Ordered Phases of non-commutative scalar field theories

Work with S. Sondhi, hep-th/0006132

- Introduction
- Non-commutative ϕ^4 theory at one loop
- Phase diagram (overview)
- Fluctuation-driven first order transition to stripes
- Various non-uniform phases
- Lifshitz point
- Scaling and renormalizability

Two motivational questions:

1. What new physics arises from non-commutativity in QFT?

   [Resolved instanton moduli spaces [Nekrasov-Schwarz]
   non-commutative solitons [Gopakumar-Minwalla-Strominger]
   UV-IR mixing [Minwalla-Seiberg-van Raamsdonk]]

2. Are non-commutative field theories renormalizable?

   [Decoupling limit, oscillatory integrands, non-local action]

We can address 1 and 2 both by examining phase transitions in non-commutative field theories.

- If a statistical mechanical system (eg cut-off field theory) exhibits a <u>2nd order</u> transition, then through scaling limits we can define continuum QFT's.

- But it might happen that the continuum QFT is trivial: eg Ising in $d>4$.

- Program: map out phase diagram for a non-commutative FT (with a cut-off), identify critical points, check for triviality.

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} = i\theta \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$$

Usually $d=4$

Wilsonian cutoff

Non-commutative ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g^2}{4}\phi\star\phi\star\phi\star\phi$$

$$\mathcal{L}_{O(N)} = \frac{1}{2}(\partial\phi_i)^2 + \frac{1}{2}m^2\phi_i^2 + \frac{g^2}{4}\phi_i\star\phi_i\star\phi_j\star\phi_j + \frac{g'^2}{4}\phi_i\star\phi_j\star\phi_i\star\phi_j$$

$$(f\star g)(x) = e^{\frac{i}{2}\Theta^{\mu\nu}\partial_{x^\mu}\partial_{y^\nu}} f(x)\, g(y)\Big|_{x=y}$$

Perturbative calculations are easiest in the disordered $(\langle\phi\rangle = 0)$ phase [Minwalla-Seiberg-van Raamsdonk + others]:

$$k\wedge p \equiv \Theta^{\mu\nu}k_\mu k^\nu$$

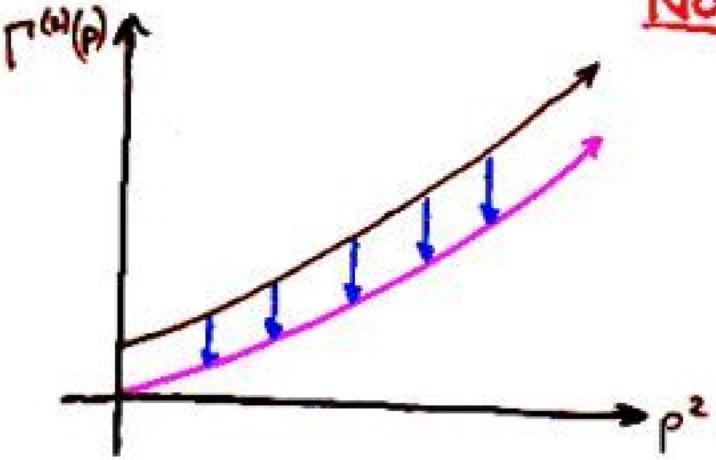
$$\Gamma^{(2)}(p) = p^2 + m^2 + 2g^2\int^\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2} + g^2\int^\Lambda \frac{d^4k}{(2\pi)^4}\frac{e^{ik\wedge p}}{k^2+m^2}$$

$$= p^2 + M^2 + \frac{m}{\sqrt{p\Theta^2p + 4/\Lambda^2}}K_1\left(m\sqrt{p\Theta^2p + \frac{4}{\Lambda^2}}\right)$$

What happens if we fix Λ, m, g, and vary θ?

Now decrease m^2

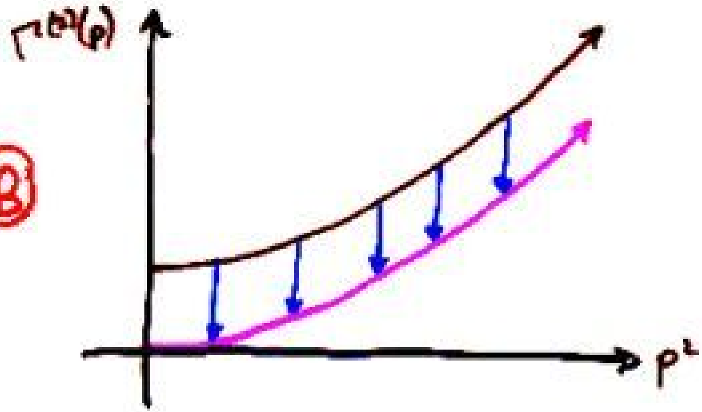
(A)



$\theta\Lambda^2 \ll 1$ has little effect on propagator

Condense $\phi(0)$

(B)

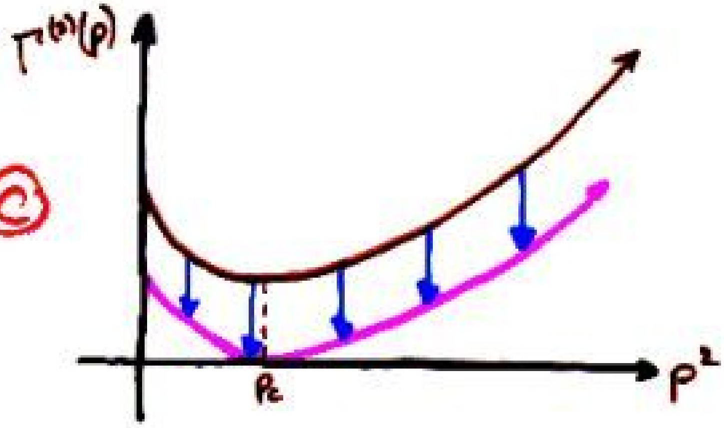


Critical $\theta\Lambda^2$:

$\Gamma^{(2)}(p) \sim p^4$ for small p.

Condense $\phi(0)$

(C)

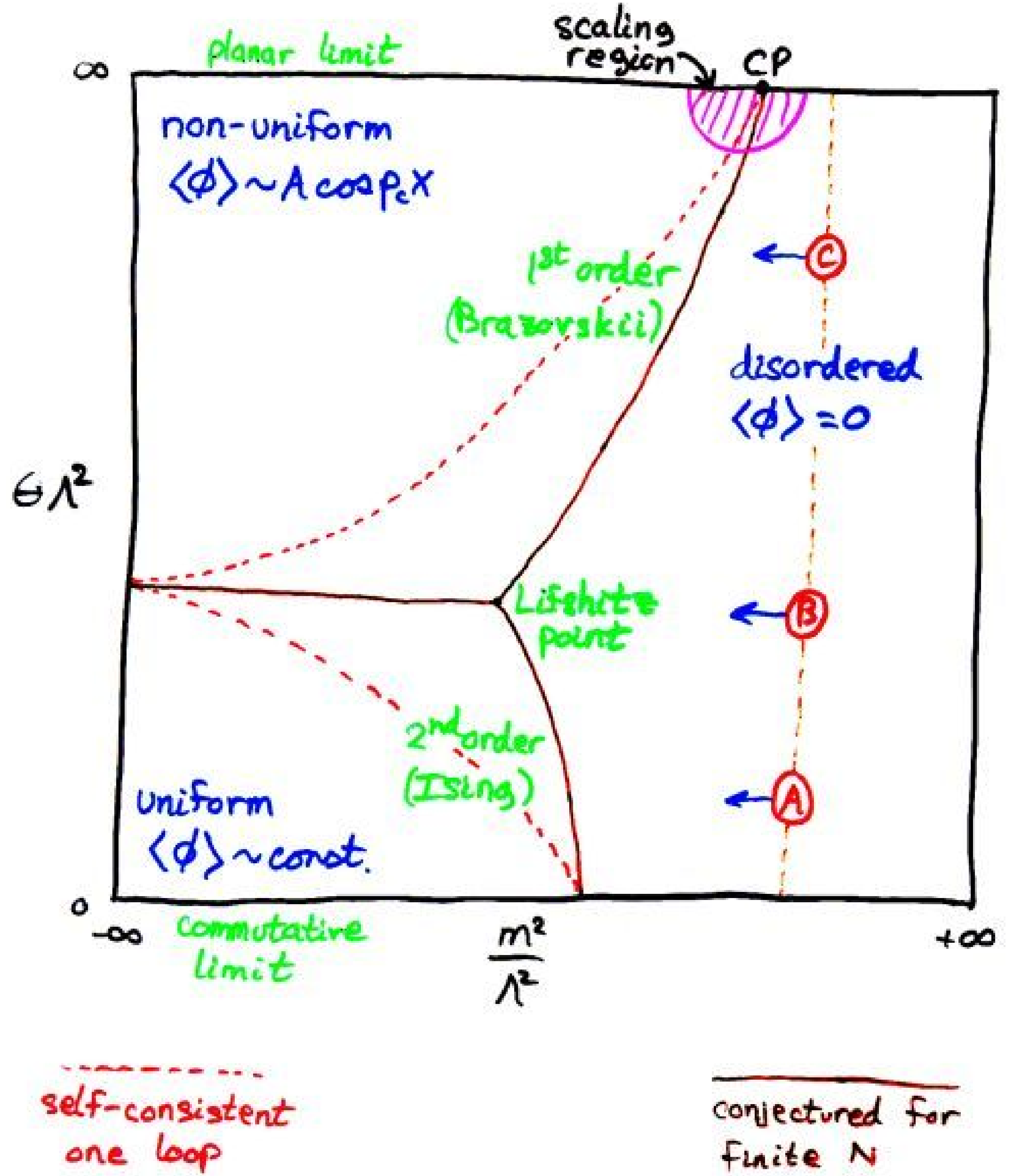


$\theta\Lambda^2 \gg 1$:

small p modes are stiffer than at $p = p_c$

Condense $\phi(p_c)$?

The phase diagram:

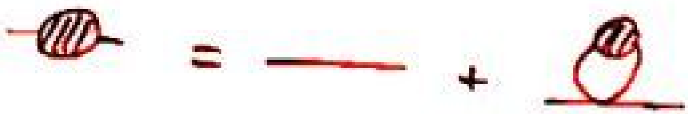


The Brazovskii transition:

Self-consistent one loop treatment:

$$\Gamma^{(2)}(p) = p^2 + m^2 + 2g^2 \int^{\Lambda} \frac{d^4k}{(2\pi)^4} \, \frac{1+\frac{1}{2}e^{ik\wedge p}}{\Gamma^{(2)}(k)}$$

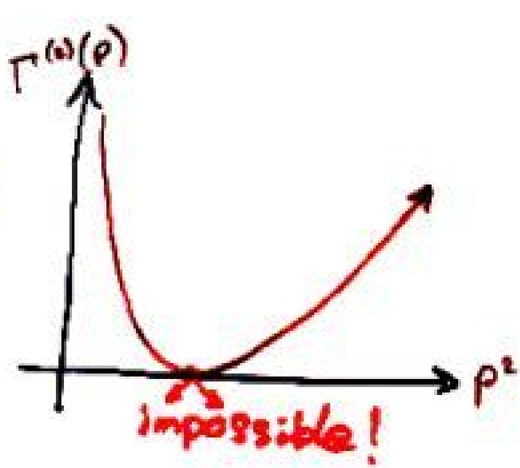
$$= p^2 + M^2 + g^2 \int_0^{\infty} k^3 dk \; \frac{J_1(\theta k p)}{\theta k p} \, \frac{1}{\Gamma^{(2)}(k)} \qquad (☆)$$



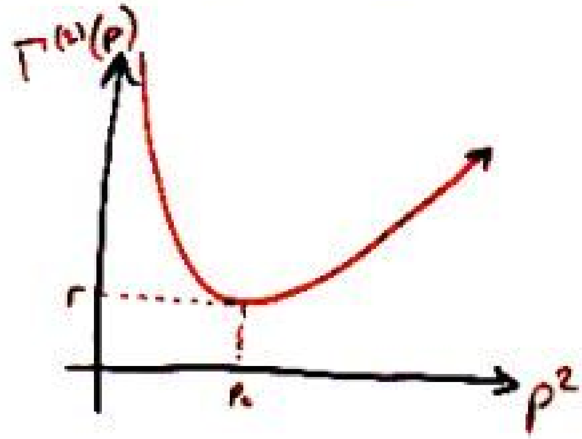
formally sums up all the superficially divergent non-planar diagrams noted in [Minwalla - Seiberg - van Raamsdonk] in a manifestly finite integral eqn.

(☆) makes a 2nd order transition to a condensate of $\phi(p_c)$ impossible.

Nevertheless claim that a condensate forms, in a first-order transition



We can approximate for small g:



$$\Gamma^{(2)}(p) \simeq p^2 + M^2 + \frac{g^2}{(\theta p)^2} + \frac{g^2 p_c^2}{\xi_0 \sqrt{r}}$$

$$r \equiv \inf_p \Gamma^{(2)}(p) = 2p_c^2 + M^2 + \frac{g^2 p_c^2}{\xi_0 \sqrt{r}}$$

$$p_c \sim \frac{1}{\xi_0} \sim \sqrt{\frac{g}{\theta}}$$

self-consistency equations from the integral equation.

and indeed, $r \to 0$ only when $M^2 \to -\infty$.

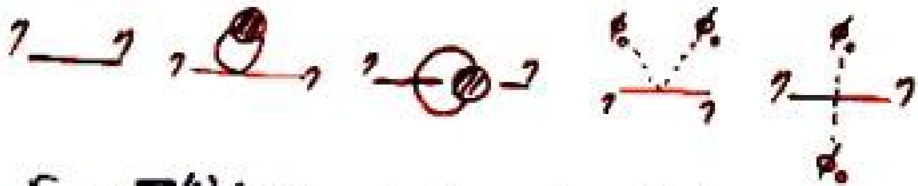
Nevertheless, condensate should form:

$$\langle \phi \rangle = A \cos p_c x \equiv \phi_0$$

breaks both translation + rotation symmetry.
Idea is to compute $F_{\phi_0} - F_{\phi=0}$
and show there is a crossover.
Analogous calculation by Brazovskii (1978).

Briefly: $\phi = \phi_0 + \eta$, background field pert. th.

$$\frac{1}{2V}\frac{dF}{dA} = \Gamma^{(2)}(p_c)A - \frac{g^2}{2}A^3$$ from these graphs;



$$\Gamma \equiv \Gamma^{(2)}(p_c) = 2p_c^2 + M^2 + \frac{g^2 p_c^2}{\xi_0\sqrt{r}} + g^2 A^2$$ from these

$$\Delta F = \int_0^A dA \, \frac{dF}{dA}$$ can be computed explicitly, with the result that ordered phase becomes preferred when

$$\boxed{M^2 < -2p_c^2 - c_2 \frac{g^{2/3}}{\Theta} + O(g^2)}$$

for some c_2 of order unity.

Stripes or squares?

So far, we just assumed $\langle\phi\rangle \sim A\cos p_c x$.

Let's contemplate next simplest possibility:

$$\langle\phi\rangle \sim A_1 \cos p_1\cdot x + A_2 \cos p_2\cdot x \qquad |p_1| = |p_2| = p_c$$

Simple test to see which is preferred:

Which soln minimizes $\mathcal{L}_{eff}$?

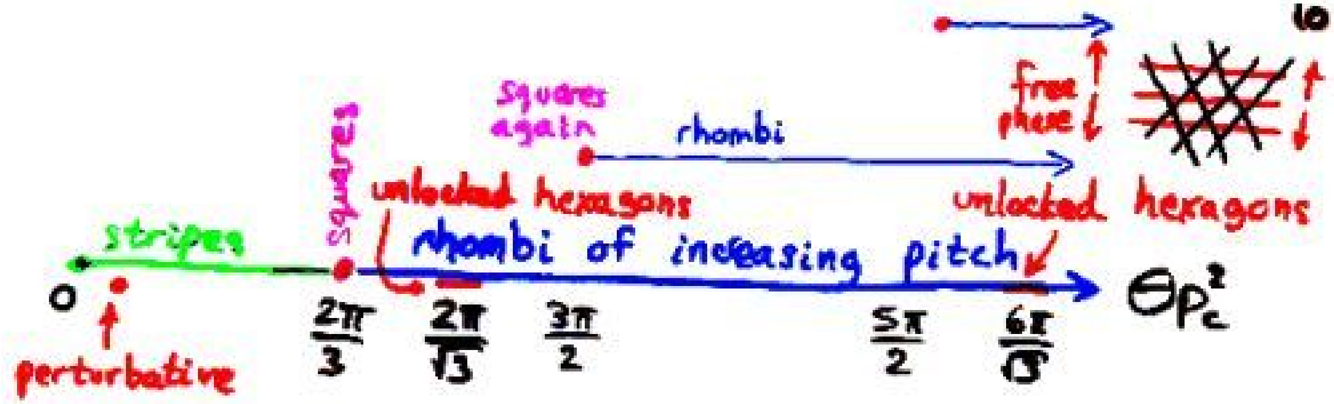
$$\mathcal{L}_{eff} \sim \frac{\chi_1}{2}\left[(\partial^2 + p_c^2)\phi\right]^2 + \frac{\chi_2}{2}\phi^2 + \frac{\chi_4}{4}\phi\star\phi\star\phi\star\phi$$

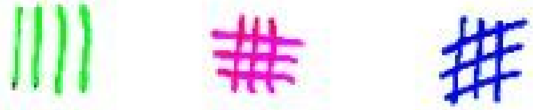
$$\langle\langle \mathcal{L}_{eff} \rangle\rangle \sim \chi_2\left(|A_1|^2 + |A_2|^2\right) + \frac{3}{2}\chi_4\left(|A_1|^2 + |A_2|^2\right)^2 + \chi_4\left(1 + 2\cos p_1\wedge p_2\right)|A_1 A_2|^2$$

spatial average

$$p_1\wedge p_2 = p_{1\mu}\theta^{\mu\nu}p_{2\nu} = \theta p_c^2 \sin \text{angle}(p_1, p_2)$$

$\chi_2 < 0 \quad \chi_4 > 0$, should be OK well inside ordered phase.





- perturbative regime has $\Theta p_c^2 \sim g \ll 1$

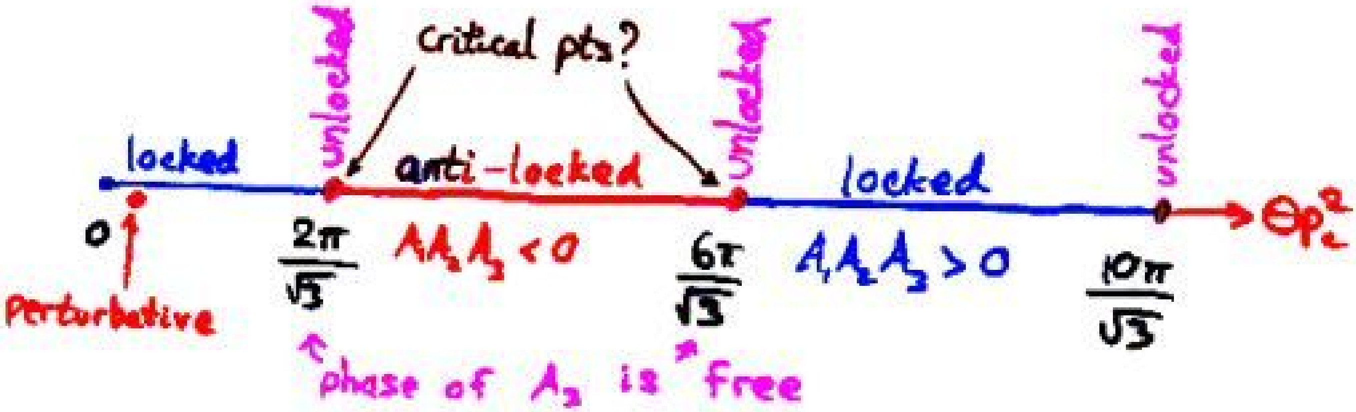
Further complications possible...

Breaking $\mathbb{Z}_2$ symmetry leads to hexagons:

$$\mathcal{L}_{eff} \sim (\text{kinetic}) + \frac{\chi_2}{2}\phi^2 + \frac{\chi_3}{3}\phi^{*3} + \frac{\chi_4}{4}\phi^{*4}$$

$$\langle\phi\rangle \sim A_1 e^{ip_1\cdot x} + A_2 e^{ip_2\cdot x} + A_3 e^{-i(p_1+p_2)\cdot x} + \text{c.c.}$$

$$|p_1| = |p_2| = |p_1+p_2| = p_c$$



Stripe phases are real:

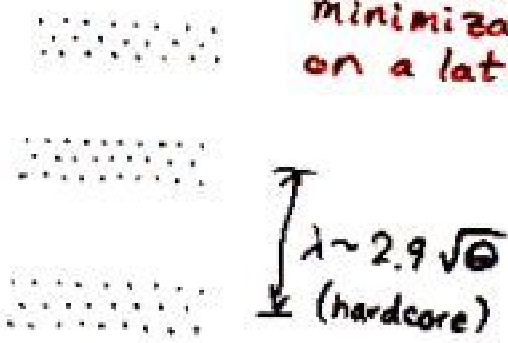
The likeliest laboratory testing ground for non-commutative field theory is quantum Hall systems. [Halperin-Lee-Read; Haldane; Read]

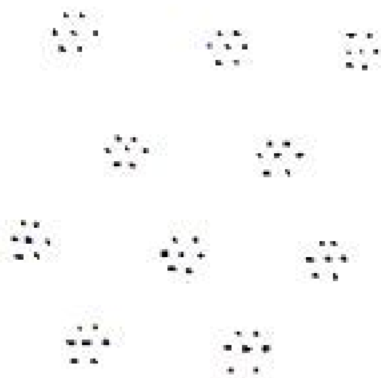
Suppose $\nu \in (n, n+1)$: $n-1$ filled Landau levels

$\theta \approx \frac{2n\hbar}{eB}$ is the scale of non-commutativity.

For large n, uniform FQH states give way to Charge Density Waves. [Fogler-Koulakov-Shklovskii, Chalker-Moessner]

numerical minimization on a lattice





$\nu \approx 1/2$

Experimentally: large anisotropy in charge transport

$\nu < 1/3$

Charge density $\rho \longrightarrow$ scalar field ϕ

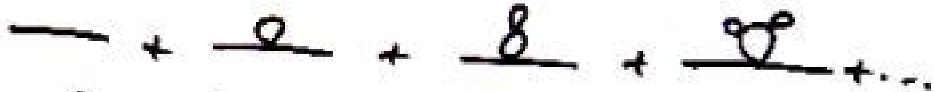
Non-commutative Landau theory?

Renormalizability again:

Critical point (CP) at $\theta\Lambda^2 = \infty$ guarantees existence of a renormalizable continuum theory, provided one can justify scaling as $\Lambda \to \infty$.

- $O(N)$ vector model simplification: bubble sum is exact in large N limit. So we only have to worry about

$\Gamma^{(2)}(p;\Lambda) =$



$$= p^2 + m^2 + O(g^2 + g'^2)$$

$$= p^2 + M^2 + \Sigma_{\text{non-planar}}(p)$$

$$M^2 = 2g^2 N \int^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + M^2 + \Sigma_{\text{non-planar}}(k)}$$

$$\Sigma_{\text{non-planar}}(p) = g'^2 N \int^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{e^{ik\wedge p}}{k^2 + M^2 + \Sigma_{\text{non-planar}}(k)}$$

CP is the 2^{nd} order transition in commutative $O(N)$ vector model:

exists for all $d > 2$.

$$\Sigma_{\text{non-planar}}(p) = M^2 f(p/M, \theta M^2) \quad \text{in } \Lambda \to \infty \text{ limit}$$

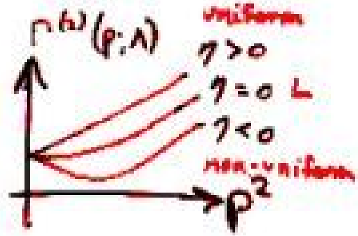
$$\sim \frac{1}{(\theta p)^{d-2}} \quad \text{for small } p.$$

Thus from self-consistency equations, conclude that

1. Scaling is justified
2. Continuum theory is non-trivial in all even $d > 2$ (!)

Lifshitz point: new scaling limits?

By definition, L-point is where transitions to uniform + non-uniform phases meet.



$$\Gamma^{(2)}(p;\Lambda) \underset{\text{small } p}{\sim} \eta p^2 + p^4 + \ldots$$

$$\underset{\substack{\text{self-consistent}\\ \text{one-loop}}}{=\!=} p^2 + m^2 + g^2 \int^{\Lambda} \frac{d^dk}{(2\pi)^d} \frac{1+\frac{1}{2}e^{ip\wedge k}}{\Gamma^{(2)}(k)}$$

Integral on RHS diverges, for $d \leq 4$, in IR when $\eta = 0$. So, no L-point at large N in $d=4$.

But in $d=6$ (or, possibly, in $d=4$ if N is finite), L-point <u>does</u> appear, at $\theta\Lambda^2 =$ finite.

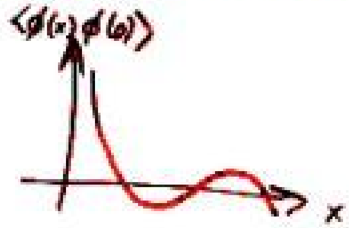
Now take $\theta \to 0$, $\Lambda \to \infty$ near L-point. Appear to find a continuum theory outside Ising universality class ($\Gamma^{(2)}(p) \sim p^4 + \text{const.}$) on <u>commutative</u> $\mathbb{R}^4$:

Non-commutativity without non-commutativity?

Conclusions

- Got a lot of mileage out of self-consistent one loop. Summed a series of superficially IR-divergent diagrams in a manifestly finite manner.

- $\Gamma^{(2)}(p) \sim \frac{1}{(\theta p)^2} + p^2 + \text{const}$ basically guarantees stripes because $\langle\phi(x)\phi(0)\rangle$ is oscillatory.



- Any stringy realization (eg unstable D-branes in type II) must involve large string loop effects at low p.

- Planar limit $(\theta \to \infty)$ provides obvious scaling limit. How about Lifshitz point?

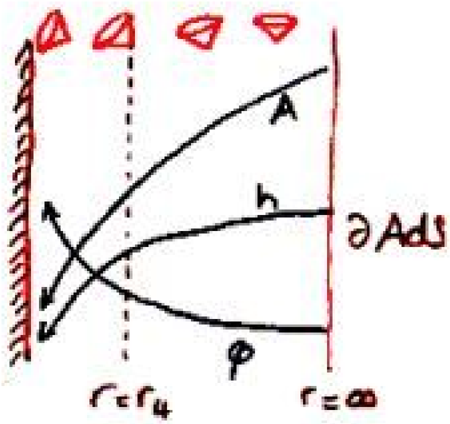
Brief Comment on Curvature Singularities:

[S.G., hep-th/0002160 ; Freedman-Gubser-Pilch-Warner; Lowe; Girardello-Petrini-Zaffaroni-Porratti; Pilch-Warner; Maldacena-Nunez] [Polchinski-Strassler]

Starting point: if a singular, asymptotically AdS spacetime is a limit of regular black holes, then it <u>should</u> admit a field theory dual.

$$ds^2 = e^{2A(r)}\left(-h(r)\,dt^2 + d\vec{x}^2\right) + \frac{dr^2}{h(r)} \qquad \varphi = \varphi(r)$$

$$\mathcal{L} = \frac{1}{4}R - \frac{1}{2}(\partial\varphi)^2 - V(\varphi)$$



$$A'' = -\frac{2}{3}\varphi'^2 \leq 0 \quad \text{so} \quad A' > 0$$

$h' \geq 0$ at horizon: $T_H = \dfrac{e^A h'(r_H)}{4\pi}$

$$G^r_r = \frac{3}{2}A'h' + 6A'^2 h = h\varphi'^2 - 2V(\varphi)$$

evaluate at $r = r_H$

$$\frac{3}{2}A'h' = -2V(\varphi) \geq 0$$

Empirical observation:

Requiring $V(\varphi)$ bounded above separates singularities corresponding to clear field theory pathologies from those which should have a field theory dual.

Examples: SUSY relevant def's of $\mathcal{N}=4$ SYM

$$\langle \mathrm{tr}\left[(X_1^2+X_2^2+X_3^2+X_4^2)-2(X_5^2+X_6^2)\right]\rangle \geq 0$$

if X_5, X_6 are massive

$$\leq 0$$

if $X_1,\dots,X_4$ are massive

$$\langle \mathrm{tr}\,\lambda\lambda\rangle \lesssim N m^3$$

if all chiral adjoints have a mass m.

Conclusion: $V(\varphi)$ bounded above to a good approximation separates good singularities in AdS from bad. Appears to work better than black hole justification warrants.