From anti-de Sitter background to superconformal world sheet symmetry

M5: hep-th/9711161

general bosonic: hep-th/9801206

black holes and superconformal mechanics: hep-th/9804177

supersymmetric theory: ongoing
In short

- **Solution** of the supergravity theory  
  $\rightarrow$ Killing vectors and spinors $\rightarrow$ supergroup

- **World-sheet** theory where the supergroup gives rigid symmetries

- **local GCT** and $\kappa$-symmetry.

- **gauge fixing** leads to  
  a **conformal** world-sheet theory.

- **Application:**  
  $d = 4$, $N = 2$ supergravity solution  
  (near horizon Reissner-Nordström black hole)  
  leads to world-line theory which is  
  *'relativistic superconformal mechanics'*.  

Plan

1. Bosonic theory
   - Brane solutions
   - Killing vector $adS$ algebra
   - world-volume action
   - conformal algebra

2. Supersymmetric world-volume theory
   - solutions of supergravity and Killing vectors/spinors
   - supergroups
   - world-volume action
   - $\kappa$-symmetry
   - gauge fixing leads to superconformal symmetry

3. Conclusions

   'relativistic superconformal mechanics': example developed in parallel to the general presentation.
1. **Brane solutions** with \( adS \) *Killing vector algebra*

\((m = 0, 1, \ldots p; \ m' = 1, \ldots d - p - 1)\): \quad \(d = 4, 10\) or \(11\)

\[
ds^2 = H_{\text{brane}}^{\frac{2}{p+1}} dx_m^2 + H_{\text{brane}}^{\frac{2}{d-p-3}} dX_{m'}^2
\]

\[
H_{\text{brane}} = 1 + \left( \frac{R}{r} \right)^{d-p-3}; \quad r^2 = X^{m'} X^{m'}
\]

\[
H_{\text{hor}} = \lim_{r \to 0} \left[ 1 + \left( \frac{R}{r} \right)^{d-p-3} \right] = \left( \frac{R}{r} \right)^{d-p-3}
\]

- **brane solution** interpolates between asymptotically flat and near horizon anti-de Sitter geometry
  
  Gibbons, Townsend

- **large N** (many branes solution) \((R^\infty \propto N)\)
  
  Maldacena

- **there is a special duality transformation** that removes the constant
  
  Hyun Boonstra, Peeters, Skenderis Cremmer, Lavrinenko, Lü, Pope, Stelle, Tran
anti-de Sitter Killing vector algebra

\[ ds^2_{hor} = \left( \frac{r}{R} \right)^{\frac{2(d-p-3)}{p+1}} dx_m^2 + \left( \frac{R}{r} \right)^2 dr^2 + R^2 d^2\Omega \]

is \( adS_{p+2} \times S^{d-p-2} \) metric.

Killing vectors \( SO(p+1,2) \times SO(d-p-1) \)

Example: \( p = 0, \ d = 4 \). Rename \( R \rightarrow M \)

\[ ds^2 = -H^{-2}_{\text{brane}} dt^2 + H^2_{\text{brane}} dX^2_{m'} \]

\[ H_{\text{brane}} = 1 + \frac{M}{r} \]

Is Reissner-Nordstrom (RN) black hole.

Near horizon geometry

\[ ds^2_{hor} = - \left( \frac{r}{M} \right)^2 dt^2 + \left( \frac{M}{r} \right)^2 dr^2 + M^2 d^2\Omega \]

which is the Bertotti–Robinson (BR) metric. \( adS_2 \times S^2 \).

\[ \delta t = a + b t + c t^2 + c(M^4/r^2) ; \quad \delta r = -r (\partial_t \delta t) \]

\( SO(1,2) \) and \( SO(3) \) isometries of \( d^2\Omega \).
This scheme works in various theories:

<table>
<thead>
<tr>
<th></th>
<th>d</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>M5</td>
<td>11</td>
<td>5</td>
</tr>
<tr>
<td>M2</td>
<td>11</td>
<td>2</td>
</tr>
<tr>
<td>D3</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>Self-dual string (D1+D5)</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Magnetic string</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>Tangerlini black hole</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>Reissner-Nordström black hole</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

All have $adS_{p+2} \times S^{d-p-2}$ geometry.

E.g. self-dual string can also be obtained from 10 dimensions with $D1 + D5$, having $adS_3 \times S^3 \times E_4$.

More generalizations, with products of spheres,

Boonstra, Peeters, Skenderis, hep-th/9803231
Bosonic world-volume theory

\[ S_{cl} = S_{\text{Born-Infeld}} + S' + S_{\text{Wess-Zumino}} \]
\[ S_{BI} = -\int d^{p+1}\sigma \sqrt{-\det g_{\mu\nu}} \]
\[ g_{\mu\nu} = g_{\mu\nu}^{\text{ind}} + T_{\mu\nu} \]
\[ g_{\mu\nu}^{\text{ind}} = \partial_\mu X^M \partial_\nu X^N G_{MN} \]

with

\[
\begin{align*}
\text{RN} & \quad T_{\mu\nu} = 0 \quad S' = 0 \\
\text{M2} & \quad T_{\mu\nu} = 0 \quad S' = 0 \\
\text{D3} & \quad T_{\mu\nu} = F_{\mu\nu} \quad S' = 0 \\
\text{M5} & \quad T_{\mu\nu} = i\mathcal{H}_{\mu\nu}^* \quad S' = \int d^6\sigma \, \mathcal{H}_{\mu\nu}^* \mathcal{H}_{\mu\nu}
\end{align*}
\]

\( a \): auxiliary field of Pasti-Sorokin-Tonin

appears in

\[
\begin{align*}
u_{\mu} & = \partial_{\mu} a; \quad H_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]} \\
H_{\mu\nu} & = \frac{u^\rho}{\sqrt{u^2}} H_{\mu\nu\rho}; \quad H_{\mu\nu}^* = \frac{u^\rho}{\sqrt{u^2}} H_{\mu\nu\rho}^*
\end{align*}
\]
Geometric input:

- $G_{MN}$, solution of supergravity

- Wess-Zumino term

**BR metric:**

$$ds^2 = - \left( \frac{r}{M} \right)^2 dt^2 + \left( \frac{M}{r} \right)^2 (dr^2 + r^2 d^2 \Omega)$$

$r = 0$ is coordinate singularity. Define $\rho$

$$\frac{r}{M} = \left( \frac{2M}{\rho} \right)^2$$

$$ds^2 = - \left( \frac{2M}{\rho} \right)^4 dt^2 + \left( \frac{2M}{\rho} \right)^2 d\rho^2 + M^2 d^2 \Omega$$

**Bosonic part of WZ:**

$$S_{WZ} = \int A ; \quad A = \frac{r}{M} i \, d\tau$$
World-line action

\[ S = mS_{BI} + qS_{WZ} \]
\[ = -m \int d\tau \sqrt{-g_{00}^{\text{ind}}} + q \int A \]

\[ g_{00}^{\text{ind}} = \left( \frac{2M}{\rho} \right)^4 \left[ -(i)^2 + \left( \frac{\rho}{2M} \right)^2 \right] + M^2 \left[ \dot{\theta}^2 + \sin^2 \theta \phi \right] \]

Gauge of time reparametrizations: \( t = \tau \).

In Hamiltonian language

\[ H = \left( \frac{2M}{\rho} \right)^2 \left[ \sqrt{m^2 + \frac{\rho^2 p_\rho^2 + 4L^2}{4M^2}} - q \right], \]

\( (L^2 = p_\theta^2 + \sin^{-2} \theta p_\phi^2) \)

is \( H = -p_0 \) solving

\[ (p_0 - qA)^2 G^{00} + p_{m'}G^{m'n'}p_{n'} + m^2 = 0. \]

charged particle in BR background.
\[ H = \frac{p^2}{2f} + \frac{mg}{\rho^2 f} \]

\[ f = \frac{1}{2} \left[ \sqrt{m^2 + (\rho^2 p^2 + 4L^2)/4M^2} + q \right] \]

\[ mg = 2M^2(m^2 - q^2) + 2L^2 \]

Limit

\( M \to \infty; \quad (m-q) \to 0; \quad M^2(m-q) \) fixed

gives \( f \to m \), and is conformal mechanics of de Alfaro, Fubini and Furlan, 1976.

*non-relativistic conformal mechanics* (large black hole mass).

The full solution we get from the 'brane-like' procedure: *relativistic conformal mechanics*.

With \( L = 0 \), force vanishes when \( m = q \).
Combination of symmetries → conformal symmetry

- **Rigid symmetries**

  - $SO(3)$ rotations on $\theta, \phi$

  - anti-de Sitter

    \[
    \delta t = a + b t + c t^2 + c \frac{M^4}{r^2} \\
    \delta r = -r (b + 2c t)
    \]

- **Local symmetry**: time reparametrizations

  \[
  \delta t = \xi(\tau) t ; \quad \delta r = \xi(\tau) \dot{r} ; \quad ...
  \]

Gauge fixing $t = \tau$

\[
0 = a + b \tau + c \tau^2 + c \frac{M^4}{r^2} + \xi(\tau)
\]

- $a = \text{translations}$
- $b = \text{dilatations}$
- $c = \text{special conformal transformations}$

$SO(2,1)$ finite conformal group in 1 dimension
PS: Notes about infinite dimensional groups in case $p = 1$

1. Brown and Henneaux, 1986: consider also other geometries which have $adS_3$ as near-horizon limit. Symmetries between such asymptotic geometries form Virasoro algebra of which $SO(2,2) = SU(1,1) \times SU(1,1)$ is finite dimensional subgroup.

2. F. Brandt, J. Gomis and J. Simón, hep-th/9707063 and 9803196: There are extra symmetries of
\[ \int d^2 \sigma \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})} \]
\[ \delta X^M = h^M(X) \lambda(\mathcal{F}) \]
\[ \delta V_\mu = -\lambda'(\mathcal{F}) \sqrt{g} (1 + \mathcal{F}^2) \epsilon_{\mu\nu} (\partial^\nu X^M) h_M(X) \]
where
\[ h^M(X) : \text{Killing vectors of the metric } G_{MN} \]
\[ \mathcal{F} = -\frac{\epsilon^{\mu\nu} F_{\mu\nu}}{2\sqrt{g}} \]
The arbitrary function $\lambda$ thus provides a sort of Kač-Moody extension of the isometry group.
2. **Supersymmetric theory**

In terms of $Z^\Lambda = \{X^M, \theta^A\}$, superspace coordinates in $d = 11, 10, 4, \ldots$ and possibly other forms.

$$ g_{\mu\nu}^{\text{ind}} = \left( \partial_\mu Z^\Lambda E^M_\Lambda \right) \left( \partial_\nu Z^\Sigma E^N_\Sigma \right) \eta_{MN} $$

(underline is flat index)

$d = 4, p = 1$ example:

comes from $N = 2, d = 4$ supergravity.

$$ A \leftarrow (\alpha i) ; \ i = 1, 2, \ \alpha \ \text{spinor index} $$

$$ S_{WZ} = \int d\tau \dot{Z}^\Lambda A_\Lambda $$

E.g. flat superspace

$$ E^M_M = \delta^M_M \ ; \ \quad E^\alpha_i M = \frac{1}{2} \left( \gamma^M \theta_i \right)_\alpha $$

$$ A_M = 0 \ ; \ \quad A^{\alpha i} = \frac{1}{2} \varepsilon_{ij} \theta^j_{\alpha} $$

leads to

$$ g_{\tau\bar{\tau}}^{\text{ind}} = \left( \dot{X}^M - \frac{1}{2} \bar{\theta} \Gamma^M \dot{\theta} \right) \left( \dot{X}^N - \frac{1}{2} \bar{\theta} \Gamma^N \dot{\theta} \right) \eta_{MN} $$

$$ S_{WZ} = \frac{1}{\sqrt{2}} \int d\tau \bar{\theta}^i \varepsilon_{ij} \theta^j + \text{h.c.} $$
Start from solutions of supergravity

\[ ds^2 \text{ as BR with mass } M \text{ (near-horizon)} \]

Soln. with electric charge \( Q \) and magnetic \( P \):

\[
F_{0r} = - \frac{Q}{M^2}; \quad F_{\theta\phi} = -P \sin \theta
\]

with \( P^2 + Q^2 = M^2 \).

For Killing spinors, define \( 2 \times 4 \) real spinors

\[
(\mathcal{P}_{\pm})_A^B = \frac{1}{2} \left( \delta_A^B \pm \frac{1}{M} (Q + i\gamma_5 P) \varepsilon^{ij} \gamma_0 \right)
\]

Killing spinors are

\[
(\mathcal{P}_+ e)^i = \left( \frac{M}{r} \right)^{1/2} \eta_+^i
\]

\[
(\mathcal{P}_- e)^i = \left( \frac{r}{M} \right)^{1/2} \left( \eta_-^i - \frac{t}{M} \gamma_0 r \eta_+^i \right)
\]

where \( \eta_{\pm}^i \) are Killing spinors of sphere \( \hat{m} = \theta, \phi \)

\[
\nabla_{\hat{m}} \eta_{\pm}^i(\theta, \phi) = \mp \frac{1}{2M} \gamma_r \gamma_{\hat{m}} \eta_{\pm}^i(\theta, \phi)
\]

4 solutions for each sign.

Lü, Pope, Rahmfeld, hep-th/9805151
Algebra of Killing spinors and vectors: \( SU(1, 1|2) \)

Commutators give \( adS \) and \( SO(3) \) of sphere.

\[
\begin{align*}
[\eta_+, \eta_+] &= P \\
[\eta_+, \eta_-] &= D + SO(3) \\
[\eta_-, \eta_-] &= K
\end{align*}
\]

or

\[
\begin{pmatrix}
SU(1, 1) & \eta^i_+ \\
\eta^i_- & SU(2)
\end{pmatrix}
\]

Should always be of a similar form, with

1. \( SO(p+1, 2) \) should appear as factor in bosonic part of the superalgebra.

2. fermionic generators in a spinorial representation of that group.

→ \textbf{more bosonic symmetries} \( (R) \) should appear as symmetries of non-adS part of the target space (here \( S^2 \))
First see isomorphisms of conformal groups

\[ p = 0 \quad SO(1, 2) \sim SU(1, 1) \sim Sp(2) \]
\[ p = 1 \quad SO(2, 2) \sim SO(1, 2) \times SO(1, 2) \sim SU(1, 1) \times SU(1, 1) \]
\[ p = 2 \quad SO(3, 2) \sim Sp(4) \]
\[ p = 3 \quad SO(4, 2) \sim SU(2, 2) \]

Results for \( p \geq 2 \):

<table>
<thead>
<tr>
<th>( p )</th>
<th>superalgebra</th>
<th>( R )</th>
<th>nr. ferm.</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( OSp(N</td>
<td>4) )</td>
<td>( SO(N) )</td>
</tr>
<tr>
<td>3</td>
<td>( SU(2, 2</td>
<td>N) )</td>
<td>( U(N) ) for ( N \neq 4 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( SU(4) ) for ( N = 4 )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( F(4) )</td>
<td>( SU(2) )</td>
<td>( 16 )</td>
</tr>
<tr>
<td>5</td>
<td>( OSp(6, 2</td>
<td>2N) )</td>
<td>( USp(2N) )</td>
</tr>
</tbody>
</table>

\( p = 0 \) (or 2 factors for \( p = 1 \))

<table>
<thead>
<tr>
<th>superalg.</th>
<th>( R )</th>
<th>nr. ferm.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( OSp(N</td>
<td>2) )</td>
<td>( O(N) )</td>
</tr>
<tr>
<td>( SU(N</td>
<td>1, 1) ) (( N \neq 2 ))</td>
<td>( U(N) )</td>
</tr>
<tr>
<td>( SU(2</td>
<td>1, 1) )</td>
<td>( SU(2) )</td>
</tr>
<tr>
<td>( OSp(4^*</td>
<td>2N) )</td>
<td>( SU(2) \times USp(2N) )</td>
</tr>
<tr>
<td>( G(3) )</td>
<td>( G_2 )</td>
<td>( 14 )</td>
</tr>
<tr>
<td>( F(4) )</td>
<td>( SO(7) )</td>
<td>( 16 )</td>
</tr>
<tr>
<td>( D^1(2, 1, \alpha) )</td>
<td>( SU(2) \times SU(2) )</td>
<td>( 8 )</td>
</tr>
</tbody>
</table>
Examples

<table>
<thead>
<tr>
<th></th>
<th>sAlg.</th>
<th>$\mathbb{G}$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>M5</td>
<td>$OSp(6,2\mid4)$</td>
<td>$SO(5)$</td>
<td>$adS_7 \times S^4$</td>
</tr>
<tr>
<td>M2</td>
<td>$OSp(8\mid4)$</td>
<td>$SO(8)$</td>
<td>$adS_4 \times S^7$</td>
</tr>
<tr>
<td>D3</td>
<td>$SU(2,2\mid4)$</td>
<td>$SO(6)$</td>
<td>$adS_5 \times S^5$</td>
</tr>
<tr>
<td>D1+D5</td>
<td>$(SU(1,1\mid2))^2$</td>
<td>$SO(4)$</td>
<td>$adS_3 \times S^3$</td>
</tr>
<tr>
<td>BR</td>
<td>$SU(1,1\mid2)$</td>
<td>$SO(3)$</td>
<td>$adS_2 \times S^2$</td>
</tr>
</tbody>
</table>
World-volume (world-line) action

BI-term, based on supervielbëins, ...

can be obtained from ‘Gauge completion’

Nath, Arnowitt, PL 65B(76)73
Cremmer, Ferrara, PL 91B(80)61
Castellani, van Nieuwenhuizen, Gates, PRD22(80)2364
de Wit, Peeters, Plefka, hep-th/9803209

WZ-term starts now from solution \( A_\mu = W_\mu \),
supergravity solution

\[
W_0 = \frac{Q}{M^2} r ; \quad W_\phi = P \cos \theta
\]

Other way: supercosets: here

\[
\frac{SU(1,1|2)}{U(1) \times U(1)}
\]

Castellani, Ceresole, D'Auria, Ferrara, Frè, Trigiante
hep-th/9803039

Matsaev, Tseytlin, hep-th/9805028 and 06095
Kallosh, Rajaraman, hep-th/9805041
\( \kappa \)-symmetry

World-volume theory has the rigid symmetries

\( N = 4, d = 2 \) has 8 real supersymmetries, and the solutions which we considered have 8 real Killing spinors (2 complex doublets of \( SU(2) \)).

But also \( \kappa \) symmetry. This imposes \( q = m \)

\[
\delta_{\kappa} \theta = (1 + \Gamma) \kappa ; \quad \Gamma = \frac{1}{\sqrt{-g_{00}}} \varepsilon^{ij} \gamma_M \Pi^M
\]

\( \Gamma \) is complicated matrix, but \( \Gamma^2 = 1 \).

At 'classical values' \( t = r; \, r = r_0; \, \theta = \theta_0; \, \phi = \phi_0; \) fermions = 0

\[
\Gamma_{cl} = \varepsilon^{ij} \gamma_0 ; \quad 1 + \Gamma_{cl} = 2P_-
\]

if \( P = 0 \) (only electr. charged \( Q = M \))

zero force if \( P = 0 \), thus \( Q = \pm M \)

Irreducible \( \kappa \) symmetry: \( P_- \kappa = 0 \).

\[
\delta_{\kappa} \theta = P_+ \kappa + ...
\]
Local symmetries

- world-volume diffeomorphisms
- Kappa symmetry

Broken by gauge choice: reformulation as $\text{ads}_{p+2} \Rightarrow \text{Conf}_{p+1}$.

Remember bosonic:

$$\delta_{\text{ads}} r = -r(b + 2c t) ; \quad \delta_{\text{gct}} r = \xi(\tau)$$

Gauge choice: $\delta t = a + b \tau + c \tau^2 + c t + \frac{M^4}{r^2} + \xi(\tau) = 0$

$r$ transforms as a scalar with Weyl weight 1.

Fermionic:

$$\delta \mathcal{P}_+ \theta^i = \left( \frac{M}{r} \right)^{1/2} \eta_+^i + \kappa^i + \ldots$$

$$\delta \mathcal{P}_- \theta = \left( \frac{r}{M} \right)^{1/2} \left( \eta_-^i - \frac{t}{M} \gamma_0 r \eta_+^i \right) + \ldots$$

Gauge fixing

$$\mathcal{P}_+ \theta = 0$$

Conformal supersymmetry: where $\eta_-$ takes role of $Q$-supersymmetry $\eta_+$ of $S$-supersymmetry.
Summary

*We establish superconformal symmetry of the gauge-fixed non-gravitational brane actions*

Starting from solution of supergravity with background $adS_{p+2} \times S^{d-p-2}$ geometry.

Non-relativistic superconformal mechanics was done by
Akulov and Pashnev, 1983.
Fubini and Rabinovici, 1984.

We obtain

'\textit{relativistic superconformal mechanics}'

having as limit $M \to \infty$ non-relativistic superconformal mechanics.

PS: there is still a simplification, keeping only the radial mode. Based on $OSp(1|2) \subset SU(2|1,1)$. 
We qualified for the case

\[
d \quad p
\]

4 - 1