x^m, \theta)$$

$$\mathcal{F} = dA + B_2$$

$$= dA + \int_0^1 dt L_t^{\hat{a}} \bar{\theta}^I \gamma^{\hat{a}} L_t^J \zeta_{IJ}$$

$$d\mathcal{F} = H_3 \leftarrow \text{same invariant form as in string action (non-U(1)-invariant)}$$

H₅: fixed uniquely from α -symmetry:

$$\delta_{\alpha} X^{\hat{a}} = 0, \quad \delta_{\alpha} \theta = \alpha$$

$$\Gamma \alpha = \alpha, \quad \Gamma^2 = 1$$

$$\Gamma = [\det(G + \mathcal{F})]^{-1/2} \varepsilon^{i_1 \dots i_4} \left(\Gamma_{i_1 \dots i_4} + \Gamma_{i_1 i_2} \mathcal{F}_{i_3 i_4} + \mathcal{F}_{i_1 i_2} \mathcal{F}_{i_3 i_4} \right)$$

$$L_t = \frac{\sinh t\sqrt{m}}{\sqrt{m}} D\theta$$

$$L_t^{\hat{a}} = e^{\hat{a}} - 2i \bar{\theta} \gamma^{\hat{a}} \frac{\cosh t\sqrt{m} - 1}{m} D\theta$$

$$m \sim \gamma^{\theta} \bar{\theta} \gamma$$

$$\hat{L}_t = L_t^{\hat{a}} \gamma^{\hat{a}}$$

$$H_5 = i \bar{L} \wedge \left(\frac{1}{6} \hat{L} \wedge \hat{L} \wedge \hat{L} + \mathcal{F} \wedge \hat{L} \right) \wedge L$$

$$+ \frac{1}{30} \left(\epsilon^{a_1 \dots a_5} L^{a_1} \wedge \dots \wedge L^{a_5} + \epsilon^{a'_1 \dots a'_5} L^{a'_1} \wedge \dots \wedge L^{a'_5} \right)$$

AdS_5 S^5

$$\hat{L} \equiv L^{\hat{a}} \gamma^{\hat{a}}$$

$$dH_5 = 0$$

(Maurer-Cartan and Fierz)

Explicit form of WZ term:

$$S_{WZ} = \int_{M_5} H_5 = \int \int_{M_4} dt \left(\frac{1}{6} \bar{\theta} \hat{L}_t \wedge \hat{L}_t \wedge \hat{L}_t \wedge L_t + \bar{\theta} \hat{L}_t \wedge \mathcal{F}_t \wedge L_t \right) + \int_{M_5} F_5$$

$$\mathcal{F}_t = dA + 2i \int_0^t dt' \bar{\theta} \hat{L}_{t'} \wedge L_{t'}$$

$$F_5 \equiv (H_5)_{\text{Bose}} = \frac{1}{30} \left(\epsilon^{a_1 \dots a_5} e^{a_1} \wedge \dots \wedge e^{a_5} + \epsilon^{a'_1 \dots a'_5} e^{a'_1} \wedge \dots \wedge e^{a'_5} \right)$$

Static gauge ($x^i = \sigma^i$) and α -gauge:
 resulting action is non-linear extension of
 $U(1)$ $N=4$ Maxwell action
 superconformal $SU(2,2|4)$ group realized non-linearly

Remarks:

- presence of F_5 background implied by α -symmetry
(Dp-brane action is α -invariant for on-shell background) Cederwall et al
Bergshoeff
Townsend
- explicit (x, θ, A) dependence:
plug in solutions of M-C eq. for $L^{\hat{a}}, L^I$
- WZ term not manifestly $SO(6)$ invariant in 4-d form (non-SYM - string correction?)

SYM effective action interpretation:

- special bosonic coordinates:
$$ds^2 = \frac{x^2}{R^2} \underbrace{dy_i dy_i}_4 + \frac{R^2}{x^2} \underbrace{dx_n dx_n}_6$$
- static gauge: $y_i = \sigma_i$
("probe parallel to the D3-brane source")
- α -symmetry gauge ($\theta^1 = 0$ or $\theta^I = \Lambda^{IJ} \theta^J$)
 $N=4$ vector multiplet

Poincare, $SO(6)$ and 16 of susy's
realized linearly in \mathcal{L}_{BI}

Non-linearly realized conformal symmetry

- Action different from flat-space one:
 $\alpha' \rightarrow x^2$: quantum SYM eff. action interpretation Maldacena
Kallush et al
Claus et al

Conclusions

- superstring action in $AdS_5 \times S^5$: $\exists!$
simple $\int L + \int L \wedge L \wedge L$ action
(same structure as in flat superspace)
- conformal field theory of novel type
 $AdS_5 \times S^5$ exact string solution
spectrum of states : $SU(2,2|4)$ reps.
- D3-brane action in $AdS_5 \times S^5$: $\exists!$
non-linear $SU(2,2|4)$ invariant generalization
of $N=4$ super Maxwell theory
static and α -gauge : effective SYM action
interpretation
susy transformation is known exactly

Open Questions :

Connection between string and D3-brane?

String in $AdS_5 \times S^5$: action on disc, b.c.'s
 $\frac{1}{2}$ susy D-brane states ?

(D3 if \parallel AdS_5 boundary, D5 over S^5, \dots)

Couple string to SYM at boundary
of disc (and boundary of AdS_5)

string in $AdS_5 \times S^5 \mapsto$ SYM connection ?

e.g. derive D3-brane action from string path integral